Chapter 4

Kinematics and relative motion

In general, kinematics is concerned with the description of moving bodies, without considering how the motion is produced. This typically involves the motion in terms of

- the position
$$\vec{r}(t),$$
- the velocity
$$\dot{\vec{r}}(t) \equiv \frac{d\vec{r}(t)}{dt} = \vec{v}(t),$$
- the acceleration
$$\ddot{\vec{r}}(t) \equiv \frac{d^2\vec{r}(t)}{dt^2} = \frac{d\vec{v}(t)}{dt} = \vec{a}(t).$$

For any given force, the motion is then determines from Newton's second law (**N2**). We have seen aspects of this already. Now we are going to take a closer look at the role of the observer. In particular, we are going to touch upon aspects relating to Newton's first law, which brings in the concept of an *inertial frame*. We know that, in a coordinate system in which there is no force acting on a body, it does not accelerate (relative to these coordinates). Let us now add the desire that different observers should infer the same physics. In order for that to be the case, the velocity of the observer (or the experimental device) must not impact on the dynamics. This gives us a hint of what we are after. In order to make the idea precise, it helps if we understand how measurements are affected by relative motion.

4.1 Frames of Reference

An event takes place at a point in space at a give time. For example, Man landed on the Moon on July 16th 1969 at the Sea of Tranquility; JFK was assassinated on 26th November 1963 in Dallas, Texas.

We can all agree that these specific events happened[1], but we may disagree on the exact coordinate location or time. We may not have synchronised our clocks and we may be using different rulers.

Events can be characterised both in time and space and for this we need a **reference frame**, which is a set of spatial coordinates plus a time coordinate. In general the motion of a body is also described by reference to a set of axes, for example the Cartesian coordinate system with a specific origin. Sometimes a set of axes, called a **frame** may itself be moving[2] and different observers may choose different points for their origin.

For example, someone sitting in a train may naturally choose to centre his (Cartesian) coordinate description of the world on his seat. Someone waiting on the platform as the train passes through the station would more likely centre her (Cartesian) coordinate description of the world on her position on the platform. If the passenger on the train throws an apple into the air as the train passes through the station, he will describe its motion according to his set of coordinates which travel with the train. The trajectory of the apple will look quite different to the person waiting on the platform as will her description of the trajectory in terms of the platform coordinates. How can we reconcile the two descriptions?

To begin to do this we need to define the concept of an inertial frame of reference.

An **inertial frame of reference** is a set of coordinate axes in which an isolated body under no forces (i.e., external influence) travels with constant velocity.

Experiment demonstrates that a frame fixed relative to the sun is inertial for all macroscopic practical purposes. However, a frame which rotates relative to another frame is **not** inertial because any motion in a curve must produce an acceleration and hence involves a non-constant velocity. Thus technically speaking a frame fixed in a laboratory on the surface of the rotating earth is **not** inertial, since the laboratory itself is rotating. However for most practical purposes the earth is spinning so slowly that the rotational complications are unimportant (the anticlockwise spinning of the water down the plug-hole in the northern hemisphere is a myth!). However for situations involving large distances or timescales, such as meteorology or intercontinental ballistic missiles, the effects of rotation do introduce additional external influences that must be taken into account.

Basically, in what follows we are always working with approximations.

4.2 Galilean Transformations

Let $\mathcal{O}$ be the origin of a coordinate frame $\mathcal{S}$ and $\mathcal{O}'$ be the origin of a coordinate frame $\mathcal{S}'$. Suppose $\mathcal{S}$ and $\mathcal{S}'$ have the same orientation[3], but that the position vector of $\mathcal{O}'$ relative to $\mathcal{O}$ be $\vec{R}(t)$.

We shall also assume that there is a universal time. In other words, in the example above say, both the commuter $\mathcal{O}$ and the passenger on the train $\mathcal{O}'$

[1] Assuming we are not conspiracy theorists.

[2] This will be important later!

[3] As we are dealing with non-accelerated motion, we can always do this.

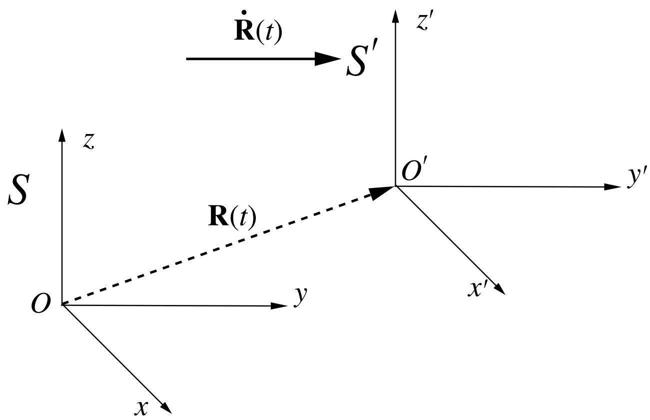

Figure 4.1: Frame $\mathcal{S}'$ moves with velocity $\dot{\vec{R}}(t)$ relative to $\mathcal{S}$. Note that, since $\mathcal{S}'$ moves with constant velocity we can always align the x-axis with that of the fixed coordinate system.

agree on the precise time at which the apple was thrown into the air and its time of flight. [4]

- Define $\vec{r}(t)$ to be the position vector of a point P in $\mathcal{S}$ relative to $\mathcal{O}$.
- Define $\vec{r}'(t)$ to be the position vector of a point P in $\mathcal{S}'$ relative to $\mathcal{O}'$.

Then it follows at time t from the law of the addition of vectors the relative positions of the point as seen by the differing observers at $\mathcal{O}$ and $\mathcal{O}'$ is instantaneously

$$\vec{r}(t) = \vec{R}(t) + \vec{r}'(t).$$

If we differentiate this with respect to time we have

$$\frac{d\vec{r}(t)}{dt} = \frac{d\vec{R}(t)}{dt} + \frac{d\vec{r}'(t)}{dt},$$

or replacing derivatives by $\vec{v}$ for velocities

$$\vec{v}_{P \text{ relative to } \mathcal{S}} = \vec{v}_{P \text{ relative to } \mathcal{S}'} + \vec{v}_{\mathcal{S}' \text{ relative to } \mathcal{S}}.$$

The last equation can be abbreviated to

$$\vec{v}_{P\mathcal{S}} = \vec{v}_{P\mathcal{S}'} + \vec{v}_{\mathcal{S}'\mathcal{S}}.$$

or (in slightly more natural notation)

$$\vec{v} = \vec{V} + \vec{v}'$$

[4]That a universal time exists irrespective of motion seems perfectly natural to us. However, it is also wrong. In the theory of relativity there is no universal measurement of time that is the same between inertial frames. If the speed of the train were near to the speed of light, the commuter and passenger would disagree as to the exact time the apple was thrown.

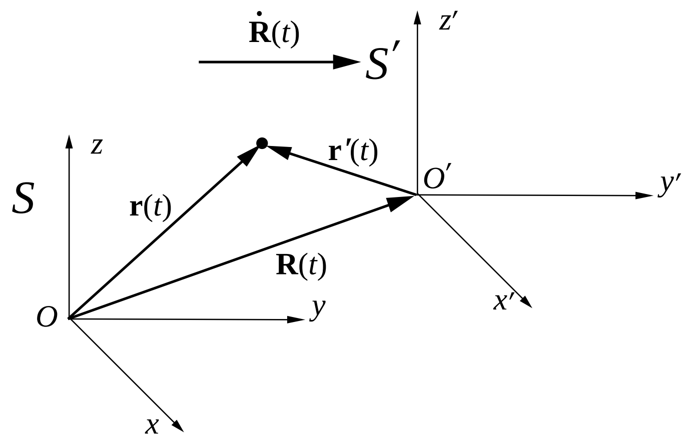

Figure 4.2: Frame $\mathcal{S}'$ moves with velocity $\dot{\vec{R}}(t)$ relative to $\mathcal{S}$.

This is the equation that links the relative velocities of the point as observed in each frame, when the two frames are moving with respect to each other. We see that velocities add, just as we expect from our everyday intuition.

Relative velocities are confusing. However the confusion can often be minimised by drawing a diagram[5]. Also, the following alternative notation might make things clearer. The crucial equation we use is of the simple form

$$\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}$$

In words, the velocity of A relative to C is equal to the sum of the velocity of A relative to B, plus the velocity of B relative to C. This follows from the equation relating displacement vectors:

$$\vec{AC} = \vec{AB} + \vec{BC}$$

or, in other notation,

$$\vec{r}_{AC} = \vec{r}_{AB} + \vec{r}_{BC}$$

The equation for relative velocities is simply the time derivative of these equations.

If $\vec{v}_{S'S}$ is a constant$=\vec{V}$, say then S' is also an inertial frame. The coordinates in each frame are them related by the following set of transformations, known as **Galilean transformations**:

$$\vec{R}(t) = \vec{V}t \Rightarrow \ \vec{r}(t) = \vec{V}t + \vec{r}'(t).$$

where we have assumed $\vec{r}(t) = \vec{r}'(t)$ at $t = 0$ (the two coordinate systems have a common origin at the initial time).

[5]In fact, you should always do this!

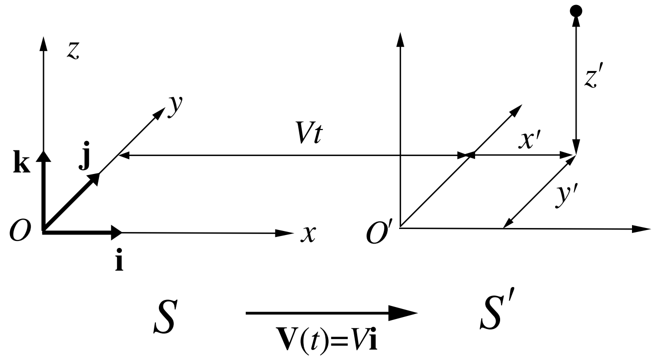

Figure 4.3: Frame $\mathcal{S}'$ moves with velocity $V\hat{\imath}$ relative to $\mathcal{S}$.

Example 1: Galilean transformation

Suppose $\mathcal{S}$ and $\mathcal{S}'$ have a uniform speed $\vec{V} = V\hat{\imath}$ relative to one another, with the respective axes aligned (see figure 4.3). The Galilean transformation between coordinates (x, y, z) relative to $\mathcal{O}$ in $\mathcal{S}$ and the coordinates (x', y', z') relative to $\mathcal{O}'$ in $\mathcal{S}'$ is then given by the set of vector equation

$$x\hat{\imath} + y\hat{\jmath} + z\hat{k} = Vt\hat{\imath} + x'\hat{\imath} + y'\hat{\jmath} + z'\vec{k}$$

or in component form

$$\begin{aligned} x &= Vt + x' \\ y &= y' \\ z &= z' \end{aligned}$$

Note that because the axes were aligned an identical set of basis vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ could be used in both frames.

Example 2: Relative motion in a straight line

A train moves at 30 ms^{-1} (about 70 mph). A boy runs to the front of the train at 2.5 ms^{-1} (about 5 mph). What is the velocity of the boy relative to the track? Let

- $\mathcal{S}$ be a coordinate frame that is stationary with respect to the track,
- $\mathcal{S}'$ be a coordinate frame that is stationary with respect to the train.

with the corresponding x and x' axes aligned in the same direction along the track/train and pointing towards the front of the train. Then let

- $\vec{v}_{PS}$ = velocity of the boy P relative to a stationary observer on the track.

- $\vec{v}_{PS'}$ = velocity of the boy P relative to a stationary observer on the train = 2.5 $\hat{\imath}$ ms^{-1}.

- $\vec{v}_{S'S}$ = velocity of the stationary observer on the train relative to the stationary observer on the track = 30 $\hat{\imath}$ ms^{-1}.

From the relative velocity formula

$$\vec{v}_{PS} = \vec{v}_{PS'} + \vec{v}_{S'S},$$

we deduce that

$$\vec{v}_{PS} = 30\ \hat{\imath} + 2.5\ \hat{\imath} = 32.5\ \hat{\imath}\ \mathrm{ms}^{-1}.$$

If the boy turns round and runs at the same speed towards the back of the train, his new velocity relative to the train is $\vec{v}_{PS'} = -2.5\ \hat{\imath}$ ms^{-1}. Consequently his velocity relative to the stationary track observer is then

$$\vec{v}_{PS} = 30\ \hat{\imath} - 2.5\ \hat{\imath} = 27.5\ \hat{\imath}\ \mathrm{ms}^{-1}.$$

Example 3: Relative motion of particles

Two particles have position vectors given by

$$\vec{r}_1(t) = 4t\hat{\imath} + 5t^2\hat{\jmath} - 2\sin\pi t\hat{k}$$

and

$$\vec{r}_2(t) = 2t\hat{\imath} - 5t\hat{\jmath} + \cos\pi t\hat{k}.$$

Find the relative velocity and the relative acceleration of the second particle with respect to the first at $t = 1$.
First define the relevant reference frames:

- Frame S: the stationary observer.
- Frame P_1: the frame of particle 1.
- Frame P_2: the frame of particle 2.

We require $\vec{v}_{P_2P_1}$, which is given by

$$\vec{v}_{P_2P_1} = \vec{v}_{P_2S} + \vec{v}_{SP_1} = \vec{v}_{P_2S} - \vec{v}_{P_1S}$$

The velocities of the particles are:

$$\vec{v}_{P_1S} = \frac{d}{dt}(\vec{r}_{P_1S}) = \frac{d}{dt}(\vec{r}_1) = \dot{\vec{r}}_1(t) = 4\hat{\imath} + 10t\hat{\jmath} - 2\pi\cos\pi t\hat{k}$$

$$\vec{v}_{P_2S} = \frac{d}{dt}(\vec{r}_{P_2S}) = \frac{d}{dt}(\vec{r}_2) = \dot{\vec{r}}_2(t) = 2\hat{\imath} - 5\hat{\jmath} - \pi\sin\pi t\hat{k}$$

The velocity of 2 relative to 1 is:

$$\vec{v}_{P_2P_1} = \dot{\vec{r}}_2(t) - \dot{\vec{r}}_1(t) = -2\hat{i} - (5+10t)\vec{j} + (2\pi\cos\pi t - \pi\sin\pi t)\,\hat{k}.$$

So at $t = 1$ we have

$$\vec{v}_{P_2P_1}(t=1) = \dot{\vec{r}}_2(1) - \dot{\vec{r}}_1(1) = -2\hat{i} - 15\vec{j} - 2\pi\hat{k}.$$

By a similar logic the acceleration of 2 relative to 1 is

$$\vec{a}_{P_2P_1} = \ddot{\vec{r}}_2(t) - \ddot{\vec{r}}_1(t) = 0\hat{i} - 10\vec{j} + \left(-2\pi^2\sin\pi t - \pi^2\cos\pi t\right)\hat{k}.$$

so at $t = 1$ we have

$$\vec{a}_{P_2P_1}(t=1) = \ddot{\vec{r}}_2(1) - \ddot{\vec{r}}_1(1) = -10\hat{j} + \pi^2\hat{k}.$$

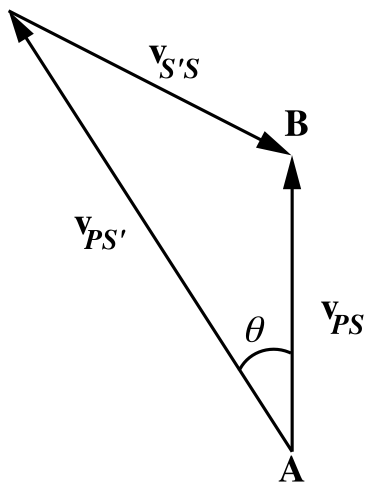

Figure 4.4: Wind velocities

Example 4: Airspeed versus ground speed

An aeroplane P is to fly from a point A to an airfield B which is 300 km due north of A. The plane can fly at a speed of 150 km h^{-1} relative to the air. A wind of speed 30 km h^{-1} blows from the north-west. Find the direction the plane should be pointing in if it is to reach B and its flight time.
Let

- $\mathcal{S}$ be a coordinate frame that is stationary with respect to the ground (A and B),
- $\mathcal{S}'$ be a coordinate frame that is stationary with respect to the wind.
- us take basis vectors $\hat{i}$ pointing east and $\hat{j}$ pointing north.

Then we have

- $\vec{v}_{P\mathcal{S}}$ = velocity of the plane P relative to a stationary observer on the ground.
- $\vec{v}_{P\mathcal{S}'}$ = velocity of the plane P relative to the wind.
- $\vec{v}_{\mathcal{S}'\mathcal{S}}$ = velocity of the wind relative to the stationary observer on the ground.

Hence we know that:

- $\vec{v}_{\mathcal{S}'\mathcal{S}} = 30 \times \frac{1}{\sqrt{2}}(\hat{i} - \hat{j})$ km h^{-1};
- $|\vec{v}_{P\mathcal{S}'}| = 150$ km h^{-1};
- if the plane is to fly from A to B, $\vec{v}_{P\mathcal{S}} = V\hat{j}$ for some V we must find.

The best way to solve this is to draw a vector diagram corresponding to the relative velocity equation (see figure 4.4)

$$\vec{v}_{P\mathcal{S}} = \vec{v}_{P\mathcal{S}'} + \vec{v}_{\mathcal{S}'\mathcal{S}}.$$

There are two unknowns in the problem, the bearing the plane must take and the speed of the plane relative to the ground. A short study of the vector triangle in the diagram shows the following.

- The unknown angle θ is the bearing (west of north) that the plane must fly, so that its velocity relative to the ground is due north. It is given by the sine rule:

$$\frac{150}{\sin 135} = \frac{30}{\sin\theta} \qquad \Rightarrow \qquad \sin\theta = \frac{1}{5\sqrt{2}} \qquad \Rightarrow \qquad \theta = \arcsin\left(\frac{1}{5\sqrt{2}}\right)$$

  or about $8^o\ 8'$ west of north. ($1' = 1$ minute $= 1/60$-th of a degree).

- The speed of the aircraft relative to the ground is also given by the sine rule

$$\frac{|\vec{v}_{PS}|}{\sin(180 - 135 - \theta)} = \frac{150}{\sin 135}$$

  Hence

$$|\vec{v}_{PS}| = 127.3 \text{km h}^{-1}$$

  and the flight time from A to B is $300/127.28 = 2$ hrs 21 min.

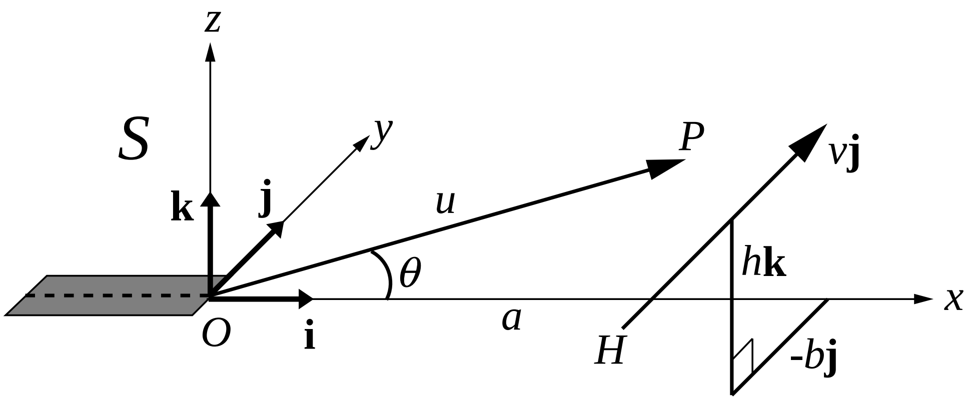

Figure 4.5: Geometry of flight paths of plane P and helicopter H.

Example 5: Closest approach

An aircraft P takes off from a runway that is aligned from west to east parallel to $\hat{\imath}$. The speed of the aircraft (relative to the stationary air) is u and after takeoff it flies due east, climbing at angle θ to the horizontal. Meanwhile a helicopter H is flying due north at speed $v\hat{\jmath}$ at a constant height $h\hat{k}$ (with $h > 0$ and $\hat{k}$ pointing upwards!).
At the moment the aircraft takes off at $t = 0$, the helicopter is at the point $a\hat{\imath} - b\hat{\jmath} + h\hat{k}$ ($a, b > 0$). Find the time at which the two machines are closest and distance between them at that point.
Clearly it makes sense to draw a diagram (figure 4.5).
Start by defining frames. Let:

- $\mathcal{S}$ be a coordinate frame that is stationary with respect to the end of the runway,
- $\mathcal{S}'$ be a coordinate frame that is stationary with respect to the helicopter.

Recall the relationship between position vectors

$$\vec{r}(t) = \vec{R}(t) + \vec{r}'(t).$$

It would be useful to calculate $\vec{r}'(t)$. Note that

- the position of the plane relative to the end of the runway at $t = 0$ is $\vec{r}(0) = \vec{0}$.
- the position of the helicopter relative to the end of the runway at $t = 0$ is

$$\vec{R}(0) = a\hat{\imath} - b\hat{\jmath} + h\hat{k}.$$

- the position of the plane at $t = 0$ relative to the helicopter at $t = 0$ is

$$\vec{r}'(0) = \vec{r}(0) - \vec{R}(0) = -a\hat{\imath} + b\hat{\jmath} - h\hat{k}.$$

The corresponding relative velocities are then given by

$$\vec{v}_{PS} = \vec{v}_{PS'} + \vec{v}_{S'S}.$$

and

- the velocity of the plane relative to the end of the runway is $\vec{v}_{PS} = u\cos\theta\hat{i} + u\sin\theta\hat{k}$.
- the velocity of the helicopter relative to the end of the runway is $\vec{v}_{S'S} = v\hat{j}$.
- the velocity of the plane relative to the helicopter is

$$\vec{v}_{PS'} = \vec{v}_{PS} - \vec{v}_{S'S} = u\cos\theta\hat{i} - v\hat{j} + u\sin\theta\hat{k}.$$

Hence we can write down a vector equation for the position of the plane relative to the helicopter. It is

$$\vec{r}'(t) = \vec{r}'(0) + \vec{v}_{PS'}t = -a\hat{i} + b\vec{j} - h\hat{k} + t\left(u\cos\theta\hat{i} - v\hat{j} + u\sin\theta\hat{k}\right)$$

which is nothing more than the equation of a line in 3D.
Now the distance of closest approach will occur when $|\vec{r}'(t)|$ or analogously $|\vec{r}'(t)|^2$ is a minimum.
This occurs when

$$\frac{d}{dt}|\vec{r}'(t)|^2 = \frac{d}{dt}\left(\vec{r}'(t)\cdot\vec{r}'(t)\right) = \frac{d\vec{r}'(t)}{dt}\cdot\vec{r}'(t) + \vec{r}'(t)\cdot\frac{d\vec{r}'(t)}{dt} = 2\vec{r}'(t)\cdot\vec{v}_{PS'} = 0$$

Geometrically this is when $\vec{v}_{PS'}$ and $\vec{r}'(t)$ are perpendicular.
Hence we have

$$\begin{aligned} 0 &= \left(u\cos\theta\hat{i} - v\hat{j} + u\sin\theta\hat{k}\right)\cdot\left(-a\hat{i} + b\hat{j} - h\hat{k} + t\left(u\cos\theta\hat{i} - v\hat{j} + u\sin\theta\hat{k}\right)\right) \\ \Rightarrow \qquad t_{\min} &= -\frac{\left(u\cos\theta\hat{i} - v\hat{j} + u\sin\theta\hat{k}\right)\cdot\left(-a\hat{i} + b\hat{j} - h\hat{k}\right)}{|u\cos\theta\hat{i} - v\hat{j} + u\sin\theta\hat{k}|^2} \\ &= \frac{au\cos\theta + bv + hu\sin\theta}{u^2+v^2}. \end{aligned}$$

The closest distance between the two machines is then

$$|\vec{r}'(t_{\min})| = \sqrt{\vec{r}'(t_{\min})\cdot\vec{r}'(t_{\min})} = \text{exercise for student!}$$

Can this ever be zero? In practice the relative speeds of aircraft are so fast (up to over 1000 mph $\approx$ 450 ms^{-1} for civil aircraft) that air traffic controllers try to allow aircraft never to come within a certain non-zero distance of each other. Usually over the Atlantic routes into the UK this is 1000 ft for vertical separation and a few miles for horizontal separation—yes, the Imperial-measures dominate air traffic control!