Chapter 9

Systems, solid bodies and moments

So far, we have only considered the motion of a single body. If our goal is to understand everything in the Universe, this is a bit limiting. So let us take a small step forwards and consider the dynamics of N interacting particles.

9.1 Many particles

The first thing that we do is put a label $i = 1, ..., N$ on everything[1]. The i^{th} particle has mass m_i, position $\vec{r}_i$ and momentum $\vec{p}_i = m_i\dot{\vec{r}}_i$.

Now **N2** states that, for each particle, $i = 1, 2, \cdots N$,

$$m_i\ddot{\vec{r}}_i(t) = \dot{\vec{p}}_i = \vec{F}_i.$$

where $\vec{F}_i$ is the force acting on the i^{th} particle. The new thing is that, for a system of particles the force $\vec{F}_i$ can be split into two parts: an external force $\vec{F}^{\text{ext}}$ (for example, if the whole system sits in a gravitational field) and a force due to the presence of the other particles. We write

$$\vec{F}_i = \vec{F}_i^{ext} + \sum_{j \neq i}^{N} \vec{F}_{ij}$$

where $\vec{F}_{ij}$ is the force on particle i due to the presence of particle j. This allows us to provide a more precise definition of Newton's third law. Recall that *every reaction has an equal and opposite reaction.* This means,

$$\vec{F}_{ij} = -\vec{F}_{ji}$$

We can (obviously) add all N equations to get

$$\sum_{i=1}^{N} m_i\ddot{\vec{r}}_i(t) = \sum_{i=1}^{N} \vec{F}_i.$$

[1]In a sense, this is the first step towards the index notation you will later see for vectors and tensors (especially in relativity). However, then we will be labelling vector components, while we are now labelling particles.

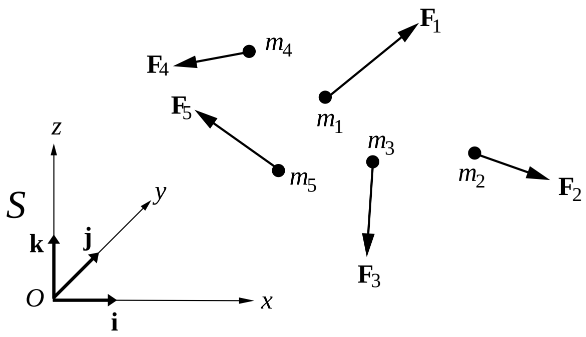

Figure 9.1: They're absolutely everywhere!

This is the equation of motion for the combined system of particles. However, it is more instructive to express the collective motion in terms of the **centre of mass**. Thus, we introduce the total mass:

$$M = \sum_{i=1}^{N} m_i$$

and the centre of mass (generalising the previous definition in the natural way)

$$\vec{R}(t) \equiv \frac{1}{M} \sum_{i=1}^{N} m_i \vec{r}_i(t)$$

The total momentum of the system, $\vec{P}$, can then be written entirely in terms of the centre of mass motion,

$$\vec{P} = \sum_{i=1}^{N} \vec{p}_i = M\dot{\vec{R}}$$

Let us now ask how the centre of mass moves. We have

$$\dot{\vec{P}} = \sum_{i=1}^{N} \dot{\vec{p}}_i = \sum_{i=1}^{N} \left(\vec{F}_i^{\text{ext}} + \sum_{j \neq i} \vec{F}_{ij} \right) = \sum_{i=1}^{N} \vec{F}_i^{\text{ext}} + \sum_{i<j} \left(\vec{F}_{ij} + \vec{F}_{ji} \right)$$

where the last term vanishes because of **N3**. That is, we are left with

$$\dot{\vec{P}} = \sum_{i=1}^{N} \vec{F}_i^{\text{ext}}$$

This is an important result. It tells us that if we just want to know the motion of the centre of mass of a system of particles, then **only the external forces count**. If you throw a wriggling, squealing cat then its internal forces $\vec{F}_{ij}$ can change its orientation, but they can do nothing to change the path of its centre of mass. That is dictated by gravity alone.

It is hard to overstate the importance of this result. Without it, the whole Newtonian framework for mechanics would come crashing down. After all, nothing that we really describe is truly a point particle. Certainly not a planet or a cat, but even something as simple as an electron has an internal spin. Yet none of these details matter because everything, regardless of the details, acts as a point particle if we just focus on the position of its centre of mass.

The centre of mass can be regarded as the **average position** of the mass. Sometimes it is convenient to consider an extended body many up of a collection of particles as but one particle with the same total mass as the system, but located at the centre of mass[2]. The idea is that the system of particles acts dynamically as if all the mass were located at the centre of mass. This is actually the idea underlying the abstraction of extended bodies into point particles that we have already been using. The centre of mass is sometimes called the *centre of gravity*[3], and represents the point through which gravity acts in a uniform gravitational field.

For future reference, it is also worth noting that if the system of particles is static, then $\ddot{\vec{R}}(t) = \mathbf{0}$ and we must have

$$\sum_{i=1}^{N} \vec{F}_i = 0.$$

This is a necessary (but not sufficient) condition for the static equilibrium for a system of particles. The system may be moving with a constant velocity, but we can always make the situation stationary by viewing from an inertial frame that moves at the same velocity as the centre of mass. This is (again) the *centre of mass frame*.

9.2 Moments

If two forces of equal magnitude act in opposite directions, then the sum of the forces vanishes. However, if these forces do not act at the same point on a body, they exert a twist, or **torque** or **moment** about an axis.

This torque will lead to a rotational acceleration of the body. Experimentally it is found that if the force is applied at a greater distance from the axis of rotation, then the torque/moment will be greater. In fact the moment can be determined by experiment as follows.

Consider particle i of constant mass m_i and suppose we decide to take the moments about the origin of the coordinate system $\mathcal{O}$ and a (resultant) force $\vec{F}_i$ is acting on a particle located at $\vec{r}_i(t)$. The moment $\vec{M}_i$ of the force about $\mathcal{O}$ is the vector product

$$\vec{M}_i(t) \equiv \vec{r}_i(t) \times \vec{F}_i(t).$$

[2]If the distribution of masses is continuous, the individual masses of particles in the definition are replaced with a local mass density and the sums are replaced with integrals over the volume of the body.

[3]Note that the centre of mass is only equivalent to the centre of gravity in a uniform gravitational field.

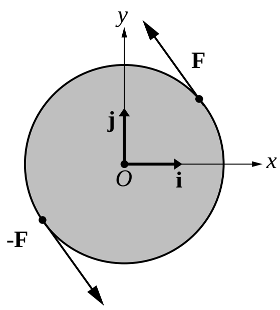

Figure 9.2: Zero-sum forces on a disc implies no translational acceleration, but there is rotational acceleration about the origin.

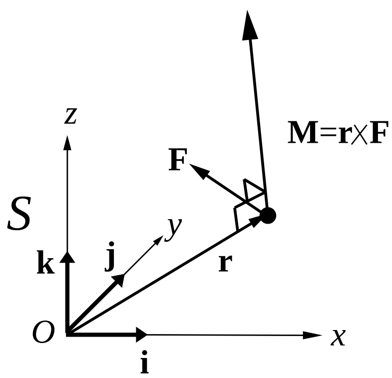

Figure 9.3: Definition of moment.

Combining the vector moment with **N2** for particle i, we obtain[4]

$$\vec{M}_i(t) \equiv \vec{r}_i(t) \times \vec{F}_i(t) = \vec{r}_i(t) \times m_i\ddot{\vec{r}}_i(t) = m_i\vec{r}_i(t) \times \ddot{\vec{r}}_i(t)$$

If you have studied mechanics before you can now see why the magnitude of a moment is said to be "force times *perpendicular* distance from the axis". The vector product ensures that only the component of the force normal to $\vec{r}_i(t)$ appears in the magnitude of the component. The definition of the vector product combined with a simple sketch shows that in magnitude, these two statements are equivalent.

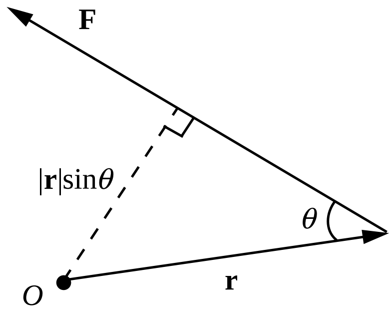

Figure 9.4: Magnitude of moment.

$$\vec{M} = \vec{r} \times \vec{F} \Rightarrow |\vec{M}| = |\vec{r}| \cdot |\vec{F}| \sin\theta \Rightarrow |\vec{M}| = \underbrace{|\vec{r}| \sin\theta}_{\perp \text{ distance}} \cdot |\vec{F}|$$

As with forces for the system of N particles, we may add the moments of individual particles about $\mathcal{O}$ to give the total moment

$$\vec{M} = \sum_{i=1}^{N} \vec{M}_i = \sum_{i=1}^{N} \vec{r}_i(t) \times \vec{F}_i(t)$$

Note that if the body is in equilibrium, each individual particle has no acceleration $\ddot{\vec{r}}_i(t) = \mathbf{0}$ and we obtain

$$\vec{M} = \sum_{i=1}^{N} \vec{r}_i(t) \times \vec{F}_i(t) = \mathbf{0}$$

That is, a system in equilibrium does not experience a torque.

9.3 Angular Momentum and Torque

The angular momentum of a particle of mass m at position $\vec{r}(t)$ is defined to be

$$\vec{L} \equiv m(\vec{r} \times \dot{\vec{r}}) = \vec{r} \times \vec{p}$$

[4]Note that the moment is a vector quantity and that the $\vec{r}$ comes before the $\vec{F}$ (or the $\ddot{\vec{r}}_i$) in the vector product.

With this definition we see that

$$\frac{d\vec{L}}{dt} = \frac{d(\vec{r} \times m\dot{\vec{r}})}{dt} = (\dot{\vec{r}} \times m\dot{\vec{r}}) + \left(\vec{r} \times \frac{d(m\dot{\vec{r}})}{dt}\right) = \mathbf{0} + \left(\vec{r} \times \frac{d\vec{p}}{dt}\right)$$

From **N2**, the rate of change of momentum is nothing other than the force $\vec{F}$ on the body. Hence we have

$$\frac{d\vec{L}}{dt} = \left(\vec{r} \times \vec{F}\right) = \vec{M}$$

which, in words, states that the moment of a force, or torque, about the origin O of a coordinate system is equal to the rate of change of the angular momentum. This is the rotational analogue of the translational form of Newton's second law:

$$\frac{d\vec{P}}{dt} = \vec{F}$$

We will return to the angular momentum later.

9.4 Energy

It is also useful to consider the energy of a system of particles. The total kinetic energy is

$$T = \frac{1}{2}\sum_i m_i \dot{\vec{r}} \cdot \dot{\vec{r}}$$

Suppose we consider the motion relative to the centre of mass, such that

$$\vec{r}_i = \vec{R} + \vec{y}_i$$

then

$$\sum_i m_i \vec{r}_i = M\vec{R} \ \Rightarrow \ \sum_i m\vec{y}_i = 0$$

This means that the kinetic energy can be written

$$\begin{aligned} T = \frac{1}{2}\sum_i m_i \left(\dot{\vec{R}} + \dot{\vec{y}}_i\right)^2 \\ = \frac{1}{2}\sum_i m_i \dot{\vec{R}}^2 + \dot{\vec{R}} \cdot \underbrace{\sum_i m_i \dot{\vec{y}}_i}_{=0} + \frac{1}{2}\sum_i m_i \dot{\vec{y}}_i^2 \\ = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2}\sum_i m_i \dot{\vec{y}}_i^2 \end{aligned}$$

Basically, the kinetic energy splits up into the kinetic energy of the centre of mass, together with the kinetic energy of the particles moving around the centre of mass.

9.5 Rigid bodies

So far, we have only discussed "particles", objects with no extended size. But what happens to more complicated objects that can twist and turn as they move? The simplest example is a rigid body. This is a collection of N particles, constrained so that the relative distance between any two points, i and j, is fixed:

$$|\vec{r}_i - \vec{r}_j| = \text{fixed}$$

A rigid body can undergo only two types of motion: its centre of mass can move and it can rotate. We will focus on the rotation.

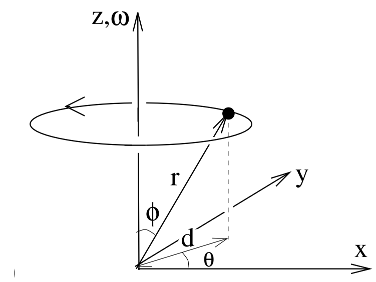

Figure 9.5: A rotating particle.

Suppose we fix some point in the rigid body and consider rotation about this point. To describe these rotations, we need the concept of angular velocity. We begin by considering a single particle which is rotating around the z-axis. The position and velocity of the particle are given by (in cylindrical coordinates)

$$\vec{r} = (d\cos\theta, d\sin\theta, z) \ \Rightarrow \ \dot{\vec{r}} = (-d\dot{\theta}\sin\theta, d\dot{\theta}\cos\theta, 0)$$

We can write this by introducing a new vector $\vec{\omega} = \dot{\theta}\hat{e}_z$:

$$\dot{\vec{r}} = \vec{\omega} \times \vec{r}$$

The vector $\vec{\omega}$ is called the angular velocity. In general we can write $\vec{\omega} = \omega\hat{n}$. Here the magnitude, $\omega = \dot{\theta}$, is the angular speed of rotation, while the unit vector $\hat{n}$ points along the axis of rotation, defined in a right-handed sense[5].

The speed of the particle is then given by

$$v = |\dot{\vec{r}}| = r\omega\sin\phi = d\omega$$

where

$$d = |\hat{n} \times \vec{r}| = r\sin\phi$$

is the perpendicular distance to the axis of rotation. Finally, we will also need an expression for the kinetic energy of this particle as it rotates about the $\hat{n}$ axis through the origin; it is

$$T = \frac{1}{2}m\dot{\vec{r}}\cdot\dot{\vec{r}} = \frac{1}{2}m(\vec{\omega}\times\vec{r})\cdot(\vec{\omega}\times\vec{r}) = \frac{1}{2}md^2\omega^2$$

[5]Curl the fingers of your right hand in the direction of rotation: your thumb points in the direction of $\vec{\omega}$.

9.6 The moment of inertia

Let us now consider a system with N particles which make up a rigid body. The fact that the body is rigid mean that all particles rotate with the same velocity:

$$\dot{\vec{r}}_i = \vec{\omega} \times \vec{r}_i$$

This ensure that the relative distance between the particles remains fixed

$$\begin{aligned}\frac{d}{dt}|\vec{r}_i - \vec{r}_j|^2 &= 2\left(\dot{\vec{r}}_i - \dot{\vec{r}}_j\right) \cdot (\vec{r}_i - \vec{r}_j) \\ &= 2\left[\vec{\omega} \times (\vec{r}_i - \vec{r}_j)\right] \cdot (\vec{r}_i - \vec{r}_j) = 0\end{aligned}$$

We can now write the kinetic energy of a rigid body as

$$T = \frac{1}{2}\sum_i m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i = \frac{1}{2} I \omega^2$$

where we have introduced

$$I = \sum_{i=1}^{N} m_i d_i^2$$

where d_i is the distance of each particle away from the rotational axis. This is the *moment of inertia*. It is worth noting the similarity between the rotational kinetic energy and the translational kinetic energy ($Mv^2/2$) – the moment of inertia is to rotation what the mass is to translation.

The moment of inertia also plays a role in the angular momentum. We have

$$\vec{L} = \sum_i m_i \vec{r}_i \times \dot{\vec{r}}_i = \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$$

Introducing $\vec{\omega} = \omega \hat{n}$ (with $\hat{n}$ a unit vector along the rotational axis), we have

$$\begin{aligned}\vec{L} \cdot \hat{n} &= \omega \sum_i m_i \left[\vec{r}_i \times (\hat{n} \times \vec{r}_i)\right] \cdot \hat{n} \\ &= \omega \sum_i m_i (\vec{r}_i \times \hat{n}) \cdot (\vec{r}_i \times \hat{n}) = I\omega\end{aligned}$$

We already know that acting with a torque $\vec{\tau}$ changes the angular momentum:

$$\dot{\vec{L}} = \vec{\tau}$$

For a rigid body, we see that if the torque is in the direction of the angular momentum (so that $\vec{\tau} = \tau \hat{n}$) we have

$$I\dot{\omega} = \tau$$

In practice, we calculate the moment of inertia by replacing the individual particles with a continuous density distribution $\rho(\vec{r})$. The total mass is given by the volume integral

$$M = \int \rho(\vec{r}) dV$$

and the moment of inertia follows as

$$I = \int \rho(\vec{r}) x_\perp^2 \, dV$$

where $x_\perp$ is the perpendicular distance from the point $\vec{r}$ to the rotation axis. For example, for a uniform density sphere (with radius a) we would have

$$I = \frac{2}{5} M a^2$$

9.7 Why the two-body problem is really a one-body problem

Solving the general problem of the dynamics of N mutually interacting particles is (very) hard. However, when there are no external forces present, the case of two particles (labelled 1 and 2) actually reduces to the kind of one particle problem we are familiar with.

We have already defined the centre of mass

$$M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2$$

Let us also introduce the relative separation

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

which allows us to write

$$\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r} \quad \text{and} \quad \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$$

As long as no external forces act on the system, we have $\ddot{\vec{R}} = 0$. Meanwhile, the relative motion is dictated by

$$\ddot{\vec{r}} = \ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = \frac{1}{m_1}\vec{F}_{12} - \frac{1}{m_2}\vec{F}_{21} = \frac{m_1 + m_2}{m_1 m_2}\vec{F}_{12}$$

after using **N3**. The equation of motion can now be written

$$\mu\ddot{\vec{r}} = \vec{F}_{12}$$

where μ is the *reduced mass*

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

This is nice. It means that we have already solved the problem of two mutually interacting particles because their centre of mass motion is trivial, while their relative separation reduces to the kind of problem we have already seen.

In the limit where one of the particles involved is very heavy, say $m_2 \gg m_1$, then $\mu \approx m_1$ and the heavy object remains essentially fixed, with the lighter object orbiting around it. For example, the centre of mass of the Earth and Sun is very close to the centre of the Sun. Even for the Earth and moon, the centre of mass is 1000 miles below the surface of the Earth.