Chapter 6

Conservation Laws

Let us now introduce the (hugely important) idea of conservation laws. As the name suggests, a conserved quantity is one that does not change throughout the motion of a system. Conservation laws can often be used as an alternative formulation of dynamical problems—they allow us to introduce suitable variables that remain fixed as the system evolves[1]. We will deal with two major conservation properties, relating to **energy** and **momentum**. These conservation laws extend the range of mechanical problems we can attack. We shall also use the concept of energy to discuss the *stability* of an static situation: if a static system is displaced slightly, will it return to the same static state, or will there be a dramatic (or even catastrophic) change in its structure. We have already alluded to these properties in the context of potential motion.

6.1 Energy

The history of the concept of energy is long and varied. We might easily be able to mathematically define an quantity that we could call the energy, and have an intuitive idea of what it means. However historically there were great issues with the concept of physical mechanisms that could convert, say, thermal energy into energy of motion. Were they the same "thing"? How was this achieved? Did they have the same origin? What was the mechanical equivalent of heat? This was not some arcane academic debate: it underlay the whole of the industrial revolution in the UK , which was largely powered by the conversion of the thermal energy of steam into the kinetic energy of pumps/wheels/looms etc. and so the industrial revolution of the whole world.

Originally heat energy was assumed to be a fluid (the caloric theory) that was transferred between bodies and could neither be created nor destroyed. This theory was demolished by the major experimental advances of James Joule who showed that heat can be produced by motion. For this reason the unit of energy is named the "joule". A joule can be defined in many equivalent ways, for example:

- The energy expended by a force of 1N moving an object a distance of 1 m in the direction of the force.

[1] It always helps to be able to with constant quantities.

- The energy expended when a current of 1 amp to pass through an (ohmic) electrical resistance of 1 ohm.

We will not be concerned with the philosophical issues of energy. Rather, we will use it as a mathematical tool.

6.2 Work and Energy

It is natural to start by introducing the idea of energy from Newton's laws of motion. We know that Newton's second law of motion states that the rate of change of momentum $\vec{p}$ with time is equal to the force applied $\vec{F}$:

$$\vec{F} = \frac{d\vec{p}}{dt}.$$

Here we shall consider this body to be of *constant* mass m and at time t at position $\vec{r}(t)$ relative to a certain intertial frame. Using the definition $\vec{p} = m\dot{\vec{r}}(t)$, we can then write **N2** as:

$$\vec{F} = m\ddot{\vec{r}}(t).$$

We can dot-product both sides with the velocity $\dot{\vec{r}}(t)$.

$$\begin{aligned} \vec{F} \cdot \dot{\vec{r}}(t) &= m \left(\ddot{\vec{r}}(t) \cdot \dot{\vec{r}}(t) \right) \\ &= \frac{1}{2} m \frac{d}{dt} \left(\dot{\vec{r}}(t) \cdot \dot{\vec{r}}(t) \right) \\ &= \frac{d}{dt} \left(\frac{1}{2} m |\dot{\vec{r}}(t)|^2 \right). \end{aligned} \tag{6.1}$$

The kinetic energy T of a particle of constant mass m and velocity $\dot{\vec{r}}(t)$ relative to an inertial frame is then defined to be:

$$T \equiv \frac{1}{2} m |\dot{\vec{r}}(t)|^2.$$

Obviously the kinetic energy of a stationary particle is zero relative to its own rest frame. Thus the quantity on the left-hand side of (6.1) is the rate of change of kinetic energy with time

$$\vec{F} \cdot \dot{\vec{r}}(t) = \frac{dT}{dt}.$$

Suppose that between $t = a$ and $t = b$, the particle moves from $r(a)$ to $r(b)$. We can then integrate both sides with respect to t to give:

$$\int_a^b \vec{F} \cdot \frac{d\vec{r}}{dt} dt = \int_a^b \frac{dT}{dt} dt = T(b) - T(a).$$

The right-hand side is the difference in kinetic energy of the body at times a and b. The left-hand side is the amount of energy that must have been expended by the force to make this change. It is defined to be the work done by the force on the particle. Hence we have:

The work W done by a force F on a particle moving along a trajectory $r(t)$ in time $a \leq t \leq b$ is the amount of energy given by the integral

$$W(\vec{r}(a), \vec{r}(b)) \equiv \int_a^b \vec{F} \cdot \frac{d\vec{r}}{dt} dt = \int_a^b \vec{F} \cdot d\vec{r} \tag{6.2}$$

and we have derived the law of conservation of energy:

$$W(a, b) = T(b) - T(a)$$

In words the law of conservation of energy states that the work done equals the change in kinetic energy of a system. If $W > 0$, the force has increased the speed of the particle. If $W < 0$, the force has resisted the motion of the particle, it has reduced in speed and dissipated energy.

Overall energy is conserved. The work done decelerating (say) a body will be dissipated as frictional heat and/or acoustic energy. The work done accelerating a body to increase its kinetic energy must come from a source, such as chemical combustion of a fuel. However for now, we will not be concerned with the form of the energy that the work done takes.

Before we can use (6.2) we need to define how we evaluate the weird looking work integral. Such integrals are called **line integrals** because they are integrals that depend on the path over which the integration is carried out.

6.3 A slight aside: Vector Fields and Line Integrals

As a useful reminder, let us outline some jargon that you will come across in reading texts about how to evaluate line integrals.

First of all, a *scalar field* is just a fancy description for an ordinary function defined at each point in space. The input is a vector $\vec{r} = x\vec{i} + y\hat{j} + z\hat{k}$, (where x, y, and z may themselves depend on another parameter, e.g., time t) the output is a number $f(\vec{r}) = f(x, y, z)$. Example of scalar fields are:

- the altitude of a landscape above (or below) sea level;
- temperature of the atmosphere;
- the pressure of the atmosphere;
- the wind *speed* at a point.

Scalar fields are often represented by drawing contour plots of curves/surfaces where $f(\vec{r})$ =constant.

Similarly, a vector field is a function whose input is a vector $\vec{r} = x\vec{i} + y\hat{j} + z\hat{k}$ while theoutput is a vector quantity. In a Cartesian frame it takes the form

$$\vec{F}(\vec{r}) = F_1(\vec{r})\hat{\imath} + F_2(\vec{r})\hat{\jmath} + F_3(\vec{r})\hat{k}.$$

Examples of vector fields are:

- the wind *velocity* on a weather map;
- the Earth's magnetic field at any point on its surface;
- the electric field near a pylon;
- the vorticity of a fluid flow.

Vector fields are usually represented with arrow plots, where the length of the arrow is proportional to the strength of the field. The forces in **N2** can be regarded as vector fields.

If we want to quantify the work done by a certain process, we need to evaluate a line integral. An ordinary 1D integral is the integration of a scalar function $f(x)$ over a straight line segment $a \le x \le b$

$$\int_a^b f(x)\, dx$$

It represents the area between $f(x)$ and the x-axis between $x = a$ and $x = b$. If $f(x)$ represents the height of a straight fence, then the integral is the area of the fence panels.

The work integral we need to evaluate

$$\int_a^b \vec{F} \cdot \frac{d\vec{r}}{dt} dt$$

is a generalisation of this definition.[2]

- First the integration is now over a given curve C, or $\vec{r}(t)$ with $a \le t \le b$. This is equivalent to working out the area of a fence that is placed along a curved base.
- Second, the integrand involves vector functions $\vec{F}$ and $\dot{\vec{r}}$. However these combine as a dot product to form a scalar function $\vec{F} \cdot \dot{\vec{r}}$.

We represent the trajectory by the curve C:

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}, \quad a \le t \le b$$

which has a tangent vector

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}, \quad a \le t \le b.$$

[2]We have introduced this type of integral using the concept of work, but it is generally defined for any vector field, independently of its physical meaning. In fact t need not be time: it is just a parameter describing positions on C. Furthermore, $\vec{F}(\vec{r})$ really only depends on the position $\vec{r}$ on the curve C, not on the parameter describing it. How you represent C or $\vec{r}$ in terms of parameters should does not affect the answer.

With these definitions, the force experienced by the particle will (in general) differ at each point of the trajectory. [3]

To evaluate the integral, we evaluate $\vec{F}$ at each point on the trajectory as a function of the curve parameter t, take the dot product

$$\vec{F} \cdot \frac{d\vec{r}}{dt},$$

and then integrate the resulting scalar function of t between a and b. Sometimes the work done depends on the path taken by the body. Sometimes it does not.

6.4 Conservation of Linear Momentum

Energy is not the only quantity that is conserved in Newtonian mechanics. We may also consider the linear momentum.

N2 tells us that

$$\vec{F} = \frac{d\vec{p}}{dt},$$

where $\vec{p} = m\dot{\vec{r}}$ is the linear momentum of the body. If there are no external forces acting, then

$$\vec{F} = 0 \Rightarrow \frac{d\vec{p}}{dt} = 0 \Rightarrow \vec{p} = \text{constant}.$$

Hence we see that the linear momentum of a body is conserved if the total force on it is zero.

Components of momentum can also be conserved. Suppose $\vec{s}$ is a constant vector such that $\vec{F} \cdot \vec{s} = \vec{0}$. Thus the force $\vec{F}$ has no component in the $\vec{s}$ direction. If we take the scalar product of both sides of $\vec{N}2$ with $\vec{s}$ we obtain:

$$\vec{F} = \frac{d\vec{p}}{dt} \Rightarrow \vec{F} \cdot \vec{s} = \frac{d\vec{p}}{dt} \cdot \vec{s} \Rightarrow \vec{0} = \frac{d(\vec{p} \cdot \vec{s})}{dt}$$

(since $\vec{s}$ is constant). Hence we deduce that

$$\vec{p} \cdot \vec{s} = \text{constant}.$$

In words, the *component* of linear momentum in a direction in which the component of force vanishes is constant in time.

Example 1: Golf on the moon

An astronaut plays golf on the surface of the moon, where there is little atmosphere (i.e., negligible air resistance). The golf ball of mass m is launched and moves with velocity $\vec{v}(t)$, subject only to the moon's gravity $-g\hat{k}$.

[3] Thus the force is not constant and so we can't apply the usual formula of
"Work done = Component of force in direction of motion × distance moved"
However we can split the trajectory into very small sections of length $d\vec{r}$ over which the force is approximately constant, apply the old formula and then sum (= integrate) over them.

N2 gives:

$$\vec{F} = \frac{d\vec{p}}{dt} \Rightarrow -mg\hat{k} = m\frac{d\vec{v}}{dt} \Rightarrow -mg\hat{k}\cdot\hat{\imath} = m\frac{d\vec{v}}{dt}\cdot\hat{\imath}$$

$$\Rightarrow 0 = m\frac{d}{dt}\left(\vec{v}\cdot\hat{\imath}\right).$$

In other words, the $\hat{\imath}$ component of $\vec{v}(t)$ is constant throughout the motion. This is simply a reflection of the fact that the force has no component in this direction.

6.5 Conservative Forces and potentials

In the context of mechanics, conservative forces represent a special class of force fields for which the work done on a body by the field is independent of the motion[4].

The idea is simple. If a force field $\vec{F}(\vec{r})$ at any point $\vec{r}(t) = x(t)\vec{i} + y(t)\hat{j} + z(t)\hat{k}$ can be represented by the gradient of a scalar function $V(\vec{r})$ of the form[5]

$$\vec{F}(\vec{r}) \equiv -\nabla V(\vec{r})$$

then it is a **conservative force field.**

The work done by such a field is independent of the path taken, it depends only on the initial and final states.

If a force is conservative such that $\vec{F} = -\nabla V$, then the function V is called the **potential energy** of the field (or **potential** for short).

Let us consider a simple example.

Example 2: Gravity

Gravity is a conservative force field. If we take the z-axis to point upwards, then

$$\vec{F} = -mg\hat{k} \Rightarrow \vec{F} = -\frac{d}{dz}(mgz)\hat{k}$$

$$\Rightarrow \vec{F} = -\frac{d}{dz}(mgz)$$

The gravitational potential is thus

$$V(z) = +mgz.$$

If the z-axis had instead pointed downwards, we would have obtained $V = -mgz$. In both cases, an arbitrary constant can be added.

[4] This is useful as it significantly simplifies the evaluation of the work integral.

[5] Note that this requires us to use partial derivatives, which we are (for the moment) trying to avoid.

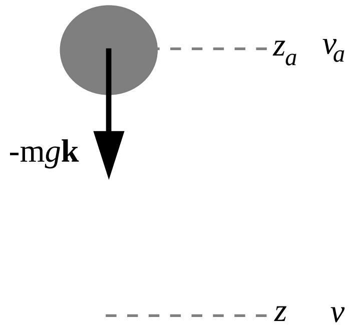

Figure 6.1: A body falling in a gravitational potential

Example 3: Motion Under Gravity

Conservation of energy can often be used as an alternative to the solution of a dynamical system using **N2**. This example provides a demonstration.
A body of mass m is thrown vertically upward under the action of a gravitational field $-mg\hat{k}$. Suppose that the particle has initial speed v_a at height $z_a\hat{k}$. Find its speed $v = |\dot{\vec{r}}|$ as a function of position
Gravity is a conservative force with potential (see above) $V(z) = mgz$. In the absence of non-conservative fields we have

$$\begin{aligned}
T(z) + V(z) &= \text{constant} = E, \quad \text{say} \\
\frac{1}{2}m|\dot{\vec{r}}|^2 + mgz &= \text{constant} = \frac{1}{2}mv_a^2 + mgz_a \\
\frac{1}{2}mv^2 + mgz &= \frac{1}{2}mv_a^2 + mgz_a \\
\Rightarrow v(z) &= \pm\sqrt{v_a^2 + 2g(z_a - z)} \quad \text{or} \\
v(z) &= \pm\sqrt{\frac{2}{m}}\sqrt{E - mgz}
\end{aligned}$$

The maximum height of the body is reached when $v(z) = 0$, i.e., $z - z_a = v_a^2/(2g)$.

6.6 One-dimensional motion

Let us now focus on a body moving along a line, so that its position is determined by a single function $x(t)$ (say). For now, suppose that the force on the particle depends only on the position, not the velocity. That is, we have $F = F(x)$. We then define the potential $V(x)$ in such a way that

$$F(x) = -\frac{dV}{dx}$$

(note the sign!). Obviously, the potential is only defined up to a constant. We can always invert the equation by integrating both sides. The integration constant is now determined by the choice of lower limit of the integral,

$$V(x) = -\int_{x_0}^{x} F(x')dx'$$

where x' is a dummy variable. Given the definition of the potential, we can write the equation of motion as

$$m\ddot{x} = -\frac{dV}{dx} \tag{6.3}$$

For any force in one dimension which depends only on the position, there exists a conserved **energy**,

$$E = \frac{1}{2}m\dot{x}^2 + V(x)$$

As before, the fact that this is conserved means that $\dot{E} = 0$ for any trajectory that obeys the equation of motion. Introducing the kinetic energy,

$$T = \frac{1}{2}m\dot{x}^2$$

it is easy to show that the energy is conserved. We only need to differentiate to get

$$\dot{E} = m\dot{x}\ddot{x} + \frac{dV}{dx}\dot{x} = \dot{x}\left(m\ddot{x} + \frac{dV}{dx}\right) = 0$$

where the last equality holds courtesy of the equation of motion (6.3). We also see why the sign of the potential in Newton's second law makes sense.

In any physical system, conserved quantities of this kind are precious—because they help us simplifying the solution. Hence, we will spend some time finding them and showing how they help us simplify various problems.

6.7 Stability of Equilibrium

If a body is stationary it is said to be in equilibrium. From **N2** we have seen that a necessary condition for this is that the sum of the forces acting on the body $\vec{F}$ must vanish. Now if all the forces acting on the system are conservative so that $\vec{F} = -V'$ we then have a necessary condition for equilibrium as

$$\vec{F} = \vec{0} \longrightarrow V' = 0.$$

Thus a system will be in equilibrium at a stationary point (or critical point) of the potential. This definition is valid in all dimensions.

To explore whether a position of equilibrium is stable, unstable (or otherwise), we will consider a 1-D potential $V(x)$.

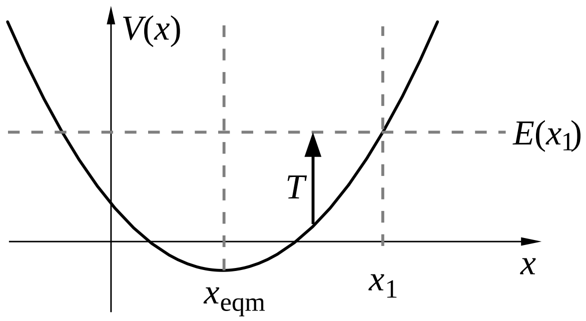

Figure 6.2: Stable equilibrium.

Example 4: Stable equilibrium

Consider a potential function $V(x)$ that behaves qualitatively like that in the diagram with a minimum of potential energy at a point x_0. Suppose a particle is released from rest a point x_1 relative to the minimum. By conservation of total energy, $E = E(x_1)$=constant, throughout the motion the kinetic, T, and potential, V, energies are related by

$$E = T + V \Rightarrow T = E - V.$$

Now since $T = m|\dot{\vec{r}}|^2/2$ we must have $T \geq 0$. Hence we can write

$$T \geq 0 \Rightarrow E - V \geq 0 \Rightarrow V \leq E$$

i.e. the motion is confined to regions where $V \leq E$. Thus the particle will oscillate in a finite region around the minimum of V and so the motion is bounded. Hence the equilibrium point is said to be stable.

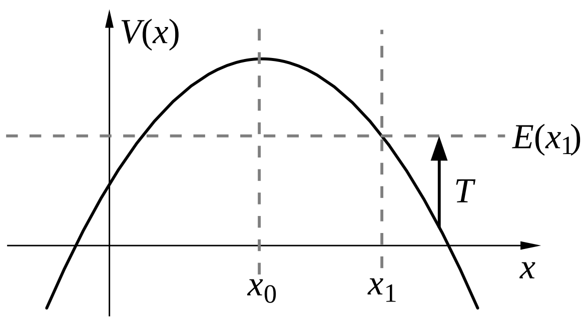

Figure 6.3: Unstable equilibrium.

Example 5: Unstable equilibrium

Consider now a potential that has a maximum at x_0. If a particle released from a small distance away, x_1, again energy is conserved $E = E(x_1)$=constant throughout the motion. Now since the kinetic energy must be non-negative, we can again write

$$T \geq 0 \Rightarrow E - V \geq 0 \Rightarrow V \leq E$$

so again, the the only classically allowed motion must have $V \leq E$. Hence the particle cannot approach the maximum and must move away from it. As it does so the kinetic energy T and hence speed will increase. Hence the equilibrium point is said to be unstable.

In general, an equilibrium point in a conservative system where $\vec{F} = -\nabla V$ occurs when

$$\nabla V = 0.$$

- It is **stable** if it is a **minimum** (in 1-D $\Rightarrow V''(x) > 0$).
- It is **unstable** if it is a **maximum** (in 1-D $\Rightarrow V''(x) < 0$).
- Otherwise it may be stable or unstable and needs further investigation (in 1-D $\Rightarrow V''(x) = 0$).

In 1-D, max/min points where $T = 0$, i.e., $E = V$ are called **turning points**. On one side of such points the classical motion is allowed, and on the other it is not. Hence a moving particle is said to "turn around" there[6].

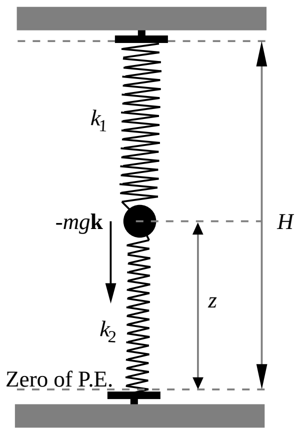

Figure 6.4: Elastic energy is stored in two springs

[6]An analogous argument can be applied in more than one dimension. In higher dimensions isolated maxima of the potential are unstable equilibrium points, isolated minima are stable points. (Just think of rolling a ball down a hill.) However, saddle points also satisfy the equilibrium condition $\nabla V = \vec{0}$ and so are also equilibrium points. Their stability or otherwise, depends on the motion perturbing the equilibrium.

Example 6: Coupled springs

A mass m is attached to two Hookean springs with spring constants k_1, k_2 and natural lengths l_1, l_2, as in figure 6.4. Determine the equilibrium point and whether it is stable or unstable.
Potential energy:

- Gravitational: mgz
- Spring 2: $\frac{1}{2}k_2(z-l_2)^2$
- Spring 1: $\frac{1}{2}k_1(H-z-l_1)^2$

$$V(z) = mgz + \frac{1}{2}k_2(z-l_2)^2 + \frac{1}{2}k_1(H-z-l_1)^2$$

Equilibrium position:

$$\begin{aligned} V'(z) &= mg + k_2(z-l_2) - k_1(H-z-l_1) = 0 \\ z_{\text{eqm}} &= \frac{k_1(H-l_1)+k_2 l_2 - mg}{k_1+k_2} \end{aligned}$$

Stability:

$$V''(z) = k_1 + k_2 > 0$$

The equilibrium point is a minimum and hence it is stable.

6.8 Moving in a potential

Let us consider the general case of a potential $V(x)$. It turns out that we can learn quite a lot about the motion without actually solving the problem. For example, the existence of a conserved energy magically allows us to turn the problem into a first order differential equation

$$E = \frac{1}{2}m\dot{x}^2 + V(x) \quad \longrightarrow \quad \frac{dx}{dt} = \pm\sqrt{\frac{2}{m}\left[E - V(x)\right]}$$

Noting that the right-hand side does not depend on t, we have our first hint of the hint of the importance of conserved quantities in solving a problem. Of course, to go from a second order equation (like **N2**) to a first order equation, we must have chosen an integration constant. In this case, this is the energy E itself. Given a first order equation, we can always write down a formal solution for the dynamics by integrating. We have

$$t - t_0 = \pm\int_{x_0}^{x} \frac{dx'}{\sqrt{\frac{2}{m}\left[E - V(x')\right]}} \tag{6.4}$$

If we can work out the integral, we are done. If we can't do the integral, you sometimes hear that the problem has been "reduced to quadrature". This really means "I can't do the integral", but it is often the case that having a solution in this form still sheds light on some of the properties. And, if nothing else, one can always evaluate the integral numerically – computers are really good at integration.

6.9 Getting a feel for the solution

Given the potential $V(x)$, it is often easy to figure out the qualitative nature of any trajectory simply by looking at the form of $V(x)$. This allows us to answer some questions with very little work[7]. For example, we may want to know whether the particle is trapped within some region of space or can escape to infinity.

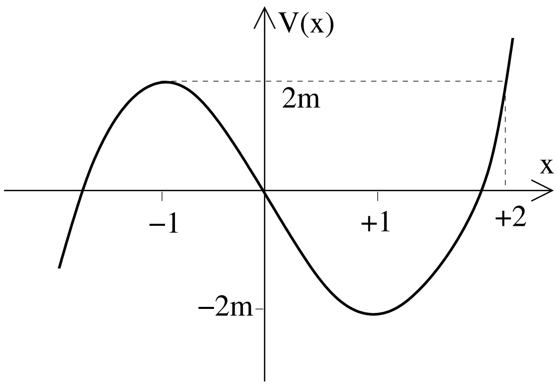

Figure 6.5: A cubic potential.

Let us illustrate this with an example. Consider the cubic potential

$$V(x) = m(x^3 - 3x)$$

If we were to substitute this into the general form (6.4), we have a fearsome looking integral which we struggle to solve[8]. Luckily, we can make progress even without solving the integral. The potential is plotted in figure 6.5. Let us start with the particle sitting stationary at some position x_0. This means that the energy is

$$E = V(x_0)$$

and we know that this must remain constant during the subsequent motion. What happens next depends only on x_0. We can identify the following possibilities:

[7]... and we like this.

[8]It can be solved – it is called an elliptic integral – but we do not yet have the tools to work this out.

- $x_0 = \pm 1$. These are the local maximum and minimum. If we drop the particle at these points, it stays there for all time.
- $x_0 \in (-1, +2)$. Here the particle is "trapped" in the dip. It oscillates backwards and forwards between the two points with potential energy $V(x_0)$. The particle cannot climb to the right because it does not have the energy. In principle, it could live off to the left where the potential energy is negative, but to get there it would have to first climb the small bump at $x = 1$ and it does not have the energy to this either[9].
- $x_0 > 2$. When released, the particle falls into the dip, climbs out the other side, before falling into the void $x \to -\infty$.
- $x_0 < -1$. The particle just falls off to the left.
- $x_0 = +2$. This is a special value, since $E = 2m$ which is the same as the potential energy at the local maximum $x = 1$. The particle falls into the dip and starts to climb up towards $x = 1$. It can never stop before it reaches $x = 1$ for at its stopping point it would have only potential energy $V < 2m$. But, similarly, it cannot arrive at $x = 1$ with any excess kinetic energy. The only option is that the particle moves towards $x = 1$ at an ever decreasing speed, only reaching the maximum at time $t \to \infty$.

One can easily play a similar game to that above if the starting speed is not zero. In general, one finds that the particle is trapped in the dip $x \in [-1, +1]$ if its energy lies in the interval $E \in [-2m, 2m]$.

Example 7: The harmonic oscillator

The harmonic oscillator is, by far, the most important dynamical system in all of mathematical physics. The good news is that it is very easy to deal with. (In fact, the reason that it is so important is precisely because it is easy!). The potential energy of the harmonic oscillator is defined to be

$$V(x) = \frac{1}{2}kx^2$$

The harmonic oscillator is a good model for, among other things, a particle attached to the end of a spring. The force resulting from the energy V is given by $F = -kx$ which, in the context of the spring, is Hooke's law. The equation of motion is (again, as before)

$$m\ddot{x} = -kx$$

with general solution

$$x(t) = A\cos\omega t + B\sin\omega t \qquad \text{with} \qquad \omega = \sqrt{\frac{k}{m}}$$

A and B are integration constants and ω is called the angular frequency (also... as before).

[9] There is an assumption here which is implicit throughout all of classical mechanics: the trajectory of the particle $x(t)$ is a continuous function.

6.10 Why (almost) everything is a harmonic oscillator

A particle placed at an **equilibrium** point will stay there for all time. In the example with a cubic potential, we had two such equilibrium points: $x = \pm 1$. In general, if we want $\dot{x} = 0$ for all time, then clearly we must have $\ddot{x} = 0$, which, from the form of the equation of motion, tells us that we can identify the equilibrium points with the critical points of the potential,

$$\frac{dV}{dx} = 0$$

What happens to a particle that is close to an equilibrium point, x_0 (say)? In this case, we can Taylor expand the potential energy about $x = x_0$. Because, by definition, the first derivative vanishes, we have

$$V(x) \approx V(x_0) + \frac{1}{2} V''(x_0)(x - x_0)^2 + \dots \tag{6.5}$$

In order to proceed, we need to know the sign of $V''(x_0)$.

- $V''(x_0) > 0$. In this case, the equilibrium is a minimum of the potential and the potential energy is that of a harmonic oscillator. The particle oscillates backwards and forwards around x_0 with frequency

  $$\omega = \sqrt{\frac{V''(x_0)}{m}}$$

  Such equilibria are stable. As long as the amplitude of the oscillations is small enough (so that we can ignore the $(x-x_0)^3$ and higher order terms in the Taylor expansion), all systems oscillating around a stable fixed point behave like a harmonic oscillator.

- $V''(x_0) < 0$. In this case, the equilibrium point is a maximum of the potential. The equation of motion again reads

  $$m\ddot{x} = -V''(x_0)(x - x_0)$$

  but if the second derivative of the potential is negative, then we have $\ddot{x} > 0$ when $x - x_0 > 0$. This means that if we displace the system a little bit away from the equilibrium point, the acceleration pushes it further away. The equilibrium is unstable.

- Finally, we could have $V''(x_0) = 0$. In this case, there is nothing we can say about the dynamics of the system without Taylor expanding the potential further.

6.11 Addendum: Taylor expansions

The Taylor series is one of the most useful mathematical tools applied to real life problems. As in the example of potential motion near equilibrium, we can often use a Taylor series to approximate a complicated function by a (simpler) polynomial function.

The starting point is Taylor's theorem (which we state without proof, as we are mainly interested in the applications):

Let $f(x)$ be continuous and differentiable an infinite number of times at point $x = a$ (i.e., f and all of its derivatives $f', f'', \ldots$ exist at $x = a$). Then $f(x)$ can be expanded about the point $x = a$ in terms of powers of $(x - a)$, and this expansion is given by the formula

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots \\ + \frac{(x-a)^n}{n!}f^{(n)}(a) + O(x-a)^{n+1}$$

This is a **Taylor expansion**. It has the form

$$f(x) = P_n(x) + O(x-a)^{n+1}$$

where $P_n(x)$ is a polynomial of degree n and $O(x-a)^{n+1}$ is a remainder term. The remainder term reminds us that in writing the Taylor expansion explicitly up to the term $\propto (x-a)^n$ ("up to order n") we are omitting terms "of order $(x-a)^{n+1}$ and higher".

For x close to a, we can expect the remainder term to be small (since $x-a$ is small), so that $f(x) \simeq P_n(x)$. Thus Taylor's theorem enables us to approximate functions by polynomials.

In many cases the remainder term goes to zero as $n \to \infty$ and the Taylor series expansion of $f(x)$ takes the form of a possibly infinite, but convergent, series.

Example 8: The exponential

As an example, let us determine the Taylor series expansion of the function $f(x) = e^x$ about $x = 0$.

We first establish the necessary data. We know that we need the derivatives:

$$\begin{array}{llll} f(x) = e^x & f'(x) = e^x & f''(x) = e^x & \ldots \\ f(0) = e^0 = 1 & f'(0) = e^0 = 1 & f''(0) = e^0 = 1 & \ldots \end{array}$$

Then we use this in the Taylor series expansion with $a = 0$

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

Similarly, we can work out the Taylor series expansion of the function $f(x) = \cos x$ about $x = 0$.

In this case, we need:

$$\begin{aligned} f(x) &= \cos x \ , \qquad f(0) = 1 \\ f'(x) &= -\sin x \ , \qquad f'(0) = 0 \\ f''(x) &= -\cos x \ , \qquad f''(0) = -1 \\ f'''(x) &= \sin x \ , \qquad f'''(0) = 0 \\ f^{(4)}(x) &= \cos x \ , \qquad f^{(4)}(0) = 1 \end{aligned}$$

and we arrive at

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

Finally, let us make contact with our discussion of potentials and equilibrium points. We see that the statement (6.5) follows immediately from Taylor's theorem if we take $f = V$, $a = x_0$ and note that $V'(x_0) = 0$ (and we neglect terms beyond quadratic order).