Chapter 5

Line Integrals and Green's Theorem

5.1 Line Integrals

At A-level and in MATH1059 you learned how to integrate a function $y(x)$ along the x-axis. In the previous chapter of this module, you learned how to integrate a function $f(x, y)$ over a 2-d domain $\mathcal{D}$ in the (x, y) plane.

In this chapter we will learn how to integrate real valued $\mathbb{R}^2 \to \mathbb{R}$ functions $P(x, y)$ or $Q(x, y)$ over a curve $\mathcal{C}$ given by a portion of the curve $y = f(x)$ in the (x, y) plane. These types of integral are called "line integrals" and can be written as:

$$\int_{\mathcal{C}} P(x, y)\, dx \qquad \text{and} \qquad \int_{\mathcal{C}} Q(x, y)\, dy,$$

or more generally as

$$\int_{\mathcal{C}} \{P(x, y)\, dx + Q(x, y)\, dy\}\, .$$

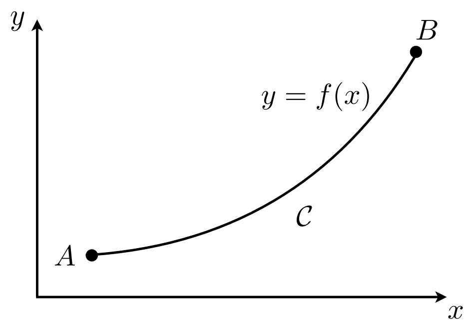

Figure 5.1: The curve $\mathcal{C}$ over which the line integral is taken is the portion of the curve $y = f(x)$ between endpoints A and B.

There is a simple analogy to help to explain what a line integral is:

- The integral

$$\int_{x=a}^{x=b} y(x)dx$$

corresponds to the area of paint you need to paint one side of a straight fence of height $y(x)$ between $x = a$ and $x = b$.

- Let $\mathcal{C}$ denote the curve along the base of a fence in the (x, y) plane given by the equation $y = f(x)$. Let ds be the elemental arc length distance along the tangential direction of a curve $\mathcal{C}$.

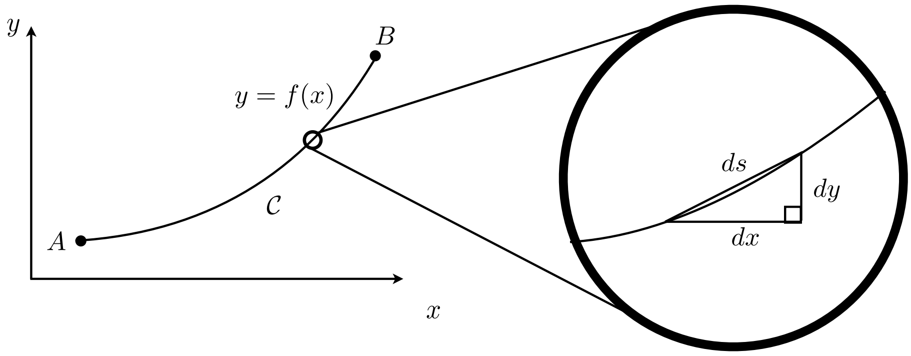

Figure 5.2: Pythagoras and arc length.

Pythagoras shows that ds and elemental steps along the curve in the x and y directions, dx and dy, respectively, are related by:

$$ds = \sqrt{(dx)^2 + (dy)^2} \qquad \Rightarrow \qquad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

The integral

$$\int_{\mathcal{C}} R(x, y) ds = \int_{\mathcal{C}} R(x, y) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{\mathcal{C}} P(x, y) dx,$$

say, corresponds to the area of paint you need to paint one side of a fence of height $R(x, y)$

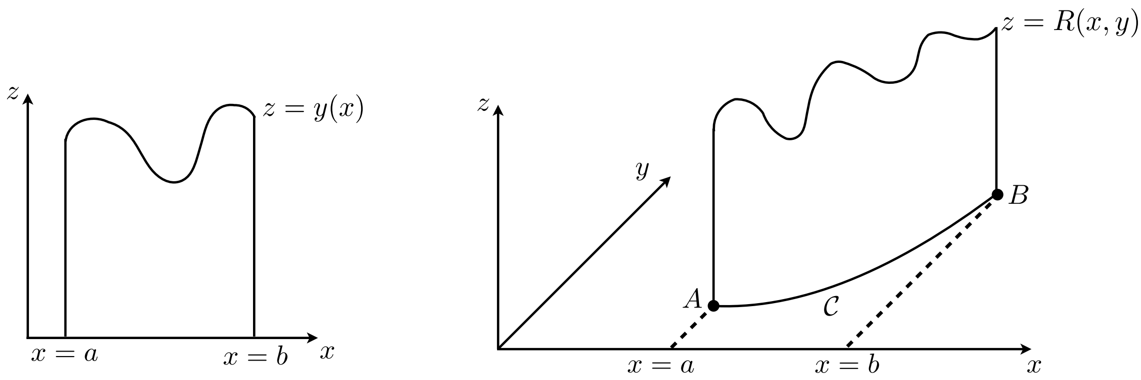

Figure 5.3: Vertical fences that need painting: one with a straight line for a base (left) and one with a curved base (right).

Note that if you set $R(x,y) = 1$ you end up with an expression for working out the length of the curve $\mathcal{C}$ between A and B:

$$\int_{\mathcal{C}} 1 ds = \int_{\mathcal{C}} ds = [\text{Length of curve}]_A^B = \text{Length of } \mathcal{C} \text{ between } A \text{ and } B.$$

Hence if we define $\mathcal{L}$ to be the length of the curve between A and B we have

$$\mathcal{L} = \int_{\mathcal{C}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

5.2 Evaluation of Line Integrals in a Plane

Consider the combined line integral

$$\int_{\mathcal{C}} \{P(x,y)\, dx + Q(x,y)\, dy\},$$

over the portion of the differentiable curve $y = f(x)$ in the (x,y) plane between $x = a$ and $x = b$.

The integral may be broken down and evaluated as:

$$\int_{\mathcal{C}} P(x,y)\, dx = \int_{x=a}^{x=b} P(x, f(x)) dx,$$

and, since "$dy = f'(x)dx$",

$$\int_{\mathcal{C}} Q(x,y)\, dy = \int_{x=a}^{x=b} Q(x, f(x)) f'(x) dx.$$

Alternatively if $\mathcal{C}$ was given as a differentiable function $x = g(y)$ between $y = c$ and $y = d$, we could evaluate the two parts of the integral as (using "$dx = g'(y)dy$")

$$\int_{\mathcal{C}} P(x,y)\, dx = \int_{y=c}^{y=d} P(g(y), y) g'(y) dy,$$

and

$$\int_{\mathcal{C}} Q(x,y)\, dy = \int_{y=c}^{y=d} Q(g(y), y) dy.$$

Example: Let $P(x,y) = x + y$. Evaluate the line integral

$$\int_{\mathcal{C}_1} P(x,y)\, dx$$

where $\mathcal{C}_1$ is the curve $y = 1 - x$ from $x = 0$ to $x = 1$.

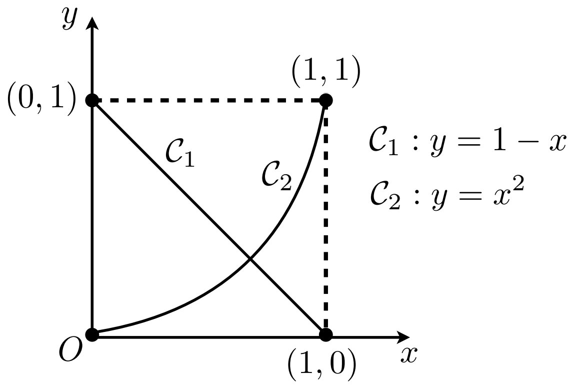

Figure 5.4: The curves $\mathcal{C}_1$ and $\mathcal{C}_2$ in the examples.

- The integrand on the curve $\mathcal{C}_1$ is given by

$$x + y = x + 1 - x = 1.$$

- The line integral then becomes

$$\int_{\mathcal{C}_1} P(x,y)\,dx = \int_{x=0}^{x=1} (x + 1 - x)\,dx = \int_{x=0}^{x=1} 1\,dx = [x]_{x=0}^{x=1} = 1$$

Example: Let $P(x,y) = x + y$. Evaluate the line integral

$$\int_{\mathcal{C}_2} P(x,y)\,dx$$

where $\mathcal{C}_2$ is the curve $y = x^2$ from $x = 0$ to $x = 1$.

- The integrand on the curve C_2 is given by

$$x + y = x + x^2.$$

- The line integral then becomes

$$\int_{\mathcal{C}_2} P(x,y)\,dx = \int_{\mathcal{C}_2} P(x,x^2)\,dx = \int_{x=0}^{x=1} (x + x^2)\,dx = \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{5}{6}.$$

For these examples, we can equally evaluate these integrals by representing the curve of integration $\mathcal{C}$ in terms of equivalent functional representations $x = g(y)$.

Example: Integrating $P(x,y) = x + y$ when $\mathcal{C}_1$ given by $y = 1 - x$ is expressed as $x = g(y)$.

- The curve $y = 1 - x$ may be rearranged as $x = 1 - y$.
- The limit of integration $x = 0$ becomes $y = 1 - 0 = 1$.
- The limit of integration $x = 1$ becomes $y = 1 - 1 = 0$.

- The integrand on the curve $\mathcal{C}_1$ when written as $x = g(y) = 1 - y$ is given by

$$x + y = 1 - y + y = 1$$

.

- If $x = 1 - y$, then $dx = -dy$.
- We thus have

$$\int_{\mathcal{C}_1} P(x,y)\,dx = \int_{y=1}^{y=0} (1 - y + y)\,(-1)dy = -\int_{y=1}^{y=0} 1\,dy = -\,[y]_{y=1}^{x=0} = 1.$$

Which is the same value as the corresponding line integral above.

Example: Integrating $P(x,y) = x + y$ when $\mathcal{C}_2$ given by $y = x^2$ is expressed as $x = g(y)$.

- The curve $y = x^2$ may be rearranged as $x = +\sqrt{y}$, since we are in the positive quadrant.
- The limit of integration $x = 0$ becomes $y = 0^2 = 0$.
- The limit of integration $x = 1$ becomes $y = 1^2 = 1$.
- The integrand on the curve $\mathcal{C}_2$ when written as $x = g(y) = +\sqrt{y}$ is given by

$$x + y = +\sqrt{y} - y$$

.

- If $x = +\sqrt{y}$, then $dx = \frac{1}{2}y^{-1/2}dy$.
- We thus have

$$\int_{\mathcal{C}_2} P(x,y)\,dx = \int_{y=0}^{y=1} (+\sqrt{y} + y)\,\frac{1}{2}y^{-1/2}dy = \frac{1}{2}\int_{y=0}^{y=1} (1 + \sqrt{y})\,dy = \frac{1}{2}\left[y + \frac{2}{3}y^{3/2}\right]_{y=0}^{y=1} = \frac{5}{6}.$$

Which is the same value as the corresponding line integral above.

Example: Evaluate the line integral

$$I = \int_{\mathcal{C}} (x^2 + 2y)\,dx + (x + y^2)\,dy$$

between the points $(0,1)$ and $(2,3)$ along the curve $\mathcal{C}$ given by $y = x + 1$.

- First, we choose to evaluate the integral by expressing it entirely in terms of x.
- If $y = x + 1$ then in the second part of the integral, $dy = dx$.
- Hence we have

$$\begin{aligned}
I &= \int_{x=0}^{x=2} (x^2 + 2(x+1))\,dx + (x + (x+1)^2)\,dx, \\
&= \int_{x=0}^{x=2} (x^2 + 2x + 2)\,dx + (x + x^2 + 2x + 1)\,dx, \\
&= \int_{x=0}^{x=2} (2x^2 + 5x + 3)\,dx, \\
&= \left[\frac{2}{3}x^3 + \frac{5}{2}x^2 + 3x\right]_{x=0}^{x=2} dx, \\
&= \frac{64}{3}.
\end{aligned}$$

Equally we can evaluate this integral by representing $\mathcal{C}$ in terms of a function $x = g(y)$ in the second part of the integral

- If $y = x + 1$ then we can rearrange this to give $x = y - 1$, hence $\Rightarrow dx = dy$.
- From the question statement, we see the limit $x = 0$ corresponds to $y = 1$ and the limit $x = 1$ corresponds to $y = 3$.
- Thus we can evaluate the integral as follows:

$$
\begin{aligned}
I &= \int_{\mathcal{C}} (x^2 + 2y)\,dx + (x + y^2)\,dy, \\
&= \int_{y=1}^{y=3} ((y-1)^2 + 2y)\,dy + \int_{y=1}^{y=3} ((y-1) + y^2)\,dy, \\
&= \int_{y=1}^{y=3} (y^2 - 2y + 1 + 2y)\,dy + \int_{y=1}^{y=3} (y^2 + y - 1)\,dy, \\
&= \int_{y=1}^{y=3} (2y^2 + y)\,dy, \\
&= \left[\frac{2}{3}y^3 + y\right]_{y=1}^{y=3} dx, \\
&= \left[\frac{54}{3} + \frac{9}{2}\right] - \left[\frac{2}{3} + \frac{1}{2}\right], \\
&= \frac{64}{3}.
\end{aligned}
$$

  Which is the same answer as we obtained before.

5.3 Properties of Line Integrals

The following properties of line integrals hold

1. **Reversal of Direction of Integration**
   Line integrals may be evaluated in either direction along a curve $\mathcal{C}$ given by $y = f(x)$. Suppose the two endpoints of the curve of integration are A and B with corresponding x coordinates $x = a$ and $x = b$ respectively. We then have

$$\int_A^B P(x,y)dx = \int_{x=a}^{x=b} P(x, f(x))dx = -\int_{x=b}^{x=a} P(x, f(x))dx = -\int_B^A P(x,y)dx.$$

   Hence the sign of the line integral changes if the direction of traversal of $\mathcal{C}$ is reversed.

2. **Integrals Parallel to the Axes**
   If the path of integration is such that $\mathcal{C}$ is a line parallel to the y-axis, say $x = a$, where a is a constant, then

$$\int_{\mathcal{C}} P(x,y)dx = 0.$$

   This follows since on $\mathcal{C}$ we have $x = a$ so "$dx = 0$."

   Similarly, if the path of integration is such that $\mathcal{C}$ is a line parallel to the x-axis, say $y = c$, where c is a constant, then

$$\int_{\mathcal{C}} Q(x,y)dy = 0.$$

   This follows since on $\mathcal{C}$ we have $y = c$ so "$dy = 0$."

3. **Addition of Paths**

   Suppose the curve of integration $\mathcal{C}$ is given by $y = f(x)$ possessing end points with x-coordinates a and b, but is divided into two parts with a common point (x_c, y_c). Then we have

$$\int_{\mathcal{C}} P(x,y)\,dx = \int_{x=a}^{x=b} P(x,f(x))\,dx = \int_{x=a}^{x=x_c} P(x,f(x))\,dx + \int_{x=x_c}^{x=b} P(x,f(x))\,dx.$$

   Similar results apply to

$$\int_{\mathcal{C}} Q(x,y)\,dy$$

   and

$$\int_{\mathcal{C}} \{P(x,y)\,dx + Q(x,y)\,dy\}.$$

   These results may also be extend to cases where there are more than two divisions of the the curve of integration.

4. **Multivalued Curves**

   If $\mathcal{C}$ is represented by the curve $y = f(x)$ but for some value(s) of x more than one value of y is obtained (see figure), then the line integral must be decomposed into a sum over portions of the curve $\mathcal{C}$ where y is single-valued.

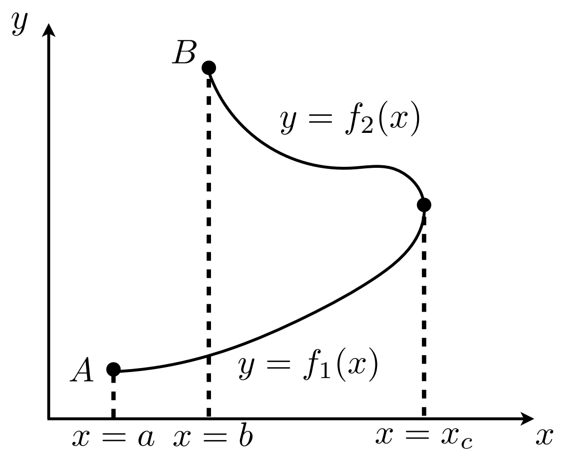

Figure 5.5: The curves $\mathcal{C}$ is multivalued for $x_b \leq x \leq x_c$ (where the curve, as traversed from A to B is first vertical). It must be split up into two single-valued portions, the lower curve given by $y = f_1(x)$, the upper curve given by $y = f_2(x)$.

   For example, in the above figure, we have:

$$\int_{\mathcal{C}} P(x,y)\,dx = \int_{x=a}^{x=x_c} P(x,f_1(x))\,dx + \int_{x=x_c}^{x=b} P(x,f_2(x))\,dx.$$

   Similar results apply to

$$\int_{\mathcal{C}} Q(x,y)\,dy$$

   for multivalued $x = g(y)$ and the combination

$$\int_{\mathcal{C}} \{P(x,y)\,dx + Q(x,y)\,dy\}.$$

Example: Multivalued Curve
Evaluate the integral

$$I = \int_{\mathcal{C}} (x+y)dx$$

from $(x, y) = (0, 1)$ to $(x, y) = (0, -1)$ where C is the semicircle $y = \sqrt{1-x^2}$

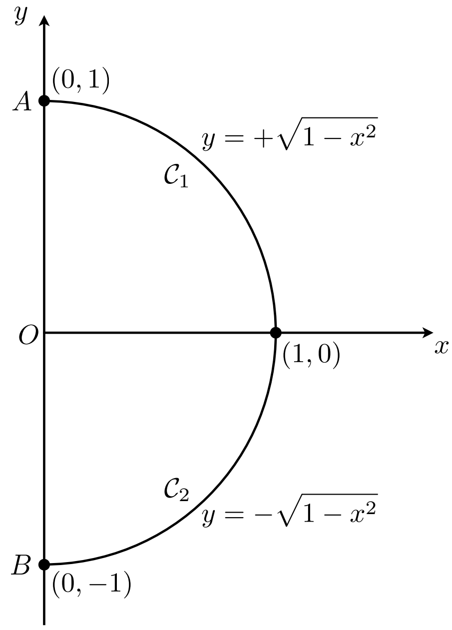

Figure 5.6: Contours $\mathcal{C}_1$ and $\mathcal{C}_2$ in the multivalued curve example.

- The issue, from point 4 above, is that the equation of the path of integration $\mathcal{C}$ is not single-valued. For every value of x on the path, there are two corresponding values of y.
- We split the path into two:
  - $\mathcal{C}_1$: $y = +\sqrt{1-x^2}$ between $(x, y) = (0, 1)$ and $(x, y) = (1, 0)$.
  - $\mathcal{C}_2$: $y = -\sqrt{1-x^2}$ between $(x, y) = (1, 0)$ and $(x, y) = (0, -1)$.
- Hence the integral becomes

$$\begin{aligned}
I &= \int_{\mathcal{C}} (x+y)dx = \int_{\mathcal{C}_1} (x+y)dx + \int_{\mathcal{C}_2} (x+y)dx, \\
&= \int_{x=0}^{x=1} (x + \sqrt{1-x^2})dx + \int_{x=1}^{x=0} (x - \sqrt{1-x^2})dx, \\
&= \int_{x=0}^{x=1} (x + \sqrt{1-x^2})dx - \int_{x=0}^{x=1} (x - \sqrt{1-x^2})dx, \\
&= 2\int_{x=0}^{x=1} \sqrt{1-x^2}dx \\
&= 2 \times \frac{1}{2}\left[x\sqrt{1-x^2} + \arcsin x\right]_{x=0}^{x=1}, \\
&= \frac{\pi}{2}.
\end{aligned}$$

A natural question arises: if the endpoints of the line integral are the same, is the value of the line integral the same regardless of which path is taken between the endpoints? In general the answer is "no", as the following example illustrates.

Example: Value of line integral along different paths

Evaluate the line integral

$$\int_{\mathcal{C}} \{(x+y)\,dx + xy\,dy\},$$

along the following curves:

1. From $(0,0)$ to $(1,1)$ along the curve $\mathcal{C}_1$ given by $y = x$.
2. From $(0,0)$ to $(1,0)$ along the curve $\mathcal{C}_2$ given by $y = 0$, then from $(1,0)$ to $(1,1)$ along the curve $\mathcal{C}_3$ given by $x = 1$.

Both these integrations have the same integrands and endpoints. They just differ by the curves over which the integrals are performed.

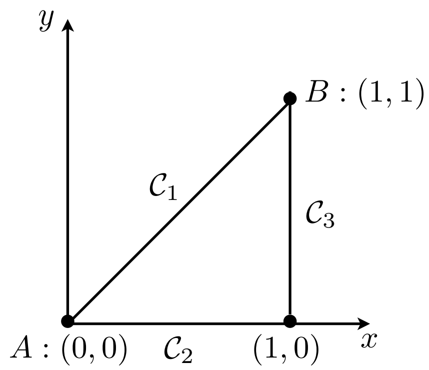

Figure 5.7: Different paths between the endpoints A and B, either the direct path $\mathcal{C}_1$ or the joined path via the intermediate point $(1,0)$, $\mathcal{C}_2 \cup \mathcal{C}_3$.

We evaluate these integrals in turn as follows:

1. From $(0,0)$ to $(1,1)$ along $y = x$.
   - Without loss of generality, we write both integrals in terms of x.
   - If $y = x$ we have $dy = dx$ in the second integral.
   - Hence the integral becomes:

$$\begin{aligned}
I_{\mathcal{C}_1} &= \int_{\mathcal{C}_1} (x+y)\,dx + xy\,dy, \\
&= \int_{x=0}^{x=1} (x+x)\,dx + \int_{x=0}^{x=1} x \times x\,dx, \\
&= \int_{x=0}^{x=1} 2x\,dx + \int_{x=0}^{x=1} x^2\,dx, \\
&= \left[x^2\right]_0^1 + \left[\frac{x^3}{3}\right]_0^1, \\
&= 1 + \frac{1}{3}, \\
&= \frac{4}{3}.
\end{aligned}$$

2. From $(0,0)$ to $(1,0)$ along the curve $\mathcal{C}_2$, $y=0$, then from $(1,0)$ to $(1,1)$ along the curve $\mathcal{C}_3$, $x=1$.

   - On the first leg $\mathcal{C}_2$ along $y=0$ we have $dy=0$. Hence we have:

$$\begin{aligned} I_{\mathcal{C}_2} &= \int_{\mathcal{C}_2} (x+y)\,dx + xy\,dy, \\ &= \int_{x=0}^{x=1} (x+0)\,dx + 0 \\ &= \left[\frac{1}{2}x^2\right]_0^1, \\ &= \frac{1}{2}. \end{aligned}$$

   - On the second leg $\mathcal{C}_3$ along $x=1$ we have $dx=0$. Hence we have:

$$\begin{aligned} I_{\mathcal{C}_3} &= \int_{\mathcal{C}_3} (x+y)\,dx + xy\,dy, \\ &= \int_{y=0}^{y=1} 0 + 1 \times y\,dy, \\ &= \left[\frac{1}{2}y^2\right]_0^1, \\ &= \frac{1}{2}. \end{aligned}$$

   - Hence the total line integral along $\mathcal{C}_2 \cup \mathcal{C}_3$ is given by

$$I_{\mathcal{C}_2 \cup \mathcal{C}_3} = I_{\mathcal{C}_2} + I_{\mathcal{C}_3} = \frac{1}{2} + \frac{1}{2} = 1.$$

Hence we see that, even though the limits of the integrals are the same, the values of the integrals across the different paths differ and $I_{\mathcal{C}_1} \neq I_{\mathcal{C}_2 \cup \mathcal{C}_3}$.

For arbitrary general integrands $P(x,y)$ and $Q(x,y)$ the integrals between the same endpoints will depend on the paths taken. Later we will examine the conditions on $P(x,y)$ and $Q(x,y)$ under which integrals are independent of the particular path taken between the same endpoints.

5.4 Line Integrals Around Closed Curves

We now consider the line integral around a closed, non-intersecting, curve, which starts and finishes at the same point.

We need to specify the direction in which we traverse the closed curve. The convention adopted is that a counterclockwise traversal is said to be in the positive direction, a clockwise traversal is said to be in the negative direction.

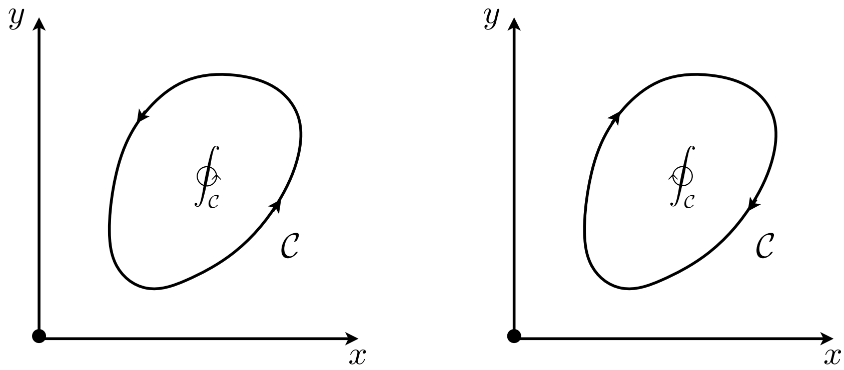

Figure 5.8: Left: counterclockwise, positive traversal of $\mathcal{C}$. Right: clockwise, negative traversal of $\mathcal{C}$.

When integrals are evaluated counterclockwise (respectively clockwise) around a closed curve $\mathcal{C}$ we denote this by

$$\oint_{\mathcal{C}}, \qquad \text{or respectively} \qquad \oint_{\mathcal{C}}.$$

Note that you might tempted to think that since the endpoints are the same, the value of line intgrals over closed paths is always zero. They can be zero, but as the following examples suggest, they are not always so.

To evaluate such integrals over closed paths, we need to break the integration contour $\mathcal{C}$ into component parts that each have a single-valued representation.

For example for the left hand (convex) curve in the diagram below, we can break the $\mathcal{C}$ up into $\mathcal{C}_1$ denoted by $y = f_1(x)$ and $\mathcal{C}_2$ denoted by $y = f_2(x)$, with endpoints A and B (where the curve is vertical).

For the right hand (non-convex) curve in the diagram below, we can break the $\mathcal{C}$ up into four parts $\mathcal{C}_i, i = 1, 2, 3, 4$ each separately denoted by $y = f_i(x), i = 1, 2, 3, 4$, along the portions AB, BC, CD and DA, respectively.

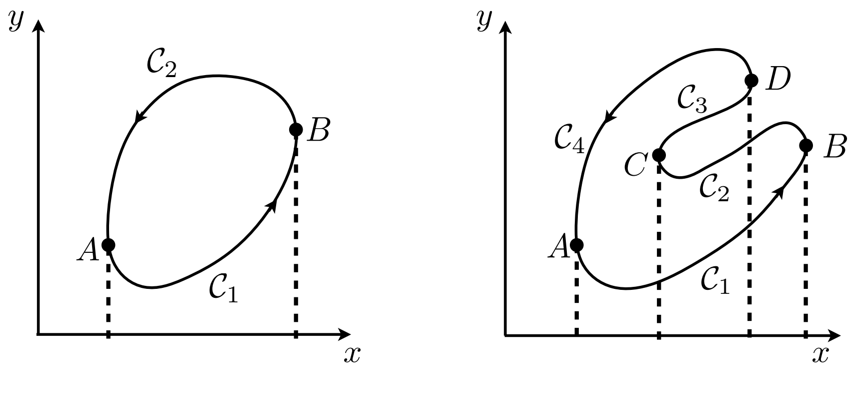

Figure 5.9: Partitioning of closed curves to obtain single-valued portions.

Example: Evaluate the line integral

$$\oint_{\mathcal{C}} \left\{ 2xydy - x^2 dx \right\},$$

where $\mathcal{C}$ is made up of the sides of the triangle with vertices $O = (0, 0), A = (1, 0), B = (1, 1)$.

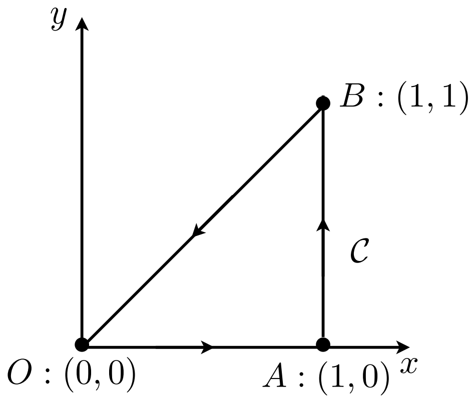

Figure 5.10: Contours in triangle example.

- If we traverse the sides of the triangle in a counterclockwise direction, then we must traverse successively OA, AB BO.
- On each of these sides the function representations of the paths are single-valued:
  - OA: $y=0 \quad \Rightarrow \quad dy=0, \quad 0 \le x \le 1,$
  - AB: $x=1 \quad \Rightarrow \quad dx=0, \quad 0 \le y \le 1,$
  - BO: $y=x, \quad \Rightarrow \quad dy=dx \quad 1 \ge x \ge 0.$
- Hence the integral is the sum of three integrals over OA, AB, BO as follows:

$$\begin{aligned}
\oint_{\mathcal{C}} \left\{2xydy - x^2dx\right\} &= \int_{OA} \left\{2xydy - x^2dx\right\} + \int_{AB} \left\{2xydy - x^2dx\right\} + \int_{BO} \left\{2xydy - x^2dx\right\}, \\
&= \int_{OA} \left\{0 - x^2dx\right\} + \int_{AB} \left\{2 \times 1 \times ydy - 0\right\} + \int_{BO} \left\{2x \times xdx - x^2dx\right\}, \\
&= -\int_{x=0}^{x=1} x^2dx + 2\int_{y=0}^{y=1} ydy + \int_{x=1}^{x=0} x^2dx, \\
&= -\left[\frac{x^3}{3}\right]_{x=0}^{x=1} + 2\left[\frac{y^2}{2}\right]_{y=0}^{y=1} + \left[\frac{x^3}{3}\right]_{x=1}^{x=0}, \\
&= -\frac{1}{3} + 1 - \frac{1}{3}, \\
&= \frac{1}{3} \neq 0.
\end{aligned}$$

Note carefully the order of the integration limits.

Example: Evaluate the line integral

$$\oint_{\mathcal{C}} ydx,$$

where $\mathcal{C}$ is the circle $x^2 + y^2 = a^2$.

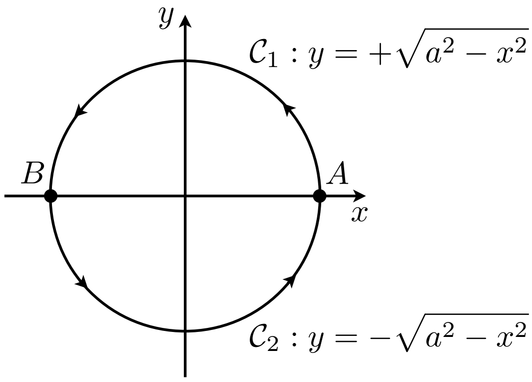

Figure 5.11: Contours in circle example.

- The representation of $\mathcal{C}$, $y = \sqrt{a^2 - x^2}$ is not single-valued.
- We split the contour up into two halves on which the contour is a single-valued function of x:
  - Upper semi-circle $\mathcal{C}_1$: $y = +\sqrt{a^2 - x^2}$ with $+a \geq x \geq -a$,
  - Lower semi-circle $\mathcal{C}_2$: $y = -\sqrt{a^2 - x^2}$ with $-a \leq x \leq a$.
- Hence the integral is the sum of two integrals as follows:

$$
\begin{aligned}
\oint_{\mathcal{C}} y dx &= \int_{\mathcal{C}_1} y dx + \int_{\mathcal{C}_2} y dx, \\
&= + \int_{x=+a}^{x=-a} \sqrt{a^2 - x^2} dx + \int_{x=-a}^{x=+a} \left(-\sqrt{a^2 - x^2}\right) dx, \\
&= - \int_{x=-a}^{x=+a} \sqrt{a^2 - x^2} dx - \int_{x=-a}^{x=+a} \sqrt{a^2 - x^2} dx, \\
&= -2 \int_{x=-a}^{x=+a} \sqrt{a^2 - x^2} dx, \\
&= -4 \int_{x=0}^{x=+a} \sqrt{a^2 - x^2} dx, \\
&= -\pi a^2.
\end{aligned}
$$

The integration can be done by a change of variables using $x = a \sin u$.

Note that, up to a sign, this line integral gives the area of the enclosed circle. We could also have found the same result by integrating

$$\oint_{\mathcal{C}} x dy.$$

In fact a general result is that

$$\oint_{\mathcal{C}} x dy = - \oint_{\mathcal{C}} y dx = \frac{1}{2} \oint_{\mathcal{C}} (x\, dy - y\, dx) = A,$$

where A is the area of the shape enclosed by the nonintersecting closed curve $\mathcal{C}$. We shall prove this result later.

5.5 Connectivity of Regions

We now introduce the concept of connectivity of a planar region. This will be used in later sections.

Definition: (Connectivity) A plane region $\mathcal{R}$ is said to be simply connected if every simple (i.e., non-intersecting) closed curve $\mathcal{C}$ lying in $\mathcal{R}$ can be continuously shrunk to a point within $\mathcal{R}$

This idea is illustrated in see figure on the left below.

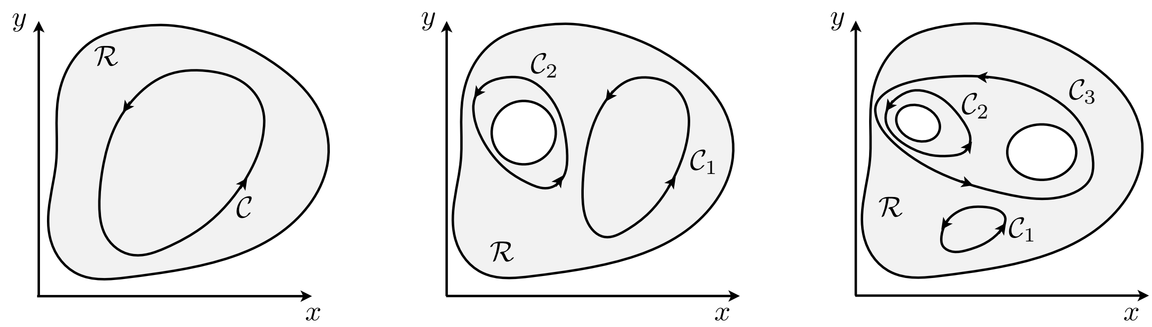

Figure 5.12: Left: Simply connected region $\mathcal{R}$. Middle: Doubly connected region $\mathcal{R}$. Right: Triply connected region $\mathcal{R}$.

In the middle diagram above the curve $\mathcal{C}_1$ may be shrunk to a point without leaving $\mathcal{R}$. However the curve $\mathcal{C}_2$ cannot be shrunk to a point in $\mathcal{R}$. There are thus two types of curves in $\mathcal{R}$, those like $\mathcal{C}_1$ and those like $\mathcal{C}_2$. $\mathcal{R}$ is this not simply connected. Due to the existence of two different types of curves in $\mathcal{R}$ it is, however, said to be doubly connected.

Similarly in the right hand diagram above, the region $\mathcal{R}$ is said to be triply connected as there are three different types of curves:

- $\mathcal{C}_1$ those that may be shrunk to a point;
- $\mathcal{C}_2$ those that may be shrunk to encircle one hole;
- $\mathcal{C}_3$ those that may be shrunk to encircle two holes.

In general if there are $n-1$ holes in region $\mathcal{R}$, the region is said to be $n-$fold connected.

5.6 Independence of Path

We now deduce the conditions under which the value of a line integral with the same endpoints is independent of the path between them.

Suppose $F(x, y)$ is a single valued continuous function of (x, y) with single valued and continuous first partial derivatives in a region $\mathcal{R}$ of the $(x, y)-$plane.

Let $\mathcal{C}$ be a curve lying entirely within $\mathcal{R}$ with endpoints $A : (x_1, y_1)$ and $B : (x_1, y_1)$. We define $\mathcal{C}$ parametrically as:

$$\mathcal{C} : x = f(t), y = g(t),$$

for $t_1 \leq t \leq t_2$ with $f(t_i) = x_i, g(t_i) = y_i, i = 1, 2$.

Consider the following line integral of $F(x, y)$ along $\mathcal{C}$:

$$\int_{\mathcal{C}} \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$$

This can be written in terms of the parameter t as follows

$$\begin{aligned}
\int_{\mathcal{C}} \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy &= \int_{\mathcal{C}} \frac{\partial F}{\partial x} \frac{dx}{dt} dt + \frac{\partial F}{\partial y} \frac{dy}{dt} dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} \right) dt \\
&= \int_{t_1}^{t_2} \frac{dF(f(t), g(t))}{dt} dt \\
&= F(f(t_2), g(t_2) - F(f(t_1), g(t_1)) \\
&= F(x_2, y_2) - F(x_1, y_1),
\end{aligned}$$

where we have used the chain rule to go between the second and third lines.

Note that the final result only depends on the endpoints and not on the actual contour of integration $\mathcal{C}$.

Thus we can now see that the condition for the value of a line integral of the form

$$I = \int_{\mathcal{C}} \{P(x, y)dx + Q(x, y)dy\},$$

to be independent of the contour of integration $\mathcal{C}$ is the existence of a function $F(x, y)$ such that

$$P(x, y) = \frac{\partial F(x, y)}{\partial x} \qquad Q(x, y) = \frac{\partial F(x, y)}{\partial y}.$$

The additional trick (*cf.* first order exact ODEs) is to observe that we have no need to actually find the function $F(x, y)$. The conditions assumed for F means that it's mixed partial derivatives exist and are equal. Thus we have:

$$\frac{\partial P(x, y)}{\partial y} = \frac{\partial^2 F(x, y)}{\partial y \partial x} = \frac{\partial^2 F(x, y)}{\partial x \partial y} = \frac{\partial Q(x, y)}{\partial x}.$$

Hence the condition for the existence of $F(x, y)$ and therefore the path independence of I is that for $(x, y) \in \mathcal{R}$

$$\boxed{\frac{\partial P(x, y)}{\partial y} = \frac{\partial Q(x, y)}{\partial x}.}$$

A caveat to this is that if $\mathcal{R}$ is not simply connected, this result does not hold.

5.6.1 Path Independence for Closed Curves

Consider the following path integral around a closed curve in a simply connected region $\mathcal{R}$:

$$I = \oint_{\mathcal{C}} \{P(x,y)dx + Q(x,y)dy\}\,.$$

If the integral is independent of path, then we have $I = 0$ for all simple closed curves in $\mathcal{R}$.

The proof is straightforward.

Consider two paths between the points A and B in a simply connected region $\mathcal{R}$. Let the endpoints be connected by two paths $\mathcal{C}_1$ and $\mathcal{C}_2$ as in the diagram

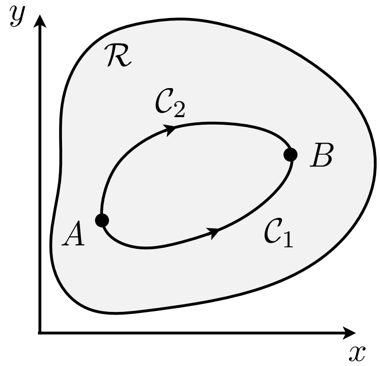

Figure 5.13: Two different paths $\mathcal{C}_1$ and $\mathcal{C}_2$ between the endpoints A and B in a simply connectd region $\mathcal{R}$.

Then we have

$$\begin{aligned} I &= \oint_{\mathcal{C}} \{P(x,y)dx + Q(x,y)dy\} \\ &= \int_{\mathcal{C}_1} \{P(x,y)dx + Q(x,y)dy\} - \int_{\mathcal{C}_2} \{P(x,y)dx + Q(x,y)dy\} \\ &= 0, \end{aligned}$$

since, by assumption the integrals are independent of path, and so the terms in the second line are identical.

The reverse also holds, namely that if

$$\oint_{\mathcal{C}} \{P(x,y)dx + Q(x,y)dy\} = 0$$

for all simple closed curves $\mathcal{C}$ in the simply connected region $\mathcal{R}$, then the integral

$$\int \{P(x,y)dx + Q(x,y)dy\}$$

is independent of the path between any two given endpoints in $\mathcal{R}$.

When you come to study complex analysis, the path independence of integrals of complex analytical functions is called Cauchy's theorem.

Example: Consider the integral

$$I = \int_A^B (y\cos x\,dx + \sin x\,dy),$$

where A is the point $(0,0)$ and B is the point $(\pi/4, \pi/4)$.

This integral is independent of the path jointing A and B in the (x,y) plane, since if we set

$$P(x,y) = y\cos x, \qquad Q(x,y) = \sin x,$$

we observe that

$$\frac{\partial P}{\partial y} = \cos x = \frac{\partial Q}{\partial x}.$$

Consequently, from above, the line integral can be written as an exact derivative of a function $F(x,y)$. We can find what $F(x,y)$ is by integration both partial derivatives and comparing the results (see Chapter 1):

$$\frac{\partial F}{\partial x} = y\cos x \quad \Rightarrow \quad F(x,y) = \int y\cos x\, dx = y\sin x + g(y); \tag{5.1}$$

$$\frac{\partial F}{\partial y} = \sin x \quad \Rightarrow \quad F(x,y) = \int \sin x\, dy = y\sin x + h(y);$$

where $g(y)$ and $h(x)$ are arbitrary functions. A comparison of the two expressions shows that they can only be equal if $g(y) = h(x) =$ constant. Hence up to an arbitrary constant (which will vanish in the final definite integration below) it is not too difficult to see that:

$$F(x,y) = y\cos x.$$

Thus the integral can be easily evaluated as:

$$I = \int_A^B (y\cos x\, dx + \sin x\, dy) = \int_A^B d(y\sin x) = [y\sin x]_{(0,0)}^{(\pi/4,\pi/4)} = \frac{\pi}{4\sqrt{2}}.$$

5.7 Green's Theorem

Green's theorem is one of the most fundamental and beautiful theorems involving line integrals.[1] It links the value of an integral over a curve to that of an integral over the region the curve bounds. It is of fundamental importance in many fields of the physical and mathematical sciences. It is a two-dimensional special case of the Kelvin- Stokes and divergence theorems and can also be used to prove results involving integrals in complex analysis, all of which you may learn next year.

[1] Green's Theorem is named after George Green (1793-1841). He was self-taught and had minimal schooling as a child. His father was a baker and Green spent much of his adult life running a windmill in Sneinton in Nottinghamshire. Nevertheless, in 1828 he privately published a text on the application of mathematics to electricity and magnetism. At the time, only 51 people bought it, but it contains results of fundamental importance in many fields of mathematics and science. Some of the subscribers encouraged Green to go to Cambridge to do an undergraduate degree. He entered in 1832, aged 39 and finally graduated in 1838 with a BA, some 10 years after his research work (thereby proving that you don't need to have been anywhere near Cambridge to change the world)! He died only 3 years later, with some rumours of alcoholism. The picture on the back of these lecture notes is of your lecturer pondering some maths on Green's grave in the churchyard in Sneinton. Green's mill still stands today and can be visited both physically and electonically

5.7.1 Statement of Green's Theorem

Theorem: (Green's Theorem) Suppose $P(x, y)$ and $Q(x, y)$ are two functions which are finite and continuous inside and on the non- self-intersecting boundary $\mathcal{C}$ of some simply connected region $\mathcal{R}$ of the (x, y)-plane. If the first partial derivatives of the these functions are continuous inside and on the boundary of $\mathcal{R}$, then

$$\boxed{\oint_{\mathcal{C}} \{P(x,y)\,dx + Q(x,y)\,dy\} = \iint_{\mathcal{R}} \left\{\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y}\right\} dx\,dy.}$$

5.7.2 Sketch Proof of Green's Theorem

The proof of the theorem when the boundary $\mathcal{C}$ is a simple (non-self-intersecting) closed curve and $\mathcal{R}$ is simply connected, is relatively straightforward. The following diagram will be helpful.

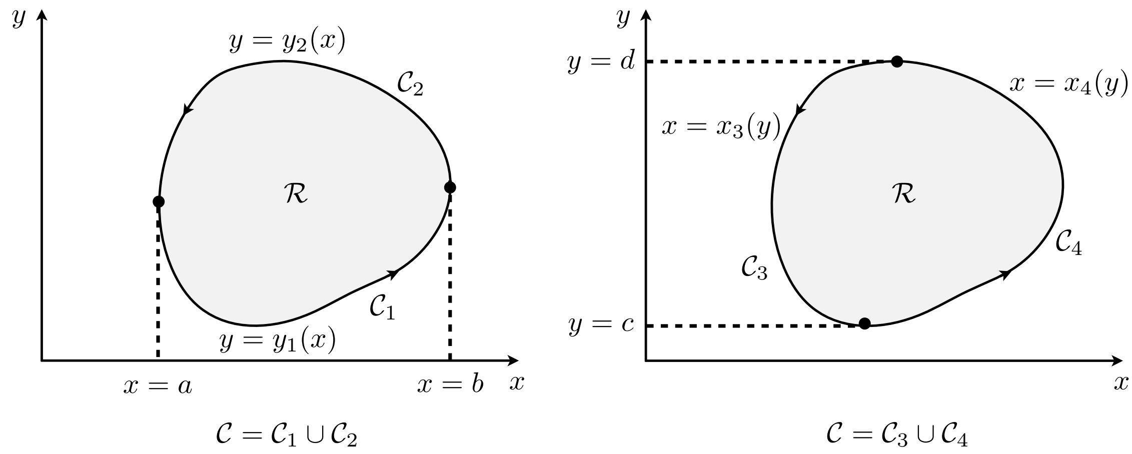

Figure 5.14: The region $\mathcal{R}$ is the same in both diagrams. The boundary curve $\mathcal{C}$ is also the same. In the left-hand diagram, $\mathcal{C}$ decomposed into two curves $\mathcal{C}_1$, $\mathcal{C}_2$, with single valued representations $y = y_1(x)$ and $y = y_2(x)$, respectively with $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. In the right-hand diagram, the same boundary $\mathcal{C}$ is now decomposed into two curves $\mathcal{C}_3$, $\mathcal{C}_4$, with single valued representations $x = x_3(y)$ and $x = x_4(y)$, respectively with $\mathcal{C} = \mathcal{C}_3 \cup \mathcal{C}_4$.

- Consider the 2-d integral

$$\begin{aligned}
\iint_{\mathcal{R}} \frac{\partial P(x,y)}{\partial y}\,dx\,dy &= \int_{x=a}^{x=b} dx \int_{y=y_1(x)}^{y=y_2(x)} \frac{\partial P(x,y)}{\partial y}\,dy \\
&= \int_{x=a}^{x=b} \Big[P(x,y)\Big]_{y=y_1(x)}^{y=y_2(x)} dx \\
&= \int_{x=a}^{x=b} \{P(x,y_2(x)) - P(x,y_1(x))\}\,dx \\
&= -\int_{x=a}^{x=b} P(x,y_1(x))\,dx + \int_{x=a}^{x=b} P(x,y_2(x))\,dx \\
&= -\int_{x=a}^{x=b} P(x,y_1(x))\,dx - \int_{x=b}^{x=a} P(x,y_2(x))\,dx
\end{aligned}$$

$$= -\oint_{\mathcal{C}} P(x,y)\,dx.$$

- Now consider the 2-d integral

$$\begin{aligned}
\iint_{\mathcal{R}} \frac{\partial Q(x,y)}{\partial x}\,dx\,dy &= \int_{y=c}^{y=d} dy \int_{x=x_3(y)}^{x=x_4(y)} \frac{\partial Q(x,y)}{\partial x}\,dx \\
&= \int_{y=c}^{y=d} \Big[Q(x,y)\Big]_{x=x_3(y)}^{x=x_4(y)}\,dy \\
&= \int_{y=c}^{y=d} \{Q(x_4(y),y) - Q(x_3(y),y)\}\,dy \\
&= \int_{y=c}^{y=d} Q(x_4(y),y)\,dy - \int_{y=c}^{y=d} Q(x_3(y),y)\,dy \\
&= \int_{y=c}^{y=d} Q(x_4(y),y)\,dy + \int_{y=d}^{y=c} Q(x_3(y),y)\,dy \\
&= \oint_{\mathcal{C}} Q(x,y)\,dy.
\end{aligned}$$

- Subtracting the first result from the second delivers the following result, as required:

$$\oint_{\mathcal{C}} \{P(x,y)\,dx + Q(x,y)\,dy\} = \iint_{\mathcal{R}} \left\{\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y}\right\} dx\,dy.$$

5.7.3 Special Case of Green's Theorem

A special case of Green's theorem leads to a surprising and immediately useful result.
Consider the special case of

$$P(x,y) = y, \qquad Q(x,y) = -x.$$

Inserting these functions into Green's theorem we obtain:

$$\begin{aligned}
\oint_{\mathcal{C}} \{P(x,y)\,dx + Q(x,y)\,dy\} &= \iint_{\mathcal{R}} \left\{\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y}\right\} dx\,dy. \\
\Rightarrow \qquad \oint_{\mathcal{C}} \{y\,dx - x\,dy\} &= \iint_{\mathcal{R}} \left\{-\frac{\partial x}{\partial x} - \frac{\partial y}{\partial y}\right\} dx\,dy. \\
\Rightarrow \qquad \oint_{\mathcal{C}} \{y\,dx - x\,dy\} &= -2\iint_{\mathcal{R}} dx\,dy. \\
\Rightarrow \qquad \frac{1}{2}\oint_{\mathcal{C}} \{x\,dy - y\,dx\} &= \iint_{\mathcal{R}} dx\,dy = A. \\
\Rightarrow \qquad A &= \frac{1}{2}\oint_{\mathcal{C}} \{x\,dy - y\,dx\}.
\end{aligned}$$

In other words, the area of the simply-connected region enclosed by the non-self-intersecting curve $\mathcal{C}$ can be given by a simple line integral. This result underpins the principle of the **planimeter**, a mechanical device for measuring the area of an arbitrary two dimensional shape.

Note that the above area result can also be simplified by making the same substitutions $P(x,y) = y, Q(x,y) = -x$ into the two integrals used to prove Green's theorem above. For then they generate:

$$\begin{aligned} A &= \iint_{\mathcal{R}} dx\, dy = -\oint_{\mathcal{C}} y\, dx, \\ A &= \iint_{\mathcal{R}} dx\, dy = \oint_{\mathcal{C}} x\, dy. \end{aligned}$$

5.7.4 Examples of the use of Green's Theorem

Example: Use Green's theorem to evaluate the line integral

$$I = \oint_{\mathcal{C}} \left\{ (x^2 + y)\, dx + (x + \sin y)\, dy \right\},$$

where $\mathcal{C}$ is the boundary of the unit square with vertices $(0,0), (1,0), (1,1), (0,1)$.

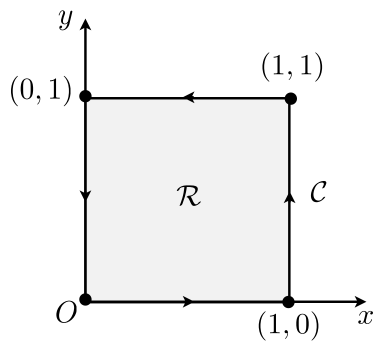

Figure 5.15: Contour $\mathcal{C}$ around unit square in example.

- From the form of the line integral, we identify

$$P(x,y) = x^2 + y, \qquad Q(x,y) = x + \sin y.$$

- The functions P and Q are continuous and possess continuous first partial derivatives (obvious).
- The integration curve $\mathcal{C}$ is not self intersecting and encircles a region $\mathcal{R}$ that is simply connected.
- Hence we may apply Green's theorem.
- Hence we have

$$\begin{aligned} I = \oint_{\mathcal{C}} \left\{ (x^2 + y)\, dx + (x + \sin y)\, dy \right\} &= \iint_{\mathcal{R}} \left\{ \frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} \right\} dx\, dy, \\ &= \iint_{\mathcal{R}} \left\{ \frac{\partial (x + \sin y)}{\partial x} - \frac{\partial (x^2 + y)}{\partial y} \right\} dx\, dy, \end{aligned}$$

$$= \int_{x=0}^{x=1} dx \int_{y=0}^{y=1} dy \, \{1 - 1\},$$
$$= 0.$$

Example: Use Green's theorem to evaluate the line integral

$$I = \oint_{\mathcal{C}} \left\{ (e^x - 2y) \, dx + (x + \cos(1457 y^{956})) \, dy \right\},$$

where $\mathcal{C}$ is the boundary of the circle, centred at the origin and radius a.

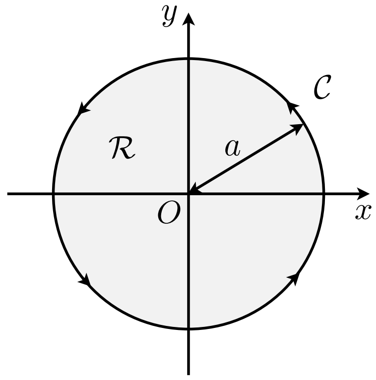

Figure 5.16: $\mathcal{C}$ is a circle of radius a in the example.

- From the form of the line integral, we identify

$$P(x, y) = e^x - 2y, \qquad Q(x, y) = x + \cos(1457 y^{956}).$$

- The functions P and Q are continuous and possess continuous first partial derivatives (obvious).
- The integration curve $\mathcal{C}$ is not self intersecting and encircles a region $\mathcal{R}$ that is simply connected.
- Hence we may apply Green's theorem.
- Hence we have

$$\begin{aligned} I = \oint_{\mathcal{C}} \left\{ (e^x - 2y) \, dx + x + \cos(1457 y^{956}) \, dy \right\} &= \iint_{\mathcal{R}} \left\{ \frac{\partial Q(x, y)}{\partial x} - \frac{\partial P(x, y)}{\partial y} \right\} dx \, dy, \\ &= \iint_{\mathcal{R}} \left\{ \frac{\partial (x + \cos(1457 y^{956}))}{\partial x} - \frac{\partial (e^x - 2y)}{\partial y} \right\} dx \, dy, \\ &= \iint_{\mathcal{R}} \{1 + 2\} \, dx \, dy, \end{aligned}$$

$$= \quad 3 \iint_{\mathcal{R}} dx\, dy,$$

$$= \quad 3 \times \text{Area of circle, radius } a,$$

$$= \quad 3\pi a^2.$$

Example: Use Green's theorem to find the area of an ellipse whose boundary is given by given by

$$x = a\cos t, \qquad y = b\sin t, \qquad 0 \le t \le 2\pi.$$

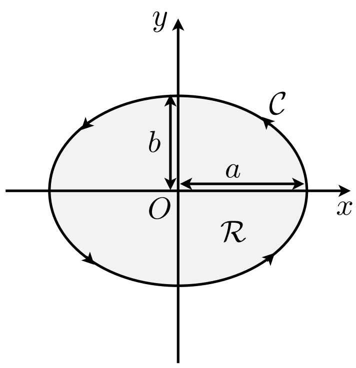

Figure 5.17: Ellipse of semi-major axis a and semi-minor axis b.

- We use the special case of Green's theorem:

$$A = \frac{1}{2}\oint_{\mathcal{C}} \{x\, dy - y\, dx\},$$

  where A is the area of the ellipse and $\mathcal{C}$ is its perimeter.

- On $\mathcal{C}$ we have

$$x = a\cos t \qquad \Rightarrow \qquad dx = -a\sin t,$$

$$y = b\sin t \qquad \Rightarrow \qquad dy = b\cos t.$$

- The integration curve $\mathcal{C}$ is not self intersecting and encircles a region $\mathcal{R}$ that is simply connected.
- Hence we may apply Green's theorem.
- Hence we have

$$\begin{aligned} A &= \frac{1}{2}\oint_{\mathcal{C}} \{x\, dy - y\, dx\}, \\ &= \frac{1}{2}\oint_{\mathcal{C}} \{a\cos t\, dy - b\sin t\, dx\}, \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{t=0}^{t=2\pi} \left\{(a\cos t)\times(b\cos t)dt - b\sin t\times(-a\sin t)dt\right\},\\
&= \frac{1}{2}\int_{t=0}^{t=2\pi} \left\{(ab\cos^2 t + ab\sin^2 t)dt\right\},\\
&= \frac{ab}{2}\int_{t=0}^{t=2\pi} \left\{(\cos^2 t + \sin^2 t)dt\right\},\\
&= \frac{ab}{2}\int_{t=0}^{t=2\pi} dt,\\
&= \frac{ab}{2}\times 2\pi,\\
&= \pi ab.
\end{aligned}
$$