Chapter 7

Collisions

As an application of the conservation of momentum, we will consider a set of problems involving the collision of two bodies. This is a common set-up in Nature, ranging from bar-room billiards to particle collisions in the Large Hadron Collider at CERN. We will discuss the classic Newtonian problem, but also take a look at how the results change if we are dealing with velocities near the speed of light.

In essence, we will make use of the concept of conservation of momentum, but before we do, we need to discuss what happens when momentum is **not** conserved. For this we need to introduce the idea of an **impulse**.

7.1 Impulses

In many situations a body is subjected to a rapidly applied force, e.g., the collision of two cars, a ball hitting the wall or even a punch in the face. In such situations, the velocity of the particle changes rapidly in a short time.

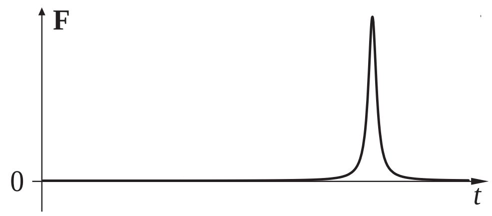

Figure 7.1: Short timescale, large amplitude force.

Consider an event where the force is applied for a brief period of time $t_1 < t < t_2$. According to **N2** the velocity change will involve a momentum change:

$$\vec{F}(t) = \frac{d\vec{p}}{dt}.$$

This momentum change is given by integrating both sides of **N2** to give

$$\vec{I} \equiv \int_{t_1}^{t_2} \vec{F}(t)dt = \vec{p}(t_2) - \vec{p}(t_1).$$

The time integral of $\vec{F}$ is called the **impulse** of the force and is denoted by $\vec{I}$. The impulse is therefore the change in momentum of the body due to the rapidly applied force.

When two particles collide, in general an impulsive force acts on each one. If the collision is rapid, the effect of forces acting over significantly longer timescales may be insignificant. We can then use the concepts of conservation of momentum and impulse to study the effect of the collision on the particles.

7.2 Conservation of Total Momentum

Consider two bodies that collide:

- Body A with momentum $\vec{p}_A = m_A \vec{v}_A$.
- Body B with momentum $\vec{p}_B = m_B \vec{v}_B$.

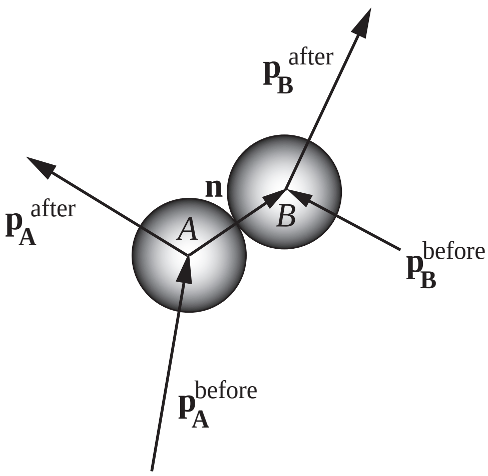

Figure 7.2: When worlds collide.

The collision is rapid and causes an impulse to each body. Assuming no other forces act during this rapid interaction, then by the time-integral of **N2** discussed above

$$\vec{p}_A^{\text{after}} - \vec{p}_A^{\text{before}} = \vec{I}_A, \qquad \vec{p}_B^{\text{after}} - \vec{p}_B^{\text{before}} = \vec{I}_B$$

Now by **N3** at the moment of contact the bodies exert equal, but opposite forces on each other, so we must have

$$\vec{I}_A = -\vec{I}_B$$

and moreover these act in the direction of $\pm\vec{n}$, where $\vec{n}$ is a unit vector along the line from the centre of one body to the centre of the other. Thus adding the two momentum equations we have

$$\left(\vec{p}_A^{\text{after}} - \vec{p}_A^{\text{before}}\right) + \left(\vec{p}_B^{\text{after}} - \vec{p}_B^{\text{before}}\right) = \vec{0}$$

$$\therefore \qquad \underbrace{\left(\vec{p}_A^{\text{after}} + \vec{p}_B^{\text{after}}\right)}_{\text{total momentum after}} = \underbrace{\left(\vec{p}_A^{\text{before}} + \vec{p}_B^{\text{before}}\right)}_{\text{total momentum before}}$$

The **total** momentum of the system is conserved in the impact.

7.3 Newton's Law of Restitution

Conservation of total momentum alone does not allow us to determine the momentum of each particle separately after the collision: it gives us but one vector equation for two vector unknowns $\vec{p}_A^{\text{after}}$ and $\vec{p}_B^{\text{after}}$. We need more information.

The nature of the forces acting between the bodies depends on their material properties. A pair of steel ball bearings will rebound in a different way than say, two cars. To assess the effect of differing material properties on the collision would be (far too) complicated. Fortunately, experiment leads to a simple empirical law.

Newton's Law of Restitution: When two bodies collide, their relative parting velocity in the direction of their line of contact $\vec{n}$ ($|\vec{n}| = 1$), is $-e$ times their relative approaching velocity in the same direction.

$$\left(\vec{v}_A^{\text{after}} - \vec{v}_B^{\text{after}}\right) \cdot \vec{n} = -e \left(\vec{v}_A^{\text{before}} - \vec{v}_B^{\text{before}}\right) \cdot \vec{n}$$

Note that this law contains the **velocities** and **not** the momenta of the bodies.

The number e is an empirically determined constant that depends on the nature of the particles and encompasses the effect of their material properties on the collision. It is called the **coefficient of restitution**.

- If $e = 1$ the collision is said to be perfectly elastic (a theoretical ideal).
- If $e = 0$, the collision is perfectly inelastic (like throwing mud at a wall).

In practice the coefficient of restitution takes a value in the range $0 < e < 1$. Only in the case of a perfectly elastic collision is the total kinetic energy of the system conserved. In an inelastic collision a proportion of the kinetic energy will be converted into another form of energy, such as heat, light or sound.

7.4 Constancy of velocities perpendicular to line of collision

We still don't have enough information to calculate the velocities of the bodies after the collision. The final ingredient follows from the definition of an impulse: *if there is no friction between the bodies*, no impulsive force will act perpendicular to their line of centres. Hence we deduce that, for frictionless collisions:

- the momentum component of each particle in a direction orthogonal to the line of centres $\vec{n}$, is conserved ;
- hence if the masses remain unchanged, the velocity of each particle in the direction orthogonal to $\vec{n}$ is unchanged.

Mathematically, we have **conservation of perpendicular velocities**: When the two bodies collide, if the unit vector $\vec{t}$ is perpendicular to the direction of their line of contact $\vec{n}$, then

$$\vec{v}_A^{\text{after}} \cdot \vec{t} = \vec{v}_A^{\text{before}} \cdot \vec{t}, \qquad \vec{v}_B^{\text{after}} \cdot \vec{t} = \vec{v}_B^{\text{before}} \cdot \vec{t}.$$

This is important: It gives us enough information to close the system of equations and solve for the motion of the bodies after the collision.

7.5 Collisions and Non-conservation of Kinetic Energy

As the ideas we have outlined are best illustrated by actual examples, we will consider a series of (increasingly involved) problem settings. It is natural to start with a one-dimensional situation.

Two particles of mass m_1 and m_2 are moving in the same direction with speeds u_1 and u_2, respectively and collide. Find their subsequent velocities u_1' and u_2' if their coefficient of restitution is e.

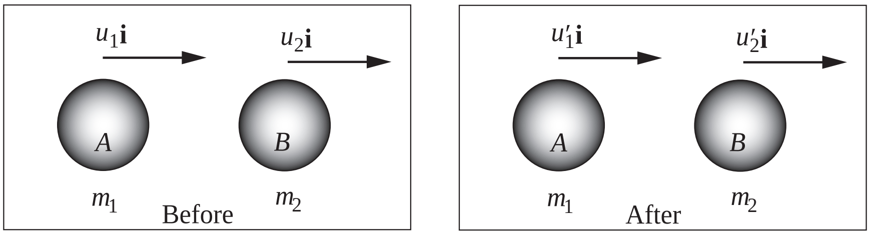

Figure 7.3: Collision parallel to the line of centres.

When the particles collide, their line of contact will be along (say) the $\vec{n} = \hat{i}$ direction and they each experience an equal impulse of magnitude I, but in opposite directions.

- Conservation of momentum for ball 1:

$$\text{(A)} \qquad m_1 u_1' \hat{i} = m_1 u_1 \hat{i} + I\hat{i},$$

  where the sign of the impulse has been chosen arbitrarily.

- Conservation of momentum for ball 2:

$$\text{(B)} \qquad m_2 u_2' \hat{i} = m_2 u_2 \hat{i} - I\hat{i},$$

  where the sign of the impulse has been determined to be opposite to the choice in (A).

- Newton's law of restitution acts in the **i** direction is:

$$\text{(C)} \qquad \left(u_2'\hat{i} - u_1'\hat{i}\right) \cdot \hat{i} = -e\left(u_2\hat{i} - u_1\hat{i}\right) \cdot \hat{i}$$

- Since there is no initial motion perpendicular to the line of contact of the bodies, there is no motion perpendicular afterwards. It is enough to consider the components u_1, u_2, u_1', u_2'.

- Resolving into components we have three equations for the three unknowns,

$$
\begin{array}{lrcl}
\text{(A)} & m_1 u_1' & = & m_1 u_1 + I \\
\text{(B)} & m_2 u_2' & = & m_2 u_2 - I \\
\text{(C)} & u_2' - u_1' & = & -e(u_2 - u_1)
\end{array}
$$

- Remove I by considering (A)+(B) $\Rightarrow$

$$m_1 u_1' + m_2 u_2' = m_1 u_1 + m_2 u_2$$

With (C), we now have two equations for the two unknowns u_1' and u_2'. These can be solved simultaneously to give:

$$
\begin{aligned}
u_1' &= \frac{u_1(m_1 - e m_2) + m_2 u_2 (1+e)}{(m_1 + m_2)} \\
u_2' &= \frac{u_2(m_2 - e m_1) + m_1 u_1 (1+e)}{(m_2 + m_2)}
\end{aligned}
$$

I may be calculated by back-substitution of u_1' or u_2' into (A) or (B). This gives

$$I = \frac{m_1 m_2}{m_1 + m_2}(1+e)(u_2 - u_1).$$

Note that often we are not so much interested in I as the velocities after the collision. Hence we do not always bother to determine it.

7.5.1 Elastic and inelastic collisions

Note that if $e = 1$ and $m_1 = m_2$, the speeds of the particles interchange at the collision, i.e., $u_1' = u_2$, $u_2' = u_1$. Calculations of this type underlie the modelling of the executive toy known as "Newton's Cradle".

Note also that when the collision is perfectly elastic the kinetic energy is conserved.

The total kinetic energy before collision

$$T_{\text{before}} = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2$$

The total kinetic energy after the collision

$$
\begin{aligned}
T_{\text{after}} &= \frac{1}{2}m_1 {u'}_1^2 + \frac{1}{2}m_2 {u'}_2^2 \\
&= \frac{1}{2}m_1 \left\{ \frac{u_1(m_1 - m_2) + 2m_2 u_2}{(m_1 + m_2)} \right\}^2 + \frac{1}{2}m_2 \left\{ \frac{u_2(m_2 - m_1) + 2m_1 u_1}{(m_2 + m_2)} \right\}^2 \\
&= \cdots \\
&= \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2
\end{aligned}
$$

If $e \neq 1$ then the total kinetic energy is **not** conserved. Where does this energy go? The **total** energy **has** been conserved, but some of the kinetic energy has been converted into other forms of energy. This could be (for example) heat, acoustic, or even light energy. The mechanical problem can predict the amount of the kinetic energy that is transformed, but not the form the energy takes.

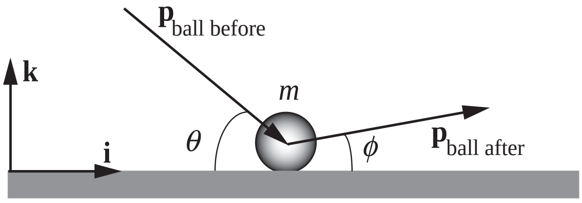

Figure 7.4: Ball bouncing off a wall.

Example 1: Bouncing ball

A ball of mass m hits the ground. The ball is travelling at speed u at an angle θ to the ground, as in figure 7.4. If the coefficient of restitution of impact is e, (and there is no spin on the ball) find the velocity after the impact.
Before impact:

$$\vec{p}_{\text{ball before}} = m(u\cos\theta\hat{i} - u\sin\theta\hat{k})$$

At impact: line of centres is along $\hat{k}$, hence the impulse takes the form

$$\vec{I} = I\hat{k}$$

After impact: assume that ground does not move,

$$\vec{p}_{\text{ball after}} = m(u'\cos\phi\hat{i} + u'\sin\phi\vec{k})$$

Conservation of Momentum for ball:

$$m(u'\cos\phi\hat{i} + u'\sin\phi\hat{k}) = m(u\cos\theta\vec{i} - u\sin\theta\hat{k}) + I\hat{k}$$

Now apply Newton's law of restitution:

$$\hat{k}\cdot[\vec{v}_{\text{ball, after}} - \vec{v}_{\text{ground, after}}] = -e\hat{k}\cdot[\vec{v}_{\text{ball, before}} - \vec{v}_{\text{ground, before}}]$$

Assume that ground does not move, so that $\Rightarrow \vec{v}_{\text{ground, before}} = \vec{v}_{\text{ground, after}} = \vec{0}$,

$$(u'\cos\phi\hat{i} + u'\sin\phi\hat{k} - \vec{0})\cdot\vec{k} = -e(u\cos\theta\hat{i} - u\sin\theta\hat{k} - \vec{0})\cdot\vec{k}$$

Resolving the conservation of momentum equations into $\hat{i}$ and $\hat{k}$ components these give 3 equations for the three unknowns, u', ϕ and I.,

$$\begin{array}{llll} \text{(A)} & mu'\cos\phi & = & mu\cos\theta \\ \text{(B)} & mu'\sin\phi & = & -mu\sin\theta + I \\ \text{(C)} & u'\sin\phi & = & eu\sin\theta \end{array}$$

(C)/(A) $\Rightarrow$

$$\tan\phi = e\tan\theta$$

$(\text{A})^2$+$(\text{C})^2$ $\Rightarrow$

$$\begin{aligned} u'^2(\cos^2\phi + \sin^2\phi) &= u^2(\cos^2\theta + e^2\sin^2\theta) \\ \Rightarrow u' &= u\sqrt{1-(1-e^2)\sin^2\theta} \end{aligned}$$

If we require it, (B) can then be used to determined the impulse as

$$I = meu\sin\theta + mu\sin\theta = (1+e)mu\sin\theta$$

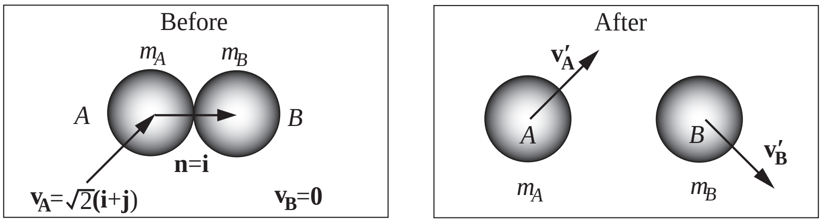

Figure 7.5: Oblique impact. (Two billiards are said to collide *obliquely*, if their initial velocities point in directions different to their line of centres at collision.)

Example 2: Oblique collision

Consider two colliding billiard balls A and B of equal mass m whose line of centres lie in the direction $\hat{\imath}$ at the point of collision, see figure 7.5. Let the velocities before the collision be $\vec{v}_A = \sqrt{2}\left(\hat{\imath}+\hat{\jmath}\right)$ and $\vec{v}_B = \vec{0}$. The coefficient of restitution between the balls is e. Find the velocities of the balls after collision.
We shall use the following notation for the post-collision velocities for A and B, respectively:

$$\vec{v}'_A = a\hat{\imath} + b\hat{\jmath}, \qquad \vec{v}'_B = \alpha\hat{\imath} + \beta\hat{\jmath}.$$

Conservation of momentum:

$$\begin{aligned} m_A\vec{v}_A + m_B\vec{v}_B &= m_A\vec{v}'_A + m_B\vec{v}'_B \\ \Rightarrow \vec{v}_A + \vec{0} &= \vec{v}'_A + \vec{v}'_B \\ \Rightarrow \sqrt{2}\left(\hat{\imath}+\hat{\jmath}\right) &= a\hat{\imath} + b\hat{\jmath} + \alpha\hat{\imath} + \beta\hat{\jmath} \\ \Rightarrow \sqrt{2} &= a+\alpha \quad \text{and} \quad \sqrt{2} = b+\beta. \end{aligned}$$

Newton's law of restitution along the line of centres $\vec{n} = \hat{\imath}$:

$$\begin{aligned} \left(\vec{v}'_A - \vec{v}'_B\right)\cdot\hat{\imath} &= -e\left(\vec{v}_A - \vec{v}_B\right)\cdot\hat{\imath} \\ \left(\left\{a\hat{\imath}+b\hat{\jmath}\right\} - \left\{\alpha\hat{\imath}+\beta\hat{\jmath}\right\}\right)\cdot\vec{\imath} &= -e\left(\sqrt{2}\left\{\hat{\imath}+\hat{\jmath}\right\} - \vec{0}\right)\cdot\hat{\imath} \\ a-\alpha &= -\sqrt{2}e. \end{aligned}$$

Conservation of (momentum) perpendicular to line of motion $\vec{t} = \hat{\jmath}$. The mass of either particle does not change (classically) as a result of the collition, so conservation of momentum perpendicular to the line of centres just becomes conservation of velocities:

$$\begin{aligned} \vec{v}'_A\cdot\hat{\jmath} = \vec{v}_A\cdot\hat{\jmath} &\quad\Rightarrow\quad b = \sqrt{2}, \\ \vec{v}'_B\cdot\hat{\jmath} = \vec{v}_B\cdot\hat{\jmath} &\quad\Rightarrow\quad \beta = 0. \end{aligned}$$

Combining the momentum and restitution equations and simultaneously solving for a and α, we have:

$$\begin{aligned}\sqrt{2} &= a + \alpha \\ -e\sqrt{2} &= a - \alpha \\ \Rightarrow a &= \frac{1}{\sqrt{2}}(1-e) \qquad \alpha = \frac{1}{\sqrt{2}}(1+e)\,.\end{aligned}$$

Hence drawing everything together, we have

$$\begin{aligned}\vec{v}'_A &= \frac{1}{\sqrt{2}}(1-e)\hat{\imath} + \sqrt{2}\hat{\jmath}, \\ \vec{v}'_B &= \frac{1}{\sqrt{2}}(1+e)\hat{\imath}.\end{aligned}$$

Note that the originally stationary ball B moves off along the line of centres at impact. If you play snooker you may already know that one aims to hit the coloured ball with the cue ball so that the line of centres points towards the pocket. This is a mathematical explanation of this phenomenon.
This is, of course, an idealised model. In practice, friction between balls and table may affect the results.

7.6 Centre of mass

Let us now consider what happens if the impact takes place at high velocities. In order to do this it is useful (as it simplifies the mathematics) to focus on elastic collisions[1]. In general all velocities are vectors, but in order to keep the discussion simple we will consider motion in one dimension, in which case the velocities can be treated as scalars.

Suppose we know the velocities of two particle before they collide, v_1 and v_2. These are measured in the "lab frame". If the two masses are m_1 and m_2 then the corresponding total (kinetic) energy is

$$T = T_1 + T_2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$

However, in many situations it is convenient to consider the problem in a different coordinate system, e.g. one that is moving along with the centre of mass. In this system, the two velocities are u_1 and u_2 (say) and, if the centre of mass moves with velocity V, we have the simple transformations

$$u_1 = v_1 - V$$

$$u_2 = v_2 - V$$

because (as we know from our everyday experience) velocities add (at least in Newtonian dynamics...).

[1] We are being lazy. It would be straightforward to extend the discussion to the general case by accounting for inelastic change in the energy of the collision products, but we would not gain much additional understanding by doing this.

We determine V by demanding that the total momentum vanishes in the centre of mass frame. With the total mass $M = m_1 + m_2$ we then have

$$MV = m_1 v_1 + m_2 v_2 \longrightarrow V = \frac{m_1 v_1 + m_2 v_2}{M}$$

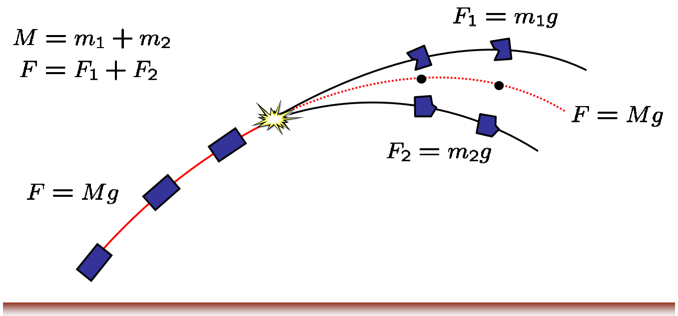

Figure 7.6: The motion of the centre-of-mass remains the same as if the object had stayed in one piece.

We can also assign a kinetic energy associated with the centre of mass:

$$T_{\text{cm}} = \frac{1}{2} M V^2$$

For an elastic collision the energy is conserved. In essence, this energy is not available for the collision, but... in general we have

$$T = T_{\text{cm}} + T_{\text{rel}}$$

where the remaining energy

$$T_{\text{rel}} = \frac{1}{2} \mu (v_2 - v_1)^2$$

is available for the collision. The quantity μ is called the reduced mass, and it is given by

$$\mu = \frac{m_1 m_2}{M}$$

We now have all the quantities we need to consider a collision in the centre of mass frame. Letting primed velocities represent the velocity of the particles after the collision, we have momentum conservation

$$m_1 u_1 + m_2 u_2 = m_1 u_1' + m_2 u_2'$$

and energy conservation

$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 (u_1')^2 + \frac{1}{2} m_2 (u_2')^2$$

We have two equations for two unknowns: u_1' and u_2'. We also know that in the centre of mass frame the total momentum is zero. So we actually have

$$m_1 u_1 + m_2 u_2 = 0 \longrightarrow u_1 = -\frac{m_2}{m_1} u_2$$

and as the same is true after the collision;

$$m_1 u_1' + m_2 u_2' = 0 \longrightarrow u_1' = -\frac{m_2}{m_1} u_2'$$

Combining these results with the conservation of energy (7.6) we see that we must have

$$u_2' = \pm u_2 \quad \text{and} \quad u_1 = \pm u_1$$

For an elastic collision (and in the centre of mass frame), there are only two options. Either the particles emerge with the same velocity as before (actually, a miss!) or the directions are reversed but the speeds stays the same.