Chapter 8

Rockets: Motion with variable mass

So far we have confined ourselves to studying the mechanics of bodies with constant mass. Let us now turn to an important variable mass problem, that of rocket motion (for which we no longer have $\vec{F} = m\vec{a}$). Although the mechanics is discussed in the context of a rocket, it could equally be applied to the acceleration of a jet fighter, the ascent of a balloon or even falling raindrops on a misty day.

We consider a rocket of mass $m(t)$ travelling at velocity $\vec{v}(t)$ in space (i.e., free from gravity and air resistance) by emitting a jet of gas at a speed $u > 0$ relative to the rocket. At an infinitesimally later time $t + dt$ suppose that the mass of the rocket is $m+dm$ due to the expulsion of a mass $-dm$ of gas through the back of the rocket.[1].

We want to use conservation of momentum to study this situation. If we denote the fixed inertial frame by S, the rocket frame by R, and the exhaust frame by E, then

$$\vec{v}_{ES} = \vec{v}_{ER} + \vec{v}_{RS}$$

We therefore have

$$\vec{v}_{ES} = -u\vec{n} + \vec{v}$$

where $\vec{n}$ is a unit vector pointing in whatever direction the rocket points.

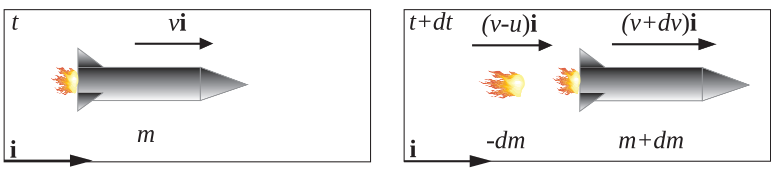

Figure 8.1: Rocket motion by ejection of gases.

[1] How can you have a negative mass of gas you ask? Well, you don't really. Note that $-dm$ is a positive quantity if dm is assumed negative. This is the convention that the most people have adopted!

At time t:

$$\text{Momentum of gas : } \vec{0} \qquad \text{Momentum of rocket : } m\vec{v}$$

At time $t + dt$:

$$\text{Momentum of gas :} -dm(-u\vec{n}+\vec{v}) \qquad \text{Momentum of rocket : } (m+dm)(\vec{v}+\vec{dv})$$

Conservation of total momentum gives:

$$m\vec{v} = (m + dm)(\vec{v} + \vec{dv}) + (-dm)(-u\vec{n} + \vec{v})$$

Expanding:

$$m\vec{v} = m\vec{v} + \vec{v}dm + md\vec{v} + u\vec{n}dm - \vec{v}dm$$

where terms beyond linear have been neglected. This can be rearranged (and turned into a differential equation by recalling that the change was associated with a time interval dt)

$$m\frac{d\vec{v}}{dt} = -u\vec{n}\frac{dm}{dt}$$

$$\Rightarrow m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt} = -u\vec{n}\frac{dm}{dt} + \vec{v}\frac{dm}{dt}$$

which has the interpretation:

$$\underbrace{\frac{d}{dt}(m\vec{v})}_{\text{rate of change of rocket momentum}} = \underbrace{(\vec{v} - u\vec{n})\frac{dm}{dt}}_{\text{force due to rocket motion}} = -\vec{v}_{ES}\frac{dm}{dt}$$

By **N2** we can generalise this to the full vector form of the equation:

$$\frac{d}{dt}(m\vec{v}) = (\vec{v} - u\vec{n})\frac{dm}{dt} + \vec{F}$$

where $\vec{F}$ is the sum of the other forces acting on the rocket, e.g., air resistance, gravity etc. Note that in the above derivation the magnitude u may also be a function of time.

Example 1: Rocket speed

Consider a rocket under gaseous propulsion in deep space, free from all external forces, travelling along the x-axis. If the initial speed of the rocket is $\vec{v}_0 = v_0\hat{i}$, its mass is m_0 and the propulsive gases are emitted at a constant speed relative to the rocket, calculate its speed as a function of mass.

Using the above calculation (with $\vec{n} = \hat{i}$) we obtain the differential equation

$$m\frac{dv}{dt} = -u\frac{dm}{dt}$$

This can be converted to a separable equation by dividing through by one of the derivatives

$$m\frac{dv}{dt}\bigg/\frac{dm}{dt} = -u \longrightarrow m\frac{dv}{dm} = -u$$
$$\int dv = -u\int\frac{dm}{m} \longrightarrow v - v_0 = u\ln\left(\frac{m_0}{m}\right).$$

Thus the speed depends only on the ratio of initial to current masses (> 1). Hence to get the rocket to travel quickly we need $m_0/m \gg 1$, i.e, to reduce the mass of the rocket by as much as possible. That's (partly) why you have multistage rockets. Once a particular fuel tank is empty, it is jettisoned to reduce the mass as much as possible. If we want to know the speed at a particular time, we need to specify the rate of burn of rocket fuel, dm/dt, which is often a constant for most of the burn time. Suppose the rate of fuel burn for the above rocket is constant rate $-\alpha$. (Remember dm was negative!)

$$\frac{dm}{dt} = -\alpha \Rightarrow m(t) = m_0 - \alpha t$$

Hence we have

$$v = v_0 + u\ln\left(\frac{m_0}{m_0 - \alpha t}\right).$$

Note that we must have $m_0 - \alpha t > 0$, or else it will most certainly be out of fuel!

Example 2: Rocket burn under gravity

Suppose the above rocket is ejecting gas at a constant rate (α) and at a constant velocity relative to it but is moving vertically upward in a constant gravitational field $\vec{g} = -g\hat{k}$ (assuming that it won't go very high...). Calculate its speed as a function of time.
We use the full expression for **N2**

$$\frac{d}{dt}(m\vec{v}) = (\vec{v} - u\vec{n})\frac{dm}{dt} + \vec{F}$$

where

$$\vec{F} = -m(t)g\hat{k}, \qquad \vec{v} = v(t)\hat{k}, \qquad \vec{n} = \hat{k}, \qquad u = \text{constant}, \qquad \frac{dm}{dt} = -\alpha.$$

The mass of the rocket is

$$\frac{dm}{dt} = -\alpha \Rightarrow m(t) = m_0 - \alpha t.$$

Inserting this into the full equation and noting that only the $\hat{k}$ component is relevant we have

$$m\frac{d\vec{v}}{dt} = -u\vec{n}\frac{dm}{dt} + \vec{F}$$

If the rocket starts at $t = 0$ with speed $v(0) = 0$, then

$$0 = -u\ln(m_0) + \text{constant} \Rightarrow \text{constant} = u\ln(m_0)$$

and we then have

$$v = u\ln\left(\frac{m_0}{m_0 - \alpha t}\right) - gt.$$

The effect of gravity is to retard the speed of the rocket by an amount gt. Integrating once more with respect to time will give the position of the rocket.

Example 3: A rocket with linear drag

Here's a slightly more involved example. The initial mass of the rocket is m_0 and we still burn fuel at a constant rate, so $\dot{m} = -\alpha$. But now the rocket is subject to linear drag, $F = -\gamma v$, presumably because it has encountered some sticky intergalactic sludge or something. If the rocket starts from rest, how fast is it going after it has burned one half of its mass as fuel? With linear drag, the rocket equation becomes

$$m\dot{v} + u\dot{m} = -\gamma v$$

We can get a feel for what's going on by looking at this equation. Since $\dot{m} = -\alpha$, rearranging we get

$$m\dot{v} = \alpha u - \gamma v$$

This means that we will continue to accelerate through the sticky stuff if we are travelling slowly and burning fuel fast enough so that $\alpha u > \gamma v$, but as our speed approaches $v = \alpha u/\gamma$ the acceleration slows down and we expect this to be the limiting velocity. However, if we were travelling too fast to begin with, so $\gamma v > \alpha u$, then we will slow down until we again hit the (same) limiting speed. These expectations are brought out by the detailed solution (which follows from solving a separable equation).