

Chapter 4

Multiple Integration

4.1 Introduction

Having dealt with the differentiation of functions of multiple variables, we now consider the integration of functions $\mathbb{R}^n \rightarrow \mathbb{R}$.

You have already seen the integral I of a (continuous) function of one variable $y = f(x)$ over and interval $a \leq x \leq b$:

$$I = \int_a^b f(x) dx.$$

The geometrical meaning of this symbol is the area of the plane region bounded by the curve $y = f(x)$, the x -axis and the line $x = a$ and $x = b$.

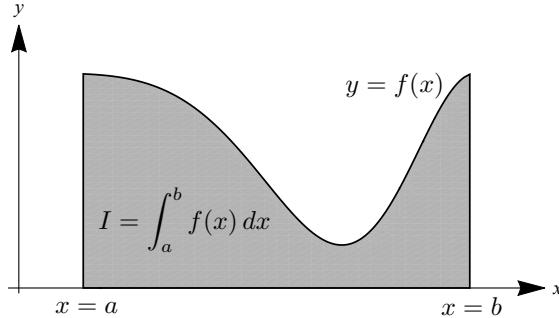


Figure 4.1: Area under a curve.

Consider now the meaning of a **double** integral of a function $\mathbb{R}^2 \rightarrow \mathbb{R}$ of two variables, $z = f(x, y)$ over a two dimensional domain \mathcal{D} in the (x, y) -plane.

By analogy, this is the **volume** of a three dimensional region \mathcal{V} , bounded above by the surface $z = f(x, y)$, below by the (x, y) -plane ($z = 0$), and the vertical (irregular) cylinder sides parallel to the z -axis, emerging from the boundary of the domain \mathcal{D} (see figure).

The region \mathcal{D} is called the **domain** of integration. We will represent the integral with the symbol

$$\mathcal{V} = \int_{\mathcal{D}} f(x, y) dA.$$

The symbol dA is the elemental area element in the (x, y) -plane. We can consider the volume integral as a limit in the following way.

- We discretise the domain \mathcal{D} into infinitesimal elements of area dA .

- If we are using cartesian coordinates, (x, y) and the infinitesimal elements are rectangles, we can write

$$dA = dx dy,$$

where dx and dy are the sides of the infinitesimal rectangles.

- Now dA at the point (x_i, y_j) can be regarded as the base of a vertical rectangular column of height $f(x_i, y_j)$
- Then the integral \mathcal{V} is the limit

$$\mathcal{V} = \lim_{dA \rightarrow 0} \sum_{i,j} f(x_i, y_j) dA$$

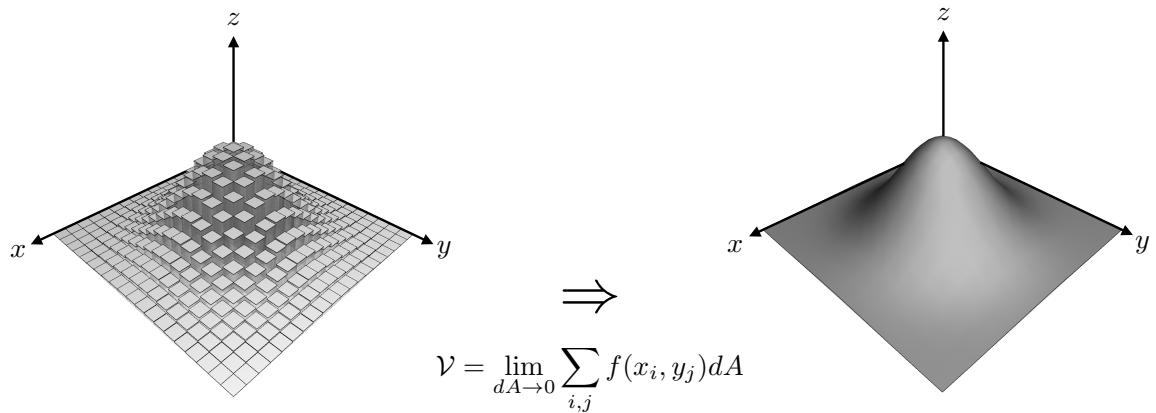


Figure 4.2: Volume under surface is the limit of the sum of the volumes of the columns of height $f(x_i, y_i)$ each with base area dA as $dA \rightarrow 0$.

When we are evaluating an integral explicitly in terms of (x, y) , it is normal to write the integral in the form

$$V = \iint_{\mathcal{D}} f(x, y) dx dy.$$

The “double” integration signs denote that we integrate both in the x direction and also the y direction. The order in which you carry out the integrations will be discussed later.

4.2 Properties of the Multiple Integral

1. **Area of a Domain:** The area of a domain \mathcal{D} in the (x, y) plane is given by

$$A = \int_{\mathcal{D}} 1 dA.$$

At first site this looks like the volume of an irregularly bounded vertical cylinder of with base \mathcal{D} and height 1, which indeed it is. But we also know that the general formula for the volume of a vertical cylinder of constant height is

$$\text{Volume} = \text{area of } \mathcal{D} \times \text{height of cylinder.}$$

Since the height of the cylinder is 1, the numerical value of the volume (forgetting the units!) is the same as the area of \mathcal{D} .

Therefore we may use double integrals to measure areas of regions of the plane.

$$\text{Area of } \mathcal{D} = \int_{\mathcal{D}} 1 dA = \int_{\mathcal{D}} dA.$$

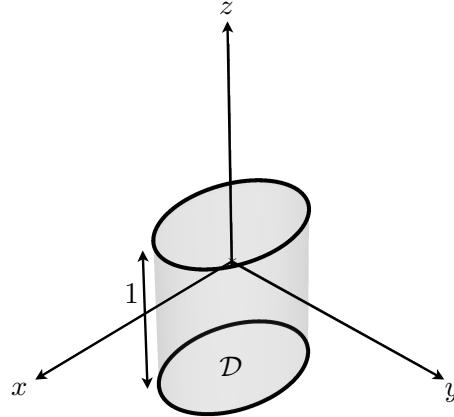


Figure 4.3: Right cylinder of height 1 and base area \mathcal{D} also has (numerical) volume \mathcal{D} .

2. **The integral is linear:** For constants λ and μ we have:

$$\int_{\mathcal{D}} (\lambda f(x, y) + \mu g(x, y)) dA = \lambda \int_{\mathcal{D}} f(x, y) dA + \mu \int_{\mathcal{D}} g(x, y) dA.$$

This just says that if you have multiple copies of two different volume cylinders, the total volume is just the sum of all the copies.

3. **Inequalities are preserved:** If we have two integrable functions f, g and that $f(x, y) \leq g(x, y)$ over the integration domain \mathcal{D} then

$$\int_{\mathcal{D}} f(x, y) dA \leq \int_{\mathcal{D}} g(x, y) dA.$$

This just says that if you have two irregular cylinders, one of which has a height smaller than or equal to the other, then the overall taller cylinder will have the greater volume.

4. **Modulus Property:**

$$\left| \int_{\mathcal{D}} f(x, y) dA \right| \leq \int_{\mathcal{D}} |f(x, y)| dA$$

This just reflects the fact that if $f(x, y)$ dips below the horizontal (x, y) axis and becomes negative in parts of the domain \mathcal{D} , then the volume becomes negative (cf. 1-D integrals and negative areas).

5. **Additivity of Domains:** If $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \dots, \mathcal{D}_k$ are non-overlapping domains in the (x, y) -plane, on each of which $f(x, y)$ is integrable, then $f(x, y)$ is integrable over the union of these domains $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \dots \cup \mathcal{D}_k$.

$$\int_{\mathcal{D}} f(x, y) dA = \sum_{j=1}^k \int_{\mathcal{D}_j} f(x, y) dA.$$

This just says the volume of a series of cylinders of height $f(x, y)$ over bases \mathcal{D}_i is just the sum of the individual volumes.

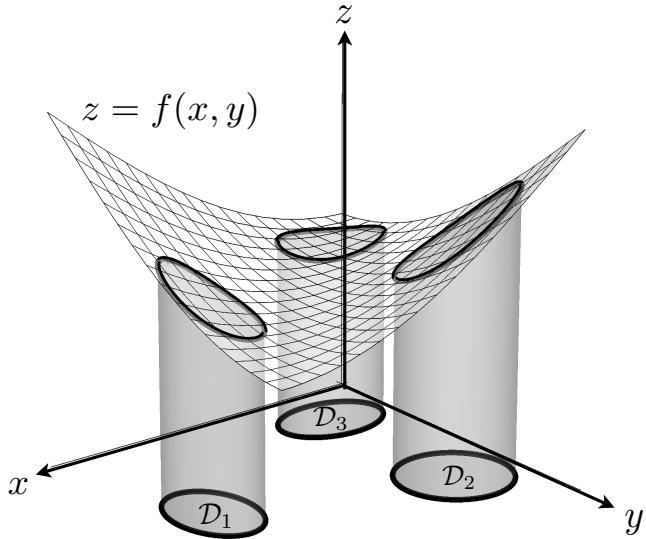


Figure 4.4: If $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$ then $\int_{\mathcal{D}} f(x, y) dA = \sum_{i=1}^3 \int_{\mathcal{D}_i} f(x, y) dA$.

4.3 Evaluation of Double Integrals

4.3.1 Rectangular Domains

Consider an integral over a rectangular domain \mathcal{R} in the (x, y) plane. How do we know whether to integrate in the x direction or the y direction first? Fortunately we are guided by a theorem due to Fubini¹ (proof not required):

Theorem 4.3.1. (Fubini) Let $\mathcal{R} = [a, b] \times [c, d]$ be a rectangle in \mathbb{R}^2 whose sides are parallel to the coordinate axes and let $f(x, y)$ be a function that is continuous on R . Then,

$$\int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy.$$

Example 1: Find the volume of the solid lying above the square in the (x, y) -plane defined by $0 \leq x \leq 1$, $1 \leq y \leq 2$ and below the plane $z = 4 - x - y$.

- The integral we want is

$$V = \int_{\mathcal{D}} z(x, y) dA = \int_{\mathcal{D}} (4 - x - y) dA.$$

- The rectangular domain \mathcal{D} is rectangular, and $4 - x - y$ is clearly continuous² so by the Fubini theorem, the double integral may be evaluated either way: you will (have to) get the same answer!
- We will evaluate the x -integral first, treating y as a constant, then integrate with respect to y , as follows:

¹Surprisingly, given the long history of calculus, dating from only 1907!

²since x is continuous y is continuous, $4 - x - y$ is continuous etc.

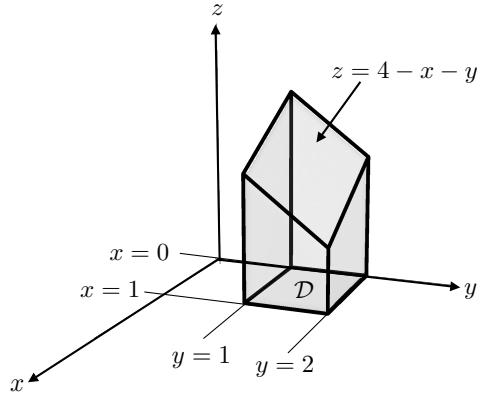
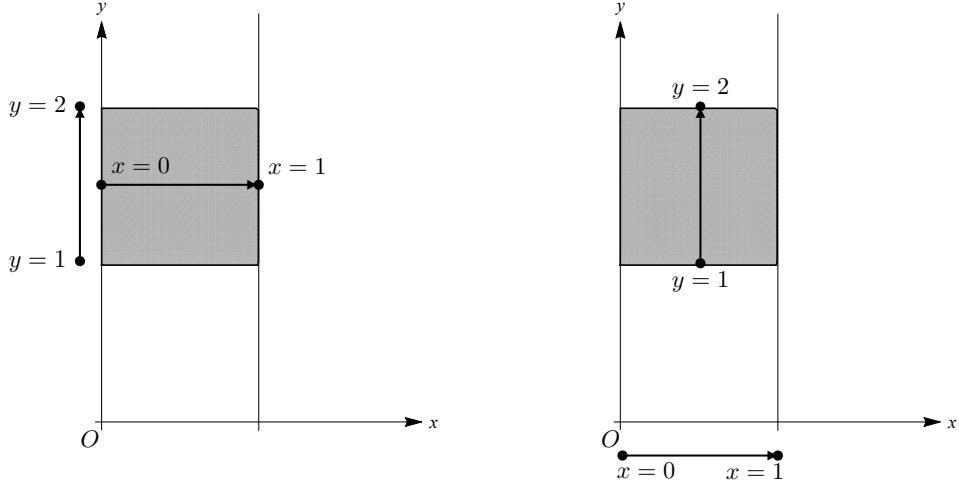


Figure 4.5: Volume: $0 \leq z \leq 4 - x - y$, $0 \leq x \leq 1$, $1 \leq y \leq 2$.

$$\begin{aligned}
 V &= \int_{y=1}^{y=2} dy \int_{x=0}^{x=1} dx (4 - x - y), \\
 &= \int_{y=1}^{y=2} dy \left[4x - \frac{x^2}{2} - yx \right]_{x=0}^{x=1}, \\
 &= \int_{y=1}^{y=2} dy \left[\left(4 \times (1) - \frac{(1)^2}{2} - y \times (1) \right) - \left(4 \times (0) - \frac{(0)^2}{2} - y \times (0) \right) \right], \\
 &= \int_{y=1}^{y=2} dy \left[4 - \frac{1}{2} - y \right], \\
 &= \int_{y=1}^{y=2} dy \left[\frac{7}{2} - y \right], \\
 &= \left[\frac{7}{2}y - \frac{y^2}{2} \right]_{y=1}^{y=2}, \\
 &= \left[\left(\frac{7}{2} \times (2) - \frac{(2)^2}{2} \right) - \left(\frac{7}{2} \times (1) - \frac{(1)^2}{2} \right) \right], \\
 &= 2.
 \end{aligned}$$

- Now to demonstrate Fubini, we first integrate with respect to y , treating x as a constant on each slice, then integrate with respect to x , as follows:

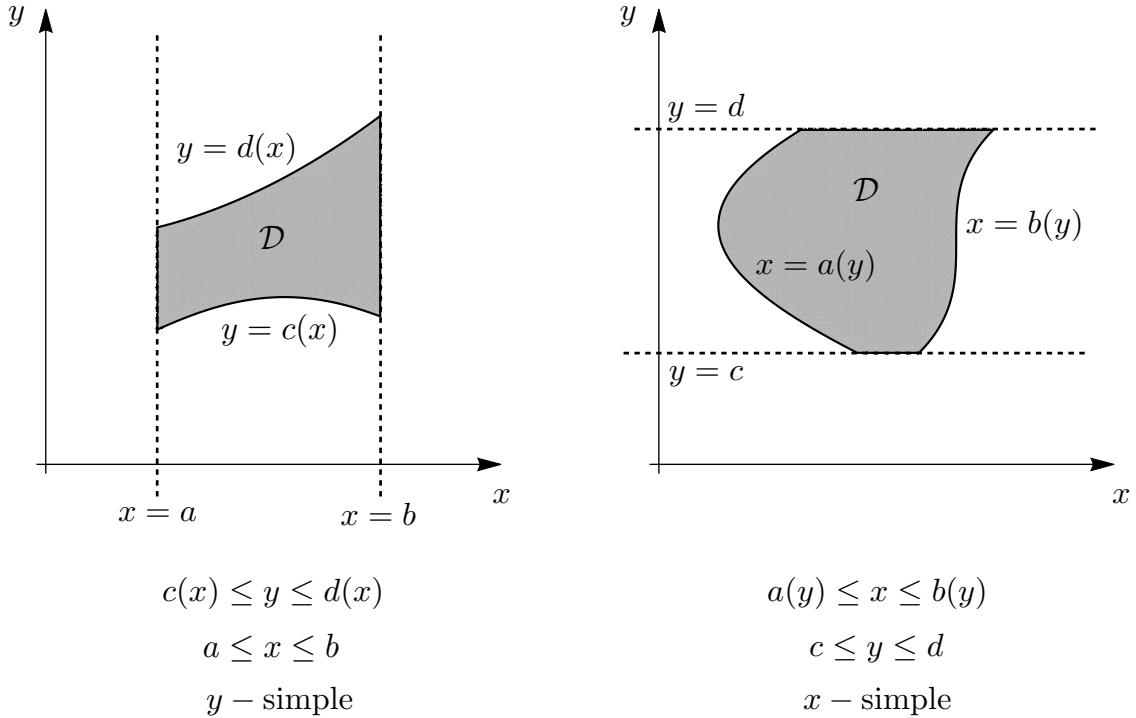
Figure 4.6: Left x -simple evaluation. Right: y -simple evaluation.

$$\begin{aligned}
 V &= \int_{x=0}^{x=1} dx \int_{y=1}^{y=2} dy (4 - x - y), \\
 &= \int_{x=0}^{x=1} dx \left[4y - xy - \frac{y^2}{2} \right]_{y=1}^{y=2}, \\
 &= \int_{x=0}^{x=1} dx \left[\left(4 \times (2) - 2x - \frac{(2)^2}{2} \right) - \left(4 \times (1) - x - \frac{(1)^2}{2} \right) \right], \\
 &= \int_{x=0}^{x=1} dx \left[8 - 2 - 4 + \frac{1}{2} - 2x + x \right], \\
 &= \int_{x=0}^{x=1} dx \left[\frac{5}{2} - x \right], \\
 &= \left[\frac{5}{2}x - \frac{x^2}{2} \right]_{x=0}^{x=1}, \\
 &= \left[\left(\frac{5}{2} \times (1) - \frac{(1)^2}{2} \right) - \left(\frac{5}{2} \times (0) - \frac{(0)^2}{2} \right) \right], \\
 &= 2.
 \end{aligned}$$

Note: The limits on the inner integral do not depend here on the integration variable of the outer limit. This is because, in this example, the domain of integration \mathcal{D} is a rectangle. If \mathcal{D} were not a rectangle, the inner limits **would** depend on the outer integration variable.

4.3.2 Simple Domains

Not all domains of integration are rectangular. However when we have non-rectangular domains, evaluation of an integral is easiest when the domain of integration is said to be **simple**, see diagram.

Figure 4.7: Left: y -simple domain \mathcal{D} . Right x -simple domain \mathcal{D} .**Definition (x - and y - Simple Domains):**

- **y -Simple:** A y -simple domain is one that is bounded by two **vertical** lines $x = a$ and $x = b$ and two continuous graphs $y = c(x)$ and $y = d(x)$.
- **x -Simple:** A x -simple domain is one that is bounded by two **horizontal** lines $y = c$ and $y = d$ and two continuous graphs $x = a(y)$ and $x = b(y)$.

4.3.3 Evaluation of an Integral over a Simple Domain

Suppose that \mathcal{D} is y -simple and is bounded by

$$x = a, \quad x = b, \quad y = c(x), \quad y = d(x),$$

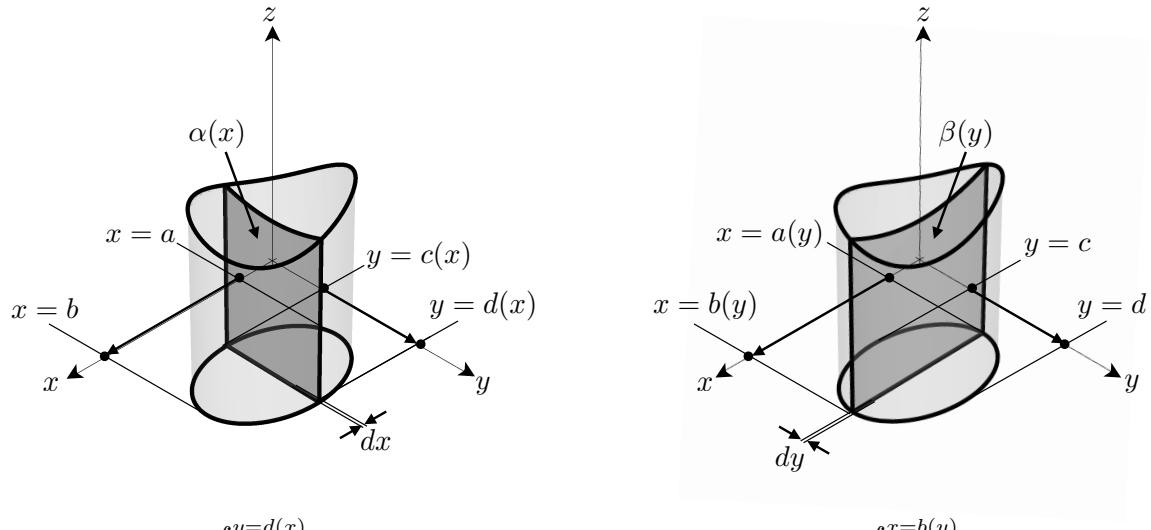
where $c(x)$ and $d(x)$ are both continuous functions. The integral

$$I = \int_D f(x, y) dA$$

is the volume of the vertical cylinder of base \mathcal{D} and bounded at the top by $z = f(x, y)$.

We shall now consider how to evaluate this multiple integral using an approach akin to the very thin slicing of a loaf of bread. The volume of the loaf is the sum of the volume all the slices (assuming no crumbs).

- Consider a very thin slice of the solid, perpendicular to the x -axis as the point x of width dx (see diagram).



$$\alpha(x) = \int_{y=c(x)}^{y=d(x)} f(x, y) dy \qquad \beta(y) = \int_{x=a(y)}^{x=b(y)} f(x, y) dx$$

Figure 4.8: Left: x -slicing. Right: y -slicing

- The slice is bounded between the values

$$c(x) \leq y \leq d(x)$$

- The area of the vertical slice is thus

$$\alpha(x) = \int_{y=c(x)}^{y=d(x)} f(x, y) dy.$$

- The volume of each vertical slice is $\alpha(x)dx$ for $a \leq x \leq b$.

- In the limit as the slice thicknesses $dx \rightarrow 0$ and $dy \rightarrow 0$ we thus have

$$I = \int_{x=a}^{x=b} \alpha(x) dx$$

- But remembering the definitions of the volume I and $\alpha(x)$ above, we have

$$I = \int_D f(x, y) dA = \int_{x=a}^{x=b} \left[\int_{y=c(x)}^{y=d(x)} f(x, y) dy \right] dx$$

We usually drop the square brackets with the understanding that the first d -something is the inner integral that is done first. Hence we can write

$$I = \int_D f(x, y) dA = \int_{x=a}^{x=b} \int_{y=c(x)}^{y=d(x)} f(x, y) dy dx$$

- Note that since the volume should not ultimately contain terms in y or x , the limits are often the key to understanding which of the integrals should be performed first. Hence, after practice, it is usually possible to leave out the explicit references to $x =$ and $y =$ in the limits and write the integral as.

$$I = \int_D f(x, y) dA = \int_a^b \int_{c(x)}^{d(x)} f(x, y) dy dx$$

- With this also in mind, an alternative representation I may use is:

$$I = \int_D f(x, y) dA = \int_a^b dx \int_{c(x)}^{d(x)} dy f(x, y).$$

The advantage of writing the dx and the dy next to the relevant integral sign is that it reminds you what what has to be substituted in as the limits after each of the integrations. The implication here is that the y integral must still be done first, since it has limits that depend on x , the integration that is here done last.

If D is x -simple, we could repeat the same procedure with slices perpendicular to to y -axis. (The volume of a loaf can't change, no matter how you slice it!).

Under those circumstances we could then evaluate the integral as (see above diagram):

$$I = \int_D f(x, y) dA = \int_{y=c}^{y=d} \beta(y) dy, \quad \beta(y) = \int_{x=a(y)}^{x=b(y)} f(x, y) dx$$

or equivalently

$$I = \int_D f(x, y) dA = \int_c^d \int_{a(y)}^{b(y)} f(x, y) dx dy,$$

or equivalently

$$I = \int_D f(x, y) dA = \int_c^d dy \int_{a(y)}^{b(y)} dx f(x, y).$$

4.3.4 Regular Domains

Definition (Regular Domain): A domain that is not simple, but which can be split up into a series of x -simple or y -simple domains is called **regular**. An integral over a regular domain is evaluated as the sum of integrals over each constituent simple domain (property 5 above).

Example: Consider a bagel-shaped domain, $1 \leq \sqrt{x^2 + y^2} \leq 2$. This is not a simple domain as it has a hole in the middle and the functions that define the boundary are multi-valued. However by careful slicing, the bagel can be turned into the sum of simple domains, each of which have boundaries that may be written in terms of single-valued functions.

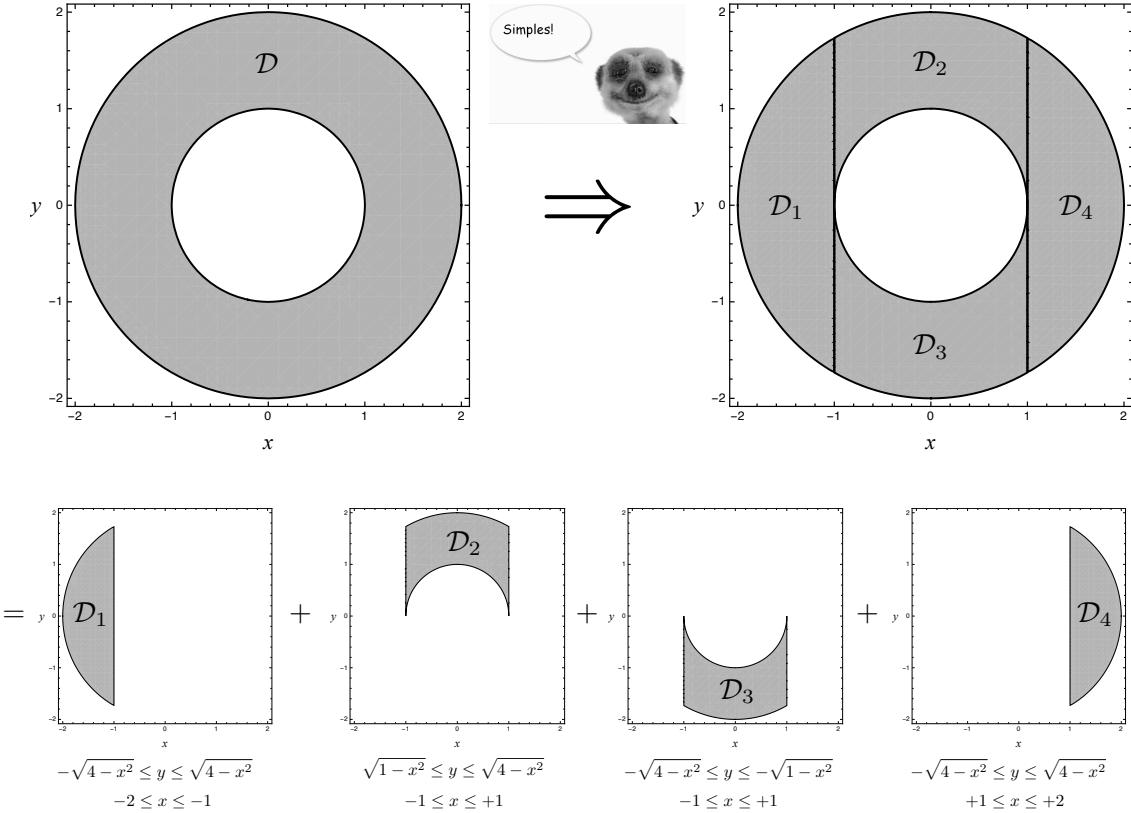


Figure 4.9: Slicing and dicing your **regular** bagel domain \mathcal{D} , to obtain, e.g., a representation of y -simple domains, such that $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4$.

The integral of a function $z = f(x, y)$ over the bagel-domain can therefore be evaluated as follows:

$$\iint_{\mathcal{D}} f(x, y) \, dx \, dy = \sum_{i=1}^4 \iint_{\mathcal{D}_i} f(x, y) \, dx \, dy.$$

4.3.5 Examples of Evaluation of Double Integrals

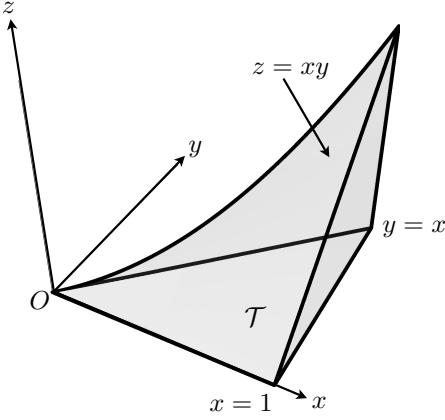
The following examples stress the importance of drawing the domain of integration in the (x, y) -plane. It gives geometrical insight that can lead to an understanding of what is going on and in some cases, considerable simplifications.

Example 2: Evaluate

$$I = \int_{\mathcal{T}} xy \, dA,$$

over the triangle \mathcal{T} in the (x, y) -plane, with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$.

The triangle is both x -simple and y -simple (see diagram), so again we can chose the order in which we perform the integrations.

Figure 4.10: Volume: $0 \leq z \leq xy$, $0 \leq y \leq x$, $0 \leq x \leq 1$.

- Evaluating the integral as y -simple we first integrate with respect to y , treating x as a constant on each slice, then integrate with respect to x .
- Consider the diagram. For each constant slice of x , we need to integrate y from 0 to the sloping side of the triangle \mathcal{T} . The sloping side of \mathcal{T} has the equation $y = x$. The coordinate of the intersection of the slice at a general point x with the sloping side of \mathcal{T} is $(x, y(x)) = (x, x)$
- Having integrated over each of these x -slices, we then have to integrate over the range $0 \leq x \leq 1$ (see diagram).
- The integral thus becomes

$$\begin{aligned}
V &= \int_{x=0}^{x=1} dx \int_{y=0}^{y=x} dy xy, \\
&= \int_{x=0}^{x=1} dx \left[x \frac{y^2}{2} \right]_{y=0}^{y=x}, \\
&= \int_{x=0}^{x=1} dx \left[\left(x \frac{(x)^2}{2} \right) - \left(x \frac{(0)^2}{2} \right) \right], \\
&= \int_{x=0}^{x=1} dx \left[\frac{x^3}{2} \right], \\
&= \left[\frac{x^4}{8} \right]_{x=0}^{x=1}, \\
&= \frac{1}{8}.
\end{aligned}$$

- Evaluating the integral as x -simple we first integrate with respect to x , treating y as a constant on each slice, then integrate with respect to y .
- Consider the diagram. For each constant slice of y , we need to integrate x from the sloping side of the triangle \mathcal{T} to $x = 1$. The sloping side of \mathcal{T} has the equation $y = x$. The coordinate of the intersection of the slice at a general point y with the sloping side of \mathcal{T} is $(x(y), y) = (y, y)$

- Having integrated over each of these y -slices, we then have to integrate over the range $0 \leq y \leq 1$ (see diagram).

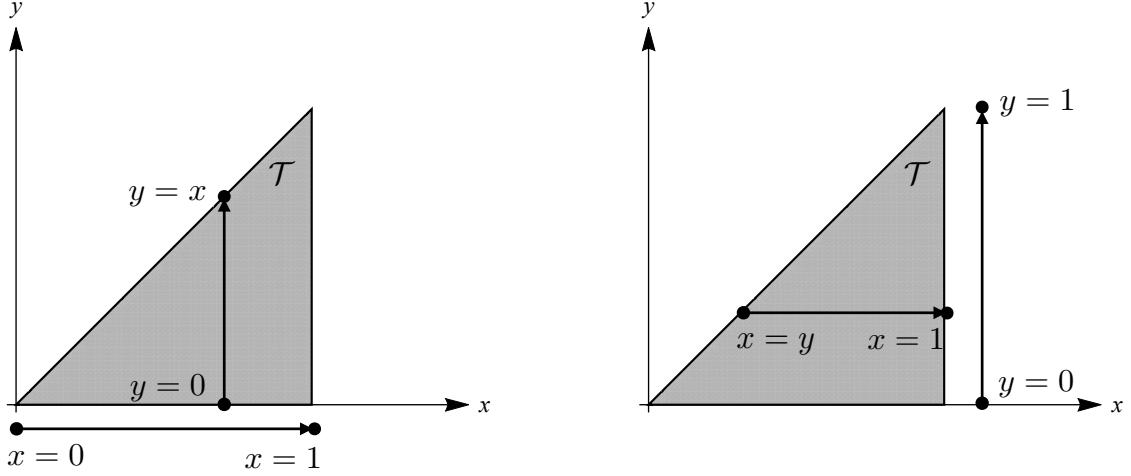


Figure 4.11: Left: y -Simple evaluation; Right: x -Simple evaluation.

- The integral thus becomes

$$\begin{aligned}
 V &= \int_{y=0}^{x=1} dy \int_{x=y}^{x=1} dx xy, \\
 &= \int_{y=0}^{y=1} dy \left[\frac{x^2}{2} y \right]_{x=y}^{x=1}, \\
 &= \int_{y=0}^{y=1} dy \left[\left(\frac{(1)^2}{2} y \right) - \left(\frac{(y)^2}{2} y \right) \right], \\
 &= \int_{y=0}^{y=1} dy \left[\frac{y}{2} - \frac{y^3}{2} \right], \\
 &= \left[\frac{y^2}{4} - \frac{y^4}{8} \right]_{y=0}^{y=1}, \\
 &= \frac{1}{8}.
 \end{aligned}$$

Sometimes altering the order of the integration can make the integral easier to evaluate.

Example 3: Evaluate

$$I = \int_{\mathcal{D}} e^{y^3} dA,$$

where \mathcal{D} is the domain in the figure below, bounded as $0 \leq x \leq 1$, $\sqrt{x} \leq y \leq 1$.

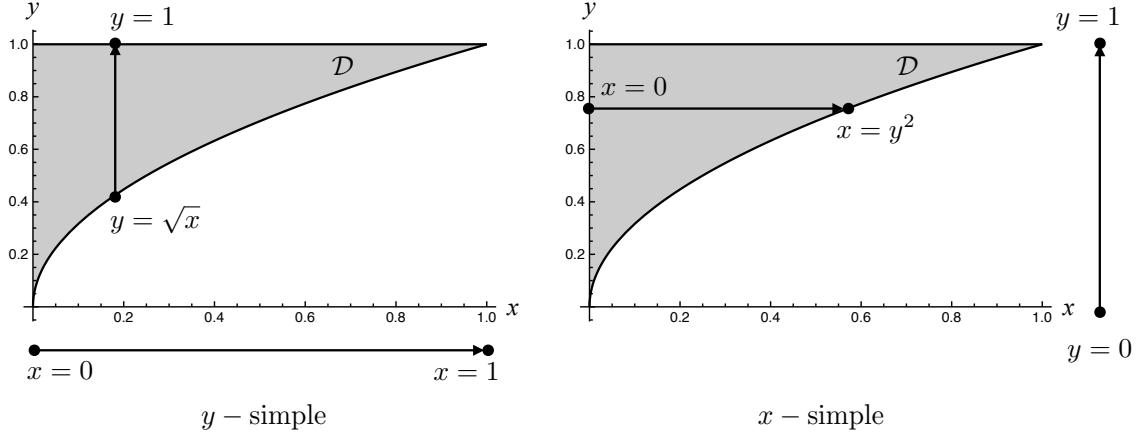


Figure 4.12: Left: y -simple evaluation of $\int_D e^{y^3} dA$. Right: x -simple evaluation of $\int_D e^{y^3} dA$

If we just plug in the limits we have

$$I = \int_{x=0}^{x=1} dx \int_{y=\sqrt{x}}^{y=1} dy e^{y^3}$$

However this leads to an immediate problem: we have no idea what the indefinite integral of e^{y^3} is. (If you don't believe me, try to find it in a table of integrals, or ask WolframAlpha to do it.)

The trick here is to **reverse** the order of integration and do the x integral **first**, and **then** the y -integral. To do this we need to work out the new limits for the integral. Hence we now consider the same integral but evaluate it as x -simple.

- We are now going to integrate first with respect to x . The limits for a constant y -slice are (see diagram) $0 \leq x \leq x(y) = y^2$. The limits for the second, y -integral are just $0 \leq y \leq 1$.

- We then have

$$I = \int_{y=0}^{y=1} dy \int_{x=0}^{x=y^2} dx e^{y^3}.$$

- The point of swapping the order of integration is that we can now easily do at least the first x integral:

$$\begin{aligned} I &= \int_{y=0}^{y=1} dy \left[xe^{y^3} \right]_{x=0}^{x=y^2}, \\ &= \int_{y=0}^{y=1} dy \left[((y^2)e^{y^3}) - ((0)e^{y^3}) \right], \\ &= \int_{y=0}^{y=1} dy y^2 e^{y^3}. \end{aligned}$$

- Now the key point here is that we are left with a y -integral that we **can** do. By inspection we see that the integrand is an exact derivative:

$$y^2 e^{y^3} = \frac{d}{dy} \left(\frac{e^{y^3}}{3} \right).$$

Hence we can now perform the (second) y -integral to obtain

$$\begin{aligned}
 I &= \int_{y=0}^{y=1} dy y^2 e^{y^3}, \\
 &= \frac{1}{3} \int_{y=0}^{y=1} dy \frac{d}{dy} (e^{y^3}) \\
 &= \frac{1}{3} [e^{y^3}]_{y=0}^{y=1} \\
 &= \frac{1}{3} (e^1 - e^0) \\
 &= \frac{1}{3} (e - 1)
 \end{aligned}$$

4.3.6 Double Integrals in Polar Coordinates

Polar Coordinates

Some integrals are much simpler to evaluate if they are expressed in alternative coordinate systems to cartesians (x, y) . For example, when you have a circular domain \mathcal{D} , or a radially-symmetric integrand, it is often easier to evaluate the integral in terms of polar coordinates.

Recall that in polar coordinates there is an origin called a **pole** and denoted by O and a **polar axis** which is a half-line extending from O .

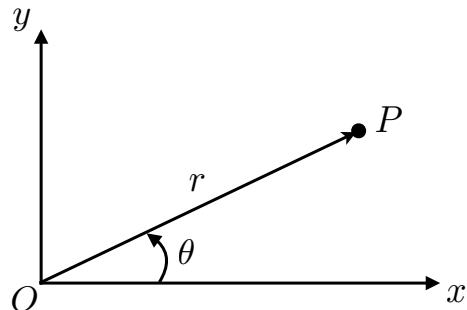


Figure 4.13: Plane polar coordinates

The position of a point P in the plane is determined by its polar coordinates (r, θ) where:

- r is the distance from O to P , $|\mathbf{OP}|$.
- θ is the angle that the vector \mathbf{OP} make with the polar axis, counterclockwise angles being counted as positive.
- Normally we fix θ in a given 2π range, e.g., $0 \leq \theta \leq 2\pi$ since $(r, \theta) = (r, \theta + 2n\pi)$, $n \in \mathbb{Z}$.

The conversion between cartesian and polar coordinates are as follows:

$$\left. \begin{array}{l} x = r \cos \theta \\ y = r \sin \theta \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} r = \sqrt{x^2 + y^2} \\ \tan \theta = y/x \end{array} \right.$$

The Use of Polar Coordinates in Double Integrals

To illustrate the use of polars, we consider the following examples:

Example 1: Find the volume V of the solid region lying above the (x, y) plane and beneath the paraboloid $z = 1 - x^2 - y^2$.

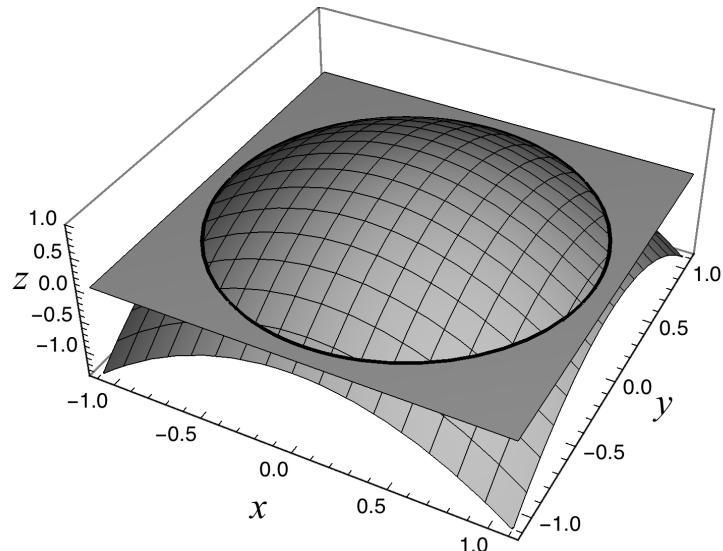


Figure 4.14: The volume we want is that above the grey horizontal (x, y) -plane, but below the meshed surface. The black circle is $x^2 + y^2 = 1$ and denotes the intersection of the two and so is the boundary of \mathcal{D} for this example. It looks rather like the O₂ in London....

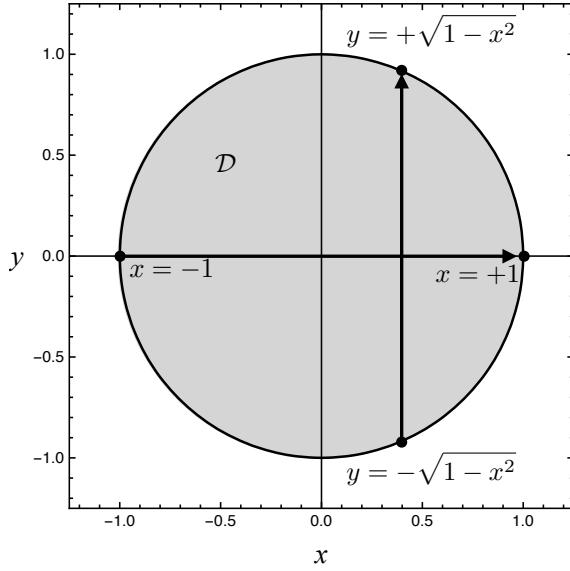


Figure 4.15: Evaluating the paraboloid integral using a y -simple approach

The paraboloid intersects the (x, y) -plane in the circle $z = 0 \Rightarrow x^2 + y^2 = 1$. The domain \mathcal{D} is thus x -simple or y -simple. We can then pick, say, the y -simple route and so in cartesian coordinates the required volume is

$$V = \int_{\mathcal{D}} (1 - x^2 - y^2) dA = \int_{x=-1}^{x=+1} dx \int_{y=-\sqrt{1-x^2}}^{y=+\sqrt{1-x^2}} dy (1 - x^2 - y^2).$$

The integral can be done, but it involves a lot of messy algebra. It is much neater to convert the problem to polar coordinates.

- The domain of integration becomes

$$x^2 + y^2 \leq 1 \quad \Rightarrow \quad r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$

- The integrand becomes

$$1 - x^2 - y^2 = 1 - (r \cos \theta)^2 - (r \sin \theta)^2 = 1 - r^2(\cos^2 \theta + \sin^2 \theta) = 1 - r^2.$$

Hence the integral becomes

$$V = \int_{\mathcal{D}} (1 - r^2) dA',$$

but where dA' is now the elemental area expressed in polar coordinates (r, θ) , **not** cartesians.

Now it is vitally important to realise that just because $dA = dx dy$, it does not mean that when expressed in polars, the equivalent infinitesimal area element dA' is equal to $dr d\theta$. For a start this is dimensionally wrong: both dx and dy have units of length, so dA has units of (length) 2 . While dr has a unit of length, $d\theta$ is just an angle, with (numerically) no units. Hence $dr d\theta$ only has units of length, not (length) 2 .

If we sketch the respective elemental areas in cartesian and polar coordinates we can deduce graphically what the representation for dA is in polar coordinates.

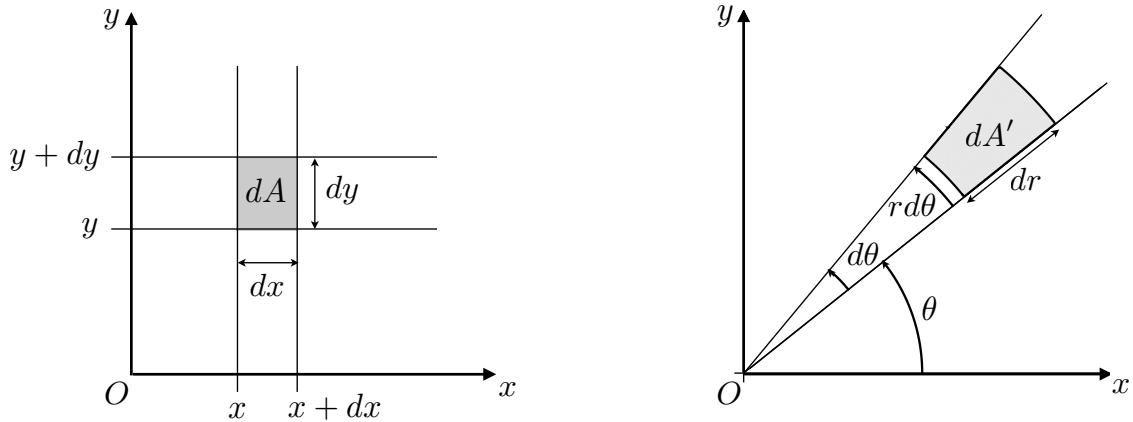


Figure 4.16: Left: $dA = dx dy$ in cartesians. Right: In the limit as dr and $d\theta \rightarrow 0$, $dA' = rdr d\theta$ in plane polars.

We find that in the limits that $dx, dy \rightarrow 0$ and $dr, d\theta \rightarrow 0$ that both elemental areas are to leading order rectangles. Hence we have:

$$dA = dx \times dy = dx dy \text{ in cartesians,} \quad dA' = dr \times rd\theta = rdr d\theta \text{ in plane polars.}$$

In general whenever you change coordinates in a multidimensional integral, you have to be very careful about what dA becomes. We study a systematic way of working this out in the next section.

Returning to the volume integral we see that in polars it becomes explicitly

$$V = \int_{\mathcal{D}} (1 - r^2) dA = \int_{r=0}^{r=1} dr \int_{\theta=0}^{\theta=2\pi} d\theta r(1 - r^2).$$

Note the extra r in the integrand that comes from the change of variable!

Modulo that complication, look now at how easy the integral is to evaluate. The limits have decoupled and the integrand only depends on one of the variables r . This is because of the radial symmetry of both the integrand (it only depends on r) and the domain of integration (it's a circle). Therefore the double integral is effectively just the product of two separate, independent integrals. We have:

$$\begin{aligned}
V &= \int_{r=0}^{r=1} dr \int_{\theta=0}^{\theta=2\pi} d\theta r(1-r^2) \\
&= \int_{\theta=0}^{\theta=2\pi} d\theta \int_{r=0}^{r=1} dr (r-r^3) \\
&= \int_{\theta=0}^{\theta=2\pi} d\theta \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_{r=0}^{r=1} \\
&= \int_{\theta=0}^{\theta=2\pi} d\theta \left(\frac{1}{2} - \frac{1}{4} \right) \\
&= \frac{1}{4} \int_{\theta=0}^{\theta=2\pi} d\theta \\
&= \frac{1}{4} [\theta]_{\theta=0}^{\theta=2\pi} \\
&= \frac{\pi}{2}.
\end{aligned}$$

Summary: To change from polar coordinates:

1. Express the integrand as a function of r and θ .
2. Express the domain of integration (limits) in polar coordinates.
3. Write the area element $dA = r dr d\theta$.
4. Evaluate the integral.

Sometimes changes of variables can be used to evaluate integrals that we otherwise would not know how to evaluate.

Example: Consider the integral

$$I = \int_{-\infty}^{+\infty} e^{-x^2} dx.$$

This integral comes up everywhere in probability theory as the normalisation term for the normal distribution. You may know the value as $\sqrt{\pi}$, but how do you obtain that? There is no antiderivative for e^{-x^2} in terms of simple functions. One trick is to use polar coordinates as follows. We consider the integral I^2 and write it as

$$I^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2} dx \right) \times \left(\int_{-\infty}^{+\infty} e^{-y^2} dy \right).$$

The second integral is exactly the same as the first, it is just that we are using a different dummy integration variable. If we do that we may rewrite the integrals as a double integral over \mathbb{R}^2

$$I^2 = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy e^{-(x^2+y^2)}.$$

Now look at the integrand. It has the tell-tale radial symmetry of polar coordinates, since x and y only appear together in combinations of the type $x^2 + y^2$.

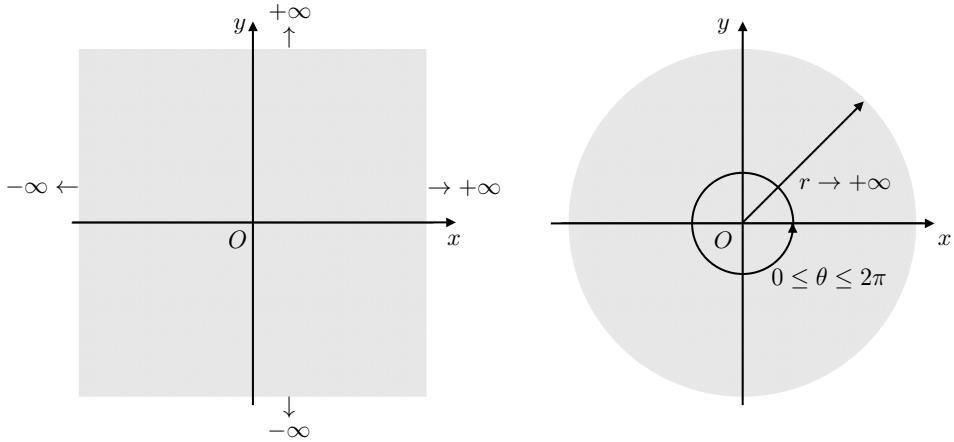


Figure 4.17: Double Gaussian integral domain in 2-D cartesians (left) and plane polars (right).

So following the summary above, we have

Summary:

1. Express the integrand as a function of r and θ : $e^{-(x^2+y^2)} \rightarrow e^{-r^2}$.
2. Express the domain of integration (limits) in polar coordinates: the \mathbb{R}^2 plane can (with a bit of care) be expressed as $0 \leq r \leq +\infty$, $0 \leq \theta \leq 2\pi$.
3. Write the area element $dA = r dr d\theta$.
4. Evaluate the integral:

$$I^2 = \int_{r=0}^{r=+\infty} dr \int_{\theta=0}^{\theta=2\pi} d\theta r e^{-r^2}.$$

Once again the integrals have decoupled: the limits do not depend on each other and the integrand is now just a function of r . We may evaluate the integral as follows:

$$\begin{aligned} I^2 &= \int_{r=0}^{r=+\infty} dr \int_{\theta=0}^{\theta=2\pi} d\theta r e^{-r^2} \\ &= \int_{\theta=0}^{\theta=2\pi} d\theta \int_{r=0}^{r=+\infty} dr r e^{-r^2} \\ &= \int_{\theta=0}^{\theta=2\pi} d\theta \int_{r=0}^{r=+\infty} dr \frac{d}{dr} \left(-\frac{e^{-r^2}}{2} \right) \\ &= \int_{\theta=0}^{\theta=2\pi} d\theta \left[\frac{-e^{-r^2}}{2} \right]_{r=0}^{r=+\infty} \\ &= \int_{\theta=0}^{\theta=2\pi} d\theta \left[\frac{1}{2} \right] \\ &= \frac{1}{2} \int_{\theta=0}^{\theta=2\pi} d\theta \\ &= \pi. \end{aligned}$$

Hence, since $I^2 = \pi$, we have that $I = \sqrt{\pi}$.

Other examples of changes of variables to polar coordinates are in the problem sheets.

4.3.7 Changes of Variables in Double Integrals

The transformation of double integrals to polar coordinates is just a special case of a general change of variables formula for double integrals. Often a general change of variables can help to simplify the algebraic representation of a domain of integration, or to deliver a simplification of the integrand.

A transformation of coordinates in a double integral is a set of two functions

$$u = U(x, y), \quad v = V(x, y),$$

that give a relationship between the old variables (x, y) and the new variables (u, v) .

We **assume** that this transformation is locally invertible, i.e., that there exists a pair of inverse functions

$$x = X(u, v), \quad y = Y(u, v).$$

These functions map the domain \mathcal{D} of integration in the (x, y) domain onto a domain \mathcal{D}' in the (u, v) plane.

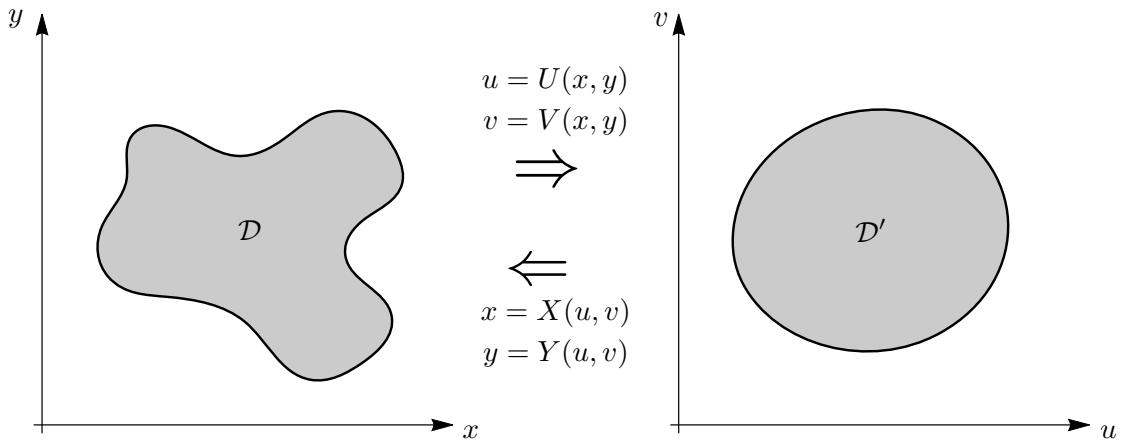


Figure 4.18: Transformations between domain \mathcal{D} in (x, y) plane and domain \mathcal{D}' in (u, v) plane.

The integral transforms as

$$\iint_{\mathcal{D}} f(x, y) dA = \iint_{\mathcal{D}'} f(X(u, v), Y(u, v)) dA'$$

The goal here is to find out how the elemental unit of area transforms from $dA = dx dy$ to dA' in the (u, v) -plane.

To do this we consider an infinitesimal rectangular patchwork $dA = dx dy$ at the point (x, y) somewhere in the domain \mathcal{D} . We label the four corners of this as A, B, C, D (see diagram).

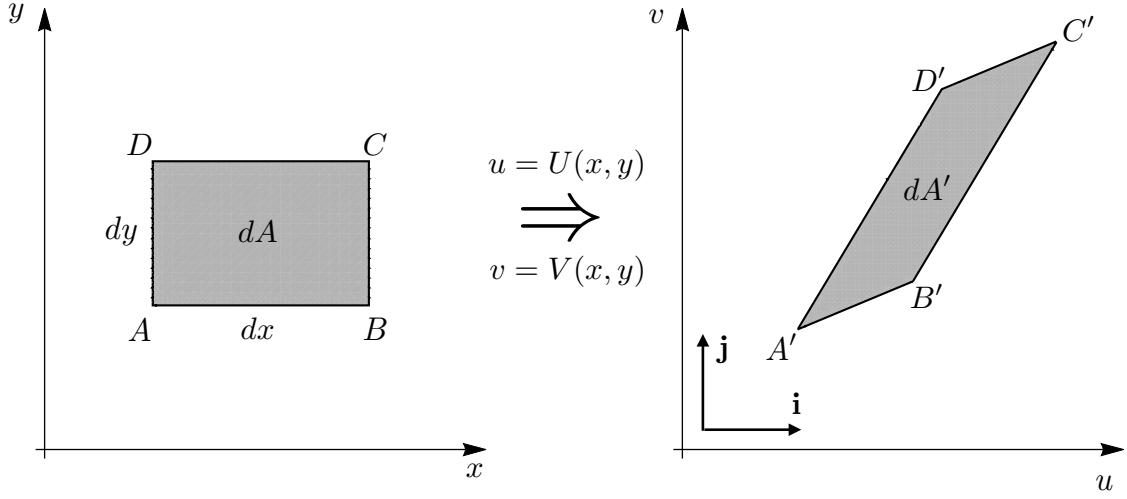


Figure 4.19: Effect of transformation $u = U(x, y)$, $v = V(x, y)$ on $dA = dx dy$

Under the transformation $u = U(x, y)$, $v = V(x, y)$, these corners map respectively to the points A' , B' , C' , D' in the (u, v) plane.

Due to the infinitesimal size, of the original rectangle in the (x, y) plane, the new infinitesimal patchwork dA' will approximately be a parallelogram. The coordinates of its vertices will be

$$A : (x, y) \rightarrow A' : (U(x, y), V(x, y))$$

$$B : (x + dx, y) \rightarrow B' : (U(x + dx, y), V(x + dx, y))$$

$$C : (x + dx, y + dy) \rightarrow C' : (U(x + dx, y + dy), V(x + dx, y + dy))$$

$$D : (x, y + dy) \rightarrow D' : (U(x, y + dy), V(x, y + dy))$$

Suppose we set up unit vectors \mathbf{i} and \mathbf{j} parallel to the u and v axes in the (u, v) plane.

Now, from the diagram above, the area of the parallelogram dA' in the (u, v) plane is given by the standard vector product result of

$$dA' = |\mathbf{A}'\mathbf{B}' \times \mathbf{A}'\mathbf{D}'|,$$

where from the coordinates above,

$$\mathbf{A}'\mathbf{B}' = (U(x + dx, y) - U(x, y))\mathbf{i} + (V(x + dx, y) - V(x, y))\mathbf{j}$$

$$\mathbf{A}'\mathbf{D}' = (U(x, y + dy) - U(x, y))\mathbf{i} + (V(x, y + dy) - V(x, y))\mathbf{j}$$

These can be expanded about (x, y) using Taylor's theorem (or use the definition of partial derivatives) to yield

$$\mathbf{A}'\mathbf{B}' = \left(\frac{\partial U}{\partial x} dx \right) \mathbf{i} + \left(\frac{\partial V}{\partial x} dx \right) \mathbf{j} + \mathcal{O}(dx)^2,$$

$$\mathbf{A}'\mathbf{D}' = \left(\frac{\partial U}{\partial y} dy \right) \mathbf{i} + \left(\frac{\partial V}{\partial y} dy \right) \mathbf{j} + \mathcal{O}(dy)^2.$$

Hence in the limit as dx and $dy \rightarrow 0$, we have (using subscripts to denote partial derivatives to save space)

$$dA' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ U_x dx & V_x dx & 0 \\ U_y dy & V_y dy & 0 \end{vmatrix} = \left\| \begin{vmatrix} U_x & U_y \\ V_x & V_y \end{vmatrix} dx dy \mathbf{k} \right\| = \left\| \begin{vmatrix} U_x & U_y \\ V_x & V_y \end{vmatrix} dx dy \right\| = \left| \frac{\partial(U, V)}{\partial(x, y)} \right| dA$$

Thus the transformed infinitesimal element depends on the modulus of the Jacobian of the transformation functions at that point. Our initial assumption that a local inversion of the functions existed means that the Jacobian is non-vanishing. Hence, using the reciprocity property of Jacobians, we have

$$dx dy = \left| \frac{\partial(X, Y)}{\partial(u, v)} \right| du dv.$$

Finally we can then write

$$\iint_{\mathcal{D}} f(x, y) dx dy = \iint_{\mathcal{D}'} f(X(u, v), Y(u, v)) \left| \frac{\partial(X, Y)}{\partial(u, v)} \right| du dv.$$

Example: We can verify that this result is consistent with the geometrical result we obtained for plane polar coordinates:

$$x = r \cos \theta, \quad y = r \sin \theta.$$

Using the formula above, we have:

$$dx dy = \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| dr d\theta = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} dr d\theta = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} dr d\theta = |r(\cos^2 \theta + \sin^2 \theta)| dr d\theta.$$

Hence, since $r > 0$ we have $|r| = r$ and so,

$$\Rightarrow dx dy = r dr d\theta.$$

4.3.8 Recipe for Change of Variables in Double Integrals

The recipe for changing variables in double integrals is as follows:

1. Identify the transformations between the old variables (x, y) and the new variables (u, v) : $x = X(u, v), y = Y(u, v)$.
2. Express the integrand as a function of the new variables (u, v) .
3. Write the boundaries of the domain of integration in terms of the new variables (u, v) .
4. Write the area element as $dA = |J(u, v)| du dv$ where

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

5. Evaluate the integral!

The motivation for changing the variable could be to exploit a symmetry to simplify the integrand, or to simplify or decouple the limits, so as to make it easier to perform the integral.

Example:

Use an appropriate change of variables to find the area of an ellipse with shape given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a, b are positive constants.

1. The key thing here is to realise that an ellipse is just a squashed circle. We can convert the ellipse to a circle by using the coordinate transformation

$$x = X(u, v) = au, \quad y = Y(u, v) = bv.$$

For then we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \frac{(au)^2}{a^2} + \frac{(bv)^2}{b^2} = 1 \quad \Rightarrow \quad u^2 + v^2 = 1.$$

In other words, in the new (u, v) plane, the ellipse becomes a circle of radius 1.

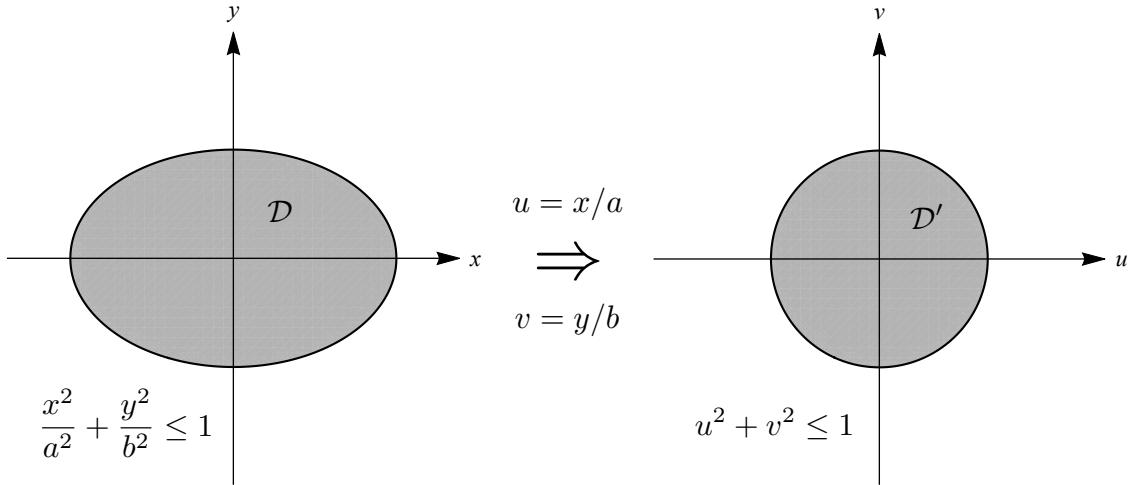


Figure 4.20: Ellipse to circle mapping.

2. The integral we want in the (x, y) -plane is actually

$$\iint_D 1 dx dy,$$

since $f(x, y) = 1$ will give the numerical value of the area of the ellipse. Obviously we then have

$$f(X(u, v), Y(u, v)) = 1,$$

i.e., the integrand in the case is a constant and so will not change under the transformation.

3. As we have seen in 1), the domain of integration changes from the ellipse in the (x, y) plane to the circle (u, v) : $u^2 + v^2 \leq 1$.

4. The Jacobian transformation is: $dA = dx dy = |J(u, v)| du dv$ where

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab.$$

Hence we have

$$dx dy = ab du dv.$$

5. The integral is thus

$$\iint_D 1 dx dy = \iint_{u^2+v^2 \leq 1} 1 ab du dv = ab \underbrace{\iint_{u^2+v^2 \leq 1} du dv}_{\text{Area of a circle of radius 1}} = \pi ab.$$

This next example shows how sometimes you are given a transformation which is easier to work with backwards.

Example:

Evaluate the integral

$$\iint_D \frac{dx dy}{xy}$$

where \mathcal{D} is the region:

$$x + y \leq 3, \quad x + y \geq 1, \quad y \geq x, \quad y \leq 2x$$

using the transformation

$$u = U(x, y) = x + y, \quad v = V(x, y) = y/x.$$

1. Note that we have been given the transformation here (and in terms of $u = U(x, y)$ and $v = V(x, y)$). (In a difficult exam problem, the exact transformation, or a strong hint would be given to you!)

The purpose of the transformation is to convert a regular, but complicated, domain \mathcal{D} in the (x, y) -plane into a much simpler, rectangular domain \mathcal{D}' in the (u, v) plane.

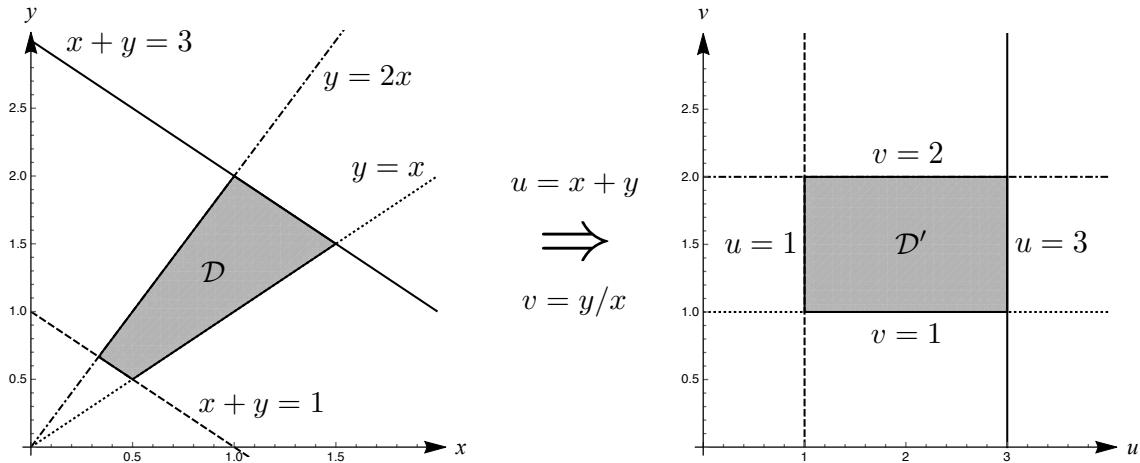


Figure 4.21: Polygon to rectangle mapping.

2. The integral we want in the (x, y) -plane transforms as

$$\iint_{\mathcal{D}} \frac{dx dy}{xy} = \iint_{\mathcal{D}'} \frac{du dv}{X(u, v)Y(u, v)} |J(u, v)|.$$

Note that we don't have explicit forms for $X(u, v)$ and $Y(u, v)$ yet. We could invert the above given transformations $u = U(x, y) = x + y, v = V(x, y) = y/x$, but this might be messy. Sometimes it pays to wait and see what form the Jacobian $J(u, v)$ takes.

3. As we have seen in 1), the domain of integration changes from the polygon in the (x, y) plane to a rectangle $(u, v): 1 \leq u \leq 3, 1 \leq v \leq 2$.

4. The Jacobian transformation is: $dA = dx dy = |J(u, v)| du dv$ where

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

However, here, since we have been given the transformation as $u = U(x, y), v = V(x, y)$, it's easier to do partial derivatives of u and v with respect to x and y . Hence we instead carry out the transformation in the backwards direction ³

$$du dv = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} dx dy = \begin{vmatrix} 1 & 1 \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} dx dy = \left| \frac{x+y}{x^2} \right| dx dy = \left(\frac{x+y}{x^2} \right) dx dy,$$

since both $x > 0$ and $y > 0$.

Solving this for $dx dy$ gives

$$dx dy = \frac{x^2}{x+y} du dv.$$

On the face of it this looks stupid as we have mixed up x, y and u, v on the RHS. But look what happens when we substitute in to the original integral:

$$\iint_{\mathcal{D}} \frac{dx dy}{xy} = \iint_{\mathcal{D}'} \frac{1}{xy} \left(\frac{x^2}{x+y} \right) du dv = \iint_{\mathcal{D}'} \frac{x}{y} \left(\frac{1}{x+y} \right) du dv = \iint_{\mathcal{D}'} \frac{1}{v} \left(\frac{1}{u} \right) du dv.$$

The form of the integrand and the Jacobian combine to give something that can be simply expressed in terms of the new variables u, v .

5. The integral is thus

$$\iint_{\mathcal{D}} \frac{dx dy}{xy} = \int_{u=1}^{u=3} du \int_{v=1}^{v=2} dv \frac{1}{uv} = \int_1^3 \frac{du}{u} \int_1^2 \frac{dv}{v} = [\ln u]_1^3 [\ln v]_1^2 = (\ln 3)(\ln 2).$$

³or by the inversion property of Jacobians from Chapter 1:

$$J(u, v) = \frac{\partial(X, Y)}{\partial(u, v)} = 1 / \frac{\partial(U, V)}{\partial(x, y)}.$$

4.4 Evaluation of Triple Integrals

Once it is clear to extend definite integration from one to two-dimensional domains, the extension to three (or more) dimensions is relatively straightforward.

If $f(x, y, z)$ is a function defined on a domain \mathcal{D} in \mathbb{R}^3 , then we denote the integral of f over \mathcal{D} as

$$\int_{\mathcal{D}} f(x, y, z) dV,$$

where dV denotes a small volume element. In cartesians, we have

$$dV = dx \, dy \, dz.$$

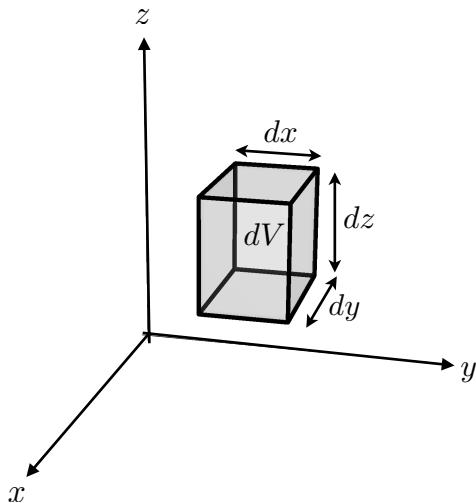


Figure 4.22: Volume element $dV = dx \, dy \, dz$.

The integral can be interpreted as a hyper-volume (the 4-dimensional volume of a function in 3-dimensional space, *cf.* the double integral case). More intuitively, for example, we could interpret $f(x, y, z)$ as the density of a 3-D object, so that the integral of f over the object's volume \mathcal{D} is the mass of the object.

$$\text{Mass} = \int_{\mathcal{D}} f(x, y, z) dV.$$

The procedure for evaluating the integrals in 3-D is effectively the same as that studied for 2-D; we integrate the three variables in succession.

Example 1: Let \mathcal{B} be the rectangular box $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$. Evaluate the integral

$$I = \int_{\mathcal{B}} (xy^2 + z^3) dV.$$

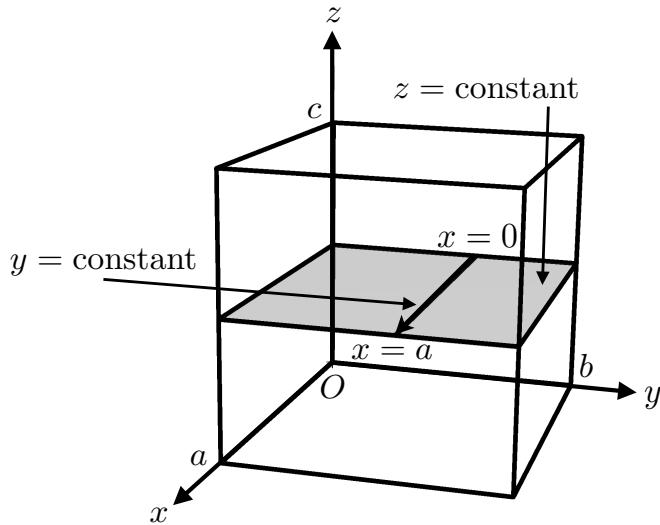


Figure 4.23: Slice with $z = \text{constant}$, then $y = \text{constant}$, then integrate with respect to x .

- We slice the box with **planes** perpendicular to the z -axis, i.e., a constant z -slice.
- The z integral is done last.
- The slices are rectangles.
- On each constant z -slice, we consider a straight line perpendicular to the y -axis and integrate first along x on that $y = \text{constant}$ line.
- So the order of integration is (chosen) here to be:

$$x - \text{first}, \quad y - \text{second}, \quad z - \text{third}.$$

- Hence we have:

$$\begin{aligned}
 I &= \int_{z=0}^{z=c} dz \int_{y=0}^{y=b} dy \int_{x=0}^{x=a} dx (xy^2 + z^3) \\
 &= \int_0^c dz \int_0^b dy \left[\frac{x^2 y^2}{2} + xz^3 \right]_{x=0}^{x=a}, \\
 &= \int_0^c dz \int_0^b dy \left[\frac{a^2 y^2}{2} + az^3 \right]_0^a, \\
 &= \int_0^c dz \left[\frac{a^2 y^3}{2 \times 3} + az^3 y \right]_{y=0}^{y=b}, \\
 &= \int_0^c dz \left[\frac{a^2 b^3}{6} + abz^3 \right], \\
 &= \left[\frac{a^2 b^3 z}{6} + ab \frac{z^4}{4} \right]_{z=0}^{z=c}, \\
 &= \left[\frac{a^2 b^3 c}{6} + ab \frac{c^4}{4} \right], \\
 &= \frac{abc}{2} \left[\frac{ab^2}{3} + \frac{c^3}{2} \right].
 \end{aligned}$$

- Of course, we could alter the order of integration and we would still obtain the same answer.

Example 2: If \mathcal{T} is the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, evaluate

$$\int_{\mathcal{T}} y dV.$$

- From the vertices of the tetrahedron, the domain of integration is the volume under $z = 1 - x - y$, for $x > 0$, $y > 0$, $z > 0$.
- We slice the pyramid with slices perpendicular to the y -axis, i.e., a constant y -slice.
- On each constant y -slice, we cut along strips perpendicular to the x -axis, i.e., integrate first along x on that $y = \text{constant}$ line.

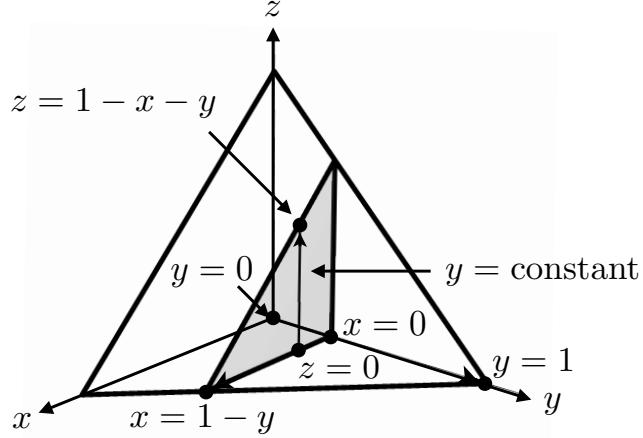


Figure 4.24: Domain \mathcal{T} . Take triangle slices: $y = \text{constant}$, $x = \text{constant}$: first integrate $0 \leq z \leq 1 - x - y$, then integrate $0 \leq x \leq y$, then integrate $0 \leq y \leq 1$.

- The ranges of integration are, in order of the performing of the integration:

$$0 \leq z \leq 1 - x - y, \quad 0 \leq x \leq 1 - y, \quad 0 \leq y \leq 1.$$

- Hence we have:

$$\begin{aligned}
J &= \int_{y=0}^{y=1} dy \int_{x=0}^{x=1-y} dx \int_{z=0}^{z=1-x-y} dz y, \\
&= \int_{y=0}^{y=1} dy \int_{x=0}^{x=1-y} dx [yz]_{z=0}^{z=1-x-y}, \\
&= \int_{y=0}^{y=1} dy \int_{x=0}^{x=1-y} dx [y(1-x-y)], \\
&= \int_{y=0}^{y=1} dy \left[y(x - \frac{x^2}{2} - yx) \right]_{x=0}^{x=1-y}, \\
&= \int_{y=0}^{y=1} dy \left[y((1-y) - \frac{(1-y)^2}{2} - y(1-y)) \right], \\
&= \int_{y=0}^{y=1} dy \left[\frac{y}{2} - y^2 + \frac{y^3}{2} \right], \\
&= \left[\frac{y^2}{4} - \frac{y^3}{3} + \frac{y^4}{8} \right]_{y=0}^{y=1}, \\
&= \left[\frac{1}{4} - \frac{1}{3} + \frac{1}{8} \right], \\
&= \frac{1}{24}.
\end{aligned}$$

4.4.1 Change of Variables in Triple Integrals

There are several commonly occurring alternative coordinate systems in \mathbb{R}^3 which may be used to simplify the evaluation of triple integrals when the domain of integration and/or the integrand possess certain symmetries.

Cylindrical Polar Coordinates

The cylindrical polar coordinate system is a straightforward extension of the 2-D polar coordinate system to 3-D. It uses the 2-D plane polar coordinate system in planes parallel to the (x, y) plane, while retaining the cartesian z -coordinate for measuring vertical distances, hence the name “cylindrical”. The relationship between cartesian (x, y, z) and cylindrical polar coordinates (r, θ, z) is given by

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z.$$

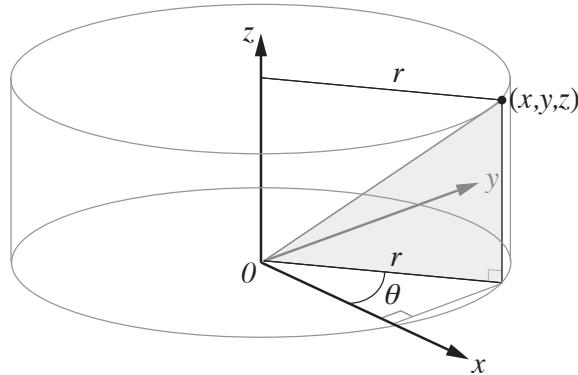


Figure 4.25: The relationship between cartesians and cylindrical polar coordinates.

As the above diagram might suggest, the cylindrical polar system is very useful for problems that have some radial symmetry about a central axis.

By comparison with plane polar coordinates, when expressed in cylindrical polars the volume element dV , it is not too difficult to see that in the limit as $dr, d\theta, dz \rightarrow 0$ it transforms as:

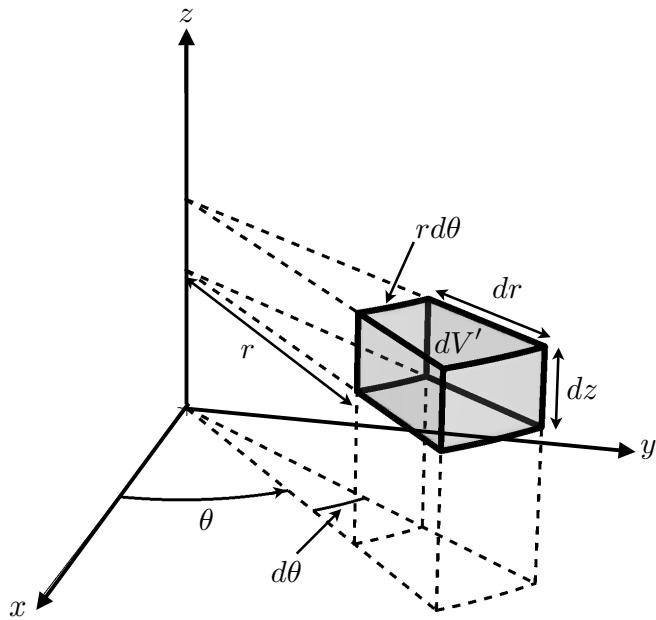


Figure 4.26: $dV' = r dr d\theta dz$ in the limit as $dr, d\theta, dz \rightarrow 0$.

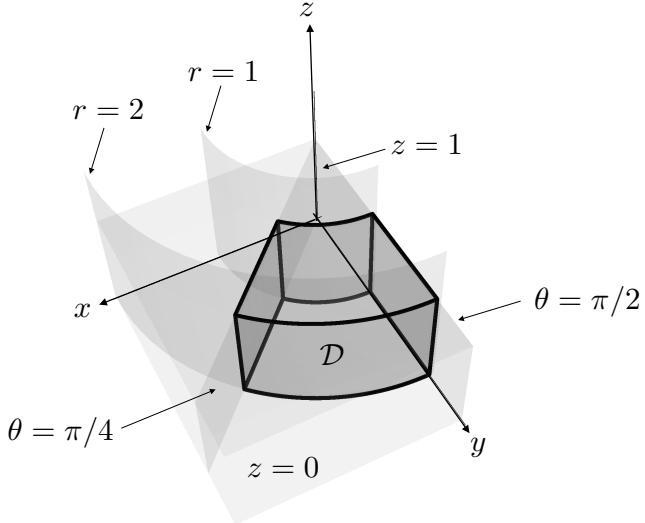
$$dV = dx dy dz \text{ in cartesians,} \quad dV' = dr \times r d\theta \times dz = r dr d\theta dz \text{ in cylindrical polars.}$$

Example 1: Evaluate

$$I = \int_{\mathcal{D}} (x^2 + y^2) dV$$

over the region bounded by the cylinders $x^2 + y^2 = 1$, and $x^2 + y^2 = 4$, the plane $z = 0$, $z = 1$, $x = 0$ and $x = y$.

- The form of the integrand $x^2 + y^2$ suggests using cylindrical polar coordinates, since it becomes $x^2 + y^2 = r^2$.
- We must then check if the domain \mathcal{D} can be represented nicely in terms of cylindrical polar coordinates. (If not, there is no real point in converting coordinate systems.)

Figure 4.27: Domain of integration \mathcal{D} .

- We have the following conversions between the bounding surfaces:

$$\text{Cartesian} \quad \Rightarrow \quad \text{Cylindrical Polar}$$

$$x^2 + y^2 = 1 \quad \Rightarrow \quad r = 1 \quad (x^2 + y^2 = r^2)$$

$$x^2 + y^2 = 4 \quad \Rightarrow \quad r = 2 \quad (x^2 + y^2 = r^2)$$

$$z = 0 \quad \Rightarrow \quad z = 0 \quad (z = z)$$

$$z = 1 \quad \Rightarrow \quad z = 1 \quad (z = z)$$

$$x = 0 \quad \Rightarrow \quad \theta = \pi/2, \quad (x = r \cos \theta = 0)$$

$$x = y \quad \Rightarrow \quad \theta = \pi/4, \quad (r \cos \theta = r \sin \theta \Rightarrow \tan \theta = \pi/4)$$

Note that all the limits will then decouple and so evaluating the integral in cylindricals in this example means just evaluating the triple integral as the product of three one-dimensional integrals.

- Hence the domain of integration becomes (see diagram):

$$1 \leq r \leq 2, \quad \pi/4 \leq \theta \leq \pi/2, \quad 0 \leq z \leq 1.$$

- The volume element dV becomes $dV = r dr d\theta dz$ in cylindrical polars.

- The integral then becomes

$$\begin{aligned}
 I &= \int_{\mathcal{D}} (x^2 + y^2) dV = \int_{z=0}^{z=1} \int_{\theta=0}^{\theta=\pi/4} \int_{r=1}^{r=2} r^2 dV, \\
 &= \int_{z=0}^{z=1} \int_{\theta=0}^{\theta=\pi/4} \int_{r=1}^{r=2} r^2 r dr d\theta dz, \\
 &= \int_{z=0}^{z=1} dz \int_{\theta=0}^{\theta=\pi/4} d\theta \int_{r=1}^{r=2} r^3 dr, \\
 &= [z]_{z=0}^{z=1} [\theta]_{\theta=0}^{\theta=\pi/4} \left[\frac{r^4}{4} \right]_{r=1}^{r=2}, \\
 &= [1 - 0] \left[\frac{\pi}{4} - 0 \right] \left[\frac{16}{4} - \frac{1}{4} \right], \\
 &= \frac{15\pi}{16}.
 \end{aligned}$$

Example 2: Evaluate the volume of a solid \mathcal{D} obtained by rotating the parabola $x = 1 - z^2$ around the z -axis.

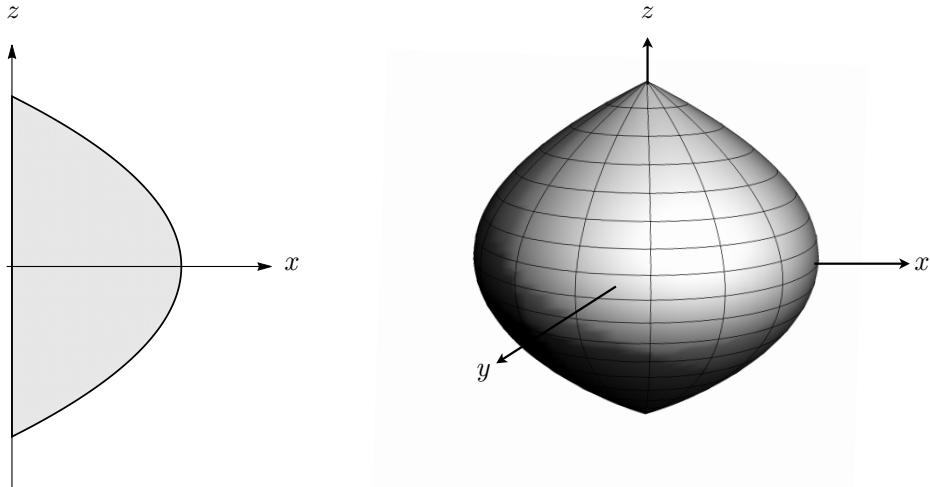


Figure 4.28: Volume of revolution obtain by rotating $x = 1 - z^2$ around the z -axis.

- The volume of \mathcal{D} is given by

$$\int_{\mathcal{D}} 1 dV.$$

(You can rationalise this as saying that if the volume has unit density, then the mass (= integral over density) of the object is numerically the same as the volume.)

- Clearly since the body is an axisymmetric solid of revolution, formed by rotating a curve around the (cylindrical) z axis. Hence it makes sense to evaluate this in terms of cylindrical polars.⁴

⁴Some of you may have treated similar problems at A-level using other methods, but we are using this example as a way to illustrate how to compute volumes of solids of revolution using cylindrical polars.

- Consider the diagram above, in cylindrical polars, the boundaries of the volume become

$0 \leq \theta \leq 2\pi$: the body makes a full rotation about the z axis.

$0 \leq r \leq 1 - z^2$: consider w.l.o.g. the x -plane, the distance of curve r from the z -axis is x .

$-1 \leq z \leq +1$: consider w.l.o.g. the x -plane, the curve intersects the z -axis at $z = \pm 1$.

- Hence the volume integral is

$$\begin{aligned} \text{Volume} &= \int_{z=-1}^{z=+1} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1-z^2} r \, dr \, d\theta \, dz, \\ &= \int_{z=-1}^{z=+1} dz \int_{\theta=0}^{\theta=2\pi} d\theta \left[\frac{r^2}{2} \right]_{r=0}^{r=1-z^2}, \\ &= \int_{z=-1}^{z=+1} dz [\theta]_{\theta=0}^{\theta=2\pi} \frac{(1-z^2)^2}{2}, \\ &= \pi \int_{z=-1}^{z=+1} dz (1-z^2)^2, \\ &= \pi \int_{z=-1}^{z=+1} dz (1-2z^2+z^4), \\ &= \pi \left[z - \frac{2z^3}{3} + \frac{z^5}{5} \right]_{z=-1}^{z=+1}, \\ &= 2\pi \left[1 - \frac{2}{3} + \frac{1}{5} \right], \\ &= \frac{16\pi}{15}. \end{aligned}$$

Spherical Polar Coordinates

The spherical polar coordinate system is similar to (but not exactly the same as⁵ the coordinate system that locates places on the surface of the earth, namely latitude and longitude. A point P in \mathbb{R}^3 is represented in spherical polar coordinates by (r, θ, ϕ) where

- r is the distance of P from the origin O of the coordinate system.
- θ is the angle that the radial line OP makes with the positive z axis, i.e., $\pi/2$ -latitude.
- ϕ is the angle between the plane containing P , the z axis and the (x, z) plane, i.e., longitude.

⁵the earth is not actually spherical but squashed at the poles to form a “oblate spheroid”.

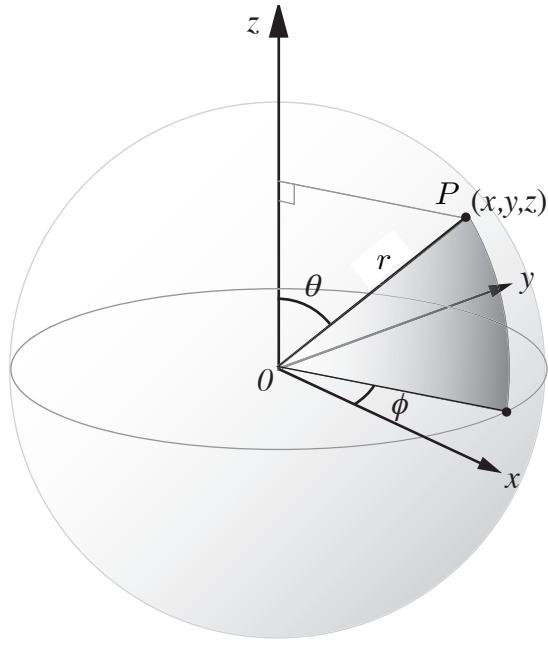


Figure 4.29: The relationship between cartesians and spherical polar coordinates.

It is conventional to consider $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. The relations between the two coordinate systems are then (see diagram):

$$\left. \begin{array}{l} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{array} \right\} \iff \left. \begin{array}{l} r = \sqrt{x^2 + y^2 + z^2} \\ \tan \theta = \sqrt{x^2 + y^2}/z \\ \tan \phi = y/x \end{array} \right\}$$

Example: Find the cartesian coordinates of P with spherical coordinates $(2, \pi/3, \pi/2)$.

- In spherical polars we have

$$r = 2, \quad \theta = \pi/3, \quad \phi = \pi/2.$$

- Hence in cartesians this transforms to:

$$x = r \sin \theta \cos \phi = 2 \sin(\pi/3) \cos(\pi/2) = 2 \times (\sqrt{3}/2) \times 0 = 0;$$

$$y = r \sin \theta \sin \phi = 2 \sin(\pi/3) \sin(\pi/2) = 2 \times (\sqrt{3}/2) \times 1 = \sqrt{3};$$

$$z = r \cos \theta = 2 \cos(\pi/3) = 2 \times (1/2) = 1.$$

- Hence the cartesian coordinates are $(0, \sqrt{3}, 1)$.

Example: Find the spherical polar coordinates of Q with cartesian coordinates $(1, 1, \sqrt{2})$.

- In cartesian polars we have

$$x = 1, \quad y = 1, \quad z = \sqrt{2}.$$

- Hence in cartesians this transforms to:

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + 1^2 + (\sqrt{2})^2} = \sqrt{4} = 2;$$

$$\tan \theta = \sqrt{x^2 + y^2}/z = \sqrt{1^2 + 1^2}/\sqrt{2} = 1 \Rightarrow \theta = \pi/4;$$

$$\tan \phi = y/x = 1/1 = 1 \Rightarrow \phi = \pi/4.$$

- Hence the spherical polar coordinates are $(2, \pi/4, \pi/4)$.

Spherical polars are very useful for problems which have a spherical symmetry, for example in the study of the gravitational force of a planet. In spherical polars the volume element dV can be represented by considering the diagram below in the limit as $dr, d\theta, d\phi \rightarrow 0$.

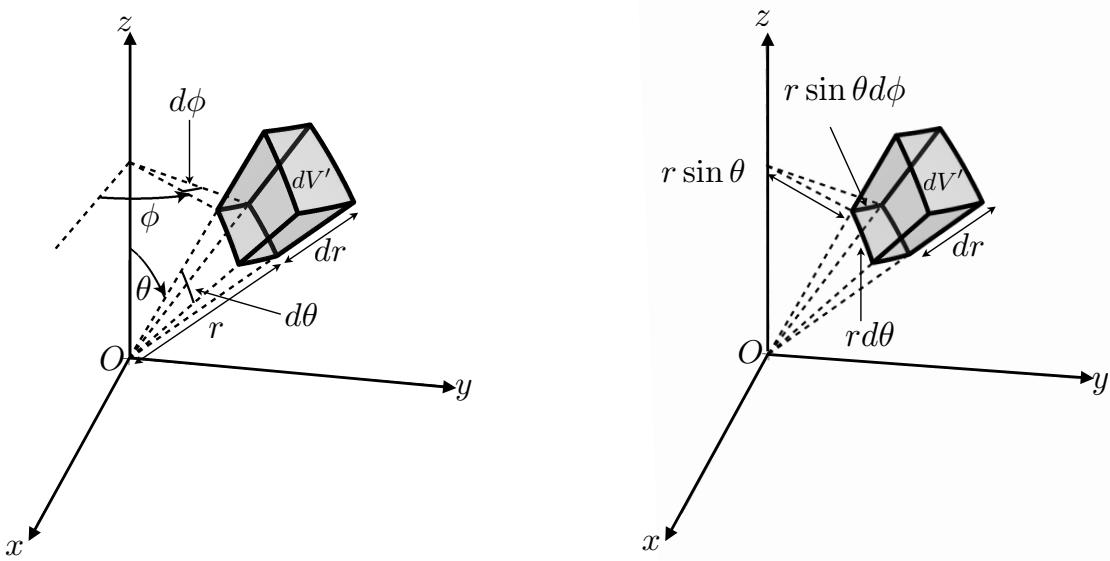


Figure 4.30: Volume element expressed in spherical polars is $dV = r^2 \sin \theta \ dr \ d\theta \ d\phi$.

We conclude that

$$dV = r^2 \sin \theta \ dr \ d\theta \ d\phi.$$

Example: Evaluate the mass M of a half-ball \mathcal{H} of radius a and of density

$$\rho(x, y, z) = x^2 + y^2 + z^2$$

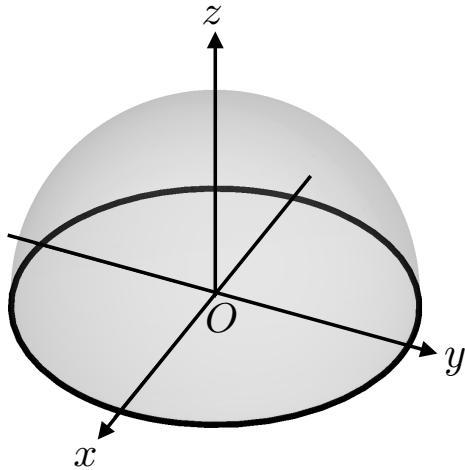


Figure 4.31: Half ball \mathcal{H} : $x^2 + y^2 + z^2 \leq 1, z \geq 0$.

- Clearly the density is spherically symmetric and, in spherical polars is given as

$$\rho = x^2 + y^2 + z^2 = r^2.$$

- The domain of integration, the half-sphere has spherical symmetry:

$$\begin{array}{ccc} \text{Cartesian} & \Rightarrow & \text{Spherical Polars} \\[1ex] 0 \leq x^2 + y^2 + z^2 \leq a^2 & \Rightarrow & 0 \leq r \leq a \quad (x^2 + y^2 + z^2 = r^2); \\[1ex] z \geq 0 & \Rightarrow & 0 \leq \theta \leq \pi/2, \quad \text{and} \quad 0 \leq \phi \leq 2\pi. \end{array}$$

- Hence it makes sense to try spherical polar coordinates.
- The volume element is then $dV = r^2 \sin \theta \ dr \ d\theta \ d\phi$.
- The required integral is then

$$\begin{aligned} M &= \int_{\mathcal{H}} \rho \, dV, \\ &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \int_{r=0}^a \rho r^2 \sin \theta \, dr \, d\theta \, d\phi, \\ &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^a r^2 r^2 \sin \theta \, dr \, d\theta \, d\phi, \\ &= \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi/2} \sin \theta \, d\theta \int_{r=0}^a r^4 \, dr, \\ &= [\phi]_0^{2\pi} [-\cos \theta]_{\theta=0}^{\pi/2} \left[\frac{r^5}{5} \right]_{r=0}^a, \\ &= 2\pi [-(0) - (-1)] \left[\frac{a^5}{5} \right], \\ &= \frac{2\pi a^5}{5}. \end{aligned}$$

General Changes of Variables in Triple Integrals

We need to know how to change the elemental volume for arbitrary changes of variables in \mathbb{R}^3 .

The formulae we studied in \mathbb{R}^2 can be easily extended to \mathbb{R}^3 . In this case the cartesian cuboid $dx \ dy \ dz$ maps onto a parallelepiped, whose volume is given by a 3×3 determinant. The idea is as follows.

A coordinate transformation is an invertible set of three $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ functions:

$$\left. \begin{array}{l} x = X(u, v, w) \\ y = Y(u, v, w) \\ z = Z(u, v, w) \end{array} \right\} \iff \left\{ \begin{array}{l} u = U(x, y, z) \\ v = V(x, y, z) \\ w = W(x, y, z) \end{array} \right.$$

The volume element dV transforms as

$$dV = dx \ dy \ dz \quad \Leftrightarrow \quad dV' = |J(u, v, w)| du \ dv \ dw,$$

where

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Example: Verify the Jacobian transformation for spherical polar coordinates:

- We have

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta.$$

- Hence the Jacobian $J(r, \theta, \phi)$ becomes:

$$\begin{aligned} J(r, \theta, \phi) &= \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}, \\ &= \begin{vmatrix} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \phi & -r \sin \theta & 0 \end{vmatrix}, \\ &= \dots, \\ &= r^2 \sin \theta. \end{aligned}$$