Chapter 3

Oscillations and resonances

We have already seen the simple example of a harmonic oscillator. With a better understanding of simple differential equations, we can turn to more concrete applications and consider more complicated oscillating systems.

3.1 Reminder: Homogeneous second order constant coefficient equations

Second order differential equations are more complicated—we can no longer get away with “integrating”—but we can readily deal with a particular class of equations: linear equations with constant coefficients.

In general we have

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

First we take a homogeneous equation (no driving force) and divide through by a (this is allowed as a – the coefficient multiplying the second derivative – is non-zero which must be true, otherwise the equation would not be second order!)

$$\frac{d^2x}{dt^2} + \frac{b}{a}\frac{dx}{dt} + \frac{c}{a}x = 0 \quad \Rightarrow \quad \frac{d^2x}{dt^2} + m\frac{dx}{dt} + nx = 0$$

where m and n are constants.

To find the general solution we make the **ansatz** that there is a solution of the form:

$$x = Ae^{kt}$$

where A and k are constants that we will try to find. Now

$$\frac{dx}{dt} = Ake^{kt}, \quad \frac{d^2x}{dt^2} = Ak^2e^{kt}$$

Thus if $x = A\exp(kt)$ is a solution of the homogeneous equation by substituting it into the equation we see that we must have

$$Ak^2e^{kt} + mAke^{kt} + nAe^{kt} = 0$$

The $\exp(kt)$ can be divided through as it is always non-zero for finite t. We can also divide through by A, since this must be non-zero for a **non-trivial** solution (a solution that is not $x(t) \equiv 0$). This leaves what is called the **auxiliary** equation of the differential equation (occasionally called the characteristic equation):

$$k^2 + mk + n = 0$$

This is a quadratic in k which has two roots k_1, k_2.

$$k_1 = \frac{-m + \sqrt{m^2 - 4n}}{2}, \quad k_2 = \frac{-m - \sqrt{m^2 - 4n}}{2}$$

Thus we typically have two possible solutions to the original equation $x_1 = A \exp(k_1 t)$ and $x_2 = A \exp(k_2 t)$.

We can check that the solutions are linearly independent[1] by computing the Wronskian[2]:

$$W[e^{k_1 t}, e^{k_2 t}] = \begin{vmatrix} e^{k_1 t} & e^{k_2 t} \\ k_1 e^{k_1 t} & k_2 e^{k_2 t} \end{vmatrix} = (k_2 - k_1) e^{(k_1 + k_2)t} \neq 0 \text{ provided } k_1 \neq k2.$$

Hence as long as the roots of the auxilliary equation are not identical, we have a set of linearly independent solutions.

It is a simple exercise of substitution to deduce that a linear combination of these two solutions is also a solution. Hence we have found that

$$x = Ae^{k_1 t} + Be^{k_2 t}$$

is a solution of the differential equation where A and B are ***any*** two (very loosely: integration) constants. This is the **general** solution for most equations of this type.

The two arbitrary constants A and B can only be determined by the **boundary** or **initial conditions**, as we shall see below.

From the formula for the solution of the auxiliary equation we can see that the nature of the solution of the equation will depend on the whether $m^2 > 4n$, $m^2 = 4n$, or $m^2 < 4n$ (in exactly the same way as the roots of a quadratic equation depend on this quantity). We now consider each of these cases in turn.

3.1.1 $m^2 > 4n$: real and distinct roots

When $m^2 > 4n$ the auxiliary equation gives two unequal real solutions for k and hence the general solution can be written as above:

$$x = Ae^{k_1 t} + Be^{k_2 t}.$$

3.1.2 $m^2 = 4n$: real and equal roots

When $m^2 = 4n$, the roots k of the auxiliary equation are real and equal. Let these roots be called k_1. In this particular case, it is not possible to form a general solution with two arbitrary constants, since we could rewrite it in the following form:

$$x = Ae^{k_1 t} + Be^{k_1 t} = (A + B)\, e^{k_1 t} = Ce^{k_1 t}$$

[1] This simply means that they are not proportional to one another.
[2] This is a new concept. We will return to it later.

So that effectively there would be only one arbitrary constant. Put another way, in this case the Wronskian vanishes everywhere, since $k_1 = k_2$. Thus we do not have the most general form (which needs two linearly independent, solutions).

We thus modify the guessed form of the solution to a new form[3]:

$$x = (A + Bt)\, e^{k_1 t}$$

Substitution into the differential equation shows that this is in fact a general solution provided that

$$k_1^2 + mk_1 + n = 0 \qquad \text{with} \qquad m^2 = 4n,$$

i.e., the auxiliary equation is satisfied.

The Wronskian of the two parts of the general solution is given by

$$W[e^{k_1 t}, te^{k_1 t}] = \begin{vmatrix} e^{k_1 t} & te^{k_1 t} \\ k_1 e^{k_1 t} & (1 + k_1 t)e^{k_1 t} \end{vmatrix} = e^{2k_1 t} \neq 0$$

for finite t, and so $e^{k_1 t}$ and $te^{k_1 t}$ are linearly independent. Hence this is a general solution of the case with equal roots.

3.1.3 $m^2 < 4n$, complex roots

When $m^2 < 4n$ the auxiliary equation has no real roots. However for m and n real the complex roots k_1 and k_2 appear as a complex conjugate pair. The roots are of the form

$$k_1 = \frac{-m + i\sqrt{4n - m^2}}{2}, \quad k_2 = \frac{-m - i\sqrt{4n - m^2}}{2}$$

where $i = \sqrt{-1}$. Hence the general solution is

$$\begin{aligned} x &= Ae^{t\left(-m+i\sqrt{4n-m^2}\right)/2} + Be^{t\left(-m-i\sqrt{4n-m^2}\right)/2} \\ &= e^{-mt/2}\left(Ae^{it\sqrt{4n-m^2}/2} + Be^{-it\sqrt{4n-m^2}/2}\right) \end{aligned}$$

Note that this looks like a complex (and complicated!) solution, even though we started off from a real equation, that only involved real numbers, but this is not necessarily the case. Now recall the following facts:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Thus we can write the above solution as

$$\begin{aligned} x &= e^{-mt/2}\left((A + B)\cos\left(\frac{\sqrt{4n - m^2}}{2}t\right) + i\,(A - B)\sin\left(\frac{\sqrt{4n - m^2}}{2}t\right)\right) \\ &= e^{-mt/2}\left(C\cos\left(\frac{\sqrt{4n - m^2}}{2}t\right) + D\sin\left(\frac{\sqrt{4n - m^2}}{2}t\right)\right). \end{aligned}$$

where C and D are arbitrary constants, formed from A and B, $C = A + B$, $D = i(A - B)$. This equation "looks" real. In fact real boundary data will generate real values of C and D.

[3] You can derive this, but we will not do that here.

3.2 Damped Oscillations

Damped oscillations play an important role in mechanical systems. A prime example is the shock absorber in a car. This can be modelled fairly accurately by combination of a spring and a “dashpot”.

A dashpot is a plunger in a pot of viscous liquid. A linear dashpot provides a resistive force proportional to the velocity of plunger, the constant of proportionality being λ. A Hookean spring with spring constant k and natural length l provides a restoring force proportional to a displacement.

As the car of mass m moves along, it encounters a bump, or variation in z in the road, the spring component compresses, and the plunger pushes into the dashpot. We want to know how the car responds to the bump so we can try to design the car to give the passengers a smooth ride.

We will take each of the car's four wheels to support a mass $m/4$. This mass will have three forces acting on it namely: gravity, the spring and the damper. We will only consider movement and forces in the vertical direction. Our coordinate system will take $\hat{k}$ to be in the vertical direction with the point $z = l$ representing the position of the car when the spring is extended to its natural length.

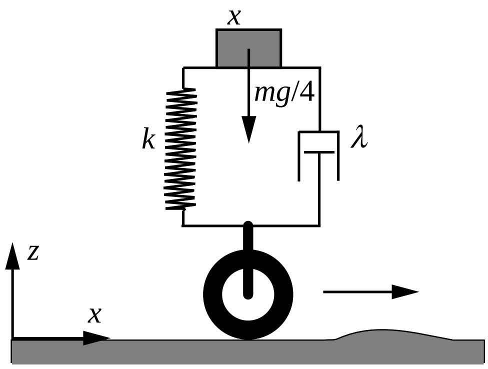

Figure 3.1: A schematic representation of one of the shock absorbers on a 4-wheeled car.

In the upwards vertical direction, **N2** then gives

$$\frac{m}{4}\ddot{z}(t) = -k(z-l) - \lambda\dot{z} - \frac{mg}{4}$$

Rearranging this we get

$$\ddot{z}(t) + \frac{4\lambda}{m}\dot{z} + \frac{4k}{m}z = \frac{4kl}{m} - g = \text{constant}$$

This is a second order linear constant coefficient ordinary differential equation. If can be represented schematically in terms of a linear operator $\mathcal{L}$ and a forcing function f.

$$\mathcal{L}z = f$$

where

$$\mathcal{L} = \left(\frac{d^2}{dt^2} + \frac{4\lambda}{m}\frac{d}{dt} + \frac{4k}{m}\right)$$

$$f = \frac{4kl}{m} - g$$

We know how to solve this type of equation. We can break the solution down into two parts.

$$z = z_{\text{CF}} + z_{\text{PI}}$$

where

- "CF" stands for "complementary function" and satisfies

$$\mathcal{L} z_{\text{CF}} = 0$$

- "PI" stands for "particular integral" and satisfies

$$\mathcal{L} z_{\text{PI}} = f$$

Clearly we still have

$$\mathcal{L} z = \mathcal{L}\left(z_{\text{CF}} + z_{\text{PI}}\right) = 0 + f = f.$$

We solve for z_{PI} by an "educated guess". Any derivative of a constant vanishes so we can guess that a possible solution to the equation is a constant. Putting this guess into the ODE we see:

$$\frac{4k z_{\text{PI}}}{m} = \frac{4kl}{m} - g \quad \Rightarrow \quad z_{\text{PI}} = l - \frac{mg}{4k}$$

This part of the solution corresponds to the position of the car if everything is stationary and the spring is compressed from its natural length due to the weight of the car.

Next we solve for z_{CF} by assuming there is a solution of the form $z = e^{\alpha t}$ where α is a constant. Substitution into the homogeneous equation

$$\ddot{z}(t) + \frac{4\lambda}{m}\dot{z} + \frac{4k}{m} z = 0$$

generates the auxiliary equation

$$\alpha^2 + \frac{4\lambda}{m}\alpha + \frac{4k}{m} = 0,$$

and hence α must be given by the solution to this quadratic

$$\alpha = -\frac{2\lambda}{m} \pm \frac{2\lambda}{m}\sqrt{1 - \frac{mk}{\lambda^2}}.$$

The complementary function is therefore of the form

$$z_{CF} = e^{-\frac{2\lambda}{m}t}\left(A e^{\frac{2\lambda}{m}\sqrt{1-\frac{mk}{\lambda^2}}\,t} + B e^{-\frac{2\lambda}{m}\sqrt{1-\frac{mk}{\lambda^2}}\,t}\right).$$

where A and B depend on initial data $z(0), \dot{z}(0)$.

Note that the precise form of this solution will depend on whether $1 - (mk)/\lambda^2$ is positive or negative. Hence the size of the positive quantity mk/λ^2 is crucial and we now discuss the different possible cases.

If $mk/\lambda^2 > 1$ the values of α are complex and so the solution can be rewritten as a decaying oscillatory one.

$$z_{CF} = e^{-\frac{2\lambda}{m}t}\left(C\cos\omega t + D\sin\omega t\right), \qquad \omega = \frac{2\lambda}{m}\sqrt{\frac{mk}{\lambda^2} - 1} > 0$$

This situation is said to be **underdamped**: the spring is dominating the response of the shock system. Any bump in the road initiates a long tail of oscillations in the shock system. It would be best not to attempt to try to drink anything while travelling in a car with this type of shock absorbing system. Indeed such a car would fail its annual MOT test of roadworthyness.

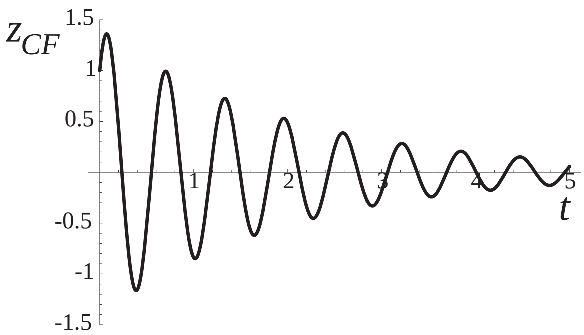

Figure 3.2: Schematic plot of response of underdamped shock system.

If $mk/\lambda^2 < 1$ then both values of α are real and negative. Hence the the solution is the sum of two exponential decaying functions.

$$z_{CF} = e^{-\frac{2\lambda}{m}t}\left(Ae^{\frac{2\lambda}{m}\sqrt{1-\frac{mk}{\lambda^2}}\,t} + Be^{-\frac{2\lambda}{m}\sqrt{1-\frac{mk}{\lambda^2}}\,t}\right),$$

where A and B are constants that depend on the initial data $z(0), \dot{z}(0)$. This situation is said to be **overdamped**: the dashpot is dominating the response of the shock system. The cushioning effect of the spring is effectively absent and the effect of bumps in the road will be felt over a long timescale.

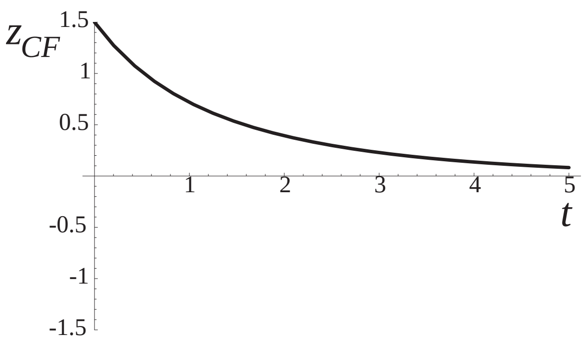

Figure 3.3: Schematic plot of response of overdamped shock system.

If $mk/\lambda^2 = 1$ mechanically the effect of the spring and dashpot are matched. Mathematically, the solution of the auxiliary equation is degenerate and only one exponential effectively exists. From our discussion of the methods used to solve the equation we know that the form of the solution will then be slightly different and takes the form

$$z_{CF} = e^{-\frac{2\lambda}{m}t}\left(At + B\right),$$

where A and B depend on initial data $z(0), \dot{z}(0)$. This situation is said to be **critical damping**: the linear term ensures that there is some cushioning, with effectively one and only one oscillation in response to a bump, but no long-term oscillatory behaviour[4]. This one-oscillation behaviour is what MOT inspectors looks for when they test the car.

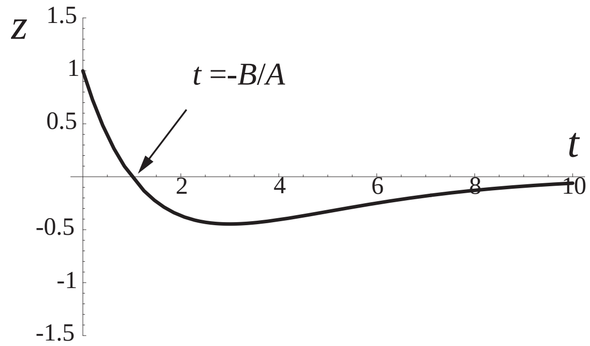

Figure 3.4: Schematic plot of response of critically damped shock system.

The ideas of underdamped, overdamped and critically damped oscillating systems governed by the second order linear constant coefficient differential equation arise in a large number of practical situations including motion of pendulums, radio waves, and water waves.

3.3 Reminder: Particular integrals and complementary functions

Let us suppose that we have found a **particular integral**, $x_P(t)$, i.e., a solution that satisfies

$$\mathcal{L}\left[x_P(t)\right] = f(t).$$

Due to the second order nature of the equation, we expect to have two arbitrary constants in the solution. Hence we might look for a second solution $x_Q(t)$ that also satisfies

$$\mathcal{L}\left[x_Q(t)\right] = f(t).$$

Then, following the ideas of the solutions of the homogeneous equation, we could construct a general solution of the form

$$x(t) = \alpha x_P(t) + \beta x_Q(t).$$

The only problem is, this is **wrong!**

[4]The exponential always wins over the linear term if you wait a while.

To see why, we substitute to the proposed solution into the left-hand side of the equation:

$$\mathcal{L}\left[\alpha x_P(t) + \beta x_Q(t)\right] = \mathcal{L}\left[\alpha x_P(t)\right] + \mathcal{L}\left[\beta x_Q(t)\right] = \alpha \mathcal{L}\left[x_P(t)\right] + \beta \mathcal{L}\left[x_Q(t)\right] = (\alpha+\beta) f(t).$$

This clearly does not balance the right-hand side of the equation unless the (assumed) arbitrary $\alpha + \beta$ is always set to be unity (which they can't be because they are arbitrary).

Hence there is **one** and **only one** particular integral of the inhomogeneous equation, called, say $x_P(t)$ which satisfies the full equation

$$\mathcal{L}\left[x_P(t)\right] = f(t).$$

We were expecting two arbitrary constants and now we have none! Where do we get them from?

The solution comes from considering the homogeneous equation

$$\mathcal{L}\left[x(t)\right] = 0,$$

i.e., the same left-hand side as the full inhomogeneous equation, but with $f(t)$ replace by 0 on the right-hand side. We know that this "reduced" equation will have a general solution involving two arbitrary constants A, B of the form

$$x_{CF}(t) = A x_1(t) + B x_2(t),$$

where the $x_j(t)$ each independent satisfy the linear homogeneous equation

$$\mathcal{L}\left[x_j(t)\right] = 0, \qquad j = 1, 2.$$

We call the general solution to the homogeneous equation $x_{CF}(t)$ the **complementary function** (hence the subscript "CF" attached to it).

Now consider what happens when we add the particular integral $x_P(t)$ and the complementary function $x_{CF}(t)$ and substitute them into the left-hand side of the **full inhomogeneous** equation and use the linearity property of the operator:

$$\mathcal{L}\left[x_P(t) + x_{CF}(t)\right] = \mathcal{L}\left[x_P(t)\right] + \mathcal{L}\left[x_{CF}(t)\right] = f(t) + 0 = f(t).$$

In other words the sum of the particular integral plus the complementary function **also** satisfies the full inhomogenous equation. Since the complementary function includes two arbitrary constants, we have found the general solution of the full inhomogeneous equation:

$$x(t) = x_P(t) + x_{CF}(t) = x_P(t) + A x_1(t) + B x_2(t),$$

where

$$\mathcal{L}\left[x_P(t)\right] = f(t), \qquad \mathcal{L}\left[x_j(t)\right] = 0,\ j = 1, 2.$$

In general the complementary function is easy to find: you just use the same techniques as we did for solving the homogeneous equations. The difficulty arises in finding the particular integral, which must satisfy the full inhomogeneous equation.

There are several ways of doing this. It is natural to first consider the **method of undetermined coefficients** (aka "educated guesswork"!).

This method considers the form of $f(t)$ and then looks for a particular integral according to the following rule of thumb:

- If $f(t)$ is an n^{th} order polynomial in t, assume a particular integral of the form
$$x_P(t) = A_n t^n + A_{n-1} t^{n-1} + \cdots A_1 t + A_0,$$
i.e., an n^{th} order polynomial with coefficients A_i that must be found by direct substitution into the inhomogeneous equation and balancing powers of t.

- If $f(t) = ke^{\alpha t}$, with k and α given constants, assume a particular integral of the form
$$x_P(t) = Ae^{\alpha t}$$
where the exponent α is the same as in $f(t)$ and A is a constant to be found by direct substitution into the inhomogeneous equation and balancing powers of $e^{\alpha t}$.

- If $f(t) = p \sin \beta t + q \cos \beta t$, with p, q and β given constants, assume a particular integral of the form
$$x_P(t) = A \sin \beta t + B \cos \beta t$$
where the frequency β is the same as in $f(t)$ and A, B are constants to be found by direct substitution into the inhomogeneous equation and balancing powers of $\sin \beta t$ and $\cos \beta t$.

- If $f(t)$ is a combination of the above three cases, e.g., an n^{th} order polynomial in t multiplied by an exponential $e^{\alpha t}$, with α a given constant, then assume a particular integral of the form
$$x_P(t) = e^{\alpha t} \left(A_n t^n + A_{n-1} t^{n-1} + \cdots A_1 t + A_0 \right),$$
and determine the unknown A_i by direct substitution into the full inhomogeneous equation and balancing of terms as above.

- If any component of the proposed particular integrals also satisfies the homogeneous equation, then this will not be a suitable particular integral, as it will only duplicate the complementary function. Hence multiply the proposed particular integral by t, to obtain a new proposed particular integral, repeating this until the guess no longer contains any terms that satisfy the homogeneous equation.

3.4 Resonance

Many oscillatory systems exhibit a phenomenon called **resonance**. This is an large (sometimes infinite) increase in the amplitude of an oscillation when the system undergoes forcing at (or near) its natural frequency.

Suppose we have a system[5]:
$$\ddot{x} + \omega^2 x = \cos \omega_0 t, \qquad x(0) = 0, \qquad \dot{x}(0) = 0.$$
This represents an oscillatory system with a natural frequency ω being forced at a frequency ω_0, chosen by us, where for the time being, we assume that $\omega \neq \omega_0$. The equation could model an undamped pendulum with periodic forcing.

We have:

[5] Note that the driving force is time dependent.

- a complementary function

$$x_{CF} = A \sin \omega t + B \cos \omega t$$

- a particular integral

$$x_P = C \sin \omega_0 t + D \cos \omega_0 t$$

 so for $\omega \neq \omega_0$ there is no duplication between the complementary function or particular integral.

We have

$$\begin{aligned} x_P(t) &= C \sin \omega_0 t + D \cos \omega_0 t, \\ \dot{x}_P(t) &= C\omega_0 \cos \omega_0 t - D\omega_0 \sin \omega_0 t \\ \ddot{x}_P(t) &= -C\omega_0^2 \sin \omega_0 t - D\omega_0^2 \cos \omega_0 t \end{aligned}$$

Hence substitution into the full inhomogeneous equation gives

$$\begin{aligned} -C\omega_0^2 \sin \omega_0 t - D\omega_0^2 \cos \omega_0 t + \omega^2 \left(C \sin \omega_0 t + D \cos \omega_0 t\right) &= \cos \omega_0 t \\ C\left(\omega^2 - \omega_0^2\right) \sin \omega_0 t + D\left(\omega^2 - \omega_0^2\right) \cos \omega_0 t &= \cos \omega_0 t \end{aligned}$$

and we have

$$C = 0, \qquad D = \frac{1}{\omega^2 - \omega_0^2}$$

Thus the particular integral is

$$x_P = \left(\frac{1}{\omega^2 - \omega_0^2}\right) \cos \omega_0 t.$$

It should be immediately obvious that if we tune the forcing frequency ω_0 to the natural frequency ω then the coefficient of the particular integral, and so the amplitude of the motion, diverges. Although catastrophic for the solution, this makes sense on both physical and mathematical grounds.

On **physical** grounds the system is receiving a forcing input at precisely the frequency it would like to oscillate. All the energy of the forcing can be transmitted into increasing the amplitude of the natural oscillation. This is very similar to pushing a child on a swing. Push the child at the wrong frequency and the amplitude will not increase (and you'll get hit in the face!). Push the child at the right frequency (just as the seat reaches its highest point in the swing) and the the amplitude of the swing will increase.

On **mathematical** grounds when $\omega_0 = \omega$ the particular integral becomes $\cos \omega t$, which identical to one of the components of the complementary function. Hence if we had started with $\omega_0 = \omega$, mathematically a particular integral of the form

$$x_P(t) = t\left(C \sin \omega t + D \cos \omega t\right),$$

 should have been used. A short exercise shows that taking this approach we obtain

$$x_P(t) = \frac{t}{2\omega_0} \sin \omega t \ .$$

However, you can see that this particular integral will also grow in time due to the presence of the multiplicative factor of t. Hence, the amplitude of the oscillation will still grow. So regardless of whether we examine the limit of $\omega_0 \to \omega$, or first take $\omega_0 = \omega$ and then consider $t \to \infty$, the amplitude will diverge. The divergence of an amplitude is often indicative of a breakdown in the assumptions underlying the actual mathematical model.

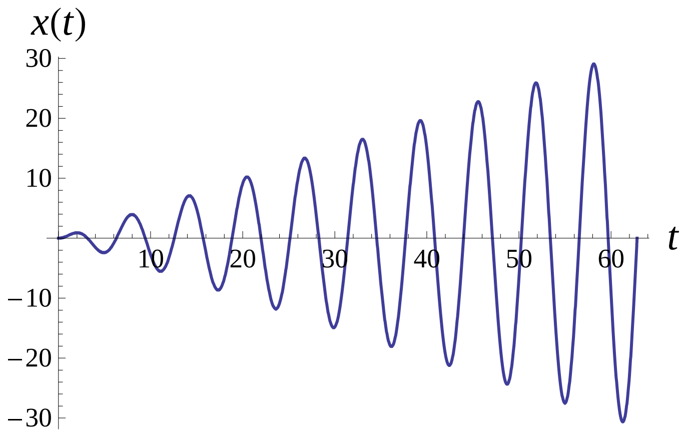

Figure 3.5: The growth of the amplitude of the oscillation when an undamped system is forced at its natural frequency ω_0 (here $= 1$), resulting in the behaviour $x(t) = t\sin(\omega t)/2\omega_0$.

For example, if x in this equation were modelling the angle of swing of a pendulum, the differential equation would only be valid for small values of $|x| \ll 1$. Hence if the amplitude of the particular integral is increasing unboundedly as $\omega_0 \to \omega$, then the assumptions that allowed us to linearise the equation from a $\sin x(t)$ gravitational component to a simple $x(t)$ term is false and hence at that frequency it would be more appropriate to consider the full nonlinear equation

$$\ddot{x} - \omega^2 \sin x(t) = \cos \omega t.$$

Another reason for a physical failure of the model might be because we have neglected the effect of damping (see below). Hence a good **mathematical** rule of thumb is:

"Divergence in a linear model for a set of parameters means you should consider a more complete nonlinear model near those parameter values".

3.4.1 Beating

We continue to analyse the same situation as above, but we use it to discuss the phenomenon of **frequency beating**.

If we now keep $\omega \neq \omega_0$, the full solution is

$$x = A\sin\omega t + B\cos\omega t + \frac{1}{\omega^2 - \omega_0^2}\cos\omega_0 t \ ,$$

which leads to

$$x(0) = B + \frac{1}{\omega^2 - \omega_0^2} = 0 \ ,$$

and

$$\dot{x}(0) = A\omega = 0 \Rightarrow A = 0 \ .$$

That is, we have

$$x(t) = x_{CF}(t) + x_P(t) = \left(\frac{1}{\omega^2 - \omega_0^2}\right)(\cos\omega_0 t - \cos\omega t)\,.$$

Subtracting the angle addition formulae

$$\cos(A \pm B) = \cos A\cos B \mp \sin A \sin B,$$

from one another, we have

$$2\sin A\sin B = \cos(A - B) - \cos(A + B).$$

Hence, seting $A - B = \omega_0$ and $A + B = \omega$, the full solution can be written as

$$x(t) = \left(\frac{2}{\omega^2 - \omega_0^2}\right)\sin\left[\left(\frac{\omega+\omega_0}{2}\right)t\right]\sin\left[\left(\frac{\omega-\omega_0}{2}\right)t\right].$$

The structure of this extremely informative. It is an amplitude, multiplied by two sines. If ω is approximately (but not exactly) equal to ω_0, the first sine oscillates with a high frequency, being the mean of ω and ω_0. The second sine oscillates at a lower frequency being the half the difference of ω and ω_0. The effect can be seen graphically:

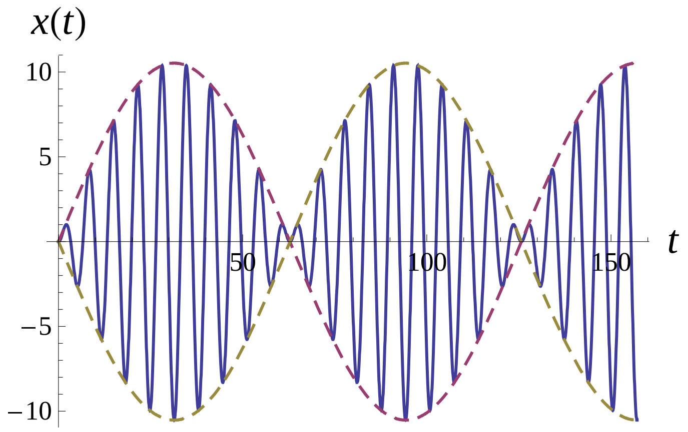

Figure 3.6: The effect of beating. Here the natural frequency is $\omega = 1$ and we have the forcing frequency set at $\omega_0 = 9/10$. The full line is the exact solution in the text. It is clear that high frequency waves, $(\omega + \omega_0)/2 = 19/20$, are modulated in amplitude by low frequency ones, $(\omega + \omega_0)/2 = 1/20$. The dashed lines denote the modulation of the amplitude given from the formula above by $\pm\sin\{(\omega - \omega_0)t/2\}$.

The amplitude higher frequency component is modulated by the lower frequency component. This is a phenomenon known as **beating**. It occurs, for example, when two musical instruments, violins say, that are almost, but not quite in tune. The audience hears a note that is at the mean of the two violins, but the sound varies in loudness at half the difference of the notes (a wowing sound).

3.4.2 Near resonance and damping.

The system we studied above had no damping $\dot{x}(t)$ term. Damping terms have the effect of "smearing out" the resonance. To see this we study the equation

$$\ddot{x} + \lambda\dot{x} + \omega^2 x = \cos\omega_0 t$$

where λ is a positive damping constant such that $\lambda^2 < 4\omega^2$ and the forcing frequency ω_0 is a parameter that can be changes to cycle through the value of the natural frequency of the undamped system ω.

We now have: a complementary function

$$x_{CF} = e^{-\lambda t/2}\left(A\cos\sigma t + B\sin\sigma t\right)$$

where

$$\sigma = \sqrt{4\omega^2 - \lambda^2}/2 > 0.$$

Clearly, this is a damped (actually underdamped) oscillatory motion, and so decreases in magnitude with time, whatever the forcing frequency ω_0 or (finite) initial conditions (i.e., the (finite) values of A and B). This type of motion is thus said to be **transient**: eventually the system will increasingly be increasingly less dominated by any component that corresponds to its **natural** frequency and will become increasingly dominated by motion at the **forced** frequency. Thus we won't bother finding A and B and pass on to the particular integral.

Due to the presence of $e^{-\lambda t/2}$ prefactor in the complementary function, the particular integral which takes the form

$$x_P = C\sin\omega_0 t + D\cos\omega_0 t$$

now has no duplication with the complementary function whether $\omega_0 = \omega$ or not. The linear growth is no longer apparent.

We have (yet again)

$$\begin{aligned}
x_P(t) &= C\sin\omega_0 t + D\cos\omega_0 t,\\
\dot{x}_P(t) &= C\omega_0\cos\omega_0 t - D\omega_0\sin\omega_0 t\\
\ddot{x}_P(t) &= -C\omega_0^2\sin\omega_0 t - D\omega_0^2\cos\omega_0 t
\end{aligned}$$

Hence substitution into the full inhomogeneous equation gives

$$\begin{aligned}
-C\omega_0^2\sin\omega_0 t - D\omega_0^2\cos\omega_0 t + \lambda\left(C\omega_0\cos\omega_0 t - D\omega_0\sin\omega_0 t\right) +&\\
+\omega^2\left(C\sin\omega_0 t + D\cos\omega_0 t\right) &= \cos\omega_0 t\\
\left\{C\left(\omega^2 - \omega_0^2\right) - \lambda D\omega_0\right\}\sin\omega_0 t + \left\{D\left(\omega^2 - \omega_0^2\right) + \lambda C\omega_0\right\}\cos\omega_0 t &= \cos\omega_0 t
\end{aligned}$$

and we have

$$C\left(\omega^2 - \omega_0^2\right) - \lambda D\omega_0 = 0, \qquad D\left(\omega^2 - \omega_0^2\right) + \lambda C\omega_0 = 1$$

which can be solved simultaneously to give

$$C = \frac{\lambda\omega_0}{\left(\omega^2 - \omega_0^2\right)^2 + \left(\lambda\omega_0\right)^2} \qquad D = \frac{\left(\omega^2 - \omega_0^2\right)}{\left(\omega^2 - \omega_0^2\right)^2 + \left(\lambda\omega_0\right)^2}$$

Thus the particular integral is

$$x_P = \frac{\lambda\omega_0}{\left(\omega^2-\omega_0^2\right)^2+\left(\lambda\omega_0\right)^2}\sin\omega_0 t + \frac{\left(\omega^2-\omega_0^2\right)}{\left(\omega^2-\omega_0^2\right)^2+\left(\lambda\omega_0\right)^2}\cos\omega_0 t$$

This can be put into the form of just a single, phase lagged, oscillatory term using the standard addition formulae for sine and cosine:

$$\begin{aligned}
C\sin\omega_0 t + D\cos\omega_0 t &= \sqrt{C^2+D^2}\left(\underbrace{\frac{C}{\sqrt{C^2+D^2}}}_{=\cos\phi}\sin\omega_0 t + \underbrace{\frac{D}{\sqrt{C^2+D^2}}}_{=\sin\phi}\cos\omega_0 t\right). \\
\Rightarrow x_P(t) &= \sqrt{C^2+D^2}\sin(\omega_0 t+\phi), \\
\tan\phi &= \frac{D}{C}.
\end{aligned}$$

Using this approach we see that the amplitude of the oscillation is $R(\omega_0) = \sqrt{C^2+D^2}$. For our C and D above this is

$$R(\omega_0) = \frac{1}{\sqrt{\left(\omega^2-\omega_0^2\right)^2+\left(\lambda\omega_0\right)^2}}, \qquad \tan\phi = \frac{\left(\omega^2-\omega_0^2\right)}{\lambda\omega}.$$

If we take the driving amplitude to be 1, the ratio of the response to the forcing is $R(\omega_0)$. A graph of this as a function of ω_0 is plotted below for different values of λ. Note that at resonance $\omega_0 = \omega$, the response climbs rapidly in amplitude, but is not infinite. The size of the response at resonance increases as the damping λ decreases.

Clearly the damping in the system means avoids a divergence in the response. This is a more physically realistic model of forced oscillatory systems.

This is a simple model of the mechanism that is responsible for the serious problems that arose with the Millennium footbridge in London. Soon after it was opened to foot passengers in June 2000, it became apparent that the bridge was undergoing a resonance swaying due to the frequency of footfalls. Extra masses and dampers had to be added to change the position and amplitude of the resonance. A more catastrophic example of resonance is the Tacoma bridge collapse of 1940. There a resonance of the bridge was excited by abnormal wind conditions. This resonance built up and ultimately led to the collapse of the bridge. Resonance is still an extremely important consideration in engineering construction (be it civil or electrical).

3.5 Going further: The Wronskian

So far we have focussed on second order constant equations. The are nice—because we can solve them–but they do not (necessarily) represent reality. In order to go further, we need to develop a bit more computational technology. As this machinery can be applied to more general problems, we take as our starting point an equation of form

$$\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} + q(t)x(t) = r(t),$$

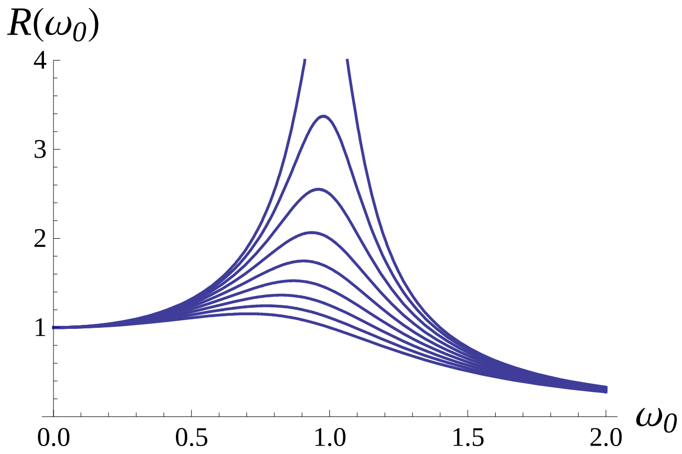

Figure 3.7: The growth of the ratio of the amplitude of response to the amplitude of forcing, $R(\omega_0)$ as the damping λ decreases, plotted as a function of the forcing frequency ω_0 near to the resonance frequency $\omega = 1$. As $\lambda \to 0$, the width of the resonance peak narrows and the amplitude at resonance $R(\omega_0 = \omega = 1) \to \infty$

where p, q and r are known functions of t. We also need initial data:

$$x(0) = x_0 \qquad \dot{x}(0) = u_0 \ .$$

We first consider the **homogeneous** case where $r(t) = 0$:

$$\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} + q(t)x(t) = 0.$$

and pose two questions we have so far (conveniently) avoided. How many solutions does this equation have? How do we know that these solutions are "independent"?

The first step towards answering these questions involves noting that, since the equation is linear, we can add solutions to get solutions. In essence, a linear combination of solutions is also a solution. This is easy to prove, bit it does not take us very far. We may also note that, since we have a second order equation, we need to "integrate" twice which means that we would expect to have two integration constants floating around. This suggests that (perhaps) we should expect that there are two solutions. This turns out to be correct, but how do we prove this?

Suppose we start by assuming that there is a single solution, $x_1(t)$, corresponding to given initial data at (say) $t = t_0$. Setting $x(t) = \alpha x_1(t)$ with α we then require

$$\begin{aligned} \alpha x_1(t_0) &= x_0 \qquad \Rightarrow \qquad \alpha = x_0/x_1(t_0) \\ \alpha \dot{x}_1(t_0) &= u_0 \qquad \Rightarrow \qquad \alpha = u_0/\dot{x}_1(t_0) \end{aligned}$$

and it is easy to see that we cannot (in general) satisfy both equations with a single value of α. Hence, a single solution is not sufficient to provide a unique solution to the second order system of the equation plus two initial conditions.

What happens if we consider two solutions, $x_1(t)$ and $x_2(t)$? Now set $x(t) = \alpha x_1(t) + \beta x_2(t)$ with α and β constants. Then the initial conditions require

$$\begin{aligned}\alpha x_1(t_0) + \beta x_2(t_0) &= x_0\\ \alpha \dot{x}_1(t_0) + \beta \dot{x}_2(t_0) &= u_0\end{aligned}$$

This can be written as a matrix equation

$$\begin{pmatrix} x_1(t_0) & x_2(t_0) \\ \dot{x}_1(t_0) & \dot{x}_2(t_0) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} x_0 \\ u_0 \end{pmatrix}$$

We see that, in order that solutions α and β exist, we require the existence of the inverse of the 2×2 matrix containing the initial data, $x_i(t_0)$.

From standard matrix theory (=linear algebra!), we know that the inverse exists if the determinant of the square matrix is non-zero, i.e.,

$$\begin{vmatrix} x_1(t_0) & x_2(t_0) \\ \dot{x}_1(t_0) & \dot{x}_2(t_0) \end{vmatrix} \neq 0.$$

We learn that two solutions will be sufficient, as long as this condition is satisfied.

Suppose that this determinant were actually zero. What would that mean? If we evaluate it explicitly we obtain[6]

$$x_1(t_0)\dot{x}_2(t_0) - \dot{x}_1(t_0)x_2(t_0) = 0 \qquad \Rightarrow \qquad \frac{x_1(t_0)}{x_2(t_0)} = \frac{\dot{x}_1(t_0)}{\dot{x}_2(t_0)} = c, \text{ say.}$$

Hence if the determinant were to vanish, we would have

$$x_1(t_0) = cx_2(t_0), \qquad \dot{x}_1(t_0) = c\dot{x}_2(t_0)$$

which implies that the two solutions $x_1(t)$ and $x_2(t)$ are proportional:

$$x_1(t) = cx_2(t).$$

Suppose instead that there are three fundamental solutions. Set $x(t) = \alpha x_1(t) + \beta x_2(t) + \gamma x_3(t)$ where $x_3(t) \not\equiv 0$ and α, β, γ constants to be found by satisfaction of the initial conditions. Then the initial conditions require the following equations to be satisfied:

$$\begin{aligned}\alpha x_1(t_0) + \beta x_2(t_0) + \gamma x_3(t_0) &= x_0\\ \alpha \dot{x}_1(t_0) + \beta \dot{x}_2(t_0) + \gamma \dot{x}_3(t_0) &= u_0\end{aligned}$$

This is an **underdetermined** system with 3 unknowns, but only 2 equations. Hence it is impossible to determine unique values of α, β and γ—we cannot have a unique solution. A similar argument applies when we have more than three candidate fundamental solutions.

We have learned something very important. A necessary and sufficient condition to find the complete solution of a second order homogenous linear ordinary differential equation with initial conditions is that we find two solutions $x_1(t)$, $x_2(t)$ such that

$$\begin{vmatrix} x_1(t_0) & x_2(t_0) \\ \dot{x}_1(t_0) & \dot{x}_2(t_0) \end{vmatrix} \neq 0.$$

[6]Assuming here that $x_2(t_0) \neq 0$, $\dot{x}_2(t_0) \neq 0$, but an analogous satisfactory argument can be found when they are.

Then the general solution of the differential equation is

$$x(t) = \alpha x_1(t) + \beta x_2(t)$$

with α and β determined by the initial conditions.

The arguments leads us to define the **Wronskian**

$$W\left[x_1(t), x_2(t)\right] = \begin{vmatrix} x_1(t) & x_2(t) \\ \dot{x}_1(t) & \dot{x}_2(t) \end{vmatrix}$$

Its non-vanishing is intimately related to the concept of **linear independence**.

Example 1: Linearly independents solutions

Consider the equation for a simple harmonic oscillator

$$x'' + x = 0$$

for which we know the solutions

$$x_1(t) = \cos t, \qquad x_2(t) = \sin t.$$

Then we have

$$x_1'(t) = -\sin t, \qquad x_2'(t) = \cos t,$$

so that

$$W[x_1(t), x_2(t)] = \begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & x_2'(t) \end{vmatrix} = \cos^2 t + \sin^2 t = 1, \forall t.$$

Hence we deduce that $\cos t$ and $\sin t$ are linearly independent, i.e., it is not possible to find non-zero constants c_1 and c_2 such that $c_1 \cos t + c_2 \sin t = 0$ on an interval of t. (Of course we can find these constants so that $c_1 \cos t + c_2 \sin t = 0$ at a point, but linear dependence would require this expression to vanish over a range of values of t.)

3.6 General force terms: Variation of parameters

Finally, let us go a step further in the direction of more realistic situations, e.g. where the involved forces do not lend themselves to "guessing" suitable particular integrals. This illustrates an important method, which "happens" to include our recent friend, the Wronskian. The key point is that it provides a systematic way to obtain a particular integral without any form of guesswork. The method of **variation of parameters** provides just such a method. It can be applied whether the coefficients in the linear equation are constant or functions of the independent variable.

Let us start from a linear second order equation of form:

$$x''(t) + p(t)x'(t) + q(t)x(t) = r(t),$$

and assume that we have found a complementary function

$$x_{CF}(t) = \alpha x_1(t) + \beta x_2(t)$$

which satisfies the homogeneous equation

$$x''(t) + p(t)x'(t) + q(t)x(t) = 0.$$

Suppose we now look for a particular integral of the form

$$x_P(t) = v_1(t)x_1(t) + v_2(t)x_2(t),$$

where the $x_i(t)$ are the complementary functions above and $v_i(t)$ are functions to be determined.

Note that, if $v_i(t)$ were constants then this would have an identical form to the complementary function, which would not be allowed. We therefore assume that $v_i(t)$ are **not** constants and that they will turn out to be such that the complementary function is linearly independent of the particular integral.

Let us further suppose that the following two conditions hold (we will soon see why these are useful):

$$\begin{aligned} v_1'(t)x_1(t) + v_2'(t)x_2(t) &= 0 \\ v_1'(t)x_1'(t) + v_2'(t)x_2'(t) &= r(t) \end{aligned}$$

We can make these two assumptions, since the two $v_i(t)$ are, as yet, undetermined.

To find the $v_i(t)$ we must differentiate $x_P(t)$ and substitute it into the inhomogeneous equation. We then have

$$x_P'(t) = \underbrace{v_1'(t)x_1(t) + v_2'(t)x_2(t)}_{=0 \text{ by assumption}} + v_1(t)x_1'(t) + v_2(t)x_2'(t) = v_1(t)x_1'(t) + v_2(t)x_2'(t)$$

$$x_P''(t) = v_1(t)x_1''(t) + v_2(t)x_2''(t) + \underbrace{v_1'(t)x_1'(t) + v_2'(t)x_2'(t)}_{=r(t) \text{ by assumption}} = v_1(t)x_1''(t) + v_2(t)x_2''(t) + r(t)$$

Substitution thus gives

$$\begin{aligned} \text{LHS} &= x_P'' + p(t)x_P' + q(t)x_P(t) \\ &= \{v_1(t)x_1''(t) + v_2(t)x_2''(t) + r(t)\} + p(t)\{v_1(t)x_1'(t) + v_2(t)x_2'(t)\} + q(t)x_P(t) \\ &= v_1(t)\underbrace{\{x_1''(t) + p(t)x_1'(t) + q(t)x_1(t)\}}_{=0} + v_2(t)\underbrace{\{x_2''(t) + p(t)x_2'(t) + q(t)x_2(t)\}}_{=0} + r(t) \\ &= r(t) = \text{RHS} \end{aligned}$$

So what? Well... we have just shown that

$$x_P(t) = v_1(t)x_1(t) + v_2(t)x_2(t)$$

is the particular integral of

$$x''(t) + p(t)x'(t) + q(t)x(t) = r(t)$$

provided that

$$x_i''(t) + p(t)x_i'(t) + q(t)x_i(t) = 0, \qquad i = 1, 2$$

and

$$\begin{aligned} v_1'(t)x_1(t) + v_2'(t)x_2(t) &= 0 \\ v_1'(t)x_1'(t) + v_2'(t)x_2'(t) &= r(t). \end{aligned}$$

How does this help to find the $v_i(t)$? We have two simultaneous equations for $v_i'(t)$. These can be rewritten as a matrix equation:

$$\begin{pmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{pmatrix} \begin{pmatrix} v_1'(t) \\ v_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ r(t) \end{pmatrix}$$

which can be solved by inverting the coefficient matrix to give

$$\begin{pmatrix} v_1'(t) \\ v_2'(t) \end{pmatrix} = \frac{\begin{pmatrix} x_2'(t) & -x_2(t) \\ -x_1'(t) & x_1(t) \end{pmatrix}}{\begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix}} \begin{pmatrix} 0 \\ r(t) \end{pmatrix} = \frac{r(t)}{W[x_1(t), x_2(t)]} \begin{pmatrix} -x_2(t) \\ x_1(t) \end{pmatrix}$$

where we have recognised $W[x_1(t), x_2(t)]$ as the Wronskian of $x_1(t)$ and $x_2(t)$. Thus we have two equations to solve

$$v_1'(t) = -\frac{x_2(t)r(t)}{W[x_1(t), x_2(t)]}, \qquad v_2'(t) = \frac{x_1(t)r(t)}{W[x_1(t), x_2(t)]}$$

These can be integrated immediately with respect to t to give the solutions

$$v_1(t) = -\int^t \frac{x_2(\zeta)r(\zeta)}{W[x_1(\zeta), x_2(\zeta)]} d\zeta \qquad v_2(t) = \int^t \frac{x_1(\zeta)r(\zeta)}{W[x_1(\zeta), x_2(\zeta)]} d\zeta$$

At the end of the day, we find that the particular integral is given by

$$\begin{aligned} x_P(t) &= v_1(t)x_1(t) + v_2(t)x_2(t) \\ &= -x_1(t)\int^t \frac{x_2(\zeta)r(\zeta)}{W[x_1(\zeta), x_2(\zeta)]} d\zeta + x_2(t)\int^t \frac{x_1(\zeta)r(\zeta)}{W[x_1(\zeta), x_2(\zeta)]} d\zeta \\ \Rightarrow x_P(t) &= \int^t \frac{\{x_2(t)x_1(\zeta) - x_1(t)x_2(\zeta)\}\, r(\zeta)}{W[x_1(\zeta), x_2(\zeta)]} d\zeta \end{aligned}$$

The final formula can be conveniently remembered as

$$x_P(t) = \int^t \frac{\begin{vmatrix} x_1(\zeta) & x_2(\zeta) \\ x_1(t) & x_2(t) \end{vmatrix}}{W[x_1(\zeta), x_2(\zeta)]} r(\zeta) d\zeta$$

- Note that the lower limit is not specified. This is because any arbitrary change in the lower limit just generates the complementary function, which adds nothing to the particular integral. Hence it is possible to ignore the arbitrary constant of integration.

- Note that the coefficients in the inhomogeneous equation $p(t)$ and $q(t)$ were not assumed to be constants. Hence this formula is valid for particular integrals of more than just inhomogeneous equations of differential equations with constant coefficients.

- An analogous formula can be derived for n^{th} order equations, but we will not consider this here.

Example 2: A simple case

Consider the following linear, inhomogeneous, second order, constant-coefficient equation:

$$\ddot{x} - \dot{x} - 2x = e^{3t}$$

for which it is easy to see that the particular integral has the form Ae^{3t} and we can substitute in to find A. Now let us instead use variation of parameters to systematically find the particular integral **without guessing**.
The complementary function is

$$x(t) = Ae^{-t} + Be^{2t} \qquad \Rightarrow \qquad x_1(t) = e^{-t}, x_2(t) = e^{2t}$$

The inhomogeneous term on the right-hand side is

$$r(t) = e^{3t}.$$

The Wronskian is

$$W[x_1(\zeta), x_2(\zeta)] = \begin{vmatrix} x_1(\zeta) & x_2(\zeta) \\ \dot{x}_1(\zeta) & \dot{x}_2(\zeta) \end{vmatrix} = \begin{vmatrix} e^{-\zeta} & e^{2\zeta} \\ -e^{-\zeta} & 2e^{2\zeta} \end{vmatrix} = 3e^{\zeta}$$

Thus the particular integral is given by:

$$\begin{aligned}
x_P(t) &= \int^t \frac{\begin{vmatrix} e^{-\zeta} & e^{2\zeta} \\ e^{-t} & e^{2t} \end{vmatrix}}{3e^{\zeta}} e^{3\zeta} d\zeta \\
&= \frac{1}{3}\int^t \left(e^{2t-\zeta} - e^{2\zeta-t}\right) e^{2\zeta} d\zeta \\
&= \frac{1}{3}\int^t \left(e^{2t+\zeta} - e^{4\zeta-t}\right) d\zeta \\
&= \frac{1}{3}e^{2t}\int^t e^{\zeta} d\zeta - \frac{1}{3}e^{-t}\int^t e^{4\zeta} d\zeta \\
&= \frac{1}{3}e^{2t+t} - \frac{1}{12}e^{-t+4t} \\
&= \frac{1}{4}e^{3t},
\end{aligned}$$

where it is worth noting that we ignored the constants of integration in the integral.

Example 3: A more complicated situation

The variation of parameters is a powerful tool for finding the particular integral of inhomogeneous equations where it is not easy to guess its form from the inhomogeneity on the right-hand side. As an example of this, we seek the particular integral of the inhomogeneous equation:

$$\ddot{x} - 2\dot{x} + x = \frac{e^t}{t^2+1}.$$

The complementary function is given in straightforward manner by

$$x(t) = (A + Bt)e^t \qquad \Rightarrow \qquad x_1(t) = e^t, x_2(t) = te^t$$

The inhomogeneous term on the RHS is

$$r(t) = \frac{e^t}{t^2+1},$$

which is clearly like nothing we have studied before. Don't panic! Just use the formula from variation of parameters.
The Wronskian is

$$W[x_1(\zeta), x_2(\zeta)] = \begin{vmatrix} x_1(\zeta) & x_2(\zeta) \\ \dot{x}_1(\zeta) & \dot{x}_2(\zeta) \end{vmatrix} = \begin{vmatrix} e^\zeta & \zeta e^\zeta \\ e^\zeta & (1+\zeta)e^\zeta \end{vmatrix} = e^{2\zeta}$$

Thus the particular integral is given by:

$$\begin{aligned}
x_P(t) &= \int^t \frac{\begin{vmatrix} e^\zeta & \zeta e^\zeta \\ e^t & te^t \end{vmatrix}}{e^{2\zeta}} \left(\frac{e^\zeta}{\zeta^2+1}\right) d\zeta \\
&= \int^t \left(te^{t+\zeta} - \zeta e^{\zeta+t}\right)\left(\frac{e^{-\zeta}}{\zeta^2+1}\right) d\zeta \\
&= \int^t \frac{(te^t - \zeta e^t)}{\zeta^2+1} d\zeta \\
&= te^t \int^t \frac{d\zeta}{\zeta^2+1} d\zeta - e^t \int^t \frac{\zeta}{\zeta^2+1} d\zeta \\
&= te^t \arctan t - \frac{1}{2} e^t \log|t^2+1|
\end{aligned}$$

Which clearly looks nothing like the original right-hand side: without the formula from the variation of parameters we would not not have realistically been able to guess the form of the PI!
Finally, the general solution of the above equation is

$$x(t) = x_{CF} + x_P = (A+Bt)e^t + te^t \arctan t - \frac{1}{2} e^t \log|t^2+1|.$$

3.7 Eigenvalues and boundary value problems (vibrating strings)

So far we have assumed that the problem we consider involves only initial conditions, which serve to fix the integration constants in the problem and/or the relation between linearly independent solutions at the initial time. There is, however, an important alternative to this, where the solution is instead constrained by set boundary conditions. This changes the nature of the problem—instead of considering the evolution of the system we would typically end up discussing eigenvalues and characteristic solutions which help us understand the behaviour. We will not go very far in this direction, but it is very important that we understand the ideas involved as the lay the foundation for the standard method of solving linear **partial differential equations**.

As a step in this direction—aimed at illustrating the principle and the fact that the specification of boundary data may lead to either no solutions or an infinite number of them—let us consider

$$x''(t) + p(t)x'(t) + \lambda q(t)x(t) = 0, \qquad x(a) = 0, \qquad x(b) = 0.$$

The key point is that we will treat λ as a parameter, which turns out to determine the existence and type of solutions. Intuitively, thinking of this as an initially unspecified parameter, we can "tune" λ (exactly as in tuning the frequency of a violin string, say) to select the type of solution we want.

Typically a solution satisfying the boundary conditions will exist for only certain values of λ. Such problems are called **eigenvalue problems** and λ is the **eigenvalue**[7]. The solution of the equation+boundary value problem corresponding to a particular value of λ is called the **eigenfunction**.

The relationship between eigenvalues of differential equations and matrices through linear algebra is intimate, but beyond the scope of the present discussion.

3.7.1 Determination of Eigenvalues.

Starting from the equation[8]

$$x''(t) + \lambda x(t) = 0, \qquad x(0) = 0, \qquad x(L) = 0, \qquad L > 0.$$

let us figure out which values of λ lead to non-trivial solutions? We have to consider all (real) possibilities.

- Suppose we have $\lambda = 0$:

[7] *Eigen* is a German word meaning "own" in the sense of "special" or "characteristic" value.

[8] This problem is motivated by the one-dimensional wave equation

$$\frac{\partial^2 Y}{\partial t^2} - \frac{\partial^2 Y}{\partial x^2} = 0$$

which, if we assume that the time dependence is "harmonic" (read: $Y \propto e^{i\omega t}$ for some frequency ω), reduces to

$$\frac{\partial^2 Y}{\partial x^2} + \omega^2 Y = 0$$

If $\lambda = 0$, the equation becomes $x''(t) = 0$, with a general solution

$$x = At + B,$$

Substitution into the endpoint conditions gives $A = B = 0$. Hence the only solution when $\lambda = 0$ is the trivial one $x(t) = 0$. Such static solutions are often rejected in modelling situations as being too boring, or irrelevant. Hence $\lambda = 0$ is rejected as an eigenvalue.

- If, instead, the parameter is negative, $\lambda < 0 = -\alpha^2$:
  Now the equation
  $$x''(t) - \alpha^2 x(t) = 0$$
  has a general solution
  $$x(t) = A \cosh \alpha t + B \sinh \alpha t.$$
  Substitution into the left endpoint conditions gives:
  $$x(0) = A \cosh 0 + B \sinh 0 = A = 0.$$
  Substitution into the right endpoint conditions gives:
  $$x(L) = A \cosh \alpha L + B \sinh \alpha L = B \sinh \alpha L = 0.$$
  Now $\sinh \alpha L$ can only vanish when $\alpha L = 0$. Hence we deduce that $B = 0$. Again the only solution when $\lambda < 0$ is the trivial one $x(t) = 0$. Hence there are no eigenvalues when $\lambda < 0$.

- Finally, the parameter may be positive, $\lambda > 0 = +\alpha^2$:
  The equation is now
  $$x''(t) + \alpha^2 x(t) = 0$$
  with a general solution
  $$x(t) = A \cos \alpha t + B \sin \alpha t.$$
  Substitution into the left endpoint conditions gives:
  $$x(0) = A \cos 0 + B \sin 0 = A = 0.$$
  Substitution into the right endpoint conditions gives:
  $$x(L) = A \cos \alpha L + B \sin \alpha L = B \sin \alpha L = 0.$$
  Clearly $B \sin \alpha L$ can vanish when $B = 0$. This would again lead to a trivial solution. However $B \sin \alpha L$ can also vanish if we tune λ so that
  $$\alpha L = n\pi, \qquad n \in \mathbb{N} \qquad \Rightarrow \qquad \alpha = \frac{n\pi}{L} \qquad \Rightarrow \qquad \lambda \equiv \lambda_n = \left(\frac{n\pi}{L}\right)^2.$$
  Hence for a set of discrete eigenvalues λ_n, we can find eigenfunctions
  $$x(t) = B \sin \lambda_n t = B_n \sin \frac{n\pi t}{L}$$
  that satisfy the equation and the boundary conditions. Note that B is still undetermined. Any possible finite value of B will do, and it could vary between the λ_n. As we shall see in the example below, additional information is often used to determine which values of B_n are appropriate.

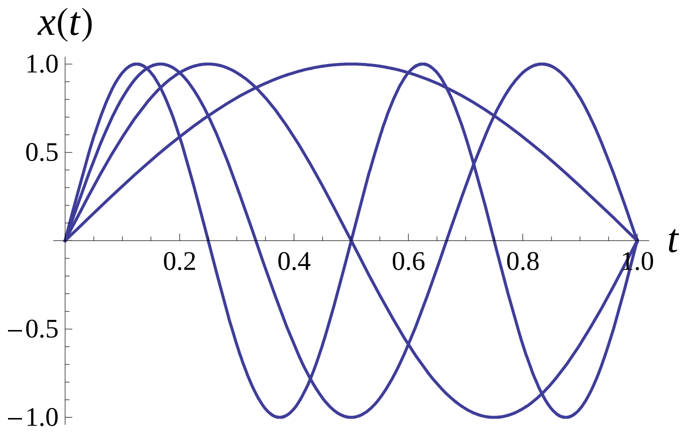

Figure 3.8: The eigenfunctions $\sin n\pi t/L$ for $L = 1$, $n = 1, 2, 3, 4$.

We have effectively just solved the problem of an oscillating violin/guitar string. The eigenvalues we have found are the frequencies you hear when the string instrument is played. The eigenfunctions describe ways in which the strings can vibrate.

3.8 Euler-type equations

An Euler-type equation[9] is the simplest type of second order differential equations that does **not** have constant coefficients. The class of equations is easy extended to higher orders, but we will only consider the second-order example:

$$a_2 t^2 \frac{d^2x}{dt^2} + a_1 t \frac{dx}{dt} + a_0 x = 0.$$

(one factor of t per derivative). The problem is easily solved because it can be turned into a second-order linear constant-coefficient equation by the change of variables:

$$z = \ln t \qquad \Rightarrow \qquad t = e^z.$$

Using the chain rule, we have

$$\begin{aligned}
\frac{dx}{dt} &= \frac{dx}{dz}\frac{dz}{dt} = \frac{1}{t}\frac{dx}{dz} = e^{-z}\frac{dx}{dz} \\
\frac{d^2x}{dt^2} &= \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\left(e^{-z}\frac{dx}{dz}\right) = e^{-z}\frac{d}{dz}\left(e^{-z}\frac{dx}{dz}\right) = e^{-2z}\frac{d^2x}{dz^2} - e^{-2z}\frac{dx}{dz}.
\end{aligned}$$

Substitution of these results into the Euler equation gives

$$\begin{aligned}
a_2 e^{2z}\left(e^{-2z}\frac{d^2x}{dz^2} - e^{-2z}\frac{dx}{dz}\right) + a_1 e^z \left(e^{-z}\frac{dx}{dz}\right) + a_0 x &= 0, \\
a_2\frac{d^2x}{dz^2} + (a_1 - a_2)\frac{dx}{dz} + a_0 x &= 0
\end{aligned}$$

[9]Note that Euler was so prolific that there are many different equations that bear the name “Euler equation”, e.g., the main equation in fluid dyanmics.

This is (obviously) a homogeneous second-order constant-coefficient equation and so can be solved by the usual method, for x as a function of z. Then x can be written in terms of t by using the substitution $z = \ln t$.

In practice, note that (at least for distinct roots) the solutions of the transformed equation will take the form $e^{k_i z}$. Hence after transforming back to the original variables x, then the solutions take the form t^{k_i}. Hence it is possible to attack Euler equations by a direct substitution of solutions of the form t^k and solving the indicial equation (=auxiliary equations) for k.

Example 4: Stellar oscillations

A typical situation where Euler equations arise is in *astero-seismology* (the oscillations of stars)—perhaps ironically, involving the Euler equations from fluid dynamics. The relevant equation relates to the dependence on the radial coordinate, which we will call t in order to stay close to the previous description. One arrives at an ordinary differential equation by the method of separation of variables (which you will find out a lot more about next year). This equation can be written

$$t^2 x'' + 2tx' - l(l+1)x = 0$$

with $l \geq 0$ and integer. We want the solution to be i) regular at the centre, $t = 0$, and ii) match smoothly to a solution in the star's exterior. We will ignore the second of these conditions for now.
Trying a power-law solution, $x = t^\alpha$, we get

$$\alpha(\alpha-1)t^2 \times t^{\alpha-2} + 2\alpha t \times t^{\alpha-1} - l(l+1)t^\alpha = 0$$

or, dividing through by the common factor t^α;

$$\alpha(\alpha-1) + 2\alpha - l(l+1) = 0$$

or

$$\alpha^2 + \alpha - l(l+1) = 0 \Longrightarrow \alpha = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + l(l+1)} = -\frac{1}{2} \pm \left(l + \frac{1}{2}\right)$$

We have two options. Either $\alpha = l$, which means that the solution behaves like t^l. That is, it satisfies the boundary condition at $t = 0$. The other option is $\alpha = -l-1$ and a solution that behaves as t^{-l-1}, which diverges as $t \to 0$ and therefore must be discarded.

3.8.1 Repeated roots of the indical equation

Suppose we want to solve

$$t^2 \frac{d^2x}{dt^2} + 3t\frac{dx}{dt} + x = 0.$$

Try a solution of the form $x(t) = t^k$, as before. Then

$$x'(t) = kt^{k-1}, \qquad x''(t) = k(k-1)^{k-2}.$$

Substitution into the equation gives

$$t^2 \times \underbrace{k(k-1)t^{k-2}}_{x''(t)} + 3t \times \underbrace{kt^{k-1}}_{x'(t)} + \underbrace{t^k}_{x(t)} = 0 \qquad \Rightarrow \qquad \{k(k-1) + 3k + 1\}\, t^k = 0$$

This equation must be valid for all t, hence we obtain the **indicial** equation as

$$k(k-1) + 3k + 1 = 0 \qquad \Rightarrow \qquad k^2 + 2k + 1 = 0 \qquad \Rightarrow \qquad k = -1, \text{twice.}$$

We obtain but one solution as $x = 1/t$, but what about the other one? The hint comes from what happens when you have equal roots in a second order linear constant coefficient equations. In that case the solution takes the form $(A + Bz)e^{k_1 z}$. We could have solved the Euler equation by a transformation $z = \ln t$ as a second order linear constant coefficient equation, in which case $k_1 = -1$. Hence the general solution takes the form:

$$x(t) = \frac{A}{t} + \frac{B}{t} \ln t.$$

In general when there are equal roots k_1 for the indicial equation of an Euler equation, the general solution is then

$$x(t) = (A + B \ln t)t^{k_1}$$

3.8.2 Complex roots of the indicial equation

It is also possible to obtain complex values for the roots of the indicial equation, say $k = \alpha \pm \beta i$. In that case, using the rules of logarithms, the general solution can be written as

$$x(t) = At^{\alpha+\beta i} + Bt^{\alpha-\beta i} = t^{\alpha} \left(Ae^{\beta i \ln t} + Be^{-\beta i \ln t} \right) = t^{\alpha} \left(C \cos(\beta \ln t) + D \sin(\beta \ln t) \right),$$

where A, B, or alternatively, C, D are arbitrary constants determined by the two initial/boundary conditions.