Chapter 5

Integer linear programs

So far we have considered linear programs of the form

$$\begin{array}{lrcl} \text{min} & cx & & \\ \text{subject to} & Ax & \geq & b \\ & x & \geq & 0 \\ & x & \in & \mathbb{R}^n. \end{array} \tag{5.1}$$

We have here made explicit that each component of x is a real number. In addition we have $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c, x \in \mathbb{R}^n$ being matrices and vectors in the appropriate real spaces.

There are a lot of applications where this does not make perfect sense. There are many situations where talking about half a person, or a third of a car, or similar, is meaningless. In these cases we instead need to consider the *integer linear program*

$$\begin{array}{lrcl} \text{min} & cx & & \\ \text{subject to} & Ax & \geq & b \\ & x & \geq & 0 \\ & x & \in & \mathbb{Z}^n, \end{array} \tag{5.2}$$

where $A \in \mathbb{Q}^{m\times n}, b \in \mathbb{Q}^m, c, x \in \mathbb{Q}^n$ are now matrices and vectors in the appropriate *rational* spaces.

Another special case is where we have to make yes/no or true/false decisions. For this we consider the *integer linear program*

$$\begin{array}{lrcl} \text{min} & cx & & \\ \text{subject to} & Ax & \geq & b \\ & x & \geq & 0 \\ & x & \in & \{0,1\}^n, \end{array} \tag{5.3}$$

where again $A \in \mathbb{Q}^{m\times n}, b \in \mathbb{Q}^m, c \in \mathbb{Q}^n$ are now matrices and vectors in the appropriate *rational* spaces.

5.1 Optimal solutions

The feasible region of an integer linear program is not a polyhedron. It is instead the intersection of a polyhedron with a lattice, where the lattice is a grid of integer points. This immediately implies that, in general, an integer linear program cannot be solved by the simplex method (as the "vertices" need not be optimal). This is illustrated in figure 5.1.

We can, however, solve the linear program *relaxation* of an integer linear program. This is obtained by dropping the restriction $x \in \mathbb{Z}^n$, replacing it with $x \in \mathbb{R}^n$. Essentially, we pretend

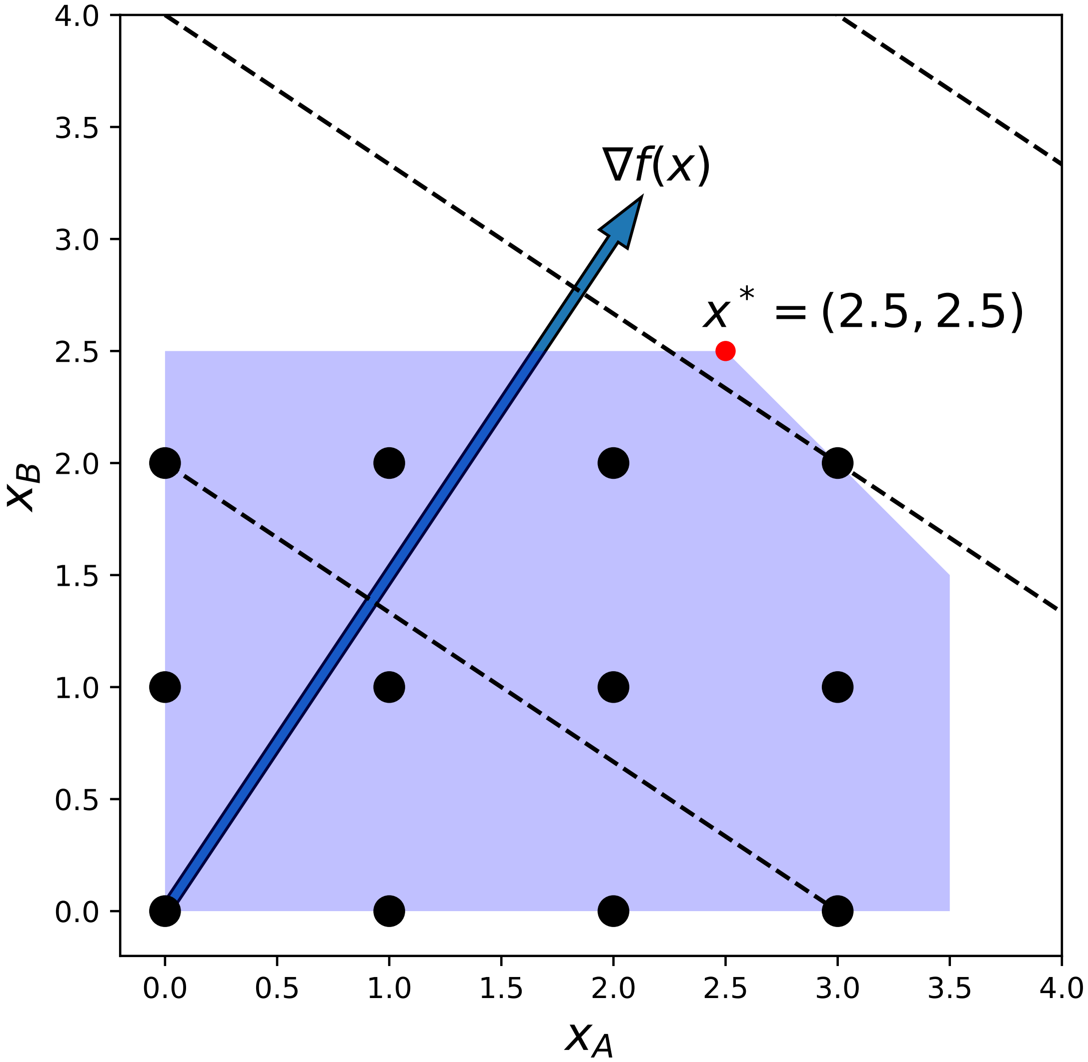

Figure 5.1: The feasible region for an integer linear program is given by the intersection of the feasible region for the standard linear program (the blue shaded region) and the integer lattice. The black dots here form that restricted feasible region. The optimal solution for the standard problem (shown by the red dot) does not lie in this feasible region, although it is close to the optimal integer solution (as seen by the level curves of the objective function, given by the dashed lines).

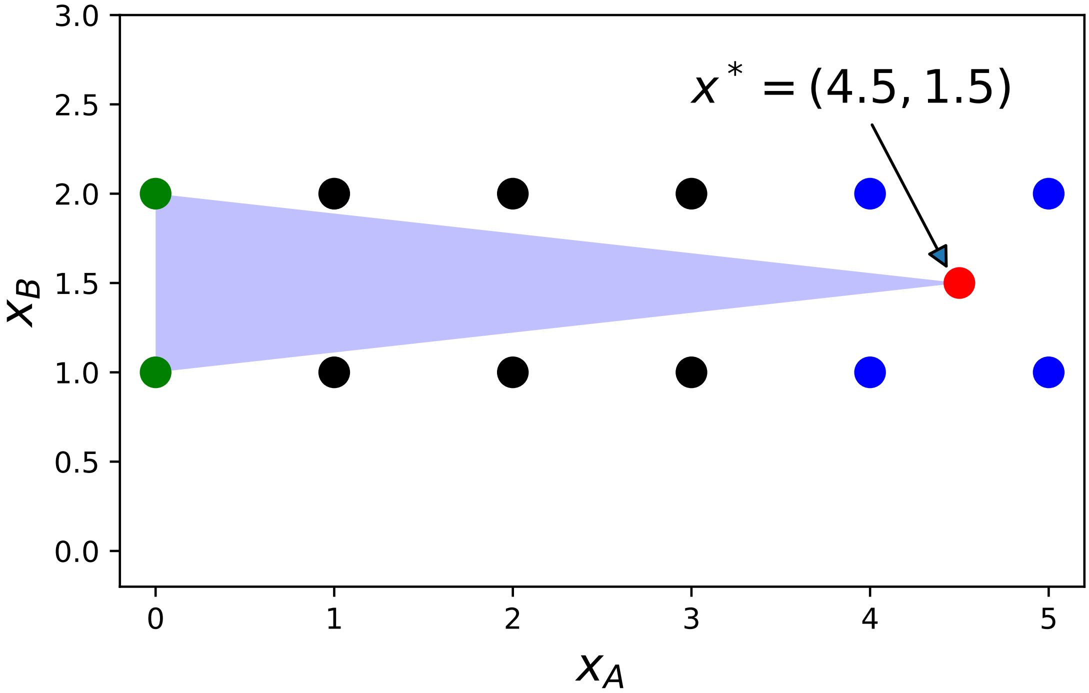

Figure 5.2: A pathological integer linear program. The feasible region of the standard problem (blue shaded region) is "long and thin". The integer solutions (blue dots) that neighbour the solution to the standard problem (red dot) are not close to any feasible solution to the integer problem (green dots).

that an integer linear program is a standard linear program. This can be solved by the simplex method as before.

Unfortunately, solutions of the relaxation problem can be a long way from the optimal solution of the integer linear program. If the feasible region near the optimal solution x^* to the relaxation problem is "long and thin", then points on the integer lattice surrounding x^* can easily be infeasible. It is possible to construct cases where the difference in the objective function value between the integer linear program and it relaxation problem are arbitrarily large. This is illustrated in figure 5.2.

Typical methods for solving integer problems do use the simplex method to solve linear programs, typically applied to many pieces of the feasible region individually (later courses will cover this). The simplex method remains a key building block.

5.2 Total unimodularity

5.2.1 When relaxation works

There are, however, problems where the relaxation problem works. Consider the problem

$$\begin{array}{llcl} \text{min} & cx & & \\ \text{subject to} & Ax & = & b \\ & x & \geq & 0 \\ & x & \in & \mathbb{Z}^n, \end{array} \tag{5.4}$$

where $b \in \mathbb{Z}^n$. This can be constructed by scaling each row of $Ax = b$ to make b integer.

Consider the relaxation problem where all restrictions are dropped. For any optimal (basic) solution of the relaxation problem we have

$$x_B = A_B^{-1}b, \quad x_N = 0. \tag{5.5}$$

We have that b is integer. If A_B^{-1} is also integer it follows that x_B is integer. Therefore the solution to the linear program is also a solution to the *integer* linear program, and the simplex method will work.

5.2.2 Total unimodularity

We want some criteria that checks when the inverse of a matrix is integer, without having to compute it and enumerate the entries. We say that a matrix $A \in \mathbb{R}^{m\times n}$ is called *total unimodular* if the determinant of all its square submatrices takes values in $\{-1, 0, 1\}$.

Theorem 5.2.1. *If A is total unimodular then A_B^{-1} is integer.*

Proof. The inverse of A_B can be written as

$$A_B^{-1} = \frac{1}{\det(A_B)} \begin{pmatrix} \alpha_{1B_1} & \cdots & \alpha_{1B_m} \\ \vdots & \ddots & \vdots \\ \alpha_{mB_1} & \cdots & \alpha_{mB_m} \end{pmatrix}, \tag{5.6}$$

where $\alpha_{ij} = (-1)^{i+j}\det(M_{ij})$ and M_{ij} is the submatrix obtained from A_B by deleting row i and column j.

Since A is total unimodular $\det(M_{ij}) \in \{-1, 0, 1\}$ and hence $\alpha_{ij} \in \{-1, 0, 1\}$, and also $\frac{1}{\det(A_B)} \in \{-1, 0, 1\}$. Therefore A_B^{-1} is integer. □

It is possible to frame the shortest path problem as an integer linear program, and to prove that the constraint matrix A is total unimodular so that it can be solved using the simplex method. However, more efficient methods are available as we will see next.