Chapter 11

Analytical Dynamics

We are now going to take a slightly different approach to dynamical problems. The starting point is a formalism developed by Lagrange to represent the behaviour of mechanical systems, irrespective of the number of constituent parts or chosen coordinate systems. At the heart of the approach is the notion of energy conservation. The advantage is that this new strategy can be applied to quite complicated systems (with extensions to particle physics, general relativity etc), to derive the equations of motion, without the need for physical intuition. The underlying physical laws are not changed by this approach, but are reduced to a set of mathematical relationships between standard functions and equations, abstracted from the physical context.

The ideas introduced here, including functions such as the Lagrangian or Hamiltonian, turn up in many other areas of applied mathematics and physics, from the quantum to the astronomical scale.

Analytical dynamics is also a powerful grounding in the more general (and abstract) theory of the dynamical systems which studies the evolution of arbitrary systems, not necessarily mechanical, as functions of their underlying governing parameters.

The concepts of conserved quantities naturally arise here, which ultimately can lead to studies of integrable and chaotic systems (which we will not discuss).

11.1 The Calculus of Variations

Lagrange's approach to mechanics is based on a method of deriving the equations of motion by finding the *minimum value* of an integral. This involves solving a problem in the calculus of variations. Before we consider the particular problem that arises in Lagrange's approach to mechanics we will start by considering some more general problems in the calculus of variations.

Example 1: The brachistochrone

Consider a bead sliding under the force of gravity. The bead starts being stationary at a point P and then slides down a smooth wire to a point Q. The points P and Q are fixed and we want to find the shape of the wire that minimises the time taken for the bead to get to Q.

Let the curve be $z = z(x)$ (where the z-axis points vertically upwards), and take P to have coordinates (x_0, z_0) to Q to have coordinates (x_1, z_1). Using the intial data at P and conservation of energy we can find the time taken to go from P to Q. We have

$$T = \int_P^Q \frac{dl}{v}$$

where dl denotes a small element of length, measured along the wire. By conservation of energy:

$$mgz_0 = \frac{1}{2}mv^2 + mgz \Rightarrow v = [2g(z_0 - z)]^{1/2}$$

while

$$dl = [(dx)^2 + (dz)^2]^{1/2} = \left[1 + \left(\frac{dz}{dx}\right)^2\right]^{1/2} dx$$

We therefore obtain

$$T = \int_{x=x_0}^{x=x_1} \sqrt{\frac{1+\dot{z}^2}{2g(z_0 - z)}}\, dx$$

where $\dot{z} = \frac{dz}{dx}$.

Again the time T depends upon the function $z(x)$ and is again a functional

$$T = T[z]$$

The problem is then to find $z(x)$ with $z(x_0) = z_0$ and $z(x_1) = z_1$ so that T is minimised. The general solution to the problem of finding the brachistochrone, or curve of quickest descent, was first found by J Bernoulli in 1696 who showed that it was a cycloid.

11.2 The Euler-Lagrange Equations

In essence, we have to minimise an integral that depends upon some function and its derivatives and has fixed end points. We therefore now consider the general case of a functional[1] that depends upon $x(t)$ and has the form

$$I[x] = \int_{t=t_0}^{t_1} f(x(t), \dot{x}(t), t)dt$$

where $f(\cdot, \cdot, \cdot)$ is a given function and $\dot{x} = \frac{dx}{dt}$.

The function $x(t)$ is taken to be fixed at the end points of the integral $x(t_0) = x_0$ and $x(t_1) = x_1$ and is required to be twice differentiable. We then want to find an expression for $x(t)$ which minimises the functional $I[x]$.

[1] A functional is simply a function of a function.

In order to find $x(t)$ it is useful to suppose we have (somehow) found an $x(t)$ which minimises $I[x]$ and then consider what happens "near" that function. Consider some nearby function $\tilde{x}(t)$ which is 'close to' $x(t)$. We can construct a general expression for such nearby functions $\tilde{x}(t)$ as follows. Let $\eta(t)$ be a twice differentiable function (it has a continuous second derivative) defined for $t_0 \leq t \leq t_1$ which also satisfies the conditions

$$\eta(t_0) = 0, \quad \eta(t_1) = 0.$$

We call such a function η a variation. Then we can define a one parameter family of functions $\tilde{x}$ close to $x(t)$ by using the variation η to define

$$\tilde{x}(t) = x(t) + \epsilon\eta(t)$$

where ϵ is a small parameter.

We now consider some fixed variation $\eta(t)$ and see how $I[\tilde{x}(t)]$ varies with ϵ. Hence we consider the function

$$\phi(\epsilon) = I[x(t) + \epsilon\eta(t)]$$

Since $x(t)$ gives a minimum for $I[x(t)]$ it follows that

$$I[x(t) + \epsilon\eta(t)] \geq I[x(t)]$$

and hence

$$\phi(\epsilon) \geq \phi(0)$$

Using results from the calculus of a function of one variable we see that in order that $\epsilon = 0$ is a minimum we must have

$$\frac{d\phi}{d\epsilon}(0) = 0$$

More generally if the above equation is true for all possible choices of variation $\eta(t)$ we say that $x(t)$ is a stationary value of $I[x(t)]$.

We now use the definition of $\phi(\epsilon)$, and the fact that $\epsilon = 0$ is a minimum of $\phi(\epsilon)$ to find equations that $x(t)$ must satisfy. We have

$$\phi(\epsilon) = \int_{t=t_0}^{t_1} f(x + \epsilon\eta, \dot{x} + \epsilon\dot{\eta}, t)dt$$

so that[2]

$$\begin{aligned}
\frac{d\phi}{d\epsilon} &= \frac{d}{d\epsilon}\int_{t=t_0}^{t_1} f(x + \epsilon\eta, \dot{x} + \epsilon\dot{\eta}, t)dt \\
&= \int_{t=t_0}^{t_1} \frac{d}{d\epsilon}\left\{f(x + \epsilon\eta, \dot{x} + \epsilon\dot{\eta}, t)\right\} dt \\
&= \int_{t=t_0}^{t_1} \left(\frac{\partial f}{\partial \tilde{x}}\frac{d\tilde{x}}{d\epsilon} + \frac{\partial f}{\partial \dot{\tilde{x}}}\frac{d\dot{\tilde{x}}}{d\epsilon}\right) dt \\
&= \int_{t=t_0}^{t_1} \left(\eta\frac{\partial f}{\partial \tilde{x}} + \dot{\eta}\frac{\partial f}{\partial \dot{\tilde{x}}}\right) dt
\end{aligned}$$

[2]There is no longer a way to avoid partial derivatives...

At $\epsilon = 0$ we have $\tilde{x} = x$ so that

$$\frac{d\phi}{d\epsilon}(0) = \int_{t=t_0}^{t_1} \left(\eta \frac{\partial f}{\partial x} + \dot{\eta} \frac{\partial f}{\partial \dot{x}} \right) dt$$

Integrating the $\dot{\eta}$ term by parts gives

$$\begin{aligned} \frac{d\phi}{d\epsilon}(0) &= \left[\frac{\partial f}{\partial \dot{x}} \eta \right]_{t_0}^{t_1} + \int_{t=t_0}^{t_1} \left\{ \eta \frac{\partial f}{\partial x} - \eta \frac{d}{dt} \left(\frac{\partial f}{\partial \dot{x}} \right) \right\} dt \\ &= -\int_{t=t_0}^{t_1} \left\{ \frac{d}{dt} \left(\frac{\partial f}{\partial \dot{x}} \right) - \frac{\partial f}{\partial x} \right\} \eta dt \end{aligned}$$

The first term vanishes because $\eta(t_0) = \eta(t_1) = 0$.

We have shown that if $x(t)$ is an stationary value of $I[x(t)]$ then we must have

$$\int_{t=t_0}^{t_1} \left\{ \frac{d}{dt} \left(\frac{\partial f}{\partial \dot{x}} \right) - \frac{\partial f}{\partial x} \right\} \eta dt = 0$$

for *all* possible choices of variation $\eta(t)$.

It follows that[3]

$$\frac{d}{dt} \left(\frac{\partial f}{\partial \dot{x}} \right) - \frac{\partial f}{\partial x} = 0$$

This is known as the Euler-Lagrange equation for $x(t)$. It is straightforward to reverse the argument to show that if $x(t)$ satisfies the Euler-Lagrange equation then $x(t)$ is an extremal for $I[x]$. We have therefore established the following result

$$x(t) \text{ is an extremal of } I[x(t)] \Longleftrightarrow \frac{d}{dt} \left(\frac{\partial f}{\partial \dot{x}} \right) - \frac{\partial f}{\partial x} = 0$$

11.3 The Lagrangian

We are now in a position to give our first example of the application of the calculus of variations to mechanics.

[3] Technically we should appeal to the so-called Euler-Lagrange Lemma here...

Example 2: Recovering Newton

Consider the motion of a particle of mass m moving in 1D along the x-axis under the influence of a force $F(x)$ due to a potential $V(x)$.
We define the *Lagrangian* L of the particle to be given by

$$L = T - V = \frac{1}{2}m\dot{x}^2 - V(x)$$

(Note the minus sign that makes this different from the energy!!!)
Note also that

$$L = L(\dot{x}, x) \ .$$

We also define the *action* $I[x]$ to be given by

$$I[x] = \int_{t=t_0}^{t_1} L(x(t), \dot{x}(t), t)dt$$

Then an extremal value of $I[x]$ occurs if

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

This is *Lagrange's equation of motion.* Let us now show that this result is equivalent to Newton's second law for this problem. Using the definition of L it follows that (hold x fixed)

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

and that (hold $\dot{x}$ fixed in the derivative!)

$$\frac{\partial L}{\partial x} = -\frac{dV}{dx} = F$$

Using these in the Euler-Lagrange equations of motion gives us

$$\frac{d}{dt}(m\dot{x}) - F = 0$$

and so

$$m\ddot{x} - F = 0$$

which is just the equation of motion in Newtonian mechanics.

11.4 Adding degrees of freedom

The advantages of the Lagrangian method become more obvious if we consider a problem with several degrees of freedom[4]. We start by considering the motion of a particle moving in three dimensions under the action of a potential V. Let (x_1, x_2, x_3) be the x, y and z coordinates of the particle.

[4]Although we need to be comfortable with partial derivatives!

We use $x = x_1$, $y = x_2$ and $z = x_3$ so that the theory can be generalised later.

Then the kinetic energy is

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)$$

Since the force may derived from a potential $V(x_1, x_2, x_3)$ the components of the force are given by

$$F_1 = -\frac{\partial V}{\partial x_1}, \quad F_2 = -\frac{\partial V}{\partial x_2} \quad \text{and} \quad F_3 = -\frac{\partial V}{\partial x_3}$$

The Lagrangian for this problem is

$$L = T - V = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - V(x_1, x_2, x_3)$$

and the action is

$$I[x_1, x_2, x_3] = \int_{t=t_0}^{t_1} L(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3, t)dt$$

minimising the action now gives three equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0, \quad \text{for} \quad i = 1, 2, 3$$

Using the definition of the Lagrangian we have

$$\frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i$$

so that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = m\ddot{x}_i$$

Also

$$\frac{\partial L}{\partial x_i} = -\frac{\partial V}{\partial x_i} = F_i$$

So that the Lagrange equations of motion give

$$m\ddot{x}_i - F_i = 0, \quad \text{for} \quad i = 1, 2, 3$$

which are simply the components of Newton's equations in three dimensions.

We have established the following result:

- Newton's equations for a particle acted on by a force due to a potential V are equivalent to the Lagrange equations of motion, which give the motion of the particle which minimises the action I.

In other words, one can derive Newton's laws of motion from the *principle of least action* (sometimes called Hamilton's principle). In fact the principle of least action is more fundamental than Newton's laws. As well as being used to derive the equations of motion for a particle it can be used to derive the equations of motion for a field. For example Maxwell's equations for electromagnetism can be derived from the principle of least action as can Einstein's general relativistic field equations.

The big advantage of the principle of least action as far as mechanics is concerned is that it is true in *any* coordinate system, since the Lagrangian is a *scalar* that only depends on the motion of the particle not the coordinates used to describe it.

Let us illustrate this by obtaining the equations of motion for a particle moving in the plane but using polar coordinates (r, θ) to describe the motion rather than the usual Cartesian coordinates (x, y).

The first step is to obtain a formula for the kinetic energy in plane polar coordinates. There are a number of ways of obtaining this but the most straightforward is to use the chain rule to differentiate the equations that relate Cartesian and polar coordinates.

$$x = r\cos\theta, \qquad y = r\sin\theta$$

Differentiating the first of these[5] with respect to t we get

$$\begin{aligned}\frac{dx}{dt} &= \frac{\partial x}{\partial r}\frac{dr}{dt} + \frac{\partial x}{\partial \theta}\frac{d\theta}{dt} \\ \dot{x} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta\end{aligned}$$

Similarly differentiating the second gives

$$\begin{aligned}\frac{dy}{dt} &= \frac{\partial y}{\partial r}\frac{dr}{dt} + \frac{\partial y}{\partial \theta}\frac{d\theta}{dt} \\ \dot{y} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta\end{aligned}$$

Squaring and adding these equations gives

$$\begin{aligned}\dot{x}^2 + \dot{y}^2 &= \dot{r}^2\cos^2\theta - 2r\dot{r}\dot{\theta}\sin\theta\cos\theta + r^2\dot{\theta}^2\sin^2\theta \\ &+ \dot{r}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta + r^2\dot{\theta}^2\cos^2\theta \\ &= \dot{r}^2 + r^2\dot{\theta}^2\end{aligned}$$

Hence in polar coordinates

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

Using the formula for x and y in terms of r and θ also allows us to write the potential V as a function of r and θ. We may therefore write the Lagrangian for this problem as

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r, \theta)$$

To find the equations of motion we simply calculate the Euler-Lagrange equations for L. We start with the r-equation

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

[5] This should be familiar from our discussion of cylindrical coordinates.

Hence

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$$

and

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r}$$

Note that this contains two terms because both T and V depend on r.

Hence the Euler-Lagrange equation for the r variable

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

gives

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$$

or

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial V}{\partial r}$$

which may be shown to be the component of Newton's equation in the radial direction (by using vectors and resolving). The $r\dot{\theta}^2$ term is the 'centripetal acceleration' term from before.

To find the θ-equation we again just look at the corresponding Euler-Lagrange equation.

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

so that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$$

Note that this contains two terms because both r and $\dot{\theta}$ depend upon t.

Differentiating L with respect to θ gives

$$\frac{\partial L}{\partial \theta} = -\frac{\partial V}{\partial \theta}$$

Hence the Euler-Lagrange equation for the θ variable

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

gives

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = -\frac{\partial V}{\partial \theta}$$

or

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = -\frac{1}{r}\frac{\partial V}{\partial \theta}$$

which may be shown to be the transverse component of Newton's equations. The $2\dot{r}\dot{\theta}$ term represent the Coriolis force (as before).

Example 3: The pendulum

As a first example, consider a simple pendulum. In this case the kinetic energy is given by

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{1}{2}ml^2\dot{\theta}^2$$

since $r = l$ and $\dot{r} = 0$, while the potential energy is

$$V = mgh = mgl(1 - \cos\theta)$$

Hence the Lagrangian is

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$$

We now calculate the Lagrange equations of motion

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}, \quad \frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

so that $ml^2\ddot{\theta} + mgl\sin\theta = 0$ or

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

One cannot solve this differential equation in closed form (without introducing so-called elliptic functions). However for small θ we have $\sin\theta \approx \theta$, so that for small oscillations the equation is approximately given by

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

writing $g/l = \omega^2$ we recognise this as simple harmonic motion with angular frequency ω so that the general solution is

$$\theta(t) = A\cos(\omega t) + B\sin(\omega t), \quad \text{where} \quad \omega = \sqrt{\frac{g}{l}}$$

So that the motion of the pendulum is described by oscillations about $\theta = 0$ with period $\tau = 2\pi\sqrt{l/g}$.

Example 4: Mass on a spring

Another example is a mass on a spring. Let x be the extension of the spring. The the force due to the spring is $F = -kx$, where k is the spring constant. Hence the potential energy stored in the spring is

$$V_{\text{spring}} = \frac{1}{2}kx^2$$

The gravitational potential energy is

$$V_{\text{grav}} = -mgx$$

So that the total potential energy is

$$V = \frac{1}{2}kx^2 - mgx$$

The kinetic energy of the mass is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$$

Hence the Lagrangian for this problem is

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + mgx$$

We now calculate the Lagrange equation of motion

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}, \quad \frac{\partial L}{\partial x} = -kx + mg$$

So that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

gives

$$m\ddot{x} + kx - mg = 0$$

or

$$\ddot{x} + \frac{k}{m}x = g$$

That is, we have recovered the usual result and we know that the mass oscillates about the equilibrium position $x = mg/k$ with angular frequency $\omega = \sqrt{k/m}$.

Example 5: Planetary motion

Lagrangian methods provide a very easy way of deriving the equations of motion of a small particle in the gravitational field of a much larger one, e.g. the motion of a planet around the Sun. We can assume that the planet's motion is confined to a plane containing the Sun, and use plane polar coordinates (r, ϕ) to label its position. The planet then has velocity

$$\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\phi}\boldsymbol{e}_\phi$$

where a dot denotes a time derivative and $\boldsymbol{e}_r, \boldsymbol{e}_\phi$ are unit vectors in the radial and angular directions. The planet's kinetic energy is then

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$$

The potential energy in this case is gravitational potetial energy, which Newton took to be given by the equation

$$V = -\frac{GMm}{r}$$

where M is the mass of the Sun (assumed fixed in space), m the mass of the planet, and G is Newton's gravitational constant. Note that this leads to a force $\boldsymbol{F}$ given by

$$\boldsymbol{F} = -\nabla V = -\frac{\partial V}{\partial r}\boldsymbol{e}_r = -\frac{GMm}{r^2}\boldsymbol{e}_r,$$

This is the famous "inverse square law".
The Lagrangian is then

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{GMm}{r}$$

The radial Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} \quad \longrightarrow \quad \frac{d}{dt}(m\dot{r}) = mr\dot{\phi}^2 - \frac{GMm}{r^2} \Rightarrow \ddot{r} = r\dot{\phi}^2 - \frac{GM}{r^2}$$

while the angular Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} \quad \longrightarrow \quad \frac{d}{dt}(mr^2\dot{\phi}) = 0$$

These are two coupled ordinary differential equations which can then be solved to give $r(t)$ and $\phi(t)$, although the solution cannot be written in simple form. Note that the angular equation particularly simple: the particle's angular momentum is given by (we will use J for the angular momentum to avoid confusion with the Lagrangian)

$$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{p} = r\boldsymbol{e}_r \times m(\dot{r}\boldsymbol{e}_r + r\dot{\phi}\boldsymbol{e}_\phi) = mr^2\dot{\phi}\boldsymbol{e}_r \times \boldsymbol{e}_\phi$$

A little thought shows that for points in the $z = 0$ plane where our planet resides, $\boldsymbol{e}_r \times \boldsymbol{e}_\phi = \boldsymbol{e}_z$, so that

$$\boldsymbol{J} = mr^2\dot{\phi}\boldsymbol{e}_z \quad \Rightarrow \quad J = mr^2\dot{\phi}$$

i.e. the angular Euler-Lagrange equation is simply J =constant (as expected from the discussion of central forces). The particle's energy is:

$$E = T + V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{GMm}{r}$$

Eliminating $\dot{\phi}$ in favour of J leads to

$$\hat{E} = \frac{1}{2}\left(\dot{r}^2 + \frac{\hat{J}^2}{r^2}\right) - \frac{GM}{r}$$

where $\hat{E} = E/m$ and $\hat{J} = J/m$, the energy and angular momentum of the particle per unit mass.