Statistical solutions and astrophysical simulations

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heavily based on work of U Fjordholm and S Mishra

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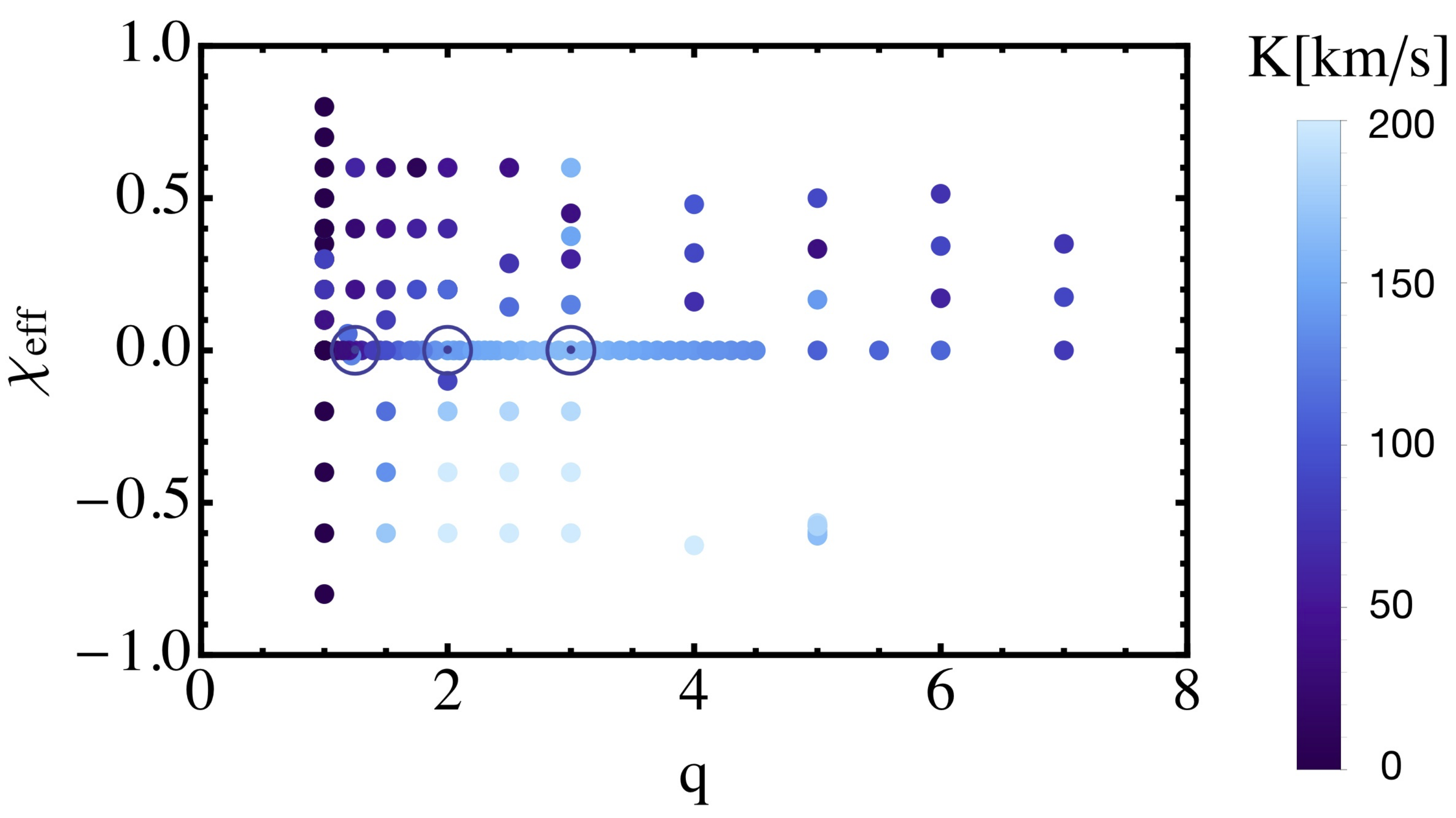
Uncertainty quantification

To match observations from simulations:

1. set initial data;
2. simulate observable;
3. repeat across parameter space;
4. interpolate.

Uncertainty from simulation and parameter space sampling.



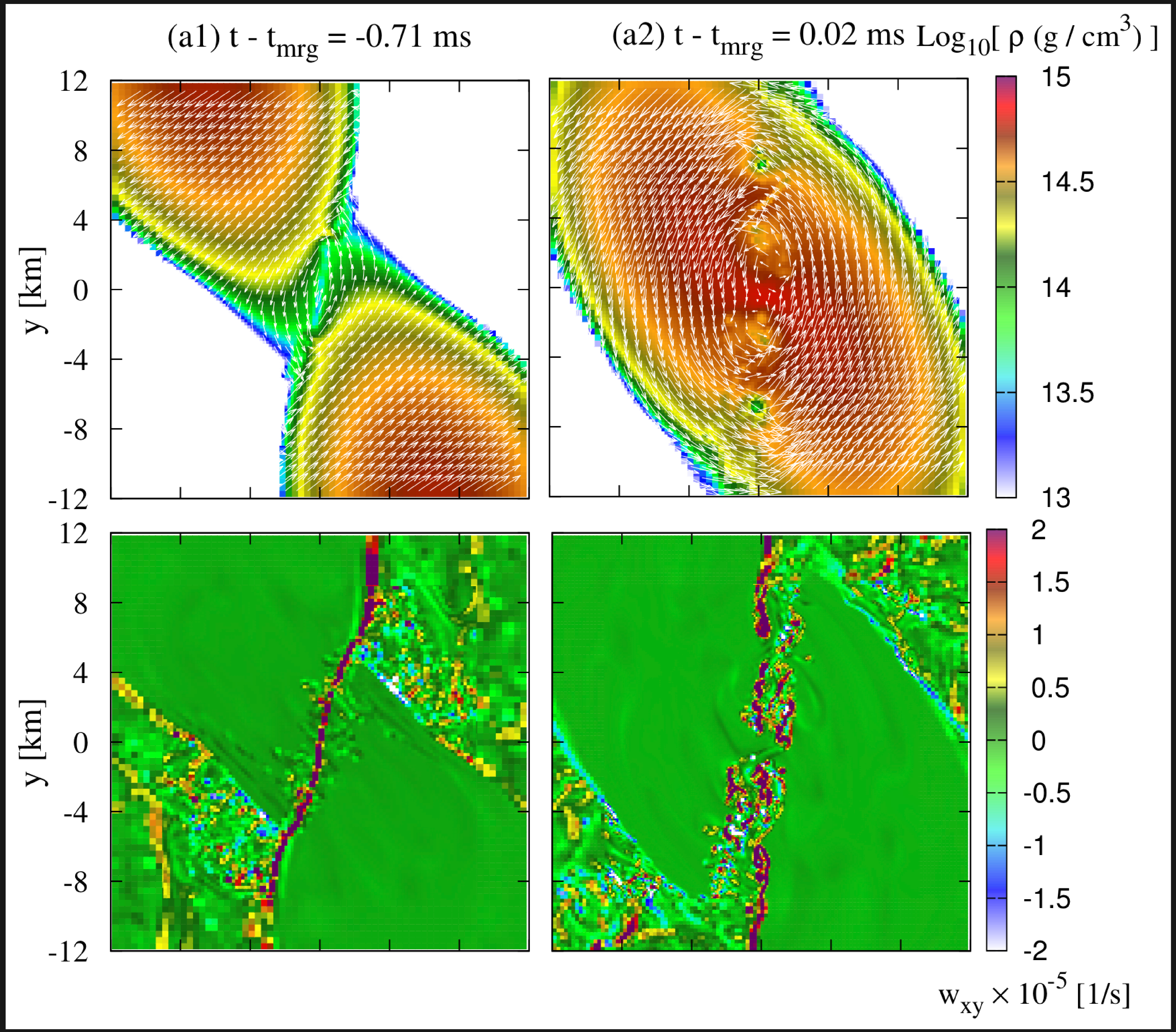
Bustillo et al, 1806.11160

Matter

Matter simulations:

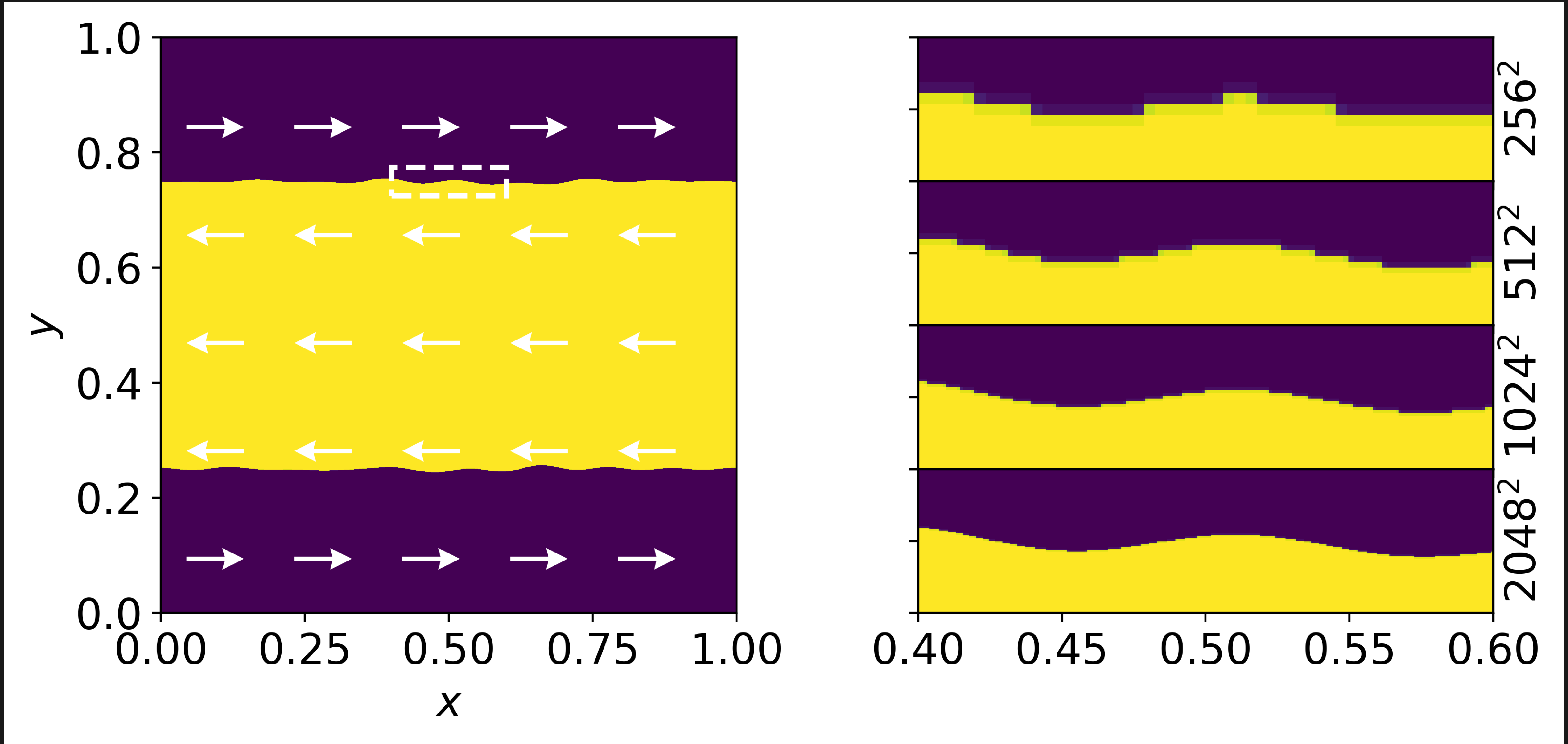
1. have shocks;
2. "turbulence";
3. more observables;
4. little on well-posedness.

Can we get reliable predictions?



Kiuchi et al, 1509.09205

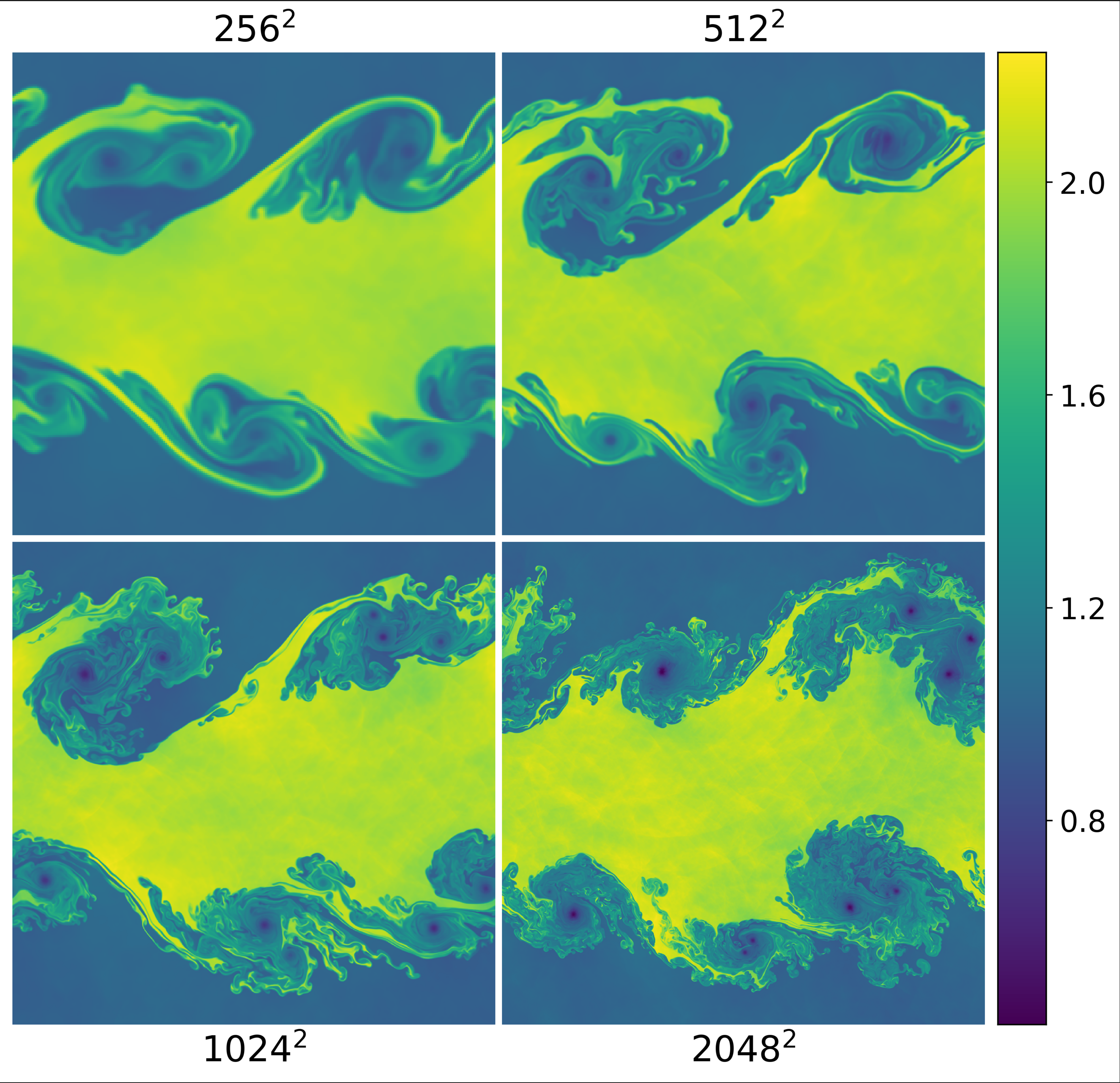
Kelvin-Helmholtz instability



Lax

- Code consistent;
- Numerics stable;
- No convergence.

Lax's theorem means the system is *not* well-posed.



Saving convergence

We don't see convergence to *weak solutions* of

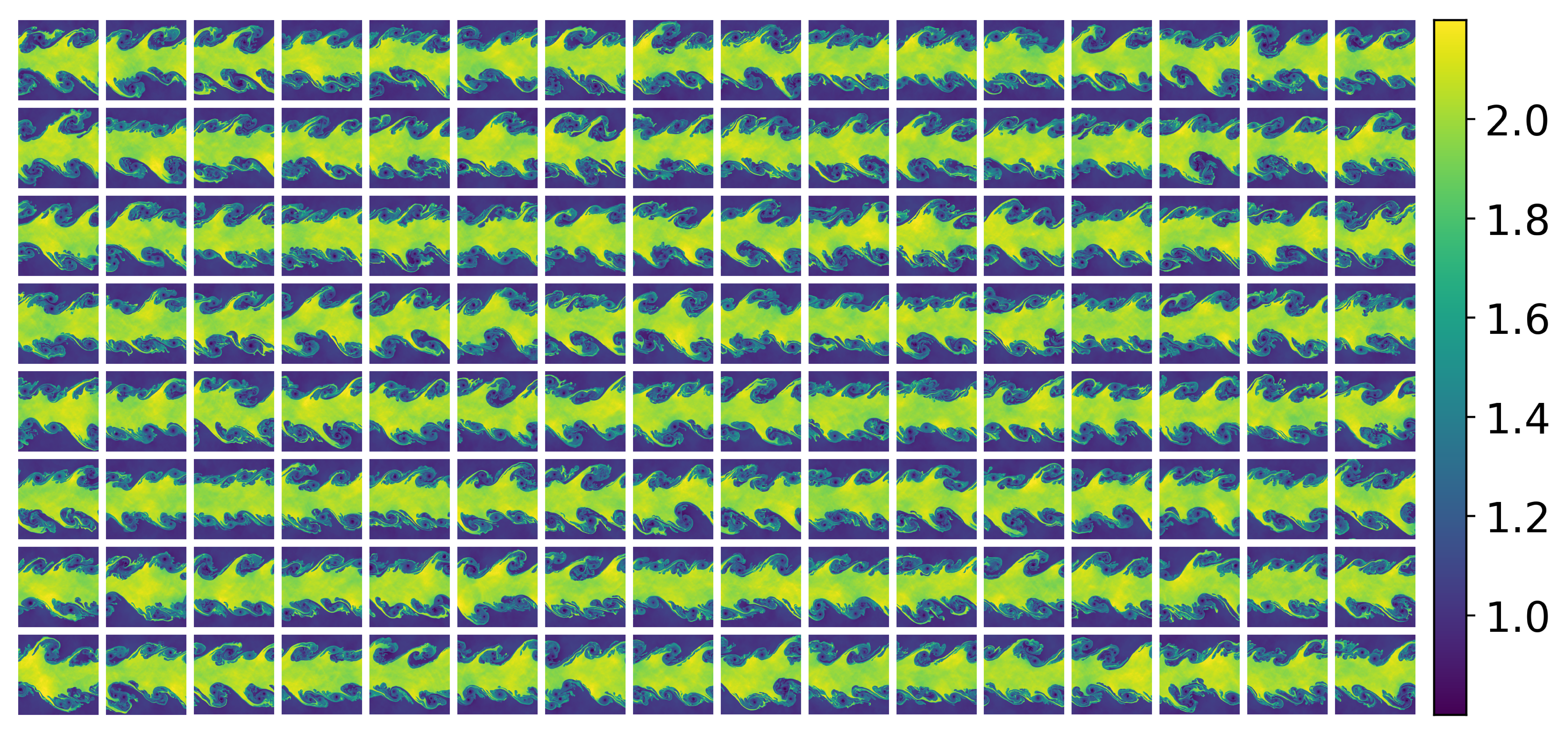
$$\partial_t q + \partial_i f^{(i)}(q) = s(q).$$

Instead use a spacetime-dependent probability measure ν solving

$$\partial_t \langle \nu, \mathrm{id} \rangle + \partial_i \langle \nu, f^{(i)} \rangle = \langle \nu, s \rangle.$$

Need to ensure appropriate entropy solution and small scale features.

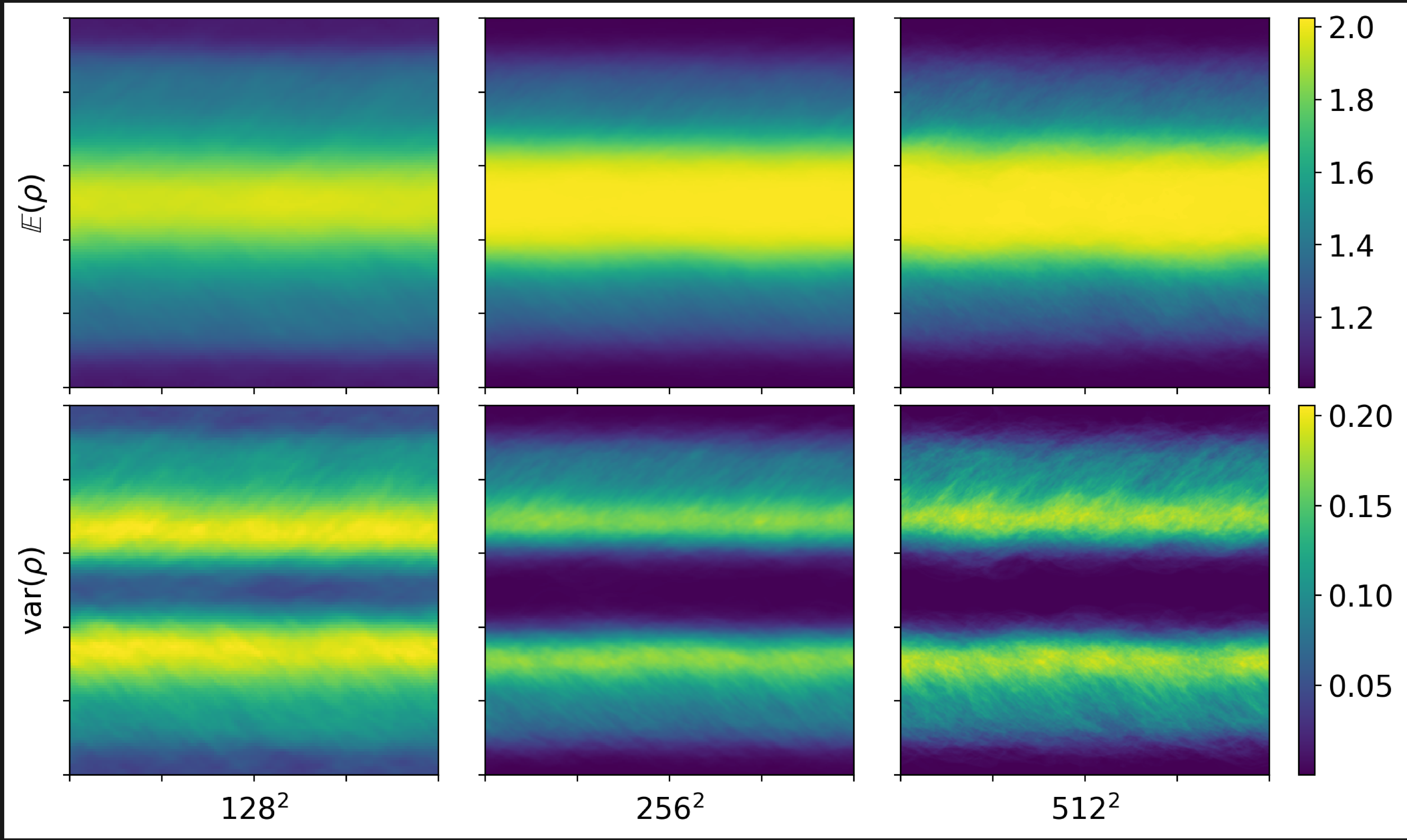
Constructing the measure



$\mathbb{E}$ & var

Both mean and variance clearly converge.

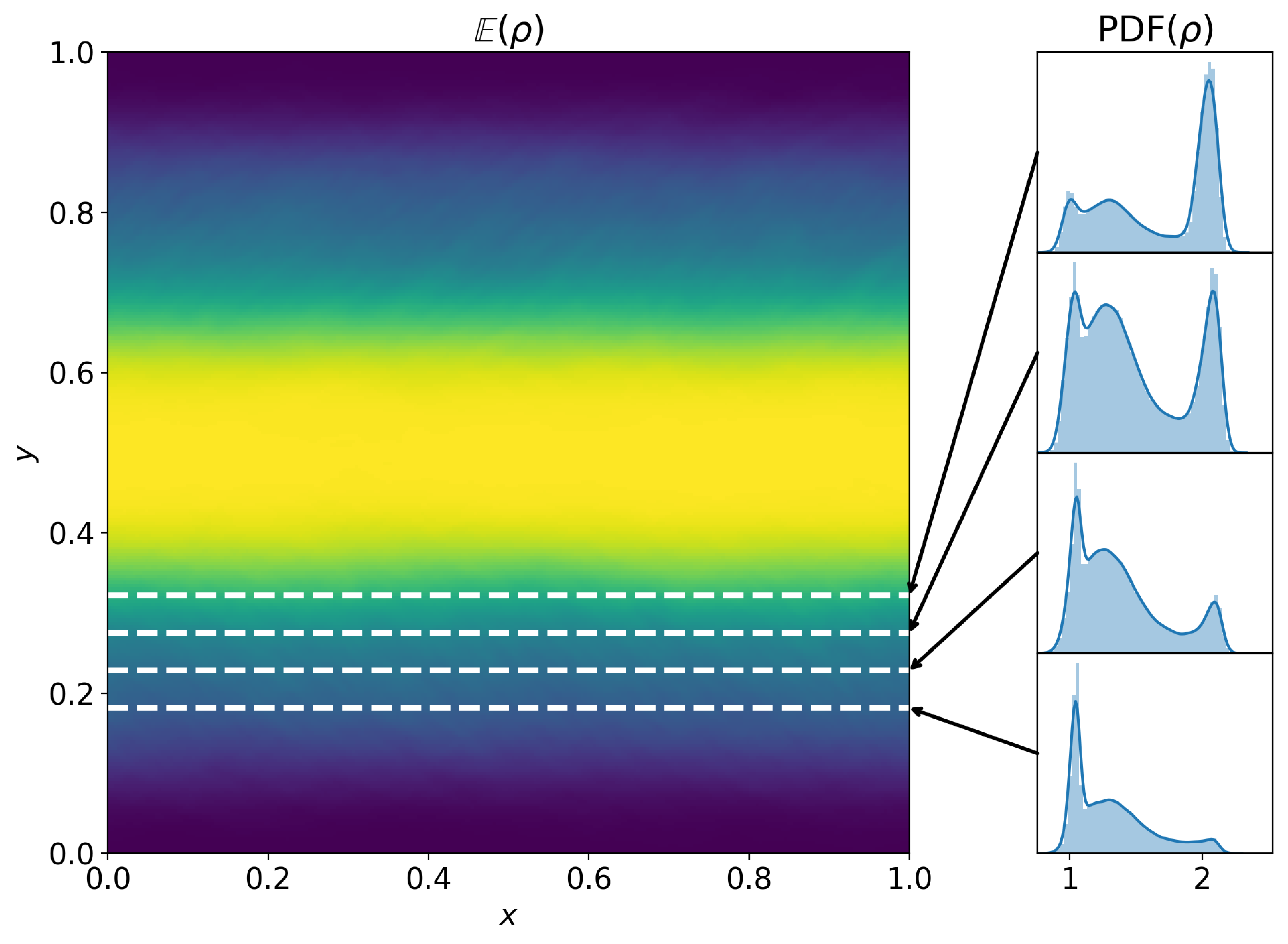
Characterize the uncertainty in the observable.

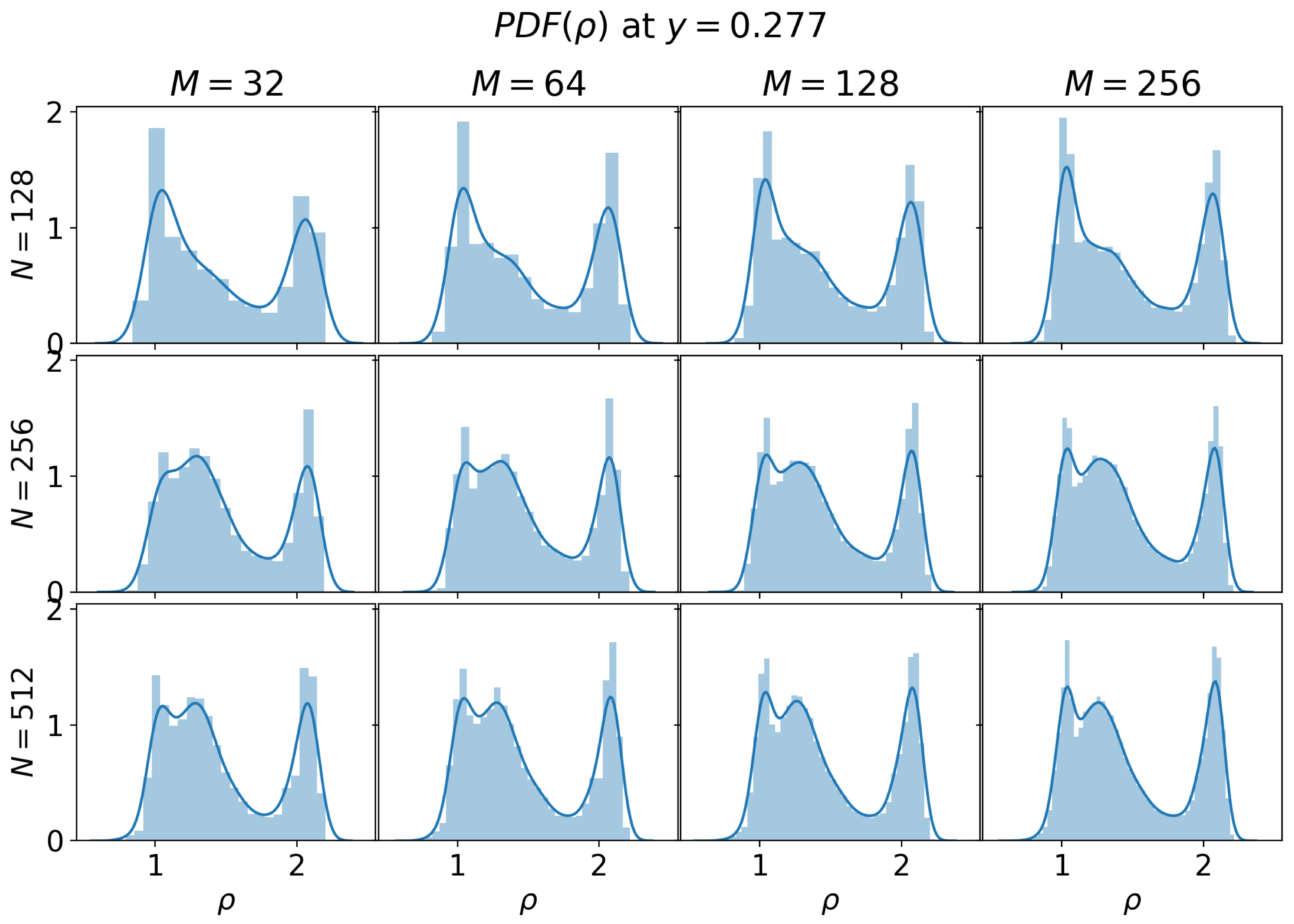


PDF

Can directly estimate the PDF, eg along a line. These also converge rapidly.

Some features not obvious from $\mathbb{E}$ and variance.





$PDF(\rho)$ at $y = 0.277$
$M = 32$
$M = 64$
$M = 128$
$M = 256$
$N = 128$
$N = 256$
$N = 512$
0
1
2
ρ

Quantifying the problem

For "standard" case, measure numerical error as

$$\mathcal{E} \simeq C_1 (\Delta x)^s.$$

For statistical solutions, simple Monte-Carlo, expect

$$\mathcal{E} \simeq C_1 (\Delta x)^s + C_2 M^{-1/2}.$$

Measure C_i's, see what error dominates. If variance large, prediction hard!

Summary

- Bad news: Euler equations not globally well-posed?
- Good news: Statistical solutions $\Rightarrow$ can still make predictions.
- Bad news: MC approach computationally expensive...
- Good news: ...but PDF may not need so high resolution.
- Bad news: High variance is crucial regions limits predictability?

Much needs doing here.

References

- Fjordholm, U. S., Lye, K., Mishra, S., & Weber, F. (2019). Statistical solutions of hyperbolic systems of conservation laws: numerical approximation. arxiv:1906.02536.
- Fjordholm, U. S., Käppeli, R., Mishra, S., & Tadmor, E. (2017). Construction of Approximate Entropy Measure-Valued Solutions for Hyperbolic Systems of Conservation Laws. Foundations of Computational Mathematics, 17(3), 763–827. DOI:10.1007/s10208-015-9299-z.
- Abgrall, R., & Mishra, S. (2017). Uncertainty Quantification for Hyperbolic Systems of Conservation Laws. In Handbook of Numerical Analysis (Vol. 18, pp. 507–544). DOI:10.1016/bs.hna.2016.11.003.