Chapter 2

Air resistance and friction

So far we have explored forces that do not *per se* resist the motion—we have assumed that there is no friction or air resistance. This is not our everyday reality. While energy is conserved at a fundamental level, it does not appear to be conserved in many things we do. At a microscopic level, kinetic energy is transferred to the atoms in the floor you may be trying to slide across, manifesting itself as heat. But if we only want to know how far our socks will slide, the details of the atomic processes are of little interest. Instead, we often summarise everything in a single, macroscopic force that we call friction. This is a messy business, but the effective description is quite simple.

In this chapter we consider the effect of such forces. Newton's laws formally remain unchanged, but the presence of resistive forces complicates the problem we have to solve.

2.1 Air Resistance

A particle moving through a fluid (or a gas, such as air!) is subject to a drag force. This arises because in moving through the liquid, the particle has to push the fluid molecules out of the way.

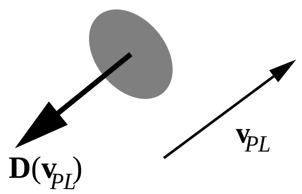

Figure 2.1: Resistive force $\vec{D}(\vec{v}_{PL})$ due to motion of body relative to liquid $\vec{v}_{PL}$.

Experimentally it has been found that a good model for the drag force $\vec{D}$ is one that depends on the velocity of the particle P relative[1] to the fluid L, $\vec{v}_{PL}$. Such a model takes the form

$$\vec{D} = -K\vec{v}_{PL}$$

[1]We will return to the issue of relative velocities later.

where K is a positive quantity that depends on the speed and shape of the particle as well as the density and viscosity of the fluid. Note that the drag force $\vec{D}$ is, at all times, in the opposite direction to the velocity. A common form for this law is

$$\vec{D} = -k|\vec{v}_{PL}|^{(\alpha-1)}\vec{v}_{PL} = -k|\vec{v}_{PL}|^{\alpha}\hat{v}_{PL}$$

where the exponent α depends on the physical situation. For "high" speed flows, such as cars or planes moving through air, $\alpha \approx 2$. This is natural when the friction is proportional to the surface area of the moving object. For "low" speed flows, such as a pebble falling though thick oil, $\alpha \approx 1$. For "intermediate" speeds α may itself be a function of the speed. A special case, valid when the particle is moving very slowly, is $\alpha = 1$ in which case

$$\vec{D} = -k\vec{v}_{PL}$$

2.2 Reminder: Separable first order equations

The simplest differential equations are those that can be solved by direct integration. As an example, consider

$$y'(x) = f\left(y\right)$$

which can be solved in the following way: First we divide both sides by $f(y)$ (assuming initially that $f(y) \neq 0$ on the range of interest)

$$\frac{1}{f\left(y\right)}\frac{dy}{dx} = 1 \quad \Rightarrow \quad \int \frac{1}{f\left(y\right)}\frac{dy}{dx}\,dx = \int 1\,dx$$

Now suppose that $H(y)$ is the integral of $1/f(y)$, i.e.,

$$H'(y) = \frac{1}{f(y)}$$

Then, using the chain rule we have

$$\frac{d}{dx}H(y(x)) = H'(y(x))\frac{dy}{dx} = \frac{1}{f(y)}\frac{dy}{dx}.$$

Hence, by the fundamental theorem of calculus, the integral on the left-hand side above becomes

$$\int \frac{1}{f\left(y\right)}\frac{dy}{dx}\,dx = \int \left(\frac{d}{dx}H(y(x))\right)dx = H(y) + c \tag{2.1}$$

If you look carefully at (2.1), effectively all we did was evaluate the integral of $1/f(y)$ with respect to y: the answer being the integral $H(y)$.

Note that you could have obtained the same result just by "cancelling" the dx top and bottom in the integral on the left-hand side and integrating with respect to dy. Hence, a short cut for the integration is as follows:

$$\frac{1}{f\left(y\right)}\frac{dy}{dx} = 1 \quad \Rightarrow \quad \int \frac{1}{f\left(y\right)}\frac{dy}{dx}\,dx = \int 1\,dx \qquad \Rightarrow \int \frac{1}{f\left(y\right)}\,dy = x + C$$

The last integral with respect to y just has to be evaluated depending on what $f(y)$ is and C is an arbitrary constant.

However, this cancellation trick is (formally) quite naughty, since the derivative dy/dx is not a fraction that can be cancelled! Still, this notational short cut is justified due to the argument outlined above. Hence everybody does it. Just remember why it is allowed!

A related kind of first order equation contains functions of x and y such that it is possible to separate them into two terms, one on each side of the equal sign, one a function of x only and the other only a function of y. In general these equations take the form.

$$\frac{dy}{dx} = g(x)f(y).$$

Separating the terms we may integrate with respect to x only:

$$\int \frac{1}{f(y)} \frac{dy}{dx}\, dx = \int g(x)\, dx \quad \Rightarrow \quad \int \frac{1}{f(y)}\, dy = \int g(x)\, dx$$

Note that we are using the naughty cancellation trick between derivatives again. This is justified by the same procedure as outlined above.

2.3 Projectile under gravity with air resistance

Suppose a particle of constant mass m is launched from the surface of the earth at $t = 0$ with initial speed U at an angle θ up from the horizontal. We shall assume that the air is stationary. Suppose the position of the particle at time t is $\vec{r}(t)$ and that the motion is very slow (this makes the mathematics easier!) so that the resistive force due to the air is $\vec{D} = -\gamma\dot{\vec{r}}(t)$. Calculate the subsequent motion of the particle as it moves under the influence of gravity $-g\hat{k}$ and air resistance.

As before we use a coordinate system with $\hat{\imath}$ horizontal and $\hat{k}$ vertical with the origin at the point of release of the particle.

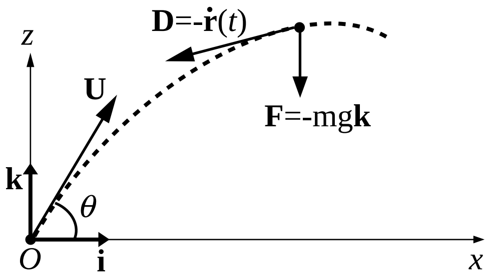

Figure 2.2: Projectile subject to drag force.

N2 then gives the equation of motion as

$$m\,\ddot{\vec{r}}(t) = -mg\,\hat{k} - \gamma\,\dot{\vec{r}}(t),$$

with

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}.$$

As in many cases, this problem can be solved using the second order equation above written in component form. However it is easier to introduce variables to indicate the velocity of the particle so we write

$$\dot{\vec{r}}(t) = u(t)\ \hat{\imath} + v(t)\ \hat{\jmath} + w(t)\ \hat{k}$$

so that **N2** is now the system of equations

$$\begin{aligned}
m\dot{u}(t) &= -\gamma u, & u(0) &= U\cos\theta \\
m\dot{v}(t) &= -\gamma v, & v(0) &= 0 \\
m\dot{w}(t) &= -\gamma w - mg, & w(0) &= U\sin\theta \\
\dot{x}(t) &= u(t) & x(0) &= 0 \\
\dot{y}(t) &= v(t) & y(0) &= 0 \\
\dot{z}(t) &= w(t) & z(0) &= 0
\end{aligned}$$

Now we solve this system of differential equations.

The first equation is separable and can be integrated:

$$\int_{U\cos\theta}^{u} \frac{du}{u} = -\frac{\gamma}{m}\int_0^t dt \quad \Rightarrow \quad [\ln u]_{U\cos\theta}^{u} = -\frac{\gamma t}{m} \quad \Rightarrow \quad u(t) = U\cos\theta\exp\left(-\frac{\gamma}{m}t\right).$$

Using the equation $u(t) = \dot{x}(t)$ we can integrate this expression once more and use the fact that $x(0) = 0$ to obtain $x(t)$

$$\dot{x}(t) = U\cos\theta\exp\left(-\frac{\gamma}{m}t\right),\ x(0) = 0 \qquad \Rightarrow \qquad x(t) = \frac{mU\cos\theta}{\gamma}\left\{1 - \exp\left(-\frac{\gamma}{m}t\right)\right\}$$

The second equation can be solved in a similar manner

$$m\frac{dv}{dt} = -\gamma v \quad \Rightarrow \quad \int\frac{dv}{v} = -\frac{\gamma}{m}\int dt$$

$$\Rightarrow \quad \ln v = -\frac{\gamma t}{m} + \ln A \quad \Rightarrow \quad v(t) = A\exp\left(-\frac{\gamma}{m}t\right)$$

$$v(0) = 0, \qquad \Rightarrow \qquad A = 0 \qquad \Rightarrow \qquad v(t) = 0.$$

Hence we have

$$\dot{y}(t) = 0, \qquad \Rightarrow \qquad y(t) = B, \qquad \text{but } y(0) = 0 \qquad \Rightarrow \qquad B = 0 \qquad \therefore\ y(t) = 0.$$

So the projectile always stays in the $y = 0$ plane and never has a non-zero $\hat{\jmath}$ component (this should have been obvious from the start!).

The third equation is also separable and can be solved as

$$\int_{U\sin\theta}^{w}\frac{dw}{(mg/\gamma + w)} = -\frac{\gamma}{m}\int_0^t dt \qquad \Rightarrow \qquad w(t) = -\frac{mg}{\gamma} + \left(U\sin\theta + \frac{mg}{\gamma}\right)\exp\left(-\frac{\gamma}{m}t\right)$$

This can be integrated to give the z-component as

$$\dot{z}(t) = -\frac{mg}{\gamma} + \left(U\sin\theta + \frac{mg}{\gamma}\right)\exp\left(-\frac{\gamma}{m}t\right)$$

$$\Rightarrow z(t) = -\frac{mgt}{\gamma} + \frac{m}{\gamma}\left(U\sin\theta + \frac{mg}{\gamma}\right)\left\{1 - \exp\left(-\frac{\gamma}{m}t\right)\right\}$$

The vector equation of the trajectory is thus

$$\vec{r}(t) = \frac{mU\cos\theta}{\gamma}\left\{1 - \exp\left(-\frac{\gamma}{m}t\right)\right\}\hat{\imath} + \left[-\frac{mgt}{\gamma} + \frac{m}{\gamma}\left(U\sin\theta + \frac{mg}{\gamma}\right)\left\{1 - \exp\left(-\frac{\gamma}{m}t\right)\right\}\right]\hat{k}.$$

which is much more complicated than the parabolic motion with no air resistance.

By considering the components $x(t)$ and $z(t)$ and eliminating t between them the Cartesian equation of the trajectory can be found to be

$$z = \left(\tan\theta + \frac{mg}{\gamma U\cos\theta}\right)x + \frac{m^2 g}{\gamma^2}\log\left(1 - \frac{\gamma x}{mU\cos\theta}\right).$$

Exercise: Take the solution above and put $\epsilon = \gamma U/mg$ and then expand the resulting expression in a Taylor series (see later) as $\epsilon \to 0$. For the first term in the series you should recover the resistance-free parabola and the next terms give you the correction to the path due to air resistance.

2.4 Reminder: Linear Equations and integrating factors

Many first order ordinary differential equations can be solved by multiplying both sides by a suitable factor, called an *integrating factor* $R(x,y)$. Finding such an integrating factor can be difficult but there is a simple rule for the case of a linear first-order ordinary differential equations.

An equation

$$\frac{dy}{dx} + p(x)\,y = q(x)$$

is called a *linear first order differential equation* and can be solved by use of an integrating factor. The integrating factor is given by

$$R(x) = \exp\left[\int^x p(s)\,ds\right].$$

Note that the integrating factor is usually taken not to contain an arbitrary constant (since this will be a multiplicative factor that can be cancelled on both sides of the equation).

We solve the equation by multiplying through by $R(x)$,

$$\exp\left(\int^x p(s)\,ds\right)\frac{dy}{dx} + \exp\left(\int^x p(s)\,ds\right)p(x)\,y = \exp\left(\int^x p(s)\,ds\right)q(x).$$

The left-hand side can then be recognised as an exact derivative:

$$\exp\left(\int^x p(s)\,ds\right)\frac{dy}{dx} + \exp\left(\int^x p(s)\,ds\right)p(x)\,y = \frac{d}{dx}\left[y\exp\left(\int^x p(s)\,ds\right)\right]$$

where we have used the fundamental theorem of calculus:

$$p(x) = \frac{d}{dx}\left(\int^x p(s)ds\right).$$

Hence, we can write the linear equation above as

$$\frac{d}{dx}\left(y \, \exp\left(\int^x p\,(s)\,ds\right)\right) = q\,(x)\exp\left(\int^x p\,(s)\,ds\right)$$

or more simply as

$$\frac{d}{dx}\Big(y\;R(x)\Big) = q(x)R(x).$$

Both sides can now be integrated directly with respect to x to obtain a solution for $y(x)$:

$$\begin{aligned} y(x)R(x) &= \int^x q(s)R(s)ds + C \\ \Rightarrow y(x) &= \frac{1}{R(x)}\int^x q(s)R(s)ds + \frac{C}{R(x)}. \end{aligned}$$

Note that if you chose not to use the initial conditions in the integration limits here, so that the arbitrary constant C is automatically found as part of the definite integration progress, it is vital that you don't forget to put C on the right hand side **before** you ultimately divide through by $R(x)$ to obtain $y(x)$. Otherwise you miss out a whole term in the solution.

Note: When seeking the integrating factor be careful. For example for the equation

$$x\frac{dy}{dx} - x^2\;y = e^{3x},$$

first put the equation in the **standard** form with the coefficient of dy/dx being unity and a function of x multiplying a single linear term in y.

$$\frac{dy}{dx} - x\;y = \frac{e^{3x}}{x},$$

and then find the integrating factor (taking care to include the correct sign of $p(x)$

$$R(x) = \exp\left(\int p\,(x)\,dx\right) = \exp\left(\int -x\;dx\right) = \exp\left(-x^2/2\right).$$

Also, you will have noticed that for a first order equation, there is one arbitrary constant in the general solution. Thus to find a specific solution you need to provide one additional bit of information, or boundary condition, to determine the specific value of the arbitrary constant.

2.5 Alternative solution to the projectile problem

While it is often natural to solve the problem in component form (as we did above), we have already seen that simple ballistic motion can also be solve directly as a vector equation. As before, we have

$$m\frac{d\vec{v}}{dt} = m\vec{g} - \gamma\vec{v}$$

This is a linear equation, so we can solve it using an integrating factor. We have

$$\frac{d}{dt}\left(e^{\frac{\gamma t}{m}}\vec{v}\right) = e^{\frac{\gamma t}{m}}\vec{g}$$

which leads to

$$\vec{v} = \frac{m}{\gamma}\vec{g} + \vec{c}e^{-\gamma t/m}$$

where the integration constant, $\vec{c}$, is fixed by the initial speed. Suppose that we have $\vec{v} = \vec{u}$ at $t = 0$. Then it follows that

$$\vec{v} = \frac{m}{\gamma}\vec{g} + \left(\vec{u} - \frac{m}{\gamma}\vec{g}\right)e^{-\gamma t/m}$$

If we want the position, we can integrate again. This leads to

$$\vec{r} = \frac{m}{\gamma}\vec{g}t - \frac{m}{\gamma}\left(\vec{u} - \frac{m}{\gamma}\vec{g}\right)e^{-\gamma t/m} + \vec{b}$$

If we let $\vec{r} = 0$ at $t = 0$, then

$$\vec{r} = \frac{m}{\gamma}\vec{g}t + \frac{m}{\gamma}\left(\vec{u} - \frac{m}{\gamma}\vec{g}\right)\left(1 - e^{-\gamma t/m}\right)$$

We can make contact with the previous calculation by expanding our result in components. Let $\vec{r} = (x, y, z)$ and send the projectile off with initial velocity $\vec{u} = (u\cos\theta, 0, u\sin\theta)$. With gravity acting downwards, so $\vec{g} = (0, 0, -g)$, our vector equation becomes three equations. One is trivial: $y = 0$. The other two are

$$x = \frac{m}{\gamma}u\cos\theta\left(1 - e^{-\gamma t/m}\right)$$

and

$$z = -\frac{mgt}{\gamma} + \frac{m}{\gamma}\left(u\sin\theta + \frac{mg}{\gamma}\right)\left(1 - e^{-\gamma t/m}\right)$$

The time scale[2] m/γ is important. For $t \gg m/\gamma$, the horizontal position is essentially constant. By this time, the particle is dropping more or less vertically. The trajectory is not symmetric.

We may also consider what happens when friction is small. A couple of terms look as if they are going to become singular (as we would appear to divide by zero) in this limit, but that seems unphysical. To resolve this, we should ask what γ is small relative to. In the present case, the answer lies in the exponential terms. To say that γ is small, really means $\gamma \ll m/t$ or, in other words, it means that we are looking at short times, $t \ll m/\gamma$. Then we can expand the exponential in a Taylor series. Reverting to the vector form of the equation, we have

$$e^{-\gamma t/m} \approx 1 - \frac{\gamma t}{m} + \dots$$

so we find

$$\begin{aligned}\vec{r} = \frac{m}{\gamma}\vec{g}t + \frac{m}{\gamma}\left(\vec{u} - \frac{m}{\gamma}\vec{g}\right)\left[1 - 1 + \gamma\frac{t}{m} - \frac{1}{2}\left(\frac{\gamma t}{m}\right)^2 + \dots\right] \\ = \vec{u}t + \frac{1}{2}\vec{g}t^2 + \mathcal{O}\left(\frac{\gamma t}{m}\right)\end{aligned}$$

This is the usual projectile result.

[2]Note that $\gamma t/m$ must be dimensionless.

2.6 Terminal velocity

Let us now consider a particle of mass m moving in a constant gravitational field, subject to quadratic friction. We measure the height z to be in the upwards direction, meaning that if $v = dz/dt > 0$, the particle is going up. We will look at the cases where the particle goes up and goes down separately.

First suppose that we drop the particle from some height. The equation of motion is given by

$$m\frac{dv}{dt} = -mg + \gamma v^2$$

It is worth commenting on the minus signs on the right-hand side. Gravity acts downwards, so comes with a minus sign. Since the particle is falling down, friction is acting upwards so comes with a plus sign. Dividing through by m, we have

$$\frac{dv}{dt} = -g + \frac{\gamma}{m}v^2$$

Integrating this, we get

$$t = -\int_0^v \frac{dv'}{g - \gamma v'^2/m}$$

This can be worked out—either through the substitution $v = \sqrt{mg/\gamma}\tanh x$ or partial fractions (which may be the easiest route)—leading to

$$t = -\sqrt{\frac{m}{\gamma g}}\tanh^{-1}\left(\sqrt{\frac{\gamma}{mg}}v\right)$$

which we can invert to get the speed as a function of time

$$v = -\sqrt{\frac{mg}{\gamma}}\tanh\left(\sqrt{\frac{\gamma g}{m}}t\right)$$

This illustrates the effect of the air resistance. As time increases, the velocity does not increase without bound. Instead, the particle reaches a maximum speed (the terminal velocity)

$$v \longrightarrow -\sqrt{\frac{mg}{\gamma}}\ , \qquad \text{as } t \to \infty$$

The sign is negative because the particle is falling downwards. Note that if all we want is the terminal velocity, then we do not need to go through the whole calculation. We can simply look for solutions with constant speed, so $dv/dt = 0$. This gives us the same answer. The advantage of going through the full calculation is that we learn how the velocity approaches its terminal value.

This example teaches us an important lesson. A small animal (like a mouse) may survive a fall where a large animal (say, an elephant) will not[3]. The point is that, if we compare objects of equal density the masses scale as the volume, meaning $m \sim L^3$ where L is the linear size of the object. In contrast, the coefficient of friction usually scales as surface area, $\gamma \sim L^2$. This means that the terminal velocity depends on size. For objects of equal density, we expect the terminal velocity to scale as $v \sim \sqrt{L}$.

[3]While it would be interesting to demonstrate this experimentally, health and safety regulations prevent us from doing so at this time.

Let us now throw the object upwards. Since both gravity and friction are then acting downwards, we get a flip of a minus sign in the equation of motion. It is now

$$\frac{dv}{dt} = -g - \frac{\gamma}{m}v^2$$

Suppose that we throw the object up with initial speed u and we want to figure out the maximum height, h, that it reaches. We could follow our earlier calculation and integrate the equation to determine $v = v(t)$. But since we are not asking about time, it is much better to instead consider velocity as a function of distance: $v = v(z)$. We write[4]

$$\frac{dv}{dt} = \frac{dv}{dz}\frac{dz}{dt} = v\frac{dv}{dz} = -g - \frac{\gamma v^2}{m}$$

or

$$\frac{1}{2}\frac{dv^2}{dz} = -g - \frac{\gamma v^2}{m}$$

Using $y = v^2$ we can integrate this to get

$$\int_{u^2}^{0} \frac{dy}{g + \gamma y/m} = -2\int_0^h dz \quad \longrightarrow \quad \frac{m}{\gamma}\left[\log\left(g + \frac{\gamma y}{m}\right)\right]_{y=u^2}^{y=0} = -2h$$

That is,

$$h = \frac{m}{2\gamma}\log\left(1 + \frac{\gamma u^2}{mg}\right)$$

Here, it is worth looking at what happens when the effect of friction is small. Naively, it looks like we are in trouble because as $\gamma \to 0$, the term in front gets large. But surely the height should not go to infinity just because the friction is small. The resolution to this is that the log is also getting small in this limit. Expanding as a Taylor series, we have

$$h = \frac{u^2}{2g}\left(1 - \frac{\gamma u^2}{2mg} + \dots\right)$$

The leading term is the answer we would get in the absence of friction; the subleading terms tell us how much the friction, γ, affects the attained height.

2.7 Friction

Friction is the resistive force that occurs when two solid surfaces pressed together attempt to slide over each other. Even the apparently macroscopically smoothest of surfaces will, in practice, give rise to frictional forces, since on the macroscopic scale the surfaces are imperfect and so are rough. The surfaces then snag on and scrap against each other's microscopic mountain ranges as they move past each other. [5]

[4] Strictly speaking, we are using $v(z(t))$ here.

[5] The effect of lubricating the surfaces reduces the friction by separating the rough surfaces with a thin layer of fluid (for example, oil). With the solid surfaces are no longer in microscopic contact the problem is effectively reduced to one of sliding a fluid under pressure over a surface, which requires less effort.

Consider two solid, reasonably flat surfaces in contact. Assume they are both stationary and are being pushed together by a normal force $\vec{N}$. This normal force is the component orthogonal to the surfaces of any force holding the surfaces together. For example you might consider the upper surface to be a book and the lower surface to be a horizontal table in which case the normal force would be due to the weight of the book.

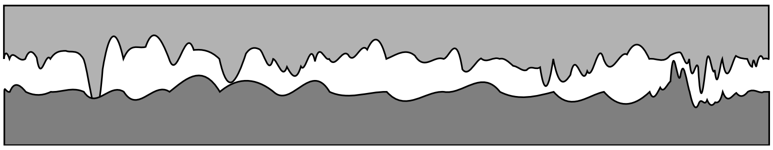

Figure 2.3: Cartoon close-up of the interface between two apparently flat surfaces.

As there is no acceleration we can apply **N2** to infer that there will be a reactive force $\vec{R}$ due to the lower surface acting on the upper surface (the table pushing on the book) of equal magnitude to $\vec{N}$, but in the opposite direction.

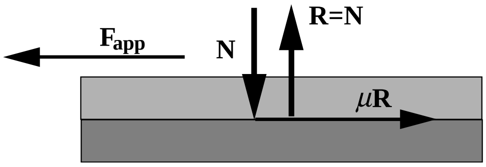

Figure 2.4: Effect of applying forces to the frictional interface between two surface forced together.

The rough surface creates a frictional force $\vec{F}_F$ that acts tangentially to each surface to oppose any translational force applied in an orthogonal direction to $\vec{R}$ (see diagram). If a translational force, $\vec{F}_{\text{app}}$, is applied to the upper surface (the book) then the surfaces will not slide over each other and hence will remain stationary only as long as

$$|\vec{F}_{\text{app}}| \leq \mu_s |\vec{R}|,$$

where μ_s is an experimentally determined positive number called the **coefficient of static friction**. If $\vec{F}_{\text{appl}}$ is greater in magnitude than the static frictional force then the surfaces will slide over each other. Once the surfaces start to slide there will still be a frictional force tangential to the surfaces but it will have a magnitude

$$|\vec{F}_F| = \mu_k |\vec{R}|,$$

and act in a direction to oppose the sliding motion. The experimentally determined positive number μ_k is called the **coefficient of kinetic friction** (also sometimes known as the coefficient of dynamic friction). Suppose the instantaneous motion of one surface relative to another in contact with it is in the direction $\hat{v} \neq \vec{0}$. In vector notation the friction force resisting the motion can then be written as

$$\vec{F}_F = -\mu_k |\vec{R}| \hat{v}.$$

This force continues to act until the surfaces become stationary at which time the static friction returns.
Several points are worth making:

- the coefficients of friction are properties of the surfaces in contact. Hence the coefficient of friction between two wooden surfaces can be expected to be different from that between a steel and wood surface.
- Frictional forces are independent of the area of the surface in contact, they depend only on the magnitude of the normal force with which the surfaces are pushed together.
- Usually $\mu_k \leq \mu_s$, however to simplify algebra they are often taken as being equal and take a value less than 1, i.e.,

$$\mu_k = \mu_s < 1.$$

  We shall follow this procedure here. Note that coefficients of friction greater than unity only occur if the surfaces interlock and that a book sliding on a table might have values around 0.2.
- Note that there is also similarly defined **coefficient of rolling friction** μ_s which is applicable to situations where two surfaces are only in minimal contact, for example a cart wheel in contact with the road. However we shall not deal so much with this situation.
- We can attack problems involving friction using Newton's laws of motion.

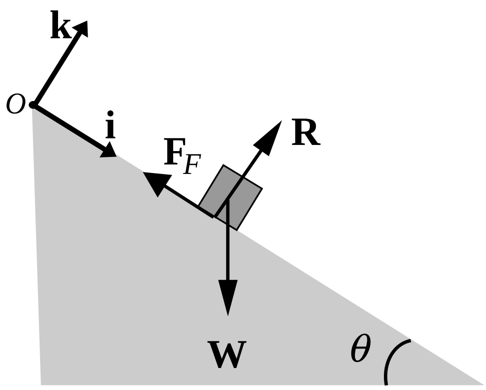

Figure 2.5: A, somewhat square, child sliding down a slide.

Example 1: A child on a slide

Consider a straight slide inclined at an angle θ to the horizontal. Suppose a child of mass m attempts to slide down the slide under gravity of magnitude g. The forces acting on the child are

- their weight $\vec{W}$ which has magnitude mg and acts vertically downwards;
- a force of reaction $\vec{R}$ that acts normal to the slide (that this exists is due to **N3** as without it the child would fall through the slide);
- the frictional force $\vec{F}_F$ acting tangentially to the slide and in a direction (up or down the slide) to oppose any motion. The coefficient of friction is supposed to be μ.

These forces are summarised in the diagram below.
We can use **N2** to analyse the situation. We fix a Cartesian coordinate frame at the base of the slide and orient the axes such that $\hat{\imath}$ points down the tangent to the slide, $\hat{\jmath}$ points into the page and $\hat{k}$ points away from the ground but normal to the slide.
Note that there is no fixed rule that says that a Cartesian coordinate system has to be oriented so that $\hat{k}$ has to point directly upwards! In fact for this problem this orientation saves a lot of algebraic effort. If you don't believe this, try orienting the coordinates with $\hat{\imath}$ horizontal and $\hat{k}$ pointing upwards and repeating the analysis.
Then we can write the normal and frictional forces as

$$\vec{F}_F = -F_F\,\hat{\imath}, \qquad \vec{R} = R\,\hat{k}.$$

If the position of the child on the slide is given by $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ with respect to this coordinate frame then **N2** says that

$$m\,\ddot{\vec{r}}(t) = \vec{F}_F + \vec{R} + \vec{W}.$$

where

$$\vec{W} = mg\sin\theta\,\hat{\imath} + 0\hat{\jmath} - mg\cos\theta\,\hat{k}$$

To solve this problem we do exactly as before and rewrite the problem in component form. By symmetry, $y(t) = 0$ for all time, so we will not need to solve for $y(t)$. We find:
$\hat{\imath}$ component:

$$m\ddot{x}(t) = mg\sin\theta - F_F.$$

$\hat{k}$ component:

$$m\ddot{z}(t) = R - mg\cos\theta.$$

Assuming that the child does not spontaneously fly off the slide (i.e., we assume no motion in the $\hat{k}$ direction) we must have no acceleration in this direction $\Rightarrow \ddot{z}(t)$=0. Hence this last equation reduces to

$$R = mg\cos\theta,$$

and gives us an expression relating the normal reaction R to the weight mg.
Two possible situations can arise.
1. The child is just stationary, on the point of sliding. In this case we must have $\ddot{x}(t) = 0$:

$$m\ddot{x}(t) = 0 \qquad \Rightarrow \qquad F_F = mg\sin\theta.$$

Using the fact that, in this case, $F_F = \mu R$ we combine this result with the z-component to arrive at:

$$R = mg\cos\theta, \quad F_F = mg\sin\theta, \quad F_F = \mu R \qquad \Rightarrow \qquad \tan\theta = \mu,$$

$$\theta = \arctan\mu.$$

This is a (rather scary for the child) way to determine μ_s. The angle of the slide θ is altered until the child first just starts to move. μ_s is then just the tangent of the angle of inclination at which slipping first occurs.

2. The child slides downwards. The equations of motion must be solved subject to some initial data, say

$$\vec{r}(0) = -L\hat{\imath}, \qquad \dot{\vec{r}}(0) = \vec{0}.$$

Once again it is best to rewrite the problem using $\dot{\vec{r}}(t) = u(t)\hat{\imath} + v(t)\hat{\jmath} + w(t)\hat{k}$ as it makes the algebra easier and leads to the following, where we have again used $F_F = \mu R = \mu mg\cos\theta$:

- $\hat{\imath}$ component:

$$m\dot{u}(t) = mg\sin\theta - \mu mg\cos\theta, \; u(0) = 0 \qquad \Rightarrow \qquad u(t) = g(\sin\theta - \mu\cos\theta)t.$$

- $\hat{k}$ component:

$$m\dot{w}(t) = R - mg\cos\theta = 0, \; w(0) = 0 \Rightarrow w(t) = 0.$$

Integrating once more with respect to t we obtain:

- $\hat{\imath}$ component:

$$\dot{x}(t) = g(\sin\theta - \mu\cos\theta)t, \; x(0) = -L \rightarrow x(t) = -L + g(\sin\theta - \mu\cos\theta)\frac{t^2}{2}$$

- $\hat{k}$ component:

$$m\dot{z}(t) = 0, \; z(0) = 0 \rightarrow z(t) = 0.$$

Hence the position as a function of time is

$$\vec{r}(t) = \left[-L + g(\sin\theta - \mu\cos\theta)\frac{t^2}{2}\right]\hat{\imath}.$$

Note: If the angle of the slope were increased to $\theta = \pi/2$ the slide would be vertical and the child would no longer held in contact with the slide by component of its weight. In the equation above, the terms involving μ vanish and the equation of motion becomes

$$\vec{r}(t) = \left(-L + g\frac{t^2}{2}\right)\hat{\imath}$$

which is just the usual equation of a body falling under gravity (with the height measured downwards!) If the slide is horizontal with $\theta = 0$, then the above dynamic analysis does not hold.

2.8 Addendum: Taylor expansions

The Taylor series is one of the most useful mathematical tools applied to real life problems. As in the example of potential motion near equilibrium, we can often use a Taylor series to approximate a complicated function by a (simpler) polynomial function.

The starting point is Taylor's theorem (which we state without proof, as we are mainly interested in the applications):

Let $f(x)$ be continuous and differentiable an infinite number of times at point $x = a$ (i.e., f and all of its derivatives $f', f'', \ldots$ exist at $x = a$). Then $f(x)$ can be expanded about the point $x = a$ in terms of powers of $(x - a)$, and this expansion is given by the formula

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^n}{n!}f^{(n)}(a) + O(x-a)^{n+1}$$

This is a **Taylor expansion**. It has the form

$$f(x) = P_n(x) + O(x-a)^{n+1}$$

where $P_n(x)$ is a polynomial of degree n and $O(x-a)^{n+1}$ is a remainder term. The remainder term reminds us that in writing the Taylor expansion explicitly up to the term $\propto (x-a)^n$ ("up to order n") we are omitting terms "of order $(x-a)^{n+1}$ and higher".

For x close to a, we can expect the remainder term to be small (since $x-a$ is small), so that $f(x) \simeq P_n(x)$. Thus Taylor's theorem enables us to approximate functions by polynomials.

In many cases the remainder term goes to zero as $n \to \infty$ and the Taylor series expansion of $f(x)$ takes the form of a possibly infinite, but convergent, series.

Example 2: The exponential

As an example, let us determine the Taylor series expansion of the function $f(x) = e^x$ about $x = 0$.

We first establish the necessary data. We know that we need the derivatives:

$$\begin{array}{llll} f(x) = e^x & f'(x) = e^x & f''(x) = e^x & \ldots \\ f(0) = e^0 = 1 & f'(0) = e^0 = 1 & f''(0) = e^0 = 1 & \ldots \end{array}$$

Then we use this in the Taylor series expansion with $a = 0$

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

Similarly, we can work out the Taylor series expansion of the function $f(x) = \cos x$ about $x = 0$.

In this case, we need:

$$\begin{aligned}
f(x) &= \cos x\ , \qquad f(0) = 1 \\
f'(x) &= -\sin x\ , \qquad f'(0) = 0 \\
f''(x) &= -\cos x\ , \qquad f''(0) = -1 \\
f'''(x) &= \sin x\ , \qquad f'''(0) = 0 \\
f^{(4)}(x) &= \cos x\ , \qquad f^{(4)}(0) = 1
\end{aligned}$$

and we arrive at

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$