Chapter 10

Central Forces

So far our main focus has been on motion that can be described in terms of a fixed Cartesian coordinate system. This takes us quite far, but it is by no means the only way we can describe a problem. Many problems have particular symmetries and dynamical problems simplify if we make use of this. To do this, we need to introduce other sets of coordinates – and these may not be fixed. For example, we can decide to use a coordinate system that is fixed in a rotating body. Then the coordinate basis vectors will vary in time. As an example of this – and an excuse to develop the required mathematics - we will discuss the motion of particles under central forces. These are forces that act towards a central point, for example the sun or an atom.

10.1 Cylindrical Polar Coordinates

When a 3D problem has a cylindrical symmetry (a rotational symmetry about a single axis), such as the magnetic field around a wire, or the flow of fluid through a circular pipe, it makes sense to use **cylindrical polar coordinates** (r, θ, z), to specify the position of an object. In such situations the equations governing the system often simplify.

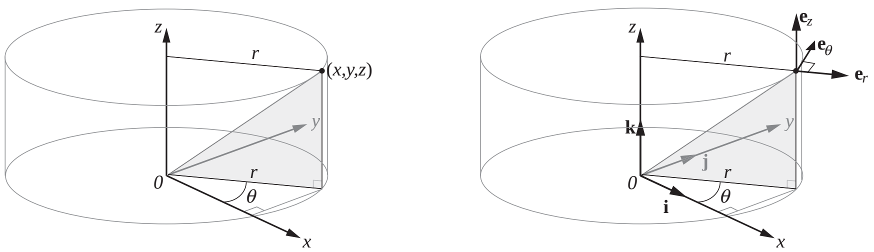

Figure 10.1: Cylindrical polar coordinate system and basis vectors

Effectively these are 2D polar coordinates (r, θ) in the (x, y) plane perpendicular to the symmetry axis, together with the usual Cartesian z-coordinate along the symmetry axis.

The transformation from cylindrical polar coordinates to Cartesian $(r, \theta, z) \to (x, y, z)$ is:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

while the inverse transformation follows from plane polar coordinates

$$r = \sqrt{x^2 + y^2},$$

with θ being the arctan of $\pm y/x$, depending on the quadrant of θ.

For motion confined to a 2D plane z =constant, it suffices to think of these coordinates as the plane 2D polar coordinates.

Instead of the unit Cartesian basis vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ the unit basis vectors in cylindrical polars are:

- $\hat{e}_r$ is in the outwards radial direction from the origin in the plane $z = 0$.
- $\hat{e}_\theta$ is in the azimuthal (increasing positive θ) direction $\hat{e}_r \times \hat{e}_z$, where
- $\hat{e}_z$ is in the Cartesian direction $\hat{k}$ (i.e., $\hat{e}_z \equiv \hat{k}$).

The relationship between cylindrical polar basis vectors at the point (r, θ, z) and Cartesian basis vectors is given by geometry as

$$\begin{aligned} \hat{e}_r &= \cos\theta\hat{\imath} + \sin\theta\hat{\jmath} \\ \hat{e}_\theta &= -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath} \\ \hat{e}_z &= \hat{k} \end{aligned}$$

In general, the position vector of a point (r, θ, z) in cylindrical polar coordinates is given by

$$\vec{r} = r\hat{e}_r + z\hat{e}_z.$$

This looks crazy! We have point in 3D but only two apparent coordinates specifying it with **no** explicit behaviour on θ. This is because, as we have seen in the last point above, $\hat{e}_r$ actually depends **itself** on θ. Thus if you use a radial basis vector $\hat{e}_r$, you are implicitly specifying a θ. Hence the third coordinate θ is actually there in the definition.

10.2 Velocity and Acceleration

The last point above shows that two of the unit vectors in cylindrical polars, $\hat{e}_r$ and $\hat{e}_\theta$, depend on their location θ. Hence as one moves around space in θ the orientation of these two unit vectors changes locally, depending on the point. This does not happen for the Cartesian basis vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$, which are independent of the position. (Note that since $\hat{e}_z = \hat{k}$ it does not change locally either.)

Suppose we wish to represent a derived quantity, such as velocity or acceleration in cylindrical polars. When we take the derivative(s) we must be sure to take into account the fact that the basis vectors are also changing from point to point, starting with the position representation.

Assume that the position of a particle at time t relative to an inertial frame can be represented by the cylindrical coordinates $(r(t), \theta(t), z(t))$. Then the vector position of the particle is

$$\vec{r} = r(t)\hat{e}_r + z(t)\hat{e}_z.$$

Since θ is a function of time, from the equation above relating polar and Cartesian bases, so will $\hat{e}_r$ and $\hat{e}_\theta$. The velocity of the particle is then given by

$$\begin{aligned}\frac{d\vec{r}}{dt} &= \frac{d}{dt}\left(r\hat{e}_r\right) + \frac{d}{dt}\left(z\hat{e}_z\right), \\ &= \dot{r}\hat{e}_r + r\dot{\hat{e}}_r + \dot{z}\hat{e}_z + z\dot{\hat{e}}_z.\end{aligned}$$

But $\dot{\hat{e}}_z = \dot{\hat{k}} = \vec{0}$ and

$$\begin{aligned}\dot{\hat{e}}_r &= \frac{d}{dt}\left(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}\right) = \dot{\theta}\left(-\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}\right) = \dot{\theta}\hat{e}_\theta, \\ \dot{\hat{e}}_\theta &= \frac{d}{dt}\left(-\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}\right) = -\dot{\theta}\left(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}\right) = -\dot{\theta}\hat{e}_r,\end{aligned}$$

where the last two derivatives also use the fact that $\dot{\hat{\imath}} = \dot{\hat{\jmath}} = \vec{0}$. Hence substitution of these results into the expression for $\dot{\vec{r}}$ gives

$$\frac{d\vec{r}}{dt} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + \dot{z}\hat{e}_z.$$

Note that unlike the position vector in cylindrical polars, the velocity vector **does** involve a $\hat{e}_\theta$ component. That this is sensible follows from a qualitative description of each term in the velocity of the particle:

- → $\dot{r}\hat{e}_r$ is the velocity in the outward radial direction;
- → $r\dot{\theta}\hat{e}_\theta$ is the velocity in the azimuthal direction, i.e., around the z-axis and also perpendicular to $\hat{e}_r$;
- → $\dot{z}\hat{e}_z$ is the velocity in the z direction.

In order to work out the acceleration, we use the results from the velocity. Thus we have

$$\begin{aligned}\frac{d^2\vec{r}}{dt^2} &= \frac{d}{dt}\left(\dot{r}\hat{e}_r\right) + \frac{d}{dt}\left(r\dot{\theta}\hat{e}_\theta\right) + \frac{d}{dt}\left(\dot{z}\hat{e}_z\right) \\ &= \ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta + \ddot{z}\hat{e}_z + z\dot{\hat{e}}_z \\ &= \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r + \ddot{z}\hat{e}_z \\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{e}_\theta + \ddot{z}\hat{e}_z\end{aligned}$$

That is,

$$\frac{d^2\vec{r}}{dt^2} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\hat{e}_\theta + \ddot{z}\hat{e}_z.$$

so the three cylindrical components of acceleration are

- → $\left(\ddot{r} - r\dot{\theta}^2\right)$ is the radial component of acceleration;
- → $\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$ is the azimuthal component of acceleration;
- → $\ddot{z}$ is the component of acceleration parallel to the z-axis.

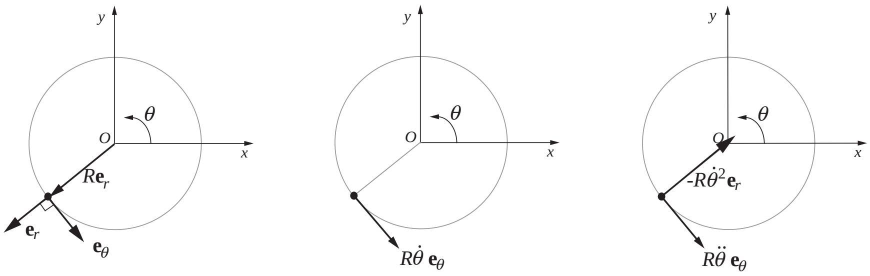

Figure 10.2: Position (left), velocity (centre) and acceleration (right) vectors for motion in a circle of radius R

Example 1: Motion in a circle

A particle moves in a circle of radius R in the $z = 0$ plane with position $\vec{r}$ at time t. We study the problem using cylindrical coordinates with origin at the centre of the circle, with $\hat{e}_z$ oriented perpendicular to the plane of the circle.
By definition we have the radial component of the position as

$$r(t) = R \Rightarrow \dot{r} = 0 \Rightarrow \ddot{r} = 0.$$

Likewise we have

$$z(t) = 0 \Rightarrow \dot{z} = 0 \Rightarrow \ddot{z} = 0.$$

Hence the velocity of the particle in radial coordinates is

$$\begin{aligned} \frac{d\vec{r}}{dt} &= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + \dot{z}\hat{e}_z = 0\hat{e}_r + R\dot{\theta}\hat{e}_\theta + 0\hat{e}_z \\ \frac{d\vec{r}}{dt} &= R\dot{\theta}\hat{e}_\theta \end{aligned}$$

It is purely tangential to the circle (as it has to be if the circular motion is to be maintained!).
The acceleration of the particle in radial components is

$$\begin{aligned} \frac{d^2\vec{r}}{dt^2} &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\hat{e}_\theta + \ddot{z}\hat{e}_z = \left(0 - R\dot{\theta}^2\right)\hat{e}_r + r\ddot{\theta}\hat{e}_\theta + 0\hat{e}_z, \\ \frac{d^2\vec{r}}{dt^2} &= -R\dot{\theta}^2\hat{e}_r + r\ddot{\theta}\hat{e}_\theta. \end{aligned}$$

Note that the radial component of acceleration is directed towards the centre of the circle. This is called the **centripetal** (literally meaning “seeking the centre”) component of acceleration. If the particle has mass m and executes the motion with a constant angular speed, i.e.,

$$\dot{\theta} = \text{constant} = \omega, \Rightarrow \theta = \omega t + \text{constant},$$

then the velocity of the particle is

$$\frac{d\vec{r}}{dt} = R\omega\hat{e}_\theta.$$

The acceleration simplifies to only a radial component and **N2** gives the force as

$$\frac{d^2\vec{r}}{dt^2} = -R\dot{\theta}^2\hat{e}_r \Rightarrow \vec{F} = -mR\omega^2\hat{e}_r.$$

Note that the particle is only accelerating towards the centre of the circle. The interpretation of $\vec{F}$ is that it is the force that is required to keep the particle in the circular orbit of the origin. Without the presence of $\vec{F}$, the particle would just continue in a straight line (**N1**). If the particle is attached to hoop or a string, by **N3** there is a balancing reactive force called the **centrifugal** force.

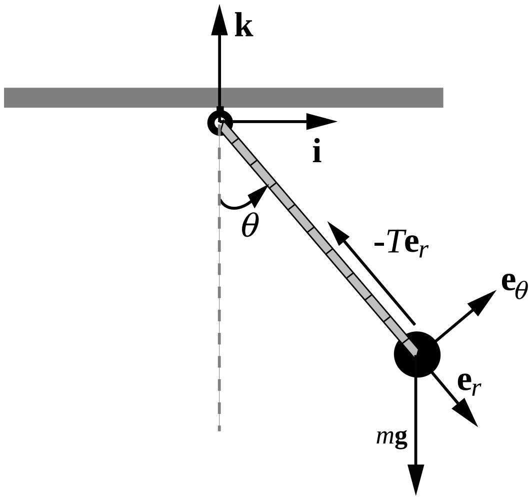

Figure 10.3: Forces acting on a pendulum represented in cylindrical polar coordinates.

Example 2: Motion of a plane pendulum revisited

Suppose a pendulum of mass m and fixed inextensible length l moves in a plane under the influence of gravity only, see figure 10.3. It is released from rest at $t = 0$ from an angle α radians, where $\alpha \ll 1$. Analyse the motion using cylindrical polar coordinates.
We fix coordinates with $\hat{e}_r$ pointing along the length of the string, from the suspension point to the bob. $\hat{e}_z$ passes horizontally through the suspension point, and points into the page. The motion of the bob will take place in the $z = 0$ plane. $\hat{e}_\theta$ is then determined by $\hat{e}_r \times \hat{e}_\theta$. The angle θ is fixed to be the angle of the string to the downwards vertical. The forces acting are the weight of the bob $m\vec{g}$ and the tension $\vec{T}$ acting along the string. **N2** gives:

$$m\ddot{\vec{r}} = \vec{T} + m\vec{g}.$$

In terms of cylindrical polar coordinates, resolving the force components we have

$$\vec{T} = -T\hat{e}_r, \qquad \vec{g} = g\left(\cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta\right)$$

Thus **N2** becomes

$$m\left[\left(\ddot{r}-r\dot{\theta}^2\right)\hat{e}_r+\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}^2\right)\hat{e}_\theta\right]=-T\hat{e}_r+mg\left(\cos\theta\hat{e}_r-\sin\theta\hat{e}_\theta\right)$$

Now as the string is inextensible, $r(t)=l$, so in component form we have

$$\begin{aligned}-ml\dot{\theta}^2 &= -T+mg\cos\theta\\ ml\ddot{\theta} &= -mg\sin\theta\end{aligned}$$

For small oscillations $|\theta|\ll 1$, $\sin\theta\approx\theta$ and $\cos\theta\approx 1$ (remember θ will be measured in radians and 1 radian $=(180/\pi)^o$). Hence the last equation becomes

$$\ddot{\theta}+\frac{g}{l}\theta=0.$$

This is simple harmonic motion, just as in the case of the vertical spring. It has solution

$$\theta(t)=C\cos\omega t+D\sin\omega t=A\cos(\omega t+B),\qquad \omega=\frac{g}{l}.$$

The initial data is $\theta(0)=\alpha$, $\dot{\theta}(0)=0$ which gives $A=\alpha$, $B=0$. Thus the motion takes the form

$$\theta(t)=\alpha\cos\omega t.$$

Note that the frequency of such small oscillations is independent of m. The tension in the string is given by the first equation above as

$$T=mg\cos\theta+ml\dot{\theta}^2\approx m\left(g+\alpha^2\omega^2 l\sin^2\omega t\right).$$

If the string is elastic, or the amplitude of the oscillations is large, the problem is much more difficult!

10.3 Spherical Polar Coordinates

When a problem has a spherical symmetry, such as the gravitational force felt by a satellite, or the precession of a spinning top, it makes sense to use **spherical polar coordinates**, (ρ,θ,ϕ) to simplify the governing equations.

Note that the θ angle does **not** have the same definition as in cylindrical coordinates.

The definitions of the coordinates are:

→ ρ is the radial (shortest) distance from the origin of coordinates $\mathcal{O}$, $0<\rho<\infty$

→ θ is the angle of latitude as measured from the z axis or $\hat{k}$ direction.

→ ϕ is the longitude as measured from the Cartesian x-axis.

Note that the latitude θ is measured from the north pole and so the geographical latitude $=\theta-\pi/2$, $0<\theta<\pi$. With these defintions we have the transformation

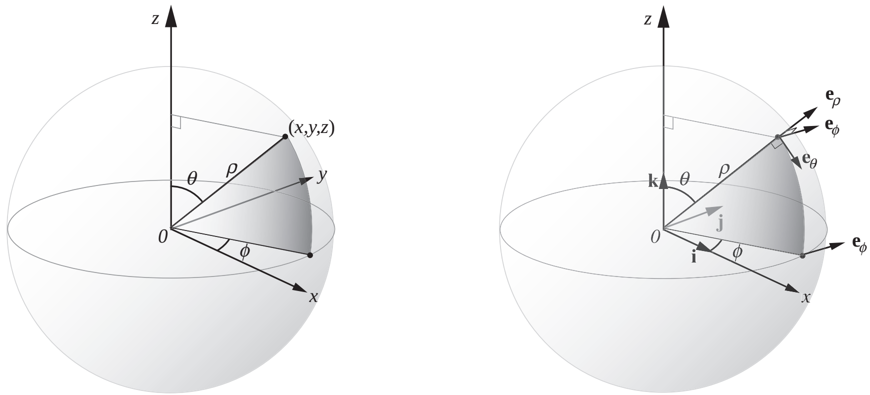

Figure 10.4: Spherical polar coordinate system and basis vectors. Note that the ϕ angle here is the same as the θ angle in the case of cylindrical coordinates.

$(r, \theta, \phi) \rightarrow (x, y, z)$

$$x = \rho \sin\theta \cos\phi \qquad y = \rho \sin\theta \sin\phi \qquad z = \rho \cos\theta.$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

The unit basis vectors at any point P are given by:

$\rightarrow$ $\hat{e}_\rho$ which is the outwards radial unit vector in the direction $\mathcal{O}P$.

$\rightarrow$ $\hat{e}_\phi = \hat{k} \times \hat{e}_\rho$.

$\rightarrow$ $\hat{e}_\theta = \hat{e}_\phi \times \hat{e}_\rho$

In general, the position vector of a point (ρ, θ, ϕ) in spherical polar coordinates is given by

$$\vec{r} = \rho \hat{e}_r$$

Now the locally defined unit vector $\hat{e}_r$ masks a dependence on both latitude θ and longitude ϕ.

Velocities and accelerations can be derived for the motion of a particle at $(\rho(t), \theta(t), \phi(t))$ in spherical polar coordinates at time t by techniques similar to those used for cylindrical polar coordinates. In summary, the results are: Velocity:

$$\dot{\vec{r}}(t) = \dot{\rho}\hat{e}_\rho + \rho\dot{\theta}\hat{e}_\theta + \rho\dot{\phi}\sin\theta\hat{e}_\phi$$

Acceleration:

$$\ddot{\vec{r}}(t) = \left(\ddot{\rho} - \rho\dot{\theta}^2 - \rho\dot{\phi}^2\sin^2\theta\right)\hat{e}_\rho + \left(2\dot{\rho}\dot{\theta} + \rho\ddot{\theta} - \rho\dot{\phi}^2\sin\theta\cos\theta\right)\hat{e}_\theta + \left(2\rho\dot{\theta}\dot{\phi}\cos\theta + 2\dot{\rho}\dot{\phi}\sin\theta + \rho\ddot{\phi}\sin\theta\right)\hat{e}_\phi.$$

10.4 Non-inertial frames

We know that inertial frames provide the setting for Newtonian dynamics, but what happens if you find yourself in a frame that is not inertial? We have already seen examples of this, expressed in terms of the changing basis vectors of cylindrical and spherical coordinates. Let us now consider what Newton's equations of motion look like in non-inertial frames. In order to be explicit, we will only consider rotating frames.

Let us start with an inertial frame S with coordinate axes x, y and z (as usual). Our goal is to understand the motion of particles as seen in a non-inertial frame S', with axes x', y' and z', which is rotating with respect to S. We denote the angle between the x-axis of S and the x'- axis of S' as θ. Since S' is rotating, we clearly have $\theta = \theta(t)$ and $\dot{\theta} \neq 0$.

Our first task is to describe the rotation of the axes. For this, we can use the angular velocity vector $\vec{\omega}$ that we introduced earlier. Consider a particle that is sitting stationary in the S' frame. Then, from the perspective of frame S it will appear to be moving with velocity

$$\dot{\vec{r}} = \vec{\omega} \times \vec{r}$$

where, in the present case $\vec{\omega} = \dot{\theta}\hat{e}_z$.

We can extend the description of the rotation to the axes of S' themselves. Let $\hat{e}'_i$, $i = 1, 2, 3$ be the unit vectors that point along the x', y' and z' directions. Then these also rotate with velocity

$$\dot{\hat{e}}'_i = \vec{\omega} \times \hat{e}'_i$$

This is the key result that will allow us to understand motion in rotating frames.

Consider now a particle which is not fixed in the S' frame, but moves along trajectory. We can measure the position of the particle in the inertial frame S, where

$$\vec{r} = \sum_i r_i \hat{e}_i$$

where $\hat{e}_i$ are the basis vectors of S. Alternatively, we can measure the position of the particle in frame S', where the position is

$$\vec{r} = \sum_i r'_i \hat{e}'_i$$

Of course, the actual position is the same in these two expressions. The coordinates differ because they are measured with respect to different axes. Let us now consider the velocity of the particle. In the fixed frame S we simply have

$$\dot{\vec{r}} = \sum_i \dot{r}_i \hat{e}_i$$

but in the rotating S' frame we have

$$\dot{\vec{r}} = \sum_i \left(\dot{r}'_i \hat{e}_i + r'_i \dot{\hat{e}}'_i \right) = \sum_i \left(\dot{r}'_i \hat{e}'_i + r'_i \vec{\omega} \times \hat{e}'_i \right) = \sum_i \dot{r}'_i \hat{e}'_i + \vec{\omega} \times \vec{r}$$

Let us introduce notation to help the intuition. We write the velocity of the particle as seen by an observer in frame S as

$$\left(\frac{d\vec{r}}{dt} \right)_S = \sum_i \dot{r}_i \hat{e}_i$$

Similarly, the velocity seen in S' is just

$$\left(\frac{d\vec{r}}{dt}\right)_{S'} = \sum_i \dot{r}'_i \hat{e}'_i$$

From our previous results, we then see that

$$\left(\frac{d\vec{r}}{dt}\right)_{S} = \left(\frac{d\vec{r}}{dt}\right)_{S'} + \vec{\omega} \times \vec{r}$$

The difference is (obviously) the relative velocity between the two frames.

We can play the same game with the acceleration. This is a bit more complicated, but after a bit of algebra[1], we find that

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{S} = \left(\frac{d^2\vec{r}}{dt^2}\right)_{S'} + 2\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{S'} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

This relation holds the key to understanding the motion of particles in a rotating frame.

Let us see what happens to **N2**. In the inertial frame S, we have

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_{S} = \vec{F}$$

This means that, in S' we have

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{S'} = \vec{F} - 2m\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{S'} - m\dot{\vec{\omega}} \times \vec{r} - m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

We learn that, in order to explain the motion of a particle an observer in S' must invoke the existence of three further terms on the right-hand side of **N2**. These are called *fictitious forces*. Viewed from S', a free particle doe not travel in a straight line and these fictitious forces are necessary to explain this departure from uniform motion. The three extra terms are called the Coriolis force, the centrifugal force and the Euler force, respectively. It is worth noting that all the fictitious forces are proportional to the inertial mass m. This is because they all originated from the "ma" side of **N2**. But, the gravitational force also appears to be proportional to the inertial mass. Is this evidence that gravity is also a fictitious force? In fact it is. Einstein's theory of general relativity recasts gravity as the fictitious force that we experience due to the curvature of space and time.

Example 3: Centrifugal force

The centrifugal force is given by

$$\vec{F}_{\text{cent}} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = -m(\vec{\omega} \cdot \vec{r})\vec{\omega} + m\omega^2\vec{r}$$

It is easy to see that this points away from the axis of rotation. Rotating bodies want to fly out from the axis of rotation to lower their energy.

[1] Kindly left as an exercise!

On Earth, the centrifugal force impacts on the (local) strength of gravity. As as example of this, consider a piece of string suspended from the ceiling. You might expect the string to point down to the centre of the Earth. But the centrifugal force due to the Earth's rotation means that this is not the case. The angle that the string makes with the line pointing to the Earth's centre depends on the latitude, θ, at which we are sitting.
The effective acceleration, due to the combination of gravity and the centrifugal force, is

$$\vec{g}_{\text{eff}} = \vec{g} - \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

We can resolve this is radial and angular (Southernly) directions using unit vectors $\hat{\vec{r}}$ and $\hat{\vec{\theta}}$. The centrifugal force then becomes

$$\vec{F}_{\text{cent}} = |\vec{F}| \cos\theta \hat{r} + |\vec{F}| \sin\theta \hat{\theta} = m\omega^2 r \cos^2\theta \hat{r} - m\omega^2 r \cos\theta \sin\theta \hat{\theta}$$

At the surface of the Earth, with $r = R$, this leads to

$$\vec{g}_{\text{eff}} = \left(-g + \omega^2 R \cos^2\theta\right) \hat{r} - \omega^2 R \cos\theta \sin\theta \hat{\theta}$$

We see that, at the equation, when $\theta = 0$, the string hangs directly towards the centre of the Earth, but we would measure gravity to be somewhat weaker than expected:

$$g_{\text{eff}} = g - \omega^2 R$$

Example 4: Coriolis force

The Coriolis force is given by

$$\vec{F}_{\text{cor}} = -2m\vec{\omega} \times \vec{v}$$

where $\vec{v}$ is the velocity measured in the rotating frame

$$\vec{v} = \left(\frac{d\vec{r}}{dt}\right)_{S'}$$

The force is velocity dependent: it is only felt by moving particles, but it is independent of the position.
We can compare the Coriolis force to one of our earlier examples, that of a charged particle moving in a magnetic field. This means that we already know what the effect of the force will be – it makes moving particles turn in circles.
On Earth, the Coriolis force is responsible for the formation of hurricanes. These rotate in an anti-clockwise direction in the Northern hemisphere and a clockwise direction in the Southern hemisphere.

10.5 Central Forces

A central force acting between two particles is a force that always acts along a line joining two the particles. Often, for convenience, one of the particles will be chosen to represent the origin of the coordinate system. This is the convention we adopt from here on.

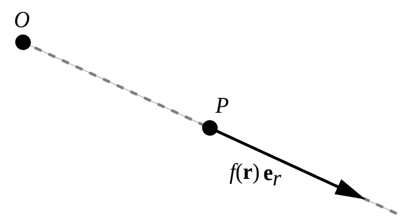

Figure 10.5: A central force $f(\vec{r})\hat{e}_r$ acting on particle P.

Examples of important central forces include:

- **Gravity:** Consider two masses m_1 and m_2 in free space. We fix the coordinate origin on mass m_1, say, and then the position vector of m_2 with respect to m_1 is $\vec{r}$. The gravitational force acting on m_2 is

$$\vec{F}(\vec{r}) = -\frac{Gm_1m_2}{|\vec{r}|^2}\hat{e}_r$$

  where $\hat{e}_r = \vec{r}/|\vec{r}|$ and $G = 6.6742 \times 10^{-11}$ $\mathrm{Nm^2kg^{-2}}$ is the universal gravitational constant. G is a fundamental physical constant and is the same for gravitational interactions between bodies with mass.[2] Note that the force is always attractive.

- **Coulomb interactions between electrical charge:** Consider two point particles in free space with electric charge q_1 and q_2. We fix the coordinate origin on charge q_1, say, and then the position vector of q_2 with respect to q_1 is $\vec{r}$. The electrical force acting on q_2 is

$$\vec{F}(\vec{r}) = \frac{q_1q_2}{4\pi\varepsilon_0|\vec{r}|^2}\hat{e}_r$$

  where $\hat{e}_r = \vec{r}/|\vec{r}|$ and $\varepsilon_0 = 8.854\cdots \times 10^{-12}$ $\mathrm{Fm^{-1}}$. ε_0 is the electrical permittivity of free space and is another fundamental constant of nature. Note that if $q_1q_2 > 0$ (like charges) the force is repulsive. If $q_1q_2 < 0$ (unlike charges) the force is attractive.

Moving on, let us represent the force acting per unit mass as $f(\vec{r})\hat{e}_r$. **N2** can then be written as

$$m\ddot{\vec{r}} = mf(\vec{r})\hat{e}_r.$$

If we take moments about the origin, using **N2** we have

$$\begin{aligned}
\vec{r}\times\ddot{\vec{r}} &= \vec{r}\times(f(\vec{r})\hat{e}_r)\,,\\
&= f(\vec{r})\,(\vec{r}\times\hat{e}_r)\,,\\
&= |\vec{r}|f(\vec{r})\,(\hat{e}_r\times\hat{e}_r)\,,\\
\Rightarrow \vec{r}\times\ddot{\vec{r}} &= \vec{0}
\end{aligned}$$

[2]Due to the extremely weak nature of gravity G is still one of the least accurately known fundamental constants.

Now, since

$$\begin{aligned}\frac{d}{dt}\left(\vec{r}\times\dot{\vec{r}}\right) &= \left(\dot{\vec{r}}\times\dot{\vec{r}}\right)+\left(\vec{r}\times\ddot{\vec{r}}\right)=\vec{0}+\left(\vec{r}\times\ddot{\vec{r}}\right)\\ \Rightarrow \qquad \vec{r}\times\ddot{\vec{r}} &= \frac{d}{dt}\left(\vec{r}\times\dot{\vec{r}}\right),\end{aligned}$$

we have

$$\begin{aligned}\frac{d}{dt}\left(\vec{r}\times\dot{\vec{r}}\right) &= \vec{0}\\ \Rightarrow \vec{r}\times\dot{\vec{r}} &= \vec{l}\end{aligned}$$

where $\vec{l}$ is a constant vector.

We learn that both the velocity and the position of a particle moving under a central force always lie in a plane. The normal to this plane is the constant vector $\vec{l}$.

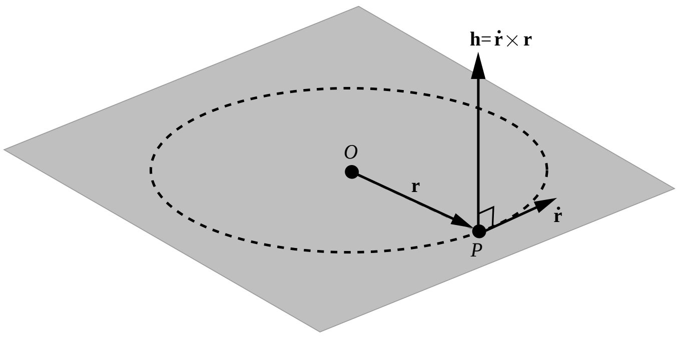

Figure 10.6: Motion under a central force is confined to a plane.

As we have seen, gravitational attraction is a central force acting between the centres of mass of two objects. This is why the Earth travels around the sun in a plane, or why a satellite moves in a planar orbit.

10.6 Conservation of Angular Momentum

Recall that the angular momentum $\vec{L}$ of a body at position $\vec{r}$ with linear momentum $\vec{p}$ about a the origin $\mathcal{O}$ is

$$\vec{L}\equiv\vec{r}\times\vec{p}=m\left(\vec{r}\times\dot{\vec{r}}\right)=m\vec{l}.$$

The angular momentum of a particle acted on by a central force is clearly conserved.

Since the motion of the particle is confined to a plane and is around a central point to which the force is directed to or from, it is natural to represent the dynamics of the particle using cylindrical polar coordinates with $z=0$, i.e., polar coordinates in the plane (the problem is effectively two dimensional). If we write **N2** in component form in these coordinates (ignoring the constant z-component), we have

$$\begin{aligned} m\ddot{\vec{r}} &= mf(\vec{r})\hat{e}_r \\ \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\hat{e}_\theta &= f(\vec{r})\hat{e}_r \end{aligned}$$

$$\Rightarrow \begin{cases} \left(\ddot{r} - r\dot{\theta}^2\right) = f(\vec{r}) \\ \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \end{cases}$$

This equation may be integrated immediately to obtain

$$r^2\dot{\theta} = l$$

where l is a (scalar) constant.

This is the explicit coordinate form for the conservation of angular momentum for a unit mass. To see this we write $\vec{r} \times \dot{\vec{r}}$ in coordinate form:

$$\begin{aligned} \vec{r} \times \dot{\vec{r}} &= r\hat{e}_r \times \left(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta\right) \\ &= r\dot{r}\underbrace{(\hat{e}_r \times \hat{e}_r)}_{\vec{0}} + r^2\dot{\theta}\underbrace{(\hat{e}_r \times \hat{e}_\theta)}_{\hat{e}_z} \\ &= l\hat{e}_z = \vec{l}. \end{aligned}$$

To determine l it therefore suffices to know the position and velocity of a particle at a single point in its motion. This information will usually be given in the form of initial data, e.g., $\vec{r}(0)$ and $\dot{\vec{r}}(0)$.

10.7 Conservative Central Forces

Let us now consider the implication of the conservation of energy for systems with central forces. We assume that the central force only depends on $r(t)$, the (scalar) distance of the body from the origin at time t. Then **N2** becomes

$$m\ddot{\vec{r}} = mf(r)\hat{e}_r.$$

Moreover, such central forces are conservative[3] The upshot of this is that we expect $f(r)$ to possess a potential and to generate an energy conservation law.

In order to demonstrate this observation law we need a couple of few preliminary calculations.

We have already seen that the motion under a central force is confined to a plane. Use this result to express the square of the speed of a body by introducing cylindrical polar coordinates.

[3] This can be checked by writing ∇ in polar coordinates and showing that the curl of $f(r)\hat{e}_r$ vanishes.

The velocity of a body in cylindrical polars is given by

$$\dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta.$$

The local basis vectors $\hat{e}_r$ and $\hat{e}_\theta$ are orthogonal unit vectors. Hence, the square of the speed v of the body is given by

$$v^2 = \dot{\vec{r}} \cdot \dot{\vec{r}} = \left(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta\right) \cdot \left(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta\right) = \dot{r}^2 + r^2\dot{\theta}^2$$

Next, we use the conservation of angular momentum, which means that

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \Rightarrow r\ddot{\theta} = -2\dot{r}\dot{\theta}$$

Hence, we have

$$\frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}^2\right) = r\dot{r}\dot{\theta}^2 + r^2\dot{\theta}\ddot{\theta} = r\dot{r}\dot{\theta}^2 + r\dot{\theta}\left(-2\dot{r}\dot{\theta}\right) = -r\dot{r}\dot{\theta}^2.$$

We shall use this result shortly.

Now since f is no longer a function of $\vec{r}$ but of r only, the $\hat{e}_r$ component of **N2** above becomes

$$\left(\ddot{r} - r\dot{\theta}^2\right) = f(r).$$

If we multiply this equation through by $\dot{r}$ we have

$$\begin{aligned}
\left(\dot{r}\ddot{r} - \dot{r}r\dot{\theta}^2\right) &= f(r)\dot{r} \\
\frac{d}{dt}\left\{\frac{1}{2}\left(\dot{r}^2\right)\right\} + \overbrace{\frac{d}{dt}\left\{\frac{1}{2}\left(r^2\dot{\theta}^2\right)\right\}}^{\text{using above result}} &= f(r)\frac{dr}{dt} \\
\frac{1}{2}\left\{\dot{r}^2 + r^2\dot{\theta}^2\right\} &= \int f(r)\frac{dr}{dt}dt + \text{constant} \\
\Rightarrow \frac{1}{2}m\left\{\dot{r}^2 + r^2\dot{\theta}^2\right\} - \int mf(r)dr &= E
\end{aligned}$$

We now identify

$$\begin{aligned}
\text{Kinetic Energy } T &= \frac{1}{2}m\left\{\dot{r}^2 + r^2\dot{\theta}^2\right\}, \\
\text{Potential Energy } V(r) &= -\int mf(r)dr, \\
\text{Total Energy} &= E
\end{aligned}$$

and deduce that a conservation of energy equation

$$T + V = E$$

holds for central forces.

We can also use conservation of angular momentum $r^2\dot{\theta}$ to simplify the above expression for energy further. In particular, we can remove the explicit dependence[4] on $\dot{\theta}$.

$$\begin{aligned}
\frac{1}{2}m\left\{\dot{r}^2 + r^2\dot{\theta}^2\right\} + V(r) &= E \\
\frac{1}{2}m\left\{\dot{r}^2 + \frac{l^2}{r^2}\right\} &= E - V(r) \\
\Rightarrow \qquad \dot{r}^2 &= \frac{2}{m}\left[E - V(r)\right] - \frac{l^2}{r^2}
\end{aligned}$$

This equation is a separable ordinary differential equation, which (at least in principle) can be integrated to give the position of the particle $r(t)$ explicitly. Once this is known then $r^2\dot{\theta} = l$ can be used to find $\theta(t)$. Of course, whether this is easily achievable depends on the exact form of $f(r)$.

10.8 Circular orbits and stability

We can use the above results to discuss the boundedness of the motion. For example, the existence of a potential (V) means that it is possible to establish the existence of a stable circular orbit, by analogy with the treatment for equilibrium states in statics.

Consider the central orbit energy equation

$$\frac{1}{2}m\left\{\dot{r}^2 + \frac{l^2}{r^2}\right\} + V(r) = E$$

This can be written in a form analogous to that of the the 1D equilibrium problem

$$\frac{1}{2}m\dot{r}^2 + U(r) = E$$

where

$$U(r) = \frac{1}{2}m\frac{l^2}{r^2} + V(r)$$

is an "effective potential", since it combines the real potential $V(r)$ with an extra term. The extra l^2/r^2 piece is often referred to as the *centrifugal barrier*. It stops the particle from getting too close to the origin – there would be a heavy price to pay in "energy".

Now, the term $\frac{1}{2}m\dot{r}^2$ looks like the usual 1D kinetic energy term. So we can pretend that this is a 1D system and treat the "effective potential" $U(r)$ as the potential in which the particle moves. This is not strictly correct, since $U(r)$ contains a term containing l, a constant that depends on the motion. However, since this term does not depend explicitly on the speed $\dot{r}$ we can use this analogy to work out where the particle comes to rest, where the equilibrium points are, and assess their stability.

Continuing with the analogy, we know from 1D equilibrium theory that for a stable equilibrium to exist, $U(r)$ must have a minimum. In the context of $U(r)$, if such a minimum exists, it will be at a value of $r = r_0$ say. This corresponds to a circular orbit about the central point of radius r_0.

[4]Recall that $l = r^2\dot{\theta}$.

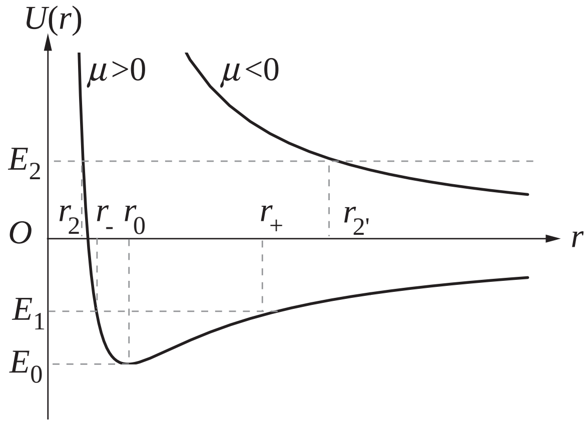

Figure 10.7: The effective potential $U(r) = \frac{ml}{2r^2} - \frac{m\mu}{r}$

Suppose we consider – as an example – an attractive inverse square law $f(r) = -\mu/r^2$. Relative to a point at infinity, this has a "proper" potential

$$V(r) = +m \int_\infty^r \frac{\mu}{\rho^2} d\rho = -\frac{m\mu}{r}.$$

This gives rise to the "effective" potential:

$$U(r) = \frac{ml^2}{2r^2} - \frac{m\mu}{r}$$

This has a minimum for $\mu > 0$ at

$$U'(r) = 0 \Rightarrow r = r_0 = \frac{l^2}{\mu}$$

Now suppose that the total energy of the system is E. Then rearranging the conservation equation we have

$$E - U(r) = \frac{1}{2} m\dot{r}^2.$$

Due to the positive quantity on the right-hand side, classically allowed motion can only take place when $E - U(r) > 0$, or $E > U(r)$. With this in mind we now examine the possibilities using figure 10.7.

- If the total energy of the system is $E = E_0 = U(r_0) = -m\mu/2l^2$, then the orbit will be a stable circle with radius r_0.
- If the system has a higher energy E_1 (see diagram) the system will oscillate between r_- and r_+.
- If the system has energy E_2 (see diagram) the system will possess no bounded orbit, and will lie in the range $r_2 < r < \infty$.
- If $\mu < 0$ there is no real minimum. All orbits are unbounded, e.g., if $E = E_2$, the orbit will lie in the range $r_{2'} < r < \infty$

It is also useful to ask if the orbits we determine are stable. As we are effectively dealing with a 1D problem, this is not a very tricky problem. Let us take as an example the case of circular orbits from above. We already know that we have a circular orbit if there is a solution, with $l \neq 0$, such that $\dot{r} = 0$ for all time. This means that we must have $\ddot{r} = 0$, which in turn leads to

$$U'(r) = 0$$

Not surprisingly, a circular orbit corresponds to a critical point r_0 of the effective potential. This orbit will be stable if a small perturbation brings us back to the original point. Basically, we have to sit at a minimum of the potential. That is, for stability we must have

$$U''(r_0) > 0$$

10.9 The orbit equation

So far we have not chosen to work out the actual orbital motion. Instead, we arrived at the energy conservation law by integrating **N2** once. Sometimes it is also possible to integrate **N2** directly to find the locus of the body as it moves under a central force.

$$\ddot{r} - r\dot{\theta}^2 = f(r), \qquad r^2\dot{\theta} = l \Rightarrow \ddot{r} - \frac{l^2}{r^3} = f(r).$$

The solution of this equation is often facilitated by using the change of variables

$$r = \frac{1}{u}, \qquad r^2\dot{\theta} = l \Rightarrow \dot{\theta} = lu^2$$

This way we obtain an ordinary differential equation for $u(\theta)$ as opposed to $r(t)$. Note that here we are solving for r as a function of θ. We are not deriving the time dependence (which in principle can be obtained from the energy equation). The procedure is as follows:

$$\dot{r} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -r^2\dot{\theta}\frac{du}{d\theta} = -l\frac{du}{d\theta}$$

$$\begin{aligned}\ddot{r} = \frac{d}{dt}(\dot{r}) = \frac{d}{dt}\left(-l\frac{du}{d\theta}\right) = \frac{d\theta}{dt}\frac{d}{d\theta}\left(-l\frac{du}{d\theta}\right) \\ = \dot{\theta}\frac{d}{d\theta}\left(-l\frac{du}{d\theta}\right) = -l\dot{\theta}\frac{d^2u}{d\theta^2} = -l^2u^2\frac{d^2u}{d\theta^2}\end{aligned}$$

where we have used

$$\frac{dl}{d\theta} = \frac{dl}{dt}\frac{dt}{d\theta} = 0$$

Making these two substitutions into **N2** we have

$$\ddot{r} - \frac{l^2}{r^3} = f(r) \Rightarrow -l^2u^2\frac{d^2u}{d\theta^2} - l^2u^3 = f\left(\frac{1}{u}\right)$$

or

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{l^2u^2}f\left(\frac{1}{u}\right)$$

This is a second order differential equation for u in terms of θ and so requires two pieces of initial/boundary data. The solution then generates $r(\theta) = 1/u(\theta)$. Alternatively, if the orbit of the particle is known experimentally as $u = u(\theta)$, this equation can be used to determine the central force $f(1/u)$ underlying the motion.

10.10 Inverse square law

Perhaps the most physically important central force is the inverse square law, which governs the gravitational attraction between masses and the Coulomb interaction of charged particles.

Let us focus on the simplest case; an attractive law

$$f(r) = -\frac{\mu}{r^2} = -\mu u^2,$$

which generates an orbit equation of the form

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{l^2u^2} f\left(\frac{1}{u}\right) \Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{\mu}{l^2}.$$

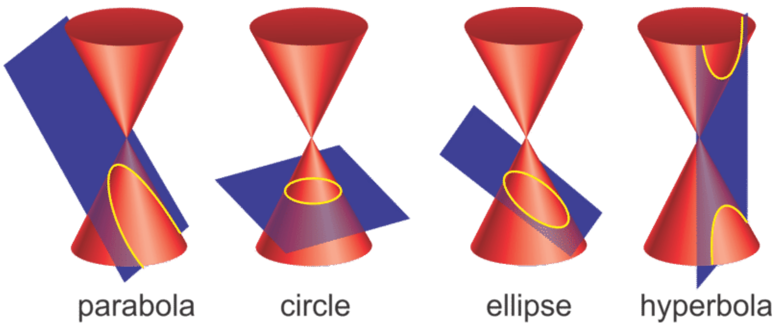

Figure 10.8: The conic sections.

This is a problem we know how to solve – it is just the constant-forcing harmonic oscillator equation. It can be solved with constant particular integral to give:

$$u(\theta) = \frac{\mu}{h^2} + A\sin\theta + B\cos\theta,$$

or in an equivalent form

$$u(\theta) = \frac{\mu}{h^2} + C\cos(\theta + D).$$

Now let us assume (actually without loss of generality for what follows) that we fix the $\theta = 0$ direction such that $D = 0$. Furthermore, we can write $l = h^2/\mu$ and rename the arbitrary constant C as e/l. Then the locus becomes:

$$u = \frac{1}{l} + \frac{e}{l}\cos\theta \Rightarrow \frac{l}{r} = 1 + e\cos\theta,$$

$$\Rightarrow r = \frac{l}{1 + e\cos\theta}.$$

This is the general equation of a **conic section**[5].

- l is called the '**semi latus rectum**'[6] and effectively determines the scale of the orbit.
- e is called the **eccentricity** of the orbit and determines whether it is circular, elliptic, parabolic, or hyperbolic. It is related to the energy of the orbiting particle.

Once we understand the motion, we can derive Kepler's three laws of planetary motion (from 1605!):

1. Each planet moves in an ellipse with the Sun at one focus.
2. The line between the planet and the Sun sweeps out equal areas in equal times.
3. The period of the orbit is proportional to $r^{3/2}$.

[5]So we are making contact with a classic bit of geometry.
[6]Yes, this is worth sniggering at.