

Chapter 3

Multivariable Functions

3.1 Functions of More than One Variable

All your previous work has been on functions of one variable, for example

$$f(x) = e^x + x^2$$

This is a map from a real number x in a domain $D \in \mathbb{R}$ of a function f to a real number $y = f(x)$ in the range $R \in \mathbb{R}$ of f :

$$f : x \rightarrow y = f(x)$$

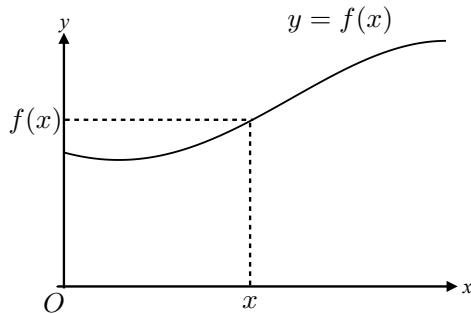


Figure 3.1: Plot of $f : x \rightarrow y = f(x)$

In this module we will generalise this to allow functions between finite real dimensional (Euclidean) spaces \mathbb{R}^n and \mathbb{R}^m for integer $n, m \geq 1$, where \mathbb{R} is the set of all real numbers.

Definition: (**Multivariate Function**) A *function* $F : D \rightarrow \mathbb{R}^m$ from a subset D of \mathbb{R}^n to \mathbb{R}^m is an assignment of a unique point $f(x_1, \dots, x_n) \in \mathbb{R}^m$ for each point $(x_1, \dots, x_n) \in D$. The set D is called the *domain* of F and the set of all points $F(x_1, \dots, x_n)$ obtained from the domain is called the *range* of F .

Example: Suppose

$$f(x, y) = \sqrt{4 - x^2 - y^2}.$$

If we restrict ourselves to only real values of the square root, the domain D is the disk of radius 2 centred at the origin:

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}.$$

The function f is thus $f : D \rightarrow \mathbb{R}$. (Strictly speaking the function f is $f : D \rightarrow \mathbb{R}^+$.)

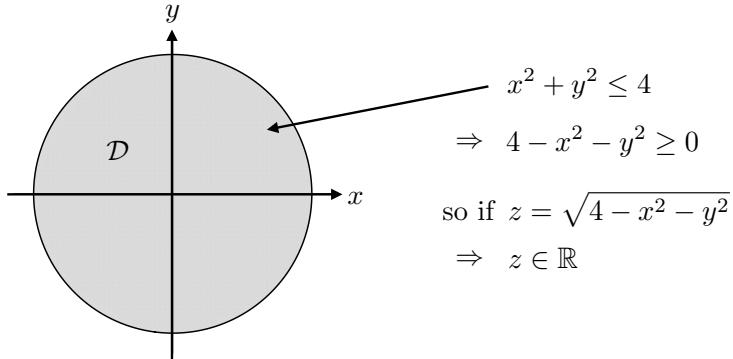


Figure 3.2: Domain of $z = \sqrt{4 - x^2 - y^2}$

Example: Suppose

$$f(x, y) = \frac{1}{x^2 + y^2}.$$

The function is well defined (i.e., has finite real values) for all points except $(x, y) = (0, 0)$, where the function has no finite value. Hence the domain D is

$$D = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\}.$$

The function f is thus $f : D \rightarrow \mathbb{R}$.

Where there is no confusion, we will often write $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ instead of $F : D \rightarrow \mathbb{R}^m$ with the understanding that the function may not be defined on all of \mathbb{R}^n but rather on its subset D , the domain.

Example: If x is the radius of a right cylinder and y is its height, the volume is given by a function $F : (x, y) \in \mathbb{R}^2 \rightarrow z \in \mathbb{R}$ such that

$$z = \pi x^2 y.$$

Clearly in the context of radius and height measurements, x and y are both here positive, so the domain D is $D = \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0\}$. However the formula for z is valid for all real values of x and y , so, in a mathematical context, we are able to drop reference to a subdomain of \mathbb{R}^2 .

Definition: (Multivariable functions between arbitrary numbers of dimensions) A function $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ can be viewed as an ordered m -tuple of functions from \mathbb{R}^n to \mathbb{R} , which we write as

$$F(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)),$$

where each $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ for $1 \leq i \leq m$ is itself a function from \mathbb{R}^n to \mathbb{R} .

Example: Suppose $F(t) = (t \cos(t), t \sin(t))$. The domain is all \mathbb{R} and $F : \mathbb{R} \rightarrow \mathbb{R}^2$ is a spiral curve in (x, y) space.

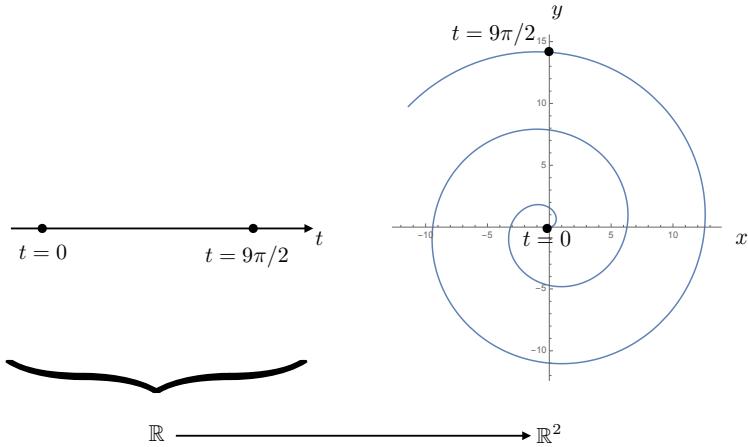


Figure 3.3: Plot of $F : \mathbb{R} \rightarrow \mathbb{R}^2$ curve with $F(t) = (t \cos(t), t \sin(t))$ (for part of the domain of \mathbb{R}).

In the notation of the above definition,

$$f_1(x_1) = x_1 \cos x_1, \quad f_2(x_1) = x_1 \sin x_1, \quad x_1 = t.$$

Example 3.1.1. If you take a Newtonian classical view of the world (see MATH1057) you may specify a particle at any point in time t by its spatial position in 3-D space (x, y, z) relative to some origin and its associated momenta (= mass \times velocity) in each of three directions (p_x, p_y, p_z) .

Hence classical particle specification consists of 6 mappings, $F : t \rightarrow (x, y, z, p_x, p_y, p_z)$, i.e., $\mathbb{R} \rightarrow \mathbb{R}^6$, i.e., you give me a time t and I have to give you back six functions $x(t), y(t), z(t), p_x(t), p_y(t), p_z(t)$ before we can define exactly what the particle is doing.

To simplify matters (and because humans can only visualise easily up to three dimensions), many of the examples in this module will focus on $\mathbb{R}^2 \rightarrow \mathbb{R}$, or $\mathbb{R}^3 \rightarrow \mathbb{R}$.

3.1.1 Explicit and Implicit Functions

Functions of a single variable have one of two basic flavours.

Explicit Function

A function is an **explicit function** if we have or are given an explicit description of $f(x)$ as a formula in terms of x , for instance a formula like

$$f(x) = x^2 \quad \text{or} \quad f(x) = \exp(\sin(x + (x^3 + 1) \cos(2x + 76))) + \ln(x^2 + 2).$$

More generally an explicit function is one where we have an explicit description of how to calculate the value of $f(x)$ given a value of the input x .

Note that sometimes this does not mean that there is an obvious formula. For example, consider the *greatest integer function*, where we set $\lfloor x \rfloor$ to be the largest integer that is less than or equal to x . We have an explicit description for $\lfloor x \rfloor$ given x from which we can calculate the function but no (immediately obvious) formula (e.g., given an input $x = 25.46$, the output $\lfloor x \rfloor$ is obviously 25).

An example of an explicit function of two variables $\mathbb{R}^2 \rightarrow \mathbb{R}$ is

$$z = x^2 - y^2 + xy \cos(x + y).$$

Here we may input any real values of x and y into the right hand side and this generates a unique value of z and we can write the function as $z = f(x, y)$, i.e., the unique output z on the LHS and all the x, y inputs on the RHS

An example of an explicit function of three variables $\mathbb{R}^3 \rightarrow \mathbb{R}$ is

$$u = x^2 + yz + e^{x+y+z}.$$

Here, we may input (the same) values of x, y and z into the right hand sides of the two (paired) equations and this will generate a pair of values of u and v .

An example of an explicit function of $\mathbb{R}^3 \rightarrow \mathbb{R}^2$ is

$$\begin{aligned} u &= x^2 + yz + e^{x+y+z}; \\ v &= \cos(x + z^2) + xyz. \end{aligned}$$

Here we input x, y and z into the right hand side and this generates a unique values of u and v .

Implicit Function

A function is an **implicit function** if we have a description of the relationship between two (or more) variables, such as x and y , but the function, either is not, or may not be, written in a form that makes one of the variables explicitly the subject in a uniquely defined way.

A simple example of an implicit function comes from the equation of a circle in the plane with centre the origin and radius 1, namely:

$$x^2 + y^2 = 1$$

The function is represented in the form $F(x, y) = x^2 + y^2 = \text{constant}$. Here, we cannot solve for y as a function of x in a way that is unique for all values x in the domain. Specifically, if we solve for y , we get $y = \pm\sqrt{x^2 - 1}$, and so for each $-1 < x < 1$, there are two possible values of y and this violates the definition of function as discussed (albeit briefly) above.

All the following are (non-specific) examples of how implicit functions are generally represented, i.e., as a function of the all variables = a constant (often zero, without loss of generality):

$$\begin{aligned} \ln(x + y) - 1 &= 0; \\ F(x, y) &= 0; \\ x^2 + y^2 + z^2 - R^2 &= 0; \\ G(x, y, z) &= 0; \end{aligned}$$

or

$$\begin{cases} U(x_1, x_2, \dots, x_n) = 0 \\ V(x_1, x_2, \dots, x_n) = 0 \end{cases}$$

In each case (and for given F, G, U, V), it may or may not be possible to solve the equational relation explicitly in terms of one of the variables. When it is possible or not will be studied later on. Actually we will not care whether an explicit representation is possible or not and we will just develop maths that can cope with both types of representations.

3.2 Graphical Representations of Multivariable Functions

Graphically we can represent $y = f(x)$ by a set of points $(x, f(x))$ in the (x, y) plane.

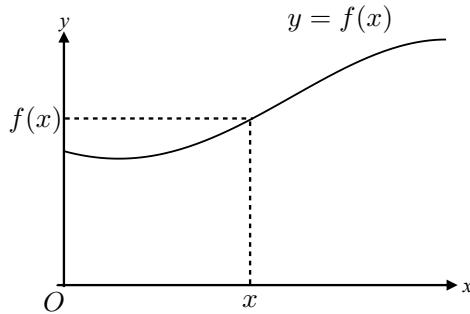


Figure 3.4: Fig to be done

By contrast the explicit function $z = f(x, y)$ can represent the height z of a surface above the (x, y) plane in the (x, y, z) space

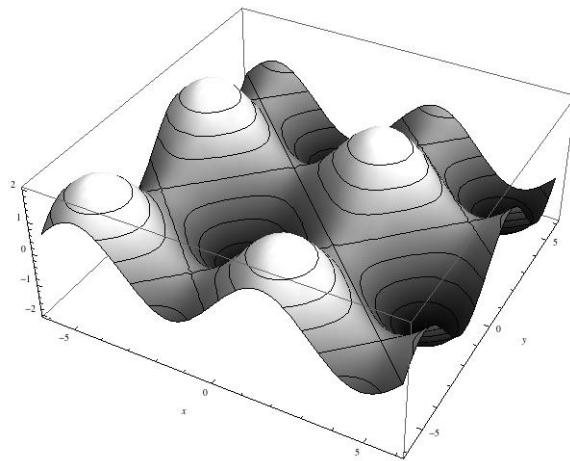


Figure 3.5: Plot of a multivariate function where the height of the function is $z = f(x, y)$ and is given by $f(x, y) = \sin(x) + \sin(y)$

Example 3.2.1. Consider the function

$$f(x, y) = 1 - x - y, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1.$$

This maps the rectangular set of points $0 \leq x \leq 1, 0 \leq y \leq 1$ to a 2-D plane in \mathbb{R}^3 passing through the (obvious) points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

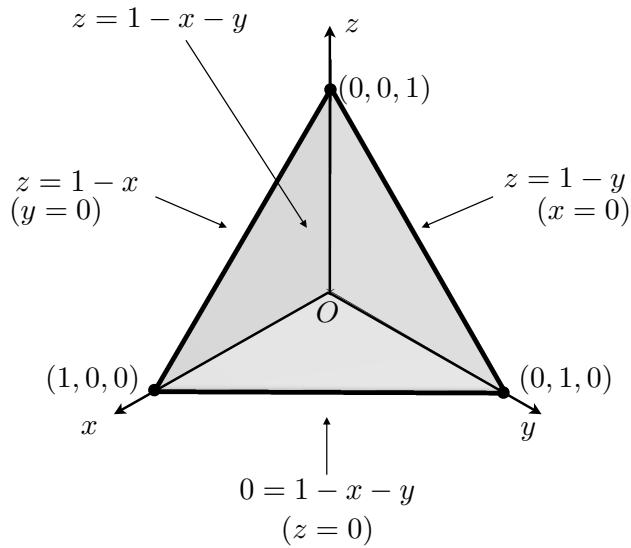


Figure 3.6: Surface $z = 1 - x - y$ plotted in positive octant $x \geq 0, y \geq 0, z \geq 0$.

Example 3.2.2. Consider the function

$$f(x, y) = \sqrt{9 - x^2 - y^2},$$

The domain of this real function is given by the circle

$$D = \{(x, y) \mid x^2 + y^2 \leq 9\},$$

since any other values of (x, y) lead to complex numbers.

Now if we set $z = f(x, y)$, then we have to plot the function $z = \sqrt{9 - x^2 - y^2}$. Squaring both sides and rearranging, we have

$$x^2 + y^2 + z^2 = 9.$$

This says that the square of the distance from the origin of all the points on this surface in (x, y, z) space must be 9. Hence this is the equation of the surface of a sphere of radius 3.

Now reverting back to the actual equation $z = \sqrt{9 - x^2 - y^2}$, if we assume that the square root function is positive only, then $f(x, y)$ represents the surface of a hemisphere, centred the origin, radius 3 **above**, the (x, y) plane.

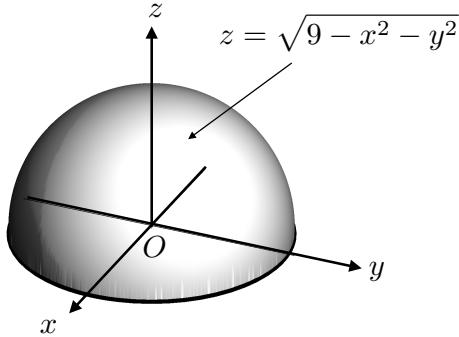


Figure 3.7: $z = \sqrt{9 - x^2 - y^2}$ where the positive square root is taken.

The corresponding function

$$f(x, y) = -\sqrt{9 - x^2 - y^2},$$

represents the surface of a hemisphere, centred the origin, radius 3, **below** the (x, y) plane.

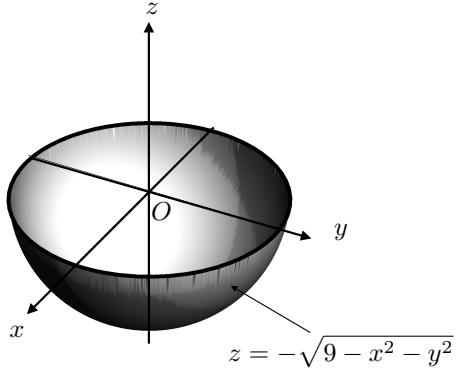


Figure 3.8: $z = -\sqrt{9 - x^2 - y^2}$ where the negative square root has been taken.

Graphs in $\mathbb{R}^n, n > 3$

All the examples above are $\mathbb{R}^2 \rightarrow \mathbb{R}$. We were able to plot these as 2D surfaces in 2+1=3D space.

Consider a function

$$w = f(x, y, z) = x^2 + y^2 + z^2$$

This is a mapping $\mathbb{R}^3 \rightarrow \mathbb{R}$. If we wanted to draw both the domain and range on the same plot, we would need to be able to visualise a 3 dimensional **hypersurface** in 3+1=4 dimensional space. Clearly this is not possible (in our concept of the world!).

Hence at best we might need to consider only partial representations of the graph. For example, we might consider what the surface $w = 9$ looks like and plotting it in 4 – 1 = 3 dimensional space (x, y, z) . We could then re-plot the graph for different chosen values of w .

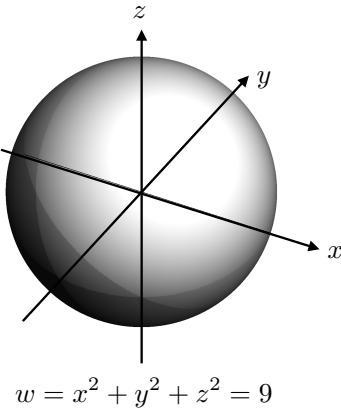


Figure 3.9: Plot of $w = 9$ where $w = x^2 + y^2 + z^2$, i.e., a sphere of radius 3 in (x, y, z) space.

Hence, to avoid brain overload, when we want to visualise graphs, we shall henceforth largely stick to mappings of $\mathbb{R}^2 \rightarrow \mathbb{R}$.

3.3 Contours

An alternative way of representing surfaces of the form $z = f(x, y)$ is to use a contour plot. This is exactly what is used to represent the height of the ground on an Ordnance Survey map¹.

The following diagram is based on data from Mathematica and depicts a relief and contour plot of Schielhallion. Schielhallion is a mountain in Perthshire in Scotland. Due to its distinctive shape, size and (lack of) proximity to other hills/mountains, it was chosen (initially by the astronomer the Reverend Nevil Maskelyne) in the 1770-1790s to verify the mean density of the earth. Those experiments required the calculation of the volume of the mountain, which in turn required the development of two vital mathematical techniques. Contour lines were invented (by Charles Hutton) to map out the level heights around the mountain so that it could be approximated as a series of column-like structures. The volume of these columns was calculated by the new technique of numerical integration.²

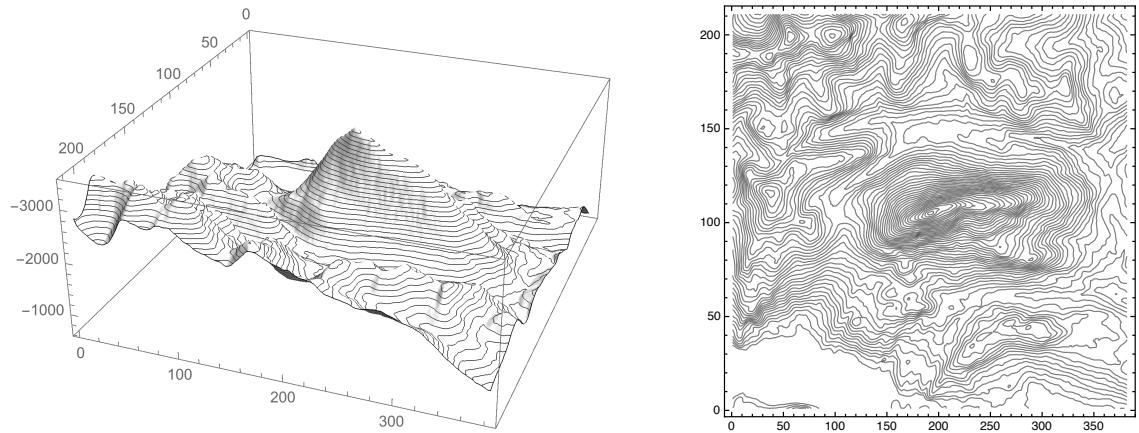


Figure 3.10: Left: Relief plot of Schielhallion. Right: Contour plot of the same region.

Definition: (Contour): A contour (or level set) of a graph of $z = f(x, y)$ is the set of points $(x, y) \in \mathbb{R}^2$ satisfying $f(x, y) = c$ for a chosen $c \in \mathbb{R}$:

$$L_c = L_c(f) = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\}.$$

In words, the contour (or level set of height c) is the set of points in the domain at which the function takes the value c . When presented in this form, the equation for contours is an implicit representation.

One immediate consequence of this is that level sets of different heights are necessarily disjoint: by the definition of function, there cannot exist a point at which the function takes on two different z -values simultaneously.

A contour plot is then the union of the contours for different values of $c \in \mathbb{R}$. It is a 2-D plot of a 3-D surface.

Example: Determine the level contours of the function

$$f(x, y) = 1 - x - y$$

¹the OS are actually based in Southampton <https://www.ordnancesurvey.co.uk/>

²You can read more about the mathematical aspects of the experiment here. Details of the theory underpinning the calculation can be found here. The original experiment was accurate to around 20% of the currently accepted value for the density of the earth. Subsequent experiments on the same mountain achieve estimates within 1%.

- The level sets are given by $c = f(x, y)$ for $c \in \mathbb{R}$.
- Hence we have $c = 1 - x - y$.
- Rearranging we obtain an explicit relation of $y = 1 - c - x$.
- These are straight lines in the (x, y) plane of slope -1 and y -intercept $1 - c$.
- The contours may then be plotted as follows:

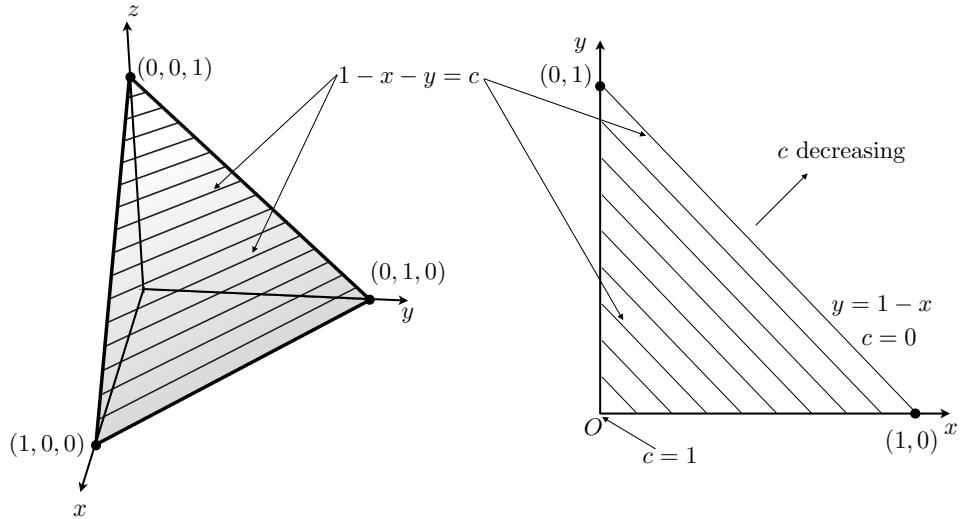


Figure 3.11: Level height contours of $z = 1 - x - y$ in the positive (x, y) -plane (and plotted on the surface in the positive octant). These are curves where $z = c$ where c is a constant, i.e., $1 - x - y = c$, or $y = -x + 1 - c$.

In practice we would only draw the 2-D plot on the right, since the whole point is to avoid having to cope with drawing 3-D surfaces. However in these first examples, we plot both the contours on the 3-D surface and in the 2-D (x, y) plane, just to give an idea of what the contours are and how they relate to the surface.

Example: Determine the level sets of the function

$$f(x, y) = x^2 + y^2$$

- The level sets are given by $c = f(x, y)$.
- Hence we have $c = x^2 + y^2$.
- We see that for real x and y , $x^2 + y^2 \geq 0$. Hence $c \geq 0$ only. So when
 - $c < 0$: No real contours exist.
 - $c = 0$: $x^2 + y^2 = 0 \Rightarrow (x, y) = (0, 0)$.
 - $c > 0$: $x^2 + y^2 = c \Rightarrow$ circles centred $(0, 0)$, radius \sqrt{c} .
- The contours may then be plotted as follows:

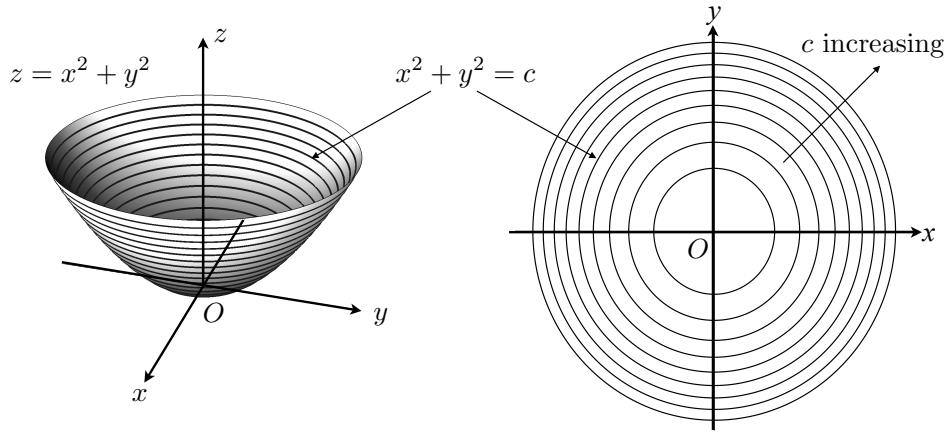


Figure 3.12: Level height contours of $z = x^2 + y^2$. These are curves where $z = c$ where c is a constant, i.e., $x^2 + y^2 = c$, circles centre the origin of radius \sqrt{c} .

- Note from the analysis above, the function $f(x, y) = x^2 + y^2$ clearly has a local **minimum** value at $(0, 0)$. If you walk away from the origin in whichever direction on the surface $z = x^2 + y^2$, you will always walk uphill.
- Clearly if you consider the function $f(x, y) = -(x^2 + y^2)$, you will have similar shapes of contours, but taking the corresponding negative values of c . Hence, for that function, the origin is a **maximum**. If you walk away from the origin in whichever direction on the surface $z = -(x^2 + y^2)$, you will always walk downhill.

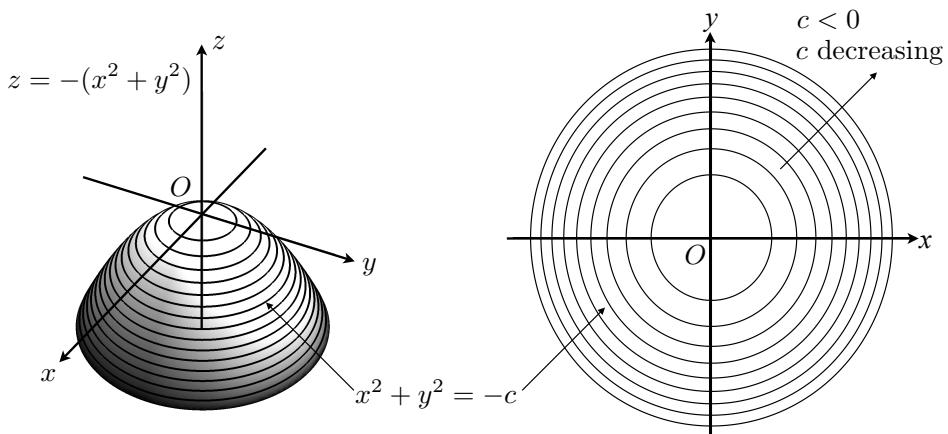


Figure 3.13: Level height contours of $z = -(x^2 + y^2)$. These are curves where $z = c$ where c is a constant, i.e., $x^2 + y^2 = -c$, but clearly $c < 0$ and decreases as $x^2 + y^2$ increases. They are still circles, centre the origin, of radius $\sqrt{-c}$.

Remark: If the contour levels c are equally spaced, clearly the closer the contour lines are together on the (x, y) plot, the “steeper” the surface is in that direction.

Example: Determine the level contours of the function

$$f(x, y) = x^2 - y^2$$

- The level sets are given by $c = f(x, y)$.
- Hence we have $c = x^2 - y^2$.
- We see that for real x and y , $x^2 - y^2 \geq 0$ may be positive, zero or negative.
- For $c = 0$ we have $x^2 = y^2$ or $y = \pm x$. These are straight lines through the origin.
- The contours may then be plotted as follows:

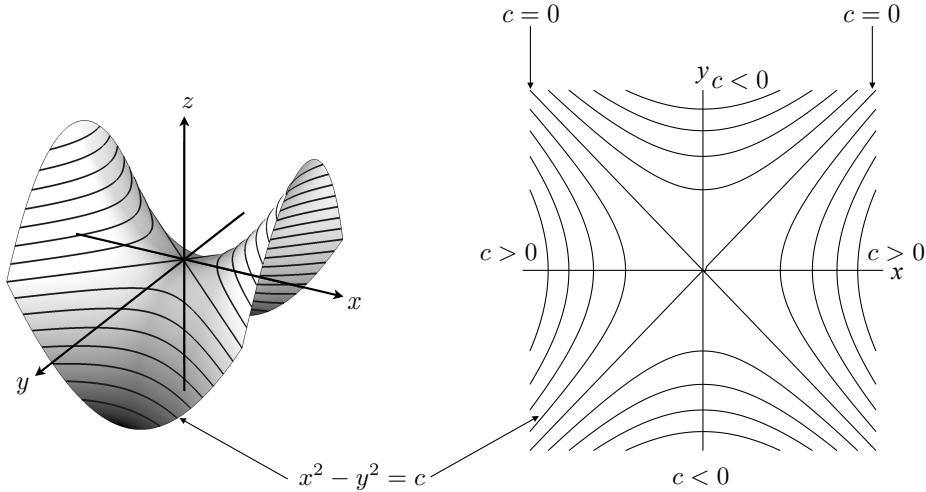


Figure 3.14: Level height contours of $z = x^2 - y^2$. These are still curves where $z = c$ where c is a constant, i.e., $x^2 - y^2 = c$. These are (for $c \neq 0$) hyperbolae. The contours corresponding to $c = 0$ are the lines $y = \pm x$. The regions $y < |x|$ have $c > 0$, the regions $y > |x|$ have $c < 0$.

- This is an example of a simple **saddle point**. The region around the origin is separated into four quadrants.
- Walking away from the origin on the surface $z = x^2 - y^2$ into two of these quadrants ($|y| > |x|$) will result in you going downhill.
- Walking away from the origin on the surface $z = x^2 - y^2$ into the other two quadrants ($|y| < |x|$) will result in you walking uphill.
- Walking away from the origin on the surface $z = x^2 - y^2$ on the lines $y = \pm x$ and you will keep the same height of 0 and neither walk up nor downhill.

More complicated contour plots can be drawn using computer packages, e.g., Maple, Mathematica or Matlab. Further examples are available on Blackboard.

3.4 Limits and Continuity

3.4.1 Definition of limits in \mathbb{R}^2

For functions of a single variable, the concept of the limit

$$\lim_{x \rightarrow a} f(x)$$

was discussed last semester.

Recall that a function is continuous at $x = a$ iff as

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = f(a).$$

In detail this means three conditions:

- First, f has to be defined at a (i.e., c must live in the domain of f).
- Second, the limits on both sides of a has to exist.
- Third, the value of this limit must equal $f(a)$.

All these concepts can be extended to multivariate functions as follows.

Definition: (Limit) Given a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. We say $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$, whenever

- (i) every neighbourhood of the point (a, b) contains a point of the domain of f different from (a, b) , and
- (ii) for every $\varepsilon > 0$, there exists $\delta > 0$ such that if (x, y) is in the domain and satisfies

$$0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta,$$

then $|f(x, y) - L| < \varepsilon$.

It's much better to draw a diagram to explain what we mean:

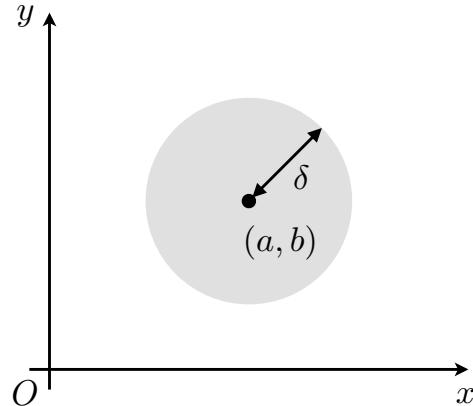


Figure 3.15: $\forall (x, y)$ in this disk, we must have $|f(x, y) - L| \leq \varepsilon$ if the limit of f at (a, b) exists.

In words this means that for a limit to exist at (a, b) , the value of the function $f(x, y)$ must approach the same value L as $(x, y) \rightarrow (a, b)$ from **whichever** route you take.

Sometimes it is possible to use the (dreaded) ε, δ definition.

Example: Let us find

$$\lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2 + 7).$$

Note that as (x, y) tends to $(0, 0)$, the function approaches the value 7. So, we shall try to prove that this is the limit. The following steps gives the standard algorithm for proving the limit.

- The domain is the (x, y) plane. So every neighbourhood of the point $(0, 0)$ is in the domain of f (condition (i) above).

- We need to show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < \sqrt{x^2 + y^2} < \delta$, then

$$|x^2 + y^2 + 7 - 7| < \varepsilon \quad \Rightarrow \quad |x^2 + y^2| < \varepsilon$$

- Obviously, if we now set $\delta = \sqrt{\varepsilon}$ for any $\varepsilon > 0$ we have found the $\delta > 0$ to satisfy the inequality condition (ii) in the definition above.
- Hence

$$\lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2 + 7) = 7.$$

This looks like a rather obvious answer. In the next example, the value of the limit at the limit point is not so obvious.

Example: Show that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^2 + y^2} = 0.$$

- The domain is the whole (x, y) -plane, excluding the origin, where the functional form is $0/0$, so ambiguous.
- Nevertheless, every neighbourhood of the point $(0, 0)$ is in the domain of f (condition (i) above).
- Now, we need to show that for $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < \sqrt{x^2 + y^2} < \delta$, then

$$\left| \frac{x^2y}{x^2 + y^2} - 0 \right| < \varepsilon.$$

- A shortcut that you can often deploy in these types of examples is to convert to polar coordinates: $x = r \cos \theta$, $y = r \sin \theta$. For then, since $|\cos \theta| \leq 1$ and $|\sin \theta| \leq 1$ and $r \geq 0$, we have

$$\begin{aligned} \left| \frac{x^2y}{x^2 + y^2} - 0 \right| &= \left| \frac{r^3 \cos^2 \theta \sin \theta}{r^2} \right| = \left| r \cos^2 \theta \sin \theta \right| < |r| = r \\ &\Rightarrow \left| \frac{x^2y}{x^2 + y^2} - 0 \right| < r \\ &\Rightarrow \left| \frac{x^2y}{x^2 + y^2} - 0 \right| < \sqrt{x^2 + y^2} \end{aligned}$$

- Hence comparing with the definition of the limit above, if we see that if there has to be a $\delta > 0$ such that the LHS has to be less than $\varepsilon > 0$ whenever $0 < \sqrt{x^2 + y^2} < \delta$, a natural choice is $\delta = \varepsilon$.

3.4.2 Sums, Products, Ratios, Composition of Limits in \mathbb{R}^2

The above formal ε, δ definition is usually too cumbersome to use in practice. Fortunately, to save a lot of pain, the following lemma holds. It speeds up limit-finding when functions are well behaved (proof of the lemma is not required).

Lemma 3.4.1. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $g : \mathbb{R}^2 \rightarrow \mathbb{R}$. Suppose

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y) \rightarrow (a,b)} g(x,y) = M.$$

Then

$$(i) \quad \lim_{(x,y) \rightarrow (a,b)} (f(x,y) \pm g(x,y)) = L \pm M,$$

$$(ii) \quad \lim_{(x,y) \rightarrow (a,b)} (f(x,y)g(x,y)) = LM,$$

$$(iii) \quad \lim_{(x,y) \rightarrow (a,b)} (f(x,y)/g(x,y)) = L/M, \text{ as long as } M \neq 0.$$

(iv) Given a function $h : \mathbb{R} \rightarrow \mathbb{R}$ that is continuous at $L \in \mathbb{R}$, then $\lim_{(x,y) \rightarrow (a,b)} h(f(x,y)) = h(L)$.

Example Find the limit of $f(x, y) = x^2 + xy + 5$ as (x, y) approaches $(1, 0)$.

- The domain contains all points of the (x, y) -plane, so the function is expected to exist at the limit point. Also we then have neighbourhood of the limit point that is in the domain. Therefore we expect that a limit may exist.
- f is composed of the sum of the functions $x^2 = x \times x$, $y = x \times y$ and 5.
- We start with the obvious results that

$$\lim_{(x,y) \rightarrow (1,0)} x = 1, \quad \lim_{(x,y) \rightarrow (1,0)} y = 0$$

This means that the limits exist for the basic building blocks of the function. We can now try to apply Lemma 3.4.1 as follows.

- We can say:

$$\lim_{(x,y) \rightarrow (1,0)} x^2 = \lim_{(x,y) \rightarrow (1,0)} (x \times x) = \underbrace{\left(\lim_{(x,y) \rightarrow (1,0)} x \right)}_{\text{(ii) of Lemma 3.4.1}} \times \underbrace{\left(\lim_{(x,y) \rightarrow (1,0)} x \right)}_{\text{(ii) of Lemma 3.4.1}} = 1 \times 1 = 1$$

$$\lim_{(x,y) \rightarrow (1,0)} xy = \lim_{(x,y) \rightarrow (1,0)} (x \times y) = \underbrace{\left(\lim_{(x,y) \rightarrow (1,0)} x \right)}_{\text{(ii) of Lemma 3.4.1}} \times \underbrace{\left(\lim_{(x,y) \rightarrow (1,0)} y \right)}_{\text{(ii) of Lemma 3.4.1}} = 1 \times 0 = 0$$

$$\lim_{(x,y) \rightarrow (1,0)} 5 = 5$$

$$\lim_{(x,y) \rightarrow (1,0)} (x^2 + xy + 5) = \underbrace{\lim_{(x,y) \rightarrow (1,0)} x^2 + \lim_{(x,y) \rightarrow (1,0)} xy}_{\text{(i) of Lemma 3.4.1}} + \lim_{(x,y) \rightarrow (1,0)} 5 = 1 + 0 + 6 = 6$$

Example: Find the limit of $f(x, y) = \frac{3x}{x+4y}$ as (x, y) approaches $(1, 1)$.

- The domain contains the point $(x, y) = (1, 1)$.
- The locus of points where the function is not defined is the straight line $y = -x/4$. This line passes nowhere near the limit point. Hence, we then have neighbourhood of the limit point that is in the domain.
- So the function is expected to exist at the limit point. We can now try to apply Lemma 3.4.1 as follows.
- We obviously have:

$$\lim_{(x,y) \rightarrow (1,1)} x = 1, \quad \lim_{(x,y) \rightarrow (1,1)} y = 1$$

- Hence we can say:

$$\lim_{(x,y) \rightarrow (1,1)} 3x = 3 \times \underbrace{\lim_{(x,y) \rightarrow (1,1)} x}_{\text{(ii) of Lemma 3.4.1}} = 3 \times 1 = 3$$

$$\lim_{(x,y) \rightarrow (1,1)} x + 4y = \underbrace{\lim_{(x,y) \rightarrow (1,1)} x}_{\text{(i) of Lemma 3.4.1}} + \underbrace{\lim_{(x,y) \rightarrow (1,1)} 4y}_{\text{(ii) of Lemma 3.4.1}} = 1 + 4 \times \underbrace{\lim_{(x,y) \rightarrow (1,1)} y}_{\text{(ii) of Lemma 3.4.1}} = 1 + 4 = 5 \neq 0$$

$$\lim_{(x,y) \rightarrow (1,1)} \frac{3x}{x+4y} = \frac{\lim_{(x,y) \rightarrow (1,1)} 3x}{\lim_{(x,y) \rightarrow (1,1)} x + 4y} = \frac{3}{5}$$

3.4.3 Showing that limits do not exist

Sometimes, you have to decide whether the limit exists or not. The following examples illustrate how the requirement that the $f(x, y)$ approaches the same limit, no matter how the limit point is approached, is actually incredibly restrictive.

Example: If

$$f(x, y) = \begin{cases} x^2/(x^2 + y^2) & (x, y) \neq 0 \\ 1 & (x, y) = 0 \end{cases}$$

does $\lim_{(x,y) \rightarrow (0,0)}$ exist?

- The functional form $x^2/(x^2 + y^2)$ is not defined when $(x, y) = (0, 0)$, since it is of the form 0/0 (and we can't use Lemma 2.4.1 part iii, since $M = 0$).
- However the function is **defined** to have the value 1 at the limit point.
- Hence the function f is defined for the entire (x, y) plane, so it makes sense to ask if a limit exists.
- The key thing we have to test is **whether the limit is the same regardless of how we approach the limit point**. We proceed as follows.
- If we approach the limit point $(0, 0)$ along $y = 0$ we have

$$f(x, 0) = \frac{x^2}{x^2 + 0^2} = 1 \quad \Rightarrow \quad \lim_{x \rightarrow 0} f(x, 0) = 1 = f(0, 0)$$

This suggests that the limit may exist and be 1. However we need to check **every possible approach!**

- If we approach the limit point $(0, 0)$ along $x = 0$ we have

$$f(0, y) = \frac{0^2}{0^2 + y^2} = 0 \quad \Rightarrow \quad \lim_{y \rightarrow 0} f(0, y) = 0$$

- Hence

$$\lim_{y \rightarrow 0} f(0, y) \neq \lim_{x \rightarrow 0} f(x, 0)$$

and so the limit depends on how the limit point is approached, so the limit is **not unique** and so the limit **does not exist**!

- In fact, consider an approach to the limit point at the origin along the straight lines $y = \alpha x$. Inserting this into the function we have:

$$f(x, \alpha x) = \frac{x^2}{x^2 + \alpha^2 x^2} = \frac{1}{1 + \alpha^2}, \quad x \neq 0$$

Hence if we approach along **any** straight line, the limit will be different, and so does not exist. In practice it suffices to show that there is ambiguity between just two different ways of approaching.

- A graph of the function is shown below, which helps to explain the ambiguity in the limit. Basically, the surface is vertical at the origin. Any curve you chose to take to walk towards the origin will therefore intersect at different heights, depending on how it enters the origin.

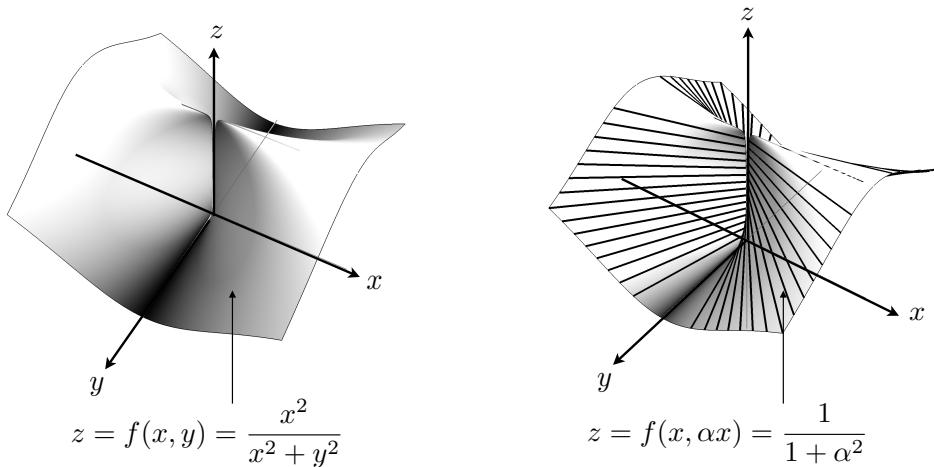


Figure 3.16: Surface $z = f(x, y) = x^2/(x^2 + y^2)$ (left). Values of function $f(x, \alpha x)$ along the lines $y = \alpha x$ (right), which take different values $\alpha^2/(1 + \alpha^2)$ at the origin $(x, y) = (0, 0)$, depending on the value of α .

This explains why the limit does not exist in this case. The definition of a limit above basically says that the value of a function must be unique at a point. Clearly in this case the function can take any value you like at $(x, y) = (0, 0)$ since it is vertical there.

This next example shows the danger of relying on approaches to the limit point along straight lines to determine whether a limit exists or not.

Example: Consider the function

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

Investigate the limiting behaviour as $(x, y) \rightarrow (0, 0)$. We can't use Lemma 2.4.1 since at the limit point we have $0/0$, i.e., $M = 0$.

- Suppose we consider the behaviour of the function along any path with $y = \alpha x$. We have

$$f(x, \alpha x) = \frac{2x^2\alpha x}{x^4 + \alpha^2 x^2} = \frac{2\alpha x^3}{x^2(x^2 + \alpha^2)} = \frac{2\alpha x}{(x^2 + \alpha^2)}$$

Hence approaching the limit point along any straight line ($\alpha \neq 0$) into the origin has

$$\lim_{x \rightarrow 0} f(x, \alpha x) = \lim_{x \rightarrow 0} \frac{2\alpha x}{(x^2 + \alpha^2)} = \frac{0}{\alpha^2} = 0$$

- This suggests that

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0.$$

- However, now consider approaching the limit point along the curve $y = x^2$. Then we have

$$f(x, x^2) = \frac{2x^2 \times x^2}{x^4 + (x^2)^2} = \frac{2x^4}{x^4(1+1)} = 1.$$

Hence approaching the limit point along any $y = x^2$ into the origin has

$$\lim_{x \rightarrow 0} f(x, x^2) = \lim_{x \rightarrow 0} 1 = 1.$$

- Hence there is not a unique limit! This underlines the fact that for the limit to exist formally, it must be the same, no matter how the limit point is approached (or how complicated the path you take to approach the limit point).
- What is actually going on with the surface near to $(x, y) = (0, 0)$ is illustrated below. The surface is vertical at the origin, but folded around the curve $y = x^2$.

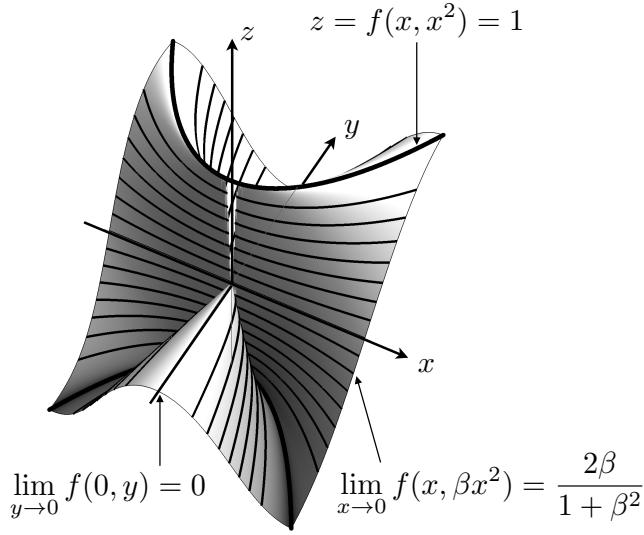


Figure 3.17: 3-D plot of $z = f(x, y) = 2x^2y/(x^4 + y^2)$. If we approach the origin along, say $x = 0$, $\lim_{y \rightarrow 0} f(0, y) = 0$. However, if we approach the origin along the curve $y = x^2$ we find $\lim_{x \rightarrow 0} f(x, x^2) = 1$. In fact if we approach the origin along any curve $y = \beta x^2$, we find $\lim_{x \rightarrow 0} f(x, \beta x^2) = \lim_{x \rightarrow 0} 2\beta x^4/(x^4 + \beta^2 x^4) = 2\beta/(1 + \beta^2)$. There is no unique value of $f(x, y)$ at the origin $(x, y) = (0, 0)$, since the surface is vertical there.

Clearly finding contradictions based on the choice of path taken to approach a limit point is a way of **disproving** the existence of a limit.³

3.4.4 Continuity of multivariable functions

Having defined the concept of a limit, we may now define continuity in multivariable functions

Definition: (Continuity) The function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is said to be *continuous at* $(a, b) \in \mathbb{R}^2$ if

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b).$$

If f is continuous at every point of a subset D of \mathbb{R}^2 , then f is said to be *continuous on* D . If D is the whole domain, then f is simply said to be *continuous*.

It follows that sums and products of continuous functions are again continuous. That is, there are analogous statements to parts (i) and (ii) of Lemma 3.4.1 for continuity at a point. For quotients one needs to proceed with a bit of care as in part (iii) of Lemma 3.4.1.

Of the examples above,

- $x^2 + y^2 + 7$ is continuous at $(0, 0)$ since the limit is the value of the function there.
- $2x^2y/(x^4 + y^2)$ is **not** continuous at $(0, 0)$ since its value at $(0, 0)$ is not unique.

3.5 Partial Derivatives

3.5.1 Definition of a Partial Derivative

Given a curve $y = F(x)$ we can work out the gradient (slope) of the curve at each point by differentiating F . The formal mathematical definition of a derivative of a function of one variable

³Don't panic! if you have a problem like this in an exam, you will be given a hint.

is

$$\text{Slope} = \frac{dF}{dx} = \lim_{h \rightarrow 0} \left\{ \frac{F(x+h) - F(x)}{h} \right\}$$

For a smooth function with no kinks, this limit exists and is unique at each point x . There is only one unique slope at each point of a smooth curve.

Now, suppose you are standing on a hill. What is the slope of the hill at that point? A little thought should tell you that it is not unique: It depends on the direction in which you are looking.

For example, you could walk around the hill, on a level contour and keeping the same height. In that case the slope is zero in the direction you walk at each point. You could also aim for the summit, in which case you have to go uphill and the gradient will be far from zero!

A hill can be represented as a function of two variables $z = f(x, y)$ (see figure 3.26) below.

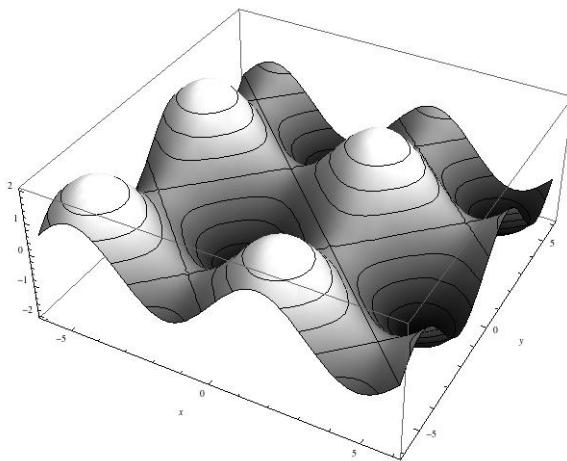


Figure 3.18: Plot of a multivariate function where the height of the function is $z = f(x, y)$ and is given by $f(x, y) = \sin(x) + \sin(y)$

Hence when we ask for the slope on a hill, this translates to asking: what is meant by the derivative of a multivariate function?

The standard way to proceed is as follows. All but one of the variables (directions of walking on the hill) are kept constant and we differentiate with respect to the remaining one. Usually we pick two orthogonal directions and work out the slope in these directions. (You will learn in a later course how to find the slope in all other directions from these two).

In two dimensions this corresponds to working out the slope looking at the slope either

- in the x direction keeping $y=\text{constant}$, or
- in the y direction keeping $x=\text{constant}$.

As these derivatives only involve differentiation with respect to one of the possible variables, they are called **partial derivatives**. They are formally defined as follows:

Definition: (Partial Derivatives) For the function $z = f(x, y)$, the partial derivatives are defined as:

$$\text{Slope in } x \text{ direction} = \left(\frac{\partial f}{\partial x} \right)_{y=\text{constant}} = \lim_{h \rightarrow 0} \left\{ \frac{f(x+h, y) - f(x, y)}{h} \right\}$$

$$\text{Slope in } y \text{ direction} = \left(\frac{\partial f}{\partial y} \right)_{x=\text{constant}} = \lim_{k \rightarrow 0} \left\{ \frac{f(x, y+k) - f(x, y)}{k} \right\}$$

provided that these limits exist.

NB: Note the curly ∂ in the derivative, rather than the usual d . This is to denote that the derivative is of a *multivariate* function and so is *partial* with respect to only one of the variables, rather than with respect to all of them. It is **incorrect** to use an ordinary d in a partial derivative. Practice drawing the ∂ !!!

In practice you compute the partial derivatives in the same way as an ordinary derivative, treating all but the relevant variable as constants. Due to this, very similar rules exist for differentiating combinations of functions.

3.5.2 Partial Derivatives of Combinations of Functions

Partial derivatives of sums, products, quotients, compositions obey similar results derivatives involving single variables.

Lemma 3.5.1. Let $f(x, y)$ and $g(x, y)$ be functions of two variables.

(i) Suppose $\frac{\partial f}{\partial x}(x, y)$ exists. Then for a constant $c \in \mathbb{R}$,

$$\frac{\partial}{\partial x}(cf(x, y)) = c \left(\frac{\partial}{\partial x}f(x, y) \right) = c \frac{\partial f}{\partial x}(x, y),$$

and similarly for the partial derivative with respect to y .

(ii) Suppose $\frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial g}{\partial x}(x, y)$ exist. Then

$$\frac{\partial}{\partial x}(f(x, y) + g(x, y)) = \frac{\partial f}{\partial x}(x, y) + \frac{\partial g}{\partial x}(x, y),$$

and similarly for the partial derivative with respect to y .

(iii) Suppose $\frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial g}{\partial x}(x, y)$ exist. For two functions $f(x, y)$ and $g(x, y)$, we have the product rule:

$$\frac{\partial}{\partial x}(f(x, y)g(x, y)) = \left(\frac{\partial f}{\partial x}(x, y) \right) g(x, y) + f(x, y) \left(\frac{\partial g}{\partial x}(x, y) \right),$$

and similarly for the partial derivative with respect to y .

(iv) Suppose $\frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial g}{\partial x}(x, y)$ exist. For two functions $f(x, y)$ and $g(x, y)$, we have the quotient rule:

$$\frac{\partial}{\partial x} \left(\frac{f(x, y)}{g(x, y)} \right) = \frac{\left(\frac{\partial f}{\partial x}(x, y) \right) g(x, y) - f(x, y) \left(\frac{\partial g}{\partial x}(x, y) \right)}{(g(x, y))^2},$$

and similarly for the partial derivative with respect to y .

Proof. We only prove part (i) to show that the proofs are analogous to those when the function has only one variable. Proofs for this lemma are **not** required.

$$\begin{aligned} \frac{\partial}{\partial x}(cf(x, y)) &= \lim_{h \rightarrow 0} \frac{cf(x+h, y) - cf(x, y)}{h} \\ &= c \left(\lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} \right) \\ &= c \left(\frac{\partial f}{\partial x}(x, y) \right). \end{aligned}$$

□

Example: $f(x, y) = x^2 - 1 + y + \sin(xy)$:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial x}(\sin(xy)) = 2x - 0 + 0 + y \cos(xy) = 2x + y \cos(xy).$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2) - \frac{\partial}{\partial y}(1) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial y}(\sin(xy)) = 0 - 0 + 1 + x \cos(xy) = 1 + x \cos(xy).$$

Example: Find the partial derivatives of

$$f(x, y) = \cos(x) \sin(y) + y \exp(x).$$

$$\frac{\partial f}{\partial x}(x, y) = \left(\frac{d}{dx} \cos(x) \right) \sin(y) + y \left(\frac{d}{dx} \exp(x) \right) = -\sin(x) \sin(y) + y \exp(x).$$

$$\frac{\partial f}{\partial y}(x, y) = \cos(x) \left(\frac{d}{dy} \sin(y) \right) + \left(\frac{d}{dy} y \right) \exp(x) = \cos(x) \cos(y) + \exp(x).$$

Example: Find the partial derivatives of

$$f(x, y) = \frac{\cos y}{x}.$$

$$\frac{\partial f}{\partial x}(x, y) = \frac{x \frac{\partial}{\partial x} (\cos(y)) - \cos(y) \frac{\partial}{\partial x} (x)}{x^2} = \frac{x \times 0 - \cos(y) \times 1}{x^2} = \frac{-\cos(y)}{x^2}.$$

$$\frac{\partial f}{\partial y}(x, y) = \frac{x \frac{\partial}{\partial y} (\cos(y)) - \cos(y) \frac{\partial}{\partial y} (x)}{x^2} = \frac{x \times (-\sin y) - \cos(y) \times 0}{x^2} = \frac{-\sin(y)}{x}.$$

3.5.3 Alternative Notation for Partial Derivatives

When a partial derivative of a function $z = f(x, y)$ is evaluated at a point $(x, y) = (a, b)$, then you often see the following equivalent notation:

$$\frac{\partial z}{\partial x} \Big|_{(a,b)} \quad \text{or} \quad \left(\frac{\partial}{\partial x} f(x, y) \right) \Big|_{(a,b)} \quad \text{or} \quad \partial_x f(x, y) \Big|_{(a,b)} \quad \text{or} \quad z_x(a, b)$$

for the x -partial derivatives and

$$\frac{\partial z}{\partial y} \Big|_{(a,b)} \quad \text{or} \quad \left(\frac{\partial}{\partial y} f(x, y) \right) \Big|_{(a,b)} \quad \text{or} \quad \partial_y f(x, y) \Big|_{(a,b)} \quad \text{or} \quad z_y(a, b)$$

for the y -partial derivatives.

Example: For the function $f(x, y) = x^2 - 1 + y + \sin(xy)$, calculate the slope in the x -direction and y -direction at the point $(x, y) = (1, 2)$.

$$\frac{\partial f}{\partial x} \Big|_{(1,2)} = 2x + y \cos(xy) \Big|_{(x,y)=(1,2)} = 2 + 2 \cos(2) = 2(1 + \cos 2).$$

$$\frac{\partial f}{\partial y} \Big|_{(1,2)} = 1 + x \cos(xy) \Big|_{(x,y)=(1,2)} = 1 + \cos(2).$$

3.5.4 Partial Derivatives in Higher Dimensions

Partial derivatives in higher dimensions are defined as follows.

Definition: For a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ given by $f(x_1, \dots, x_n)$, we have n first order partial derivatives, one with respect to each variable x_1, \dots, x_n , is given by

$$\frac{\partial f}{\partial x_j}(x_1, \dots, x_n) = \lim_{h \rightarrow 0} \left[\frac{f(x_1, \dots, x_{j-1}, x_j + h, x_{j+1}, \dots, x_n) - f(x_1, \dots, x_n)}{h} \right].$$

provided the limit exists.

We don't normally use the definition to evaluate the partial derivatives of particular functions, any more than we use the definition for a function of one variable to calculate the derivative of that function.

For combinations of functions in higher dimensions, the following lemma, analogous to Lemma 2.5.1, holds:

Lemma 3.5.2. Let $f(x_1, \dots, x_n)$ and $g(x_1, \dots, x_n)$ be functions of n -variables. Then

1. For a constant $c \in \mathbb{R}$ and for each $1 \leq j \leq n$,

$$\frac{\partial}{\partial x_j}(cf(x_1, \dots, x_n)) = c \frac{\partial f}{\partial x_j}(x_1, \dots, x_n);$$

2. The derivative of a sum is the sum of the derivatives, namely

$$\frac{\partial}{\partial x_j}(f(x_1, \dots, x_n) + g(x_1, \dots, x_n)) = \frac{\partial f}{\partial x_j}(x_1, \dots, x_n) + \frac{\partial g}{\partial x_j}(x_1, \dots, x_n)$$

for each $1 \leq j \leq n$;

3. We have the product rule:

$$\begin{aligned} & \frac{\partial}{\partial x_j}(f(x_1, \dots, x_n)g(x_1, \dots, x_n)) \\ &= \left(\frac{\partial f}{\partial x_j}(x_1, \dots, x_n) \right) g(x_1, \dots, x_n) + f(x_1, \dots, x_n) \left(\frac{\partial g}{\partial x_j}(x_1, \dots, x_n) \right) \end{aligned}$$

for each $1 \leq j \leq n$;

4. We have the quotient rule:

$$\begin{aligned} & \frac{\partial}{\partial x_j} \left(\frac{f(x_1, \dots, x_n)}{g(x_1, \dots, x_n)} \right) \\ &= \frac{\left(\frac{\partial f}{\partial x_j}(x_1, \dots, x_n) \right) g(x_1, \dots, x_n) - f(x_1, \dots, x_n) \left(\frac{\partial g}{\partial x_j}(x_1, \dots, x_n) \right)}{(g(x_1, \dots, x_n))^2} \end{aligned}$$

for each $1 \leq j \leq n$.

Really the only difference between this and the lemma for 2 variables (above), is the number of variables in the arguments of the function, and you remember that you treat all the variables as constants, except the one you are differentiating with respect to.

Example: Find all the partial derivatives of the function

$$f(x, y, z) = 5x^3y^2z - y^2z^2 + 3x - 4$$

at $(-1, 0, 2)$.

Recall that when we differentiate with respect to x , we treat all other variables as constants and then differentiate normally. So, to calculate $\frac{\partial f}{\partial x}$, we treat y and z as constants and see that

$$\begin{aligned}\frac{\partial f}{\partial x}(x, y, z) &= 5 \frac{\partial}{\partial x}(x^3 y^2 z) - \frac{\partial}{\partial x}(y^2 z^2) + 3 \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial x}(4) \\ &= 15x^2 y^2 z + 3.\end{aligned}$$

Similarly, to calculate $\frac{\partial f}{\partial y}$, we treat x and z as constants and see that

$$\begin{aligned}\frac{\partial f}{\partial y}(x, y, z) &= 5 \frac{\partial}{\partial y}(x^3 y^2 z) - \frac{\partial}{\partial y}(y^2 z^2) + 3 \frac{\partial}{\partial y}(x) - \frac{\partial}{\partial y}(4) \\ &= 10x^3 y z - 2y z^2.\end{aligned}$$

Similarly, to calculate $\frac{\partial f}{\partial z}$, we treat x and y as constants and see that

$$\begin{aligned}\frac{\partial f}{\partial z}(x, y, z) &= 5 \frac{\partial}{\partial z}(x^3 y^2 z) - \frac{\partial}{\partial z}(y^2 z^2) + 3 \frac{\partial}{\partial z}(x) - \frac{\partial}{\partial z}(4) \\ &= 5x^3 y^2 - 2y^2 z.\end{aligned}$$

Then $\frac{\partial f}{\partial x}(-1, 0, 2) = 3$, $\frac{\partial f}{\partial y}(-1, 0, 2) = 0$, and $\frac{\partial f}{\partial z}(-1, 0, 2) = 0$.

3.5.5 Chain rule

Recall that in the single variable case, if we were given a function $z = z(x)$ where $x = x(t)$, then we could express the derivative of the composite function $z = z(x(t))$ with respect to t using the chain rule:

$$\frac{dz}{dt} = \lim_{h \rightarrow 0} \frac{z(x(t+h)) - z(x(t))}{h} = \lim_{h \rightarrow 0} \left(\frac{z(x(t+h)) - z(x(t))}{x(t+h) - x(t)} \right) \left(\frac{x(t+h) - x(t)}{h} \right) = \frac{dz}{dx} \frac{dx}{dt}.$$

(This follows from the formula for the product of individually well-defined limits.)

What is the analogous situation for multivariable functions? We can illustrate this with a definite example.

Example: Suppose you are walking in a mountainous region with a map. Your horizontal coordinates on the map are given by (x, y) and your vertical height above sea level is $z = f(x, y)$.

Now suppose you take a path so that your horizontal coordinates are given at time t by $(x, y) = (x(t), y(t))$. Your height at time t is then given by

$$z = f(x(t), y(t)).$$

So it is possible to now express z as a function of t only, i.e., $z = z(t)$. The question we pose is, give this, how fast is your height changing with time, i.e., what is dz/dt ? ⁴

⁴Note that in this example, when z is expressed in terms of the ultimate independent variables we find that there is only 1 independent variable. Therefore the final derivative with respect to that variable is a total one.

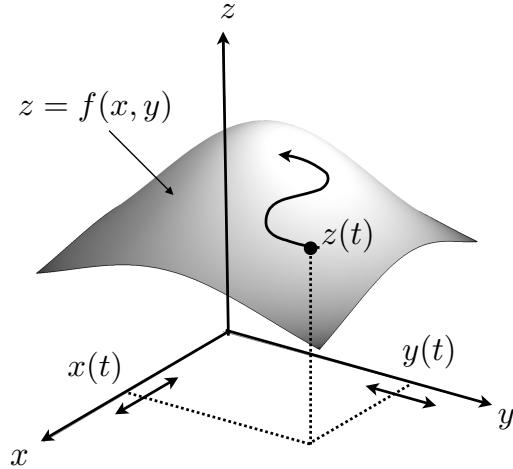


Figure 3.19: A schematic representation of the motion of a point on the surface $z = f(x, y)$ that is given when we allow the coordinates of the point (x, y, z) to depend on an additional variable t so that the x coordinate moves as $x = x(t)$ and $y = y(t)$. The chain rule can then be used to calculate how the rate of change of z with respect to t may be expressed. For example if t is time, then dz/dt gives the vertical speed.

The solution is obtained by a series of careful steps.

If we treat z as a function of just t we can write down the definition of dz/dt as

$$\begin{aligned}
 \frac{dz}{dt} &= \lim_{h \rightarrow 0} \frac{z(t+h) - z(t)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{z(x(t+h), y(t+h)) - z(x(t), y(t))}{h} \\
 &= \lim_{h \rightarrow 0} \frac{z(x(t+h), y(t+h)) + [-z(x(t), y(t+h)) + z(x(t), y(t+h))] - z(x(t), y(t))}{h} \\
 &= \lim_{h \rightarrow 0} \frac{z(x(t+h), y(t+h)) - z(x(t), y(t+h))}{h} + \lim_{h \rightarrow 0} \frac{z(x(t), y(t+h)) - z(x(t), y(t))}{h} \\
 &= \frac{\partial z}{\partial x}(x(t), y(t)) \frac{dx}{dt}(t) + \frac{\partial z}{\partial y}(x(t), y(t)) \frac{dy}{dt}(t).
 \end{aligned}$$

The last line follows by comparison with the 1-d chain rule above.

Of course the result we have derived here is general and not dependent on the mountain walk context.

So, the Chain Rule says that if $z = f(x, y)$ and $x = x(t)$ and $y = y(t)$ for differentiable f, x and y , and we compose the functions as $\mathbb{R}_{(t)} \rightarrow \mathbb{R}_{(x,y)}^2 \rightarrow \mathbb{R}_{(z)}$, we have the following result:

Lemma 3.5.3. *If $z = z(x, y)$ and $x = x(t)$ and $y = y(t)$ for differentiable f, x and y , we have*

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}.$$

(3.1)

Example: Let $z = \sin(x + y^2)$, and $x = t^2$, $y = e^t$. What is dz/dt ?

We have

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = e^t$$

$$\frac{\partial z}{\partial x} = \cos(x + y^2), \quad \frac{\partial z}{\partial y} = 2y \cos(x + y^2)$$

Hence

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = 2t \cos(x + y^2) + 2ye^t \cos(x + y^2) = \cos(t^2 + e^{2t}) (2t + 2e^{2t})$$

Note that the last result could have been obtained by directly substituting first for $x = t^2$ and $y = e^t$ to get $z(t) = \sin(t^2 + e^t)$ and then differentiating z as a simple 1-d function with respect to t .

So in this type of problem we see that x and y are **intermediate variables** and t is the actual independent variable.

Suppose now that we have a composition $\mathbb{R}_{(s,t)}^2 \rightarrow \mathbb{R}_{(x,y)}^2 \rightarrow \mathbb{R}_{(z)}$, i.e., we have a function $z = f(x, y)$, but that we have intermediate variable $x = x(s, t)$ and $y = y(s, t)$ with the independent variables being now s and t , so that

$$z = f(x(s, t), y(s, t)) = z(s, t).$$

The function z is now a multivariable function, depending on two independent variables s and t . The derivatives of z with respect to s and t will now be partials. Hence the analogous chain rule questions to ask now is what are $\partial z / \partial s$ and $\partial z / \partial t$?

A derivation of the chain rule (not required) delivers the following results:

Lemma 3.5.4. If $z = z(x, y)$ and $x = x(s, t)$ and $y = y(s, t)$ for differentiable f, x and y , we have

$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}.} \quad (3.2)$$

Example: Consider the functions:

$$z(x, y) = xe^y,$$

with

$$x = s + 3t, \quad y = s - 2t.$$

- We have:

$$\frac{\partial z}{\partial x} = e^y, \quad \frac{\partial z}{\partial y} = xe^y,$$

$$\frac{\partial x}{\partial s} = 1, \quad \frac{\partial x}{\partial t} = 3, \quad \frac{\partial y}{\partial s} = 1, \quad \frac{\partial y}{\partial t} = -2.$$

- Hence the chain rule becomes:

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \\ &= \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = e^y + xe^y = e^y(1 + x) \\ &= e^{s-2t}(1 + s + 3t); \end{aligned}$$

- and

$$\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \\
&= 3 \frac{\partial z}{\partial x} - 2 \frac{\partial z}{\partial y} = 3e^y - 2xe^y = e^y(3 - 2x) \\
&= e^{s-2t}(3 - 2s - 6t).
\end{aligned}$$

Example: Transformation to polar coordinates. Let $z = z(x, y)$ where $x = r \cos \theta$ and $y = r \sin \theta$. Find $\partial z / \partial r$ and $\partial z / \partial \theta$.

We can use the chain rule from equation (3.2) with $s = r$ and $t = \theta$. Then we have

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r}$$

and

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta}$$

But

$$\frac{\partial x}{\partial r} = \cos \theta \quad \frac{\partial y}{\partial r} = \sin \theta$$

$$\frac{\partial x}{\partial \theta} = -r \sin \theta \quad \frac{\partial y}{\partial \theta} = r \cos \theta$$

Hence the r and θ partial derivatives of z may be expressed in terms of the x and y partial derivatives of z as

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta$$

and

$$\frac{\partial z}{\partial \theta} = -\frac{\partial z}{\partial x} r \sin \theta + \frac{\partial z}{\partial y} r \cos \theta$$

Note: A convenient way to express the chain rule (3.2), that will be useful later, is to recognise that it can be written as a matrix multiplication:

$$\left(\begin{array}{cc} \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{array} \right) = \left(\begin{array}{cc} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{array} \right) \left(\begin{array}{cc} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{array} \right) \quad (3.3)$$

3.5.6 Graphical Way to Work out the Chain Rule

There is a simplified graphical way to remember the chain rules for arbitrary compositions $\mathbb{R}_{(t_1, t_2, \dots, t_m)} \rightarrow \mathbb{R}_{(x_1, x_2, \dots, x_n)} \cdots \rightarrow \mathbb{R}_{(z)}$. For the cases studied above we can construct a chart of chain links that indicates how all the variables depend on one another.

Example: When $z = z(x, y)$ and $x = x(t)$, $y = y(t)$ we have:

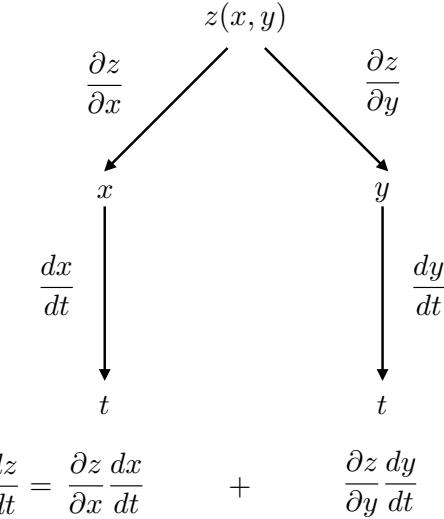


Figure 3.20: Chain rule for $z = z(x, y)$, when $x = x(t)$ and $y = y(t)$.

The rules are simple:

- Each link in the chain is a partial derivative.
- Successive links on a particular chain are multiplied.
- Overall contributions from each chain are added.

Example: $z = z(x, y) = x^2y + y^3$ with $y = \cos x$.

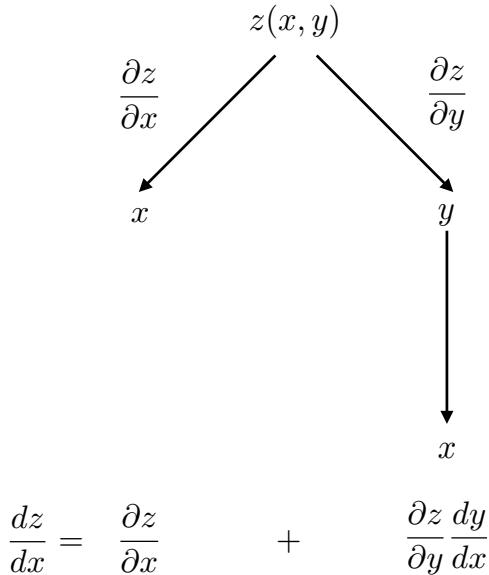


Figure 3.21: Chain rule for $z = z(x, y)$, when $y = y(x)$

Note that on substituting $y = \cos x$ into $z = f(x, y)$ we end up with $z = f(x, \cos x) = z(x)$ only. In other words, if we make the substitution for y in the first place, we find that z is just a function of a single variable x . Hence the final derivative we seek using the chain rule will just be a total derivative dz/dx .

However, if we just use the chain rule, on the way we do need to differentiate with respect to each of the x and y variables partially. Hence we will see $\partial z/\partial x$ appear within the chain rule: this just means that we must differentiate with respect to the first of the original variables x , keeping the other one, y , as a constant. However since $y = y(x)$ only, any derivatives of y with respect to x will also be total and so we can expect dy/dx to appear too.

Hence, when we apply the diagram to derive the chain rule, we find:

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \frac{dy}{dx}.$$

Thus we have

$$\frac{\partial z}{\partial x} = 2xy, \quad \frac{\partial z}{\partial y} = x^2 + 3y^2.$$

We also have

$$\frac{dy}{dx} = -\sin x.$$

So

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \frac{dy}{dx} = 2xy(x) + (x^2 + 3y(x)^2)(-\sin x) = 2x \cos x - (x^2 + 3 \cos^2 x) \sin x.$$

Exercise 4 U: Check that if you do make the substitution $y = \cos x$ and then differentiate $z(x) = f(x, \cos x)$ with respect to x , that you obtain the same answer as above.

Example: Suppose we have $z = z(x, y)$, but $x = x(s, t, u)$ but $y = y(s, u)$ only, and $u = u(v)$. We can represent the functional relationships using the chain diagram below:

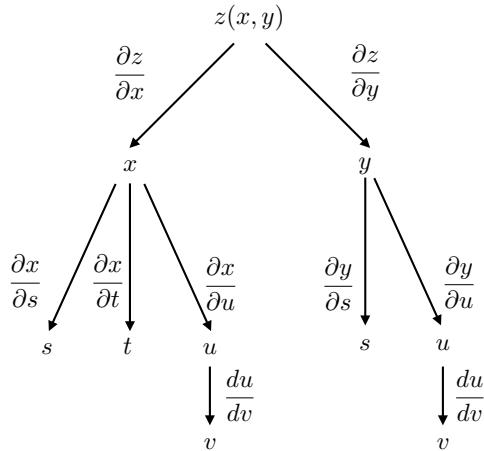


Figure 3.22: Chain rule for $z = z(x, y)$, when $x = x(s, t, u)$ and $y = y(s, u)$.

Hence following the above shortcut rules, it is easy to see that ultimately $z = z(s, t, v)$ and that.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} \frac{\partial u}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \frac{\partial u}{\partial v}$$

3.5.7 Integration of Partial Derivatives

When we integrate an ordinary derivative, from the Fundamental Theorem of Calculus we have:

$$\int \frac{dy}{dx} dx = y(x) + C,$$

where C is an arbitrary constant. This is because

$$\frac{d}{dx} (y(x) + C) = \frac{dy}{dx},$$

i.e., the derivative of the constant with respect to x is zero.

When we integrate a partial derivative, we get something similar, but the role of the constant is taken by an arbitrary function of the variable(s) which has (have) been held constant in the partial differentiation. Consider a function $z(x, y)$ which has partial derivatives:

$$\int \left(\frac{\partial z}{\partial x} \right)_{y=\text{const}} dx = z(x, y) + g(y),$$

where $g(y)$ is an arbitrary function of y . This is because, when we differentiate the right hand side, $z(x, y) + g(y)$, partially with respect to x , keeping y constant, any arbitrary function of y has a zero partial derivative with respect to x :

$$\left(\frac{\partial}{\partial x} (z(x, y) + g(y)) \right)_{y=\text{const}} = \left(\frac{\partial z}{\partial x} \right)_{y=\text{const}} + 0 = \left(\frac{\partial z}{\partial x} \right)_{y=\text{const}}.$$

Similarly we have:

$$\int \left(\frac{\partial z}{\partial y} \right)_{x=\text{const}} dy = z(x, y) + f(x),$$

where $f(x)$ is an arbitrary function of x . This is because, when we differentiate the right hand side, $z(x, y) + f(x)$, partially with respect to y , keeping x constant, any arbitrary function of x has a zero partial derivative with respect to y :

$$\left(\frac{\partial}{\partial y} (z(x, y) + f(x)) \right)_{x=\text{const}} = \left(\frac{\partial z}{\partial y} \right)_{x=\text{const}} + 0 = \left(\frac{\partial z}{\partial y} \right)_{x=\text{const}}.$$

Note that when we are given two partial derivatives of the same function z we can integrate both derivatives separately as above with respect to the relevant variable to obtain two apparently different representations for z . The resolution of this paradox is that by exploit the arbitrary nature of the functions $g(y)$ and $f(x)$ to “pattern match” the two expressions $z(x, y) + g(y)$ and $z(x, y) + f(x)$ to obtain identical expressions for both integrals:

Example: Suppose you are given that a function $z(x, y)$ exists such that

$$\left(\frac{\partial z}{\partial x} \right)_{y=\text{const}} = 2x + 2y, \quad \left(\frac{\partial z}{\partial y} \right)_{x=\text{const}} = 3y^2 + 2x.$$

Then we have, respectively:

$$\int \left(\frac{\partial z}{\partial x} \right)_{y=\text{const}} dx = x^2 + 2xy + g(y),$$

$$\int \left(\frac{\partial z}{\partial y} \right)_{x=\text{const}} dy = y^3 + 2xy + f(x).$$

Given that we are told that $z(x, y)$ exists, both of these expressions should be the same. Clearly both have a $2xy$, but otherwise they look different. However, a careful term-by-term comparison shows that can be made the same if we match up using the arbitrary functions as:

$$f(x) = x^2 + C, \quad g(y) = y^3 + C,$$

where C is an arbitrary constant. Note that the arbitrariness of the constant C maintains the arbitrariness of g and f . Whence we have a single expression for $z(x, y)$, unique up to arbitrary constants (as in the single variable integration case):

$$z(x, y) = x^2 + y^3 + 2xy + C.$$

You can check for yourselves that if you differentiate this function $z(x, y)$ partially with respect to x and y independently that you cover the partial derivatives that you started off with above.

We can use integration of partial derivatives to arrive at the following neat result:

Theorem 3.5.5. *Let $u(x, y)$ be a function of the variables x and y , and assume that*

$$\frac{\partial u}{\partial x}(x, y) = \frac{\partial u}{\partial y}(x, y) = 0$$

for all $(x, y) \in \mathbb{R}^2$. Then, $u(x, y)$ is constant on \mathbb{R}^2 .

Proof. We start from the assumption that $\frac{\partial u}{\partial x}(x, y) = 0$ for all $(x, y) \in \mathbb{R}^2$ (i.e., everywhere and not just at a point).

- Integrating with respect to x , we see that $u(x, y) = g(y)$, since (from the way integration works above) the constant of integration with respect to x must depend only on y .
- Using now the assumption that $\frac{\partial u}{\partial y}(x, y) = 0$ everywhere, this means that $\frac{\partial u}{\partial y}(x, y) = g'(y) = 0$ everywhere.
- Hence, integrating $g'(y) = 0$ with respect to y , since $g'(y)$ is just a function of y , the ordinary 1-D rules of integration apply. Hence $g(y) = \text{constant}$ everywhere, and hence that $u(x, y)$ must be constant everywhere.

□

3.5.8 Example of Use of Chain Rule: First Order PDEs

The chain rule is used in the solution of first order exact ordinary differential equations. Chain rules are also useful in the solution of partial differential equations (PDE) by converting the equations from something complicated when expressed in one set of variables, to something simpler to solve in another set.

Example: Suppose you are looking for a solution to the first order PDE:⁵

$$\frac{\partial z}{\partial t} + c \frac{\partial z}{\partial x} = 0,$$

where c is a constant.

⁵DON'T PANIC: you won't be expected to do this until MATH2038! This is just an example to show you how the chain rule is useful.

Suppose (Rabbit!) the we make a change of variable

$$\xi = x - ct, \quad \eta = x + ct$$

so that $z(x, t) = z(x(\xi, \eta), t(\xi, \eta)) = z(\xi, \eta)$. We can use the chain rule to rewrite the partial derivatives with respect to x and t in the following form,

$$\frac{\partial z}{\partial t} = \frac{\partial \xi}{\partial t} \frac{\partial z}{\partial \xi} + \frac{\partial \eta}{\partial t} \frac{\partial z}{\partial \eta} = -c \frac{dz}{d\xi} + c \frac{dz}{d\eta},$$

and

$$\frac{\partial z}{\partial x} = \frac{\partial \xi}{\partial x} \frac{\partial z}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial z}{\partial \eta} = \frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}.$$

Hence the equation simplifies to

$$\frac{\partial z}{\partial t} + c \frac{\partial z}{\partial x} = -c \frac{dz}{d\xi} + c \frac{dz}{d\eta} + c \left(\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta} \right) = 2c \frac{\partial z}{\partial \eta} = 0.$$

Hence the PDE simplifies greatly and becomes:

$$\frac{\partial z}{\partial \eta} = 0.$$

Thus if we integrate with respect to η and remember that z also depends on ξ we see that

$$z = F(\xi),$$

In other words z is just an arbitrary function of ξ only.

Converting back to the original (x, t) coordinates we have $z(x, t) = F(x - ct)$. The actual function F could be determined by specifying the initial profile of the wave at $t = 0$. If we set this to be $z(x, 0) = z_0(x)$ we would then find $z(x, 0) = F(x) = z_0(x)$. Hence the solution would then be

$$z(x, t) = z_0(x - ct).$$

The solution of this equation represents a wave of amplitude z traveling to the right (positive x) with speed c at time t .

3.5.9 Jacobian Matrix and the Chain Rule

In this section we consider the so-called **Jacobian matrices** of partial derivatives and show how they can be used to encode and to represent the chain rule. We then on to examine the relationship between the inverses of these matrices and so establish conditions for an inverse of a multivariable function to have a local inverse.

Consider the $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ mapping

$$\mathbf{f} : (x, y) \rightarrow (f_1(x, y), f_2(x, y)).$$

Here \mathbf{f} is called a **vector** function, since the result is an ordered set of two functions $f_1(x, y)$ and $f_2(x, y)$. These could be the component of a vector whose direction and magnitude depends on the (x, y) position in space, e.g.,

$$\mathbf{f}(x, y) = f_1(x, y)\mathbf{i} + f_2(x, y)\mathbf{j}.$$

The function \mathbf{f} could represent, say, the wind velocity at sea level at any point on a map.

Definition: (Jacobian Matrix) The Jacobian Matrix of $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined to be

$$J_{\mathbf{f}}(x, y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

This is a matrix of partial derivatives. Note the order of the entries. The independent variables (x, y) go on the bottom of each partial derivative as appropriate. Partial derivatives of the function f_i are in the i^{th} row. The j^{th} column should contain the same independent variable.

Now suppose that we make a change of variable from $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$\mathbf{g} : (s, t) \rightarrow (x(s, t), y(s, t)).$$

Then we can use the chain rule to write

$$\frac{\partial f_1}{\partial s} = \frac{\partial f_1}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f_1}{\partial y} \frac{\partial y}{\partial s}. \quad (3.4)$$

$$\frac{\partial f_1}{\partial t} = \frac{\partial f_1}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f_1}{\partial y} \frac{\partial y}{\partial t}. \quad (3.5)$$

A little bit of thought shows that we can combine these two equations and represent them as components in the following matrix multiplication:

$$\begin{pmatrix} \frac{\partial f_1}{\partial s} & \frac{\partial f_1}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} \quad (3.6)$$

Likewise we could do the same for f_2 and obtain

$$\begin{pmatrix} \frac{\partial f_2}{\partial s} & \frac{\partial f_2}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} \quad (3.7)$$

Due to the identity of the final 2×2 matrix in both these expressions, we could again combine these results and write them as a 2×2 matrix multiplication:

$$\begin{pmatrix} \frac{\partial f_1}{\partial s} & \frac{\partial f_1}{\partial t} \\ \frac{\partial f_2}{\partial s} & \frac{\partial f_2}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix}. \quad (3.8)$$

Or in terms of the definition of the Jacobian Matrix the chain rule for the mapping

$$\mathbb{R}_{(s,t)}^2 \xrightarrow{\mathbf{g}} \mathbb{R}_{(x,y)}^2 \xrightarrow{\mathbf{f}} \mathbb{R}_{(f_1,f_2)}^2,$$

may be summarised as:

$$J_{\mathbf{f} \circ \mathbf{g}}(s, t) = J_{\mathbf{f}}(x, y) J_{\mathbf{g}}(s, t) \quad (3.9)$$

This can be generalised to arbitrary dimensional functions as follows.

Definition: (Jacobian Matrix in Higher Dimensions) Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^p$ be a function where $n \geq 1$ and $p \geq 1$. Assume $\frac{\partial f_i}{\partial x_j}$ exists for all $1 \leq i \leq p$ and $1 \leq j \leq n$ at a point $(x_1, \dots, x_n) \in D$. The *Jacobian matrix* at (x_1, \dots, x_n) is the $p \times n$ matrix of first order partial derivatives of the component functions of $F(x_1, \dots, x_n)$. Specifically, for

$$F(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_p(x_1, \dots, x_n)),$$

we set

$$J_F(x_1, \dots, x_n) = \left(\frac{\partial f_i}{\partial x_j} \right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_1, \dots, x_n) & \cdots & \frac{\partial f_1}{\partial x_n}(x_1, \dots, x_n) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \frac{\partial f_p}{\partial x_1}(x_1, \dots, x_n) & \cdots & \frac{\partial f_p}{\partial x_n}(x_1, \dots, x_n) \end{pmatrix}.$$

Next, we state the most general form of the Chain Rule.

Theorem 3.5.6 (The Chain Rule). *Let $G : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be differentiable at a point $\mathbf{a} \in \mathbb{R}^m$ and $F : \mathbb{R}^n \rightarrow \mathbb{R}^p$ be differentiable at the point $\mathbf{b} = G(\mathbf{a})$. Then $F \circ G : \mathbb{R}^m \rightarrow \mathbb{R}^p$ is differentiable at \mathbf{a} and*

$$J_{F \circ G}(\mathbf{a}) = J_F(\mathbf{b}) \cdot J_G(\mathbf{a}).$$

Example: Jacobians need not be square matrices! For example, if we have a function:

$$\begin{aligned} F : \mathbb{R}^n &\rightarrow \mathbb{R}^p & F &= (f_1(x_1, \dots, x_n), \dots, f_p(x_1, \dots, x_n)) \\ G : \mathbb{R}^p &\rightarrow \mathbb{R}^q & G &= (g_1(f_1, \dots, f_p), \dots, g_q(f_1, \dots, f_p)) \end{aligned}$$

for arbitrary integer p, q, n , not necessarily equal, is then the chain rule is the matrix product

$$\underbrace{\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_q}{\partial x_1} & \cdots & \frac{\partial g_q}{\partial x_n} \end{pmatrix}}_{q \times n} = \underbrace{\begin{pmatrix} \frac{\partial g_1}{\partial f_1} & \cdots & \frac{\partial g_1}{\partial f_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_q}{\partial f_1} & \cdots & \frac{\partial g_q}{\partial f_p} \end{pmatrix}}_{q \times p} \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial x_1} & \cdots & \frac{\partial f_p}{\partial x_n} \end{pmatrix}}_{p \times n}$$

Example: The ordinary total derivative for compositions of single variable $\mathbb{R} \rightarrow \mathbb{R}$ functions $g(x) = g(f(x))$ is a special case of this result with $q = n = p = 1$. For then the above matrices degenerate to a single entry (with partial derivatives becoming total derivatives):

$$\frac{dg}{dx} = \frac{dg}{df} \frac{df}{dx}.$$

3.5.10 Relationships between Partial Derivatives and Jacobians

Recall that in 1 dimension, if $y = y(x)$, then the inverse function, if it exists, may be denoted by $x = x(y)$. Inserting the definition of y into the inverse function, we obtain the (tautological) result:

$$x = x(y(x)).$$

Assuming both the function and its inverse to be differentiable, if we differentiate both sides with respect to x we end up with

$$\frac{dx}{dx} = \frac{dx}{dy} \frac{dy}{dx} \quad 1 = \frac{dx}{dy} \frac{dy}{dx} \quad \frac{dy}{dx} = 1 \Big/ \left(\frac{dx}{dy} \right).$$

Does the same result hold for partial derivatives, i.e., given a differentiable $z = f(x, y)$, does

$$\frac{\partial z}{\partial x} = 1 \Big/ \left(\frac{\partial x}{\partial z} \right)?$$

In general, the answer is an emphatic **NOOOOOOOOOOOOOOO!**

The analogous answer (and question!) is obtained by considering the following example:

Example: Consider the following $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ function

$$\mathbf{f} : (x, y) \rightarrow (f_1, f_2) = (f_1(x, y), f_2(x, y)) = \mathbf{f}(\mathbf{x})$$

and suppose that locally, it has an inverse function

$$\mathbf{x} : (f_1, f_2) \rightarrow (x, y) = (x(f_1, f_2), y(f_1, f_2)) = \mathbf{x}(\mathbf{f})$$

Now, substituting the inverse function into the function (as in the 1-D case) we have

$$\mathbf{x} = \mathbf{x}(\mathbf{f}(\mathbf{x})).$$

The 2-D chain rule with $n = p = q = 2$ and $g_1 \rightarrow x$ and $g_2 \rightarrow y$ gives

$$\begin{pmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial f_1} & \frac{\partial x}{\partial f_2} \\ \frac{\partial y}{\partial f_1} & \frac{\partial y}{\partial f_2} \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

Now since x and y are independent variables,

$$\begin{pmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial f_1} & \frac{\partial x}{\partial f_2} \\ \frac{\partial y}{\partial f_1} & \frac{\partial y}{\partial f_2} \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

Since the LHS is the identity matrix, the matrices on the RHS must be inverses of each other, i.e.,

$$\begin{pmatrix} \frac{\partial x}{\partial f_1} & \frac{\partial x}{\partial f_2} \\ \frac{\partial y}{\partial f_1} & \frac{\partial y}{\partial f_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}^{-1}. \quad (3.10)$$

Hence the matrices of the Jacobians of the inverse functions are inverses of each other. This is a general result for $n = p = q$ maps leading to square Jacobian matrices.

So in general, for multivariable functions, you can see that **not much** can be said about the relationship between individual partial derivatives $\partial f / \partial x$ and $\partial x / \partial f$.

What can be said is that the Jacobian matrices of the function and the inverse function are inverses of each other!

Remark: Note that in the case of $n = p = q = 1$ for single variable functions, the matrices collapse to just as single entry, so the matrix inverse is just the inverse of a scalar (i.e., the reciprocal), the partials become total derivatives and so we recover the single variable ratio result above:

$$\frac{dx}{df} = 1 / \left(\frac{df}{dx} \right).$$

A rather useful result follows from the above derivation.

Definition: (Jacobian Determinant) Let $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n = (f_1(x_1, \dots, x_n), f_n(x_1, \dots, x_n))$, then the **Jacobian Determinant** is just the determinant of the Jacobian matrix J_f and is written as:

$$\det J_{\mathbf{f}} = \frac{\partial(f_1 \dots f_n)}{\partial(x_1 \dots x_n)}$$

Then, e.g., in the case of $n = 2$, the following theorem holds.

Theorem 3.5.7. (Inverse Functions, Implicit Function Theorem) If $f_1 = f_1(x, y)$ and $f_2 = f_2(x, y)$ have continuous partial derivatives at a point $P : (x, y) = (a, b)$, then provided that

$$\frac{\partial(f_1, f_2)}{\partial(x, y)} \Big|_P \neq 0,$$

there exists locally a set of inverse functions $x = x(f_1, f_2), y = y(f_1, f_2)$.

Very Sketchy Proof: (Not required, but outlined here for information. Note that we are using $x_1 = x, x_2 = y$ in the $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ version of the theorem.) The result (3.10) assumed that inverse functions existed. Hence in that instance, the Jacobian matrices will have inverses. For a 2×2 matrix, a necessary condition for the inverse to exist is that its determinant does not vanish. Hence if the Jacobian determinant does not vanish, the inverse matrix will exist, and so by implication the inverse functions.

Note that the theorem does not find the inverse functions, it just guarantees that they exist. They may be complicated or even impossible to write down explicitly. Hence the use of the term **implicit** function theorem.

Example: Given $x = r \cos \theta$ and $y = r \sin \theta$, find where the inverse functions $r(x, y)$ and $\theta(x, y)$ exist.

From the Implicit Function Theorem, we require the Jacobian determinant to be non-vanishing. We have

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r$$

Hence we see that inverse functions exist locally wherever $r \neq 0$. This is the polar origin, at which θ is ambiguous: it can take any value from 0 to 2π .

The determinant of a Jacobian matrix will return again later when we consider changes of variables in integrals over multivariate functions.

And finally, Jacobians should not be confused with Jacobites (who were 18th supporters of the claim of the descendants of James II of England (or VII of Scotland) to the throne of England and Scotland). :-) Jacobians are in fact, named after the mathematician Carl Gustav Jacobi (1804-1851).

3.6 Calculation of Derivatives of Implicit Functions.

We can use the chain rule and Jacobians to calculate (partial) derivatives associated with implicit functions.

3.6.1 Calculation of a Total Derivative

Consider the implicit representation:

$$F(x, y) = 0,$$

for a known, given, real function F for which the partial derivatives with respect to the real variables x and y may be found. Suppose that, in principle, it is possible to rewrite this function with y as the subject, i.e., $y = y(x)$, locally in some interval around $x = x_0$. What is the derivative of the function y with respect to x , i.e., dy/dx ?

We can work this out using the chain rule *without needing to even solve initially and explicitly for $y = y(x)$!*

- First, if $y = y(x)$ (whatever $y(x)$ may be - we don't care), then

$$F(x, y) = F(x, y(x)) = 0$$

- Using the chain rule, we can differentiate this with respect to x :

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0.$$

- Solving this we can obtain:

$$\frac{dy}{dx} = -\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial y}.$$

Note that for the total derivative dy/dx to exist the partial derivatives of F need to exist and that $\partial F/\partial y \neq 0$.

Example: Consider the implicit function representation of the unit circle:

$$F(x, y) = x^2 + y^2 - 1 = 0.$$

- We can compute the partial derivatives:

$$\frac{\partial F}{\partial x} = 2x \quad \frac{\partial F}{\partial y} = 2y.$$

- Hence

$$\frac{dy}{dx} = -\frac{\partial F}{\partial x} / \frac{\partial F}{\partial y} = -\frac{2x}{2y} = -\frac{x}{y}, \quad y \neq 0.$$

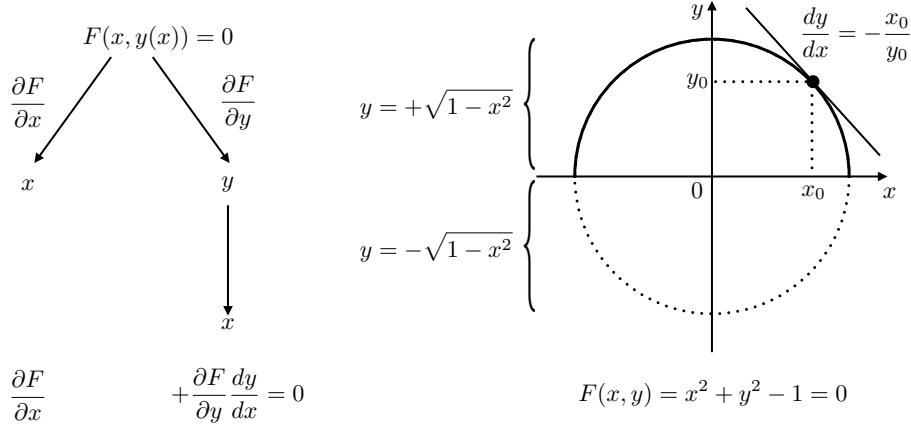


Figure 3.23: Chain rule for implicit function $F(x, y(x)) = 0$ (left). The derivative dy/dx is the slope of the tangent line to the curve $F(x, y) = x^2 + y^2 - 1 = 0$ in the vicinity of the point (x_0, y_0) when, e.g., when solving the implicit representation and taking $y = +\sqrt{1 - x^2}$ as the explicit, single valued, representation (right). The value of dy/dx can also be checked in this case by differentiating the explicit function representation $y = +\sqrt{1 - x^2}$. In more complicated problems, this latter direct approach may not be possible if no explicit representation can be found.

3.6.2 Calculation of Partial Derivatives from an Implicit Function

Consider the implicit function representation:

$$F(x, y, z) = 0,$$

for a known, given, real function F for which the partial derivatives with respect to the real variables x , y and z may be found. Suppose that, in principle, we could solve this locally in some domain around $(x, y) = (x_0, y_0)$ to rewrite this with z as the subject, i.e., $z = z(x, y)$. What are the partial derivatives of the function z with respect to x and y , i.e., $\partial z/\partial x$, and $\partial z/\partial y$?

We can do this using the chain rule, *without needing to even solve initially and explicitly for $z = z(x, y)$!*

- First, if $z = z(x, y)$, then

$$F(x, y, z(x, y)) = 0$$

- Using the chain rule, we can differentiate this with respect to x and solve for $\partial z / \partial x$:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \left(\frac{\partial z}{\partial x} \right)_y = 0 \quad \Rightarrow \quad \left(\frac{\partial z}{\partial x} \right)_y = -\frac{\partial F}{\partial x} / \frac{\partial F}{\partial z}.$$

- Similarly, using the chain rule, we can differentiate this with respect to y and solve for $\partial z / \partial y$:

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \left(\frac{\partial z}{\partial y} \right)_x = 0 \quad \Rightarrow \quad \left(\frac{\partial z}{\partial y} \right)_x = -\frac{\partial F}{\partial y} / \frac{\partial F}{\partial z}.$$

In both cases we see that locally around $(x, y) = (x_0, y_0)$, the partials of z with respect to the independent variables (x, y) exist provided that the partial derivatives of F exist and that $\partial F / \partial z \neq 0$.

Example: Consider the implicit function:

$$F(x, y) = y \sin(x + z) + \ln(y) = 0.$$

- We can compute the partial derivatives at $(x, y, z) = (0, 1, 0)$ as follows (note that $F(0, 1, 0) = 0!$):

$$\frac{\partial F}{\partial x} = y \cos(x + z) \quad \frac{\partial F}{\partial y} = \sin(x + z) + 1/y, \quad \frac{\partial F}{\partial z} = y \cos(x + z).$$

- Hence

$$\left(\frac{\partial z}{\partial x} \right)_y = -\frac{\partial F}{\partial x} / \frac{\partial F}{\partial z} = -\frac{y \cos(x + z)}{y \cos(x + z)} = -\frac{\cos(0)}{\cos(0)} = -1.$$

$$\left(\frac{\partial z}{\partial y} \right)_x = -\frac{\partial F}{\partial y} / \frac{\partial F}{\partial z} = -\frac{\sin(x + z) + 1/y}{y \cos(x + z)} = -\frac{\sin(0) + 1}{\cos(0)} = -1.$$

3.6.3 Generalisation to Systems of Implicit Functions

This section is included for completeness. It is not examinable, but is useful background material.

Consider the implicit function representation:

$$F(x, y, z, w) = 0, \quad G(x, y, z, w) = 0.$$

for known, given, real functions F and G for which the partial derivatives with respect to the real variables x, y, z, w may be found. Here we may have multiple ways that we might try to solve these relations to obtain an explicit relationships. We may seek solutions at a point in terms of pairs of independent and dependent variables as follows (!!):

$$\begin{aligned} \begin{cases} x = x(z, w) \\ y = y(z, w), \end{cases} \quad \begin{cases} x = x(y, w) \\ z = z(y, w), \end{cases} \quad \begin{cases} x = x(y, z) \\ w = w(y, z), \end{cases} \\ \begin{cases} y = y(x, w) \\ z = z(x, w), \end{cases} \quad \begin{cases} y = y(x, z) \\ w = w(x, z), \end{cases} \quad \begin{cases} z = z(x, y) \\ w = w(x, y), \end{cases} \end{aligned}$$

Suppose we are asked to find $\partial x / \partial z$.

If we had been dealing with a single function $F(x, y, z) = 0$, this **partial** derivative automatically would imply that x was going to be a function of the remaining variables y and z , i.e., was $(\partial x / \partial z)_y$, the derivative of $x = x(y, z)$ with respect to z , keeping y constant.

When we were dealing with a more than one function $F(x, y, z, w) = 0$, $G(x, y, z, w)$, asking for $\partial x / \partial z$ alone is ambiguous. Looking at the possible pairs of relations between independent and dependent variables above, we see that x could be expressed in terms of z and either y or w depending on which pair of independent variables are chosen:

$$\begin{cases} x = x(z, w) \\ y = y(z, w), \end{cases} \quad \text{or} \quad \begin{cases} x = x(y, z) \\ w = w(y, z), \end{cases}$$

In such cases, we therefore have to specify explicitly the variables we are keeping constant in the partial derivatives:

$$\begin{aligned} (\partial x / \partial z)_w &\Rightarrow \begin{cases} x = x(z, w) \\ y = y(z, w), \end{cases} \\ (\partial x / \partial z)_y &\Rightarrow \begin{cases} x = x(y, z) \\ w = w(y, z). \end{cases} \end{aligned}$$

Suppose we now look for $(\partial x / \partial z)_w$. We are treating the independent variables then as z and w . We therefore consider F and G as:

$$F(x(z, w), y(z, w), z, w) = 0, \quad G(x(z, w), y(z, w), z, w) = 0,$$

and partially differentiate them with respect to z using the chain rule:

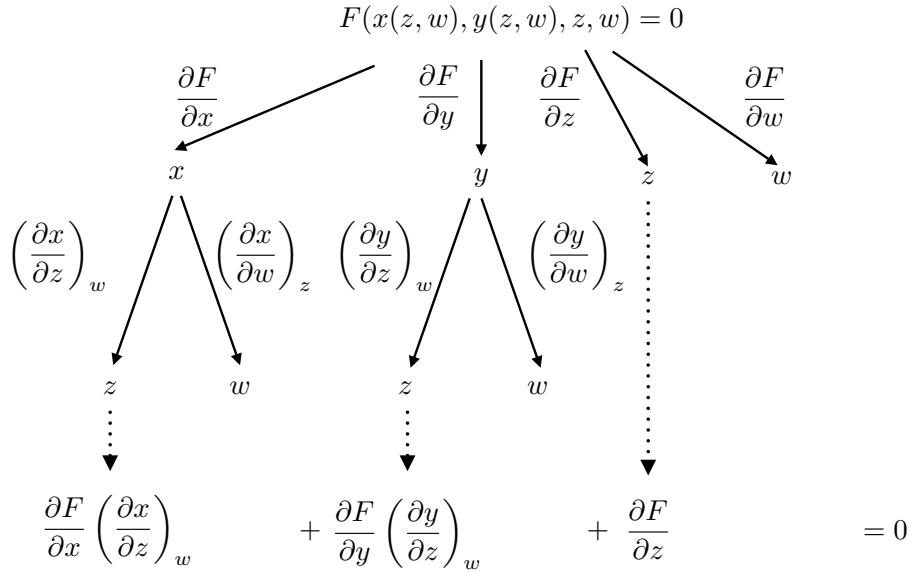


Figure 3.24: Chain rule for implicit function $F(x(z, w), y(z, w), z, w) = 0$. A parallel diagram for $G(x(z, w), y(z, w), z, w) = 0$ exists, identical in structure, after replacing F with G .

$$\frac{\partial F}{\partial x} \left(\frac{\partial x}{\partial z} \right)_w + \frac{\partial F}{\partial y} \left(\frac{\partial y}{\partial z} \right)_w + \frac{\partial F}{\partial z} = 0, \quad \frac{\partial G}{\partial x} \left(\frac{\partial x}{\partial z} \right)_w + \frac{\partial G}{\partial y} \left(\frac{\partial y}{\partial z} \right)_w + \frac{\partial G}{\partial z} = 0.$$

Now for given, known, F and G , all the partial derivatives of F and G with respect to x , y and z will be known. The thing we want to find is $(\partial x / \partial z)_w$. However we also see that each of these expressions involve the also unknown $(\partial y / \partial z)_w$. This is not a problem, since we now have

two linear algebraic equations in two unknowns, $(\partial x/\partial z)_w$ and $(\partial y/\partial z)_w$, with known coefficients $\partial F/\partial x$, $\partial F/\partial y$, $\partial F/\partial z$, $\partial G/\partial x$, $\partial G/\partial y$, $\partial G/\partial z$. This can be solved easily, (for example using Cramer's rule or directly by algebraic elimination) to give:

$$\left(\frac{\partial x}{\partial z}\right)_w = -\frac{(\partial F/\partial z)(\partial G/\partial y) - (\partial F/\partial y)(\partial G/\partial z)}{(\partial F/\partial x)(\partial G/\partial y) - (\partial F/\partial y)(\partial G/\partial x)} = -\frac{\partial(F, G)}{\partial(z, y)} / \frac{\partial(F, G)}{\partial(x, y)}.$$

Hence the partial derivative $(\partial x/\partial z)_w$ is expressed as the ratio of two Jacobian determinants of F, G with respect to the other variables x, y, z . The corresponding other solution of the algebraic equation gives the other partial derivative as

$$\left(\frac{\partial y}{\partial z}\right)_w = -\frac{(\partial F/\partial z)(\partial G/\partial x) - (\partial F/\partial x)(\partial G/\partial z)}{(\partial F/\partial y)(\partial G/\partial x) - (\partial F/\partial x)(\partial G/\partial y)} = -\frac{\partial(F, G)}{\partial(z, x)} / \frac{\partial(F, G)}{\partial(y, x)}.$$

Here we see that a condition for the two functions F and G to be solvable locally at a point $(x, y, z, w) = (x_0, y_0, z_0, w_0)$ in the form of $x = x(z, w)$ and $y = y(z, w)$ is that the Jacobian determinant of F and G with respect to the newly chosen dependent variables x and y does not vanish, i.e.,

$$\frac{\partial(F, G)}{\partial(x, y)} \neq 0.$$

Generalisations to n functions $F_i, i = 1, \dots, n$ in terms of $n+m$ variables $x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n$ are possible. This leads to expressions for partial derivatives $(\partial y_i/\partial x_j)_{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m}$ expressed in terms of ratios of $n \times n$ Jacobian determinants. However, that is an exercise in extreme tedium.

3.7 Definition of Gradient

We have seen that for a multivariable function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, we have, in general, two “obvious” partial derivatives that we may take, namely $\partial f/\partial x$ and $\partial f/\partial y$. A combination of these that is extremely useful, and which will come up time and time again in equations is the **gradient** of f .

Definition: (Gradient) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function. Assume that both partial derivatives exist at a point $(x, y) \in \mathbb{R}^2$. We define the *gradient* of $f(x, y)$ to be the vector of partial derivatives

$$\nabla f(x, y) = \left(\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right).$$

One important thing to note is that, while $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, its gradient $\nabla f(x, y)$ is a **vector** function $\nabla f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. For this reason in \mathbb{R}^2 , if we assign unit vectors \mathbf{i} and \mathbf{j} to the positive x, y directions respectively, we can write ∇f as:

$$\nabla f(x, y) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j},$$

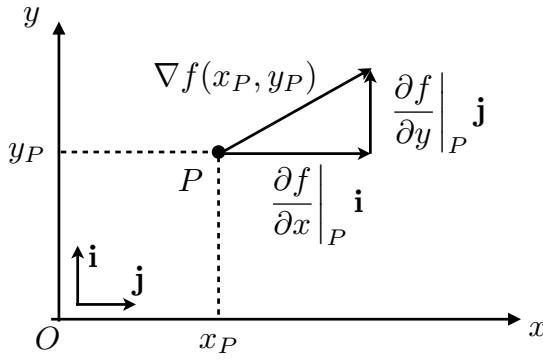


Figure 3.25: Representation of the vector which is the gradient of a function $f(x, y)$ in the (x, y) plane at a point P , (x_p, y_p) .

More generally, we can define the *gradient* of a function $f(x_1, \dots, x_n)$ of n variables to again be the vector of partial derivatives

$$\nabla f(x_1, \dots, x_n) = \left(\frac{\partial f}{\partial x_1}(x_1, \dots, x_n), \dots, \frac{\partial f}{\partial x_n}(x_1, \dots, x_n) \right).$$

For $f(x, y, z) : \mathbb{R}^3 \rightarrow \mathbb{R}$ the gradient can be written as

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k},$$

where \mathbf{k} is the unit vector pointing in the positive z direction.

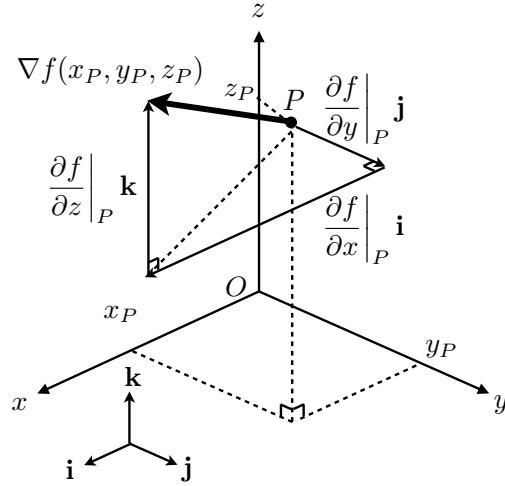


Figure 3.26: Representation of the vector which is the gradient of a function $f(x, y, z)$ in (x, y, z) space at a point P , (x_p, y_p, z_p) .

Example: Find the gradients of the functions $f(x, y) = x + y - 1$, $g(x, y) = e^x + xe^y$, and $h(x, y, z) = x^2 + y^2 + z^2 + 3$.

$$\nabla f(x, y) = \left(\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right) = (1, 1),$$

$$\nabla g(x, y) = (e^x + e^y, xe^y),$$

$$\nabla h(x, y, z) = (2x, 2y, 2z).$$

3.8 Tangent Planes

A differentiable curve in one dimension has a tangent line at each point. This is the straight line that passes through the chosen point on the curve, but which has the same slope of the curve at that point.

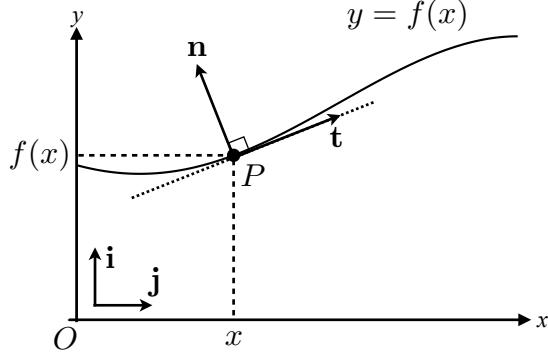


Figure 3.27: Tangent line (dotted), tangent vector \mathbf{t} and normal vector \mathbf{n} at a point P on the (differentiable) curve $y = f(x)$.

We can set up unit vectors \mathbf{i}, \mathbf{j} , in the x, y , directions respectively.

Note that a vector that is tangential to the curve at a chosen point P is given by (see figure 3.27),

$$\mathbf{t} = \mathbf{i} + \left(\frac{df}{dx} \right)_P \mathbf{j}.$$

A vector \mathbf{n} normal to this curve satisfies $\mathbf{t} \cdot \mathbf{n} = 0$. A little bit of thought gives this as

$$\mathbf{n} = - \left(\frac{df}{dx} \right)_P \mathbf{i} + \mathbf{j}.$$

Suppose we now consider the smooth surface $z = f(x, y)$ (we return to what we mean by “smooth” later, but for now just think of it having no kinks or creases). At each point on this surface, there will be two normal directions to the surface (parallel, but in opposite directions to each other).

It is then possible to define a **tangent plane** \mathcal{T} at a chosen point P on the surface. This is the plane that passes through the point P , but is everywhere perpendicular to either of the normal directions $\pm \mathbf{n}$ (see figure 3.28).

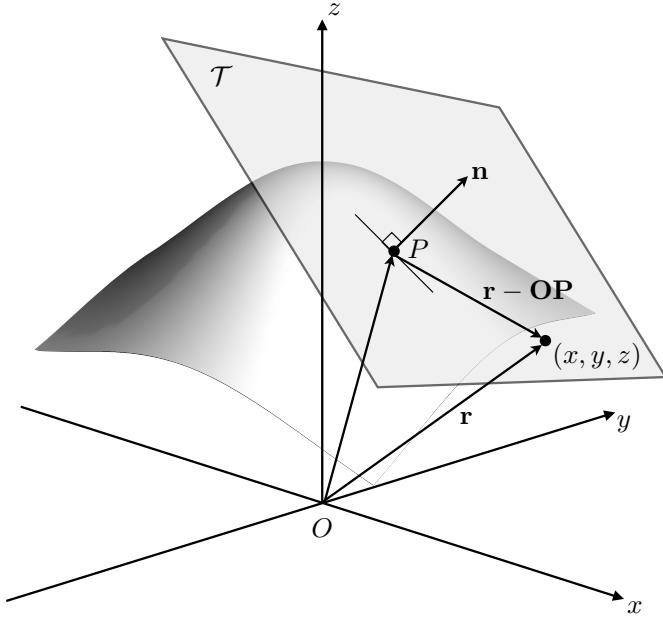


Figure 3.28: Tangent plane \mathcal{T} at the point $P : (a, b, f(a, b))$ that is everywhere normal to one of the normal vectors $\pm \mathbf{n}$. Also shown is a general point \mathbf{r} in the tangent plane and the vector $\mathbf{r} - \mathbf{OP}$ that lies wholly within the tangent plane.

What is the equation of this tangent plane at P ? We can proceed as follows:

- We can set up unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in the x, y , and z directions respectively.
- Now suppose that the chosen point P on the surface has (x, y, z) coordinates given by $(a, b, f(a, b))$. It will have a position vector relative to the origin given by \mathbf{OP} as

$$\mathbf{OP} = a\mathbf{i} + b\mathbf{j} + f(a, b)\mathbf{k}.$$

- We know the slope of the surface in the x direction. It is given by $(\partial f / \partial x)_P$. This is the rate of change of the height of the surface in the x direction. Hence, by analogy with the 1-D case above, a vector tangential to the surface in the x direction is given by

$$\mathbf{t}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x} \right)_P \mathbf{k}$$

- We also know the slope of the surface in the y direction. It is given by $\partial f / \partial y$. This is the rate of change of the height of the surface in the y direction. Hence, by analogy with the 1-D case above, a vector tangential to the surface in the y direction is given by

$$\mathbf{t}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y} \right)_P \mathbf{k}$$

- We can now find a normal \mathbf{n} to the surface by taking the vector product of the two tangent vectors.

$$\mathbf{n} = \mathbf{t}_x \times \mathbf{t}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \left(\frac{\partial f}{\partial x} \right)_P \\ 0 & 1 & \left(\frac{\partial f}{\partial y} \right)_P \end{vmatrix} = - \left(\frac{\partial f}{\partial x} \right)_P \mathbf{i} - \left(\frac{\partial f}{\partial y} \right)_P \mathbf{j} + \mathbf{k}$$

- The other normal in the other direction is then

$$-\mathbf{n} = \left(\frac{\partial f}{\partial x} \right)_P \mathbf{i} + \left(\frac{\partial f}{\partial y} \right)_P \mathbf{j} - \mathbf{k}$$

- Now suppose that the chosen point P on the surface has (x, y, z) coordinates given by $(a, b, f(a, b))$. It will have a position vector relative to the origin given by \mathbf{OP} as

$$\mathbf{OP} = a\mathbf{i} + b\mathbf{j} + f(a, b)\mathbf{k}.$$

- Likewise, suppose a general point on the tangent plane is given by the position vector \mathbf{r} where

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

- The vector $\mathbf{r}-\mathbf{OP}$ is a vector that joins the point P to any other general point on the tangent plane at P . The vector $\mathbf{r}-\mathbf{OP}$ therefore lies wholly within the tangent plane. Consequently it must be perpendicular to either of the normals $\pm\mathbf{n}$ at P . Hence we must have

$$(\mathbf{r} - \mathbf{OP}) \cdot \mathbf{n} = 0.$$

- Substituting for \mathbf{r} , \mathbf{OP} and \mathbf{n} this equation becomes

$$\begin{aligned} & \{x\mathbf{i} + y\mathbf{j} + z\mathbf{k} - (a\mathbf{i} + b\mathbf{j} + f(a, b)\mathbf{k})\} \cdot \mathbf{n} = 0. \\ & \{(x-a)\mathbf{i} + (y-b)\mathbf{j} + (z-f(a, b))\mathbf{k}\} \cdot \mathbf{n} = 0. \\ & \{(x-a)\mathbf{i} + (y-b)\mathbf{j} + (z-f(a, b))\mathbf{k}\} \cdot \left\{ -\left(\frac{\partial f}{\partial x} \right)_P \mathbf{i} - \left(\frac{\partial f}{\partial y} \right)_P \mathbf{j} + \mathbf{k} \right\} = 0. \\ & -(x-a) \left(\frac{\partial f}{\partial x} \right)_P - (y-b) \left(\frac{\partial f}{\partial y} \right)_P + (z-f(a, b)) = 0. \\ & \boxed{z = f(a, b) + (x-a) \left(\frac{\partial f}{\partial x} \right)_P + (y-b) \left(\frac{\partial f}{\partial y} \right)_P}. \end{aligned}$$

Note that we can express the equation of the plane in terms of the gradient of the function evaluated at P in the following way:

$$\boxed{z = f(a, b) + \nabla f(a, b) \cdot ((x-a)\mathbf{i} + (y-b)\mathbf{j})}.$$

Example: Compute the tangent plane to the surface $z = x^2 + y^2$ at the point P above $(x, y) = (1, 2)$.

- Note that $z = x^2 + y^2$, so that in the notation of derivation above we have $f(x, y) = x^2 - y^2$.
- We also have the coordinates of the point P on the surface as

$$(a, b, f(a, b)) = (1, 2, f(1, 2)) = (1, 2, 1^2 + 2^2) = (1, 2, 5)$$

- Hence the ingredients for the formula for the tangent surface are:

$$\frac{\partial f}{\partial x} = 2x, \quad \Rightarrow \quad \left(\frac{\partial f}{\partial x} \right)_P = 2 \times 1 = 2$$

$$\frac{\partial f}{\partial y} = 2y, \quad \Rightarrow \quad \left(\frac{\partial f}{\partial y} \right)_P = 2 \times 2 = 4$$

- Hence the tangent surface at P is given by

$$z = f(a, b) + (x - a) \left(\frac{\partial f}{\partial x} \right)_P + (y - b) \left(\frac{\partial f}{\partial y} \right)_P$$

$$\Rightarrow z = 5 + 2(x - 1) + 4(y - 2)$$

$$\Rightarrow z = 2x + 4y - 5$$

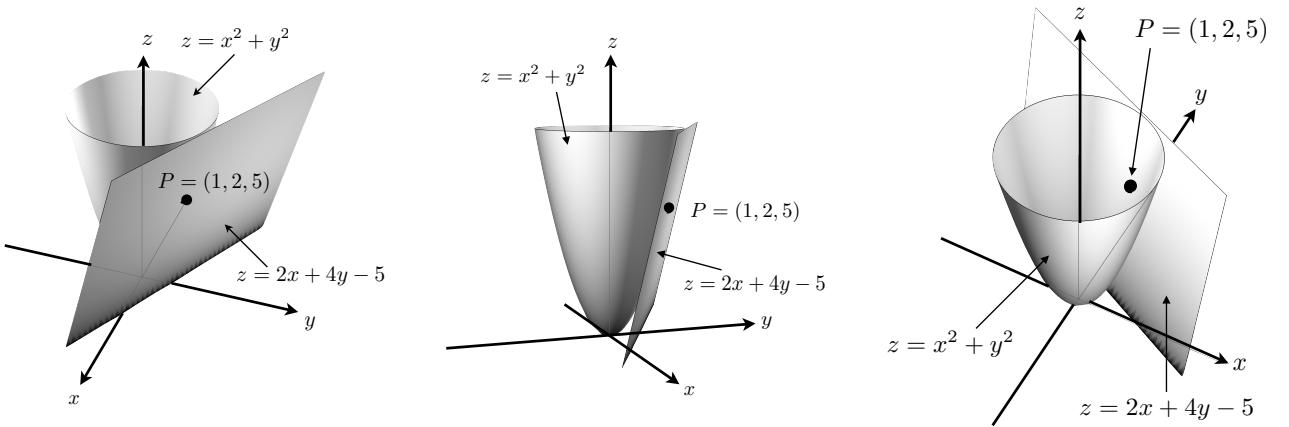


Figure 3.29: Different views of the tangent plane $z = 2x + 4y - 5$ to the surface $z = x^2 + y^2$ at $P = (1, 2, 5)$.

Recall that for a curve $y = F(x)$ in 1-D, the tangent line at the point $(a, F(a))$ is given by

$$y = F(a) + \left(\frac{df}{dx} \right)_{x=a} (x - a).$$

The tangent line gives a linear approximation to the line and is the first two terms of the Taylor series expansion of $y = F(x)$ about $(a, F(a))$.

It is then not too difficult to see that the tangent plane is a linear expansion of the function $z = f(x, y)$ about the point $P = (a, b, f(a, b))$. In fact it is the first two terms in the multivariable Taylor series expansion of $z = f(x, y)$ at P , which we will study later.

Tangent hyper-planes can also be derived for hypersurfaces $f : \mathbb{R}^n \rightarrow \mathbb{R}$. The formula take a similar form to that derived above: a sum of linear differences in the variable from the tangent point multiplied by the appropriate partial derivative at that point plus the value of the function at that point.

3.9 Differentiability

We now examine what it means for a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ to be differentiable. We again recall the definition of a derivative of a function $f(x)$ in one variable at a point $x = a$:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}.$$

Since $f'(a)$ is a constant, we can carry it over to the right hand side and use basic properties of limits to see that we can rewrite this equation as

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a) - f'(a)h}{h} = 0. \quad (3.11)$$

We can then say that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ is *differentiable* at a point $x = a$ if there exists $f'(a) \in \mathbb{R}$ so that

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a) - f'(a)h}{h} = 0.$$

We also know that the derivative $f'(a)$ has a geometric interpretation as the slope of the line tangent to the graph of the function at the point $(a, f(a)) \in \mathbb{R}^2$. So, the function is differentiable when such a tangent line exists and hence its slope is well-defined.

If we now consider a function of two variables $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ instead, then the tangent line is replaced by the tangent plane at a given point $(a, b, f(a, b))$. So, the function would be differentiable at (a, b) when such a tangent plane exists.

The tangent plane does not have a single slope but rather many slopes coming from different directions. We then compute the slopes in the x and y directions, that is, when we fix $y = b$ and $x = a$ respectively and consider the slopes of the lines $(x, b, f(x, b))$ and $(a, y, f(a, y))$ in the tangent plane. These slopes are precisely the partial derivatives of $f(x, y)$ at (a, b) .

So, for a function $f(x, y)$ to be differentiable at (a, b) , the gradient $\nabla f(a, b)$ needs to exist.⁶

3.9.1 Differentiability in the case of two variable functions

Now, let us make the precise definition of differentiability. We will only do this for a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$:

Definition: (Differentiability) A function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is *differentiable* at a point $(a, b) \in \mathbb{R}^2$ if there is a vector $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ so that

$$\lim_{(h,k) \rightarrow (0,0)} \frac{f(a+h, b+k) - f(a, b) - \mathbf{v} \cdot (h, k)}{\sqrt{h^2 + k^2}} = 0. \quad (3.12)$$

The vector \mathbf{v} is a bit mysterious. However, the following theorem shows why it is natural in the definition of differentiability and explains precisely what it is.

Theorem 3.9.1. *If the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is differentiable at a point $(a, b) \in \mathbb{R}^2$, then it is continuous at (a, b) . Moreover, the vector \mathbf{v} defined by (3.12) is unique and is equal to $\nabla f(a, b)$.*

Proof:

- To see the first assertion note that $\lim_{(h,k) \rightarrow (0,0)} \sqrt{h^2 + k^2} = 0$. This together with (3.12) and the limit rules in lemma 1.4.1, gives us

$$\lim_{(h,k) \rightarrow (0,0)} \sqrt{h^2 + k^2} \cdot \lim_{(h,k) \rightarrow (0,0)} \frac{f(a+h, b+k) - f(a, b) - \mathbf{v} \cdot (h, k)}{\sqrt{h^2 + k^2}} = 0,$$

$$\Rightarrow \lim_{(h,k) \rightarrow (0,0)} f(a+h, b+k) - f(a, b) - \mathbf{v} \cdot (h, k) = 0,$$

$$\Rightarrow \lim_{(h,k) \rightarrow (0,0)} f(a+h, b+k) - f(a, b) = 0,$$

⁶One should expect the converse not to be true as we only choose two specific lines on the tangent plane in order to describe the gradient vector. But as we shall see, this is almost true.

$$\Rightarrow \lim_{(h,k) \rightarrow (0,0)} f(a+h, b+k) = f(a, b).$$

This is the definition of continuity (see section 1.4.4) and so a differentiable function at a point is continuous there also. (NB the converse is **not** true! Just fold a piece of paper and make a crease: the surface is continuous, but not differentiable for the reason we now see below.)

- To see the second assertion, note that we get the same limit if we let $k = 0$ in (3.12), since for the limit to exist, which it does by assumption, the value must be the same, regardless of the direction of approach. We can pick to approach the limit from a direction where $k = 0$, without loss of generality. Hence we have

$$\lim_{(h,0) \rightarrow (0,0)} \frac{f(a+h, b) - f(a, b) - (v_1, v_2) \cdot (h, 0)}{|h|} = 0.$$

Since the limit exists by assumption and equals zero by definition, it does not depend on whether $h > 0$ or $h < 0$ (any extra negative sign (-1) from the modulus may be divided away into the zero). Hence we can write:

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b) - (v_1, v_2) \cdot (h, 0)}{h} = 0, \\ & \Rightarrow \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h} - v_1 = 0, \\ & \Rightarrow v_1 = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h} = \frac{\partial f}{\partial x}(a, b). \end{aligned}$$

Similarly, by letting $h = 0$, we get

$$v_2 = \lim_{k \rightarrow 0} \frac{f(a, b+k) - f(a, b)}{h} = \frac{\partial f}{\partial y}(a, b).$$

Note that the vector

$$\mathbf{v} = (v_1, v_2) = \left(\frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b) \right)$$

is thus just the gradient of $f(x, y)$ at (a, b) .

We have just shown that if a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is differentiable at a point $(a, b) \in \mathbb{R}^2$, then it is continuous at (a, b) and its partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at that point. Equivalently, if f is differentiable, ∇f exists at (a, b) .

There is a useful criteria that almost states the converse, guaranteeing that a function is differentiable at a given point. The proof of the theorem is not required, but we will use it in the subsequent examples.

Theorem 3.9.2. *If both partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ exist and are continuous on a neighbourhood of a point (a, b) then f is differentiable at (a, b) .*

Example: Show that the function

$$f(x, y) = x^2 + y^2 + xy \exp(x^2 + y^2)$$

is differentiable.

First, we compute the partial derivatives.

$$\frac{\partial f}{\partial x}(x, y) = 2x + y \exp(x^2 + y^2) + 2x^2y \exp(x^2 + y^2),$$

$$\frac{\partial f}{\partial y}(x, y) = 2y + x \exp(x^2 + y^2) + 2xy^2 \exp(x^2 + y^2).$$

As these partial derivatives are formed from continuous functions via products, sums and compositions they are also continuous. So, by Theorem 3.9.2, $f(x, y)$ is differentiable.

The definition of differentiability leads us on to a really neat trick.

3.10 Directional derivative

3.10.1 Definition of Directional Derivative

For a function of two or more variables, we have already seen that its partial derivatives describe the rate of change of the function in either x or y directions depending which partial derivative we consider. In this section we discuss how one can measure the rate of change of such a function in **any** chosen direction.

We can appeal to the definitions of partial derivatives of a function $f(x, y)$ in the x and y directions and write down a definition of the derivative of a function in the direction of a unit vector \mathbf{w} emerging from the point $(a, b, f(a, b))$ and tangential to the surface in that direction.

Definition: (Directional Derivative) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and \mathbf{u} be a unit vector in \mathbb{R}^2 . The *directional derivative* (or *rate of change*) of f in the direction \mathbf{u} at the point $(a, b) \in \mathbb{R}^2$ is defined to be

$$D_{\mathbf{u}}(f)(a, b) = \lim_{h \rightarrow 0} \frac{f((a, b) + h\mathbf{u}) - f(a, b)}{h}.$$

provided this limit exists.

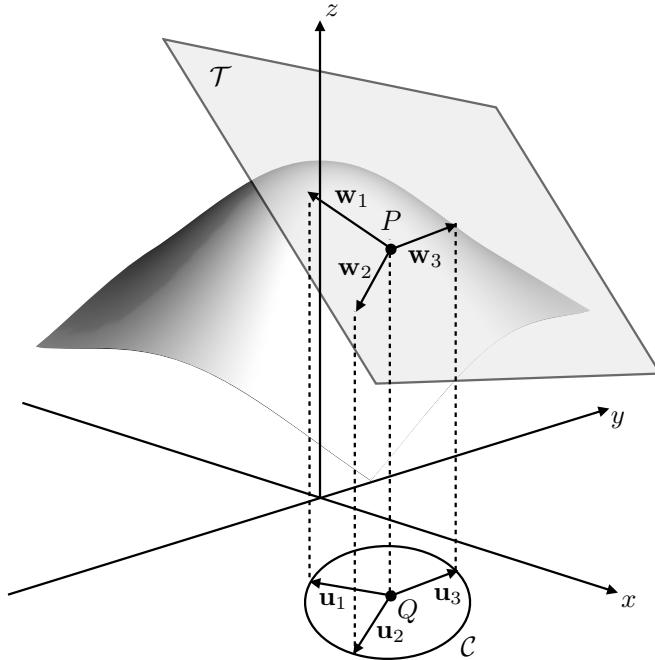


Figure 3.30: The tangent plane \mathcal{T} to surface $z = f(x, y)$ at point $P : (a, b, f(a, b))$ contains an infinite number of vectors \mathbf{w}_i . The derivative of the function $f(x, y)$ is, in general, different in each of these directions. The projection of these vectors \mathbf{w}_i onto a unit disk \mathcal{C} centred at $Q : (a, b)$ are unit vectors \mathbf{u}_i . (x, y) plane are unit vectors \mathbf{u}_i

Actually, by working through the definition of the derivative, we can come up with a much simpler formula for the directional derivative that is easy to use in practice.

Theorem 3.10.1. Suppose $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is differentiable at $(a, b) \in \mathbb{R}^2$ and \mathbf{u} is a unit vector. Then the directional derivative of f in the direction \mathbf{u} at (a, b) is given by

$$D_{\mathbf{u}}(f)(a, b) = \nabla f(a, b) \cdot \mathbf{u}.$$

Proof. We can write $\mathbf{u} = (\alpha, \beta) \in \mathbb{R}^2$ where $\alpha^2 + \beta^2 = 1$, because \mathbf{u} is a unit vector. Since $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is differentiable at (a, b) , by (3.12) and Theorem 3.9.1, we have

$$\lim_{h \rightarrow 0} \frac{f(a + h\alpha, b + h\beta) - f(a, b) - \nabla f(a, b) \cdot (h\alpha, h\beta)}{\sqrt{(h\alpha)^2 + (h\beta)^2}} = 0$$

Since $\alpha^2 + \beta^2 = 1$, it follows that

$$\lim_{h \rightarrow 0} \frac{f(a + h\alpha, b + h\beta) - f(a, b) - \nabla f(a, b) \cdot (h\alpha, h\beta)}{h} = 0.$$

This gives us

$$\lim_{h \rightarrow 0} \frac{f(a + h\alpha, b + h\beta) - f(a, b)}{h} = \nabla f(a, b) \cdot (\alpha, \beta).$$

But the left-hand side of the above identity is exactly the directional derivative and thus

$$D_{\mathbf{u}}(f)(a, b) = \nabla f(a, b) \cdot \mathbf{u}$$

as desired. □

3.10.2 Geometric Meaning of Directional Derivative

We are now in a position to deduce the geometric meaning of the gradient that we introduced above, from the definition of the directional derivative. We use the property of dot products that

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos(\theta)$$

where θ is the angle between \mathbf{a} and \mathbf{b} . Letting θ be the angle between $\nabla f(a, b)$ and \mathbf{u} , we see that

$$D_{\mathbf{u}}(f)(a, b) = \nabla f(a, b) \cdot \mathbf{u} = |\nabla f(a, b)| |\mathbf{u}| \cos(\theta) = |\nabla f(a, b)| \cos(\theta),$$

since \mathbf{u} is a unit vector.

We see that since $\nabla f(a, b)$ is given constant vector when evaluated at the point (a, b) , to maximise $D_{\mathbf{u}}(f)(a, b)$, we only need to maximise $\cos(\theta)$. Therefore, we obtain the following geometric properties of the gradient.

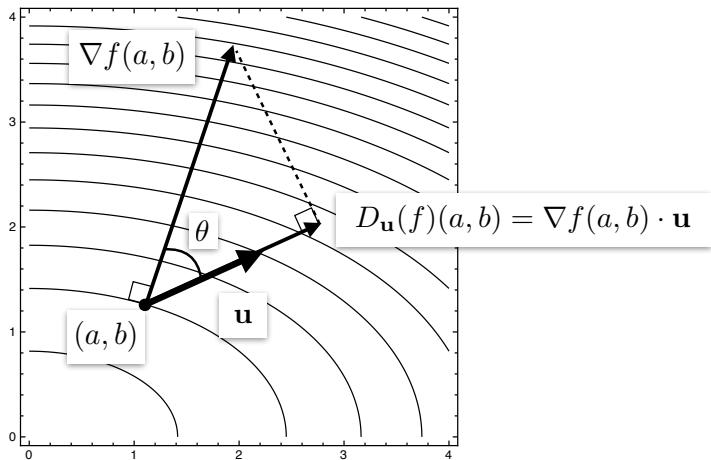


Figure 3.31: Geometric meaning of directional derivative relative to a unit vector \mathbf{u} and the gradient vector ∇f at a point (a, b) , superimposed on a contour plot of a $\mathbb{R}^2 \rightarrow \mathbb{R}$ function.

- (i) The function f increase the most rapidly at (a, b) when $\theta = 0$, which is when \mathbf{u} is pointing in the same direction as $\nabla f(a, b)$.
- (ii) The function f decrease the most rapidly at (a, b) when $\theta = \pi$, which is when \mathbf{u} is pointing in the opposite direction as $\nabla f(a, b)$.
- (iii) The rate of change of the function f is zero at (a, b) when $\theta = \frac{\pi}{2}$, which is when \mathbf{u} is perpendicular to the direction of $\nabla f(a, b)$.
- (iv) Choosing $\mathbf{u}=\mathbf{i}$ we obtain

$$D_{\mathbf{i}}(f)(x, y) = \nabla f \cdot \mathbf{i} = \frac{\partial f}{\partial x}$$

- (iv) Choosing $\mathbf{u}=\mathbf{j}$ we obtain

$$D_{\mathbf{j}}(f)(x, y) = \nabla f \cdot \mathbf{j} = \frac{\partial f}{\partial y}$$

For a given function $f(x, y)$, the sorts of calculations we do with the directional derivative involve solving the equation $D_{\mathbf{u}}(f)(a, b) = c$ when two of the parameters \mathbf{u} , (a, b) and c are specified.

3.10.3 Examples of Directional Derivative

Examples:

Evaluate the directional derivative of the function $z = f(x, y)$ where $f(x, y) = e^{x^2+y^2}$ at the point $(x, y) = (1, 1)$ in the direction

(a) $\mathbf{i} + \mathbf{j}$.

- Remember that to find the directional derivative \mathbf{u} has to be a **unit vector**. Hence we must have that

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{1}{\sqrt{2}} (\mathbf{i} + \mathbf{j}).$$

- We calculate the gradient of $f(x, y)$ as

$$\nabla f(x, y) = \frac{\partial}{\partial x} (e^{x^2+y^2}) \mathbf{i} + \frac{\partial}{\partial y} (e^{x^2+y^2}) \mathbf{j} = 2xe^{x^2+y^2} \mathbf{i} + 2ye^{x^2+y^2} \mathbf{j}.$$

- Therefore we have

$$D_{\mathbf{u}}(f)(x, y) = \nabla f(x, y) \cdot \mathbf{u} = (2xe^{x^2+y^2} \mathbf{i} + 2ye^{x^2+y^2} \mathbf{j}) \cdot \frac{1}{\sqrt{2}} (\mathbf{i} + \mathbf{j}) = (\sqrt{2}xe^{x^2+y^2} + \sqrt{2}ye^{x^2+y^2}).$$

- Hence when $(x, y) = (1, 1)$ we have

$$D_{\mathbf{u}}(f)(1, 1) = (\sqrt{2}e^2 + \sqrt{2}e^2) = 2\sqrt{2}e^2.$$

(b) $\mathbf{i} - \mathbf{j}$

- Remember that to find the directional derivative \mathbf{u} has to be a **unit vector**. Hence we must have that

$$\mathbf{u} = \frac{\mathbf{i} - \mathbf{j}}{|\mathbf{i} - \mathbf{j}|} = \frac{1}{\sqrt{2}} (\mathbf{i} - \mathbf{j}).$$

- We calculate the gradient of $f(x, y)$ as

$$\nabla f(x, y) = \frac{\partial}{\partial x} (e^{x^2+y^2}) \mathbf{i} + \frac{\partial}{\partial y} (e^{x^2+y^2}) \mathbf{j} = 2xe^{x^2+y^2} \mathbf{i} + 2ye^{x^2+y^2} \mathbf{j}.$$

- Therefore we have

$$D_{\mathbf{u}}(f)(x, y) = \nabla f(x, y) \cdot \mathbf{u} = (2xe^{x^2+y^2} \mathbf{i} + 2ye^{x^2+y^2} \mathbf{j}) \cdot \frac{1}{\sqrt{2}} (\mathbf{i} - \mathbf{j}) = (\sqrt{2}xe^{x^2+y^2} - \sqrt{2}ye^{x^2+y^2}).$$

- Hence when $(x, y) = (1, 1)$ we have

$$D_{\mathbf{u}}(f)(1, 1) = (\sqrt{2}e^2 - \sqrt{2}e^2) = 0.$$

So we may also conclude that

$$\mathbf{u} = \frac{1}{\sqrt{2}} (\mathbf{i} - \mathbf{j})$$

is in the direction of the level contour of $f(x, y)$ at the point $(x, y) = (1, 1)$.

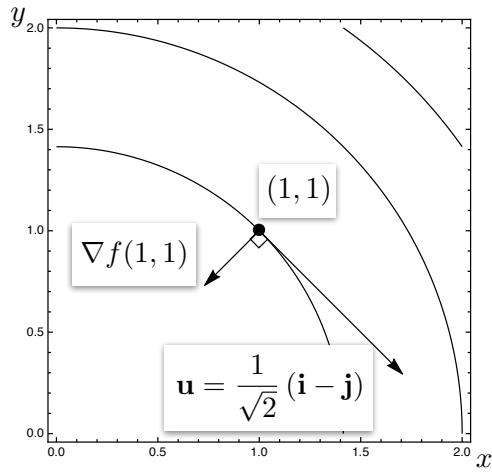


Figure 3.32: The level contour lines of $f(x, y) = e^{-(x^2+2y^2)}$ with gradient vectors at the point $(x, y) = (1, 1)$ superimposed. The unit vector $\mathbf{u} = (\mathbf{i} - \mathbf{j})/\sqrt{2}$ extending from $(1, 1)$ is also shown. Since $\nabla f(1, 1) \cdot \mathbf{u} = 0$, the vector \mathbf{u} is perpendicular to the gradient vector and is thus locally parallel to the level contour line.

Example:

Evaluate the Directional derivative of the above function at $(x, y) = (0, 0)$ in every direction. Comment on the result:

- Let $\mathbf{u} = \alpha\mathbf{i} + \beta\mathbf{j}$ where $\alpha^2 + \beta^2 = 1$, so that \mathbf{u} is a unit vector.
- We know that

$$\nabla f(x, y) = 2xe^{x^2+y^2}\mathbf{i} + 2ye^{x^2+y^2}\mathbf{j}$$

- Hence

$$D_{\mathbf{u}}(f)(x, y) = (2xe^{x^2+y^2}\mathbf{i} + 2ye^{x^2+y^2}\mathbf{j}) \cdot (\alpha\mathbf{i} + \beta\mathbf{j}) = (2\alpha xe^{x^2+y^2} + 2\beta ye^{x^2+y^2})$$

- Hence when $(x, y) = (1, 1)$ we have

$$D_{\mathbf{u}}(f)(0, 0) = (2\alpha \times 0 \times e^{0^2+0^2} + 2\beta \times 0 \times e^{0^2+0^2}) = 0, \quad \forall(\alpha, \beta).$$

Hence the directional derivative at $(0, 0)$ is everywhere zero in no matter what direction you look in.

- Does this mean the function is everywhere constant? Clearly from the previous examples, we know that the directional derivatives do vary, depending on the evaluation point. The actual reason is easily seen by realising that at $(x, y) = (0, 0)$ the surface has a local maximum. This gives a hint towards how we are going to find local maxima and minima in the next couple of sections.

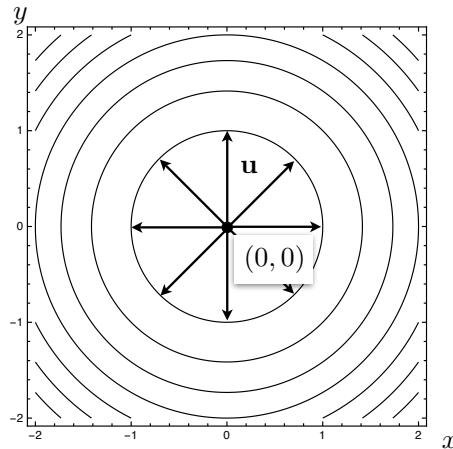


Figure 3.33: Value of directional derivative of $z = e^{-(x^2+y^2)}$ at $(x, y) = (0, 0)$ in any direction \mathbf{u} is zero, since the function is locally horizontal at that point, and is in fact a local maximum. This can also be seen from the fact that the contour at that point is just an isolated point and not a line.

Example:

Given the function $z = f(x, y) = x^2 + 2y^2$, work out the direction of the steepest ascent and the steepest descent at the point $(x, y) = (2, 1)$.

- We know that the function increases most rapidly at $(2, 1)$ when \mathbf{u} is pointing in the same direction as $\nabla f(2, 1)$.
- We also know that at that point f decreases most rapidly in the direction $-\nabla f(2, 1)$.
- We find that $\nabla f(x, y)$ is

$$\nabla f = \frac{\partial}{\partial x} (x^2 + 2y^2) \mathbf{i} + \frac{\partial}{\partial y} (x^2 + 2y^2) \mathbf{j} = 2x\mathbf{i} + 4y\mathbf{j}$$

- Hence we have

$$\nabla f(2, 1) = (2 \times 2)\mathbf{i} + (4 \times 1)\mathbf{j} = 4\mathbf{i} + 4\mathbf{j}.$$

- Hence the path of steepest ascent at $(x, y) = (2, 1)$ is in the direction $4\mathbf{i} + 4\mathbf{j}$, or $(\mathbf{i} + \mathbf{j})/\sqrt{2}$ if you want to express it as a unit vector.
- Hence the path of steepest descent at $(x, y) = (2, 1)$ is in the direction $-4\mathbf{i} - 4\mathbf{j}$ or $-(\mathbf{i} + \mathbf{j})/\sqrt{2}$ if you want to express it as a unit vector.

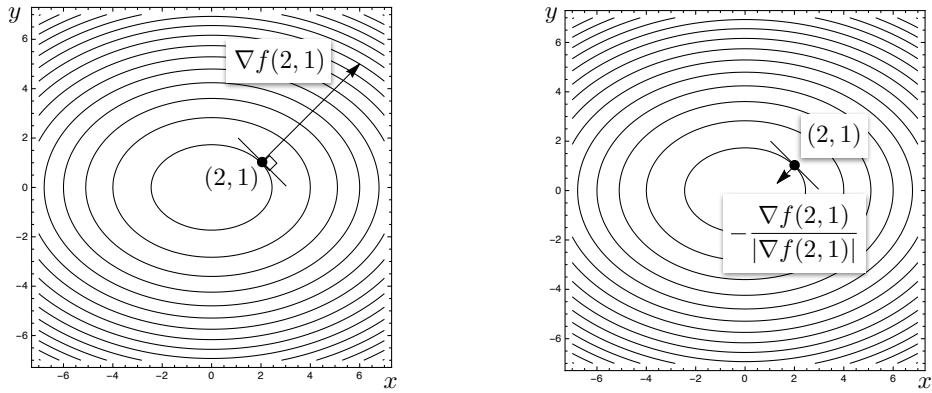


Figure 3.34: The level contour lines of $f(x, y) = x^2 + 2y^2$ with gradient vectors at the point $(x, y) = (2, 1)$ superimposed. Left: The vector $\nabla f(2, 1)$ gives the direction of steepest *ascent* at the point $(2, 1)$, i.e., the direction perpendicular to the level contours lines of $f(x, y)$ at $(2, 1)$ with a positive gradient. The value of $|\nabla f(2, 1)|$ gives the slope of the surface at that point in the steepest ascent direction. Right: The **unit** vector $-\nabla f(2, 1)/|\nabla f(2, 1)|$ gives the direction of steepest *descent* at the point $(2, 1)$, i.e., the direction perpendicular to the level contours lines of $f(x, y)$ at $(2, 1)$ with a negative gradient. The value of $-|\nabla f(2, 1)|$ gives the slope of the surface at that point in the steepest descent direction (although we have chosen only to draw the unit vector on the diagram).

Example:

For the same function calculate the direction of the contour lines through $(x, y) = (2, 1)$.

- The direction of the level contour lines \mathbf{u} through a point satisfy

$$\nabla f(2, 1) \cdot \mathbf{u} = 0,$$

i.e., the directional derivative in that direction is zero.

- Let $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$, where θ is the angle in the horizontal (x, y) plane to the positive real axis of the level lines at $(x, y) = (2, 1)$.
- We thus have

$$\nabla f(2, 1) \cdot \mathbf{u} = (4\mathbf{i} + 4\mathbf{j}) \cdot (\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) = 4(\cos \theta + \sin \theta) = 0.$$

- Hence θ must satisfy:

$$\cos \theta + \sin \theta = 0 \quad \Rightarrow \quad \sin \theta = -\cos \theta \quad \Rightarrow \quad \tan \theta = -1 \quad \Rightarrow \quad \theta = -\frac{\pi}{4} \text{ or } \frac{3\pi}{4}.$$

- Hence the level lines are in the directions

$$\mathbf{u} = \cos\left(-\frac{\pi}{4}\right)\mathbf{i} + \sin\left(-\frac{\pi}{4}\right)\mathbf{j} = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$$

$$\mathbf{u} = \cos\left(\frac{3\pi}{4}\right)\mathbf{i} + \sin\left(\frac{3\pi}{4}\right)\mathbf{j} = \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j})$$

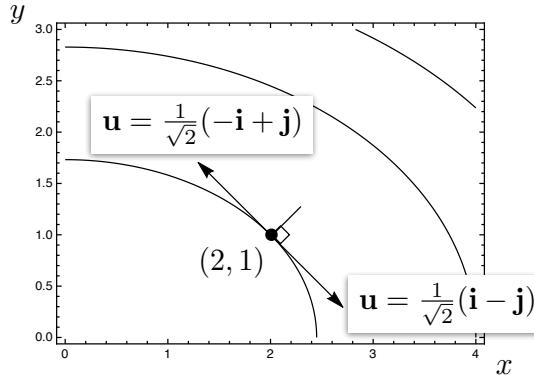


Figure 3.35: The level contour lines of $f(x, y) = x^2 + 2y^2$ with the direction vectors $\mathbf{u} = \pm(\mathbf{i} - \mathbf{j})/\sqrt{2}$ superimposed as $(x, y) = (2, 1)$, where they are parallel to the level contours through that point.

3.11 Computation of Steepest Paths and Contour Lines

Having seen that we can derive the direction of steepest ascent/descent or level lines at any particular point, we now examine how to compute the corresponding paths of steepest ascent/descent or level lines as you take an extended walk away from any particular point.

Example: The height on a hill is given by

$$z = f(x, y) = \frac{1}{\sqrt{(1 + x^2 + 2y^2)}}.$$

A stream starts at the point $(2, 1, f(2, 1))$ follows the path of steepest descent away from that point. Compute the trajectory of stream.

- We know that the path of steepest descent is in the direction of

$$-\nabla f = -\frac{\partial f}{\partial x} \mathbf{i} - \frac{\partial f}{\partial y} \mathbf{j} = -\frac{x}{(1+x^2+2y^2)^{3/2}} \mathbf{i} - \frac{2y}{(1+x^2+2y^2)^{3/2}} \mathbf{j}.$$

- Let the path of steepest descent in projection be given by $y = y(x)$. The slope $y'(x)$ must then be parallel at each point to $-\nabla f$.

- Hence we have

$$\frac{dy}{dx} = \frac{y \text{ component of } (-\nabla f)}{x \text{ component of } (-\nabla f)} = \frac{-2y}{(1+x^2+2y^2)^{3/2}} \times \frac{(1+x^2+2y^2)^{3/2}}{-x} = \frac{2y}{x}.$$

- Thus the steepest path satisfies

$$\begin{aligned} \frac{dy}{dx} = \frac{2y}{x} &\Rightarrow \int \frac{dy}{y} = 2 \int \frac{dx}{x} &\Rightarrow \ln y = 2 \ln x + \ln C \\ &\Rightarrow \ln y = \ln(Cx^2) &\Rightarrow y = Cx^2 \end{aligned}$$

- Inserting the initial point of $(x, y) = (2, 1)$ we see that

$$1 = C \times 2^2 \Rightarrow C = 1/4.$$

- Hence the steepest descent path from $(2, 1)$ follows the path

$$y = x^2/4$$

in the (x, y) plane.

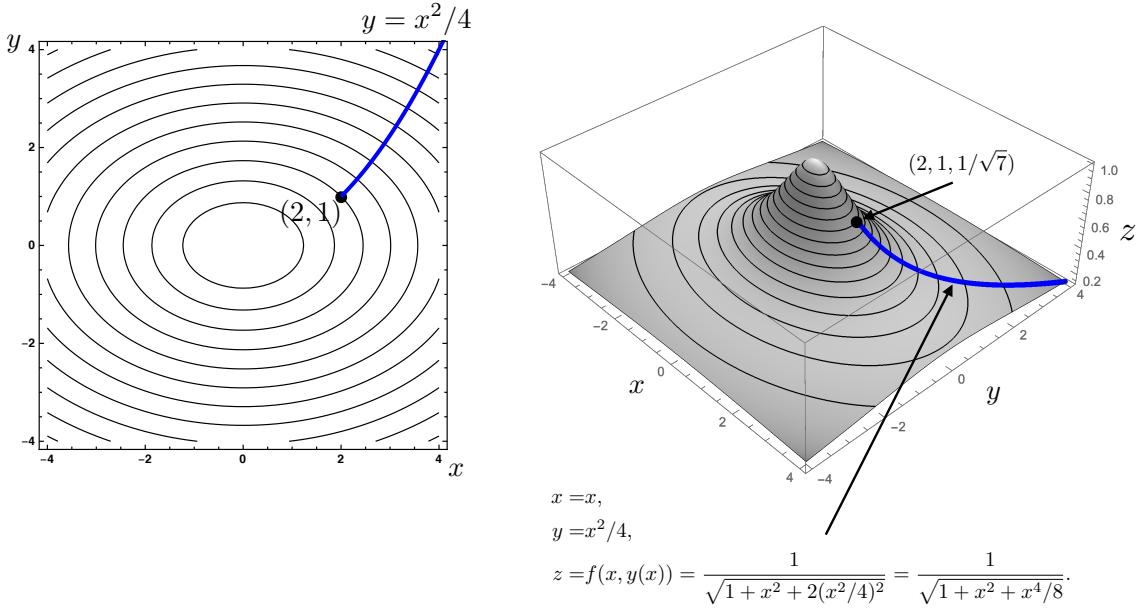


Figure 3.36: Steepest descent path from point above $(2, 1)$ on surface $z = \frac{1}{\sqrt{1+x^2+2y^2}}$. The left hand diagram shows the contour plot of $f(x, y) = \text{constant}$, together with the projection of the steepest descent path from $(2, 1)$ in the (x, y) plane. The right hand diagram is a plot of the surface $(x, y, z = f(x, y))$ in \mathbb{R}^3 . The steepest descent path from $(2, 1)$ is also show. It is the three dimensional curve with coordinates given by $(x, y(x), z = f(x, y(x)))$ as explained in the text.

- In $(x, y, z) \in \mathbb{R}^3$, the steepest descent curve of course sits on the surface $z = f(x, y)$. Its coordinates are thus given parametrically by $(x, y(x), z = f(x, y(x)))$, where

$$\begin{aligned} x &= x, \\ y &= y(x) = x^2/4, \\ z &= f(x, y(x)) = \frac{1}{\sqrt{1+x^2+2(x^2/4)^2}} = \frac{1}{\sqrt{1+x^2+x^4/8}}. \end{aligned}$$

Example: The height on a hill is given by

$$z = f(x, y) = e^{-(x^2+3y^2)}.$$

Find the equation of the level contours, i.e., $z(x, y) = c$ (without doing the obvious!).

- We know that the contour is in the direction \mathbf{v} , where $\mathbf{v} \cdot \nabla f = 0$.

- We have

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} = -2xe^{-(x^2+3y^2)} \mathbf{i} - 6ye^{-(x^2+3y^2)} \mathbf{j}.$$

- Hence a suitable \mathbf{v} would be:

$$\mathbf{v} = 6ye^{-(x^2+3y^2)} \mathbf{i} - 2xe^{-(x^2+3y^2)} \mathbf{j}.$$

- Let the level contour in projection be given by $y = y(x)$. The slope $y'(x)$ must then be parallel at each point to \mathbf{v} .

- Hence we have

$$\frac{dy}{dx} = \frac{\text{y component of } \mathbf{v}}{\text{x component of } \mathbf{v}} = \frac{-2xe^{-(x^2+3y^2)}}{6ye^{-(x^2+3y^2)}} = \frac{-x}{3y}.$$

- Thus the level contour satisfies

$$\begin{aligned} \frac{dy}{dx} = \frac{-x}{3y} &\Rightarrow 3 \int y \, dy = - \int x \, dx \Rightarrow \frac{3}{2}y^2 = -\frac{x^2}{2} + C \\ &\Rightarrow \frac{x^2}{2} + \frac{3}{2}y^2 = C \Rightarrow \frac{x^2}{2C} + \frac{3}{2C}y^2 = 1. \end{aligned}$$

- These are confocal ellipses of semi major/minor axes $\sqrt{2C}$ and $\sqrt{2C/3}$.
- Inserting an initial point of $(x, y) = (1, 1)$, say, we see that

$$\frac{1^2}{2} + \frac{3}{2}1^2 = C \Rightarrow C = 2.$$

- Hence the level contour through $(1, 1)$ follows the path

$$\frac{x^2}{4} + \frac{3}{4}y^2 = 1$$

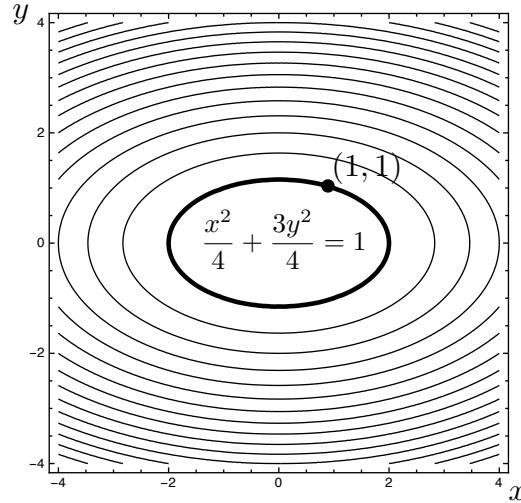


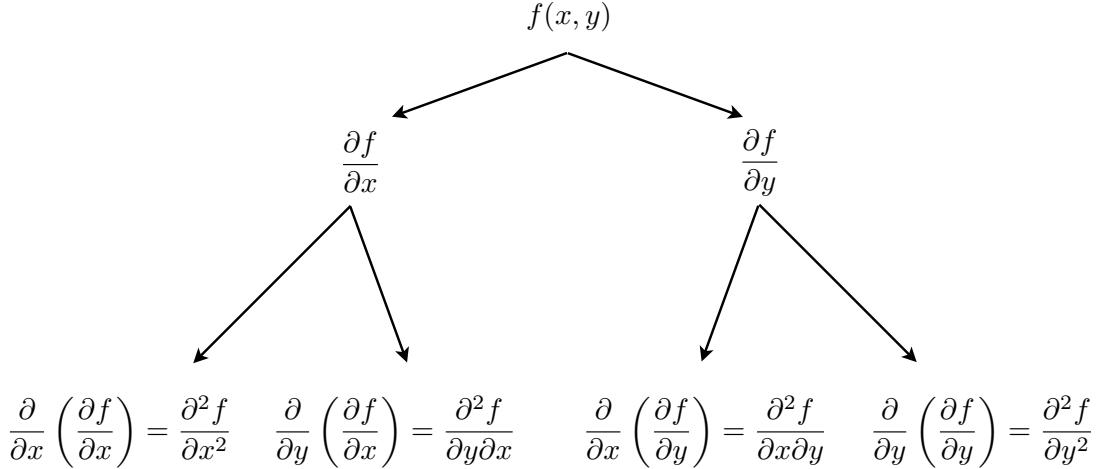
Figure 3.37: Level contours $\frac{x^2}{4C} + \frac{3}{4C}y^2 = 1$ of surface $z = e^{-(x^2+3y^2)}$. The bold contour is the level contour through $(1, 1)$, i.e., $\frac{x^2}{4} + \frac{3}{4}y^2 = 1$.

Of course, we could have just spotted what the level contours were by just observing that $z = \text{constant}$ when $x^2 + 3y^2 = \text{constant}$. However this example is to demonstrate how curves other than steepest descent ones may be calculated.

3.12 Higher Order Partial Derivatives

So far we have just looks at the first partial derivatives of multivariable functions. Now we consider higher order partial derivatives. Again we initially focus on $f : \mathbb{R}^2 \rightarrow \mathbb{R}$.

Suppose that a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ may be differentiated twice. We then have the following sequence of partial derivatives.



Thus we have up to four second-order partial derivatives. At higher orders the pyramid structure propagates. (Question: How many partial derivatives are there at order n ?)

For ease of writing the subscript notation is often used:

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}, \quad \frac{\partial^2 f}{\partial y \partial x} = f_{yx}, \quad \frac{\partial^2 f}{\partial x \partial y} = f_{xy}, \quad \frac{\partial^2 f}{\partial y^2} = f_{yy}.$$

Example: $f(x, y) = x^2 - 1 + y + \sin(xy)$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} (2x + y \cos(xy)) = 2 - y^2 \sin(xy).$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} (2x + y \cos(xy)) = \cos(xy) - xy \sin(xy).$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} (1 + x \cos(xy)) = \cos(xy) - xy \sin(xy).$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} (1 + x \cos(xy)) = -x^2 \sin(xy).$$

Example: Given $w = f(x, y, z) = x \cos(z) + \sin(xy)$, calculate $\frac{\partial^3 w}{\partial z \partial x^2}$, $\frac{\partial^3 w}{\partial x \partial z \partial x}$, and $\frac{\partial^3 w}{\partial x^2 \partial z}$.

$$\frac{\partial^3 w}{\partial z \partial x^2} = \frac{\partial^2 w}{\partial z \partial x} (\cos(z) + y \cos(xy)) = \frac{\partial w}{\partial z} (-y^2 \sin(x)) = 0,$$

$$\frac{\partial^3 w}{\partial x \partial z \partial x} = \frac{\partial^2 w}{\partial x \partial z} (\cos(z) + y \cos(xy)) = \frac{\partial w}{\partial x} (-\sin(z)) = 0,$$

$$\frac{\partial^3 w}{\partial x^2 \partial z} = \frac{\partial^2 w}{\partial x^2} (-x \sin(z)) = \frac{\partial w}{\partial x} (-\sin(z)) = 0.$$

The above examples suggest that under certain circumstances the order in which the partial derivatives are performed is irrelevant. In fact the following theorem holds.

Theorem 3.12.1. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function. Suppose for some positive integer k , there exists a neighbourhood U of a point $P \in \mathbb{R}^n$ such that f and all its partial derivatives of order less than k are continuous on U . Then any two mixed partial derivatives of order k involving the same variables but in different sequential orders are equal at P , provided those partial derivatives are continuous at P .*

Boiled down to something digestible for $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and for second order partial derivatives, we also have the theorem:

Theorem: If f_{xy} and f_{yx} are continuous at $(x, y) = (a, b)$ then they are equal there.

Proof: Uses the mean value theorem, but is not required.

So provided the mixed partial derivatives of order up to k are continuous at a point, then all the mixed partial derivatives of order k will all be equal. This reduces dramatically the number of partial derivatives at higher order, so that for example, sufficiently continuous functions and their partial derivatives, $f_{xxy} = f_{xyx} = f_{yxx}$ etc. Hence for such functions, the order in which you do the partial derivatives doesn't matter.

3.13 Changes of Variables in PDES

We have seen above that combining the chain rule and changes of variables can be used to simplify first order (partial) differential equations and make them easier to solve. It may be that when expressed in a new set of coordinates that respect the symmetry of the geometry, such an approach can also either simplify the form of PDEs or express the equation in a form that is more amenable to solution in certain geometries.

Example: (Solving the wave equation in cartesian coordinates) Consider the second order wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

We shall see below that the solution of this equation is a superposition of waves travelling to the left and right with speed c . We shall use the intermediate variables:

$$\xi = x - ct, \quad \eta = x + ct,$$

to write

$$u = u(\xi(x, t), \eta(x, t))$$

For then we have

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial t} = c \left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} \right), \\ \Rightarrow \quad \frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t} \right) = c \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right) c \left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} \right) \\ &= c^2 \left(\frac{\partial^2 u}{\partial \xi^2} - 2 \frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \eta^2} \right)\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} = \left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \right), \\ \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) \left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \right) \\ &= \left(\frac{\partial^2 u}{\partial \xi^2} + 2 \frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \eta^2} \right)\end{aligned}$$

So now substituting these two expressions back into the wave equation we obtain:

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= c \frac{\partial^2 u}{\partial x^2} \\ c^2 \left(\frac{\partial^2 u}{\partial \xi^2} - 2 \frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \eta^2} \right) &= c^2 \left(\frac{\partial^2 u}{\partial \xi^2} + 2 \frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \eta^2} \right) \\ \Rightarrow \frac{\partial^2 u}{\partial \eta \partial \xi} &= 0 \\ \Rightarrow u(\xi, \eta) &= F(\xi) + G(\eta), \\ \Rightarrow u(x, t) &= F(x - ct) + G(x + ct),\end{aligned}$$

for arbitrary functions F and G . These might be determined by giving the initial values of the function $u(x, 0) = F(x) + G(x)$ and $u_x(x, 0) = F'(x) + G'(x)$. You will learn about how to solve such PDEs in second year modules.

Example: (Rewriting Laplace's equation in polar coordinates) Consider Laplace's equation in cartesian coordinates (x, y) :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Laplace's equation is widespread in mathematics and its applications. Its solution in 2-D represents, for example, the shape formed by a soap bubble, the electrostatic potential away from a charge distribution, the gravitational potential away from a mass, the real (or imaginary) part of an analytic complex function, the components of velocity of a steady 2-D incompressible, irrotational fluid flow, etc. In this example we transform the equation from cartesian to plane polar

coordinates. This is used to solve the equation whenever there is cylindrical symmetry in the problem.⁷

Consider the transformations:

$$x = r \cos \theta, \quad y = r \sin \theta \quad \Rightarrow \quad r = \sqrt{x^2 + y^2}, \quad \tan \theta = y/x.$$

The chain rule is then about transforming from (x, y) to (r, θ) . We have

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial x}, \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial y}. \end{aligned}$$

Hence we are going to need the following partial derivatives:

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2}} = \frac{x}{r} = \frac{r \cos \theta}{r} = \cos \theta.$$

$$\frac{\partial r}{\partial y} = \frac{2y}{2\sqrt{x^2 + y^2}} = \frac{y}{r} = \frac{r \sin \theta}{r} = \sin \theta.$$

The derivatives of θ with respect to x and y are a little more tricky:

$$\frac{\partial(\tan \theta)}{\partial x} = \sec^2 \theta \frac{\partial \theta}{\partial x} \quad \text{but} \quad \frac{\partial(y/x)}{\partial x} = -\frac{y}{x^2},$$

Combining these two identities, using the fact that $y/x = \tan \theta$, we have in (r, θ) coordinates:

$$\frac{\partial \theta}{\partial x} = -\cos^2 \theta \frac{y}{x^2} = -\cos^2 \theta \frac{r \sin \theta}{r^2 \cos^2 \theta} = -\frac{\sin \theta}{r}.$$

Similarly we have:

$$\frac{\partial(\tan \theta)}{\partial y} = \sec^2 \theta \frac{\partial \theta}{\partial y} \quad \text{but} \quad \frac{\partial(y/x)}{\partial y} = \frac{1}{x},$$

hence combining these two identities, we have (r, θ) coordinates:

$$\frac{\partial \theta}{\partial y} = \frac{\cos^2 \theta}{x} = \frac{\cos^2 \theta}{r \cos \theta} = \frac{\cos \theta}{r}$$

Combining these results with the chain rule above we obtain:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial x} = \cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta}, \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial y} = \sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta}. \end{aligned}$$

Hence in polar coordinates we can see that the partial derivatives can be written as the following operators:

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}, \\ \frac{\partial}{\partial y} &= \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}. \end{aligned}$$

⁷Such problems will be discussed next year, but the transformation here serves as a good exercise in use of the chain rule.

Thus we can now write down the second derivatives with respect to x and y :

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \left(\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) \left(\cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta} \right), \\
&= \cos \theta \frac{\partial}{\partial r} \left(\cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta} \right) - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta} \right), \\
&= \cos^2 \theta \frac{\partial^2 u}{\partial r^2} - \cos \theta \sin \theta \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial u}{\partial \theta} \right) - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\cos \theta \frac{\partial u}{\partial r} \right) + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right), \\
&= \cos^2 \theta \frac{\partial^2 u}{\partial r^2} + \frac{\cos \theta \sin \theta}{r^2} \frac{\partial u}{\partial \theta} - \frac{\cos \theta \sin \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\sin^2 \theta}{r} \frac{\partial u}{\partial r} - \frac{\cos \theta \sin \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} \\
&\quad + \frac{\sin \theta \cos \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 u}{\partial \theta^2}, \\
&= \cos^2 \theta \frac{\partial^2 u}{\partial r^2} + 2 \frac{\cos \theta \sin \theta}{r^2} \frac{\partial u}{\partial \theta} - 2 \frac{\cos \theta \sin \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\sin^2 \theta}{r} \frac{\partial u}{\partial r} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 u}{\partial \theta^2}.
\end{aligned}$$

Similarly we have:

$$\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) = \left(\sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \right) \left(\sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta} \right), \\
&= \sin \theta \frac{\partial}{\partial r} \left(\sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta} \right) + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta} \right), \\
&= \sin^2 \theta \frac{\partial^2 u}{\partial r^2} + \sin \theta \cos \theta \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial u}{\partial \theta} \right) + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial r} \right) + \frac{\cos \theta}{r^2} \frac{\partial}{\partial \theta} \left(\cos \theta \frac{\partial u}{\partial \theta} \right), \\
&= \sin^2 \theta \frac{\partial^2 u}{\partial r^2} - \frac{\sin \theta \cos \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{\sin \theta \cos \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\cos^2 \theta}{r} \frac{\partial u}{\partial r} + \frac{\cos \theta \sin \theta}{r} \frac{\partial^2 u}{\partial \theta \partial r} \\
&\quad - \frac{\cos \theta \sin \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 u}{\partial \theta^2}, \\
&= \sin^2 \theta \frac{\partial^2 u}{\partial r^2} - 2 \frac{\cos \theta \sin \theta}{r^2} \frac{\partial u}{\partial \theta} + 2 \frac{\cos \theta \sin \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\cos^2 \theta}{r} \frac{\partial u}{\partial r} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 u}{\partial \theta^2}.
\end{aligned}$$

Now we add the left and right and sides together:

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \cos^2 \theta \frac{\partial^2 u}{\partial r^2} + 2 \frac{\cos \theta \sin \theta}{r^2} \frac{\partial u}{\partial \theta} - 2 \frac{\cos \theta \sin \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\sin^2 \theta}{r} \frac{\partial u}{\partial r} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 u}{\partial \theta^2}, \\
&\quad + \sin^2 \theta \frac{\partial^2 u}{\partial r^2} - 2 \frac{\cos \theta \sin \theta}{r^2} \frac{\partial u}{\partial \theta} + 2 \frac{\cos \theta \sin \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\cos^2 \theta}{r} \frac{\partial u}{\partial r} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 u}{\partial \theta^2}, \\
&= (\cos^2 \theta + \sin^2 \theta) \frac{\partial^2 u}{\partial r^2} + \frac{(\cos^2 \theta + \sin^2 \theta)}{r} \frac{\partial u}{\partial r} + \frac{(\cos^2 \theta + \sin^2 \theta)}{r^2} \frac{\partial^2 u}{\partial \theta^2},
\end{aligned}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}.$$

Hence in plane polar coordinates, Laplace's equation can be written as

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.$$

Phew!

This result actually has a simpler form: multiply first through by r , then note that the sum of the partial derivatives in r can be written as a derivative

$$r \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} = \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right).$$

Hence Laplace's equation can be written as

$$r \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial \theta^2} = 0,$$

or

$$r \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) = -\frac{\partial^2 u}{\partial \theta^2}.$$

The key point here is that everything on the LHS involves derivatives wrt r and everything on the RHS, derivatives wrt to θ . This is a key step in solving the equation by separation of variables in plane polar coordinates: see second year modules!

As an aside, suppose we want to find a solution of Laplace's equation that is a function of r , i.e., radial distance from the origin. Then we have $u = u(r)$ only and

$$\frac{\partial u}{\partial \theta} = \frac{\partial^2 u}{\partial \theta^2} = 0.$$

Laplace's equation then becomes:

$$r \frac{d}{dr} \left(r \frac{du}{dr} \right) = 0,$$

i.e., all reference to θ vanishes and the partial derivatives with respect to r become total.

After dividing by r (for $r \neq 0$) we can integrate this, now, ODE, twice to obtain:

$$\left(r \frac{du}{dr} \right) = A,$$

$$\frac{du}{dr} = \frac{A}{r}$$

$$u(r) = A \log r + B.$$

for arbitrary constants A and B . This shows, for example, that the electrostatic potential due to a straight line source of charge decays logarithmically with radial distance r from the line source.

In three (cartesian) dimensions, Laplace's equation becomes

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

An exercise for you is to show that a radially symmetric solution in 3-D is given by:

$$u(r) = \frac{A}{r} + B$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

More generally, in arbitrary coordinates and dimensions, Laplace's equation is written as

$$\nabla^2 u = 0.$$

The symbol “ ∇ ” is pronounced “del” (short for “upside down delta”), so that ∇^2 is pronounced “del-squared”. It can be thought of $\nabla \cdot \nabla$, i.e., the result of taking the scalar product of two gradient operators. (Try this for yourself using the cartesian definition $\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j}$. More of this type of thing will come in second year modules.)

3.14 Multivariable Taylor Series

We saw above that the tangent plane could be interpreted as a linear approximation to a multivariate function around a given point P . We can improve on that approximation using the chain rule and higher order derivatives.

Recall that for a single-variable function $g(t)$, if all the derivatives g exist at a point, say $t = 0$ (that is $g(0), g'(0), g''(0), \dots$ all exist), then g can be expanded as a power series in a neighbourhood of $t = 0$ via the Taylor-Maclaurin Series expansion

$$g(t) = \sum_{r=0}^{\infty} \frac{g^{(k)}(0)}{r!} t^r = g(0) + g'(0)t + \frac{g''(0)}{2!}t^2 + \frac{g'''(0)}{3!}t^3 + \dots$$

Now suppose that $g(t)$ is defined as

$$g(t) = f(a + ht, b + kt),$$

where a, b, h, k are real and f is a multivariate function $f(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}$, with independent variables (x, y) with

$$x = x(t) = a + ht, \quad y = y(t) = b + kt.$$

But if we want to write down a Taylor series for $g(t)$ about $t = 0$, then we need to compute the derivatives of g at $t = 0$. To do that we will need to use the chain rule:

$$g'(t) = \frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y}.$$

So far so good, now we need the higher derivatives. From above we can see that a derivative with respect to t is, by the chain rule, in this case equivalent to the following linear combination of partial derivatives with respect to x and y acting on f :

$$\frac{d}{dt} = h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}$$

We can apply this to $g'(t)$ to get $g''(t)$ as follows:

$$\begin{aligned} g''(t) &= \frac{d}{dt} g'(t) = \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) \left(h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y} \right) \\ &= \left(h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2} \right) \end{aligned}$$

We can repeat this for $g'''(t)$ and we find

$$g'''(t) = \frac{d}{dt} g''(t) = \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) \left(h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2} \right)$$

$$= \left(h^3 \frac{\partial^3 f}{\partial x^3} + 3h^2 k \frac{\partial^3 f}{\partial^2 x \partial y} + 3hk^2 \frac{\partial^3 f}{\partial x \partial^2 y} + k^3 \frac{\partial^3 f}{\partial y^3} \right)$$

Going one step further we find,

$$g^{(iv)}(t) = \left(h^4 \frac{\partial^4 f}{\partial x^4} + 4h^3 k \frac{\partial^4 f}{\partial^3 x \partial y} + 6h^2 k^2 \frac{\partial^4 f}{\partial x^2 \partial^2 y} + 4hk^3 \frac{\partial^4 f}{\partial x \partial^3 y} + k^4 \frac{\partial^4 f}{\partial y^4} \right)$$

The pattern should now be a bit clear:

The n^{th} derivative of $g(t) = f(a + ht, b + kt)$ can be represented as the sum of the n^{th} mixed partial derivatives of f , with respect to x and y , weighted by the Pascal triangle combinatorial coefficients multiplied by the powers of h and k corresponding to the respective order of the partial derivatives (h with x and k with y).

The final bit is to realise that if we insert this into the Taylor series for $g(t)$, then we have to evaluate all the derivatives at $t = 0$. Hence we have

$$g(0) = f(x(0), y(0)) = f(a, b),$$

$$g'(0) = h \left(\frac{\partial f}{\partial x} \right)_{(a,b)} + k \left(\frac{\partial f}{\partial y} \right)_{(a,b)},$$

$$g''(0) = h^2 \left(\frac{\partial^2 f}{\partial x^2} \right)_{(a,b)} + 2hk \left(\frac{\partial^2 f}{\partial x \partial y} \right)_{(a,b)} + k^2 \left(\frac{\partial^2 f}{\partial y^2} \right)_{(a,b)},$$

$$g'''(0) = \dots$$

So finally we can insert the above expressions into the Taylor series for $g(t)$, but then chose to set $t = 1$ on the LHS. We have $g(1) = f(a + h, b + k)$ so the Taylor series becomes

$$\begin{aligned} f(a + h, b + k) &= f(a, b) + h \left(\frac{\partial f}{\partial x} \right)_{(a,b)} + k \left(\frac{\partial f}{\partial y} \right)_{(a,b)} + \\ &+ \frac{1}{2!} \left(h^2 \left(\frac{\partial^2 f}{\partial x^2} \right)_{(a,b)} + 2hk \left(\frac{\partial^2 f}{\partial x \partial y} \right)_{(a,b)} + k^2 \left(\frac{\partial^2 f}{\partial y^2} \right)_{(a,b)} \right) \\ &+ \frac{1}{3!} \left(h^3 \left(\frac{\partial^3 f}{\partial x^3} \right)_{(a,b)} + 3h^2 k \left(\frac{\partial^3 f}{\partial^2 x \partial y} \right)_{(a,b)} + 3hk^2 \left(\frac{\partial^3 f}{\partial x \partial^2 y} \right)_{(a,b)} + k^3 \left(\frac{\partial^3 f}{\partial y^3} \right)_{(a,b)} \right) \\ &+ \dots \end{aligned}$$

We may substitute for $h = x - a$ and $k = y - b$ and use the more compact subscript notation partial derivatives to obtain

$$\begin{aligned} f(x, y) &= f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b) + \\ &+ \frac{1}{2!} \{(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b)f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)\} \\ &+ \frac{1}{3!} \{(x - a)^3 f_{xxx}(a, b) + 3(x - a)^2(y - b)f_{xxy}(a, b) + 3(x - a)(y - b)^2 f_{yxx}(a, b) + (y - b)^3 f_{yyy}(a, b)\} \\ &+ \dots \end{aligned}$$

Notes:

- This gives a way of approximating the function f at points $x = a + h$ and $y = b + k$ given the value of the function and all its partial derivatives at $(x, y) = (a, b)$.
- The first line of the formula is equivalent to the tangent plane approximation of the function $z = f(x, y)$ at the point $(x, y, z) = (a, b, f(a, b))$.

- The “...” denotes an infinite sum of terms of increasingly higher order partial derivatives. We make no statement about whether this sum converges (i.e., adds up to something finite). This type of question will be explored in MATH2039.
- Nevertheless for the functions we shall deal with here, we can assume that the more terms we take in the multivariate Taylor series, the better the approximation to the function $f(x, y)$, given the data at $(x, y) = (a, b)$.
- An analogous argument may be deployed to find the Taylor series for a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ about a point $(x_1, x_2, \dots, x_n) = (a_1, a_2, \dots, a_n)$, but we will restrict ourselves in this module to only $n = 2$.

Example: Calculate the Taylor series expansion of $f(x, y) = e^x \sin y$ about the point $(1, 0)$ up to third order in x and y :

$$f(x, y) = e^x \sin y \quad \Rightarrow \quad f(1, 0) = e^1 \sin 0 = 0$$

$$f_x = e^x \sin y \quad \Rightarrow \quad f_x(1, 0) = e^1 \sin 0 = 0$$

$$f_y = e^x \cos y \quad \Rightarrow \quad f_y(1, 0) = e^1 \cos 0 = e$$

$$f_{xx} = e^x \sin y \quad \Rightarrow \quad f_{xx}(1, 0) = e^1 \sin 0 = 0$$

$$f_{xy} = e^x \cos y \quad \Rightarrow \quad f_{xy}(1, 0) = e^1 \cos 0 = e$$

$$f_{yy} = -e^x \sin y \quad \Rightarrow \quad f_{yy}(1, 0) = -e^1 \sin 0 = 0$$

$$f_{xxx} = e^x \sin y \quad \Rightarrow \quad f_{xxx}(1, 0) = e^1 \sin 0 = 0$$

$$f_{xxy} = e^x \cos y \quad \Rightarrow \quad f_{xxy}(1, 0) = e^1 \cos 0 = e$$

$$f_{xyy} = -e^x \sin y \quad \Rightarrow \quad f_{xyy}(1, 0) = -e^1 \sin 0 = 0$$

$$f_{yyy} = -e^x \cos y \quad \Rightarrow \quad f_{yyy}(1, 0) = -e^1 \cos 0 = -e$$

Substituting into the Taylor series and retaining terms up to third order in x and y we have

$$\begin{aligned} e^x \sin y &= 0 + 0(x - 1) + e(y - 0) + \\ &+ \frac{1}{2!} \{ 0(x - 1)^2 + 2e(x - 1)(y - 0) + 0(y - 0)^2 \} + \\ &+ \frac{1}{3!} \{ 0(x - 1)^3 + 3e(x - 1)^2(y - 0) + 3.0(x - 1)(y - 0)^2 - e(y - 1)^3 \} + \dots \\ e^x \sin y &= e \left\{ (y + 2(x - 1)y + \frac{1}{2}(x - 1)^2y - \frac{1}{6}y^3 + \dots) \right\}. \end{aligned}$$

3.15 The Differential

Definition: Given a function $z = f(x, y)$ we may define the **differential** dz as:

$$dz = df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

For a (differentiable) function that possesses a Taylor series expansion, if we interpret dx and dy as small shifts away from the expansion point (a, b) , so that $dx = x - a$ and $dy = y - b$ it can be seen from the definition of the multivariable Taylor series that the differential dz is an approximation to the corresponding change in f , i.e., $f(x, y) - f(a, b)$.

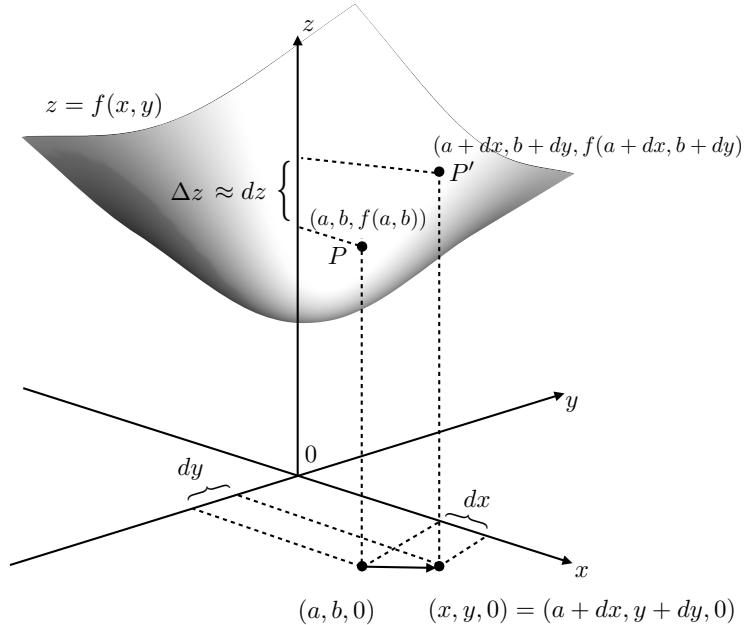


Figure 3.38: The graphical interpretation of the differential dz for a function $z = f(x, y)$. A point P on the surface is given by the coordinates $(a, b, f(a, b))$. If the x coordinate a is shifted by an amount dx and the y coordinate by an amount dy , P moves to become P' on the surface with coordinates $(a+dx, b+dy, f(a+dx, b+dy))$. For (vanishingly) small values of dx and dy , the change in the height of P' relative to P is given by $dz = (\partial f / \partial x)dx + (\partial f / \partial y)dy \approx \Delta z = f(a+dx, b+dy) - f(a, b)$.

Example: Consider $z = f(x, y) = x^2 + y^3$. Find the differential dz and if x changes from 1 to 1.05 and from 2 to 1.95, compare the values of dz and $\Delta z = f(1.05, 1.95) - f(1, 2)$.

- The differential is given by

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 2x dx + 3y dy.$$

- We have the increments dx and dy given by

$$dx = 1.05 - 1.00 = 0.05 \quad dy = 1.95 - 2 = -0.05$$

- Hence at the point $(x, y) = (1, 2)$ we have

$$dz = 2 \times 1 \times 0.05 + 3 \times 2^2 \times (-0.05) = 0.1 - 0.3 = -0.5.$$

- Finally evaluating $\Delta z = f(1.05, 1.95) - f(1, 2)$ we have:

$$\Delta z = f(1.05, 1.95) - f(1, 2) = 1.05^2 + 1.95^3 - 1^2 - 2^3 = -0.48.$$

Hence for these small changes dx and dy in the location of $(1, 2)$ we see that $dz \approx \Delta z$.

Differentials are often useful for estimating the dependence of a result on a small change in the underlying independent variables.

Example: The volume V of a right cylinder of radius r and height h is given by $V = \pi r^2 h$. Suppose that r and h are measured as being 20 cm and 10 cm respectively, with a maximum error of 0.1 cm. Use differentials to estimate the corresponding maximum error in the volume of the cylinder.

- The differential dV is given by

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh = 2\pi rh dr + \pi r^2 dh.$$

- The maximum possible errors in the measurements of r and h are given by dr and dh .

$$dr = |0.1|, \quad dh = |0.1|.$$

We take these to be positive, since this from the form of the differential, and the fact that both $r > 0$ and $h > 0$, this choice will give the largest value of the differential, and hence an estimate of the maximum possible error (overestimate) of the volume.

- We could take them both negative too, but this just gives an identical maximum **underestimate** of the volume. Taking opposite signs in dr and dh will not give either the maximum value of $|dV|$.
- Hence we have

$$dV = 2\pi rh dr + \pi r^2 dh = 2\pi \times 20 \times 10 \times |0.1| + \pi \times 20^2 \times 10 \times |0.1| = 251.327 \text{ cm}^3.$$

- This seems like a lot, but bear in mind that $V = \pi r^2 h = 4000\pi = 12,566.4\text{cm}^3$. Hence, given a relative errors in r and h of

$$dr/r = 0.1/20 = 0.5\% \quad dh/h = 0.1/10 = 1\%,$$

the relative error in V , is then

$$dV/V = 251.327/12,566.4 = 0.02 = 2\%.$$

The definition of a differential generalises easily and obviously to $\mathbb{R}^n \rightarrow \mathbb{R}$ functions:

$$z = f(x_1, x_2, \dots, x_n) \quad \Rightarrow \quad dz = df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \dots + \frac{\partial f}{\partial x_n} dx_n.$$

3.16 Critical Points of Multivariate Functions

Recall that for a function of one variable, $y = f(x)$ the stationary points are located at the solutions of

$$\frac{df}{dx} = 0$$

The type of stationary point is characterised according to the value of the second derivative.

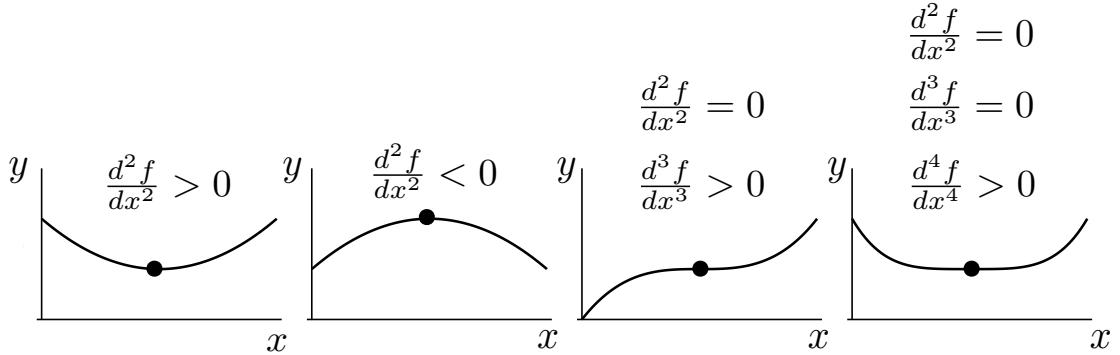


Figure 3.39: Different types of stationary points for $\mathbb{R} \rightarrow \mathbb{R}$ functions.

$$\frac{d^2 f}{dx^2} > 0 : \text{Minimum} \quad \frac{d^2 f}{dx^2} < 0 : \text{Maximum} \quad \frac{d^2 f}{dx^2} = 0 : \text{Test Further}$$

What is the corresponding set of tests for the existence of a minimum or maximum in multivariate functions? We shall again focus on $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, but the condition we derive is valid for $f : \mathbb{R}^n \rightarrow \mathbb{R}$.

3.16.1 Conditions for a Local Isolated Critical Point

Suppose we **assume** there is a local isolated maximum of $z = f(x, y)$ at $(x, y) = (a, b)$ and that the partial derivatives of f exist there. Let us consider the directional derivatives at (a, b) .

$$D_{\mathbf{u}}(f) = \frac{\partial f}{\partial \mathbf{u}} = \nabla f \cdot \mathbf{u},$$

in the direction $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$, where $0 \leq \theta \leq 2\pi$.

Let us suppose that $\nabla f \neq \mathbf{0}$ at this maximum. Then for θ such that \mathbf{u} is in the direction of ∇f we have $\mathbf{u} = \nabla f / |\nabla f|$ and

$$\frac{\partial f}{\partial \mathbf{u}} = \nabla f \cdot \frac{\nabla f}{|\nabla f|} = \frac{|\nabla f|^2}{|\nabla f|} = |\nabla f| > 0.$$

Thus in the direction $\mathbf{u} = \nabla f / |\nabla f|$, the function is **increasing**, which contradicts that the assumption that (a, b) was a local maximum. Hence if the partial derivatives exist at a maximum, then the only other possible situation is that $\nabla f = \mathbf{0}$ there.

An analogous argument holds for a local minimum by considering the directional derivative in the direction of $-\nabla f$.

Hence, by contradiction, a local maximum or minimum in an interior part of a domain must have $\nabla f = \mathbf{0}$. Points at which this occurs are called **critical points**.

A little more work (not required here) leads to a theorem that holds for $f : \mathbb{R}^n \rightarrow \mathbb{R}$:

Theorem 3.16.1. A function $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ has an absolute or local minimum or maximum value at $\mathbf{a} \in D$ only if either

- (i) \mathbf{a} is a **critical point** of f , that is, when $\nabla f(\mathbf{a}) = \mathbf{0}$, or
 - (ii) \mathbf{a} is a **singular point** of f , that is, when $\nabla f(\mathbf{a})$ does not exist, or
 - (iii) \mathbf{a} is a **boundary point** of the domain D .
-

- Part (i) of the theorem is easy to visualise in $\mathbb{R}^2 \rightarrow \mathbb{R}$.

Example: Consider the function $z = f(x, y) = 1 - x^2 - y^2$. This has a critical point when $\nabla f = \mathbf{0}$, i.e., when

$$\nabla f = -2x\mathbf{i} - 2y\mathbf{j} = \mathbf{0}.$$

Note that this implies that both components have to be equal to 0 **simultaneously**. Hence we have a critical point when both

$$-2x = 0, \quad \Rightarrow \quad x = 0,$$

$$-2y = 0, \quad \Rightarrow \quad y = 0,$$

i.e., when $(x, y) = (0, 0)$, see figure 3.40.

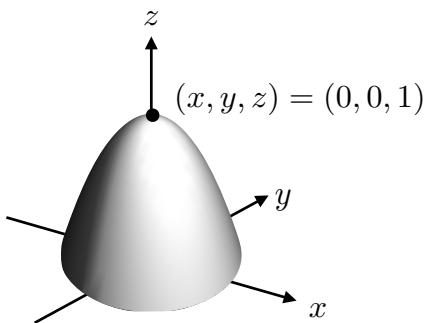


Figure 3.40: Plot of $z = f(x, y) = 1 - x^2 - y^2$ showing the point $(x, y, z) = (0, 0, 1)$ at which $\nabla f = \mathbf{0}$: which is horizontal at that point and a maximum.

- Part (ii) in the theorem holds, for example if there is no critical point, but the function is singular and diverges.

Example: Consider the function $z = f(x, y) = 1/(x^2 + y^2)$ when evaluated over any domain $D : (x, y)$ including $(x, y) = (0, 0)$.

We have

$$\nabla f(x, y) = \frac{2x}{(x^2 + y^2)^2}\mathbf{i} + \frac{2y}{(x^2 + y^2)^2}\mathbf{j}.$$

At the origin, $(x, y) = (0, 0)$ a little limit exercise⁸ shows that both components in $\nabla f(0, 0)$ diverges and so does not exist.

A sketch of the function $z = 1/(x^2 + y^2)$ shows that it too diverges to $+\infty$ at $(x, y) = (0, 0)$ but nowhere else, and so it is a local maximum (see figure 3.41).

⁸Use $x = r \cos \theta$ and $y = r \sin \theta$ etc.

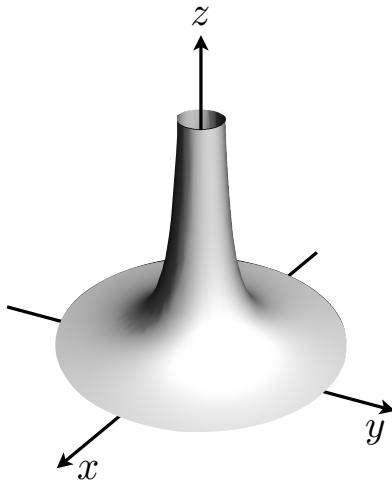


Figure 3.41: Plot of $z = 1/(x^2 + y^2)$.

- Part (iii) in the theorem holds, for example if there is no critical point, but there is a finite bounded domain.

Example: Consider the plane

$$z = f(x, y) = 1 - x - y$$

when evaluated over the domain $D : (x, y) : \{x \geq 0, y \geq 0, y \geq 1 - x\}$.

There is no point on that domain which has $\nabla f = \mathbf{0}$, since the partial derivatives are both just a constant -1 .

But clearly on the boundary of that domain, the point $(x, y, z) = (0, 0, 1)$ is a local maximum (see figure 3.42).

This is an example of how when no point interior to a domain satisfies $\nabla f = \mathbf{0}$, the maxima or minima of a function may be found on the boundary. This is an example of a **constrained** maximum

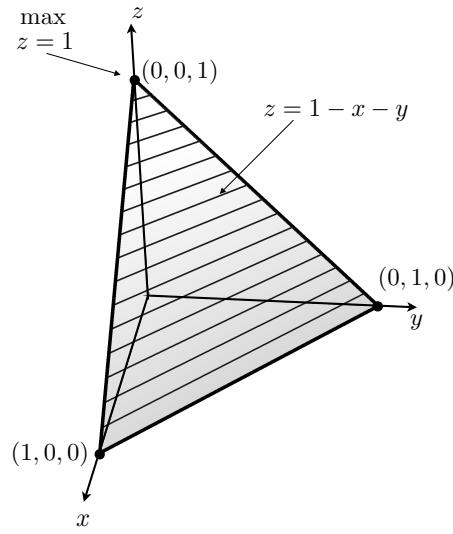


Figure 3.42: Constrained maximum on a boundary

Note that **critical points** are not just local **maxima** or **minima**!

Example: Consider the function

$$z = f(x, y) = x^2 - y^2.$$

This has a critical point when $\nabla f = \mathbf{0}$, i.e., when

$$\nabla f = 2x\mathbf{i} - 2y\mathbf{j} = \mathbf{0}.$$

Note that this implies that both components have to be equal to 0 **simultaneously**. Hence we have a critical point when both

$$2x = 0, \quad \Rightarrow \quad x = 0,$$

$$2y = 0, \quad \Rightarrow \quad y = 0,$$

i.e., when $(x, y) = (0, 0)$. However if we plot the function $z = f(x, y)$ near to the origin, see figure 3.43, we see that even though $\nabla f = 0$ there, that point is neither a maximum nor a minimum.

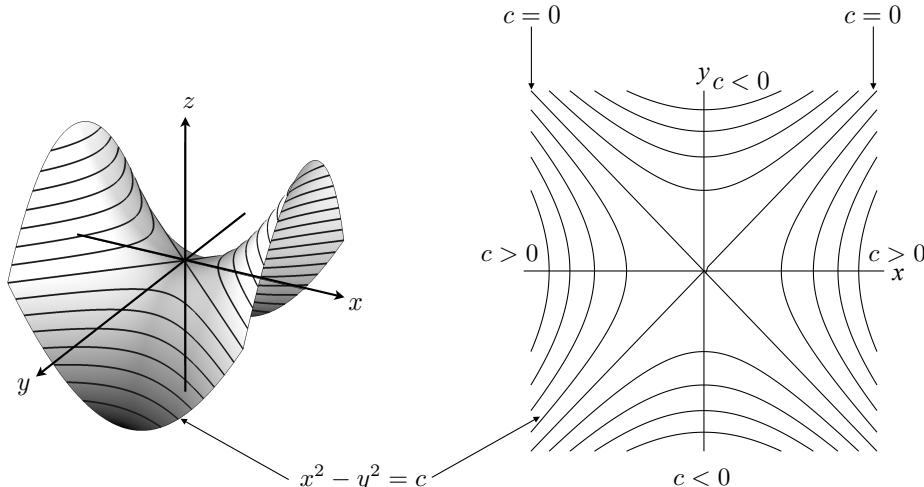


Figure 3.43: Level height contours of $z = x^2 - y^2$. These are still curves where $z = c$ where c is a constant, i.e., $x^2 - y^2 = c$. These are (for $c \neq 0$) hyperbolae. The contours corresponding to $c = 0$ are the lines $y = \pm x$. The regions $y < |x|$ have $c > 0$, the regions $y > |x|$ have $c < 0$.

For:

- If you take a vertical section through the origin along the x axis, the origin looks like a minimum, $z = f(x, 0) = x^2$.
- If you take a vertical section through the origin along the y axis, the origin looks like a maximum, $z = f(0, y) = -y^2$.

The origin is the typical example of the third type of critical point that satisfies $\nabla f = \mathbf{0}$, namely a **saddle point**. It is called a saddle point because the local shape of the surface $z = f(x, y)$ looks like a horse saddle.

Note that it is also possible to have **non-isolated critical points**. For these situations, $\nabla f = \mathbf{0}$ along whole curves, not just at single points. An easy visualisation is to take a piece of paper and curve it over to create a “U” or “∩” shape. We do not consider such cases in this module.

3.16.2 Higher Dimensions

In higher dimensions the same condition for the existence of a critical point holds.

Example: Consider the function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ given by

$$f(x, y, z) = 5 + x^2 + y^2 + z^2 + xy.$$

The critical points are given by

$$\nabla f = f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k} = \mathbf{0}.$$

Hence we simultaneously need:

$$f_x = 2x + y = 0, \quad f_y = 2y + x = 0, \quad f_z = 2z = 0.$$

The third equation immediate gives $z = 0$. Solving the first two as simultaneous equations for x and y gives $x = 0, y = 0$. hence there is a critical point at $(x, y, z) = (0, 0, 0)$.

3.16.3 Characterisation of an Isolated Interior Critical Point: Hessian Matrix

We focus on case (i) in the theorem above, i.e., critical points for which $\nabla f = \mathbf{0}$. Having seen that this can lead to a maximum, a minimum or a saddle, how can we systematically tell the difference between them algebraically? As in the 1-d case, this will use the second (here partial) derivatives. We shall again restrict an explicit discussion to $\mathbb{R}^2 \rightarrow \mathbb{R}$.

Consider the Taylor expansion of the function $z = f(x, y)$ around an isolated interior critical point P at the point $(x, y) = (a, b)$. From above, we know that $\nabla f(a, b) = \mathbf{0}$. So the Taylor series expansion about P becomes

$$\begin{aligned} f(x, y) &= f(a, b) + \left\{ \cancel{f'_x(a, b)(x - a)}^0 + \cancel{f'_y(a, b)(y - b)}^0 \right\} + \\ &+ \frac{1}{2!} \{ f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2 \} + \dots \end{aligned}$$

which on using the shorthand $h = (x - a)$ and $k = (y - b)$ becomes

$$\Rightarrow f(x, y) = f(a, b) + \frac{1}{2!} \{ h^2 f_{xx}(a, b) + 2hk f_{xy}(a, b) + k^2 f_{yy}(a, b) \} + \dots$$

Let us now suppose that we are in the immediate neighbourhood N of the critical point P , such that the orders higher than quadratic in the Taylor series are negligible.

Thus whether the critical point is a maximum or minimum will depend on the sign of the quadratic terms, or quadratic form:

$$Q = \{ h^2 f_{xx}(a, b) + 2hk f_{xy}(a, b) + k^2 f_{yy}(a, b) \}.$$

Clearly we have:

- If $Q > 0 \forall (h, k) \neq (0, 0) \in N$, $\Rightarrow P$ is a **minimum** (f is greater than $f(a, b)$ in N).
- If $Q < 0 \forall (h, k) \neq (0, 0) \in N$, $\Rightarrow P$ is a **maximum** (f is less than $f(a, b)$ in N).

We don't really want to have to check this for all $(h, k) \in N$ as we would be here forever. Fortunately, we can rearrange the formula to put it in terms of sums of squared quantities.

$$Q = \frac{1}{f_{xx}} \left\{ (hf_{xx} + kf_{xy})^2 + (f_{xx}f_{yy} - f_{xy}^2)k^2 \right\}$$

It's now not too difficult to see that the sign of Q depends on the sign of f_{xx} and the sign of the combination $f_{xx}f_{yy} - f_{xy}^2$, since all the other terms in the expression are squared and so are positive. Since these are independent of h and k we don't have to check the sign of Q throughout the neighbourhood, but just by evaluating these quantities at P .

Hence we can state the following.

For a critical point P with $\nabla f(a, b) = \mathbf{0}$,

- $f_{xx}(a, b) > 0$ and $f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b) > 0 \Rightarrow Q > 0$, so **minimum**;
- $f_{xx}(a, b) < 0$ and $f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b) > 0 \Rightarrow Q < 0$, so **maximum**.

What happens if $f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b) < 0$?

For the sake of brevity, if this is the case we define $f_{xx}f_{yy} - f_{xy}^2 = -D^2$, where D is real, so that $-D^2$ is always negative. Then the equation $Q = 0$ has two real roots and Q may be rearranged as (Gory and we'll just quote and use the result!):

$$Q = \gamma(k - \alpha h)(k - \beta h); \quad \alpha = \frac{-f_{xx}}{f_{xy} - D}, \quad \beta = \frac{-f_{xx}}{f_{xy} + D}, \quad \gamma = f_{yy}.$$

Recalling the definitions of $h = x - a$ and $k = y - b$, we can sketch the following diagrams in the neighbourhood of P , i.e., $(h, k) = (0, 0)$:

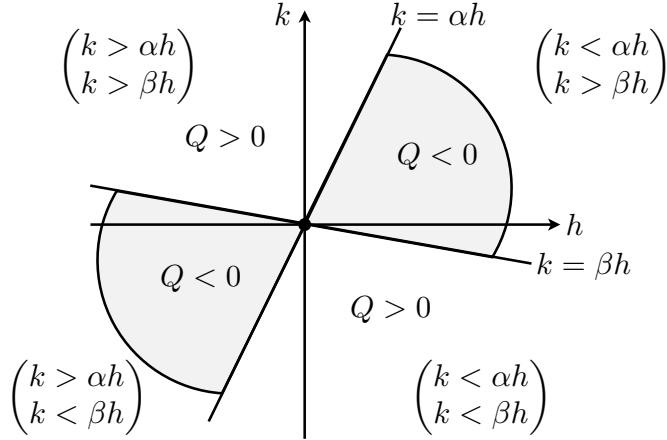


Figure 3.44: Schematic representation of the domains giving the signs of Q in the neighbourhood of $h = x - a$ and $k = y - b$.

The diagram shows that in two sectors, f increases away from the critical point, whereas in two others, it decreases. The critical point is neither a maximum nor a minimum, but the **saddle point** we saw above.

Notice that the quantity $f_{xx}f_{yy} - f_{xy}^2$ is a major player in determining the nature of the critical point. Notice that it can also be written as the determinant of a matrix of partial derivatives. This is called a Hessian determinant:

Definition: Assuming that the function $z = f(x,y)$ possesses continuous second partial derivatives and that all the first partial derivatives exist. We define the **Hessian matrix** of f as

$$H(f)(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

The **Hessian determinant** Δ is defined as

$$\Delta(x,y) = \det H(f)(x,y) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}.$$

All of the above can now be summarised in the following theorem (no proof required):

Theorem 3.16.2. (Classification of Critical points in \mathbb{R}^2): Let $f(x,y)$ be a function of two variables and let (a,b) be a critical point. Assume that $f(x,y)$ has continuous second order partial derivatives on a neighbourhood of (a,b) . Set $\Delta(a,b) = \det H(f)(a,b)$.

- (i) If $\Delta(a,b) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a,b) > 0$, then $f(x,y)$ has a local minimum at (a,b) .
- (ii) If $\Delta(a,b) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a,b) < 0$, then $f(x,y)$ has a local maximum at (a,b) .
- (iii) If $\Delta(a,b) < 0$, then $f(x,y)$ has a saddlepoint at (a,b) .
- (iv) If $\Delta(a,b) = 0$, then we have no information.

NB: You may be worrying about what happens when $\Delta(a,b) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a,b) = 0$. The answer is that it can never occur because the condition on the continuity of the second order partial derivatives means that the mixed partials are equal: $f_{xy}(a,b) = f_{yx}(a,b)$. Hence $\Delta(a,b) = f_{xx}(a,b)f_y(a,b) - f_{xy}(a,b)^2$ and so if $f_{xx} = 0$, $\Delta(a,b) \leq 0$.

Example: Classify the critical points of the function $z = f(x, y)$ where

$$f(x, y) = x^2 + y^2$$

- From an example above, we know that there is a single critical point at $(x, y) = (0, 0)$. Evaluating the second partial derivatives we have:

$$f_{xx}(x, y) = 2, \quad f_{xy}(x, y) = 0, \quad f_{yy}(x, y) = 2,$$

so,

$$f_{xx}(0, 0) = 2 > 0, \quad \Delta(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}^2(0, 0) = 2 \cdot 2 - 0 = 4 > 0,$$

so $(0, 0)$ is a minimum.

Example: Classify the critical points of the function $z = f(x, y)$ where

$$f(x, y) = 1 - x^2 - xy - y^2$$

.

- This has a critical point when

$$\nabla f = f_x \mathbf{i} + f_y \mathbf{j} = (-2x - y)\mathbf{i} + (-x - 2y)\mathbf{j} = \mathbf{0}.$$

- This gives us two simultaneous equations to solve:

$$-2x - y = 0 \quad \Rightarrow \quad y = -2x$$

$$-x - 2y = 0 \quad \Rightarrow \quad y = -x/2$$

- Solving these simultaneously we obtain that f has a critical point at $(x, y) = (0, 0)$.

- To determine the nature of the critical point we evaluate the second partial derivatives:

$$f_{xx}(x, y) = -2, \quad f_{xy}(x, y) = -1, \quad f_{yy}(x, y) = -2,$$

so,

$$f_{xx}(0, 0) = -2 < 0, \quad \Delta(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}^2(0, 0) = (-2).(-2) - (-1)^2 = 3 > 0,$$

so $(0, 0)$ is a maximum.

Example: Classify the critical points of the function $z = f(x, y)$ where

$$f(x, y) = x + y - x^2 - xy$$

- This has a critical point when

$$\nabla f = f_x \mathbf{i} + f_y \mathbf{j} = (1 - 2x - y)\mathbf{i} + (1 - x)\mathbf{j} = \mathbf{0}.$$

- This gives us two simultaneous equations to solve:

$$1 - 2x - y = 0 \quad \Rightarrow \quad y = -2x + 1$$

$$1 - x = 0 \quad \Rightarrow \quad x = 1$$

- Solving these simultaneously we obtain that f has a critical point at $(x, y) = (1, -1)$
- To determine the nature of the critical point we evaluate the second partial derivatives:

$$f_{xx}(x, y) = -2, \quad f_{xy}(x, y) = -1, \quad f_{yy}(x, y) = 0,$$

so,

$$f_{xx}(1, -1) = -2 < 0, \quad \Delta(1, -1) = f_{xx}(1, -1)f_{yy}(1, -1) - f_{xy}^2(1, -1) = -2.0 - (-1)^2 = -1 < 0,$$

so $(1, -1)$ is a saddlepoint.

Example: Classify the critical points of the function $z = f(x, y)$ where

$$f(x, y) = x + y^2 - x^3/3$$

- This has a critical point when

$$\nabla f = f_x \mathbf{i} + f_y \mathbf{j} = (1 - x^2)\mathbf{i} + (2y)\mathbf{j} = \mathbf{0}.$$

- This gives us two simultaneous equations to solve:

$$1 - x^2 = 0 \quad \Rightarrow \quad x = \pm 1$$

$$y = 0 \quad \Rightarrow \quad y = 0$$

- Solving these simultaneously we obtain that f has a critical point at $(x, y) = (1, 0)$ and $(x, y) = (-1, 0)$
- To determine the nature of the critical point we evaluate the second partial derivatives:

– At $(1,0)$:

$$f_{xx}(x,y) = -2x, \quad f_{xy}(x,y) = 0, \quad f_{yy}(x,y) = 2,$$

so,

$$f_{xx}(1,0) = -2 < 0, \quad \Delta(1,0) = f_{xx}(1,0)f_{yy}(1,0) - f_{xy}^2(1,0) = -2.2 - (0)^2 = -4 < 0,$$

so $(1,0)$ is a saddlepoint.

– At $(-1,0)$:

$$f_{xx}(x,y) = -2x, \quad f_{xy}(x,y) = 0, \quad f_{yy}(x,y) = 2,$$

so,

$$f_{xx}(-1,0) = 2 > 0, \quad \Delta(-1,0) = f_{xx}(-1,0)f_{yy}(-1,0) - f_{xy}^2(-1,0) = 2.2 - (0)^2 = +4 > 0,$$

so $(-1,0)$ is a minimum.

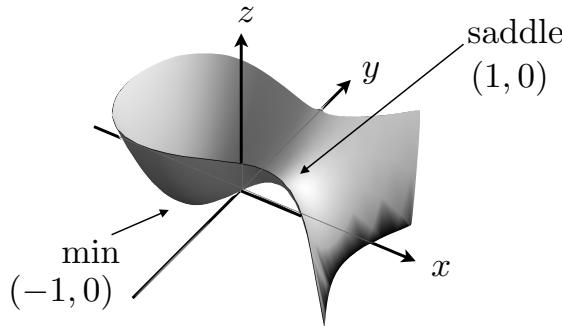


Figure 3.45: The surface $z = x + y^2 - x^3/3$ and its critical points.

Example: Classify the critical points of the function $z = f(x,y)$ where

$$f(x,y) = x^2 - 3xy^2 + 2y^4$$

- This has a critical point when

$$\nabla f = f_x \mathbf{i} + f_y \mathbf{j} = (2x - 3y^2)\mathbf{i} + (-6xy + 8y^3)\mathbf{j} = \mathbf{0}.$$

- This gives us two simultaneous equations to solve:

$$2x - 3y^2 = 0 \quad \Rightarrow \quad y^2 = 2x/3$$

$$-6xy + 8y^3 = 0 \quad \Rightarrow \quad y = 0, y^2 = 3x/4$$

- Solving these simultaneously we obtain that f has a critical point at $(x,y) = (0,0)$.

- To determine the nature of the critical point we evaluate the second partial derivatives:

– At (0,0):

$$f_{xx}(x, y) = 2, \quad f_{xy}(x, y) = -6y, \quad f_{yy}(x, y) = -6x + 24y^2,$$

so,

$$f_{xx}(0, 0) = 2 > 0, \quad \Delta(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}^2(0, 0) = 2.(-6.0+24.0)-(-6.0)^2 = 0,$$

so the Hessian matrix test fails at (0,0). More information is needed. (In fact looking at contour plots around (0,0) suggests that the critical point is a saddlepoint.)

3.16.4 Higher Dimensions

The classification of critical points in higher dimensions follows from a simple extension of the above results. It proceeds by calculation of successive sub-determinants of the relevant Hessian matrix.

Example: The Hessian matrix of a doubly differentiable $\mathbb{R}^3 \rightarrow \mathbb{R}$ function $f(x, y, z)$ is

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}$$

Definition: Given an n by n matrix with elements a_{ij} , we define a sequence of k by k determinants ($k = 1$ to n) of upper-left sub-determinants as:

$$\Delta_0 = 1; \quad \Delta_1 = a_{11}; \quad \Delta_2 = \left| \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right|; \quad \dots \quad \Delta_k = \left| \begin{array}{ccc} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{array} \right|.$$

Given this definition, the following theorem can be proved (proof not required):

Theorem 3.16.3. (Classification of Critical Points in \mathbb{R}^n): Suppose that P is a critical point of a multivariate function f interior to the domain of f . Suppose that the second partial derivatives of f are continuous in the neighbourhood of P . Then

- (a) If $Hf|_P$ has $\Delta_i > 0, i = 0, \dots, n$, then f has a local **minimum** at P ;
- (b) If $Hf|_P$ has $\Delta_i \neq 0, i = 0, \dots, n$, with $\Delta_1 < 0, \Delta_2 > 0, \Delta_3 < 0, \Delta_4 > 0 \dots$, etc., then f has a local **maximum** at P ;
- (c) If $Hf|_P$ has $\Delta_i \neq 0, i = 0, \dots, n$ but the Δ_i have neither of the above sign patterns, then f has a **saddlepoint** at P ;
- (d) If $Hf|_P$ has one or more $\Delta_i = 0$, then no information about P may be obtained from this test. there may be a maximum, minimum or saddlepoint at P .

Example: Classify the critical points of

$$w = f(x, y, z) = 5 + x^2 + y^2 + z^2 + xy$$

- The critical point is at $\nabla f = \mathbf{0}$. So we have

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} = (2x+y)\mathbf{i} + (2y+x)\mathbf{j} + 2z\mathbf{k} = \mathbf{0}.$$

From which we immediately see that $z = 0$.

The x and y coordinates of P are given by the simultaneous solution of

$$2x + y = 0, \quad x + 2y = 0, \quad z = 0,$$

from which we see that $x = y = z = 0$.

Hence P is at $(0, 0, 0)$.

- The Hessian matrix and its sub-determinants are:

$$Hf|_P = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}_P = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\Delta_0 = 1; \quad \Delta_1 = 2; \quad \Delta_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 2; \quad \Delta_3 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 6.$$

Hence all the $\Delta_i, i = 0, 1, 2, 3$ are greater than zero. Hence the critical point P is a minimum.

Hessians should not be confused with mercenaries used in the American War of Independence, or Deputy-Führers of the same names.

3.17 Maximisation and Minimisation with a Constraint: Lagrange Multipliers

Consider the following problem:

Example: Find the dimensions and area of the largest rectangle inside the ellipse

$$\frac{x^2}{4} + y^2 = 1.$$

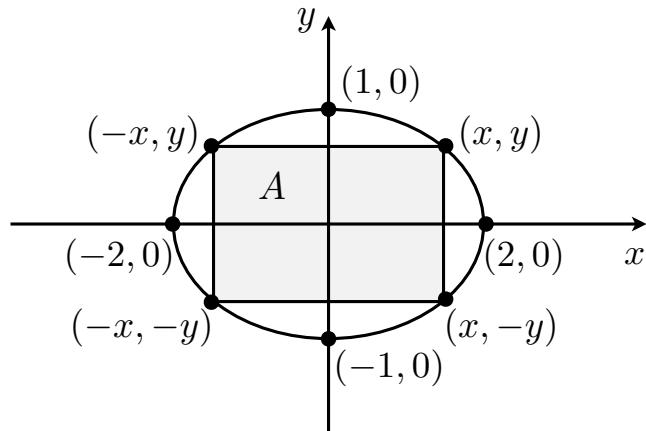


Figure 3.46: Rectangle of area $A = 4xy$ embedded within the ellipse $x^2/4 + y^2 = 1$.

- Consider a rectangle that lies within the ellipse with the top right corner being at the point (x, y) that lies on the ellipse (see diagram). By symmetry the other three corners of the rectangle are at $(x, -y)$, $(-x, y)$ and $(-x, -y)$.
- The lengths of the sides are $2x$ and $2y$, so the area of the rectangle within the ellipse is $A = 4xy$.
- The goal is then to **maximise** $A = 4xy$ subject to the **constraint** that $x^2/4 + y^2 = 1$.
- In this example it is easy to see that we could eliminate y by solving the constraint for y and substituting the resulting expression⁹ $y = \sqrt{4 - x^2}/2$ in terms of x into the formula for A to obtain

$$A = 2x\sqrt{4 - x^2}.$$

- The finding of the critical point x corresponding to the maximum of A would then take place by just differentiating A with respect to x and setting

$$\frac{dA}{dx} = 2\sqrt{4 - x^2} - \frac{2x^2}{\sqrt{4 - x^2}} = 0 \quad \Rightarrow \quad x = \sqrt{2}.$$

where we have taken the positive value of x since it is the length of a rectangle.

- Hence we have $y = \sqrt{4 - 2}/2 = 1/\sqrt{2}$ and so $A = 4\sqrt{2}/\sqrt{2} = 4$.

In this case we were able to reduce the problem to one of single variable calculus by being able to solve the constraint equation ($x^2/4 + y^2 = 1$) exactly for one of the variables. What happens when the constraint is not solvable for one of the variables, e.g., $(x + y) = e^{-xy}$? In the general case, we shall use the **Method of Lagrange Multipliers**.

3.17.1 General Statement of Problem of Optimisation with an Algebraic Constraint

We will restrict ourselves to $\mathbb{R}^2 \rightarrow \mathbb{R}$. We require the solution of the following problem.

Problem: Find the maximum or minimum of $f(x, y)$, subject to $g(x, y) = 0$.

- To do this we form what is called the the **Lagrangian** \mathcal{L} :

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda g(x, y),$$

where λ is a constant, called the **Lagrange multiplier**.

- Now $g(x, y) = 0$ defines a curve \mathcal{C} in \mathbb{R}^2 .
- We seek to optimise \mathcal{L} with respect to all three of its variables x, y, λ . The critical point is when $\nabla_{x,y,\lambda}\mathcal{L} = \mathbf{0}$, i.e., when

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0, \\ \frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0, \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= g(x, y) = 0. \end{aligned}$$

⁹We take the positive square root as y is a length.

- Clearly from the third line, if we find the critical point of \mathcal{L} we automatically satisfy the constraint $g(x, y) = 0$. Hence the critical points of \mathcal{L} lie on \mathcal{C} .
- But the Lagrangian \mathcal{L} is identical to f on \mathcal{C} (where $g(x, y) = 0$).
- Hence if we are on \mathcal{C} and we find the critical point of \mathcal{L} from the above three equations, we are also finding the critical point of f on \mathcal{C} , i.e., we are finding the max/min of $f(x, y)$ subject to the constraint $g(x, y) = 0$.
- We can go further, and eliminate the Lagrange multiplier λ from the first two equations:

$$\begin{aligned}\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} &= 0, & \Rightarrow & \lambda = -\frac{\partial f}{\partial x} / \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} &= 0 & \Rightarrow & \lambda = -\frac{\partial f}{\partial y} / \frac{\partial g}{\partial y} \\ \Rightarrow \frac{\partial f}{\partial x} / \frac{\partial g}{\partial x} &= \frac{\partial f}{\partial y} / \frac{\partial g}{\partial y}\end{aligned}$$

which may be rearranged to give:

$$\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial y} = 0$$

or, in terms of the Jacobian determinant,

$$\frac{\partial(f, g)}{\partial(x, y)} = 0.$$

- Hence the critical point will be given by the simultaneous solution of

$$\frac{\partial(f, g)}{\partial(x, y)} = 0, \quad g(x, y) = 0.$$

- Note that the actual value of the Lagrange multiplier is therefore **irrelevant!**

We may now repeat the above example as follows:

Example: We have the area function

$$f(x, y) = 4xy$$

and the constraint function

$$g(x, y) = x^2/4 + y^2 - 1 = 0$$

- Using the above method, the relevant partial derivatives are:

$$\frac{\partial f}{\partial x} = 4y, \quad \frac{\partial f}{\partial y} = 4x, \quad \frac{\partial g}{\partial x} = \frac{x}{2}, \quad \frac{\partial g}{\partial y} = 2y.$$

- Substituting into the Jacobian determinant we have:

$$\frac{\partial(f, g)}{\partial(x, y)} = \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial y} \right) = (4y)(2y) - (4x)(x/2) = 8y^2 - 2x^2 = 0$$

- Hence we have a critical point when

$$y^2 = x^2/4$$

and (from the third equation $\partial\mathcal{L}/\partial\lambda = 0$ above)

$$g(x, y) = x^2/4 + y^2 - 1 = 0.$$

Hence we can substitute for y in g to obtain

$$x^2/4 + x^2/4 - 1 = 0 \quad \Rightarrow x^2 = 2 \quad \Rightarrow x = \pm\sqrt{2}.$$

Hence using $y^2 = x^2/4$ we obtain

$$y^2 = 2/4 \quad \Rightarrow x = \pm 1/\sqrt{2}.$$

- Finally since x and y are meant to be the lengths of sides of a rectangle, they both have to be positive. So the critical point is when

$$(x, y) = (\sqrt{2}, 1/\sqrt{2})$$

and the area is

$$4\sqrt{2}/\sqrt{2} = 4.$$

- Note that we don't choose to show that this is a maximum here.

Example: A farmer wants to enclose a rectangle of land next to a stream (see diagram) She will build 3 fences, the two perpendicular to the stream will be of length x , the one parallel to the stream will be of length y . If the farmer has only 100m length of fencing, what is the maximum area that she may enclose?

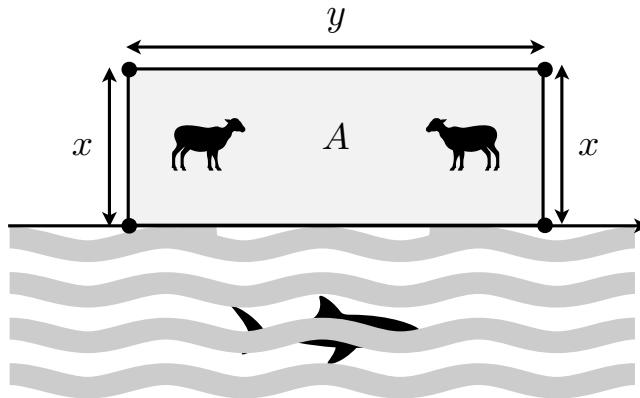


Figure 3.47: Area of enclosed rectangular field is $A = xy$. Since it is adjacent to the stream only three fences are built with total perimeter $2x + y = 100$ m.

- The problem is to maximise

$$f(x, y) = xy$$

subject to

$$g(x, y) = 2x + y - 100 = 0.$$

- Using the above method, the relevant partial derivatives are:

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\partial g}{\partial x} = 2, \quad \frac{\partial g}{\partial y} = 1.$$

- Substituting into the Jacobian determinant we have:

$$\frac{\partial(f, g)}{\partial(x, y)} = \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial y} \right) = (y)(1) - (x)(2) = y - 2x = 0$$

- Hence we have a critical point when

$$y = 2x$$

and (from the third equation $\partial\mathcal{L}/\partial\lambda = 0$ above)

$$g(x, y) = 2x + y - 100 = 0.$$

Hence we can substitute for y in g to obtain

$$2x + 2x - 100 = 0 \quad \Rightarrow x = 25.$$

Hence using $y = 2x$ we obtain

$$y = 50.$$

- So the critical point is when

$$(x, y) = (25, 50)$$

and the area is

$$25 \times 50 = 1,250 \text{m}^2.$$

Other types of problems involving optimisation with a constraint are possible (e.g., there are multiple constraints and/or at least one of the constraints is an inequality), but we will leave these for later modules.

One interesting use of optimisation with a set of algebraic constraints is in the selection of an optimal portfolio of stocks to maximise the expected return. This problem is dealt with in MATH3022.