Frisch-Waugh-Lovell Theorem

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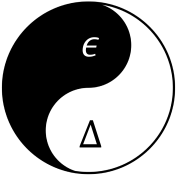

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Overview I

The **Frisch-Waugh-Lovell (FWL) theorem**, named after the econometricians Ragnar Frisch, Frederick V. Waugh, and Michael C. Lovell, might be one of the most useful theorems in Econometrics. Its key idea can be stated in two ways.

Theorem 1 (FWL Theorem, Three-Step Version)

In the model $\mathbf{Y} = \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2 + \mathbf{e}$, the OLS estimator of β_2 may be computed by either the OLS regression $\mathbf{Y} = \mathbf{X}_1\hat{\beta}_1 + \mathbf{X}_2\hat{\beta}_2 + \hat{\mathbf{e}}$ or via the following algorithm:

1) Regress $\mathbf{Y}$ on $\mathbf{X}_1$, obtain residuals $\tilde{\mathbf{e}}_1$;
2) Regress $\mathbf{X}_2$ on $\mathbf{X}_1$, obtain residuals $\tilde{\mathbf{e}}_2$;
3) Regress $\tilde{\mathbf{e}}_1$ on $\tilde{\mathbf{e}}_2$, obtain OLS estimates $\hat{\beta}_2$.

Overview II

Theorem 2 (FWL Theorem, Two-Step Version)

In the model $\mathbf{Y} = \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2 + \mathbf{e}$, the OLS estimator of β_2 may be computed by either the OLS regression $\mathbf{Y} = \mathbf{X}_1\hat{\beta}_1 + \mathbf{X}_2\hat{\beta}_2 + \hat{\mathbf{e}}$ or via the following algorithm:

1) Regress $\mathbf{X}_2$ on $\mathbf{X}_1$, obtain residuals $\tilde{\mathbf{e}}_2$;
2) Regress $\mathbf{Y}$ on $\tilde{\mathbf{e}}_2$, obtain OLS estimates $\hat{\beta}_2$.

The two-step version has the exactly same meaning as the regression anatomy formula introduced by Angrist and Pischke (2009).

$$\hat{\beta}_2 = \frac{Cov(Y, \tilde{e}_2)}{Var(\tilde{e}_2)}$$

What's the Usage of the FWL Theorem?

To my knowledge, the FWL Theorem is most often used in the derivations of estimators and in the proofs for the equivalence between two estimators.

For example,

- In Chapter 4 of Angrist and Pischke (2009) and Chapter 12 of Hansen (2022), the authors used the FWL Theorem to derive the two-stage least squares (2SLS) estimator.
- In Wooldridge (2021), the author used the FWL Theorem to show that the coefficient of the independent variable X_{it} in the two-way Mundlak (TWM) regression is the same as the one in the traditional two-way fixed effects (TWFE) regression.

Mathematical Proof

Step 1: Partition the Regressors

Consider a model $\mathbf{Y} = \mathbf{X}\beta + \mathbf{e}$, where

- $\mathbf{Y}$ is an $n \times 1$ vector of outcomes,
- $\mathbf{X}$ is an $n \times k$ matrix of regressors,
- β is a $k \times 1$ vector of coefficients of interest, and
- $\mathbf{e}$ is an $n \times 1$ vector of error terms.

The regressors and coefficients can be partitioned as

$$\underset{n\times k}{\mathbf{X}} = \begin{pmatrix} \underset{n\times k_1}{\mathbf{X}_1} & \underset{n\times k_2}{\mathbf{X}_2} \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}_{(k_1+k_2)\times 1}$$

where $k_1 + k_2 = k$. Then the model can be rewritten as

$$\mathbf{Y} = \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2 + \mathbf{e}$$

Step 2: Construct a System of Equations

We are interested in algebraic expressions for β_1 and β_2 in the model $\mathbf{Y} = \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2 + \mathbf{e}$.

Recall that the OLS estimator is $\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$. Thus,

$$(\mathbf{X}'\mathbf{X})\hat{\beta} = \mathbf{X}'\mathbf{Y}$$

$$\Rightarrow \left[\begin{pmatrix} \mathbf{X}_1' \\ \mathbf{X}_2' \end{pmatrix} \begin{pmatrix} \mathbf{X}_1 & \mathbf{X}_2 \end{pmatrix}\right] \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1' \\ \mathbf{X}_2' \end{pmatrix} \mathbf{Y}$$

$$\Rightarrow \begin{pmatrix} \mathbf{X}_1'\mathbf{X}_1 & \mathbf{X}_1'\mathbf{X}_2 \\ \mathbf{X}_2'\mathbf{X}_1 & \mathbf{X}_2'\mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1'\mathbf{Y} \\ \mathbf{X}_2'\mathbf{Y} \end{pmatrix}$$

Step 3: Solve the System I

The matrix form above can be written as a system of two equations:

$$\begin{pmatrix} \mathbf{X}_1'\mathbf{X}_1 & \mathbf{X}_1'\mathbf{X}_2 \\ \mathbf{X}_2'\mathbf{X}_1 & \mathbf{X}_2'\mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1'\mathbf{Y} \\ \mathbf{X}_2'\mathbf{Y} \end{pmatrix} \iff \begin{cases} \mathbf{X}_1'\mathbf{X}_1\hat{\beta}_1 + \mathbf{X}_1'\mathbf{X}_2\hat{\beta}_2 = \mathbf{X}_1'\mathbf{Y} & (1) \\ \mathbf{X}_2'\mathbf{X}_1\hat{\beta}_1 + \mathbf{X}_2'\mathbf{X}_2\hat{\beta}_2 = \mathbf{X}_2'\mathbf{Y} & (2) \end{cases}$$

To solve the system, let's first focus on $\hat{\beta}_1$. From (1), we have

$$\mathbf{X}_1'\mathbf{X}_1\hat{\beta}_1 = \mathbf{X}_1'\mathbf{Y} - \mathbf{X}_1'\mathbf{X}_2\hat{\beta}_2$$

$$\Rightarrow \hat{\beta}_1 = (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\left(\mathbf{Y} - \mathbf{X}_2\hat{\beta}_2\right) \quad (3)$$

which shows that we can get $\hat{\beta}_1$ by regressing $\left(\mathbf{Y} - \mathbf{X}_2\hat{\beta}_2\right)$ on $\mathbf{X}_1$.

Step 3: Solve the System II

Next, substituting (3) into (2) gives us

$$\mathbf{X}_2'\mathbf{X}_1\left[(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\left(\mathbf{Y}-\mathbf{X}_2\hat{\beta}_2\right)\right]+\mathbf{X}_2'\mathbf{X}_2\hat{\beta}_2=\mathbf{X}_2'\mathbf{Y}$$

$$\Rightarrow \mathbf{X}_2'\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{Y}-\mathbf{X}_2'\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2\hat{\beta}_2+\mathbf{X}_2'\mathbf{X}_2\hat{\beta}_2=\mathbf{X}_2'\mathbf{Y}$$

$$\Rightarrow \mathbf{X}_2'\left[\mathbf{I}-\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\right]\mathbf{X}_2\hat{\beta}_2=\mathbf{X}_2'\left[\mathbf{I}-\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\right]\mathbf{Y}$$

$$\Rightarrow (\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2)\hat{\beta}_2=\mathbf{X}_2'\mathbf{M}_1\mathbf{Y}$$

where $\mathbf{I}$ is an $n \times n$ identity matrix, and $\mathbf{M}_1=\mathbf{I}-\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'$ is the **annihilator matrix**, also called the **residual maker matrix**. appendix

Therefore,

$$\hat{\beta}_2=(\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2)^{-1}(\mathbf{X}_2'\mathbf{M}_1\mathbf{Y})$$

Step 3: Solve the System III

To summarize, if we partition a regression to $\mathbf{Y} = \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2 + \mathbf{e}$, then the OLS estimators can be expressed as

$$\hat{\beta}_1 = (\mathbf{X}_1'\mathbf{M}_2\mathbf{X}_1)^{-1}(\mathbf{X}_1'\mathbf{M}_2\mathbf{Y})$$

$$\hat{\beta}_2 = (\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2)^{-1}(\mathbf{X}_2'\mathbf{M}_1\mathbf{Y})$$

Explaining Our Results I

Since the annihilator matrix is idempotent, $\mathbf{M}_1 = \mathbf{M}_1\mathbf{M}_1$ and $\mathbf{M}_2 = \mathbf{M}_2\mathbf{M}_2$, then we have

$$\begin{aligned}\hat{\beta}_2 &= (\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{M}_1\mathbf{Y} \\ &= \left[(\mathbf{M}_1\mathbf{X}_2)'(\mathbf{M}_1\mathbf{X}_2)\right]^{-1}(\mathbf{M}_1\mathbf{X}_2)'\mathbf{M}_1\mathbf{Y}\end{aligned}$$

where $\mathbf{M}_1\mathbf{X}_2$ is the residual from the regression of $\mathbf{X}_2$ on $\mathbf{X}_1$, and $\mathbf{M}_1\mathbf{Y}$ is the residual from the regression of $\mathbf{Y}$ on $\mathbf{X}_1$.

Thus, in words, $\hat{\beta}_2$ is the coefficient from the regression of $\mathbf{M}_1\mathbf{Y}$ on $\mathbf{M}_1\mathbf{X}_2$. Similarly, $\hat{\beta}_1 = (\mathbf{X}_1'\mathbf{M}_2\mathbf{X}_1)^{-1}(\mathbf{X}_1'\mathbf{M}_2\mathbf{Y})$ tells us that $\hat{\beta}_1$ is the coefficient from the regression of $\mathbf{M}_2\mathbf{Y}$ on $\mathbf{M}_2\mathbf{X}_1$.

This is the proof for the **three-step** version of FWL Theorem.

Explaining Our Results II

Now let's consider our results in another way: The expressiong of $\hat{\beta}_2$ can also be written as

$$\hat{\beta}_2 = (\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{M}_1\mathbf{Y} = [(\mathbf{M}_1\mathbf{X}_2)'\mathbf{M}_1\mathbf{X}_2]^{-1}(\mathbf{M}_1\mathbf{X}_2)'\mathbf{Y}$$

Thus, we can understand the coefficient estimator in another way: $\hat{\beta}_2$ is from a regression of $\mathbf{Y}$ on $\mathbf{M}_1\mathbf{X}_2$ (i.e., a residual from the regression of $\mathbf{X}_2$ on $\mathbf{X}_1$).

This is the proof corresponding to the **two-step** version of FWL Theorem.

Appendix

Annihilator Matrix

The annihilator matrix is defined as

$$\mathbf{M} = \mathbf{I}_n - \mathbf{P} = \mathbf{I}_n - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$$

where $\mathbf{I}_n$ is an $n \times n$ identity matrix and $\mathbf{P}$ is the projection matrix. The annihilator matrix $\mathbf{M}$ has similar properties with $\mathbf{P}$, including that $\mathbf{M}$ is **symmetric** ($\mathbf{M}' = \mathbf{M}$) and **idempotent** ($\mathbf{M}\mathbf{M} = \mathbf{M}$). back

$\mathbf{M}$ is called the **annihilator matrix**, because

$$\mathbf{M}\mathbf{X} = (\mathbf{I}_n - \mathbf{P})\mathbf{X} = \mathbf{X} - \mathbf{P}\mathbf{X} = \mathbf{X} - \mathbf{X} = \mathbf{0}$$

$\mathbf{M}$ is also called the **residual maker matrix**, because it creates least squares residuals:

$$\mathbf{M}\mathbf{Y} = \mathbf{Y} - \mathbf{P}\mathbf{Y} = \mathbf{Y} - \mathbf{X}\hat{\beta} = \hat{\mathbf{e}}$$

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