Characterizing a Distribution by Distribution Functions

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Table of Contents

Overview I

One crucial feature of the variables we study in Econometrics is that the outcome (or payoff) of interest is stochastic, random, and uncertain. Our goal is to study the **distribution** behind such outcome, instead of the probability that a specific outcome appears.

We have multiple approaches to characterizing the distribution of a random variable. Those approaches, in my opinion, can be divided into two big groups:

- Probability-based (direct) approach;
- Expectation-based (indirect) approach.

Here my focus is the first one.

Overview II

	Discrete Variable	**Continuous Variable**
Probability	PMF	PDF
Cumulative Probability	CDF	CDF

- **PMF**: probability mass function.
- **PDF**: probability density function.
- **CDF**: cumulative probability function.

Probability Mass Function

Discrete Random Variable

Definition 1

A random variable, X, is a **discrete random variable** if there is a discrete set $\mathcal{X} \subseteq \mathbb{R}$ such that $Pr(X \in \mathcal{X}) = 1$. A **discrete set** $\mathcal{X}$ is a set that contains a finite or countably infinite number of elements.

An **indicator function**, $\mathbb{1}\{\cdot\}$, can be used to construct a discrete random variable. For example, for an event A,

$$\mathbb{1}\{a \in A\} = \begin{cases} 1 & \text{if } a \in A \\ 0 & \text{otherwise} \end{cases}$$

Definition of PMF

Definition 2

The **probability mass function (PMF)** of a discrete random variable X is a function $f_X : \mathcal{X} \to [0, 1]$ such that $f_X(x) = Pr(X = x)$, the probability that X equals x.

If f_X is the PMF of X, we can write $X \sim f_X$, which is read as "X follows distribution f_X". When it is clear from the context, we can omit the subscript and write only $X \sim f$.

Properties of PMF

The PMF, $p_i \equiv Pr(X = x_i)$, must satisfy two properties:

- $0 \leq p_i \leq 1$;
- $\sum_i p_i = 1$.

These two properties can be easily proved by the three axioms of probability.

Reporting the PMF I

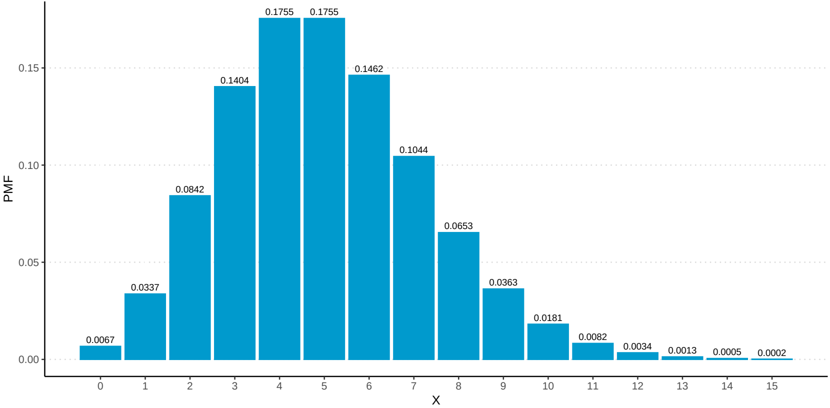
There are three common ways of reporting the PMF (or any probability distribution function) of a random variable:

1) **Mathematical form.** For example, if X follows a Poisson distribution, then its PMF is $f_X(x) = \frac{\lambda^x e^{-\lambda}}{x!}$ with parameter $\lambda > 0$.
2) **Table.** For example, a fair coin is flipped until it comes up heads the first time. At that point, the player win 2^n where n is the number of times the coin was flipped. The PMF for possible payoffs is

Payoff	**Probability**
2	$\frac{1}{2}$
2^2	$\frac{1}{2^2}$
$\vdots$	$\vdots$
2^n	$\frac{1}{2^n}$

Reporting the PMF II

3) **Histogram.** For example, the histogram below shows the PMF of a random variable $X \sim \mathit{Poisson}(5)$.



Cumulative Distribution Function

Definition of CDF

Definition 3

The **cumulative distribution function (CDF)** of X is a function $F_X : \mathbb{R} \to [0, 1]$ such that

$$F_X(x) = Pr(X \leq x)$$

If the CDF of X is F_X, we can write $X \sim F_X$, which is read as "X follows the distribution F_X".

Properties of CDF

A CDF has the following unique properties:

- $\lim_{x\to\infty} F_X(x) = 1$ and $\lim_{x\to-\infty} F_X(x) = 0$.
- $F_X(x)$ is non-decreasing in x. proof
- $F_X(x)$ is right-continuous: $\lim_{x\to x_0^+} F_X(x) = F_X(x_0)$. continuity
- $F_X(x)$ has a limit from the left, i.e., $\lim_{x\to x_0^-} F_X(x)$ exists.

Note: The above only says that $F_X(x)$ has a limit from the left but doesn't say $\lim_{x\to x_0^-} F_X(x) = F_X(x_0)$. Actually, it is possible that $\lim_{x\to x_0^-} F_X(x) \neq F_X(x_0)$; thus, a CDF may be discontinuous.

CDF versus PMF

For a discrete random variable X, we can obtain the CDF from the PMF by summation:

$$F_X(x) = \sum_{x' \in \mathcal{X} : x' \leq x} f_X(x')$$

Conversely, we can obtain the PMF from the CDF by subtraction:

$$f_X(x) = Pr(X = x) = Pr(X \leq x) - Pr(X < x) = F_X(x) - \lim_{x' \to x^-} F_X(x')$$

Note: CDF is well defined for discrete random variable and non-discrete random variables, but PMF is only defined for discrete random variables (detailed later).

Quantile: The Inverse of CDF

Definition 4

For a $\tau \in (0,1)$, the $100\tau\%$ **quantile** of a random variable $X \sim F_X$, denoted by q_τ or $Q_X(\tau)$, is defined to be

$$F_X^{-1}(\tau) = \inf\{x \in \mathcal{X} : F_X(x) \geq \tau\}$$

where inf is the abbreviation of the "infimum".

As you may know, the inverse function of a CDF F_X doesn't always exist because F_X may not be a bijective (one-to-one) function. However, in the definition of quantile shown above, we use a *specific* inverse function of F_X, which always exists. Actually, there are various algorithms for quantile in statistical software (e.g., there are 9 types in R and type 1 is exactly what we saw above).

Probability Density Function

Continuous Random Variable

Definition 5

A **continuous random variable** is one which takes an infinite number of possible values.

If $X \sim F_X(x)$ and $F_X(x)$ is continuous, then X is a continuous random variable.

PMF Is Not Defined

PMF (defining **absolute probability**) is not defined for continuous variable. Why? Because we always have $Pr(X = x) = 0$ for a continuous random variable X.

Proof:

$$
\begin{aligned}
Pr(X = x) &= \lim_{\epsilon \to 0} Pr(x \leq X \leq x + \epsilon) \\
&= \lim_{\epsilon \to 0} [F_X(x + \epsilon) - F_X(x)] \\
&= 0
\end{aligned}
$$

$\blacksquare$

Thus, we resort to a concept defining **relative probability**: PDF!

Definition of PDF

Definition 6

The **probability density function (PDF)** for a continuous random variable X with CDF F_X is a function that satisfies the following integral equation for all values of $x \in \mathbb{R}$:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\ dt$$

It is clear from the definition of PDF that

$$Pr(a \leq X \leq b) = Pr(X \leq b) - Pr(X \leq a) = \int_a^b f_X(x)\ dx$$

Propeties of PDF

A PDF has the following unique properties:

- $f_X(x) \geq 0$ for all x. This property follows from the fact that $F_X(x)$ is non-decreasing in x.
- $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$. Geometrically, this means that the area between the PDF and the x-axis must equal 1.

Note: In contract with PMF, PDF $f_X(x)$ can be greater than 1! Recall that a PDF defines the relative probability rather than the absolute probability.

Link between PDF and CDF

Definition 7

A continuous F_X is **absolutely continuous** if it is differentiable at all points in $\mathbb{R}$ except a finite or countably infinite number of them.

If a continuous random variable X has an absolutely continuous CDF $F_X(x)$, then its PDF is of the form

$$f_X(x) = \begin{cases} \frac{dF_X(x)}{dx} & \text{if } x \in \mathcal{X}' \\ \text{arbitrary value} & \text{if } x \notin \mathcal{X}' \end{cases}$$

where $\mathcal{X}'$ is the set of $x \in \mathbb{R}$ at which F_X is differentiable.

Empirical Distribution

Emprical PMF, CDF, PDF, and Quantile I

The definitions of PMF, CDF, PDF, and quantile apply to the **population**. In real life, we cannot observe them, because we only have a **sample** of random variables (i.e., data). Thus, we can only *estimate* them. The estimated ones are called **empirical** PMF, PDF, CDF, and quantile.

Let $(x_1, x_2, ..., x_n)$ be independent and identically distributed samples drawn from some unknown univariate distribution.

- Empirical PMF can be obtained from dividing frequency by number of payoffs:

$$\hat{f}(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{x_i = x\}$$

Emprical PMF, CDF, PDF, and Quantile II

- Empirical CDF can be obtained similarly from

$$\widehat{F}(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{x_i \leq x\}$$

- Empirical PDF can be obtained by kernel density estimation (KDE):

$$\hat{f}(x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right)$$

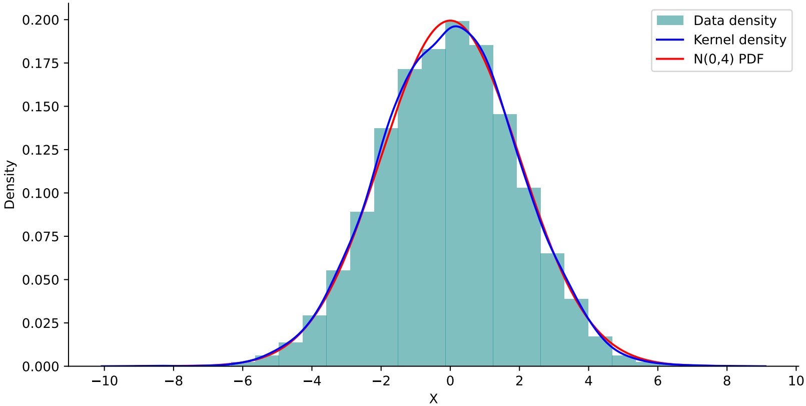
  where h is bandwidth and $K(\cdot)$ is a kernel function.

- Empirical quantile can be obtained from empirical CDF:

$$\widehat{Q}(\tau) = \inf\left\{x : \widehat{F}(x) \geq \tau\right\} = \inf\left\{x : \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{x_i \leq x\} \geq \tau\right\}$$

Emprical PMF, CDF, PDF, and Quantile III

Example: Empirical PDF versus True PDF of $N(0, 4)$



Appendix

Proof of Non-Decreasing CDF

Consider a randon variable X with support set $\mathcal{X}$. Suppose that $x_1 < x_2$ without loss of generality. Set

$$A = \{x \in \mathcal{X} : x \leq x_1\} \quad \text{and} \quad B = \{x \in \mathcal{X} : x \leq x_2\}$$

Then, $A \subseteq B$. Using the implication of probability theory, we have $Pr(A) \leq Pr(B)$; that is,

$$Pr(X \leq x_1) \leq Pr(X \leq x_2)$$

By definition of CDF, we have

$$F_X(x_1) \leq F_X(x_2)$$

Thus, we've proved that CDF is non-decreasing.

back

Limit and Continuity of a Function I

Assume the domain of function f contains an interval (c, d) to the right of c. We say that $f(x)$ has **right-hand limit** L at c, and write

$$\lim_{x \to c^+} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon$$

whenever $c < x < c + \delta$.

Limit and Continuity of a Function II

Assume the domain of function f contains an interval (b, c) to the left of c. We say that $f(x)$ has **left-hand limit** L at c, and write

$$\lim_{x \to c^-} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon$$

whenever $c - \delta < x < c$.

Limit and Continuity of a Function III

Let c be a real number that is either an interior point or an endpoint of an interval in the domain of f.

The function f is **right-continuous** at c if

$$\lim_{x \to c^+} f(x) = f(c)$$

The function f is **left-continuous** at c if

$$\lim_{x \to c^-} f(x) = f(c)$$

The function f is **continuous** at c if

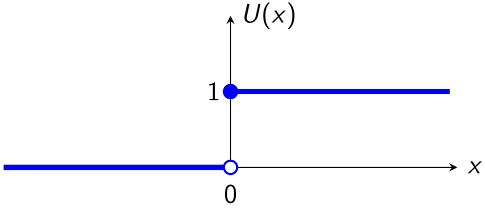
$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = f(c)$$

Limit and Continuity of a Function IV

We say that a function is **continuous** over a closed interval $[a, b]$ if it is right-continuous at a, left-continuous at b, and continuous at all interior points of the interval. This definition applies to the infinite closed intervals $[a, \infty)$ and $(-\infty, b]$ as well, but only one endpoint is involved.

back

For example, $U(x) = \mathbb{1}\{x \geq 0\}$ is right-continuous at $x = 0$, but is neither left-continuous nor continuous there. It has a jump discontinuity at $x = 0$.



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