Information Theory for Prediction Markets

Notes and Extension by Ian Joffe
Based on *False Consensus, Information Theory, and Prediction Markets*
by Yuqing Kong and Grant Schoenebeck, 2022

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1 Round Robin Reaches Consensus

In an agreement game, agents aim to accurately predict the probability of event W based on incomplete information. Each agent starts with a prior credence for W and additional private information X. K&S assume that agents are rational Bayesian aggregators of information. They credit the results of this section originally to Aaronson (2005) and Geanakoplos & Polemarchakis (1982), but K&S uniquely apply the information theoretic framework reviewed here.

The round robin protocol is a format where n agents, indexed by i, take turns sharing announcements related to W. Agent's i's announcement in round t is called h_i^t, and we call the history of all announcements up to but excluding that one H_i^t. Then before the next announcement h_i^{t+1}, each agent uses the previous announcement to update their credence in W to p_i^{t+1}

K&S consider the setting where all agents have a common prior for W. Let p_i^t denote the credence of agent i as the agent announces their h_i^t. For any random variables X and Y, let $H(X)$ denote the entropy of X, $H(X|Y)$ denote the entropy of X conditional on Y, and $I(X;Y) = \sum_{x,y} P(x,y)\log(\frac{P(x,y)}{P(x)P(y)})$ denote the mutual information of X and Y. One can verify that $I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$. The multivariable extension of mutual information is called co-information, and has analogous properties. The authors also adopt a convenient convention where ";" represents intersection of events, "," represents union, and "+" represents strictly disjoint union.

Round t achieves ε-MI ("ε-Mutual Information") if, for all i,

$$I(X_i; W|H^t) \leq \varepsilon$$

meaning no one's private information is any more informative about W than the public information (approximately). In other words, the marginal value of each agent's information is no greater than ε. We say that a protocol reaches consensus if, for some finite t, it guaranteed to achieve ε-MI by the of round t.

The authors make a couple observations about the public information on W, namely $I(H_i^t|W)$, at the end of each round.

- $I(H_i^t; W)$ in monotonically increasing with t
- The growth of the mutual information between H and W with each announcement is the mutual information between the agent's announcement and the and W conditional on the previous history - the announcement's "marginal mutual information"

$$I(H_{i+1}^t; W) - I(H_i^t; W) = I(h_i^t; W|H_i^t) \tag{1}$$

We can use this to show that certain agreement games do reach consensus. Take the round robin protocol. First, consider the game where each announcement is the agent's true credence, so $p_i^t = h_i^t$. Denoting the history of announcements *after* round t as H^t, Then the information gained in *each round* before consensus can be decomposed into the marginal information of the announcement of agent i, and the marginal information of the round's other announcements:

$$\begin{aligned}
I(H^{t+1}; W) - I(H^t; W) &\geq I(H_{i+1}^{t+1}; W) - I(H^t; W) && (H_{i+1}^{t+1} \text{ is before } H^{t+1}) \\
&= I(h_i^{t+1}, H_i^{t+1}; W) - I(H^t; W) \\
&= I(h_i^{t+1}, H_i^{t+1}; W|H^t) \\
&= I(H_i^{t+1}; W|H^t) + I(h_i^{t+1}; W|H_i^{t+1}) && (2)
\end{aligned}$$

Specifically, the authors aim to prove that **consensus is reached in less than $\frac{2}{\varepsilon}$ rounds**. To do this, we can show that in each round before consensus, at least $\frac{\varepsilon}{2}$ information is made public. Without loss of generality, say that before round $t+1$, agent i's marginal information is still at least ε. Then, either agent i's announcement will contribute $\frac{\varepsilon}{2}$ information, or the announcements in round $t+1$ before agent i's announcement will contribute $\frac{\varepsilon}{2}$ information. Mathematically, this is an application of mutual information's chain rule

$$\begin{aligned} I(X_i, H_i^{t+1}; W|H^t) &= I(H_i^{t+1}; W|H^t) + I(X_i; W|H_i^{t+1}) \\ &= \varepsilon \qquad \text{(so one of the terms above must be at least } \frac{\varepsilon}{2}) \end{aligned} \tag{3}$$

Clearly, if the first term is $\geq \frac{\varepsilon}{2}$, then at least $\frac{\varepsilon}{2}$ information is contributed in the round. To see that the second term being at least at least $\frac{\varepsilon}{2}$ leads to the same conclusion, we must understand that $I(X_i; W|H_i^{t+1}) \geq \frac{\varepsilon}{2} \implies I(h_i^{t+1}; W|H_i^{t+1}) \geq \frac{\varepsilon}{2}$, i.e. that marginally informative private information implies a marginally informative announcement. Since in round robin, agents simply announce their posteriors p_i^{t+1}, this is true if an agent's posterior has the same marginal information as their private X_i, which is in turn true due to the information processing inequality - the posterior is a function of X_i, and given the history we can also solve for X_i and a function of the posterior.

2 Extension to Random Participation

Kong and Schoenebeck's work concerns games where agents take turns making announcement, but what if instead agent's made announcements at random times? To avoid confusion, I'll index time with τ. We are interested in information content of all announcements made before time $\mathcal{T}$. Using the marginal information of an announcement,

$$\begin{aligned} I(H^{\mathcal{T}}; W) &= \sum_{\text{all } h_i^\tau : \tau < \mathcal{T}} I(h_i^\tau; W|H^\tau) \\ &= \sum_i \sum_{h_i^\tau : \tau < \mathcal{T}} I(h_i^\tau; W|H^\tau) \end{aligned} \tag{4}$$

To calculate the expected information value of the history at $\mathcal{T}$,

$$\begin{aligned} \mathbb{E}(I(H^{\mathcal{T}}; W)) &= \mathbb{E}\Big(\int_0^{\mathcal{T}} \sum_i \mathbb{P}(i \text{ makes announcement at } \tau) I(h_i^\tau; W|H^\tau) d\tau\Big) \\ &= \int_0^{\mathcal{T}} \sum_i \mathbb{P}(i \text{ makes announcement at } \tau) \mathbb{E}\Big(I(h_i^\tau; W|H^\tau)\Big) d\tau \\ &\qquad\qquad \text{(I hope idk Fubini's Thm)} \\ &= \int_0^{\mathcal{T}} \sum_i \mathbb{P}(i \text{ makes announcement at } \tau) \big(\sum_H I(h_i^\tau; W|H) \mathbb{P}(H^\tau = H)\big) d\tau \end{aligned} \tag{5}$$

Let's assume that agents are equally likely to make announcements at any time, and we expect agent i to make λ_i announcements over each unit of time. Note this is equivalent to saying that the waiting times between i's announcements are memoryless and $Exponential(\lambda_i)$ distributed, and that i's total number of announcements is $Poisson(\lambda_i)$ distributed. Now,

$$\mathbb{E}(I(H^{\mathcal{T}}; W)) = \int_0^{\mathcal{T}} \sum_i \frac{\lambda_i}{\mathcal{T}} \big(\sum_H I(h_i^\tau; W|H) \mathbb{P}(H^\tau = H)\big) d\tau \tag{6}$$

Looking at the inner summation, we are only interested in the history since agent i's last announcement, because for Bayesian agents i's last announcement contains all the information from announcements before it. Still, an infinite number of histories are possible, and even with truncated histories for a medium number of agents, the combinatorics quickly become intractable.

So to simplify, we'll assume very few agents. The case of two agents A and B is especially basic, since there are only two effective histories to account to account for. Say A is making an announcement at time τ. If A made the previous announcement, then the new one will not increase public information. If B made the previous announcement, then A's announcement will increase public information, but by the same amount as if B had made the last several announcements. Therefore, **for two agents**,

$$\begin{aligned}
\mathbb{E}(I(H^{\mathcal{T}};W)) &= \int_0^{\mathcal{T}} \left(\frac{\lambda_A}{\mathcal{T}}(0 + I(h_A^\tau; W|\text{B's announcement})P(\text{B made the last announcement})\right) \\
&\quad + \left(\frac{\lambda_B}{\mathcal{T}}(0 + I(h_B^\tau; W|\text{A's announcement})P(\text{A made the last announcement})\right) d\tau \\
&= \int_0^{\mathcal{T}} \frac{\lambda_A}{\mathcal{T}} I(h_A^\tau; W|\text{B's announcement})(\frac{\lambda_B}{\lambda_A+\lambda_B}) + \frac{\lambda_B}{\mathcal{T}} I(h_B^\tau; W|\text{A's announcement})(\frac{\lambda_A}{\lambda_A+\lambda_B})\, d\tau \\
&= \frac{\lambda_A \lambda_B}{\lambda_A+\lambda_B} \cdot \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} \qquad (7)
\end{aligned}$$

3 Application to Prediction Markets