Question 1)

a) Find the expected value and Variance of the sample mean

$\bar{x} = \frac{1}{n}\sum x_i$

$E(\bar{x}) = \frac{1}{n}\sum E(x_i)$

Since x_i are iid with mean μ

$E(\bar{x}) = \frac{1}{n} n \mu$ $\quad E(x_i) = \mu$ and $\sum \mu = n\mu$

$E(\bar{x}) = \mu$

$Var(\bar{x}) = var\left(\frac{1}{n}\sum x_i\right)$

$= \frac{1}{n^2} var\left(\sum x_i\right)$

x_i has a var σ^2, $\sum \sigma^2 = n\sigma^2$

$= \frac{1}{n^2} n\sigma^2$

$= \frac{1}{n}\sigma^2$

$var(\bar{x}) = \sigma^2/n$

$E(\bar{X}) = \mu$ and $Var(\bar{x}) = \sigma^2/n$

b) Show that $\sum x_i^2 - n\bar{x}^2 = \sum (x_i - \bar{x})^2$

$\sum (x_i - \bar{x})^2 = \sum (x_i^2 - 2x_i\bar{x} + \bar{x}^2)$

$= \sum x_i^2 - 2\bar{x}\sum x_i + n\bar{x}^2$

$\sum x_i = n\bar{x}$

$= \sum x_i^2 - 2\bar{x}n\bar{x} + n\bar{x}^2$

$= \sum x_i^2 - 2n\bar{x}^2 + n\bar{x}^2$

$\sum (x_i - \bar{x})^2 = \sum x_i^2 - n\bar{x}^2$

c) Show that s^2 is an unbiased estimator of σ^2.

$s^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$

$\sum (x_i - \bar{x})^2 = \sum x_i^2 - n\bar{x}^2$ ← From previous question

$= \sum E(x_i^2) - nE(\bar{x}^2)$ Taking expection

$\cancel{E}$ $Var(x_i) = E(x_i^2) - (E(x_i))^2$

$E(x_i^2) = var(x_i) + \{E(x_i)\}^2 = \sigma^2 - N^2$ ← From question

$Var(\bar{x}) = E(\bar{x}^2) - \{E(\bar{x})\}^2$

$E(\bar{x}^2) = var(\bar{x}) - \{E(\bar{x})\}^2 = \frac{\sigma^2}{n} \cancel{+} N^2$ ← from previous question

$= \sum (\sigma^2 - N^2) - n(\frac{\sigma^2}{n} - N^2)$

$= n\sigma^2 - nN^2 - \sigma^2 + nN^2$

$= n\sigma^2 - \sigma^2$

$= \sigma^2(n-1)$

$E(s^2) = \frac{1}{n-1} \sigma^2(n-1)$

$E(s^2) = \sigma^2$

Question 2)

a) Show $x'x = \sum x_i^2$

$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $X' = (x_1, \dots, x_n)$

$x'x = (x_1, \dots, x_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \cdot x_1 + x_2 \cdot x_2 + \dots + x_n \cdot x_n$

$= \cancel{x} \sum_{i=1}^{n} x_i^2$

Show $\frac{1}{n} 1_n' x = \bar{x}$

$= \frac{1}{n} \overset{1 \times n}{(1, 1, \dots, 1)} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$= (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$= \frac{x_1}{n} + \frac{x_2}{n} + \dots + \frac{x_n}{n}$

$= \frac{1}{n}(x_1 + x_2 + \dots + x_n)$

$= \frac{1}{n} \sum x_i$

$= \bar{x}$

Q2b) $\sum(x_i-\bar{x})^2$

$= \sum x_i^2 - n\bar{x}^2$

$= x'x - n\bar{x}\bar{x}$

$= x'x - n\bar{x}(\frac{1}{n} l_n' x)$

$= x'x - \bar{x}^2 l_n' x$

$= x(x' - \bar{x} l_n')$

2c) $x'x - n\bar{x}^2$ ~~using the proof~~ from 1b

$= x'x - (\frac{1}{n} l_n' x)^2$

$\Rightarrow \sum x_i^2 - 2\bar{x}n\bar{x} + n\bar{x}^2$

$= x'x - 2n(\frac{1}{n} l_n' x)^2 + n(\frac{1}{n} l_n' x)^2$

$= x'x - 2n(\frac{1}{n} l_n' x)(\frac{1}{n} l_n' x) + n\frac{1}{n}(l_n' x)^2$

$= \sum x_i^2 - 2\sum\bar{x}\sum x_i + \sum\bar{x}^2$

$= \sum(x_i-\bar{x})^2$