Q5 a) $y_t = \mu + \phi(y_{t-1} - \mu) + \epsilon_t$

$y_{t+1} = \mu + \phi(y_t - \mu) + \epsilon_{t+1}$

$y_{t+2} = \mu + \phi(y_{t+1} - \mu) + \epsilon_{t+2}$

$y_{t+2} = \mu + \phi(\mu + \phi(y_t - \mu) + \epsilon_{t+1}) - \mu) + \epsilon_{t+2}$

$y_{t+2} = \mu + \phi(\mu + \phi y_t - \phi\mu + \epsilon_{t+1} - \mu) + \epsilon_{t+2}$

$y_{t+2} = \mu + \phi^2 y_t - \phi^2\mu + \underbrace{\phi\epsilon_{t+1}}_{E(\epsilon)=0} + \underbrace{\epsilon_{t+2}}_{E(\epsilon)=0}$

$E(y_{T+2}/I_T, \theta) = \mu + \phi^2 E(y_t) - \phi^2 (\mu)$

$= \mu + \phi^2 \mu - \phi^2\mu$

$E(y_{T+2}/I_T, \theta) = \mu$

The $E(y_{T+2}|I_T,\theta)$ is equal to the mean of the series.
This makes sense as it is a mean reverting process

b) $L(y_{T+2} - \hat{y}_{T+2}) = (y_{T+2} - \hat{y}_{T+2})^2$

$E(y_{T+2} - y_{T+2})^2 = E(y^2_{T+2}) + \hat{y}^2_{T+2} - 2\hat{y}_{T+2} E(y_{T+2})$

Taking the FOC w.r.t $\hat{y}_{T+2}$ $\Rightarrow$ $2\hat{y}_{T+2} - 2E(y_{T+2}) = 0$

$\hat{y}_{T+2} = E(y_{T+2})$

$\therefore$ $\hat{y}_{T+2}$ minimizes the expected loss function.

c) $f(y_{T+2} | I_T, \theta)$

~~$E(y_{T+2}) = \mu$~~, $\epsilon_t \sim N(0, \sigma^2)$ $y_{t+2} = \mu + \phi^2 y_t - \phi\mu + \phi\epsilon_{t+1} + \epsilon_{t+2}$

~~$y_{T+2} \sim N(\mu, \sigma^2)$~~ $var(y_{t+2}) = \phi\sigma^2 + \sigma^2 = \sigma^2(\phi + 1)$

~~$y_{T+2} \sim N(\mu, \sigma^2)$~~ $f(y_{T+2} | I_T, \theta) \sim N(\mu, \sigma^2(\phi+1))$

d) As θ is unknown, ϕ and μ are unknown. A way to still estimate this equation is to provide sample estimates for ϕ and μ withe the current samples.

As $E(\hat{\phi}) = \phi$ and $E(\hat{\mu}) = \mu$