Algorithmic Game Theory and Applications
Study Note

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Winter, Spring 2021

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0 Useful Notations

1. A mixed strategy $\boldsymbol{x_i} \in X_i$ is **pure** if $!\exists j \in S_i$ s.t. $x_i(j) = 1, x_i(j') = 0\ \forall j' \neq j$. Such strategy is denoted $\pi_{i,j}$.
2. Given a mixed strategies $x = (x_1, ..., x_n) \in X$, we denote $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ as everybody's but player i's strategies.
3. For Profile of mixed strategies $x \in X$ and mix strategy $y_i \in X_i$, we denote the new profile $(x_{-i}; y_i) = (x_i, ..., x_{i-1}, y_i, x_{i+1}, x_n)$, where the new profile replaces x_i with y_i.

1 Basics of Game Theory

DEFINITION (Game Theory)**.** **Game Theory** is the formal study if interaction between *goal-oriented agents (players)* and the strategic scenarios that arise such settings.

DEFINITION (Algorithmic Game Theory)**.** **Algorithmic Game Theory** is concerned with the computational questions that arise in game theory, ans that enlighten game theory. In particular, questions about finding efficient algorithms to 'solve' games.

DEFINITION (Zero-sum Game)**.** Total payoff of all player is zero, for all possible outcomes.

Nash Equilibria

DEFINITION (Nash Equilibria)**.** A pair (n-tuple) of strategies for the 2 players (n players) such that no player can benefit by only changing his/her own strategy.

THEOREM (Nash's Theorem)**.** Every (finite) game has a mixed Nash Equilibrium.

Form of Game

- **Normal Form/Strategic Form**: all players choose strategies simultaneously
- **Extensive Form**: thew game is played by a sequence of move (eg. take turns), might be showned as a game tree (See lecture 1 page 9)

Perfect Information

A game tree is made up by numbers of nodes, which are connected by a set of strategies/moves. Some nodes are controlled by a player, and some neither, which are called **chance nodes**. The set of possible strategies/moves-lead nodes from the same nodes is the **information set**. A game where every information set have only 1 node is called a **game of perfectr information**.

THEOREM. Any finite n-person extensive game of perfect information has an **equilibruim in pure strategies**

Strategic form Game

DEFINITION. A **strategic form game** Γ with n players, consists of:

- A set of players $N = \{1, ..., n\}$

- Set of pure strategies S_i $\forall i \in N$, The set of all possible combinations of strategies is denoted $S = \prod_{i \in N} S_i$
- A payoff function(utility) function $u_i : S \mapsto \mathbb{R}$ for each $i \in N$ describes the payoff $u_i(s_1, ..., s_n)$ to player i ubder each combination of strategies.

DEFINITION. A **finite strategic form game** Γ with n players, consists of:

- A set of players $N = \{1, ..., n\}$
- Set of pure strategies $S_i = \{1, ..., m_i\}$ $\forall i \in N$, The set of all possible combinations of pure strategies is denoted $S = \prod_{i \in N} S_i$
- A payoff function(utility) function $u_i : S \mapsto \mathbb{R}$ for each $i \in N$ describes the payoff $u_i(s_1, ..., s_n)$ to player i under each combination of strategies.

DEFINITION (Zero-sum Game)**.**

$$\sum_{i \in N} u_i(s) = 0 \ \forall s \in S \Leftrightarrow \ \Gamma \text{ is a zero-sum game}$$

Mixed (Ramdomized) Strategies

DEFINITION (Mixed Strategy)**.** A **mixed strategy** $\boldsymbol{x_i}$ for player i with $S_i = \{1, ..., m_i\}$ is a probability distribution over S_i. In other words $\boldsymbol{x_i} = (x_i(1), ..., x_i(m_i))$, where $x_i(s) \in [1, 0]$ $\forall s \in S_i$ and $\sum_{s \in S_i} x_i(s) = 0$

Let X_i be the set of all possible mixed strategies $\boldsymbol{x_i}$ for player i, then for an n-player game, $X = X_1 \times ... \times X_n$ denotes the set of all possible combinations/profiles of mixed strategies.

Expected Payoffs

Here we let $x = (\boldsymbol{x_1}, ..., \boldsymbol{x_n}) \in X$ a profile of mixed strategies. For $s = (s_1, ..., s_n) \in S$ a combination of pure strategies, let $x(s) = \prod_{i \in N} x_i(s_i)$ be the probability of combination s ubder mixed profile x.

DEFINITION (Expected Payoff)**.** The expected playoff of player i under a mixed strategy profile $x = (\boldsymbol{x_1}, ..., \boldsymbol{x_n}) \in X$, is: $U_i(x) = \sum_{s \in S} x(s) u_i(s)$ where $u_i(s)$ denoted the payoff of player $i \in N$ with pure strategy $s \in S$.

In fact, this is the same as $\mathbb{E}\left[u|i, x\right] = \sum_{s \in S} \mathbb{P}(s) u_i(s)$, where $U_i(x) = \mathbb{E}\left[u|i, x\right]$ and $\mathbb{P}(s) = x(s)$.

Best Responses

DEFINITION (Best Response)**.** A (mixed) strategy $z_i \in X_i$ is a **best response** for player i to x_{-i} if
$U_i(x_{-1}; z_i) \geq U_i(x_{-i}; y_i)$ $\forall y_i \in X_i$

2 Nash Equilibrium

DEFINITION (Mixed Nash Equilibrium)**.** For a strategic game Γ, a strategy profile $x = (x_1, ..., x_n) \in X$ is a **mixed Nash Equilibrium** if for every player i, x_i is the best response to x_{-i}.

A mixed Nash Equilibrium x is a Nash Equilibrium if every $x_i \in x$ is a pure strategy $\pi_{i,j}$ for some $j \in S_i$