Algorithmic Game Theory and Applications Study Note

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0 Useful Notations

1. A mixed strategy $\boldsymbol{x_i} \in X_i$ is **pure** if $!\exists j \in S_i$ s.t. $x_i(j) = 1, x_i(j') = 0\ \forall j' \neq j$. Such strategy is denoted $\pi_{i,j}$.
2. Given a mixed strategies $x = (x_1, ..., x_n) \in X$, we denote $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ as everybody's but player i's strategies.
3. For Profile of mixed strategies $x \in X$ and mix strategy $y_i \in X_i$, we denote the new profile $(x_{-i}; y_i) = (x_i, ..., x_{i-1}, y_i, x_{i+1}, x_n)$, where the new profile replaces x_i with y_i.

1 Basics of Game Theory

DEFINITION (Game Theory)**.** **Game Theory** is the formal study if interaction between *goal-oriented agents (players)* and the strategic scenarios that arise such settings.

DEFINITION (Algorithmic Game Theory)**.** **Algorithmic Game Theory** is concerned with the computational questions that arise in game theory, ans that enlighten game theory. In particular, questions about finding efficient algorithms to 'solve' games.

DEFINITION (Zero-sum Game)**.** Total payoff of all player is zero, for all possible outcomes.

Nash Equilibria

DEFINITION (Nash Equilibria)**.** A pair (n-tuple) of strategies for the 2 players (n players) such that no player can benefit by only changing his/her own strategy.

THEOREM (Nash's Theorem)**.** Every (finite) game has a mixed Nash Equilibrium.

Form of Game

- **Normal Form/Strategic Form**: all players choose strategies simultaneously
- **Extensive Form**: thew game is played by a sequence of move (eg. take turns), might be showned as a game tree (See lecture 1 page 9)

Perfect Information

A game tree is made up by numbers of nodes, which are connected by a set of strategies/moves. Some nodes are controlled by a player, and some neither, which are called **chance nodes**. The set of possible strategies/moves-lead nodes from the same nodes is the **information set**. A game where every information set have only 1 node is called a **game of perfectr information**.

THEOREM. Any finite n-person extensive game of perfect information has an **equilibrium in pure strategies**

Strategic form Game

DEFINITION. A **strategic form game** Γ with n players, consists of:

- A set of players $N = \{1, ..., n\}$
- Set of pure strategies $S_i\ \forall i \in N$, The set of all possible combinations of strategies is denoted $S = \prod_{i \in N} S_i$
- A payoff function(utility) function $u_i : S \mapsto \mathbb{R}$ for each $i \in N$ describes the payoff $u_i(s_1, ..., s_n)$ to player i ubder each combination of strategies.

DEFINITION. A **finite strategic form game** Γ with n players, consists of:

- A set of players $N = \{1, ..., n\}$
- Set of pure strategies $S_i = \{1, ..., m_i\}\ \forall i \in N$, The set of all possible combinations of pure strategies is denoted $S = \prod_{i \in N} S_i$
- A payoff function(utility) function $u_i : S \mapsto \mathbb{R}$ for each $i \in N$ describes the payoff $u_i(s_1, ..., s_n)$ to player i under each combination of strategies.

DEFINITION (Zero-sum Game)**.**

$$\sum_{i \in N} u_i(s) = 0\ \forall s \in S \Leftrightarrow\ \Gamma \text{ is a zero-sum game}$$

Mixed (Ramdomized) Strategies

DEFINITION (Mixed Strategy)**.** A **mixed strategy $\boldsymbol{x_i}$** for player i with $S_i = \{1, ..., m_i\}$ is a probability distribution over S_i. In other words $\boldsymbol{x_i} = (x_i(1), ..., x_i(m_i))$, where $x_i(s) \in [1, 0]\ \forall s \in S_i$ and $\sum_{s \in S_i} x_i(s) = 0$

Let X_i be the set of all possible mixed strategies $\boldsymbol{x_i}$ for player i, then for an n-player game, $X = X_1 \times ... \times X_n$ denotes the set of all possible combinations/profiles of mixed strategies.

Expected Payoffs

Here we let $x = (\boldsymbol{x_1}, ..., \boldsymbol{x_n}) \in X$ a profile of mixed strategies. For $s = (s_1, ..., s_n) \in S$ a combination of pure strategies, let $x(s) = \prod_{i \in N} x_i(s_i)$ be the probability of combination s ubder mixed profile x.

DEFINITION (Expected Payoff)**.** The expected playoff of player i under a mixed strategy profile $x = (\boldsymbol{x_1}, ..., \boldsymbol{x_n}) \in X$, is: $U_i(x) = \sum_{s \in S} x(s) u_i(s)$ where $u_i(s)$ denoted the payoff of player $i \in N$ with pure strategy $s \in S$.

In fact, this is the same as $\mathbb{E}[u|i, x] = \sum_{s \in S} \mathbb{P}(s) u_i(s)$, where $U_i(x) = \mathbb{E}[u|i, x]$ and $\mathbb{P}(s) = x(s)$.

Best Responses

DEFINITION (Best Response)**.** A (mixed) strategy $z_i \in X_i$ is a **best response** for player i to x_{-i} if
$U_i(x_{-1}; z_i) \geq U_i(x_{-i}; y_i)\ \forall y_i \in X_i$

Social Walfare and Pareto Optimal/Efficient

DEFINITION (Social Walfare). Given a profile $x \in X$ in an n-player game, the **social walfare** is $\sum_{i=1}^{n} U_i(x)$.

DEFINITION (Pareto Optimal/ Pareto Efficient). Given an n-player game, a profile $x \in X$ is a **pareto optimal/efficient** if there is no other profile x' such that $U_i(x) \leq U_i(x')$ for all players i, and $U_k(x) < U_k(x')$ for some player k.

2 Nash Equilibrium

DEFINITION (Mixed Nash Equilibrium). For a strategic game Γ, a strategy profile $x = (x_1, ..., x_n) \in X$ is a **mixed Nash Equilibrium** if for every player i, x_i is the best response to x_{-i}.

A mixed Nash Equilibrium x is a Nash Equilibrium if every $x_i \in x$ is a pure strategy $\pi_{i,j}$ for some $j \in S_i$.

Nash's Theorem

Before understanding Nash's Theorem, we first take a look to the **Brouwer Fixed Point Theorem**.

THEOREM (Brouwer Fixed Point Theorem). Every continous $f : D \to D$ mapping a compact and convex, nonempty subset $D \subseteq \mathbb{R}^m$ to itself has a "fixed point". In other words: $\exists x^*, f(x^*) = x^*$

Nash's theorem is as follow:

THEOREM (Nash's Theorem). Every (finite) game has a mixed Nash Equilibrium.

Using Brouwer Fixed Point Theorem, we can prove Nash's Theorem:

PROOF (Proof of Nash's theorem). We define a continous finction $f : X \to X$, where $X = X_1 \times ... \times X_n$, and we show that if $f(x^*) = x^*$ then $x^* = (x_1^*, ..., x_n^*)$ must be a Nash Equilibrium.

We now start a claim:

Claim: A profile $x^* = (x_1^*, ..., x_n^*) \in X$ is a Nash Equilibrium i.f.f. for every player i and pure strategy $\pi_{i,j}, j \in S_i$:

$$U_i(x^*) \geq U_i(x_{-i}^*; \pi_{i,j})$$

Proof: If x^* is a NE then obviously $U_i(x^*) \geq U_i(x_{-i}^*; \pi_{i,j})$ by definition.

Now we have to proof the other direction: By calculation we can see that for any mixed strategy $x_i \in X_i$: $U_i(x_{-i}^*; x_i) = \sum_{j \in S_i} x_i(j) U_i(x_{-i}^*; \pi_{i,j})$.

By assumption, $U_i(x^*) \geq U_i(x^*_{-i};\pi_{i,j})\ \forall j \in S_i$ So, clearly: $U_i(x^*) \geq U_i(x^*_{-i};\pi_{i,j})$ as :

$$U_i(x^*_{-i};x_i) = \sum_{j\in S_i} x_i(j)U_i(x^*_{-i};\pi_{i,j}) \leq \sum_{j\in S_i} x_i(j)U_i(x^*) = U_i(x^*)$$

Hence each x^*_i is the best response to x^*_{-i}, therefore a NE.

$\Rightarrow$ Claim true ■

Now rephrasing the goal, we want to find $x^* = (x^*_1, ..., x^*_n)$ such that $U_i(x^*_{-i};\pi_{i,j}) \leq U_i(x^*)$ ie. $U_i(x^*_{-1};\pi_{i,j}) - U_i(x^*) \leq 0$ for all players $i \in N$, and $j \in S_i$.

For a mixed profile $x = (x_1, ..., x_n) \in X$: let $\varphi_{i,j}(x) = \max\{0, U_i(x^*_{-i};\pi_{i,j})\}$ measures "how much better off" player i would be if picked $\pi_{i,j}$ instead of x_i.

Define $f : X \to X$ as follows: For $x = (x_1, ..., x_n) \in X$, let $f(x) = (x'_1, ..., x'_n)$ where for all i and $j \in S_i$:

$$x'_i(j) = \frac{x_i(j) + \varphi_{i,j}(x)}{1 + \sum_{k\in S_i} \varphi_{i,k}(x)}$$

By Brouwer, $\exists x^* \in X, f(x^*) = x^*$.

Now we have to show x^* is a NE.

For each $i \in N$, and $j \in S_i$, $x^*_i(j) = \dfrac{x^*_i(j) + \varphi_{i,j}(x^*)}{1 + \sum_{k\in S_i} \varphi_{i,k}(x^*)}$,
thus $x^*_i(j)\left(1 + \sum_{k\in S_i} \varphi_{i,k}(x^*)\right) = x^*_i(j) + \varphi_{i,j}(x^*)$
hence $x^*_i(j)\sum_{k\in S_i} \varphi_{i,k}(x^*) = \varphi_{i,j}(x^*)$

We now have to show that this implies that $\varphi_{i,j}(x^*)$ must be equal to 0 $\forall j$

Claim: For any mixed profile x,, for each player i, $\exists j$ such that $x_i(j) > 0$ and $\varphi_{i,j}(x) = 0$

Proof: For any $x \in X$, $\varphi_{i,j} = \max\{0, U_i(x_{-i};\pi_{i,j}) - U_i)(x)\}$

Since $U_i(x)$ is the "weighted average" of $U_i(x_{-i};\pi_{i,j})$'s, based on the weights $(x_i(j))$ in x_i, there must be some j used in x_i, ie, with $x_i(j) > 0$, such that $U_i(x_{-1};\pi_{i,j})$ is no more than the average $(U_i(x_{-i};\pi_{i.j}) \leq U_i(x))$, therefore $\varphi_{i,j}(x) = 0$. ■

Thus, for such j, $x^*_i(j)\sum_{k\in S_i} \varphi_{i,k}(x^*) = \varphi_{i,j}(x^*) = 0$.

But, sice $\varphi_{i,k}(x^*) \geq 0\ \forall k \in S_i$, this means $\varphi_{i,k}(x^*) \geq 0\ \forall k \in S_i = 0$. Thus, for all $i \in N$, and $j \in S_i$: $U_i(x^*) \geq U_i(x^*_{-i};\pi_{i.j})$ **Q.E.D**

COROLLARY. $U_i(x^*) = U_i(x^*_{-i};\pi_{i,j})$ whenever $x^*_i(j) > 0$

3 Symmetric Game

DEFINITION (Symmetric Game)**.** A 2-player game is **symmetric** if $S_1 = S_2$ and $u_1(s_1, s_2) = u_2(s_2, s_1)\ \forall s_1, s_2 \in S_1$

DEFINITION (Evolutionarity Stable Strategy(ESS))**.** In a 2-player symmetric game, a mixed strategy x^*_1 is an **evolutionarity stable strategy** if:

1. x^*_1 is a Nash Equilibrium, and

2. If $\exists x'_1 \neq x^*_1$ another best response to x^*_1, then $U_1(x'_1, x'_1) < U_1(x_1*, x'_1)$

Although, Nash provided, that every symmetric gamer has a symmetric Nash Equilibrium. However, not every symmetric game has a ESS