Honours Algebra
Quick Notes

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1 Vector Spaces

1.1 Solution of Simultianeous Linear Equations

Assume $F = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$, where $a_{ij}, b_i \in F$, then

$$\sum_{j=1}^{m} a_{ij}x_j = b_i \quad \forall i \in [1, n] : i \in \mathbb{Z}$$

is a **system of linear Equations**

- if all b's are 0 then the system is **homogenous**
- $L = \{x_1, ..., x_m\}$ is the **solution set** of Equations

1.2 Fields and Vector Spaces

DEFINITION 1.2.1.

1. A **field** F is a set with functions:

   - **addition** $= + : F \times F \to F; (\lambda, \mu) \mapsto \lambda + \mu$
   - **multiplication** $= \cdot : F \times F \to F; (\lambda, \mu) \mapsto \lambda\mu$

   such that $(F, +)$ ans $(F\backslash\{0\}, \cdot)$ are abelian groups, with:

   $$\lambda(\mu + \nu) = \lambda\mu + \lambda\nu \in F \quad \forall \lambda, \mu, \nu \in F$$

   The neutral elements are called $0_F, 1_F$, in particular for all $\lambda, \mu \in F$

   - $\lambda + \mu = \mu + \lambda \in F$
   - $\lambda \cdot \mu = \mu \cdot \lambda \in F$
   - $\lambda + 0_F = \lambda \in F$
   - $\lambda \cdot 1_F = \lambda \in F$

   For all $\lambda \in F$ there exists $-\lambda \in F$ such that $\lambda + (-\lambda) = 0_F \in F$
   For all $\lambda \neq 0 \in F$ tehre exists $\lambda^{-1} \neq 0 \in F$ such that $\lambda(\lambda^{-1}) = 1_F \in F$

2. A **vector space** V **over a field** F is a pair consisting of an abelian group $V = (V, \dot{+})$ and a mapping
$$F \times V \to V; (\lambda, \overrightarrow{v})$$
such that for all $\lambda, \mu \in F$ and $\overrightarrow{v}, \overrightarrow{w} \in V$ the following identities hold:
   - $\lambda(\overrightarrow{v} + \overrightarrow{w}) = \lambda\overrightarrow{v} + \lambda\overrightarrow{w}$ **Distributive Law**
   - $(\lambda + \mu)\overrightarrow{v} = \lambda\overrightarrow{v} + \mu\overrightarrow{v}$ **Distributive Law**
   - $\lambda(\mu\overrightarrow{v}) = (\lambda\mu)\overrightarrow{v}$ **Associativity Law**
   - $1_F\overrightarrow{v} = \overrightarrow{v}$

LEMMA 1.2.1. If V is a vector space and $\overrightarrow{v} \in V$ then $0\overrightarrow{v} = \overrightarrow{0}$

LEMMA 1.2.2. If V is a vector space and $\overrightarrow{v} \in V$ then $(-1)\overrightarrow{v} = -\overrightarrow{v}$

LEMMA 1.2.3. If V is a vector space over a field F then $\lambda\overrightarrow{0} = \overrightarrow{0} \quad \forall \lambda \in F$

1.3 Product of Sets and of Vector Spaces

- **Cartesian product** of sets: $X_1 \times ... \times X_n := \{(x_1, ..., x_n) : x_i \in X_i \text{ for } 1 \leq i \leq n\}$, an element of this product is known as a product n-tuples.

There are special mappings called **projections** for a cartesian product

$$\begin{aligned} \mathrm{pr}_i : X_i \times ... \times X_n &\to X_i \\ (x_i, ..., x_n) &\mapsto x_i \end{aligned}$$

The cartesian product of n copies of a set X is written in short as X^n

$$\forall n, m \geq 0,\ X^n \times X^m \xrightarrow{\sim} X^{n+m}; ((x_1, ..., x_n), (x_{n+1}, ..., x_{n+m})) \mapsto (x_1, ..., x_n, x_{n+1}, x_{n+m})$$

1.4 Vector Subspaces

DEFINITION 1.4.1. A subset U of a vector space V is called a **vector subspace** or **subspace** if U contains the zero vector ($\overrightarrow{0}$) and whenever $\overrightarrow{u}, \overrightarrow{v}$ and $\lambda \in F$ we have $\overrightarrow{u} + \overrightarrow{v} \in U$ and $\lambda\overrightarrow{u} \in U$

PROPOSITION 1.4.1. Let T be a subset od vector space V over a field F. Then amongst all vector subspaces of V that include T there is a smallest vector subspace

$$\langle T \rangle = \langle T \rangle_F \subseteq V$$

DEFINITION 1.4.2. A subset S of a vector space V is called a **generating set** of a V if its span is all of the vector space. A vector space that has a finite generating set is **finitely generated**

DEFINITION 1.4.3. Check Definition 1.4.9 on Iain's(Not me I am Ian) note I dun think it's English

1.5 Linear Independence and Bases

DEFINITION 1.5.1. A subset $L = \{\overrightarrow{v_1}, ..., \overrightarrow{v_n}\}$ of a vector subspace V is **linearly independent** if for all arbitrary scalars $\alpha_1, ..., \alpha_n \in F$:

$$\sum_{i=1}^{n} \alpha_i \overrightarrow{v_i} = 0 \to \alpha_1 = ... = \alpha_n = 0$$

DEFINITION 1.5.2. A subset $L = \{\overrightarrow{v_1}, ..., \overrightarrow{v_n}\}$ of a vector subspace V is **linearly dependent** if it's not **linearly independent**, whcih mean there exist some $\alpha_j \in \{a_1, ..., a_n\}, \alpha_j \neq 0$ such that

$$\sum_{i=1}^{n} \alpha_i \overrightarrow{v_i} = 0$$

DEFINITION 1.5.3. A **basis of a vector space** B of a vector space V is a linearly independent generating set in V

DEFINITION 1.5.4. Let F be a field, V a vector space over F and $\overrightarrow{v_1}, ..., \overrightarrow{v_r} \in V$ vectors. The *family* $(\overrightarrow{v_i})_{1 \leq i \leq r}$ is a basis of V if and only if the following "*evaluation*"

$$\Phi : F^r \to V$$
$$(\alpha_1, ..., \alpha_r) \mapsto \alpha_1 \overrightarrow{v}_1 + ... + \alpha_r \overrightarrow{v}_r$$

is a bijection

DEFINITION 1.5.5. The following for a subset E of a vector space V are equivalent:An isomorphism of a vector space to itself is called **anautomorphism** of our vector space.

- Our subset E is a basis, ie. a linearly independent generating set;
- Our subset E is a minimal among all generating sets, meaning that $E \backslash \{\overrightarrow{v}\}$ does not generate V
- Our subset E is maximal among all linearly independent subsets, meaning that $E \cup \{\overrightarrow{v}\}$ is not linearly independent for any $\overrightarrow{v} \in V$

COROLLARY 1.5.1. Let V be a finitely generated vector space over a vield F, then V is a basis

THEOREM 1.5.1. Let V be a vector space.

- If $L \subset V$ is a linearly independent subset and E is a minimal amongst all generating sets of our vector with $L \subseteq E$, then E isn a basis.
- If $L \subseteq V$ is a generating set and if L is maximal amongstall linearly independent subsets of vector space with $L \subseteq E$, then L is a basis.

THEOREM 1.5.2. Let field F, F-vector space V and family of vectors $(\overrightarrow{v_i})_{i\in I}$ from V, The following are equivalent:

- The family $(\overrightarrow{v_i})_{i\in I}$ is a basis of V;
- For each vector $\overrightarrow{v} \in V$ there is percisely one family $(a_i)_{i\in I}$ od elements of field F, almost all of which are zero and such that:
$$\overrightarrow{v} = \sum_{i\in I} a_i \overrightarrow{v_i}$$

1.6 Dimension of a vector space

THEOREM 1.6.1 (Fundamental Estimate of Linear Algebra)**.** No linearly independent subset of a given vector space has more elements then a generating set. Thus if V is a vector space, $L \subset V$ a linearly independent subset and $E \subseteq V$ a generating set, then $|L| \leq |E|$

THEOREM 1.6.2 (Steinitz Exchange Theorem)**.** Let V be a vector space, $L \subset V$ a finite linearly independent subset and $E \subseteq V$ a generating set. Then there is an injection $\phi : L \hookrightarrow E$ such that $(E\backslash\phi(L)) \cup L$ is also a generating set for V

LEMMA 1.6.1 (Exchange Lemma)**.** Let V be a vector space, $M \subseteq V$ a linearly independent subset, adn $E \subseteq V$ a generating subset, such that $M \subseteq E$. if $\overrightarrow{w} \in V\backslash M$ is a vector not belonging to M such that $M \cup \{\overrightarrow{w}\}$ is linearly independent, then there exists $\overrightarrow{e} \in E\backslash M$ such that $\{E\backslash\{\overrightarrow{e}\}\} \cup \{\overrightarrow{w}\}$ is a generating set for V

COROLLARY 1.6.1 (Cardinality of Bases)**.** Let V be a finitely generated vector space.

- V has a finite basis
- V cannot have an infinite basis
- Any two bases of V have the same number of elements

DEFINITION 1.6.1. The cardinality of one (and by Cardinality of Bases each) basis of a finitely generated vector space V is called the **dimension** of V and will be denoted by $\dim(V)$. If the vector space is not finitely generated, then we write $\dim(V) = \infty$ and call V infinite dimensional. As usual, we will ignore the difference between infinities.

COROLLARY 1.6.2 (Cardinality Criterion for Bases)**.** Let V be a finitely generated vector space.

- Each linearly independent subset $L \subset V$ has at most $\dim(V)$ elements, and if $|L| = \dim(V)$ then L is actually a basis
- Each generating set $E \subseteq V$ has at least $\dim(V)$ elements, and if $|E| = \dim(V)$ the E is actually a basis.

COROLLARY 1.6.3 (Dimension Estimate for Vector Subspaces)**.** A proper vector subspace of a finitedimensional vector space has itself a strictly smaller dimension.

THEOREM 1.6.3 (The Dimension Theorem)**.** Let V be vectpr space containing vector subspaces $U, W \subseteq V$. Then
$$\dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W)$$

1.7 Linear Mappings

DEFINITION 1.7.1. Let V, W be a vector spaces over a field F. A mapping $f : V \to W$ is called **linear** or more percisely **F-linearly** or even a **homomorphism of F-vector spaces** if for all $\overrightarrow{v}_1, \overrightarrow{v}_2 \in V$ and $\lambda \in F$ we have

$$f(\overrightarrow{v}_1 + \overrightarrow{v}_2) = f(\overrightarrow{v}_1) + f(\overrightarrow{v}_2)$$
$$f(\lambda \overrightarrow{v}_1) = \lambda f(\overrightarrow{v}_1)$$

- A bijective linear mapping is called as **isomorphism** of vector spaces. If there is an isomorphism bwetween two vector spaces we say them **isomorphic**.
- A **homomorphism** from one vector space to itself is called anendomorphismof our vector space.
- An isomorphism of a vector space to itself is called **anautomorphism** of our vector space.

DEFINITION 1.7.2. A point that is sent to itself br a mapping is called a **fixed point** of the mapping. Given a mapping $f : X \to X$, we donote the set of fixed points by

$$X^f = \{x \in X : f(x) = x\}$$

DEFINITION 1.7.3. Two vector subspaces V_1, V_2 of a vector space V are called **complementary** of addition defines a bijection $V_1 \times V_2 \xrightarrow{\sim} V$

THEOREM 1.7.1 (The Classification of Vector Spaces by their Dimension)**.** Let $n \in \mathbb{N}$. Then a vector space over a field F is isomorphic to F^n i.f.f it has dimension n

LEMMA 1.7.1 (Linear Mappings and Bases)**.** Let V, W be vector spaces over F and let $B \subset V$ be a basis. Then restriction of a mapping gives a bijection

$$\mathrm{Hom}_F(V, W) \xrightarrow{\sim} \mathrm{Maps}(B, W)$$
$$f \mapsto f|_B$$

PROPOSITION 1.7.1.
- Every injective linar mapping $f : V \hookrightarrow W$ has a **left inverse**, in other words a linear mapping $g : W \to V$ such that $g \circ f = \mathrm{id}_V$
- Every surjective linear mapping $f : V \twoheadrightarrow W$ has a **right inverse**, in other words a linear mapping $g : W \to V$ such that $g \circ f = \mathrm{id}_W$

1.8 Rank-Nullity Throrem

DEFINITION 1.8.1. The **image** of a linear mapping $f : V \to W$ is the subset $\mathrm{im}(f) = f(V) \subseteq W$. The **preimage** of the zero vector (**kernel**) of a linear mapping $f : V \to W$ is denoted by

$$\ker(f) := f^{-1}(0) = \{v \in V : f(v) = 0\}$$

The kernel is a vector subspace if V

LEMMA 1.8.1. A linear mapping $f : V \to W$ is injective if an only if it's kernel is zero.

THEOREM 1.8.1 (Rank-Nullity Theorem)**.** Let $f : V \to W$ be a linear mapping between vector spaces, then $\dim(V) = \dim(\ker f) + \dim(\mathrm{im}(f))$

2 Linear Mappings and Matrices