# 1

Given:

$$f(y;\theta) = \frac{y^{\phi-1}\theta^{\phi}e^{-y\theta}}{\Gamma(\phi)} \quad (\phi \text{ known}) \quad [1]$$
$$f(y;\theta) = \binom{y-r+1}{r-1}\theta^{r}(1-\theta)^{y} \quad (r \text{ known}) \quad [2]$$

## 1.a

[1]:
Likelihood function:

$$\begin{aligned} L(\theta; y_1, ..., y_n) &= \prod_{i=1}^{n} \frac{y_i^{\phi-1}\theta^{\phi}e^{-y_i\theta}}{\Gamma(\phi)} \\ &= \frac{\left(\prod_{i=1}^{n} y_i\right)^{\phi-1}\theta^{n\phi}\exp\left(-\sum_{i-1}^{n} y_i\theta\right)}{\Gamma(\phi)^{n}} \end{aligned}$$

Log- likelihood:

$$\begin{aligned} l(\theta; y_1, ..., y_n) &= \ln\left(L(\theta; y_1, ..., y_n)\right) \\ &= (\phi-1)\sum_{i=1}^{n}\ln(y_i) + n\phi\ln(\theta) - \sum_{i=1}^{n} y_i\theta - n\ln\Gamma(\phi) \end{aligned}$$

Taking the derivative of the log- likelihood:

$$\begin{aligned} U(\theta) &= \frac{dl}{d\theta} \\ &= \frac{n\phi}{\theta} - \sum_{i=1}^{n} y_i \end{aligned}$$

For MLE of $\theta$, $\hat{\theta}$:

$$\begin{aligned} \frac{dl}{d\theta} &= 0 \\ \frac{n\phi}{\theta} - \sum_{i=1}^{n} y_i &= 0 \\ \frac{\theta}{n\phi} &= \frac{1}{\sum_{i=1}^{n} y_i} \\ \theta &= \frac{n\phi}{\sum_{i=1}^{n} y_i} \end{aligned}$$

$$\therefore \hat{\theta} = \frac{n\phi}{\sum_{i=1}^{n} y_i}$$

[2]:
Likelihood function:

$$\begin{aligned}
L(\theta; y_1, ..., y_n) &= \prod_{i=1}^{n} \left[ \binom{y_i - r + 1}{r - 1} \theta^r (1-\theta)^{y_i} \right] \\
&= \left[ \prod_{i=1}^{n} \binom{y_i - r + 1}{r - 1} \right] \theta^{nr} (1-\theta)^{\sum_{i=1}^{n} y_i}
\end{aligned}$$

Log- likelihood:

$$\begin{aligned}
l(\theta; y_1, ..., y_n) &= \ln(L(\theta; y_1, ..., y_n)) \\
&= const. + nr \ln\theta + \sum_{i=1}^{n} y_i \ln(1-\theta)
\end{aligned}$$

Taking the derivative of the log- likelihood:

$$\begin{aligned}
U(\theta) &= \frac{dl}{d\theta} \\
&= \frac{nr}{\theta} - \frac{\sum_{i=1}^{n} y_i}{1-\theta}
\end{aligned} \tag{1}$$

For MLE of $\theta$, $\hat{\theta}$:

$$\begin{aligned}
\frac{dl}{d\theta} &= 0 \\
\frac{nr}{\theta} - \frac{\sum_{i=1}^{n} y_i}{1-\theta} &= 0 \\
(1-\theta)nr - \theta \sum_{i=1}^{n} y_i &= 0 \\
\theta \left( nr + \sum_{i=1}^{n} y_i \right) &= nr \\
\theta &= \frac{nr}{nr + \sum_{i=1}^{n} y_i}
\end{aligned}$$

$$\therefore \hat{\theta} = \frac{nr}{nr + \sum_{i=1}^{n} y_i}$$

### 1.b

[1]:
Given:

$$\begin{aligned} U(\theta) &= \frac{n\phi}{\theta} - \sum_{i=1}^{n} y_i \\ U'(\theta) &= \frac{-n\phi}{\theta^2} \end{aligned} \tag{2}$$

The Fisher information:

$$\begin{aligned} I_\theta &= -\mathbb{E}(U') \\ &= -\mathbb{E}\left(\frac{-n\phi}{\theta^2}\right) \\ &= \frac{n\phi}{\theta^2} \end{aligned}$$

$$\begin{aligned} \therefore Var(\hat{\theta}) &= I_\theta^{-1} \\ &= \frac{\theta^2}{n\phi} \end{aligned}$$

[2]:
Given:

$$\begin{aligned} U(\theta) &= \frac{nr}{\theta} - \frac{\sum_{i=1}^{n} y_i}{1-\theta} \\ U'(\theta) &= \frac{-nr}{\theta^2} - \frac{\sum_{i=1}^{n} y_i}{(1-\theta)^2} \end{aligned}$$

For Fisher information:

$$\begin{aligned} I_\theta &= -\mathbb{E}(U') \\ &= -\mathbb{E}\left(\frac{-nr}{\theta^2} - \frac{\sum_{i=1}^{n} y_i}{(1-\theta)^2}\right) \\ &= \frac{nr}{\theta^2} + \frac{\sum_{i=1}^{n} \mathbb{E}(y_i)}{(1-\theta)^2} \\ &= \frac{nr}{\theta^2} + \frac{\sum_{i=1}^{n} \theta r}{(1-\theta)^3} \\ &= \frac{nr}{\theta^2} + \frac{nr\theta}{(1-\theta)^3} \\ &= \frac{nr(1-\theta)^3 + nr\theta^3}{\theta^2(1-\theta)^3} \end{aligned}$$

$$\begin{aligned} \therefore Var(\hat{\theta}) &= I_\theta^{-1} \\ &= \frac{\theta^2(1-\theta)^3}{nr(1-\theta)^3 + nr\theta^3} \end{aligned}$$

## 2

Given:

$$f(y;\theta) = \theta e^{-y\theta}$$

Likelihood:

$$\begin{aligned} L(\theta; y_1, ..., y_n) &= \prod_{i=1}^{n} \left[\theta e^{-y_i\theta}\right] \\ &= \theta^n \exp\left(-\theta \sum_{i=1}^{n} y_i\right) \end{aligned}$$

Log- likelihood:

$$\begin{aligned} l(\theta; y_1, ..., y_n) &= \ln(L(\theta; y_1, ..., y_n)) \\ &= n \ln\theta - \theta \sum_{i=1}^{n} y_i \end{aligned}$$

Score:

$$\begin{aligned} U(\theta) &= \frac{dl}{d\theta} \\ &= \frac{n}{\theta} - \sum_{i=1}^{n} y_i \end{aligned}$$

Claim: $\mathbb{E}(U) = 0$
Proof:

$$\begin{aligned} \mathbb{E}(U) &= \mathbb{E}\left(\frac{n}{\theta} - \sum_{i=1}^{n} y_i\right) \\ &= \frac{n}{\theta} - \sum_{i=1}^{n} \mathbb{E}(y_i) \\ &= \frac{n}{\theta} - \sum_{i=1}^{n} \frac{1}{\theta} \\ &= \frac{n}{\theta} - \frac{n}{\theta} \\ &= 0 \end{aligned}$$

$$\therefore \mathbb{E}(U) = 0$$

Claim: $Var(U) = \mathbb{E}(U^2) = -\mathbb{E}(U')$
Proof:

$$\begin{aligned}
Var(U) &= Var\left(\frac{n}{\theta} - \sum_{i=1}^{n} y_i\right) \\
&= Var\left(\sum_{i=1}^{n} y_i\right) \\
&= \sum_{i=1}^{n} Var(y_1) \\
&= \sum_{i=1}^{n} \frac{1}{\theta^2} \\
&= \frac{n}{\theta^2}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(U^2) &= Var(U) + \mathbb{E}(U)^2 \\
&= \frac{n}{\theta^2} + 0^2 \\
&= \frac{n}{\theta^2}
\end{aligned}$$

$$\begin{aligned}
-\mathbb{E}(U') &= -\mathbb{E}(\frac{-n}{\theta^2}) \\
&= \frac{n}{\theta^2}
\end{aligned}$$

$$\therefore Var(U) = \mathbb{E}(U^2) = -\mathbb{E}(U')$$