1

Given $y_1, ..., y_n$ are all independent realizations of random variavles Y with the p.d.f.

$$f(y;\theta) = \theta^2 y \exp(-\theta y) \quad (y > 0)$$

1.a

The Likelihood:

$$\begin{aligned} L(\theta; y_1, ..., y_n) &= \prod_{i=1}^{n} f(y_i;\theta) \\ &= \prod_{i=1}^{n} \left[\theta^2 y_i \exp(-\theta y_i)\right] \\ &= \theta^{2n} \left(\prod_{i=1}^{n} y_i\right) \exp\left(-\theta \sum_{i=1}^{n} y_i\right) \end{aligned}$$

The Log-Likelihood:

$$\begin{aligned} l(\theta, y_1, ..., y_n) &= \ln(L) \\ &= \ln\left[\theta^{2n} \left(\prod_{i=1}^{n} y_i\right) \exp\left(-\theta \sum_{i=1}^{n} y_i\right)\right] \\ &= 2n \ln\theta + \sum_{i=1}^{n} \ln y_i - \theta \sum_{i=1}^{n} y_i \end{aligned}$$

The Score function:

$$\begin{aligned} U(\theta) &= \frac{dl}{d\theta} \\ &= \frac{d}{d\theta}\left[2n \ln\theta + \sum_{i=1}^{n} \ln y_i - \theta \sum_{i=1}^{n} y_i\right] \\ &= \frac{2n}{\theta} - \sum_{i=1}^{n} y_i \end{aligned}$$

$$U' = \frac{d^2 l}{d\theta^2} = \frac{-2n}{\theta^2}$$

For MLE of $\theta, \hat{\theta}$:

$$\begin{aligned} U &= 0 \\ \frac{2n}{\theta} - \sum_{i=1}^{n} y_i &= 0 \\ \frac{2n}{\theta} &= \sum_{i=1}^{n} y_i \\ \theta &= \frac{2n}{\sum_{i=1}^{n} y_i} = \frac{2}{\bar{y}} \end{aligned}$$

As $U' < 0$, $\hat{\theta} = \frac{2n}{\sum_{i=1}^{n} y_i} = \frac{2}{\bar{y}}$ maximises the likelihood function.

1.b

The Fisher Information:

$$\begin{aligned} I_\theta &= -\mathbb{E}(U') \\ &= -\mathbb{E}\left(\frac{2n}{\theta^2}\right) \\ &= \frac{2n}{\theta^2} \end{aligned}$$

$$\therefore Var(\hat{\theta}) = I_\theta^{-1} = \frac{\theta^2}{2n}$$

1.c

For likelihood-ratio test:

$$\begin{cases} H_0 : \theta = \theta_0 \\ H_0 : \theta \neq \theta_0 \end{cases}$$

The test statistic:

$$\begin{aligned} z_{LR} &= -2\left[\ln \frac{L(\theta_0)}{L(\hat{\theta})}\right] \\ &= -2\ln\left[\frac{\theta_0^{2n}\left(\prod_{i=1}^n y_i\right)\exp\left(-\theta_0\sum_{i=1}^n y_i\right)}{\hat{\theta}^{2n}\left(\prod_{i=1}^n y_i\right)\exp\left(-\hat{\theta}\sum_{i=1}^n y_i\right)}\right] \\ &= -2\left[2n(\ln\theta_0 - \ln\hat{\theta}) - (\theta_0 - \hat{\theta})\sum_{i=1}^n y_i\right] \end{aligned}$$

When $\theta = \theta_0$, so H_0 is not rejected,

$$z_{LR} = -2\left[\ln\frac{L(\theta_0)}{L(\hat{\theta})}\right] = -2\ln(LR) \sim \chi_1^2$$

2

Givien $y_1, ..., y_n \sim Y$ where Y has p.d.f.

$$f(y;\mu) = \frac{1}{\sqrt{2\pi y^3}} \exp\left\{-\frac{(y-\mu)^2}{2\mu^2 y}\right\} \quad (y > 0)$$

2.a

The Likelihood:

$$\begin{aligned}
L(\mu;\mathbf{y}) &= L(\mu; y_1, ..., y_n) \\
&= \prod_{i=1}^{n} f(y_1;\mu) \\
&= \prod_{i=1}^{n} \left\{ \frac{1}{\sqrt{2\pi y_i^3}} \exp\left[-\frac{(y_i-\mu)^2}{2\mu^2 y_i}\right] \right\} \\
&= (2\pi)^{-n/2} \prod_{i=1}^{n} \left(y_i^{-3/2}\right) \exp\left[-\sum_{i=1}^{n} \frac{(y_i-\mu)^2}{2\mu^2 y_i}\right]
\end{aligned}$$

The Log-Likelihood:

$$\begin{aligned}
l(\mu;\mathbf{y}) &= \ln(L) \\
&= \ln\left\{(2\pi)^{-n/2} \prod_{i=1}^{n} \left(y_i^{-3/2}\right) \exp\left[-\sum_{i=1}^{n} \frac{(y_i-\mu)^2}{2\mu^2 y_i}\right]\right\} \\
&= \frac{-n}{2}\ln(2\pi) - \frac{3}{2}\sum_{i=1}^{n} y^{-3/2} - \sum_{i=1}^{n} \frac{(y_i-\mu)^2}{2\mu^2 y_i}
\end{aligned}$$

The Score:

$$\begin{aligned}
U(\mu) &= \frac{dl}{d\mu} \\
&= \frac{d}{d\mu}\left[\frac{-n}{2}\ln(2\pi) - \frac{3}{2}\sum_{i=1}^{n} y^{-3/2} - \sum_{i=1}^{n} \frac{(y_i-\mu)^2}{2\mu^2 y_i}\right] \\
&= \frac{d}{d\mu}\left[-\sum_{i=1}^{n} \frac{(y_i-\mu)^2}{2\mu^2 y_i}\right] \\
&= -\sum_{i=1}^{n} \frac{-2(y_1-\mu)(2\mu^2 y_i) - (y_i-\mu)^2(4\mu y_i)}{4\mu^4 y_i^2} \\
&= \sum_{i=1}^{n} \frac{2(y_1-\mu)(2\mu^2 y_i) + (y_i-\mu)^2(4\mu y_i)}{4\mu^4 y_i^2} \\
&= \sum_{i=1}^{n} \frac{4\mu^2 y_i^2 - 4\mu^3 y_i + (y_i^2 - 2\mu y_i + \mu^2)(4\mu y_i)}{4\mu^4 y_i^2} \\
&= \sum_{i=1}^{n} \frac{4\mu y_i^3 - 4\mu^2 y_i^2}{4\mu^4 y_i^2} \\
&= \sum_{i=1}^{n} \frac{y_i - \mu}{\mu^3} \\
&= \frac{1}{\mu^3}\sum_{i=1}^{n} (y_i - \mu)
\end{aligned}$$

For MLE of $\theta, \hat{\theta}$:

$$
\begin{aligned}
U &= 0 \\
\frac{1}{\mu^3}\sum_{i=1}^{n}(y_i - \mu) &= 0 \\
\sum_{i=1}^{n}(y_i - \mu) &= 0 \\
n\mu &= \sum_{i=1}^{n} y_i \\
\mu &= \bar{y}
\end{aligned}
$$

$$
\begin{aligned}
\therefore \hat{\mu} &= \bar{y} \\
\mathbb{E}(\hat{\mu}) &= \mu
\end{aligned}
$$

$$
\begin{aligned}
U'(\mu) &= \frac{d}{d\mu}\sum_{i=1}^{n}\left(\frac{y_i}{\mu^3} - \frac{1}{\mu^2}\right) \\
&= \sum_{i=1}^{n}\left(\frac{2}{\mu^3} - \frac{3y_i}{\mu^4}\right)
\end{aligned}
$$

Fisher Information:

$$
\begin{aligned}
I_\mu &= -\mathbb{E}(U') \\
&= -\mathbb{E}\left[\sum_{i=1}^{n}\left(\frac{2}{\mu^3} - \frac{3y_i}{\mu^4}\right)\right] \\
&= -\sum_{i=1}^{n}\mathbb{E}\left(\frac{2}{\mu^3} - \frac{3y_i}{\mu^4}\right) \\
&= -\sum_{i=1}^{n}\left[\frac{2}{\mu^3} - \frac{3\mathbb{E}(y_i)}{\mu^4}\right] \\
&= -\sum_{i=1}^{n}\left[\frac{2}{\mu^3} - \frac{3\mu}{\mu^4}\right] \\
&= -\sum_{i=1}^{n}\frac{-1}{\mu^3} \\
&= \frac{n}{\mu^3}
\end{aligned}
$$

Asymptotic variance:

$$
Var(\hat{\mu}) = I_\mu^{-1} = \frac{\mu^3}{n}
$$

$\therefore \hat{\mu}$ is unbiased with asymptotic variance of $\dfrac{\mu^3}{n}$

2.b

$$\begin{cases} H_0 : \mu = \mu_0 \\ H_1 : \mu \neq \mu_0 \end{cases}$$

For score test, test statistic:

$$z_{score} = \frac{U(\theta_0)}{\sqrt{I(\hat{\theta})}} = \frac{U(\theta_0)}{\sqrt{I(\theta_0)}} \sim \mathcal{N}(0,1)$$

For Ratio test, test statistic:

$$z_{LR} = -2\left(l(\theta_0) - l(\hat{\theta})\right)$$

As $z_{score} \sim \mathcal{N}(0,1)$, therefore $z_{score}^2 = \frac{U^2(\theta_0)}{I(\theta_0)} \sim \chi_1^2$ Using taylor expansion for $l(\hat{\theta})$:

$$l(\hat{\theta}) \simeq l(\theta_0) + (\hat{\theta} - \theta_0)l'(\theta_0) + (\hat{\theta} - \theta_0)^2 l''(\theta_0) = l(\theta_0) + (\hat{\theta} - \theta_0)U(\theta_0) - \frac{1}{2}(\hat{\theta} - \theta_0)^2 I(\theta_0) \quad [1]$$

Using taylor expansion for $U(\hat{\theta})$:

$$0 = U(\hat{\theta}) \simeq U(\theta_0) + (\hat{\theta} - \theta_0)U'(\theta_0) = U(\theta_0) - (\hat{\theta} - \theta_0)I(\theta_0)$$

$$\Rightarrow \begin{cases} U(\theta_0) \simeq (\hat{\theta} - \theta_0)I(\theta_0) \\ \hat{\theta} - \theta_0 \simeq \frac{U(\theta_0)}{I(\theta_0)} \end{cases} \qquad [2]$$

Sub [2] to [1]:

$$\begin{aligned} l(\hat{\theta}) &\simeq l(\theta_0) + U(\theta_0)\frac{U(\theta_0)}{I(\theta_0)} - \frac{1}{2}\left(\frac{U(\theta_0)}{I(\theta_0)}\right)^2 I(\theta_0) \\ l(\hat{\theta}) - l(\theta_0) &\simeq \frac{1}{2}\frac{U^2(\theta_0)}{I(\theta_0)} \\ -2\left[l(\theta_0) - l(\hat{\theta})\right] &\simeq \frac{U^2(\theta_0)}{I(\theta_0)} \\ z_{LR} &\simeq z_{score} \end{aligned}$$

therefore the two tets statistic are asymptotically equivalent.