Statistical Methodology
Quick Revision Note

Basic things to remember

Likelihood: $L(\theta;\mathbf{x}) = \prod_{i=1}^{n} f(x_i;\theta)$ Log-Likelihood: $l(\theta;\mathbf{x}) = \sum_{i=1}^{n} \ln f(x_i;\theta)$

For MLE of θ, $\hat{\theta}$: $L' = 0$ or $l' = 0$

Score Function: $U(\theta) = l'(\theta)$
Fisher Information: $I(\theta) = \mathbb{E}(U^2) = -\mathbb{E}(U')$
Asympototic Varince of MLE: $var(\hat{\theta}) \approx I^{-1}(\theta)$

Observed Information: $J = -l''$ Expected Information: $I = \mathbb{E}(J)$

Score Test:

$$z_{score} = \frac{U(\theta_0)}{\sqrt{I(\theta_0)}} = \frac{U(\theta_0)}{\sqrt{I(\hat{\theta})}} = \frac{U(\theta_0)}{\sqrt{J(\theta_0)}} = \frac{U(\theta_0)}{\sqrt{J(\hat{\theta})}} \sim N(0,1)$$

Wald Test:

$$z_{wald} = \sqrt{I(\theta_0)}(\hat{\theta}-\theta_0) = \sqrt{I(\hat{\theta})}(\hat{\theta}-\theta_0) = \sqrt{J(\theta_0)}(\hat{\theta}-\theta_0) = \sqrt{J(\hat{\theta})}(\hat{\theta}-\theta_0) \sim N(0,1)$$

Likelihood Ratio Test:

$$-2\ln(LR) = -2[l(\theta_0) - l(\hat{\theta})] \sim \chi^2_1 \qquad LR = \frac{L(\theta_0)}{L(\hat{\theta})}$$

Normal Distribution: $N(\mu,\sigma)$

$$f(x) = \frac{\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]}{\sqrt{2\pi\sigma^2}}$$

Multiparameter Problems ($\boldsymbol{\theta} = (\theta_1, ..., \theta_n)^T$))

Likelihood: $L(\boldsymbol{\theta};\mathbf{y}) = \prod_{i=1}^{n} f(y_i;\boldsymbol{\theta})$ Log-Likelihood: $l(\boldsymbol{\theta};\mathbf{y}) = \sum_{i=1}^{n} \ln f(y_i;\boldsymbol{\theta})$

Score vector: $\mathbf{U}(\boldsymbol{\theta}) = \begin{pmatrix} \partial_{\theta_1} l \\ \vdots \\ \partial_{\theta_n} l \end{pmatrix}$ $J(\boldsymbol{\theta}) = \begin{bmatrix} -\partial_{\theta_1}\partial_{\theta_1} l & \dots & -\partial_{\theta_1}\partial_{\theta_n} l \\ \vdots & \ddots & \vdots \\ -\partial_{\theta_n}\partial_{\theta_1} l & \dots & -\partial_{\theta_n}\partial_{\theta_n} l \end{bmatrix}$ $I(\boldsymbol{\theta}) = \mathbb{E}(J)$

Likelihood Ration Test:

$$-2[l(\boldsymbol{\theta}_0) - l(\hat{\boldsymbol{\theta}})] \sim \chi^2_n(\alpha)$$

Score Test:

$$\mathbf{U}(\boldsymbol{\theta}_0)^T I^{-1} \mathbf{U}(\boldsymbol{\theta}_0) \sim \chi^2_n(\alpha)$$

Wald Test:

$$(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)^T I^{-1} (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \sim \chi^2_n(\alpha)$$

Bayes Theorem

- $\mathbb{P}(A_j|B) = \frac{\mathbb{P}(B|A_j)\mathbb{P}(A_j)}{\sum_{i=1}^{n}\mathbb{P}(B|A_i)\mathbb{P}(A_i)} = \frac{\mathbb{P}(B\cap A_j)}{\mathbb{P}(B)}$
- $\mathbb{P}(B) = \sum_{i=1}^{n}\mathbb{P}(B\cap A_i) = \sum_{i=1}^{n}\mathbb{P}(B|A_i)\mathbb{P}(A_i)$

Baysian Modelling

$$p(\boldsymbol{\theta}|\mathbf{y}) = \frac{p(\mathbf{y};\boldsymbol{\theta})}{f(\mathbf{y})} = \frac{f(\mathbf{y}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{\int f(\mathbf{y}|\boldsymbol{\theta})p(\boldsymbol{\theta})d\boldsymbol{\theta}}$$

Iteratove Maximum Likelihood Estimation

Newton-Raphson iterative formula:

$$\theta_{r+1} = \theta_r - \frac{U(\theta_r)}{l''(\theta_r)} = \theta_r + \frac{U(\theta_r)}{J(\theta_r)}$$

Fisher's method of scoring:

$$\theta_{r+1} = \theta_r - \frac{U(\theta_r)}{\mathbb{E}(l''(\theta_r))} = \theta_r + \frac{U(\theta_r)}{I(\theta_r)}$$

Simple linear regression

Assumption:

- All random variables are uncorrelated
- All random variables have common variance σ^2
- $\mathbb{E}(Y_i|x_i) = \beta_0 + \beta_1 x_i$

$$S_{xx} = \sum (x_i - \bar{x})^2 \equiv \sum (x_i - \bar{x}) x_i \equiv \sum x_i^2 - n\bar{x}^2$$
$$S_{xy} = \sum (x_i - \bar{x})(y_i - \bar{y}) \equiv \sum (x_i - \bar{x}) y_i = \sum x_i y_i - n\bar{x}\bar{y}$$

For all linear regression model $\mathbb{E}(Y_i|x_i) = \beta_0 + \beta_1 x_i$:

- $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 x_i$
- $\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$

Redidual and Regression Sum of Squares:

- Residual: $e_i = y_i - \hat{y}_i = y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i = (y_i - \hat{y}) - \hat{\beta}_1 (x_i - \bar{x})$
- Residual Sum of Squares:

$$\begin{aligned} RSS &= \sum e_i^2 \\ &= \sum [(y_i - \hat{y}) - \hat{\beta}_1 (x_i - \bar{x})]^2 \\ &= S_{yy} - 2\hat{\beta}_1 S_{xy} - \hat{\beta}_1^{\,2} S_{xx} \\ &= S_{yy} - \hat{\beta}_1 S_{xy} \\ &= S_{yy} - \hat{\beta}_1^{\,2} S_{xx} \\ &= S_{yy} - \frac{S_{xy}^2}{S_{xx}} \end{aligned}$$

- $\sigma^2 = \frac{RSS}{n-2}$