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Physics 441: Assignment #3 - Electro-statics

Due on Friday, May 24, 2013

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Problem 2.6

Find the electric field a distance z above the center of a circular loop of radius r that carries a uniform line charge λ

We will be using the following two equations

$$\boldsymbol{E}(\boldsymbol{r})_{\text{line}} = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\boldsymbol{r}')}{r^2}\hat{\boldsymbol{r}}\,da'$$

$$\boldsymbol{E}(\boldsymbol{r})_{\text{surf}} = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\boldsymbol{r}')}{r^2}\hat{\boldsymbol{r}}\,da'$$

The easiest way to do this problem is to break the disk up into a series of circular rings, each with a radius r. We see that $\boldsymbol{r}'$ can be found using the Pythagorean theorem: $\boldsymbol{r}' = \sqrt{\boldsymbol{r}^2 + z^2}\cos\theta\hat{\boldsymbol{z}}$, where $\cos\theta = \frac{z}{r}$. Using symmetry we can say that $dl = 2\pi$Putting all this together we can get the electric field for each ring is

$$\begin{aligned}
E_{ring} &= \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\boldsymbol{r}')}{r^2}\hat{\boldsymbol{r}}\,dl \\
&= \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\boldsymbol{r}')z}{(r^2+z^2)^{3/2}}\hat{\boldsymbol{z}}\,dl \\
&= \frac{1}{4\pi\varepsilon_0}\frac{2\pi r z\lambda(\boldsymbol{r}')}{(r^2+z^2)^{3/2}}\hat{\boldsymbol{z}}
\end{aligned}$$

We are given that the charge is a uniform σ, so $\lambda(\boldsymbol{r}') = \sigma dr$. Ready to plug that in and integrate from 0 to R (Note I let the computer do the actual integral for me).

$$E_{ring} = \int_0^R \frac{1}{4\pi\varepsilon_0} \frac{2\pi r z \lambda(\boldsymbol{r}')}{(r^2+z^2)^{3/2}} \hat{\boldsymbol{z}}$$
$$= \frac{\sigma z}{2\varepsilon_0} \int_0^R \frac{r}{(r^2+z^2)^{3/2}} dr$$
$$= \frac{\sigma z}{2\varepsilon_0} \left[\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}} \right] \hat{\boldsymbol{z}}$$

In the limit as $R \to \infty$ the second term goes to zero and we get $E = \frac{\sigma}{2\varepsilon_0} \hat{\boldsymbol{z}}$
When $z >> R$ we need to expand the square root in the denominator. $\frac{1}{\sqrt{R^2+z^2}} = 1/z\left(1 + \frac{R^2}{z^2}\right) \approx 1/z\left(1 - \frac{R^2}{2z^2}\right) \approx \frac{R^2}{2z^3}$ so we can say that $E = \frac{\sigma R^2}{4\varepsilon_0 z^2}$ □

Problem 2.10

A charge q site at the back corner of a cube. What is the flux of $\boldsymbol{E}$ through a side not touching that corner?

We will be using Gauss' law. To do that we need to think of a convenient gaussian surface such that we can use symmetry arguments to make this problem easier. One such surface is a larger cube that puts the charge at the very center. We can then think of our original surface as being one of 4 panels on a side of the larger cube. This "pabel" is one of 24 similar panels that evenly share the total flux caused by our point charge. It is then simple to say

$$\text{flux}_{\text{panel}} = \frac{q}{24\varepsilon_0}$$

□

Problem 2.16

A long coaxial cable carries a uniform volume charge density ρ on the inner cylinder (radius a), and a uniform surface charge density on the outer cylindrical shell (radius b). This surface charge is negative and is of just the right magnitude that the cable as a whole is electrically neutral. Find the electric field in each of the three regions:

1. inside the inner cylinder ($s < a$)
2. Between the cylinders ($a < s < b$)
3. Outside the cable ($s < b$)

Plot $|\boldsymbol{E}|$ as a function of s.

1. For the first part we will take as a Gaussian surface a cylinder of radius a. We will be using $\oint \boldsymbol{E} \cdot d\boldsymbol{a}$, but due to symmetry simplifies down to $E \oint d\boldsymbol{a} = E 2\pi s l$. We now apply Gauss' Law and say that this must be equal to $\frac{Q}{\varepsilon_0}$. Doing so we can get an expression and solve for E

$$2E\pi s l = \frac{Q}{\varepsilon_0}$$
$$2E\pi s l = \frac{\rho \pi s^2 l}{\varepsilon_0}$$
$$E = \frac{\rho s}{2\varepsilon_0}$$

This points radially outward in he $\hat{\mathbf{s}}$ direction.

2. For the second part we choose as Gaussian surface a cylinder of radius s, where $a < s < b$.In this case we end up with the exact same expressions, except $Q = \rho \pi a^2 l$ (sub a for s, because when $s > a$ no additional charge is enclosed). This allows us to substitute a for s in the final part and get that

$$E = \frac{\rho a^2}{2 s \varepsilon_0} \hat{\mathbf{s}}$$

3. Finally for the third part we recognize that anywhere on or beyond the surface of the outer cylinder has a neutral charge, therefore

$$E = 0$$

In Figure 1 I have plotted the magnitude of the electric field. Below is the code used to make the plot.

```
import numpy as np
import matplotlib.pyplot as plt

def e2_16(s, a, b, rho):
    """
    Electric field as a function of distance from center of coaxial
    cable as described in problem 2.16

    Parameters
    ----------
    s : array_like, dtype=float
        A numpy array of s values interpreted as linear distances from
        the center of the coaxial cable

    a, b, rho : float
        The constants a, b, and rho that appear in expression for E

    Returns
    -------
    e : array_like, dtype=float, shape=s.shape
        The value of the magnitude of the electric field at each point
        in s

    """
    if s < a:
        return rho * s / (2 * eps0)

    elif a < s < b:
        return rho * a ** 2 / (2 * eps0 * s)

    else:
        return 0

a = 1.
b = 2.
rho = 1.
eps0 = 8.8542e-12

e2_16 = np.vectorize(e2_16)
s = np.linspace(.01, 2.5, 150)
e = e2_16(s, a, b, rho)
plt.xticks([0, a, b], ['0', 'a', 'b'])
plt.plot(s, e)
plt.title('E as a function of radial distance (s) for coaxial cable')
plt.xlabel('s')
plt.ylabel(r'$|E|$')
plt.savefig('./E2_16.eps', format='eps', dpi=1000)
plt.show()
```

Problem 2.16 continued on next page. . .

□

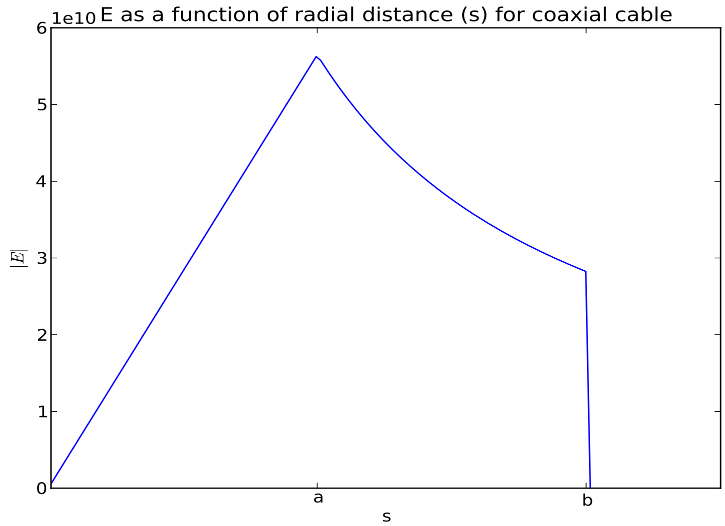

Figure 1: The magnitude of the electric field for the coaxial cable in problem 2.16

Problem 2.24

For the configuration of problem 2.16, find the potential difference between a point on the axis and a point on the outer cylinder. Note that it its not necessary to commit yourself to a particular reference point if you use

$$
\begin{aligned}
V(\boldsymbol{b}) - V(\boldsymbol{a}) &= -\int_{\mathbb{O}}^{\boldsymbol{b}} \boldsymbol{E}\cdot d\boldsymbol{l} + \int_{\mathbb{O}}^{\boldsymbol{a}} \boldsymbol{E}\cdot d\boldsymbol{l} \\
&= -\int_{\mathbb{O}}^{\boldsymbol{b}} \boldsymbol{E}\cdot d\boldsymbol{l} - \int_{\boldsymbol{a}}^{\mathbb{O}} \boldsymbol{E}\cdot d\boldsymbol{l} \\
&= -\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{E}\cdot d\boldsymbol{l}
\end{aligned}
$$

For the point on the cylinder $s = b$ and for the point on the axis $s = 0$. We now need to integrate from $0 \to b$. As we learned in previous problem, the field changes so we will break the integral up into into two integrals going from $0 \to a$ and $a \to b$. I show this below.

$$
\begin{aligned}
-\int_0^b \boldsymbol{E}\cdot d\boldsymbol{a} &= -\left(\int_0^a \boldsymbol{E}\cdot d\boldsymbol{a} + \int_a^b \boldsymbol{E}\cdot d\boldsymbol{a}\right) \\
&= -\left(\int_0^a E ds + \int_a^b E ds\right) \\
&= -\left(\frac{\rho}{2\varepsilon_0}\int_0^a s ds + \frac{\rho a^2}{2\varepsilon_0}\int_a^b \frac{1}{s} ds\right) \\
&= -\left(\frac{\rho}{2\varepsilon_0}\frac{s^2}{2}\bigg|_0^a + \frac{\rho a^2}{2\varepsilon_0}\ln s\bigg|_a^b\right) \\
&= -\frac{\rho a^2}{2\varepsilon_0}\left(\frac{b}{a} + \ln\left(\frac{b}{a}\right)\right)
\end{aligned}
$$

□

Problem 2.29

Check that

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\boldsymbol{r}')}{r} d\tau'$$

satisfies Poisson's equation, by applying the Laplacian and using

$$\nabla^2 \frac{1}{r} = -4\pi\delta^3(\boldsymbol{r})$$

Poisson's equation is

$$\nabla^2\phi = f$$

We want to show that $\nabla^2 V$ ends up being a scalar function in r. We do this below. Note that we apply the hint given about the Laplacian of $\frac{1}{r}$

$$
\begin{aligned}
\nabla^2 V &= \frac{1}{4\pi\varepsilon_0}\nabla^2 \int \frac{\rho(\boldsymbol{r}')}{r} d\tau' \\
&= \frac{1}{4\pi\varepsilon_0}\int \rho(\boldsymbol{r}')\nabla^2\left(\frac{1}{r}\right) d\tau' \\
&= \frac{1}{4\pi\varepsilon_0}\int \rho(\boldsymbol{r}')\left[-4\pi\delta^3(\boldsymbol{r}-\boldsymbol{r}')\right] d\tau' \\
&= -\frac{\rho(\boldsymbol{r})}{\varepsilon_0}
\end{aligned}
$$

I made the last simlification by canceling out the 4π in the numerator and denominator and using identities for integrals of products, where one of the things being multiplied is a delta function. □

Problem 2.38

A metal sphere of radius R, carrying charge q, is surrounded by a thick concentric metal shell (inner radius a, outer radius b). The shell carries no net charge.

1. Find the surface charge density σ at R, a, and b.

2. Find the potential at the center, using infinity as a reference point
3. Now the outer surface is touched to a grounding wire, which drains off charge and lowers its potential to zero (same as infinity). How do your answers to the previous parts change?

1. Equation 2.10 in the book says

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}}$$

Then equation 2.48 says

$$\boldsymbol{E} = \frac{\sigma}{\varepsilon_0}\hat{\boldsymbol{n}}$$

Putting these equations together we see that

$$\sigma = \frac{q}{4\pi r^2}\hat{\boldsymbol{n}}$$

We will apply this expression to the three cases

- at R: $r = R$ and $\hat{\boldsymbol{n}} = 1$ so $\sigma = \frac{q}{4\pi R^2}$
- at a: $r = a$ and $\hat{\boldsymbol{n}} = -1$ so $\sigma = -\frac{q}{4\pi a^2}$. Note that it is negative at this time because it is pointing into the middle layer of the shell, not outside.
- at b: $r = b$ and $\hat{\boldsymbol{n}} = 1$ so $\sigma = \frac{q}{4\pi b^2}$

2. We will need to integrate $\boldsymbol{E}\cdot d\boldsymbol{l} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr$ from $-\infty$ to 0. To do this we will break the integral up into the following integrals: $[-\infty, b], [b, a], [a, R], [R, 0]$. From the problem description we know that $\boldsymbol{E} = 0$ inside the inner radius R (interval $[R, 0]$) and in between the shell (interval $[b, a]$).

$$\begin{aligned}
V(0) &= -\int_{-\infty}^{0} \boldsymbol{E}\cdot d\boldsymbol{l} \\
&= -\int_{-\infty}^{0} \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr \\
&= -\left(\int_{-\infty}^{b} \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr + \int_{b}^{a} 0\,dr + \int_{a}^{R} \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr + \int_{R}^{0} 0\,dr\right) \\
&= -\left(\left(-\frac{q}{4\varepsilon_0\pi r}\right)\Bigg|_{-\infty}^{b} + 0 + \left(-\frac{q}{4\varepsilon_0\pi r}\right)\Bigg|_{a}^{R} + 0\right) \\
&= -\left(-\frac{q}{4b\varepsilon_0\pi} + \frac{q}{4a\varepsilon_0\pi} - \frac{q}{4R\varepsilon_0\pi}\right) \\
&= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{b} + \frac{q}{R} - \frac{q}{a}\right)
\end{aligned}$$

3. It is easy to answer this part qualitatively. For part a, If the outer surface at $r = b$ is grounded, $E = 0$ there which means that $\sigma = 0$ there. For part b, we just have no contribution from the electric field at b so the answer would be $\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{R} - \frac{q}{a}\right)$.

□

Problem 2.43

Find the capacitance per unit length of two coaxial metal cylindrical tubes of radii a and b.

We looked at coaxial cables in problems 2.16 and 2.24. In those problems we allied Gauss' law and derived the expression $E 2\pi s L = \frac{Q}{\varepsilon_0}$ which leads us to

$$\boldsymbol{E} = \frac{Q}{2\pi\varepsilon_0 L}\frac{1}{s}\hat{\boldsymbol{s}}$$

. If we apply this to our problem and we say that the charge on the inner cable is Q we can get an expression for V.

$$\begin{aligned} V(b) - V(a) &= -\int_a^b \boldsymbol{E}\cdot d\boldsymbol{l} \\ &= -\int_a^b \frac{Q}{2\pi\varepsilon_0 L}\frac{1}{s}\hat{\boldsymbol{s}}\cdot d\boldsymbol{l} \\ &= -\frac{Q}{2\pi\varepsilon_0 L}\int_a^b \frac{1}{s} ds \\ &= -\frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right) \end{aligned}$$

The way we have set this problem up $V = V(a) - V(b) = -(V(b) - V(a)) = \frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right)$. We can now apply the identity for capacitance and divide by length to get capacitance per unit length:

$$\begin{aligned} \frac{C}{L} &= \left(\frac{Q}{V}\right)\frac{1}{L} \\ &= \left(\frac{Q}{\frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right)}\right)\frac{1}{L} \\ &= \frac{2\pi\varepsilon_0}{\ln\left(\frac{b}{a}\right)} \end{aligned}$$

□

Problem 2.50

The electric potential of some configuration is given by the expression

$$V(\boldsymbol{r}) = A\frac{e^{-\lambda r}}{r}$$

where A and λ are constants. Find the electric field $\boldsymbol{E}(\boldsymbol{R})$, the charge density $\rho(r)$, and the total charge Q. [Answer: $\rho = \varepsilon_0 A(4\pi\delta^3(\boldsymbol{r}) - \lambda^2 e^{-\lambda r}/r]$

We know that $E = -\nabla V$. We use this identity to solve for the electric field.

$$\begin{aligned}
\boldsymbol{E} &= -\nabla V \\
&= -A\left(\frac{\partial}{\partial r}\frac{e^{-\lambda r}}{r}\hat{\boldsymbol{r}}\right) \\
&= -A\left(-\frac{\lambda e^{-\lambda r}}{r} - \frac{e^{-\lambda r}}{r^2}\right) \\
&= Ae^{-\lambda r}\left(\frac{\lambda}{r} + \frac{1}{r^2}\right)\hat{\boldsymbol{r}}
\end{aligned}$$

Equation 2.14 gives us an expression for ρ in terms of $\boldsymbol{E}$:

$$\nabla\cdot\boldsymbol{E} = \frac{1}{\varepsilon_0}\rho$$

which we will apply to solve for ρ. Before doing do a little algebra on $\boldsymbol{E}$ to make it in a more friendly form: $\boldsymbol{E} = A(1+\lambda r)e^{\lambda r}\frac{\hat{\boldsymbol{r}}}{r^2}$. We will also apply the equation 1.102 to get $\nabla\cdot\frac{\hat{\boldsymbol{r}}}{r^2}$.

$$\begin{aligned}
\rho &= \varepsilon_0\nabla\cdot\boldsymbol{E} \\
&= \varepsilon_0\nabla\cdot\left(A(1+\lambda r)e^{\lambda r}\frac{\hat{\boldsymbol{r}}}{r^2}\right) \\
&= A\varepsilon_0\left((1+\lambda r)e^{\lambda r}\nabla\cdot\frac{\hat{\boldsymbol{r}}}{r^2} + \frac{\hat{\boldsymbol{r}}}{r^2}\cdot\nabla\left((1+\lambda r)e^{\lambda r}\right)\right) \\
&= A\varepsilon_0\left((1+\lambda r)e^{\lambda r}4\pi\delta^3(\boldsymbol{r}) + \frac{\hat{\boldsymbol{r}}}{r^2}\left[-\lambda^2 re^{-\lambda r}\right]\right) \\
&= A\varepsilon_0\left(4\pi\delta^3(\boldsymbol{r}) - \frac{\lambda^2}{r}e^{-\lambda r}\right)
\end{aligned}$$

We now solve for Q using the identity that $Q = \int\rho\, d\tau$.

$$\begin{aligned}
Q &= \int\rho\, d\tau \\
&= \int A\varepsilon_0\left(4\pi\delta^3(\boldsymbol{r}) - \frac{\lambda^2}{r}e^{-\lambda r}\right)4\pi dr \\
&= A\varepsilon_0\left[4\pi\int\delta^3(r)d\tau - 4\pi\lambda^2\int\frac{e^{-\lambda r}}{r}dr\right] \\
&= A\varepsilon_0\left(4\pi - 4\pi\lambda^2\int_0^\infty re^{-\lambda r}dr\right) \\
&= A\varepsilon_0\left(4\pi - 4\pi\lambda^2/\lambda^2\right) \\
&= 0
\end{aligned}$$

□