A Simple Real Business Cycle Model

Major Features of the Model

One source of uncertainty: z

Stochastic technology growth about a deterministic trend

We first define the important terms that will be used throughout:

Endogenous variables that change over time:

z	productivity (temporary or permanent)
K	capital stock owned by household
C	consumption
w	wage rate
r	interest rate
Y	output of final goods

Parameters:

α	capital share in output from a Cobb-Douglas production function
δ	rate of depreciation
β	time discount factor; $\beta<1$
a	trend in z
γ	elasticity of substitution, $\gamma>0$
ρ	autocorrelation parameter for z; $0<\rho<1$
σ	standard deviations of the shocks to z; $0<\sigma$
H	fixed labor endowment

We next setup the timing of the model, where a prime on a variable indicates its value next period.

1) Beginning of period –z known
2) Factor markets open & clear
   K is loaned out to production firms and r is determined
   H is hired out to production firms and w is determined
3) Production of goods occurs
4) Factor payments made (wH, rK)
5) K' is chosen
   Consumption, C, occurs
6) Temporary shocks, z' revealed
   End of period

Nonstationary Model
Households
Given information on prices and shocks, $\mathbf{\Omega} = \{w, r, z\}$, the household solves the following non-linear program when the factor markets clear.
$$V(K, \mathbf{\Omega}) = \underset{K'}{Max}\, u(C) + \beta E\{V(K', \mathbf{\Omega}')\}$$
where:
$$C = wH + (1-\delta + r)K - K' \tag{1.1}$$
The first-order condition is:
One condition for K':
$$u'(C)(-1) + \beta E\{V_K^i(K', \Omega')\} = 0$$
The envelope condition from this problem is as follows.
One condition for K:
$$V_K(K, \Omega) = u'(C)(1-\delta + r)$$
Picking functional form of $u(C) = \frac{1}{1-\gamma}(C^{1-\gamma} - 1)$
$$u'(C) = C^{-\gamma}$$
The Euler equation from combining the first-order condition with next period's version of the envelope condition is:
$$1 = \beta E\{(\tfrac{C}{C'})^{\gamma}(1-\delta + r')\} \tag{1.2}$$

Additional Behavioral Equations
The law of motion for z is:
$z' = \rho z + \varepsilon'$; where ε' is distributed normal with a mean of 0 and a variance of σ^2 (1.3)

An assumption of competition in the goods market along with a Cobb-Douglas production function gives the following shares in output for labor & capital. Market clearing conditions have already been imposed.
$$Y = K^{\alpha}(e^{at+z}H)^{1-\alpha} \tag{1.4}$$
$$wH = (1-\alpha)Y \tag{1.5}$$
$$rK = \alpha Y \tag{1.6}$$

Definitions for Later Use
$$I \equiv K' - (1-\delta)K \tag{1.7}$$
$$A \equiv e^{at+z} \tag{1.8}$$

Eqs (1.1)-(1.8) are the system.

Transformation & Simplifications

1) If z is stationary ($\rho < 1$):

Transform the problem by dividing all growing variables by e^{at}, denoting with a carat.

$z' = \rho z + \varepsilon'$; where ε' is distributed normal with a mean of 0 and a variance of σ^2 (2.1)

$$\hat{C} = \hat{w}H + (1-\delta+r)\hat{K} - (1+a)\hat{K}' \quad (2.2)$$

$$1 = \beta E\left\{\left(\frac{\hat{C}}{(1+a)\hat{C}'}\right)^{\gamma}(1-\delta+r')\right\} \quad (2.3)$$

$$\hat{Y} = \hat{K}^{\alpha}(e^{z}H)^{1-\alpha} \quad (2.4)$$

$$\hat{w}H = (1-\alpha)\hat{Y} \quad (2.5)$$

$$r\hat{K} = \alpha\hat{Y} \quad (2.6)$$

$$\hat{I} = (1+a)\hat{K}' - (1-\delta)\hat{K} \quad (2.7)$$

$$\hat{A} = e^{z} \quad (2.8)$$

2) If z is non-stationary ($\rho = 1$):

Transform the problem by dividing all growing variables by e^{at+z}, denoting with a carat.

$\Delta z' = \varepsilon'$; where ε' is distributed normal with a mean of 0 and a variance of σ^2 (2.1)

$$\hat{C} = \hat{w}H + (1-\delta+r)\hat{K} - (1+a+\Delta z')\hat{K}' \quad (2.2)$$

$$1 = \beta E\left\{\left(\frac{\hat{C}}{(1+a+\Delta z')\hat{C}'}\right)^{\gamma}(1-\delta+r')\right\} \quad (2.3)$$

$$\hat{Y} = \hat{K}^{\alpha}H^{1-\alpha} \quad (2.4)$$

$$\hat{w}H = (1-\alpha)\hat{Y} \quad (2.5)$$

$$r\hat{K} = \alpha\hat{Y} \quad (2.6)$$

$$\hat{I} = (1+a+\Delta z')\hat{K}' - (1-\delta)\hat{K} \quad (2.7)$$

$$\hat{A} = 1 \quad (2.8)$$

These are the equations we will use in Dynare.

The endogenous variables are $\hat{C}, \hat{K}, \hat{Y}, \hat{w}, r, \hat{I}, \hat{A},$ & Δz or z.

The exogenous variable is ε.

The parameters are $\alpha, \delta, \beta, a, \gamma, \rho, \sigma$ & H.

Steady State
System of 6 equations in 6 unknowns, $\gamma, \bar{z}$ or $\Delta\bar{z}, \overline{C}, \overline{K}, \overline{w}, \overline{Y}$
Parameters are $\bar{r}, H, \alpha, \beta, \delta, a$
Note that we could switch $\bar{r}$, γ and a, but by (3.3) any two determine the remaining one.

$$\bar{z} = 0 \text{ or } \Delta\bar{z} = 0 \tag{3.1}$$
$$\overline{C} = \overline{w}H + (\bar{r} - \delta - g)\overline{K} \tag{3.2}$$
$$(1+a)^{\gamma} = \beta(1 - \delta + \bar{r}) \tag{3.3}$$
$$\overline{Y} = \overline{K}^{\alpha} H^{1-\alpha} \tag{3.4}$$
$$\overline{w}H = (1-\alpha)\overline{Y} \tag{3.5}$$
$$\bar{r}\overline{K} = \alpha\overline{Y} \tag{3.6}$$

(3.1) eliminates $\bar{z}$ or $\Delta\bar{z}$
(3.3) eliminates σ

$$\gamma = \frac{\ln\beta + \ln(1 - \delta + \bar{r})}{\ln(1+a)} \cong \frac{\ln\beta + \bar{r} - \delta}{a} \tag{3.3'}$$

(3.4) into (3.5) & (3.6) and eliminate $\overline{Y}$

$$\overline{w} = (1-\alpha)\overline{K}^{\alpha} H^{-\alpha} \tag{3.5'}$$
$$\bar{r} = \alpha\overline{K}^{\alpha-1} H^{1-\alpha} \tag{3.6'}$$

(3.6') yields a solution for $\overline{K}$

$$\overline{K} = \left(\frac{\alpha}{\bar{r}}\right)^{\frac{1}{1-\alpha}} H \tag{3.7}$$

Parameterization
We need to choose parameters that are reasonable given evidence from micro studies or from other sources. Picking parameters so that the model matches empirical evidence is the simulation version of data mining.

α	.3 – average observed US capital share
δ	.02 – quarterly rate of depreciation
β	.995 – quarterly time discount factor
a	.008341 – average quarterly growth of GDP in post-war USA
$\bar{r}$	.026214 – implies a user cost of capital of 2.5% APR
H	normalize to one

Using equations (3.3') & (3.7) these give:
$\gamma = 0.1423$
$\overline{K} = 32.5288$

Parameterizing the law of motion

ρ	.9
σ	.02

A 2nd Real Business Cycle Model

Major Features of the Model

Add a labor-leisure decision with continuous hours to model 1
One source of uncertainty: z
Stochastic technology growth about a deterministic trend

We first define the important terms that will be used throughout:
Endogenous variables that change over time:

z	productivity (temporary or permanent)
K	capital stock owned by household
H	labor supplied by household
C	consumption
w	wage rate
r	interest rate
Y	output of final goods

Parameters:

α	capital share in output from a Cobb-Douglas production function
δ	rate of depreciation
β	time discount factor; β<1
a	trend in z
γ	elasticity of substitution, γ >0
D	leisure weight in utility
ρ	autocorrelation parameter for z; 0<ρ<1
σ	standard deviations of the shocks to z; 0<σ

Nonstationary Model

Given information on prices and shocks, $\mathbf{\Omega} = \{w, r, z\}$, the household solves the following non-linear program when the factor markets clear.

$$V(K, \mathbf{\Omega}) = \underset{K', H}{Max}\, u(C, 1-H) + \beta E\{V(K', \mathbf{\Omega}')\}$$

where:

$$C = wH + (1-\delta + r)K - K' \tag{1.1}$$

The first-order conditions are:

$$u_c(C, 1-H)(-1) + \beta E\{V_K^i(K', \Omega')\} = 0$$

$$u_c(C, 1-H)w - u_h(C, 1-H) = 0$$

The envelope condition from this problem is as follows.

$$V_K(K, \Omega) = u_c(C, 1-H)(1-\delta + r)$$

The Euler equations are:

$$u_c(C, 1-H) = \beta E\{u_c(C', 1-H')(1-\delta + r')\} \tag{1.2}$$

$$u_c(C, 1-H)w = u_h(C, 1-H) \tag{1.3}$$

Picking functional form of $u(C,L)=\frac{1}{1-\gamma}(C^{1-\gamma}-1)+D\frac{1}{1-\gamma}[(e^{at}L)^{1-\gamma}-1]$

$u_c(C,L)=C^{-\gamma}$ & $u_h(C,L)=BL^{-\gamma}$

Rewriting (1.2) & (1.3)

$$1=\beta E\left\{\left(\tfrac{C}{C'}\right)^{\gamma}(1-\delta+r')\right\} \tag{1.2'}$$

$$C^{-\gamma}w=De^{at}(1-H)^{-\gamma} \tag{1.3'}$$

Additional Behavioral Equations

The law of motion for *z* is:

$z'=\rho z+\varepsilon'$; where ε' is distributed normal with a mean of 0 and a variance of σ^2 (1.4)

An assumption of competition in the goods market along with a Cobb-Douglas production function gives the following shares in output for labor & capital. Market clearing conditions have already been imposed.

$$Y=K^{\alpha}(e^{at+z}H)^{1-\alpha} \tag{1.5}$$

$$wH=(1-\alpha)Y \tag{1.6}$$

$$rK=\alpha Y \tag{1.7}$$

Definitions for Later Use

$$I\equiv K'-(1-\delta)K \tag{1.8}$$

$$A\equiv e^{at+z} \tag{1.9}$$

Eqs (1.1)-(1.9) are the system.

Transformation & Simplifications

If z is stationary ($\rho < 1$):

Transform the problem by dividing all growing variables by $A \equiv e^{at}$, denoting with a carat.

$$z' = \rho z + \varepsilon' \text{; where } \varepsilon' \text{ is distributed normal with a mean of 0 and a variance of } \sigma^2 \tag{2.1}$$

$$\hat{C} = \hat{w}H + (1-\delta+r)\hat{K} - (1+a)\hat{K}' \tag{2.2}$$

$$1 = \beta E\left\{\left(\frac{\hat{C}}{(1+a)\hat{C}'}\right)^{\gamma}(1-\delta+r')\right\} \tag{2.3}$$

$$\hat{C}^{-\gamma}\hat{w} = D(1-H)^{-\gamma} \tag{2.4}$$

$$\hat{Y} = \hat{K}^{\alpha}(e^{z}H)^{1-\alpha} \tag{2.5}$$

$$\hat{w}H = (1-\alpha)\hat{Y} \tag{2.6}$$

$$r\hat{K} = \alpha\hat{Y} \tag{2.7}$$

Variables are

These are the equations we will use in Dynare.

The endogenous variables are $\hat{C}, \hat{K}, H, \hat{Y}, \hat{w}, r, \hat{I}, \hat{A}$ & z .

The exogenous variable is ε.

The parameters are $\alpha, \delta, \beta, a, \gamma, \rho, \sigma$ & D .

A 3rd Real Business Cycle Model

Major Features of the Model

Add a labor-leisure decision with indivisible labor hours to model 1
One source of uncertainty: z
Stochastic technology growth about a deterministic trend

We first define the important terms that will be used throughout:
Endogenous variables that change over time:

z	productivity (temporary or permanent)
K	capital stock owned by households
H	labor supplied by households
C	consumption
w	wage rate
r	interest rate
Y	output of final goods

Parameters:

α	capital share in output from a Cobb-Douglas production function
δ	rate of depreciation
β	time discount factor; $\beta<1$
a	trend in z
γ	elasticity of substitution, $\gamma>0$
B	leisure weight in utility
ρ	autocorrelation parameter for z; $0<\rho<1$
σ	standard deviations of the shocks to z; $0<\sigma$
H_0	hours worked by household that have a job

Nonstationary Model

Households *ex ante* enter a labor lottery with probability π of working H_0 hours. Consumption is fully insured, so all households have the same consumption. Households that "win" the lottery work, those that lose do not work.

Using the same utility function as before,
$u(C,1-H)=\frac{1}{1-\gamma}(C^{1-\gamma}-1)+B\frac{1}{1-\gamma}\{[e^{at}(1-H)]^{1-\gamma}-1\}$. The *ex ante* expected utility is

$$u(C,1-H)=\tfrac{1}{1-\gamma}(C^{1-\gamma}-1)+\pi B\tfrac{1}{1-\gamma}\{[e^{at}(1-H_0)]^{1-\gamma}-1\}+(1-\pi)B\tfrac{1}{1-\gamma}\{[e^{at}(1-0)]^{1-\gamma}-1\}$$
$$u(C,1-H)=\tfrac{1}{1-\gamma}(C^{1-\gamma}-1)+B\tfrac{1}{1-\gamma}\{e^{(1-\gamma)at}[\pi(1-H_0)^{1-\gamma}+(1-\pi)]-1\}$$
$$u(C,1-H)=\tfrac{1}{1-\gamma}(C^{1-\gamma}-1)+\pi e^{(1-\gamma)at}B[(1-H_0)^{1-\gamma}-1]+B\tfrac{1}{1-\gamma}[e^{(1-\gamma)at}-1]$$

In equilibrium all households enter the lottery, so $H=\pi H_0$. Households choose the level of consumption and whether to enter the lottery or not. The expected utility from entering the lottery must be greater than or equal to the utility from not entering. Since all households are identical, they must all enter the lottery in order for any labor to be supplied at all. If expected utility from entering the lottery is strictly greater than that from not entering, firms can reduce the wage and get the same amount of labor. In equilibrium the households will be indifferent between entering the lottery or not. The effect of this in the aggregate is that households collectively behave as if they were choosing the probability of winning the lottery. Since $\pi = \frac{H}{H_0}$ this is the same as choosing the aggregate hours worked, H.

Given information on prices and shocks, $\mathbf{\Omega} = \{w, r, z\}$, the household solves the following non-linear program when the factor markets clear.

$$V(K, \mathbf{\Omega}) = \underset{K', H}{Max} \tfrac{1}{1-\gamma}(C^{1-\gamma} - 1) + He^{(1-\gamma)at}\widetilde{D} + \widetilde{F}e^{(1-\gamma)at} - \widetilde{F} + \beta E\{V(K', \mathbf{\Omega}')\}$$

where $\widetilde{D} \equiv \frac{1}{H_0} D[(1-H_0)^{1-\gamma} - 1] < 0$, $\widetilde{F} \equiv D\frac{1}{1-\gamma}$, and

$$C = wH + (1-\delta+r)K - K' \tag{1.1}$$

The first-order conditions are:

$$C^{-\gamma}(-1) + \beta E\{V_K(K', \Omega')\} = 0$$

$$C^{-\gamma}w + e^{(1-\gamma)at}\widetilde{D} = 0$$

The envelope condition from this problem is as follows.

$$V_K(K, \Omega) = C^{-\gamma}(1-\delta+r)$$

The Euler equations are:

$$C^{-\gamma} = \beta E\{C'^{-\gamma}(1-\delta+r')\} \tag{1.2}$$

$$C^{-\gamma}w = -e^{(1-\gamma)at}\widetilde{D} \tag{1.3}$$

Additional Behavioral Equations

The law of motion for z is:

$$z' = \rho z + \varepsilon'; \text{ where } \varepsilon' \text{ is distributed normal with a mean of 0 and a variance of } \sigma^2 \tag{1.4}$$

An assumption of competition in the goods market along with a Cobb-Douglas production function gives the following shares in output for labor & capital. Market clearing conditions have already been imposed.

$$Y = K^{\alpha}(e^{at+z}H)^{1-\alpha} \tag{1.5}$$

$$wH = (1-\alpha)Y \tag{1.6}$$

$$rK = \alpha Y \tag{1.7}$$

Definitions for Later Use

$$I \equiv K' - (1-\delta)K \tag{1.8}$$

$$A \equiv e^{at+z} \tag{1.9}$$

Eqs (1.1)-(1.9) are the system.

Transformation & Simplifications

If z is stationary ($\rho < 1$):

Transform the problem by dividing all growing variables by $A \equiv e^{at}$, denoting with a carat.

$z' = \rho z + \varepsilon'$; where ε' is distributed normal with a mean of 0 and a variance of σ^2 (2.1)

$$\hat{C} = \hat{w}H + (1-\delta + r)\hat{K} - (1+a)\hat{K}' \tag{2.2}$$

$$1 = \beta E\left\{\left(\frac{\hat{C}}{(1+a)\hat{C}'}\right)^{\gamma}(1-\delta + r')\right\} \tag{2.3}$$

$$\hat{C}^{-\gamma}\hat{w} = \widetilde{D} \tag{2.4}$$

$$\hat{Y} = \hat{K}^{\alpha}(e^{z}H)^{1-\alpha} \tag{2.5}$$

$$\hat{w}H = (1-\alpha)\hat{Y} \tag{2.6}$$

$$r\hat{K} = \alpha\hat{Y} \tag{2.7}$$

$$\hat{I} = (1+a)\hat{K}' - (1-\delta)K \tag{2.8}$$

$$\hat{A} \equiv e^{z} \tag{2.9}$$

These are the equations we will use in Dynare.

The endogenous variables are $\hat{C}, \hat{K}, H, \hat{Y}, \hat{w}, r, \hat{I}, \hat{A}$ & z .

The exogenous variable is ε.

The parameters are $\alpha, \delta, \beta, a, \gamma, \rho, \sigma, D$ & H_0 .

A 4th Real Business Cycle Model

Major Features of the Model

Add population growth that follows a deterministic trend to model 3
One source of uncertainty: z
Stochastic technology growth about a deterministic trend
Labor-leisure decision with indivisible labor hours

We first define the important terms that will be used throughout:
Endogenous variables that change over time:

z	productivity (temporary or permanent)
K	capital stock owned by households
h	labor supplied by a single individual
H	total labor supplied
c	consumption by a single individual
w	wage rate
r	interest rate
Y	output of final goods
N	number of persons per household

Parameters:

α	capital share in output from a Cobb-Douglas production function
δ	rate of depreciation
β	time discount factor; β<1
a	trend in z
n	trend in N
γ	elasticity of substitution, γ >0
D	leisure weight in utility
ρ	autocorrelation parameter for z; 0<ρ<1
σ	standard deviations of the shocks to z; 0<σ
h_0	hours worked by household that have a job

Nonstationary Model

Households have increasing numbers of members, denoted N.
The law of motion for N is:

$$N' = e^n N \text{ or } N = e^{nt} N_0 \tag{1.1}$$

Given information on prices and shocks, $\mathbf{\Omega} = \{w, r, z\}$, the household solves the following non-linear program when the factor markets clear.

$$V(K, \mathbf{\Omega}) = \underset{K', m', h}{Max} \left[\tfrac{1}{1-\gamma}(c^{1-\gamma} - 1) + he^{(1-\gamma)at}\widetilde{D} + \widetilde{F}e^{(1-\gamma)at} - \widetilde{F}\right] N + \beta E\{V(K', \mathbf{\Omega}')\}$$

$$\widetilde{D} \equiv \tfrac{1}{H_0} D[(1-h_0)^{1-\gamma} - 1] < 0, \quad \widetilde{F} \equiv D\tfrac{1}{1-\gamma}$$

$$c = wh + (1 - \delta + r)\tfrac{K}{N} - \tfrac{K'}{N} \tag{1.2}$$

The first-order conditions are:

$c^{-\gamma}(-\tfrac{1}{N})N + \beta E\{V_K(K',\Omega')\} = 0$

$c^{-\gamma}wN + e^{(1-\gamma)at}\widetilde{D}N = 0$

The envelope condition from this problem is as follows.

$V_K(K,\Omega) = c^{-\gamma}(1-\delta+r)\tfrac{1}{N}N$

The Euler equations are:

$$c^{-\gamma} = \beta E\{c'^{-\gamma}(1-\delta+r')\} \tag{1.3}$$

$$c^{-\gamma}w = -e^{(1-\gamma)at}\widetilde{D} \tag{1.4}$$

Additional Behavioral Equations

The law of motion for z is:

$z' = \rho z + \varepsilon'$; where ε' is distributed normal with a mean of 0 and a variance of σ^2 (1.5)

An assumption of competition in the goods market along with a Cobb-Douglas production function gives the following shares in output for labor & capital. Market clearing conditions have already been imposed.

$$Y = K^{\alpha}(e^{at+z}H)^{1-\alpha} \tag{1.6}$$

$$wH = (1-\alpha)Y \tag{1.7}$$

$$rK = \alpha Y \tag{1.8}$$

Aggregating over household members gives:

$$H = Nh \tag{1.9}$$

Definitions:

$$I \equiv K' - (1-\delta)K \tag{1.10}$$

$$A \equiv e^{at+z} \tag{1.11}$$

Eqs (1.1)-(1.11) are the system.

Transformation & Simplifications

Without loss of generalization set $\hat{N} = N_0 = 1$, and eliminate it from the system.
Use (1.11) to eliminate H from the system.

Transform the problem by dividing:
c, w, A by e^{at}
K, Y, I by $e^{(a+n)t}$

$$z' = \rho z + \varepsilon'\text{; where } \varepsilon' \text{ is distributed normal with a mean of 0 and a variance of } \sigma^2 \tag{2.1}$$
$$\hat{c} = \hat{w}h + (1-\delta+r)\hat{K} - \hat{K}'(1+a+n) \tag{2.2}$$
$$1 = \beta E\left\{\left(\tfrac{\hat{c}}{(1+a)\hat{c}'}\right)^{\gamma}(1-\delta+r')\right\} \tag{2.3}$$
$$\hat{c}^{-\gamma}\hat{w} = \widetilde{D} \tag{2.4}$$
$$\hat{Y} = \hat{K}^{\alpha}(e^{z}h)^{1-\alpha} \tag{2.5}$$
$$\hat{w}h = (1-\alpha)\hat{Y} \tag{2.6}$$
$$r\hat{K} = \alpha\hat{Y} \tag{2.7}$$
$$\hat{I} = (1+a+n)\hat{K}' - (1-\delta)K \tag{2.8}$$
$$\hat{A} \equiv e^{z} \tag{2.9}$$

These are the equations we will use in Dynare.
The endogenous variables are $\hat{c}, \hat{K}, h, \hat{Y}, \hat{w}, r, \hat{I}, \hat{A}$ & z.
The exogenous variable is ε.
The parameters are $\alpha, \delta, \beta, a, \gamma, \rho, \sigma, D$ & h_0.

A 5th Real Business Cycle Model

Major Features of the Model

Add money and a cash-in-advance constraint to model 4
Add stochastic money growth about a deterministic trend
Two sources of uncertainty: z and g
Stochastic technology growth about a deterministic trend
Labor-leisure decision with indivisible labor hours
Population growth follows a deterministic trend

We first define the important terms that will be used throughout:
Endogenous variables that change over time:

z	productivity
g	money growth
K	capital stock owned by households
h	labor supplied by a single individual
c	consumption by a single individual
w	wage rate
r	interest rate
Y	output of final goods
N	number of persons per household
m	money balances per household
M	aggregate money supply
P	price of good in terms of money

Parameters:

α	capital share in output from a Cobb-Douglas production function
δ	rate of depreciation
β	time discount factor; β<1
a	trend in z
μ	trend in g
n	trend in N
γ	elasticity of substitution, γ >0
D	leisure weight in utility
h_0	hours worked by household that have a job
ρ_i	autocorrelation parameter for i=z,g; 0<ρ_i <1
σ_z	standard deviations of the shocks to i=z,g; 0< σ_i

Nonstationary Model

Households have increasing numbers of members, denoted N.
The law of motion for N is:

$$N' = e^n N \text{ or } N = e^{nt} N_0 \tag{1.1}$$

Households face both a budget constraint and a cash-in-advance constraint. These are:

$c = \frac{m}{PN} + \frac{(\mu+g)M}{PN}$

$wh + (1-\delta + r)\frac{K}{N} + \frac{m}{PN} + \frac{gM}{PN} = c + \frac{K'}{N} + \frac{m'}{PN}$

Substituting the first into the second and solving for c & h gives the definitions in the problem below.

Given information on prices and shocks, $\mathbf{\Omega} = \{w, r, z, g\}$, the household solves the following non-linear program when the factor markets clear.

$$V(K, m, \mathbf{\Omega}) = \underset{K', m', h}{Max}\left[\tfrac{1}{1-\gamma}(c^{1-\gamma} - 1) + he^{(1-\gamma)at}\widetilde{D} + \widetilde{F}e^{(1-\gamma)at} - \widetilde{F}\right]N + \beta E\{V(K', m', \mathbf{\Omega}')\}$$

$\widetilde{D} \equiv \frac{1}{H_0}D[(1-h_0)^{1-\gamma} - 1] < 0$, $\widetilde{F} \equiv D\frac{1}{1-\gamma}$

$$c = \frac{m}{PN} + \frac{(\mu+g)M}{PN} \tag{1.2}$$

$$h = \frac{K'}{wN} + \frac{m'}{wPN} - (1 + r - \delta)\frac{K}{wN} \tag{1.3}$$

The first-order conditions are:

$e^{(1-\gamma)at}\widetilde{D}(-\frac{1}{wN})N + \beta E\{V_K(K', \Omega')\} = 0$

$e^{(1-\gamma)at}\widetilde{D}(-\frac{1}{wPN})N + \beta E\{V_m(K', \Omega')\} = 0$

The envelope conditions are:

$V_K(K, \Omega) = e^{(1-\gamma)at}\widetilde{D}(-\frac{1}{wPN})(1-\delta + r)N$

$V_m(K, \Omega) = c^{-\gamma}\frac{1}{PN}N$

The Euler equations are:

$e^{(1-\gamma)at}\widetilde{D}\frac{1}{w} = \beta E\{e^{(1-\gamma)a(t+1)}\widetilde{D}(\frac{1}{w'})(1-\delta + r')\}$

$e^{(1-\gamma)at}\widetilde{D}(-\frac{1}{wP}) = \beta E\{c'^{-\gamma}\frac{1}{P'}\}$

Simplifying:

$$1 = \beta E\{e^{(1-\gamma)a}\frac{w}{w'}(1-\delta + r')\} \tag{1.4}$$

$$-e^{(1-\gamma)at}\widetilde{D} = \beta E\{c'^{-\gamma}\frac{wP}{P'}\} \tag{1.5}$$

Additional Behavioral Equations

Money changes over time according to:

$$M' = e^{\mu+g}M = M_0\prod_{s=1}^{t+1}e^{\mu+g_s} \tag{1.6}$$

The law of motion for g is:

$g' = \rho_g g + \varepsilon_g'$; where ε_g' is distributed normal with a mean of 0 and a variance of σ_g^2 (1.7)

The law of motion for z is:

$z' = \rho_z z + \varepsilon_z'$; where ε_z' is distributed normal with a mean of 0 and a variance of σ_z^2 (1.8)

An assumption of competition in the goods market along with a Cobb-Douglas production function gives the following shares in output for labor & capital. Market clearing conditions have already been imposed.

$$Y = K^{\alpha}(e^{at+z}H)^{1-\alpha} \tag{1.9}$$

$$wH = (1-\alpha)Y \tag{1.10}$$

$$rK = \alpha Y \tag{1.11}$$

Aggregating over household members gives:

$$H = Nh \tag{1.12}$$

Money market clearing gives:

$$M = m \tag{1.13}$$

Definitions:

$$I \equiv K' - (1-\delta)K \tag{1.14}$$

$$A \equiv e^{at+z} \tag{1.15}$$

Eqs (1.1)-(1.15) are the system.

Transformation & Simplifications

Without loss of generalization set $\hat{N} = N_0 = 1$, and eliminate it from the system.
Use (1.12) to eliminate *H* from the system.
Use (1.13) to eliminate *m* from the system.

Transform the problem by dividing:
c, w, A by e^{at}
K, Y, I by $e^{(a+n)t}$
M by $e^{\mu t + G_t}$; $G_t \equiv \sum_{s=1}^{t} g_s$ *M* has a unit root.
P by $e^{(\mu - a - n)t + G_t}$
r & *h* do not need to be transformed.

$$g' = \rho_g g + \varepsilon_g' \tag{2.1}$$

$$z' = \rho_z z + \varepsilon_z' \tag{2.2}$$

$$\hat{M}' = \hat{M} = M_0 \tag{2.3}$$

$$\hat{c} = \tfrac{(1+\mu+g)\hat{M}}{\hat{P}} \tag{2.4}$$

$$h = \tfrac{\hat{K}'(1+a+n)}{\hat{w}} + \tfrac{\hat{M}'(1+\mu+g')}{\hat{w}\hat{P}} - (1 + r - \delta)\tfrac{\hat{K}}{\hat{w}} \tag{2.5}$$

$$1 = \beta E\{e^{-\gamma a}\tfrac{\hat{w}}{\hat{w}'}(1 - \delta + r')\} \tag{2.6}$$

$$-\widetilde{D} = \beta E\{[\hat{c}' e^{a}]^{-\gamma} \tfrac{\hat{w}\hat{P}}{\hat{P}'(1+\mu-a-n+g')}\} \tag{2.7}$$

$$\hat{Y} = \hat{K}^{\alpha}(e^{z}h)^{1-\alpha} \tag{2.8}$$

$$\hat{w}h = (1-\alpha)\hat{Y} \tag{2.9}$$

$$r\hat{K} = \alpha\hat{Y} \tag{2.10}$$

$$\hat{I} = (1 + a + n)\hat{K}' - (1-\delta)K \tag{2.11}$$

$$\hat{A} \equiv e^{z} \tag{2.12}$$

These are the equations we will use in Dynare.
The endogenous variables are $\hat{c}, \hat{K}, h, \hat{Y}, \hat{w}, r, \hat{M}, \hat{P}, \hat{I}, \hat{A}, g$ & z.
The exogenous variables are ε_z & ε_g.
The parameters are $\alpha, \delta, \beta, a, \mu, \gamma, \rho_z, \sigma_z, \rho_g, \sigma_g, D, h_0$ & M_0.

A 6th Real Business Cycle Model

Major Features of the Model

Add money and utility from money to model 4
Add stochastic money growth about a deterministic trend
Two sources of uncertainty: z and g
Stochastic technology growth about a deterministic trend
Labor-leisure decision with indivisible labor hours
Population growth follows a deterministic trend

We first define the important terms that will be used throughout:
Endogenous variables that change over time:

z	productivity
g	money growth
K	capital stock owned by households
h	labor supplied by a single individual
c	consumption by a single individual
w	wage rate
r	interest rate
Y	output of final goods
N	number of persons per household
m	money balances per household
M	aggregate money supply
P	price of good in terms of money

Parameters:

α	capital share in output from a Cobb-Douglas production function
δ	rate of depreciation
β	time discount factor; β<1
a	trend in z
μ	trend in g
n	trend in N
γ	elasticity of substitution, γ >0
D	leisure weight in utility
J	money weight in utility
ρ_i	autocorrelation parameter for i=z,g; 0<ρ_i <1
σ_z	standard deviations of the shocks to i=z,g; 0< σ_i

Nonstationary Model

Households have increasing numbers of members, denoted N.
The law of motion for N is:

$$N' = e^n N \text{ or } N = e^{nt} N_0 \tag{1.1}$$

Households face only a budget constraint:

$$wh + (1-\delta + r)\tfrac{K}{N} + \tfrac{m}{PN} + \tfrac{(g+\mu)M}{PN} = c + \tfrac{K'}{N} + \tfrac{m'}{PN}$$

Given information on prices and shocks, $\mathbf{\Omega} = \{w, r, z, g\}$, the household solves the following non-linear program when the factor markets clear.

$$V(K, m, \mathbf{\Omega}) = \underset{K', m', h}{Max}\left[\tfrac{1}{1-\gamma}(c^{1-\gamma} - 1) + he^{(1-\gamma)at}\widetilde{D} + \widetilde{F}e^{(1-\gamma)at} - \widetilde{F} + J\tfrac{1}{1-\gamma}[(\tfrac{m}{PN})^{1-\gamma} - 1]\right]N$$
$$+ \beta E\{V(K', m', \mathbf{\Omega}')\}$$

$$\widetilde{D} \equiv \tfrac{1}{H_0}D[(1-h_0)^{1-\gamma} - 1] < 0, \quad \widetilde{F} \equiv D\tfrac{1}{1-\gamma}$$

$$c = wh + (1-\delta + r)\tfrac{K}{N} + \tfrac{m}{PN} + \tfrac{(\mu+g)M}{PN} - \tfrac{K'}{N} - \tfrac{m'}{PN} \tag{1.2}$$

The first-order conditions are:

$$c^{-\gamma}(-\tfrac{1}{N})N + \beta E\{V_K(K', \Omega')\} = 0$$
$$c^{-\gamma}(-\tfrac{1}{PN})N + \beta E\{V_m(K', \Omega')\} = 0$$
$$c^{-\gamma}wN + e^{(1-\gamma)at}\widetilde{D}N = 0$$

The envelope conditions are:

$$V_K(K, \Omega) = c^{-\gamma}(-\tfrac{1}{N})(1-\delta + r)N$$
$$V_m(K, \Omega) = c^{-\gamma}\tfrac{1}{PN}N + F(\tfrac{m}{PN})^{-\gamma}\tfrac{1}{PN}N$$

The Euler equations are:

$$c^{-\gamma} = \beta E\{c'^{-\gamma}(1-\delta + r')\} \tag{1.3}$$
$$c^{-\gamma} = \beta E\{[c'^{-\gamma} + F(\tfrac{m'}{P'N'})^{-\gamma}]\tfrac{P}{P'}\} \tag{1.4}$$
$$c^{-\gamma}w = -e^{(1-\gamma)at}\widetilde{D} \tag{1.5}$$

Additional Behavioral Equations

Money changes over time according to:

$$M' = e^{\mu+g}M = M_0\prod_{s=1}^{t+1}e^{\mu+g_s} \tag{1.6}$$

The law of motion for g is:

$g' = \rho_g g + \varepsilon_g'$; where ε_g' is distributed normal with a mean of 0 and a variance of σ_g^2 (1.7)

The law of motion for z is:

$z' = \rho_z z + \varepsilon_z'$; where ε_z' is distributed normal with a mean of 0 and a variance of σ_z^2 (1.8)

An assumption of competition in the goods market along with a Cobb-Douglas production function gives the following shares in output for labor & capital. Market clearing conditions have already been imposed.

$$Y = K^{\alpha}(e^{at+z}H)^{1-\alpha} \tag{1.9}$$
$$wH = (1-\alpha)Y \tag{1.10}$$
$$rK = \alpha Y \tag{1.11}$$

Aggregating over household members gives:

$$H = Nh \tag{1.12}$$

Money market clearing gives:

$$M = m \tag{1.13}$$

Definitions:

$$I \equiv K' - (1-\delta)K \tag{1.14}$$

$$A \equiv e^{at+z} \tag{1.15}$$

Eqs (1.1)-(1.15) are the system.

Transformation & Simplifications

Without loss of generalization set $\hat{N} = N_0 = 1$, and eliminate it from the system.
Use (1.12) to eliminate *H* from the system.
Use (1.13) to eliminate *m* from the system.

Transform the problem by dividing:
c, w, A by e^{at}
K, Y, I by $e^{(a+n)t}$
M by $e^{\mu t + G_t}$; $G_t \equiv \sum_{s=1}^{t} g_s$ *M* has a unit root.
P by $e^{(\mu - a - n)t + G_t}$
r & *h* do not need to be transformed.

$$g' = \rho_g g + \varepsilon_g' \tag{2.1}$$

$$z' = \rho_z z + \varepsilon_z' \tag{2.2}$$

$$\hat{M}' = \hat{M} = M_0 \tag{2.3}$$

$$\hat{c} = \hat{w}h + (1 - \delta + r)\hat{K} + \tfrac{(1+\mu+g)\hat{M}}{\hat{P}} - \hat{K}(1 + a + n) - \tfrac{\hat{M}'(1+\mu+g')}{\hat{P}} \tag{2.4}$$

$$1 = \beta E\{(\tfrac{\hat{c}}{\hat{c}' e^{a}})^{-\gamma}(1 - \delta + r')\} \tag{2.5}$$

$$\hat{c}^{-\gamma} = \beta E\{e^{-\gamma a}[\hat{c}'^{-\gamma} + F(\tfrac{\hat{M}'}{\hat{P}'})^{-\gamma}]\tfrac{\hat{P}}{\hat{P}'(1+\mu-a-n+g')}\} \tag{2.6}$$

$$\hat{c}^{-\gamma}\hat{w} = -\widetilde{D} \tag{2.7}$$

$$\hat{Y} = \hat{K}^{\alpha}(e^{z}h)^{1-\alpha} \tag{2.8}$$

$$\hat{w}h = (1 - \alpha)\hat{Y} \tag{2.9}$$

$$r\hat{K} = \alpha\hat{Y} \tag{2.10}$$

$$\hat{I} = (1 + a + n)\hat{K}' - (1 - \delta)K \tag{2.11}$$

$$\hat{A} \equiv e^{z} \tag{2.12}$$

These are the equations we will use in Dynare.
The endogenous variables are $\hat{c}, \hat{K}, h, \hat{Y}, \hat{w}, r, \hat{M}, \hat{P}, \hat{I}, \hat{A}, g$ & z.
The exogenous variables are ε_z & ε_g.
The parameters are $\alpha, \delta, \beta, a, \mu, \gamma, \rho_z, \sigma_z, \rho_g, \sigma_g, D, F$ & h_0.

A Sticky-Price Real Business Cycle Model

Major Features of the Model

Add Calvo-style sticky prices to model 5
We also add intermediate goods with monopoly producers
Two sources of uncertainty: z and g
Stochastic technology growth about a deterministic trend
Labor-leisure decision with indivisible labor hours
Population growth follows a deterministic trend

We first define the important terms that will be used throughout:
Endogenous variables that change over time:

z	productivity
g	money growth
K	capital stock owned by households
h	labor supplied by a single individual
c	consumption by a single individual
w	wage rate
r	interest rate
Y	output of final goods
$Y(i)$	output of intermediate goods firm i
N	number of persons per household
m	money balances per household
M	aggregate money supply
P	price of final good in terms of money
$P(i)$	price of intermediate good i

Parameters:

α	capital share in output from a Cobb-Douglas production function
δ	rate of depreciation
β	time discount factor; β<1
a	trend in z
μ	trend in g
n	trend in N
γ	elasticity of substitution, γ >0
ψ	elasticity of substitution between different intermediate goods
λ	Calvo probability of being unable to adjust prices
D	leisure weight in utility
h_0	hours worked by household that have a job
ρ_i	autocorrelation parameter for i=z,g; 0<ρ_i <1
σ_z	standard deviations of the shocks to i=z,g; 0< σ_i

Nonstationary Model

Households

Households have increasing numbers of members, denoted N.
The law of motion for N is:

$$N' = e^{n}N \text{ or } N = e^{nt}N_0 \tag{1.1}$$

Households face both a budget constraint and a cash-in-advance constraint. These are:

$$c = \tfrac{m}{PN} + \tfrac{(\mu+g)M}{PN}$$

$$wh + (1-\delta+r)\tfrac{K}{N} + \tfrac{m}{PN} + \tfrac{gM}{PN} = c + \tfrac{K'}{N} + \tfrac{m'}{PN}$$

Substituting the first into the second and solving for c & h gives the definitions in the problem below.
Given information on prices and shocks, $\mathbf{\Omega} = \{w, r, z, g\}$, the household solves the following non-linear program when the factor markets clear.

$$V(K, m, \mathbf{\Omega}) = \underset{K', m', h}{Max}\left[\tfrac{1}{1-\gamma}(c^{1-\gamma}-1) + he^{(1-\gamma)at}\widetilde{D} + \widetilde{F}e^{(1-\gamma)at} - \widetilde{F}\right]N + \beta E\{V(K', m', \mathbf{\Omega}')\}$$

$$\widetilde{D} \equiv \tfrac{1}{H_0}D[(1-h_0)^{1-\gamma}-1] < 0\,,\quad \widetilde{F} \equiv D\tfrac{1}{1-\gamma}$$

$$c = \tfrac{m}{PN} + \tfrac{(g+\mu)M}{PN} \tag{1.2}$$

$$h = \tfrac{K'}{wN} + \tfrac{m'}{wPN} - (1+r-\delta)\tfrac{K}{wN} \tag{1.3}$$

The first-order conditions are:

$$e^{(1-\gamma)at}\widetilde{D}(-\tfrac{1}{wN})N + \beta E\{V_K(K', \Omega')\} = 0$$

$$e^{(1-\gamma)at}\widetilde{D}(-\tfrac{1}{wPN})N + \beta E\{V_m(K', \Omega')\} = 0$$

The envelope conditions are:

$$V_K(K, \Omega) = e^{(1-\gamma)at}\widetilde{D}(-\tfrac{1}{wPN})(1-\delta+r)N$$

$$V_m(K, \Omega) = c^{-\gamma}\tfrac{1}{PN}N$$

The Euler equations are:

$$1 = \beta E\{e^{(1-\gamma)a}\tfrac{w}{w'}(1-\delta+r')\} \tag{1.4}$$

$$-e^{(1-\gamma)at}\widetilde{D} = \beta E\{c'^{-\gamma}\tfrac{wP}{P'}\} \tag{1.5}$$

Final Goods Producers

Each period a set of final goods producers spontaneously organizes and maximizes profits. Their production function is an intermediate goods aggregator:

$$Y = \left[\int_0^1 Y(i)^{\frac{\psi-1}{\psi}}\,di\right]^{\frac{\psi}{\psi-1}} \tag{1.6}$$

Their problem is:

$$\underset{\{Y(i)\}}{Max}\, P\left[\int_0^1 Y(i)^{\frac{\psi-1}{\psi}}\,di\right]^{\frac{\psi}{\psi-1}} - \int_0^1 P(i)Y(i)\,di$$

The generic first-order condition is:

$$P\frac{\psi}{\psi-1}\left[\int_0^1 Y(i)^{\frac{\psi-1}{\psi}}\,di\right]^{\frac{\psi}{\psi-1}-1}\frac{\psi-1}{\psi}Y(i)^{\frac{\psi-1}{\psi}-1}-P(i)=0$$

Which gives the following demand function for intermediate good *i*:

$$Y(i)=\left(\frac{P}{P(i)}\right)^{\psi}Y \tag{1.7}$$

It also gives the following relation between final goods and intermediate goods prices from a zero profit condition:

$$P=\left[\int_0^1 P(i)^{1-\psi}\,di\right]^{\frac{1}{1-\psi}} \tag{1.8}$$

Intermediate Goods Producers

There is a continuum of producers indexed on the unit interval. Each period a fraction, $1-\lambda$, of these producers are allowed to change their price. The remaining fraction retain the price from last period. This setup is known as Calvo pricing.

Intermediate goods are produced using the following production function.

$$Y(i)=K(i)^{\alpha}[e^{at+z}H(i)]^{1-\alpha} \tag{1.9}$$

The firms who cannot change their prices solve the following problem.

$$\underset{K(i),H(i)}{Max}\; P^{*}(i)K(i)^{\alpha}[e^{at+z}H(i)]^{1-\alpha}-P[rK(i)-wH(i)]$$

The first-order conditions are:

$$\alpha P^{*}(i)K(i)^{\alpha-1}[e^{at+z}H(i)]^{1-\alpha}-P\,r=0$$

$$(1-\alpha)P^{*}(i)K(i)^{\alpha}[e^{at+z}H(i)]^{-\alpha}-P\,w=0$$

These give the following capital to labor ratio.

$$\frac{K(i)}{H(i)}=\frac{\alpha w}{(1-\alpha)r}$$

And they can be rewritten in conjunction with the production function as:

$$K(i)=Y(i)e^{(\alpha-1)(at+z)}[\tfrac{r(1-\alpha)}{w\alpha}]^{\alpha-1} \tag{1.10}$$

$$H(i)=Y(i)e^{\alpha(at+z)}[\tfrac{r(1-\alpha)}{w\alpha}]^{\alpha} \tag{1.11}$$

Note how the factor demands are proportional to output.

Substituting these into the cost function $TC=P[rK(i)-wH(i)]$ gives:

$$TC=PY(i)e^{(\alpha-1)(at+z)}[\tfrac{r(1-\alpha)}{w\alpha}]^{\alpha}\tfrac{w}{1-\alpha} \tag{1.12}$$

The firms that can change their prices solve (incorporating time subscripts here):

$$\underset{P_t^{*}(i)}{Max}\,E\left\{\sum_{s=0}^{\infty}(\beta\lambda)^{s}[P_t^{*}(i)Y_{t+s}(i)-TC_{t+s}]\right\}$$

Using (1.7) & (1.10):

$$\underset{P_t^{*}(i)}{Max}\,E\left\{\sum_{s=0}^{\infty}(\beta\lambda)^{s}\left(\tfrac{P_{t+s}}{P_t^{*}(i)}\right)^{\psi}Y_{t+s}\left[P_t^{*}(i)-P_{t+s}e^{(\alpha-1)[a(t+s)+z_{t+s}]}[\tfrac{r_{t+s}(1-\alpha)}{w_{t+s}\alpha}]^{\alpha}\,\tfrac{w_{t+s}}{1-\alpha}\right]\right\}$$

The first-order condition is:

$$E\left\{\sum_{s=0}^{\infty}(\beta\lambda)^s\psi\left(\tfrac{P_{t+s}}{P_t^*(i)}\right)^{\psi-1}\left(\tfrac{-1}{P_t^*(i)^2}\right)Y_{t+s}\left[P_t^*(i)-P_{t+s}e^{(\alpha-1)[a(t+s)+z_{t+s}]}\left[\tfrac{r_{t+s}(1-\alpha)}{w_{t+s}\alpha}\right]^{\alpha}\tfrac{w_{t+s}}{1-\alpha}\right]+\sum_{s=0}^{\infty}(\beta\lambda)^s\left(\tfrac{P_{t+s}}{P_t^*(i)}\right)^{\psi}Y_{t+s}\right\}=0$$

$$E\left\{\sum_{s=0}^{\infty}(\beta\lambda)^s\psi\left(\tfrac{P_{t+s}}{P_t^*(i)}\right)^{\psi}Y_{t+s}\left[-1+\tfrac{P_{t+s}}{P_t^*(i)}e^{(\alpha-1)[a(t+s)+z_{t+s}]}\left[\tfrac{r_{t+s}(1-\alpha)}{w_{t+s}\alpha}\right]^{\alpha}\tfrac{w_{t+s}}{1-\alpha}\right]+\sum_{s=0}^{\infty}(\beta\lambda)^s\left(\tfrac{P_{t+s}}{P_t^*(i)}\right)^{\psi}Y_{t+s}\right\}=0$$

$$E\left\{\sum_{s=0}^{\infty}(\beta\lambda)^s\left(\tfrac{P_{t+s}}{P_t^*(i)}\right)^{\psi}Y_{t+s}\left[1-\psi+\psi\tfrac{P_{t+s}}{P_t^*(i)}e^{(\alpha-1)[a(t+s)+z_{t+s}]}\left[\tfrac{r_{t+s}(1-\alpha)}{w_{t+s}\alpha}\right]^{\alpha}\tfrac{w_{t+s}}{1-\alpha}\right]\right\}=0$$

$$P_t^*(i)=\frac{\psi}{1-\psi}\frac{E\left\{\sum_{s=0}^{\infty}(\beta\lambda)^s\left(\tfrac{P_{t+s}}{P_t^*(i)}\right)^{\psi}Y_{t+s}P_{t+s}\left[e^{(\alpha-1)[a(t+s)+z_{t+s}]}\left[\tfrac{r_{t+s}(1-\alpha)}{w_{t+s}\alpha}\right]^{\alpha}\tfrac{w_{t+s}}{1-\alpha}\right]\right\}}{E\left\{\sum_{s=0}^{\infty}(\beta\lambda)^s\left(\tfrac{P_{t+s}}{P_t^*(i)}\right)^{\psi}Y_{t+s}\right\}}$$

Note that the above implies $P_t^*(i)=P_t^*$ for all firms that choose prices in the current period.
Using (1.7):

$$P_t^*=\frac{\psi}{1-\psi}\frac{E\left\{\sum_{s=0}^{\infty}(\beta\lambda)^sY_{t+s}(i)P_{t+s}\left[e^{(\alpha-1)[a(t+s)+z]}\left[\tfrac{r_{t+s}(1-\alpha)}{w_{t+s}\alpha}\right]^{\alpha}\tfrac{w_{t+s}}{1-\alpha}\right]\right\}}{E\left\{\sum_{s=0}^{\infty}(\beta\lambda)^sY_{t+s}(i)\right\}} \tag{1.13}$$

Aggregating Price Movements
Using (1.8) and noting that a fraction λ of prices do not change, while the remaining prices are set to P^*:

$$P'=\left[\int_0^1 P'(i)^{1-\psi}\,di\right]^{\frac{1}{1-\psi}}=\left[\int_0^{\lambda}P(i)^{1-\psi}\,di+\int_{\lambda}^{1}P^{*\prime 1-\psi}\,di\right]^{\frac{1}{1-\psi}}$$

$$P'^{1-\psi}=\lambda P^{1-\psi}+(1-\lambda)P^{*\prime 1-\psi} \tag{1.8'}$$

Additional Behavioral Equations
Money changes over time according to:

$$M'=e^{\mu+g}M=M_0\prod_{s=1}^{t+1}e^{\mu+g_s} \tag{1.14}$$

The law of motion for g is:

$$g'=\rho_g g+\varepsilon_g';\ \varepsilon_g'\sim N(0,\sigma_g^2) \tag{1.15}$$

The law of motion for z is:

$$z'=\rho_z z+\varepsilon_z';\quad \varepsilon_z'\sim N(0,\sigma_z^2) \tag{1.16}$$

Aggregating over household members gives:

$$H=Nh \tag{1.17}$$

Money market clearing gives:

$$M=m \tag{1.18}$$

Capital and labor market clearing give:

$$K = \int_0^1 K(i)\, di \tag{1.19}$$

$$H = \int_0^1 H(i)\, di \tag{1.20}$$

Definitions:

$$I \equiv K' - (1-\delta)K \tag{1.21}$$

$$A \equiv e^{at+z} \tag{1.22}$$

Eqs (1.1)-(1.22) are the system.

Transformation & Simplifications

Without loss of generalization set $\hat{N} = N_0 = 1$, and eliminate it from the system.
TC has already been eliminated by substitution of (1.12) to get (1.13).
(1.7) is also incorporated in (1.13).
Use (1.17) to eliminate *H* from the system.
Use (1.18) to eliminate *m* from the system.

Transform the problem by dividing:

c, w, A by e^{at}

$K(i), K, Y(i), Y, I$ by $e^{(a+n)t}$

M by $e^{\mu t + G_t}$; $G_t \equiv \sum_{s=1}^{t} g_s$ M has a unit root.

P by $e^{(\mu - a - n)t + G_t}$

r & *h* do not need to be transformed.

$$g' = \rho_g g + \varepsilon_g' ; \varepsilon_g' \sim N(0, \sigma_g^2) \tag{2.1}$$

$$z' = \rho_z z + \varepsilon_z' ;\ \ \varepsilon_z' \sim N(0, \sigma_z^2) \tag{2.2}$$

$$\hat{M}' = \hat{M} = M_0 \tag{2.3}$$

$$\hat{c} = \tfrac{e^{\mu+g}\hat{M}}{\hat{P}} \tag{2.4}$$

$$h = \tfrac{\hat{K}' e^{a+n}}{\hat{w}} + \tfrac{\hat{M}' e^{\mu+g'}}{\hat{w}\hat{P}} - (1 + r - \delta)\tfrac{\hat{K}}{\hat{w}} \tag{2.5}$$

$$1 = \beta E\{e^{-\gamma a} \tfrac{\hat{w}}{\hat{w}'} (1 - \delta + r')\} \tag{2.6}$$

$$-\widetilde{D} = \beta E\{(\hat{c}' e^{a})^{-\gamma} \tfrac{\hat{w}\hat{P}}{\hat{P}' e^{\mu - a - n + g'}}\} \tag{2.7}$$

$$\hat{Y} = \left[\int_0^1 \hat{Y}(i)^{\frac{\psi-1}{\psi}}\ di \right]^{\frac{\psi}{\psi-1}} \tag{2.8}$$

$$[\hat{P}' e^{\mu - a - n + g'}]^{1-\psi} = \lambda \hat{P}^{1-\psi} + (1-\lambda)[\hat{P}^{*\prime} e^{\mu - a - n + g'}]^{1-\psi} \tag{2.9}$$

$$\hat{Y}(i) = \hat{K}(i)^{\alpha} [e^{z} \hat{H}(i)]^{1-\alpha} \tag{2.10}$$

$$\hat{P}^* = \frac{\psi}{1-\psi} \frac{E\left\{\sum_{s=0}^{\infty} (\beta\lambda e^{\mu})^s e^{G_{t+s}} \hat{Y}_{t+s}(i) \hat{P}_{t+s} \left[e^{(\alpha-1)z_{t+s}} \left[\frac{r_{t+s}(1-\alpha}{\hat{w}_{t+s}\alpha}\right]^{\alpha} \frac{\hat{w}_{t+s}}{1-\alpha}\right]\right\}}{E\left\{\sum_{s=0}^{\infty} (\beta\lambda e^{\mu})^s e^{G_{t+s}} \hat{Y}_{t+s}(i)\right\}} \tag{2.11}$$

$$\hat{K}(i) = \hat{Y}(i) e^{(\alpha-1)z} [\tfrac{r(1-\alpha)}{\hat{w}\alpha}]^{\alpha-1} \tag{2.12}$$

$$\hat{H}(i) = \hat{Y}(i) e^{\alpha z} [\tfrac{r(1-\alpha)}{w\alpha}]^{\alpha} \tag{2.13}$$

$$\hat{K} = \int_0^1 \hat{K}(i)\, di \tag{2.14}$$

$$h = \int_0^1 \hat{H}(i)\, di \tag{2.15}$$

$$\hat{I} = e^{a+n} \hat{K}' - (1-\delta)K \tag{2.16}$$

$$\hat{A} \equiv e^{z} \tag{2.17}$$

One of these equations is redundant by Walras' Law. We could eliminate (2.5) or (2.8). We choose (2.8).

These are the 16 equations we will log-linearize and then use in Dynare.
The 16 endogenous variables are $\hat{c}, \hat{K}, \hat{K}(i), h, \hat{H}(i), \hat{Y}, \hat{Y}(i), \hat{w}, r, \hat{M}, \hat{P}, \hat{P}^*, \hat{I}, \hat{A}, g$ & z.
The exogenous variables are ε_z & ε_g.
The parameters are $\alpha, \delta, \beta, a, n, \mu, \gamma, \psi, \lambda, \rho_z, \sigma_z, \rho_g, \sigma_g, D$ & h_0.

The model as written above cannot be solved because it includes variables that are a function of *i*. These need to be substituted out before we can numerically solve and simulate.

Eliminating $\hat{M}$ & $\hat{H}$ and equations (2.3) & (2.17).

$$g' = \rho_g g + \varepsilon_g'; \ \varepsilon_g' \sim N(0, \sigma_g^2) \tag{3.1}$$

$$z' = \rho_z z + \varepsilon_z'; \quad \varepsilon_z' \sim N(0, \sigma_z^2) \tag{3.2}$$

$$\hat{c} = \tfrac{e^{\mu+g}\hat{M}}{\hat{P}} \tag{3.3}$$

$$h = \tfrac{\hat{K}' e^{a+n}}{\hat{w}} + \tfrac{M_0 e^{\mu+g'}}{\hat{w}\hat{P}} - (1 + r - \delta)\tfrac{\hat{K}}{\hat{w}} \tag{3.4}$$

$$1 = \beta E\{e^{-\gamma a}\tfrac{\hat{w}}{\hat{w}'}(1 - \delta + r')\} \tag{3.5}$$

$$-\widetilde{D} = \beta E\{(\hat{c}' e^a)^{-\gamma}\tfrac{\hat{w}\hat{P}}{\hat{P}' e^{\mu-a-n+g'}}\} \tag{3.6}$$

$$[\hat{P}' e^{\mu-a-n+g'}]^{1-\psi} = \lambda \hat{P}^{1-\psi} + (1-\lambda)[\hat{P}^{*\prime} e^{\mu-a-n+g'}]^{1-\psi} \tag{3.7}$$

$$\hat{P}^* = \frac{\psi}{1-\psi}\frac{E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^s e^{G_{t+s}}\hat{Y}_{t+s}(i)\hat{P}_{t+s}\left[e^{(\alpha-1)z_{t+s}}[\tfrac{r_{t+s}(1-\alpha}{\hat{w}_{t+s}\alpha}]^{\alpha}\ \tfrac{\hat{w}_{t+s}}{1-\alpha}\right]\right\}}{E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^s e^{G_{t+s}}\hat{Y}_{t+s}(i)\right\}} \tag{3.8}$$

$$\hat{Y}(i) = \hat{K}(i)^{\alpha}[e^z \hat{H}(i)]^{1-\alpha} \tag{3.9}$$

$$\hat{K}(i) = \hat{Y}(i)e^{(\alpha-1)z}[\tfrac{r(1-\alpha)}{\hat{w}\alpha}]^{\alpha-1} \tag{3.10}$$

$$\hat{H}(i) = \hat{Y}(i)e^{\alpha z}[\tfrac{r(1-\alpha)}{w\alpha}]^{\alpha} \tag{3.11}$$

$$\hat{K} = \int_0^1 \hat{K}(i)\, di \tag{3.12}$$

$$h = \int_0^1 \hat{H}(i)\, di \tag{3.13}$$

$$\hat{I} = e^{a+n}\hat{K}' - (1-\delta)K \tag{3.14}$$

14 equations in 14 unknowns

$\hat{c}, \hat{K}, h, \hat{Y}, \hat{Y}(i), \hat{K}(i), \hat{H}(i), \hat{w}, r, \hat{P}, \hat{P}^*, \hat{I}, g$ & z.

We need to eliminate $\hat{Y}(i), \hat{K}(i), \hat{H}(i)$ from the system.

Define $\phi(i) \equiv \frac{\hat{Y}(i)}{\hat{Y}}$ and note that by (3.10) & (3.11) we get $\phi(i) = \frac{\hat{K}(i)}{\hat{K}} = \frac{\hat{H}(i)}{h}$.

Using the intermediate goods production function:

$$\hat{Y}(i) = e^{(1-\alpha)z} \hat{K}(i)^{\alpha} \hat{H}(i)^{1-\alpha}$$

$$\phi(i)\hat{Y} = e^{(1-\alpha)z} \left[\phi(i)\hat{K}\right]^{\alpha} \left[\phi(i)h\right]^{1-\alpha}$$

$$\phi(i)\hat{Y} = e^{(1-\alpha)z} \phi(i) \hat{K}^{\alpha} h^{1-\alpha}$$

$$\hat{Y} = e^{(1-\alpha)z} \hat{K}^{\alpha} h^{1-\alpha}$$

Use the above in place of (3.9) with (3.4) becoming redundant

Also note that $\int_0^1 \phi(i)\, di = 1$, making (3.12) & (3.13) redundant.

(3.10) and (3.11) become:

$$\hat{K} = \hat{Y} e^{(\alpha-1)z} [\tfrac{r(1-\alpha)}{\hat{w}\alpha}]^{\alpha-1}$$

$$h = \hat{Y} e^{\alpha z} [\tfrac{r(1-\alpha)}{\hat{w}\alpha}]^{\alpha}$$

Which with a little algebra reduce to:

$$r\hat{K} = \alpha \hat{Y}$$

$$\hat{w}h = (1-\alpha)\hat{Y}$$

Lastly, we need to remove $\hat{Y}(i)$ from (3.8):

Let's start with the nonstationary version, noting that $P_t^*(i) = P_t^* \ \forall i$.

$$P_t^* = \frac{\psi}{1-\psi} \frac{E\left\{\sum_{s=0}^{\infty} (\beta\lambda)^s \left(\frac{P_{t+s}}{P_t^*}\right)^{\psi} Y_{t+s} P_{t+s} \left[e^{(\alpha-1)[a(t+s)+z_{t+s}]} [\frac{r_{t+s}(1-\alpha)}{w_{t+s}\alpha}]^{\alpha} \frac{w_{t+s}}{1-\alpha}\right]\right\}}{E\left\{\sum_{s=0}^{\infty} (\beta\lambda)^s \left(\frac{P_{t+s}}{P_t^*}\right)^{\psi} Y_{t+s}\right\}}$$

Stationarizing:

$$\frac{P_t^* e^{(\mu-a-n)t+G_t}}{e^{(\mu-a-n)t+G_t}} = \frac{\psi}{1-\psi} \frac{E\left\{\sum_{s=0}^{\infty} (\beta\lambda)^s \left(\frac{P_{t+s}}{P_t^*}\right)^{\psi} \frac{Y_{t+s} e^{(a+n)(t+s)}}{e^{(a+n)(t+s)}} \frac{P_{t+s} e^{(\mu-a-n)(t+s)+G_{t+s}}}{e^{(\mu-a-n)(t+s)+G_{t+s}}} \left[e^{(\alpha-1)z_{t+s}} [\frac{r_{t+s}(1-\alpha)}{\hat{w}_{t+s}\alpha}]^{\alpha} \frac{\hat{w}_{t+s}}{1-\alpha}\right]\right\}}{E\left\{\sum_{s=0}^{\infty} (\beta\lambda)^s \left(\frac{P_{t+s}}{P_t^*}\right)^{\psi} \frac{Y_{t+s} e^{(a+n)(t+s)}}{e^{(a+n)(t+s)}}\right\}}$$

$$\left(\frac{P_{t+s}}{P_t^*}\right)^{\psi} = \left(\frac{P_{t+s} e^{(\mu-a-n)(t+s)+G_{t+s}} e^{(\mu-a-n)t+G_t}}{P_t^* e^{(\mu-a-n)t+G_t} e^{(\mu-a-n)(t+s)+G_{t+s}}}\right)^{\psi} = \left(\frac{\hat{P}_{t+s} e^{(\mu-a-n)(t+s)+G_{t+s}}}{\hat{P}_t^* e^{(\mu-a-n)t+G_t}}\right)^{\psi} = \left(\frac{\hat{P}_{t+s}}{\hat{P}_t^*}\right)^{\psi} e^{\psi[(\mu-a-n)s+G_{t+s}-G_t]}$$

$$\hat{P}_t^* e^{(\mu-a-n)t+G_t} = \frac{\psi}{1-\psi} \frac{e^{\mu t+G_t} E\left\{\sum_{s=0}^{\infty} (\beta\lambda e^{\mu+\psi(\mu-a-n)})^s e^{(1+\psi)(G_{t+s}-G_t)} \left(\frac{\hat{P}_{t+s}}{\hat{P}_t^*}\right)^{\psi} \hat{Y}_{t+s} \hat{P}_{t+s} \left[e^{(\alpha-1)z_{t+s}} \left[\frac{r_{t+s}(1-\alpha)}{\hat{w}_{t+s}\alpha}\right]^{\alpha} \frac{\hat{w}_{t+s}}{1-\alpha}\right]\right\}}{e^{(a+n)t} E\left\{\sum_{s=0}^{\infty} (\beta\lambda e^{a+n+\psi(\mu-a-n)})^s e^{\psi(G_{t+s}-G_t)} \left(\frac{\hat{P}_{t+s}}{\hat{P}_t^*}\right)^{\psi} \hat{Y}_{t+s}\right\}}$$

$$\hat{P}_t^* = \frac{\psi}{1-\psi} \frac{\hat{P}_t^{*-\psi} E\left\{\sum_{s=0}^{\infty} (\beta\lambda e^{\mu+\psi(\mu-a-n)})^s e^{(1+\psi)(G_{t+s}-G_t)} \hat{P}_{t+s}{}^{\psi} \hat{Y}_{t+s} \hat{P}_{t+s} \left[e^{(\alpha-1)z_{t+s}} \left[\frac{r_{t+s}(1-\alpha)}{\hat{w}_{t+s}\alpha}\right]^{\alpha} \frac{\hat{w}_{t+s}}{1-\alpha}\right]\right\}}{\hat{P}_t^{*-\psi} E\left\{\sum_{s=0}^{\infty} (\beta\lambda e^{a+n+\psi(\mu-a-n)})^s e^{\psi(G_{t+s}-G_t)} \hat{P}_{t+s}{}^{\psi} \hat{Y}_{t+s}\right\}}$$

$$\hat{P}_t^* = \frac{\psi}{1-\psi} \frac{E\left\{\sum_{s=0}^{\infty} (\beta\lambda e^{(1+\psi)\mu-\psi(a+n)})^s e^{(1+\psi)(G_{t+s}-G_t)} \hat{P}_{t+s}{}^{1+\psi} \hat{Y}_{t+s} \left[e^{(\alpha-1)z_{t+s}} r_{t+s}{}^{\alpha} \hat{w}_{t+s}{}^{1-\alpha} \alpha^{-\alpha} (1-\alpha)^{\alpha-1}\right]\right\}}{E\left\{\sum_{s=0}^{\infty} (\beta\lambda e^{(1-\psi)(a+n)+\psi\mu})^s e^{\psi(G_{t+s}-G_t)} \hat{P}_{t+s}{}^{\psi} \hat{Y}_{t+s}\right\}}$$

Now transform sums in equation above into state variables.

Define $S_D = \sum_{s=0}^{\infty} (\beta\lambda e^{(1-\psi)(a+n)+\psi\mu})^s e^{\psi(G_{t+s}-G_t)} \hat{P}_{t+s}{}^{\psi} \hat{Y}_{t+s}$ and

$$S_N = \sum_{s=0}^{\infty} (\beta\lambda e^{(1+\psi)\mu-\psi(a+n)})^s e^{(1+\psi)(G_{t+s}-G_t)} \hat{P}_{t+s}{}^{1+\psi} \hat{Y}_{t+s} \left[e^{(\alpha-1)z_{t+s}} r_{t+s}{}^{\alpha} \hat{w}_{t+s}{}^{1-\alpha} \alpha^{-\alpha} (1-\alpha)^{\alpha-1}\right]$$

Rewrite these as:

$S_D' = \dfrac{S_D - \hat{P}^{\psi} \hat{Y}}{\beta\lambda e^{(1-\psi)(a+n)+\psi\mu} e^{\psi g}}$ and

$$S_N' = \frac{S_N - \hat{P}^{1+\psi} \hat{Y} \left[e^{(\alpha-1)z} r^{\alpha} \hat{w}^{1-\alpha} \alpha^{-\alpha} (1-\alpha)^{\alpha-1}\right]}{\beta\lambda e^{(1+\psi)\mu-\psi(a+n)} e^{(1+\psi)g}}$$

$$g' = \rho_g g + \varepsilon_g'; \ \varepsilon_g' \sim N(0, \sigma_g^2) \tag{4.1}$$

$$z' = \rho_z z + \varepsilon_z'; \quad \varepsilon_z' \sim N(0, \sigma_z^2) \tag{4.2}$$

$$\hat{c} = \tfrac{e^{\mu+g}\hat{M}}{\hat{P}} \tag{4.3}$$

$$1 = \beta E\{e^{-\gamma a} \tfrac{\hat{w}}{\hat{w}'}(1-\delta+r')\} \tag{4.4}$$

$$-\widetilde{D} = \beta E\{(\hat{c}' e^{a})^{-\gamma} \tfrac{\hat{w}\hat{P}}{\hat{P}' e^{\mu-a-n+g'}}\} \tag{4.5}$$

$$[\hat{P}' e^{\mu-a-n+g'}]^{1-\psi} = \lambda \hat{P}^{1-\psi} + (1-\lambda)[\hat{P}^{*}{}' e^{\mu-a-n+g'}]^{1-\psi} \tag{4.6}$$

$$\hat{P}^{*} = \frac{\psi}{1-\psi}\frac{E\{S_N\}}{E\{S_D\}} \tag{4.7}$$

$$S_D' = \frac{S_D - \hat{P}^{\psi}\hat{Y}}{\beta\lambda e^{(1-\psi)(a+n)+\psi(\mu+g)}} \tag{4.8}$$

$$S_N' = \frac{S_N - \hat{P}^{1+\psi}\hat{Y}\left[e^{(\alpha-1)z} r^{\alpha} \hat{w}^{1-\alpha} \alpha^{-\alpha} (1-\alpha)^{\alpha-1}\right]}{\beta\lambda e^{(1+\psi)(\mu+g)-\psi(a+n)}} \tag{4.9}$$

$$\hat{Y} = e^{(1-\alpha)z}\hat{K}^{\alpha} h^{1-\alpha} \tag{4.10}$$

$$r\hat{K} = \alpha\hat{Y} \tag{4.11}$$

$$\hat{w}h = (1-\alpha)\hat{Y} \tag{4.12}$$

$$\hat{I} = e^{a+n}\hat{K}' - (1-\delta)K \tag{4.13}$$

13 equations in 13 unknowns:
$\hat{c}, \hat{K}, h, \hat{Y}, \hat{w}, r, \hat{P}, \hat{P}^{*}, \hat{I}, S_N, S_D, g \ \& \ z$.

Steady State

$\bar{g} = 0$ (5.1)

$\bar{z} = 0$ (5.2)

$\overline{M}$ is an arbitrary constant pinned down by its initial value of 1. (5.3)

$\bar{c} = \frac{(1+\mu)\overline{M}}{\overline{P}}$ (5.4)

$\bar{h} = \frac{(a+n-r+\delta)}{\overline{w}} + \frac{\overline{M}(1+\mu)}{\overline{w}\overline{P}}$ (5.5)

$1 = \beta e^{-\gamma a}(1-\delta+\bar{r})$ (5.6)

$-\widetilde{D} = \beta \bar{c}^{-\gamma} \frac{\overline{w}}{(1+\mu-a-n)}$ (5.7)

$[\overline{P}(1+\mu-a-n)]^{1-\psi} = \lambda \overline{P}^{1-\psi} + (1-\lambda)[\overline{P}^*(1+\mu-a-n)]^{1-\psi}$ (5.9)

$\overline{Y}(i) = \overline{K}(i)^{\alpha}\,\overline{H}(i)^{1-\alpha}$ (5.10)

$\overline{P}^* = \frac{\psi}{1-\psi}\frac{\overline{Y}(i)\overline{P}[\frac{\bar{r}(1-\alpha}{\overline{w}\alpha}]^{\alpha}\frac{\overline{w}}{1-\alpha}}{\overline{Y}(i)}$ (5.11)

$\overline{K}(i) = \overline{Y}(i)[\frac{\bar{r}(1-\alpha)}{\overline{w}\alpha}]^{\alpha-1}$ (5.12)

$\overline{H}(i) = \overline{Y}(i)[\frac{\bar{r}(1-\alpha)}{\overline{w}\alpha}]^{\alpha}$ (5.13)

$\overline{K} = \int_0^1 \overline{K}(i)\,di$ (5.14)

$\bar{h} = \int_0^1 \overline{H}(i)\,di$ (5.15)

$\bar{I} = (a+n+\delta)\overline{K}$ (5.16)

$\overline{A} = 1$ (5.17)

Manipulating (5.9) gives:

$$\overline{P}^{1-\psi}(1+\mu-a-n)^{1-\psi} = \lambda \overline{P}^{1-\psi}\frac{(1+\mu-a-n)^{1-\psi}}{(1+\mu-a-n)^{1-\psi}} + (1-\lambda)\overline{P}^{*1-\psi}(1+\mu-a-n)^{1-\psi}$$

$$(1-\lambda\frac{1}{(1+\mu-a-n)^{1-\psi}})\overline{P}^{1-\psi}(1+\mu-a-n)^{1-\psi} = (1-\lambda)\overline{P}^{*1-\psi}(1+\mu-a-n)^{1-\psi}$$

$$(1-\lambda\frac{1}{(1+\mu-a-n)^{1-\psi}})\overline{P}^{1-\psi} = (1-\lambda)\overline{P}^{*1-\psi}$$

$$(\frac{(1+\mu-a-n)^{1-\psi}-\lambda}{(1+\mu-a-n)^{1-\psi}})\overline{P}^{1-\psi} = (1-\lambda)\overline{P}^{*1-\psi}$$

$$\overline{P}^{*1-\psi} = \frac{(1+\mu-a-n)^{1-\psi}-\lambda}{(1+\mu-a-n)^{1-\psi}(1-\lambda)}$$

$\overline{P}^* = \Phi\overline{P};\ \ \Phi \equiv \left(\frac{(1+\mu-a-n)^{1-\psi}-\lambda}{(1+\mu-a-n)^{1-\psi}(1-\lambda)}\right)^{\frac{1}{1-\psi}}$ (5.9')

Putting this into the SS version of (1.7) gives:

$\overline{Y}(i) = \overline{Y}$

(5.12) - (5.15) then give:

$\overline{K} = \overline{K}(i)$ (5.14')

$\bar{h} = \overline{H}(i)$ (5.15')

$\overline{K} = \overline{Y}[\frac{\bar{r}(1-\alpha)}{\overline{w}\alpha}]^{\alpha-1}$ (5.12')

$$\overline{h} = \overline{Y}[\tfrac{\bar{r}(1-\alpha)}{\overline{w}\alpha}]^{\alpha} \tag{5.13'}$$

(5.10) gives:

$$\overline{Y} = \overline{K}^{\alpha}\overline{h}^{1-\alpha} \tag{5.10'}$$

The SS is found by noting that (5.6) implies:

$$\bar{r} = \tfrac{1}{\beta} e^{\gamma a} + \delta - 1 \tag{5.6''}$$

Using (5.11):

$$\bar{w} = [\tfrac{1-\psi}{\psi} \bar{r}^{-\alpha} (1-\alpha)^{1-\alpha} \alpha^{\alpha}]^{\frac{1}{1-\alpha}} \tag{5.11''}$$

(5.7) gives:

$$\bar{c} = [\beta \tfrac{\bar{w}}{-\widetilde{D}(1+\mu-a-n)}]^{\frac{1}{\gamma}} \tag{5.7''}$$

$\overline{M}$ is an arbitrary constant pinned down by its initial value.

(5.4) gives:

$$\overline{P} = \tfrac{(1+\mu)\overline{M}}{\bar{c}} \tag{5.4''}$$

Then (5.9') gives:

$$\overline{P}^{*} = \Phi \overline{P} \tag{5.9''}$$

Use (5.5) as is with (5.15'):

$$\bar{h} = \overline{H}(i) = \tfrac{(a+n-r+\delta)}{\bar{w}} + \tfrac{\overline{M}(1+\mu)}{\bar{w}\overline{P}} \tag{5.5}$$

(5.12') & (5.13') together with (5.14') give:

$$\overline{K} = \overline{K}(i) = \tfrac{\bar{w}\alpha}{\bar{r}(1-\alpha)} \bar{h} \tag{5.12''}$$

(5.10') gives:

$$\overline{Y} = \overline{Y}(i) = \overline{K}^{\alpha} \bar{h}^{1-\alpha} \tag{5.10'}$$

Finally, use (5.16) as is.

$$\bar{I} = (a+n+\delta)\overline{K} \tag{5.16}$$

Log-Linearization Approach

The equations (2.1) – (2.7) & (2.9) – (2.17) need to be log linearized.
(2.1) & (2.2) are already log-linearized.

Examine (2.9):

$$(\overline{P}e^{\widetilde{P}'}e^{\mu-a-n})^{1-\psi} = \lambda(\overline{P}e^{\widetilde{P}})^{1-\psi} + (1-\lambda)(\overline{P}^{*}e^{\widetilde{P}^{*}{}'}e^{\mu-a-n})^{1-\psi}$$

Rearranging:

$$(\overline{P}e^{\mu-a-n})^{1-\psi}e^{\widetilde{P}'(1-\psi)} = \lambda\overline{P}^{1-\psi}e^{\widetilde{P}(1-\psi)} + (1-\lambda)(\overline{P}^{*}e^{\mu-a-n})^{1-\psi}e^{\widetilde{P}^{*}{}'(1-\psi)}$$

$$(\overline{P}e^{\mu-a-n})^{1-\psi}[1+\widetilde{P}'(1-\psi)] = \lambda\overline{P}^{1-\psi}[1+\widetilde{P}(1-\psi)] + (1-\lambda)(\overline{P}^{*}e^{\mu-a-n})^{1-\psi}[1+\widetilde{P}^{*}{}'(1-\psi)]$$

$$(\overline{P}e^{\mu-a-n})^{1-\psi} + (\overline{P}e^{\mu-a-n})^{1-\psi}\widetilde{P}'(1-\psi) =$$

$$\lambda\overline{P}^{1-\psi} + \lambda\overline{P}^{1-\psi}\widetilde{P}(1-\psi) + (1-\lambda)(\overline{P}^{*}e^{\mu-a-n})^{1-\psi} + (1-\lambda)(\overline{P}^{*}e^{\mu-a-n})^{1-\psi}\widetilde{P}^{*}{}'(1-\psi)$$

Using (5.9) & (5.9'):

$$(\overline{P}e^{\mu-a-n})^{1-\psi}\widetilde{P}'(1-\psi) = \lambda\overline{P}^{1-\psi}\widetilde{P}(1-\psi) + (1-\lambda)(\Phi\overline{P}e^{\mu-a-n})^{1-\psi}\widetilde{P}^{*}{}'(1-\psi)$$

$$\widetilde{P}' = \tfrac{\lambda}{e^{(\mu-a-n)(1-\psi)}}\widetilde{P} + (1-\lambda)\Phi^{1-\psi}\widetilde{P}^{*}{}'$$

$$\widetilde{P}' = \tfrac{\lambda}{e^{(\mu-a-n)(1-\psi)}}\widetilde{P} + (1-\lambda)\tfrac{e^{(\mu-a-n)(1-\psi)}-\lambda}{e^{(\mu-a-n)(1-\psi)}(1-\lambda)}\widetilde{P}^{*}{}'$$

$$\widetilde{P}' = \tfrac{\lambda}{e^{(\mu-a-n)(1-\psi)}}\widetilde{P} + \left(\tfrac{e^{(\mu-a-n)(1-\psi)}-\lambda}{e^{(\mu-a-n)(1-\psi)}}\right)\widetilde{P}^{*}{}' \qquad (6.9)$$

The most challenging equation is (2.11):

$$\hat{P}_t^{*}E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}\hat{Y}_{t+s}(i)\right\} = \frac{\psi}{1-\psi}E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}\hat{Y}_{t+s}(i)\hat{P}_{t+s}\left[e^{(\alpha-1)z_{t+s}}\left[\tfrac{r_{t+s}(1-\alpha)}{\hat{w}_{t+s}\alpha}\right]^{\alpha}\tfrac{\hat{w}_{t+s}}{1-\alpha}\right]\right\}$$

Linearize LHS first:

$$E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}\hat{P}_s^{*}\hat{Y}_{t+s}(i)\right\} = E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}\overline{P}\,\overline{Y}e^{\widetilde{P}_t^{*}+\widetilde{Y}_{t+s}}\right\}$$

$$= \overline{P}\,\overline{Y}E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}(1+\widetilde{P}_t^{*}+\widetilde{Y}_{t+s})\right\}$$

$$= \overline{P}\,\overline{Y}(1+\widetilde{P}^{*})\frac{1}{1-\beta\lambda e^{\mu}} + \overline{P}\,\overline{Y}E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}\widetilde{Y}_{t+s}\right\} \qquad (6.11\ \text{LHS})$$

Next linearize RHS:

$$\frac{\psi}{1-\psi}E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}\overline{P}\,\overline{Y}e^{\widetilde{P}_{t+s}+\widetilde{Y}_{t+s}}e^{(\alpha-1)z_{t+s}}(\overline{r}e^{\widetilde{r}_{t+s}})^{\alpha}(\overline{w}e^{\widetilde{w}_{t+s}})^{1-\alpha}(1-\alpha)^{\alpha-1}\alpha^{\alpha}\right\}$$

$$\frac{\psi}{1-\psi}E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}\overline{P}\,\overline{Y}\overline{r}^{\alpha}\overline{w}^{1-\alpha}(1-\alpha)^{\alpha-1}\alpha^{\alpha}e^{\widetilde{P}_{t+s}+\widetilde{Y}_{t+s}+\alpha\widetilde{r}_{t+s}+(1-\alpha)\widetilde{w}_{t+s}-(1-\alpha)z_{t+s}}\right\}$$

$$\frac{\psi}{1-\psi}\overline{P}\,\overline{Y}\overline{r}^{\alpha}\overline{w}^{1-\alpha}(1-\alpha)^{\alpha-1}\alpha^{\alpha}E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}[1+\widetilde{P}_{t+s}+\widetilde{Y}_{t+s}+\alpha\widetilde{r}_{t+s}+(1-\alpha)\widetilde{w}_{t+s}-(1-\alpha)z_{t+s}]\right\}$$

Noting from (5.11) that $\frac{\psi}{1-\psi}\overline{r}^{\alpha}\overline{w}^{1-\alpha}(1-\alpha)^{\alpha-1}\alpha^{\alpha} = 1$

$$\overline{P}\,\overline{Y}E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}[1+\widetilde{P}_{t+s}+\widetilde{Y}_{t+s}+\alpha\widetilde{r}_{t+s}+(1-\alpha)\widetilde{w}_{t+s}-(1-\alpha)z_{t+s}]\right\} \qquad (6.11\ \text{RHS})$$

Combining the two sides again:

$$(1+\widetilde{P}_t^{*})\frac{1}{1-\beta\lambda e^{\mu}}+E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}\widetilde{Y}_{t+s}\right\}$$
$$=E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}[1+\widetilde{P}_{t+s}+\widetilde{Y}_{t+s}+\alpha\widetilde{r}_{t+s}+(1-\alpha)\widetilde{w}_{t+s}-(1-\alpha)z_{t+s}]\right\}$$

$$\widetilde{P}_t^{*}=(1-\beta\lambda e^{\mu})E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}[\widetilde{P}_{t+s}+\alpha\widetilde{r}_{t+s}+(1-\alpha)\widetilde{w}_{t+s}-(1-\alpha)z_{t+s}]\right\} \qquad (6.11)$$

Substitute this into lagged version of (6.9) and add current period subscripts back to notation:

$$\widetilde{P}_t=\tfrac{\lambda}{e^{(\mu-a-n)(1-\psi)}}\widetilde{P}_{t-1}+(\tfrac{e^{(\mu-a-n)(1-\psi)}-\lambda}{e^{(\mu-a-n)(1-\psi)}})(1-\beta\lambda e^{\mu})E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}[\widetilde{P}_{t+s}+\alpha\widetilde{r}_{t+s}+(1-\alpha)\widetilde{w}_{t+s}-(1-\alpha)z_{t+s}]\right\}$$

Quasi-difference by $1-\beta\lambda e^{\mu}L^{-1}$:

$$\widetilde{P}_t-\beta\lambda e^{\mu}\widetilde{P}_{t+1}=\tfrac{\lambda}{e^{(\mu-a-n)(1-\psi)}}\widetilde{P}_{t-1}-\tfrac{\beta\lambda^{2}e^{\mu}}{e^{(\mu-a-n)(1-\psi)}}\widetilde{P}_t$$
$$+\tfrac{e^{(\mu-a-n)(1-\psi)}-\lambda}{e^{(\mu-a-n)(1-\psi)}}(1-\beta\lambda e^{\mu})E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}[\widetilde{P}_t+\alpha\widetilde{r}_{t+s}+(1-\alpha)\widetilde{w}_{t+s}-(1-\alpha)z_{t+s}]\right\}$$
$$-\tfrac{e^{(\mu-a-n)(1-\psi)}-\lambda}{e^{(\mu-a-n)(1-\psi)}}(1-\beta\lambda e^{\mu})E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}[\widetilde{P}_{t+1}+\alpha\widetilde{r}_{t+s+1}+(1-\alpha)\widetilde{w}_{t+s+1}-(1-\alpha)z_{t+s+1}]\right\}$$

Cancelling:

$$\widetilde{P}_t-\beta\lambda e^{\mu}\widetilde{P}_{t+1}=\tfrac{\lambda}{e^{(\mu-a-n)(1-\psi)}}\widetilde{P}_{t-1}-\tfrac{\beta\lambda^{2}e^{\mu}}{e^{(\mu-a-n)(1-\psi)}}\widetilde{P}_t+\tfrac{e^{(\mu-a-n)(1-\psi)}-\lambda}{e^{(\mu-a-n)(1-\psi)}}(1-\beta\lambda e^{\mu})[\widetilde{P}_t+\alpha\widetilde{r}_t+(1-\alpha)\widetilde{w}_t-(1-\alpha)z_t]$$

Collecting terms:

$$-\beta\lambda e^{\mu}\widetilde{P}_{t+1}+[1+\tfrac{\beta\lambda^{2}e^{\mu}}{e^{(\mu-a-n)(1-\psi)}}-\tfrac{(e^{(\mu-a-n)(1-\psi)}-\lambda)(1-\beta\lambda e^{\mu})}{e^{(\mu-a-n)(1-\psi)}}]\widetilde{P}_t-[\tfrac{\lambda}{e^{(\mu-a-n)(1-\psi)}}]\widetilde{P}_{t-1}$$
$$=\tfrac{(e^{(\mu-a-n)(1-\psi)}-\lambda)(1-\beta\lambda e^{\mu})}{e^{(\mu-a-n)(1-\psi)}}[\alpha\widetilde{r}_t+(1-\alpha)\widetilde{w}_t-(1-\alpha)z_t]$$

Defining $RMC_t\equiv\alpha\widetilde{r}_t+(1-\alpha)\widetilde{w}_t-(1-\alpha)z_t$

$$-\beta\lambda e^{\mu}\widetilde{P}_{t+1}+[1+\tfrac{\beta\lambda^{2}e^{\mu}}{e^{(\mu-a-n)(1-\psi)}}-\tfrac{e^{(\mu-a-n)(1-\psi)}-\lambda}{e^{(\mu-a-n)(1-\psi)}}]\widetilde{P}_t-[\tfrac{\lambda}{e^{(\mu-a-n)(1-\psi)}}]\widetilde{P}_{t-1}=\tfrac{(e^{(\mu-a-n)(1-\psi)}-\lambda)(1-\beta\lambda e^{\mu})}{e^{(\mu-a-n)(1-\psi)}}RMC_t$$

Working on the coefficient on $\widetilde{P}_t$:

$$\tfrac{e^{(\mu-a-n)(1-\psi)}}{e^{(\mu-a-n)(1-\psi)}}+\tfrac{\beta\lambda^{2}e^{\mu}}{e^{(\mu-a-n)(1-\psi)}}-\tfrac{(e^{(\mu-a-n)(1-\psi)}-\lambda)(1-\beta\lambda e^{\mu})}{e^{(\mu-a-n)(1-\psi)}}$$
$$\tfrac{e^{(\mu-a-n)(1-\psi)}}{e^{(\mu-a-n)(1-\psi)}}+\tfrac{\beta\lambda^{2}e^{\mu}}{e^{(\mu-a-n)(1-\psi)}}-\tfrac{(e^{(\mu-a-n)(1-\psi)}-\lambda-\beta\lambda e^{\mu}e^{(\mu-a-n)(1-\psi)}+\beta\lambda^{2}e^{\mu})}{e^{(\mu-a-n)(1-\psi)}}$$
$$\tfrac{\lambda+\beta\lambda e^{\mu}e^{(\mu-a-n)(1-\psi)})}{e^{(\mu-a-n)(1-\psi)}}$$

$$-\beta\lambda e^{\mu}\widetilde{P}_{t+1}+[\tfrac{\lambda+\beta\lambda e^{\mu}e^{(\mu-a-n)(1-\psi)})}{e^{(\mu-a-n)(1-\psi)}}]\widetilde{P}_t-[\tfrac{\lambda}{e^{(\mu-a-n)(1-\psi)}}]\widetilde{P}_{t-1}=\tfrac{(e^{(\mu-a-n)(1-\psi)}-\lambda)(1-\beta\lambda e^{\mu})}{e^{(\mu-a-n)(1-\psi)}}RMC_t$$

$$\tfrac{\lambda}{e^{(\mu-a-n)(1-\psi)}}(\widetilde{P}_t-\widetilde{P}_{t-1})=\beta\lambda e^{\mu}(\widetilde{P}_{t+1}-\widetilde{P}_t)-\tfrac{(e^{(\mu-a-n)(1-\psi)}-\lambda)(1-\beta\lambda e^{\mu})}{e^{(\mu-a-n)(1-\psi)}}RMC_t$$

$$(\widetilde{P}_t - \widetilde{P}_{t-1}) = e^{(\mu-a-n)(1-\psi)}\beta e^{\mu}(\widetilde{P}_{t+1} - \widetilde{P}_t) - \tfrac{1}{\lambda}(e^{(\mu-a-n)(1-\psi)} - \lambda)(1-\beta\lambda e^{\mu})RMC_t \tag{6.9'}$$

Or defining inflation as $\widetilde{\pi}_t \equiv \widetilde{P}_t - \widetilde{P}_{t-1}$

$$\widetilde{\pi}_t = \beta e^{(\mu-a-n)(1-\psi)+\mu}\widetilde{\pi}_{t+1} - \tfrac{1}{\lambda}(e^{(\mu-a-n)(1-\psi)} - \lambda)(1-\beta\lambda e^{\mu})[\alpha\widetilde{r}_t + (1-\alpha)\widetilde{w}_t - (1-\alpha)z_t]$$

$$\widetilde{M}_{t+1} = g_t + \widetilde{M}_t \tag{6.3}$$

$$\widetilde{c}_t = g + \widetilde{M}_t - \widetilde{P}_t \tag{6.4}$$

$$\overline{h}\widetilde{h}_t = \tfrac{\overline{K}e^{a+n}}{\overline{w}}(\widetilde{K}_{t+1} - \widetilde{w}_t) + \tfrac{\overline{M}e^{\mu}}{\overline{w}\overline{P}}(\widetilde{M}_{t+1} - \widetilde{w}_t - \widetilde{P}_t) + \tfrac{\overline{K}}{\overline{w}}\overline{r}\widetilde{r}_t \tag{6.5}$$

$$\widetilde{w}_t - \widetilde{w}_{t+1} + \overline{r}\widetilde{r}_{t+1} = 0 \tag{6.6}$$

$$-\gamma\widetilde{c}_{t+1} + \widetilde{w}_t + \widetilde{P}_t - \widetilde{P}_{t+1} = 0 \tag{6.7}$$

$$\widetilde{Y}_t = \int_0^1 \overline{Y}\widetilde{Y}_t(i)\, di \tag{6.8}$$

$$\widetilde{Y}_t(i) = \alpha\widetilde{K}_t(i) + (1-\alpha)[z_t + \widetilde{H}_t(i)] \tag{6.10}$$

$$\widetilde{K}_t(i) = \widetilde{Y}_t(i) + (1-\alpha)(\widetilde{w}_t - \widetilde{r}_t - z_t) \tag{6.12}$$

$$\widetilde{H}_t(i) = \widetilde{Y}_t(i) + \alpha(\widetilde{r}_t + z_t - \widetilde{w}_t) \tag{6.13}$$

$$\widetilde{K}_t = \int_0^1 \overline{K}\widetilde{K}_t(i)\, di \tag{6.14}$$

$$\widetilde{h}_t = \int_0^1 \overline{h}\widetilde{H}_t(i)\, di \tag{6.15}$$

$$\overline{II}\widetilde{II}_t = e^{a+n}\overline{K}\widetilde{K}_{t+1} - (1-\delta)\overline{K}\widetilde{K}_t \tag{6.16}$$

$$\widetilde{A}_t = z_t \tag{6.17}$$

We can remove $\widetilde{Y}_t(i), \widetilde{K}_t(i)$ & $\widetilde{H}_t(i)$ from the system as follows.

Linearizing stationary versions of (1.7) and (1.8):

$$\widetilde{Y}_t(i) = \psi\widetilde{P}_t - \psi\widetilde{P}_t(i) + \widetilde{Y}_t \tag{6.18}$$

$$\widetilde{P}_t = \int_0^1 \overline{P}\widetilde{P}_t(i)\, di \tag{6.19}$$

Using (6.12) with (6.18):

$$\widetilde{K}_t(i) = \psi\widetilde{P}_t - \psi\widetilde{P}_t(i) + \widetilde{Y}_t + (1-\alpha)(\widetilde{w}_t - \widetilde{r}_t - z_t)$$

Substituting this into (6.14):

$$\widetilde{K}_t = \int_0^1 \overline{K}\Big[\psi\widetilde{P}_t - \psi\widetilde{P}_t(i) + \widetilde{Y}_t + (1-\alpha)(\widetilde{w}_t - \widetilde{r}_t - z_t)\Big] di$$

$$\widetilde{K}_t = \overline{K}[\widetilde{Y}_t + (1-\alpha)(\widetilde{w}_t - \widetilde{r}_t - z_t)] \tag{6.14'}$$

Similarly:

$$\widetilde{h}_t = \widetilde{Y}_t + \alpha(\widetilde{r}_t + z_t - \widetilde{w}_t) \tag{6.15'}$$

Combining (6.8) & (6.10):

$$\widetilde{Y}_t = \int_0^1 \overline{Y}\{\alpha\widetilde{K}_t(i) + (1-\alpha)[z_t + \widetilde{H}_t(i)]\}\, di$$

$$\widetilde{Y}_t = \overline{Y}[\alpha\widetilde{K}_t + (1-\alpha)z_t + (1-\alpha)\widetilde{h}_t] \tag{6.8'}$$

Since we have used (6.8) which was our redundant equation by Walras Law, we need to drop (6.5) instead.

The system is defined by the 11 equations: (2.1), (2.2), (6.3), (6.4), (6.6), (6.7), (6.8'), (6.9'), (6.14'), (6.15') & (6.16).
The 11 variables are $z_t, g_t, \widetilde{Y}_t, \widetilde{K}_t, \widetilde{h}_t, \widetilde{r}_t, \widetilde{w}_t, \widetilde{P}_t, \widetilde{c}_t, \widetilde{M}_t, \widetilde{I}_t$.