Spencer Lyon

Physics 441: Assignment #5 - Electric Fields in Matter

Due on Monday, June 10, 2013

June 19, 2013

Problem 4.2

According to quantum mechanics, the electron could for a hydrogen atom in the ground state has a charge density

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a},$$

where q is the charge of the electron and a is the Bohr radius. Find the atomic polarizability of such an atom [Hint: First calculate the electric field for the electron could, $E_e(r)$; then expand the exponential assuming that $r >> a$]

I will use Gauss' Law to find an expression for E. Recall that Gauss' Law is $\int \mathbf{E} \cdot d\mathbf{a} = \frac{Q}{/ve_0}$. We need to find Q, which we can do by integrating the expression for charge density.

$$\begin{aligned} Q &= \int_0^r \rho d\tau \\ &= \frac{4\pi q}{\pi a^3} \int_0^r e^{-2r/a} r^2 dr \\ &= -\frac{q\left(-a^2 e^{2\frac{r}{a}} + a^2 + 2ar + 2r^2\right) e^{-2\frac{r}{a}}}{a^2} \end{aligned}$$

Now that we have Q we just need to find E from Gauss' law.

$$
\begin{aligned}
E &= \frac{1}{4\pi\varepsilon_0 r^2} Q \\
&= -\frac{q\left(-a^2 e^{2\frac{r}{a}} + a^2 + 2ar + 2r^2\right) e^{-2\frac{r}{a}}}{4\pi a^2 \epsilon_0 r^2} \\
&= \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \left[1 - e^{-2r/a}\left(1 + 2\frac{r}{a} + 2\left(\frac{d}{a}\right)^2\right)\right]
\end{aligned}
$$

We now need to expand the exponential term in E. I do this below

$$
e^{-2r/a} = -\frac{4}{3}\frac{r^3}{a^3} + 2\frac{r^2}{a^2} - 2\frac{r}{a} + 1 + \mathcal{O}\left(\frac{r^4}{a^4}\right)
$$

If we plug this into the solution for E, we get the following:

$$
\begin{aligned}
E &= \frac{q}{4\pi\varepsilon_0 r^2}\left[1 - e^{-2r/a}\left(1 + 2\frac{r}{a} + 2\left(\frac{r}{a}\right)^2\right)\right] \\
&= \frac{q}{4\pi\varepsilon_0 r^2}\left[1 - 1 - 2\frac{r}{a} - 2\frac{r^2}{a^2} + 2\frac{r}{a} + 4\frac{r^2}{a^2} + 4\frac{r^3}{a^3} - 2\frac{r^2}{a^2} - 4\frac{r^3}{a^3} - \frac{4}{3}\frac{r^3}{a^3}\right] \\
&= \frac{q}{4\pi\varepsilon_0 r^2}\left[\frac{4}{3}\frac{r^3}{a^3}\right] \\
&= \frac{1}{3\pi\varepsilon_0 a^3} qr \\
&= \alpha p
\end{aligned}
$$

where $\alpha = 3\pi\varepsilon_0 a^3$. □

Problem 4.5

In Figure 4.6, $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are (perfect) dipoles at a distance r apart. What is the torque on $\boldsymbol{p}_1$ due to $\boldsymbol{p}_2$? What is the torque on $\boldsymbol{p}_2$ due to $\boldsymbol{p}_1$? [In each case, I want the torque on the dipole about its own center]

For this problem we will use equation 3.103: $\boldsymbol{E}_{\text{dipole}} = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}})$ and equation 4.4: $\boldsymbol{N} = \boldsymbol{p} \times \boldsymbol{E}$
We can find the torque of $\boldsymbol{p}_1$ on $\boldsymbol{p}_2$ by finding $\boldsymbol{E}_1$, which is what we get when $\theta = \pi/2$ in equation 3.103 and plugging the result into equation 4.4

$$
\begin{aligned}
\boldsymbol{E}_1 &= \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}) \\
&= \frac{p_1}{4\pi\varepsilon_0 r^3}(2\cos\pi/2\hat{\boldsymbol{r}} + \sin\pi/2\hat{\boldsymbol{\theta}}) \\
&= \frac{p_1}{4\pi\varepsilon_0 r^3}\hat{\boldsymbol{\theta}} \\
\boldsymbol{N}_2 &= \boldsymbol{p}_2 \times \boldsymbol{E}_1 \\
&= p_2 E_1 \\
&= \frac{p_1 p_2}{4\pi\varepsilon_0 r^3}
\end{aligned}
$$

We now repeat the analysis above using $\theta = \pi$ for $\boldsymbol{p}_2$:

$$
\begin{aligned}
\boldsymbol{E}_2 &= \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}) \\
&= \frac{p_2}{4\pi\varepsilon_0 r^3}(2\cos\pi\hat{\boldsymbol{r}} + \sin\pi\hat{\boldsymbol{\theta}}) \\
&= \frac{p_2}{4\pi\varepsilon_0 r^3} - 2\hat{\boldsymbol{r}} \\
\boldsymbol{N}_1 &= \boldsymbol{p}_1 \times \boldsymbol{E}_2 \\
&= p_1 E_2 \\
&= \frac{2p_1 p_2}{4\pi\varepsilon_0 r^3}
\end{aligned}
$$

□

Problem 4.10

A sphere of radius R carries a polarization

$$\boldsymbol{P}(\boldsymbol{r}) = k\boldsymbol{r}$$

where k is a constant and $\boldsymbol{r}$ is the vector from the center.

1. Calculate the bound of charges σ_b and ρ_b
2. Find the field inside and outside the sphere

1.
   - σ_b is found using equation 4.11: $\sigma_b = \boldsymbol{P}\cdot\hat{\boldsymbol{n}} = k\hat{r}\cdot\hat{\boldsymbol{n}} = kR$
   - ρ_b is found using equation 4.12 (Note I use the expression for the gradient in spherical coordinates as found in the front cover of the book): $\rho_b = -\nabla\cdot\boldsymbol{P} = -\nabla\cdot k\boldsymbol{r} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 kr\right) = -\frac{1}{r^2}3kr^2 = -3k$
2.
   - For inside the sphere ($r < R$) we will use Gauss' law to find an expression for E in terms of ρ.

$$
\begin{aligned}
\oint \boldsymbol{E}\cdot d\boldsymbol{a} = Er\pi r^2 = \frac{1}{\varepsilon_0}Q = \frac{1}{\varepsilon_0}\frac{4}{3}\pi r^3\rho \\
\boldsymbol{E} = \frac{1}{3\varepsilon_0}\rho r\hat{\boldsymbol{r}}
\end{aligned}
$$

     We simply plug our ρ in to get:

$$\boldsymbol{E} = \frac{1}{3\varepsilon_0} - 3kr\boldsymbol{r} = -(kr/\varepsilon_0)\hat{\boldsymbol{r}}$$

   - Outside the sphere ($r > R$) we can treat it as if all the charge were at the center. This makes $Q = (kR)(4\pi R^2) + (-3k)(\frac{4}{3}\pi R^3 = 0$ so $\boldsymbol{E} = 0$. Gauss' law can help is verify this intuitively.

□

Problem 4.15

A thick spherical shell (inner radius a, outer radius b) is made of dielectric material with "frozen-in" polarization

$$\boldsymbol{P} = \frac{k}{r}\hat{\boldsymbol{r}}$$

where k is a constant and r is the distance from the center. (There is no free charge in this problem.) Find the electric field in all three regions by two different methods:

1. Locate all the bound charge, and use Gauss' law (Equation 2.13) to calculate the field it produces
2. Use equation 4.23 to find $\boldsymbol{D}$, and then get $\boldsymbol{E}$ from equation 4.21. [Notice that the second method is much faster and avoids any reference to bound charges.]

1. We start by finding σ_b and ρ_b like we did in the previous problem. $\sigma_b = \boldsymbol{P}\cdot\hat{\boldsymbol{n}} = \begin{cases} \boldsymbol{P}\cdot\hat{\boldsymbol{r}} = k/b & \text{r=b} \\ -\boldsymbol{P}\cdot\hat{\boldsymbol{r}} = -k/a & \text{r=a} \end{cases}$ and $\rho_b = -\nabla\cdot\boldsymbol{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2(k/r)\right) = -\frac{k}{r^2}$.

   We will now apply Gauss' law ($\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$) to three different regions

   (a) $r < a$. Here $Q = 0$ so $\boldsymbol{E} = 0$

   (b) $a < r < b$: Here we need to calculate $Q = \sigma_b A + \int \rho_b dv$:

   $$Q = \left(\frac{-k}{a}\right)(4\pi a^2) + \int_a^r \left(\frac{-k}{r^2}\right) 4\pi r^2 dr = -4\pi ka - 4\pi k(r-1) = -4\pi kr$$

   We plug this in to get that $\boldsymbol{E} = -(k/\varepsilon_0 r)\hat{\boldsymbol{r}}$

   (c) $r > b$: Here $Q = 0$ so $\boldsymbol{E} = 0$

2. Equation 4.23 says that $\oint D\cdot d\boldsymbol{a} = Q_{f_{enc}}$. In our case there are not free charges so $Q_{f_{enc}} = 0 \to \boldsymbol{D} = 0$. We now say that

   $$\boldsymbol{D} = \varepsilon_0\boldsymbol{E} + \boldsymbol{P} \to \boldsymbol{E} = (-1/\varepsilon_0)\boldsymbol{P}$$

   We get the same answer as before because inside a and outside b, $\boldsymbol{P} = 0$ and plugging $\boldsymbol{P}$ into our expression above yields:

   $$\boldsymbol{E} = (-1/\varepsilon_0)k/r\hat{\boldsymbol{r}} = -(k/\varepsilon_0 r)\hat{\boldsymbol{r}}$$

□

Problem 4.19

Suppose you have enough linear dielectric material, of dielectric constant ε_r, to half-fill a parallel-plat capacitor. By what fraction is the capacitance increased when you distribute the material as shown in figure 4.25(a)? How about figure 4.25(b)? For a given potential difference V between the plates, find $\boldsymbol{E}$, $\boldsymbol{D}$, and $\boldsymbol{P}$ in each region and the free and bound charge on all surfaces, for both cases.

We begin with equation 2.54 and say that $C = \frac{A\varepsilon_0}{d}$ for a parallel plate capacitor with no dielectric. We then want to apply equation 2.53 ($C = Q/V$) to find the relative capacitance in each configuration. We will do this one at at time below.

1. Configuration (a): We know that $E = \sigma/\varepsilon$ so for the part without dielectric we have $E = \sigma/\varepsilon_0$ and the part with the dielectric we have that $E = \sigma/\varepsilon_r$. I then integrate E to get an expression for the potential. (Note that I make the substitution $\sigma = \frac{Q}{\varepsilon_0^2 A}$)

Problem 4.19 continued on next page...

$$\begin{aligned}
V &= -\int \boldsymbol{E}\cdot d\boldsymbol{l} \\
&= -\left(\int_0^{d/2} \sigma/\varepsilon_0 d\boldsymbol{l} + \int_{d/2}^{d} \sigma/\varepsilon_0 d\boldsymbol{l}\right) \\
&= \frac{\sigma}{\varepsilon_0}\frac{d}{2} + \frac{\sigma}{\varepsilon_r}\frac{d}{2} \\
&= \frac{Q}{\varepsilon_0 A}\frac{d}{2}(1+\varepsilon_0/\varepsilon_r)
\end{aligned}$$

Now I use the formula $C = \frac{Q}{V}$ to solve for C_a

$$\begin{aligned}
C_a &= \frac{Q}{V} \\
&= \frac{Q}{}\frac{Q}{\varepsilon_0 A}\frac{d}{2}(1+\varepsilon_0/\varepsilon_r) \\
&= \varepsilon_0 A\frac{2}{d}\left(\frac{1}{1+1/\varepsilon_r}\right) \\
\frac{C_a}{C_0} &= \frac{\varepsilon_0 A\frac{2}{d}\left(\frac{1}{1+1/\varepsilon_r}\right)}{A\varepsilon_0/d} \\
&= \frac{2\varepsilon_r}{1+\varepsilon_r}
\end{aligned}$$

Problem 4.26

A spherical conductor of radius a, carries a charge Q. It is surrounded by linear dielectric material of susceptibility χ_e, out to radius b. Find the energy of this configuration.

This is a lot like example 4.5. That's nice. We will start there and display some results (note that both $\boldsymbol{D}$ and $\boldsymbol{E}$ are 0 when $r < a$).

$$\boldsymbol{D} = \frac{Q}{4\pi r^2}\hat{\boldsymbol{r}}, \quad \boldsymbol{E} = \begin{cases} \frac{Q}{4\pi\varepsilon r^2}\hat{\boldsymbol{r}} & \text{for } a < r < b \\ \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}} & \text{for } r > b \end{cases}$$

We can find the energy of the configuration by finding the work needed to hold it together. I do this below using the equation given on page 197. Also note that I use the expression for tau in spherical coordinates given in the front cover ($d\tau = r^2 \sin\theta dr d\theta d\phi$), but because our $\boldsymbol{D}$ and $\boldsymbol{E}$ only depend on r, I can only use that part and say that $d\tau = 4\pi r^2 dr$

Problem 4.26 continued on next page...

$$
\begin{aligned}
W &= \frac{1}{2}\int \boldsymbol{D}\cdot\boldsymbol{E}d\tau \\
&= \frac{1}{2}4\pi\left(\int_a^b\left(\frac{Q}{4\pi r^2}\right)\left(\frac{Q}{4\pi\varepsilon r^2}\right)r^2dr + \int_b^\infty\left(\frac{Q}{4\pi r^2}\right)\left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)r^2dr\right) \\
&= \frac{1}{2}\left(\frac{Q}{4\pi}\right)^2 4\pi\left(\int_a^b\frac{1}{r^2}dr + \int_b^\infty\frac{1}{r^2}dr\right) \\
&= \frac{Q^2}{8\pi}\left(\frac{1}{\varepsilon}\left(\frac{-1}{r}\right)\Big|_a^b + \frac{1}{\varepsilon_0}\left(\frac{-1}{r}\right)\Big|_b^\infty\right) \\
&= \frac{Q^2}{8\pi}\left(\frac{1}{\varepsilon}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{\varepsilon_0}\frac{1}{b}\right)
\end{aligned}
$$

□

Problem 4.30

An electric dipole $\boldsymbol{p}$, pointing in the y direction, is placed midway between two large conducting plates, as shown in figure 4.33. Each plate makes a small angle θ with respect to the x axis, and they are maintained at potentials $\pm V$. What is the direction of the net force on $\boldsymbol{p}$? (There's nothing to calculate, but explain your answer qualitatively.)

This one was a bit tricky, but thinking about field lines helps a lot. We know that the field lines are always perpendicular to the surface of a conductor, in this case our two plates. That being said, we would have to draw them bowed out to the $+\hat{\boldsymbol{x}}$ direction. We then need to decide which way things flow. This is easy, from positive to negative. If that is the case we are going down from the $+V$ plate (at a positive θ) to the $-V$ plate. Following the right hand rule around those field lines points us to the $+\hat{\boldsymbol{x}}$ direction, so that is the answer.

□

Problem 4.36

At the interface between one linear dielectric and another, the electric field lines bend(see figure 4.34). Show that

$$\tan_{\theta_2} / \tan_{\theta_1} = \varepsilon_2\varepsilon_1$$

assuming there is no free charge at the boundary.

We need to use 4 equations:

1. Equation 4.26: $D^\perp_{\text{above}} - D^\perp_{\text{below}} = \sigma_f$, in our case there are no free charges so $\sigma_f = 0$, which tells us that $D^\perp_{\text{above}} = D^\perp_{\text{below}} \to D_{y1} = D_{y2}$
2. Equation 4.28 $E^\parallel_{\text{above}} - E^\parallel_{\text{below}} = 0$, which means that $E^\parallel_{\text{above}} = E^\parallel_{\text{below}} \to E_{x1} = E_{x2}$
3. Equation 4.32: $\boldsymbol{D} = \varepsilon\boldsymbol{E}$, which with the above equations tells us that $\varepsilon_1 E_{y1} = \varepsilon_2 E_{y2}$
4. Just some geometry from Figure 4.34: $\tan\theta_1 = \frac{E_{x1}}{E_{y1}}$ and $\tan\theta_2 = \frac{E_{x2}}{E_{y2}}$

If we put all those things together we can get our solution

Problem 4.36 continued on next page. . .

$$\begin{aligned}
\frac{\tan\theta_2}{\tan\theta_1} &= \frac{\frac{E_{x2}}{E_{y2}}}{\frac{E_{x1}}{E_{y1}}} \\
&= \frac{E_{y1}}{E_{y2}} \\
&= \frac{\varepsilon_2}{\varepsilon_1}
\end{aligned}$$

□