A 4th Real Business Cycle Model

Major Features of the Model

Add population growth that follows a deterministic trend to model 3
One source of uncertainty: z
Stochastic technology growth about a deterministic trend
Labor-leisure decision with indivisible labor hours

We first define the important terms that will be used throughout:
Endogenous variables that change over time:

z	productivity (temporary or permanent)
K	capital stock owned by households
h	labor supplied by a single individual
H	total labor supplied
c	consumption by a single individual
w	wage rate
r	interest rate
Y	output of final goods
N	number of persons per household

Parameters:

α	capital share in output from a Cobb-Douglas production function
δ	rate of depreciation
β	time discount factor; β<1
a	trend in z
n	trend in N
γ	elasticity of substitution, γ >0
D	leisure weight in utility
ρ	autocorrelation parameter for z; 0<ρ<1
σ	standard deviations of the shocks to z; 0<σ
h_0	hours worked by household that have a job

Nonstationary Model

Households have increasing numbers of members, denoted N.
The law of motion for N is:

$$N' = e^{n} N \text{ or } N = e^{nt} N_0 \tag{1.1}$$

Given information on prices and shocks, $\mathbf{\Omega} = \{w, r, z\}$, the household solves the following non-linear program when the factor markets clear.

$$V(K, \mathbf{\Omega}) = \underset{K', m', h}{Max} \left[\tfrac{1}{1-\gamma}(c^{1-\gamma} - 1) + h e^{(1-\gamma)at} \widetilde{D} + \widetilde{F} e^{(1-\gamma)at} - \widetilde{F} \right] N + \beta E\{V(K', \mathbf{\Omega}')\}$$

$$\widetilde{D} \equiv \tfrac{1}{H_0} D[(1-h_0)^{1-\gamma} - 1] < 0, \quad \widetilde{F} \equiv D \tfrac{1}{1-\gamma}$$

$$c = wh + (1 - \delta + r)\tfrac{K}{N} - \tfrac{K'}{N} \tag{1.2}$$

The first-order conditions are:

$c^{-\gamma}(-\tfrac{1}{N})N + \beta E\{V_K(K',\Omega')\} = 0$

$c^{-\gamma}wN + e^{(1-\gamma)at}\widetilde{D}N = 0$

The envelope condition from this problem is as follows.

$V_K(K,\Omega) = c^{-\gamma}(1-\delta+r)\tfrac{1}{N}N$

The Euler equations are:

$$c^{-\gamma} = \beta E\{c'^{-\gamma}(1-\delta+r')\} \tag{1.3}$$

$$c^{-\gamma}w = -e^{(1-\gamma)at}\widetilde{D} \tag{1.4}$$

Additional Behavioral Equations

The law of motion for z is:

$z' = \rho z + \varepsilon'$; where ε' is distributed normal with a mean of 0 and a variance of σ^2 (1.5)

An assumption of competition in the goods market along with a Cobb-Douglas production function gives the following shares in output for labor & capital. Market clearing conditions have already been imposed.

$$Y = K^{\alpha}(e^{at+z}H)^{1-\alpha} \tag{1.6}$$

$$wH = (1-\alpha)Y \tag{1.7}$$

$$rK = \alpha Y \tag{1.8}$$

Aggregating over household members gives:

$$H = Nh \tag{1.9}$$

Definitions:

$$I \equiv K' - (1-\delta)K \tag{1.10}$$

$$A \equiv e^{at+z} \tag{1.11}$$

Eqs (1.1)-(1.11) are the system.

Transformation & Simplifications

Without loss of generalization set $\hat{N} = N_0 = 1$, and eliminate it from the system.
Use (1.11) to eliminate H from the system.

Transform the problem by dividing:
c, w, A by e^{at}
K, Y, I by $e^{(a+n)t}$

$$z' = \rho z + \varepsilon' \text{; where } \varepsilon' \text{ is distributed normal with a mean of 0 and a variance of } \sigma^2 \tag{2.1}$$
$$\hat{c} = \hat{w}h + (1-\delta + r)\hat{K} - \hat{K}'(1+a+n) \tag{2.2}$$
$$1 = \beta E\left\{\left(\tfrac{\hat{c}}{(1+a)\hat{c}'}\right)^{\gamma}(1-\delta + r')\right\} \tag{2.3}$$
$$\hat{c}^{-\gamma}\hat{w} = \widetilde{D} \tag{2.4}$$
$$\hat{Y} = \hat{K}^{\alpha}(e^{z}h)^{1-\alpha} \tag{2.5}$$
$$\hat{w}h = (1-\alpha)\hat{Y} \tag{2.6}$$
$$r\hat{K} = \alpha\hat{Y} \tag{2.7}$$
$$\hat{I} = (1+a+n)\hat{K}' - (1-\delta)K \tag{2.8}$$
$$\hat{A} \equiv e^{z} \tag{2.9}$$

These are the equations we will use in Dynare.
The endogenous variables are $\hat{c}, \hat{K}, h, \hat{Y}, \hat{w}, r, \hat{I}, \hat{A}$ & z.
The exogenous variable is ε.
The parameters are $\alpha, \delta, \beta, a, \gamma, \rho, \sigma, D$ & h_0.