Problem 1

Expand $(A+B)(A-B)$ and $(A-B)(A+B)$. Are these expansions the same? If not, why not?

These two matrix expressions can be expanded as follows:

$$(A+B)(A-B) = AA - AB + BA - BB$$
$$(A-B)(A+B) = AA + AB - BA - BB$$

As can be seen these expressions are not equal. This is because matrix multiplication is not commutative. $\square$

Problem 2

Given $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 4 & 1 \\ 0 & -1 & 5 \\ 0 & 2 & -2 \end{pmatrix}$, and $C = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$. Calculate $(AB)'$, $B'A'$, $C'A'$, and $(AC)'$.

- $AB = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 & 1 \\ 0 & -1 & 5 \\ 0 & 2 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 10 & 11 \\ -5 & -5 \end{pmatrix}$
- $B'A' = \begin{pmatrix} 3 & 0 & 0 \\ 4 & -1 & 2 \\ 1 & 5 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 10 & 11 \\ -5 & -5 \end{pmatrix}$
- $C'A' = \begin{pmatrix} 2 & -1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 9 \end{pmatrix}$
- $(AC)' = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 14 & 9 \end{pmatrix}$ $\square$

Problem 3

Prove that diagonal matrices of the same order are commutative in multiplication with each other.

Let A_n, B_n be nxn diagonal matrices. In this case the product $AB = \begin{pmatrix} A_{22}B_{11} & 0 & 0 & \dots & 0 \\ 0 & A_{22}B_{22} & 0 & \dots & 0 \\ 0 & 0 & A_{33}B_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & & \\ & & & \dots & A_{nn}B_{nn} \end{pmatrix}$. The product BA would be very similar: $BA = \begin{pmatrix} B_{22}A_{11} & 0 & 0 & \dots & 0 \\ 0 & B_{22}A_{22} & 0 & \dots & 0 \\ 0 & 0 & B_{33}A_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & & \\ & & & \dots & B_{nn}A_{nn} \end{pmatrix}$. These two are clearly equal because in each case there is a scalar on the diagonal. That scalar is the result of scalar multiplication between elements of A and B that are in the same position. Scalar multiplication is commutative so $AB = BA$ for diagonal matrices. $\square$

Problem 4

Prove that $\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & a_{nn} \end{vmatrix} = \prod_{i=1}^{n} a_{ii}$

Problem 4 continued on next page...

We will prove this by using the method of cofactor expansion to find the determinant. Let A_{ij} stand for the cofactor about the i, j element of the matrix A. We will start by taking the cofactor expansion along the first column. This results in the following

$$det(A) = a_{11} \begin{vmatrix} a_{22} & \dots & a_{2n} \\ \vdots & & \vdots \\ 0 & \dots & a_{nn} \end{vmatrix} + 0A_{21} + \dots + 0A_{n1} = a_{11}A_{11}$$

The next step would be to do a cofactor expainsion about the first column of A_{11}. By induction, we can see that this would be

$$det(A) = a_{11}\left(a_{22}A_{22} + 0A_{32} + \dots + 0A_{n2}\right)$$

If we follow this inductive argument we will arrive at the answer which is that

$$det(A) = a_{11}a_{22}\dots a_{nn} = \prod_{i=1}^{n} a_{iI}) \quad \square$$

Problem 5

Show that the matrix $Q = \begin{pmatrix} 1/\sqrt{6} & 2/\sqrt{5} & 1/\sqrt{30} \\ -2/\sqrt{6} & 1/\sqrt{5} & -2/\sqrt{30} \\ 1/\sqrt{6} & 0 & -5/\sqrt{30} \end{pmatrix}$ **is orthogonal.**

To do this we will simply show that $QQ' = I$

$$QQ' = \begin{pmatrix} 1/\sqrt{6} & 2/\sqrt{5} & 1/\sqrt{30} \\ -2/\sqrt{6} & 1/\sqrt{5} & -2/\sqrt{30} \\ 1/\sqrt{6} & 0 & -5/\sqrt{30} \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 1/\sqrt{30} & -2/\sqrt{30} & -5/\sqrt{30} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I \quad \square$$

Problem 6

Determine whether the following quadratic forms are positive definite:

- $6x_1^2 + 49x_2^2 + 51x_3^2 - 82x_2x_3 + 20x_1x_3 - 4x_1x_2$
- $4x_1^2 + 9x_2^2 + 2x_3^2 + 8x_2x_3 + 6x_1x_3 + 6x_1x_2$

In order to solve this problem I am assuming (as Dr. McDonald did in his notes) that the quadratic form Q is constructed using some symmetric matrix A and a non-zero vector x. If that is the case then we can use the expression for Q to back into the matrix A. For each case, we know that the diagonal entries are simply the coefficients on the squared terms. This means that for the first problem $a_{11} = 6, a_{22} = 49, a_{33} = 51$.
To find the off-diagonal entries we know that the coefficient in front of $x_i x_j$ must be $a_{ij} + a_{ji}$. Knowing this, and that A must by symmetric, we can fill in all remaining entries of the matrices A. I will write them below and note that A_1 is the matrix for the first quadratic form above and A_2 is the matrix for the second one.

$$A_1 = \begin{pmatrix} 6 & -2 & 10 \\ -2 & 49 & -41 \\ 10 & -41 & 51 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 4 & 3 & 3 \\ 3 & 9 & 4 \\ 3 & 4 & 2 \end{pmatrix}$$

The conditions for a symmetric matrix to be positive definite are

- $a_{11} > 0$
- $|A_{2x2}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$
- $|A_{3x3}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0$

I will take it one case at a time and start with A_1

- $a_{11} = 6 > 0$
- $A_{(1,2x2)} = (6)(49) - (-2)(-2) = 290 > 0$
- $A_{(1,3x3)} = (6)(49)(51) - (6)(-41)(-41) - (-2)(-2)(51) + (-2)(-41)(10) + (10)(-2)(-41) - (10)(49)(10) = 1444 > 0$

For A_2 we get

- $a_{11} = 4 > 0$
- $A_{(1,2x2)} = (4)(9) - (3)(3) = 27 > 0$
- $A_{(1,3x3)} = (4)(9)(2) - (4)(4)(4) - (3)(3)(2) + (3)(4)(3) + (3)(3)(4) - (3)(9)(3) = -19 < 0$

This means that A_1 is positive definite, but A_2 is not. □

Problem 7

Prove that A = $\begin{pmatrix} 1/2 & 1 \\ 1/4 & 1/2 \end{pmatrix}$ is a nonsymmetric, idempotent matrix.

It is clear that A is non symmetric. For a matrix to be idempotent $AA = A$. I will show that below

$$AA = \begin{pmatrix} 1/2 & 1 \\ 1/4 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 1 \\ 1/4 & 1/2 \end{pmatrix} = \begin{pmatrix} (1/2)(1/2) + (1)(1/4) & (1)(1/2) + (1/2)(1) \\ (1/2)(1/2) + (1/4)(1/2) & (1/4)(1) + (1/2)(1/2) \end{pmatrix} = \begin{pmatrix} 1/2 & 1 \\ 1/4 & 1/2 \end{pmatrix} \tag{1}$$

Problem 8

Let X denote an NxK ($N > K$) matrix. Demonstrate that $B = I_N - X(X'X)^{-1}X'$ is symmetric and idempotent.

We will use some rules of the transpose to get this. First I define another matrix $P \equiv X(X'X)^{-1}X'$ and then I will proceed to show symmetry.

$$P' = (X(X'X)^{-1}X')' = X''(X'X)^{-1'}X' = X(X'X'')^{-1}X' = X(X'X)^{-1}X' = P$$

$$B' = (I-P)' = I' - P' = I - P = B$$

Now to show that B is Idempotent I will do something similar.

$$PP = X(X'X)^{-1}X'X(X'X)^{-1}X' = XI(X'X)^{-1}X' = X(X'X)^{-1}X' = P$$

$$BB = (I-P)(I-P) = II - 2IP + PP = I - 2P + P = I - P = B$$

□

Problem 9

Obtain the characteristics roots of $C = \left(\begin{smallmatrix} 5/2 & 1/2 \\ 1/2 & 5/2 \end{smallmatrix}\right)$. Can you determine the sign of $(x_1, x_2)C(x_1, x_2)'$ for an arbitrary $X = (x_1, x_2) \neq 0$?

The characteristic roots of a matrix are found by solving the equation $|C - \lambda I| = 0$. For our matrix this becomes

$$0 = |C - \lambda I| = \begin{vmatrix} 5/2 - \lambda & 1/2 \\ 1/2 & 5/2 - \lambda \end{vmatrix} = (5/2 - \lambda)(5/2 - \lambda) - (1/2)(1/2) = \lambda^2 - 5\lambda - 6 = 0$$

The roots of the characteristic equation are $\lambda_1 = 2, \lambda_2 = 3$.
Because both roots are positive we can say that $XCX' > 0 \quad \forall X \neq 0$.

□

Problem 10

Using the technique of inverses of partitioned matrices, determine the inverse of $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 5 & 2 \\ \hline 0 & 2 & 1 \end{array}\right] = \begin{bmatrix} D_{11} & D_{12} \\ D_{21}, & D_{22} \end{bmatrix}$

Problem 10 continued on next page...

To solve this we will need the inverse of D_{11} and $_D 22$. The inverse of a diagonal matrix is simply the inverse of each scalar element along the diagonal so $D_{11}^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/5 \end{smallmatrix}\right)$ and $D_{22}^{-1} = 1$.
We will also define $B \equiv D^{-1} = \left(\begin{smallmatrix} B_{11} & B_{12} \\ B_{21} & B22 \end{smallmatrix}\right)$.
We are now ready to use the expressions on pages 5-6 of the notes

- $B_{11} = (D_{11} - D_{12} D_{22}^{-1} D_{21})^{-1} = (\left(\begin{smallmatrix} 1 & 0 \\ 0 & 5 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right)(1)\left(\begin{smallmatrix} 0 & 2 \end{smallmatrix}\right))^{-1} = (\left(\begin{smallmatrix} 1 & 0 \\ 0 & 5 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 0 & 0 \\ 0 & 4 \end{smallmatrix}\right))^{-1} = (\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right))^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$
- $B_{12} = -B_{11} D_{12} D_{22}^{-1} = -\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right)(1) = \left(\begin{smallmatrix} 0 \\ -2 \end{smallmatrix}\right)$
- $B_{21} = -D_{22}^{-1} D_{21} B_{11} = -(1)\left(\begin{smallmatrix} 0 & 2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -2 \end{smallmatrix}\right)$
- $B_{22} = D_{22}^{-1} + D_{22}^{-1} D_{21} B_{11}^{-1} D_{12} D_{22} = (1) + (1)\left(\begin{smallmatrix} 0 & 2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right)(1) = 1 + 4 = 5$

We are now ready to define D^{-1}

$$D^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & -2 \\ \hline 0 & -2 & 5 \end{array}\right] \quad \square$$

Problem 11

Let $A = \left(\begin{smallmatrix} 5 & 2 \\ 2 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 3 & 5 \\ 1 & 2 \end{smallmatrix}\right)$ Find

- $A \otimes B$
- $(A \otimes B)^{-1}$
- $(A \otimes B)'$
- $Tr(A \otimes B)$
- $|A \otimes B|$

We will take them one at a time:

- $A \otimes B = \begin{pmatrix} 5\left(\begin{smallmatrix} 3 & 5 \\ 1 & 2 \end{smallmatrix}\right) & 2\left(\begin{smallmatrix} 3 & 5 \\ 1 & 2 \end{smallmatrix}\right) \\ 2\left(\begin{smallmatrix} 3 & 5 \\ 1 & 2 \end{smallmatrix}\right) & 1\left(\begin{smallmatrix} 3 & 5 \\ 1 & 2 \end{smallmatrix}\right) \end{pmatrix} = \left(\begin{smallmatrix} 15 & 25 & 6 & 10 \\ 5 & 10 & 2 & 4 \\ 6 & 10 & 3 & 5 \\ 2 & 4 & 1 & 2 \end{smallmatrix}\right)$
- $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1} = \left(\begin{smallmatrix} 1 & -2 \\ -2 & 5 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 2 & -5 \\ -1 & 3 \end{smallmatrix}\right) = \begin{pmatrix} 1\left(\begin{smallmatrix} 2 & -5 \\ -1 & 3 \end{smallmatrix}\right) & -2\left(\begin{smallmatrix} 2 & -5 \\ -1 & 3 \end{smallmatrix}\right) \\ -2\left(\begin{smallmatrix} 2 & -5 \\ -1 & 3 \end{smallmatrix}\right) & 5\left(\begin{smallmatrix} 2 & -5 \\ -1 & 3 \end{smallmatrix}\right) \end{pmatrix} = \left(\begin{smallmatrix} 2 & -5 & -4 & 10 \\ -1 & 3 & 2 & -6 \\ -4 & 10 & 10 & -25 \\ 2 & -6 & -5 & 15 \end{smallmatrix}\right)$
- $(A \otimes B)' = \left(\begin{smallmatrix} 5 & 2 \\ 2 & 1 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 3 & 1 \\ 5 & 2 \end{smallmatrix}\right) = \begin{pmatrix} 5\left(\begin{smallmatrix} 3 & 1 \\ 5 & 2 \end{smallmatrix}\right) & 2\left(\begin{smallmatrix} 3 & 1 \\ 5 & 2 \end{smallmatrix}\right) \\ 2\left(\begin{smallmatrix} 3 & 1 \\ 5 & 2 \end{smallmatrix}\right) & 1\left(\begin{smallmatrix} 3 & 1 \\ 5 & 2 \end{smallmatrix}\right) \end{pmatrix} = \left(\begin{smallmatrix} 15 & 5 & 6 & 2 \\ 25 & 10 & 10 & 4 \\ 6 & 2 & 3 & 1 \\ 10 & 4 & 5 & 2 \end{smallmatrix}\right)$
- $Tr(A \otimes B) = Tr(A)Tr(B) = (5+1)(3+2) = (6)(5) = 30$
- $|A \otimes B| = |A|^2 |B|^2 = (1)^2 (1)^2 = 1$

$\square$

Problem 12

Let A and B be square matrices. Prove that $Tr(AB) = Tr(BA)$

We know that the i,j element of the matrix product AB is $AB_{ij} = \sum_{k=1}^{n} a_{ik} b_{ki}$.
We also know that the trace of a matrix is the sum of the diagonal, or $Tr(a) = \sum_{i=1}^{n} a_{ii}$.
Using these two facts we can get the trace of a matrix product by $\sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki}$. We can use this to write an expression for $Tr(AB)$ and $Tr(BA)$:

$$Tr(AB) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki}$$
$$Tr(BA) = \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik} a_{ki}$$

Because scalar multiplication and addition are commutative, these two expressions are equal. □

Problem 16

Prove that $\frac{\partial trace(A)}{A} = I$

The following is an expression for the derivative of a scalar y with respect to a (pxq) matrix A

$$\frac{\partial y}{\partial A} = \begin{pmatrix} \frac{\partial y}{\partial A_{11}} & \frac{\partial y}{\partial A_{21}} & \cdots & \frac{\partial y}{\partial A_{p1}} \\ \frac{\partial y}{\partial A_{12}} & \frac{\partial y}{\partial A_{22}} & \cdots & \frac{\partial y}{\partial A_{p1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y}{\partial A_{1q}} & \frac{\partial y}{\partial A_{2q}} & \cdots & \frac{\partial y}{\partial A_{pq}} \end{pmatrix}$$

In our case $y = Tr(A) = \sum_{i=1}^{p} A_{ii}$. This means that the partial derivative of y with respect to any non-diagonal entry of A will be zero. It also means that the partial derivative of y with respect to any diagonal entry of A will be equal to 1. Combining these two facts with the expression above we get our answer:

$$\frac{\partial trace(A)}{A} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} = I \quad \square$$