A 5th Real Business Cycle Model

Major Features of the Model

Add money and a cash-in-advance constraint to model 4
Add stochastic money growth about a deterministic trend
Two sources of uncertainty: z and g
Stochastic technology growth about a deterministic trend
Labor-leisure decision with indivisible labor hours
Population growth follows a deterministic trend

We first define the important terms that will be used throughout:
Endogenous variables that change over time:

z	productivity
g	money growth
K	capital stock owned by households
h	labor supplied by a single individual
c	consumption by a single individual
w	wage rate
r	interest rate
Y	output of final goods
N	number of persons per household
m	money balances per household
M	aggregate money supply
P	price of good in terms of money

Parameters:

α	capital share in output from a Cobb-Douglas production function
δ	rate of depreciation
β	time discount factor; β<1
a	trend in z
μ	trend in g
n	trend in N
γ	elasticity of substitution, γ >0
D	leisure weight in utility
h_0	hours worked by household that have a job
ρ_i	autocorrelation parameter for i=z,g; 0<ρ_i <1
σ_z	standard deviations of the shocks to i=z,g; 0< σ_i

Nonstationary Model

Households have increasing numbers of members, denoted N.
The law of motion for N is:

$$N' = e^n N \text{ or } N = e^{nt} N_0 \tag{1.1}$$

Households face both a budget constraint and a cash-in-advance constraint. These are:

$c = \frac{m}{PN} + \frac{(\mu+g)M}{PN}$

$wh + (1-\delta+r)\frac{K}{N} + \frac{m}{PN} + \frac{gM}{PN} = c + \frac{K'}{N} + \frac{m'}{PN}$

Substituting the first into the second and solving for c & h gives the definitions in the problem below.

Given information on prices and shocks, $\mathbf{\Omega} = \{w, r, z, g\}$, the household solves the following non-linear program when the factor markets clear.

$$V(K, m, \mathbf{\Omega}) = \underset{K', m', h}{Max}\left[\tfrac{1}{1-\gamma}(c^{1-\gamma} - 1) + he^{(1-\gamma)at}\widetilde{D} + \widetilde{F}e^{(1-\gamma)at} - \widetilde{F}\right]N + \beta E\{V(K', m', \mathbf{\Omega}')\}$$

$$\widetilde{D} \equiv \tfrac{1}{H_0} D[(1-h_0)^{1-\gamma} - 1] < 0, \quad \widetilde{F} \equiv D\tfrac{1}{1-\gamma}$$

$$c = \tfrac{m}{PN} + \tfrac{(\mu+g)M}{PN} \tag{1.2}$$

$$h = \tfrac{K'}{wN} + \tfrac{m'}{wPN} - (1+r-\delta)\tfrac{K}{wN} \tag{1.3}$$

The first-order conditions are:

$$e^{(1-\gamma)at}\widetilde{D}(-\tfrac{1}{wN})N + \beta E\{V_K(K', \Omega')\} = 0$$

$$e^{(1-\gamma)at}\widetilde{D}(-\tfrac{1}{wPN})N + \beta E\{V_m(K', \Omega')\} = 0$$

The envelope conditions are:

$$V_K(K, \Omega) = e^{(1-\gamma)at}\widetilde{D}(-\tfrac{1}{wPN})(1-\delta+r)N$$

$$V_m(K, \Omega) = c^{-\gamma}\tfrac{1}{PN}N$$

The Euler equations are:

$$e^{(1-\gamma)at}\widetilde{D}\tfrac{1}{w} = \beta E\{e^{(1-\gamma)a(t+1)}\widetilde{D}(\tfrac{1}{w'})(1-\delta+r')\}$$

$$e^{(1-\gamma)at}\widetilde{D}(-\tfrac{1}{wP}) = \beta E\{c'^{-\gamma}\tfrac{1}{P'}\}$$

Simplifying:

$$1 = \beta E\{e^{(1-\gamma)a}\tfrac{w}{w'}(1-\delta+r')\} \tag{1.4}$$

$$-e^{(1-\gamma)at}\widetilde{D} = \beta E\{c'^{-\gamma}\tfrac{wP}{P'}\} \tag{1.5}$$

Additional Behavioral Equations

Money changes over time according to:

$$M' = e^{\mu+g}M = M_0\prod_{s=1}^{t+1} e^{\mu+g_s} \tag{1.6}$$

The law of motion for g is:

$g' = \rho_g g + \varepsilon_g'$; where ε_g' is distributed normal with a mean of 0 and a variance of σ_g^2 (1.7)

The law of motion for z is:

$z' = \rho_z z + \varepsilon_z'$; where ε_z' is distributed normal with a mean of 0 and a variance of σ_z^2 (1.8)

An assumption of competition in the goods market along with a Cobb-Douglas production function gives the following shares in output for labor & capital. Market clearing conditions have already been imposed.

$$Y = K^\alpha(e^{at+z}H)^{1-\alpha} \tag{1.9}$$

$$wH = (1-\alpha)Y \tag{1.10}$$

$$rK = \alpha Y \tag{1.11}$$

Aggregating over household members gives:

$$H = Nh \tag{1.12}$$

Money market clearing gives:

$$M = m \tag{1.13}$$

Definitions:

$$I \equiv K' - (1-\delta)K \tag{1.14}$$

$$A \equiv e^{at+z} \tag{1.15}$$

Eqs (1.1)-(1.15) are the system.

Transformation & Simplifications

Without loss of generalization set $\hat{N} = N_0 = 1$, and eliminate it from the system.
Use (1.12) to eliminate *H* from the system.
Use (1.13) to eliminate *m* from the system.

Transform the problem by dividing:
c, w, A by e^{at}
K, Y, I by $e^{(a+n)t}$
M by $e^{\mu t + G_t}$; $G_t \equiv \sum_{s=1}^{t} g_s$ *M* has a unit root.
P by $e^{(\mu - a - n)t + G_t}$
r & *h* do not need to be transformed.

$$g' = \rho_g g + \varepsilon_g' \tag{2.1}$$

$$z' = \rho_z z + \varepsilon_z' \tag{2.2}$$

$$\hat{M}' = \hat{M} = M_0 \tag{2.3}$$

$$\hat{c} = \tfrac{(1+\mu+g)\hat{M}}{\hat{P}} \tag{2.4}$$

$$h = \tfrac{\hat{K}'(1+a+n)}{\hat{w}} + \tfrac{\hat{M}'(1+\mu+g')}{\hat{w}\hat{P}} - (1 + r - \delta)\tfrac{\hat{K}}{\hat{w}} \tag{2.5}$$

$$1 = \beta E\{e^{-\gamma a} \tfrac{\hat{w}}{\hat{w}'}(1 - \delta + r')\} \tag{2.6}$$

$$-\widetilde{D} = \beta E\{[\hat{c}' e^{a}]^{-\gamma} \tfrac{\hat{w}\hat{P}}{\hat{P}'(1+\mu-a-n+g')}\} \tag{2.7}$$

$$\hat{Y} = \hat{K}^{\alpha}(e^{z}h)^{1-\alpha} \tag{2.8}$$

$$\hat{w}h = (1-\alpha)\hat{Y} \tag{2.9}$$

$$r\hat{K} = \alpha\hat{Y} \tag{2.10}$$

$$\hat{I} = (1 + a + n)\hat{K}' - (1-\delta)K \tag{2.11}$$

$$\hat{A} \equiv e^{z} \tag{2.12}$$

These are the equations we will use in Dynare.
The endogenous variables are $\hat{c}, \hat{K}, h, \hat{Y}, \hat{w}, r, \hat{M}, \hat{P}, \hat{I}, \hat{A}, g$ & z.
The exogenous variables are ε_z & ε_g.
The parameters are $\alpha, \delta, \beta, a, \mu, \gamma, \rho_z, \sigma_z, \rho_g, \sigma_g, D, h_0$ & M_0.