A Sticky-Price Real Business Cycle Model

Major Features of the Model

Add Calvo-style sticky prices to model 5
We also add intermediate goods with monopoly producers
Two sources of uncertainty: z and g
Stochastic technology growth about a deterministic trend
Labor-leisure decision with indivisible labor hours
Population growth follows a deterministic trend

We first define the important terms that will be used throughout:
Endogenous variables that change over time:

z	productivity
g	money growth
K	capital stock owned by households
h	labor supplied by a single individual
c	consumption by a single individual
w	wage rate
r	interest rate
Y	output of final goods
$Y(i)$	output of intermediate goods firm i
N	number of persons per household
m	money balances per household
M	aggregate money supply
P	price of final good in terms of money
$P(i)$	price of intermediate good i

Parameters:

α	capital share in output from a Cobb-Douglas production function
δ	rate of depreciation
β	time discount factor; $\beta<1$
a	trend in z
μ	trend in g
n	trend in N
γ	elasticity of substitution, $\gamma > 0$
ψ	elasticity of substitution between different intermediate goods
λ	Calvo probability of being unable to adjust prices
D	leisure weight in utility
h_0	hours worked by household that have a job
ρ_i	autocorrelation parameter for $i=z,g$; $0<\rho_i<1$
σ_z	standard deviations of the shocks to $i=z,g$; $0<\sigma_i$

Nonstationary Model
Households
Households have increasing numbers of members, denoted N.
The law of motion for N is:
$$N' = e^{n}N \text{ or } N = e^{nt}N_0 \tag{1.1}$$

Households face both a budget constraint and a cash-in-advance constraint. These are:
$$c = \tfrac{m}{PN} + \tfrac{(\mu+g)M}{PN}$$
$$wh + (1-\delta+r)\tfrac{K}{N} + \tfrac{m}{PN} + \tfrac{gM}{PN} = c + \tfrac{K'}{N} + \tfrac{m'}{PN}$$
Substituting the first into the second and solving for c & h gives the definitions in the problem below.
Given information on prices and shocks, $\mathbf{\Omega} = \{w, r, z, g\}$, the household solves the following non-linear program when the factor markets clear.
$$V(K, m, \mathbf{\Omega}) = \underset{K', m', h}{Max}\left[\tfrac{1}{1-\gamma}(c^{1-\gamma}-1) + he^{(1-\gamma)at}\widetilde{D} + \widetilde{F}e^{(1-\gamma)at} - \widetilde{F}\right]N + \beta E\{V(K', m', \mathbf{\Omega}')\}$$
$$\widetilde{D} \equiv \tfrac{1}{H_0}D[(1-h_0)^{1-\gamma}-1] < 0\,, \quad \widetilde{F} \equiv D\tfrac{1}{1-\gamma}$$
$$c = \tfrac{m}{PN} + \tfrac{(g+\mu)M}{PN} \tag{1.2}$$
$$h = \tfrac{K'}{wN} + \tfrac{m'}{wPN} - (1+r-\delta)\tfrac{K}{wN} \tag{1.3}$$
The first-order conditions are:
$$e^{(1-\gamma)at}\widetilde{D}(-\tfrac{1}{wN})N + \beta E\{V_K(K', \Omega')\} = 0$$
$$e^{(1-\gamma)at}\widetilde{D}(-\tfrac{1}{wPN})N + \beta E\{V_m(K', \Omega')\} = 0$$
The envelope conditions are:
$$V_K(K, \Omega) = e^{(1-\gamma)at}\widetilde{D}(-\tfrac{1}{wPN})(1-\delta+r)N$$
$$V_m(K, \Omega) = c^{-\gamma}\tfrac{1}{PN}N$$
The Euler equations are:
$$1 = \beta E\{e^{(1-\gamma)a}\tfrac{w}{w'}(1-\delta+r')\} \tag{1.4}$$
$$-e^{(1-\gamma)at}\widetilde{D} = \beta E\{c'^{-\gamma}\tfrac{wP}{P'}\} \tag{1.5}$$

Final Goods Producers
Each period a set of final goods producers spontaneously organizes and maximizes profits. Their production function is an intermediate goods aggregator:
$$Y = \left[\int_0^1 Y(i)^{\frac{\psi-1}{\psi}}\, di\right]^{\frac{\psi}{\psi-1}} \tag{1.6}$$
Their problem is:
$$\underset{\{Y(i)\}}{Max}\; P\left[\int_0^1 Y(i)^{\frac{\psi-1}{\psi}}\, di\right]^{\frac{\psi}{\psi-1}} - \int_0^1 P(i)Y(i)\, di$$
The generic first-order condition is:

$$P\frac{\psi}{\psi-1}\left[\int_0^1 Y(i)^{\frac{\psi-1}{\psi}}\ di\right]^{\frac{\psi}{\psi-1}-1}\frac{\psi-1}{\psi}Y(i)^{\frac{\psi-1}{\psi}-1}-P(i)=0$$

Which gives the following demand function for intermediate good *i*:

$$Y(i)=\left(\frac{P}{P(i)}\right)^{\psi}Y \tag{1.7}$$

It also gives the following relation between final goods and intermediate goods prices from a zero profit condition:

$$P=\left[\int_0^1 P(i)^{1-\psi}\ di\right]^{\frac{1}{1-\psi}} \tag{1.8}$$

Intermediate Goods Producers

There is a continuum of producers indexed on the unit interval. Each period a fraction, $1-\lambda$, of these producers are allowed to change their price. The remaining fraction retain the price from last period. This setup is known as Calvo pricing.

Intermediate goods are produced using the following production function.

$$Y(i)=K(i)^{\alpha}[e^{at+z}H(i)]^{1-\alpha} \tag{1.9}$$

The firms who cannot change their prices solve the following problem.

$$\underset{K(i),H(i)}{Max}\ P^{*}(i)K(i)^{\alpha}[e^{at+z}H(i)]^{1-\alpha}-P[rK(i)-wH(i)]$$

The first-order conditions are:

$$\alpha P^{*}(i)K(i)^{\alpha-1}[e^{at+z}H(i)]^{1-\alpha}-P\,r=0$$

$$(1-\alpha)P^{*}(i)K(i)^{\alpha}[e^{at+z}H(i)]^{-\alpha}-P\,w=0$$

These give the following capital to labor ratio.

$$\frac{K(i)}{H(i)}=\frac{\alpha w}{(1-\alpha)r}$$

And they can be rewritten in conjunction with the production function as:

$$K(i)=Y(i)e^{(\alpha-1)(at+z)}[\tfrac{r(1-\alpha)}{w\alpha}]^{\alpha-1} \tag{1.10}$$

$$H(i)=Y(i)e^{\alpha(at+z)}[\tfrac{r(1-\alpha)}{w\alpha}]^{\alpha} \tag{1.11}$$

Note how the factor demands are proportional to output.

Substituting these into the cost function $TC=P[rK(i)-wH(i)]$ gives:

$$TC=PY(i)e^{(\alpha-1)(at+z)}[\tfrac{r(1-\alpha)}{w\alpha}]^{\alpha}\ \tfrac{w}{1-\alpha} \tag{1.12}$$

The firms that can change their prices solve (incorporating time subscripts here):

$$\underset{P_t^{*}(i)}{Max}\,E\left\{\sum_{s=0}^{\infty}(\beta\lambda)^{s}[P_t^{*}(i)Y_{t+s}(i)-TC_{t+s}]\right\}$$

Using (1.7) & (1.10):

$$\underset{P_t^{*}(i)}{Max}\,E\left\{\sum_{s=0}^{\infty}(\beta\lambda)^{s}\left(\tfrac{P_{t+s}}{P_t^{*}(i)}\right)^{\psi}Y_{t+s}\left[P_t^{*}(i)-P_{t+s}e^{(\alpha-1)[a(t+s)+z_{t+s}]}[\tfrac{r_{t+s}(1-\alpha)}{w_{t+s}\alpha}]^{\alpha}\ \tfrac{w_{t+s}}{1-\alpha}\right]\right\}$$

The first-order condition is:

$$E\left\{\sum_{s=0}^{\infty}(\beta\lambda)^s\psi\left(\tfrac{P_{t+s}}{P_t^*(i)}\right)^{\psi-1}\left(\tfrac{-1}{P_t^*(i)^2}\right)Y_{t+s}\left[P_t^*(i)-P_{t+s}e^{(\alpha-1)[a(t+s)+z_{t+s}]}\left[\tfrac{r_{t+s}(1-\alpha)}{w_{t+s}\alpha}\right]^{\alpha}\tfrac{w_{t+s}}{1-\alpha}\right]+\sum_{s=0}^{\infty}(\beta\lambda)^s\left(\tfrac{P_{t+s}}{P_t^*(i)}\right)^{\psi}Y_{t+s}\right\}=0$$

$$E\left\{\sum_{s=0}^{\infty}(\beta\lambda)^s\psi\left(\tfrac{P_{t+s}}{P_t^*(i)}\right)^{\psi}Y_{t+s}\left[-1+\tfrac{P_{t+s}}{P_t^*(i)}e^{(\alpha-1)[a(t+s)+z_{t+s}]}\left[\tfrac{r_{t+s}(1-\alpha)}{w_{t+s}\alpha}\right]^{\alpha}\tfrac{w_{t+s}}{1-\alpha}\right]+\sum_{s=0}^{\infty}(\beta\lambda)^s\left(\tfrac{P_{t+s}}{P_t^*(i)}\right)^{\psi}Y_{t+s}\right\}=0$$

$$E\left\{\sum_{s=0}^{\infty}(\beta\lambda)^s\left(\tfrac{P_{t+s}}{P_t^*(i)}\right)^{\psi}Y_{t+s}\left[1-\psi+\psi\tfrac{P_{t+s}}{P_t^*(i)}e^{(\alpha-1)[a(t+s)+z_{t+s}]}\left[\tfrac{r_{t+s}(1-\alpha)}{w_{t+s}\alpha}\right]^{\alpha}\tfrac{w_{t+s}}{1-\alpha}\right]\right\}=0$$

$$P_t^*(i)=\frac{\psi}{1-\psi}\frac{E\left\{\sum_{s=0}^{\infty}(\beta\lambda)^s\left(\tfrac{P_{t+s}}{P_t^*(i)}\right)^{\psi}Y_{t+s}P_{t+s}\left[e^{(\alpha-1)[a(t+s)+z_{t+s}]}\left[\tfrac{r_{t+s}(1-\alpha)}{w_{t+s}\alpha}\right]^{\alpha}\tfrac{w_{t+s}}{1-\alpha}\right]\right\}}{E\left\{\sum_{s=0}^{\infty}(\beta\lambda)^s\left(\tfrac{P_{t+s}}{P_t^*(i)}\right)^{\psi}Y_{t+s}\right\}}$$

Note that the above implies $P_t^*(i)=P_t^*$ for all firms that choose prices in the current period.

Using (1.7):

$$P_t^*=\frac{\psi}{1-\psi}\frac{E\left\{\sum_{s=0}^{\infty}(\beta\lambda)^sY_{t+s}(i)P_{t+s}\left[e^{(\alpha-1)[a(t+s)+z]}\left[\tfrac{r_{t+s}(1-\alpha)}{w_{t+s}\alpha}\right]^{\alpha}\tfrac{w_{t+s}}{1-\alpha}\right]\right\}}{E\left\{\sum_{s=0}^{\infty}(\beta\lambda)^sY_{t+s}(i)\right\}}\qquad(1.13)$$

Aggregating Price Movements

Using (1.8) and noting that a fraction λ of prices do not change, while the remaining prices are set to P^*:

$$P'=\left[\int_0^1 P'(i)^{1-\psi}\,di\right]^{\frac{1}{1-\psi}}=\left[\int_0^{\lambda}P(i)^{1-\psi}\,di+\int_{\lambda}^{1}P^{*\prime 1-\psi}\,di\right]^{\frac{1}{1-\psi}}$$

$$P'^{1-\psi}=\lambda P^{1-\psi}+(1-\lambda)P^{*\prime 1-\psi}\qquad(1.8')$$

Additional Behavioral Equations

Money changes over time according to:

$$M'=e^{\mu+g}M=M_0\prod_{s=1}^{t+1}e^{\mu+g_s}\qquad(1.14)$$

The law of motion for g is:

$$g'=\rho_g g+\varepsilon_g';\ \varepsilon_g'\sim N(0,\sigma_g^2)\qquad(1.15)$$

The law of motion for z is:

$$z'=\rho_z z+\varepsilon_z';\ \ \varepsilon_z'\sim N(0,\sigma_z^2)\qquad(1.16)$$

Aggregating over household members gives:

$$H=Nh\qquad(1.17)$$

Money market clearing gives:

$$M=m\qquad(1.18)$$

Capital and labor market clearing give:

$$K = \int_0^1 K(i)\, di \tag{1.19}$$

$$H = \int_0^1 H(i)\, di \tag{1.20}$$

Definitions:

$$I \equiv K' - (1-\delta)K \tag{1.21}$$

$$A \equiv e^{at+z} \tag{1.22}$$

Eqs (1.1)-(1.22) are the system.

Transformation & Simplifications

Without loss of generalization set $\hat{N} = N_0 = 1$, and eliminate it from the system.

TC has already been eliminated by substitution of (1.12) to get (1.13).

(1.7) is also incorporated in (1.13).

Use (1.17) to eliminate *H* from the system.

Use (1.18) to eliminate *m* from the system.

Transform the problem by dividing:

c, w, A by e^{at}

$K(i), K, Y(i), Y, I$ by $e^{(a+n)t}$

M by $e^{\mu t + G_t}$; $G_t \equiv \sum_{s=1}^{t} g_s$ M has a unit root.

P by $e^{(\mu - a - n)t + G_t}$

r & *h* do not need to be transformed.

$$g' = \rho_g g + \varepsilon_g'; \varepsilon_g' \sim N(0, \sigma_g^2) \tag{2.1}$$

$$z' = \rho_z z + \varepsilon_z'; \ \ \varepsilon_z' \sim N(0, \sigma_z^2) \tag{2.2}$$

$$\hat{M}' = \hat{M} = M_0 \tag{2.3}$$

$$\hat{c} = \tfrac{e^{\mu+g}\hat{M}}{\hat{P}} \tag{2.4}$$

$$h = \tfrac{\hat{K}' e^{a+n}}{\hat{w}} + \tfrac{\hat{M}' e^{\mu+g'}}{\hat{w}\hat{P}} - (1 + r - \delta)\tfrac{\hat{K}}{\hat{w}} \tag{2.5}$$

$$1 = \beta E\{e^{-\gamma a} \tfrac{\hat{w}}{\hat{w}'}(1 - \delta + r')\} \tag{2.6}$$

$$-\widetilde{D} = \beta E\{(\hat{c}' e^{a})^{-\gamma} \tfrac{\hat{w}\hat{P}}{\hat{P}' e^{\mu - a - n + g'}}\} \tag{2.7}$$

$$\hat{Y} = \left[\int_0^1 \hat{Y}(i)^{\frac{\psi-1}{\psi}}\, di \right]^{\frac{\psi}{\psi-1}} \tag{2.8}$$

$$[\hat{P}' e^{\mu - a - n + g'}]^{1-\psi} = \lambda \hat{P}^{1-\psi} + (1-\lambda)[\hat{P}^{*}{}' e^{\mu - a - n + g'}]^{1-\psi} \tag{2.9}$$

$$\hat{Y}(i) = \hat{K}(i)^{\alpha}[e^{z}\hat{H}(i)]^{1-\alpha} \tag{2.10}$$

$$\hat{P}^* = \frac{\psi}{1-\psi} \frac{E\left\{\sum_{s=0}^{\infty} (\beta\lambda e^{\mu})^s e^{G_{t+s}} \hat{Y}_{t+s}(i) \hat{P}_{t+s} \left[e^{(\alpha-1)z_{t+s}} [\frac{r_{t+s}(1-\alpha}{\hat{w}_{t+s}\alpha}]^{\alpha} \frac{\hat{w}_{t+s}}{1-\alpha} \right] \right\}}{E\left\{\sum_{s=0}^{\infty} (\beta\lambda e^{\mu})^s e^{G_{t+s}} \hat{Y}_{t+s}(i) \right\}} \tag{2.11}$$

$$\hat{K}(i) = \hat{Y}(i) e^{(\alpha-1)z} [\tfrac{r(1-\alpha)}{\hat{w}\alpha}]^{\alpha-1} \tag{2.12}$$

$$\hat{H}(i) = \hat{Y}(i) e^{\alpha z} [\tfrac{r(1-\alpha)}{w\alpha}]^{\alpha} \tag{2.13}$$

$$\hat{K} = \int_0^1 \hat{K}(i)\, di \tag{2.14}$$

$$h = \int_0^1 \hat{H}(i)\, di \tag{2.15}$$

$$\hat{I} = e^{a+n} \hat{K}' - (1-\delta)K \tag{2.16}$$

$$\hat{A} \equiv e^{z} \tag{2.17}$$

One of these equations is redundant by Walras' Law. We could eliminate (2.5) or (2.8). We choose (2.8).

These are the 16 equations we will log-linearize and then use in Dynare.
The 16 endogenous variables are $\hat{c}, \hat{K}, \hat{K}(i), h, \hat{H}(i), \hat{Y}, \hat{Y}(i), \hat{w}, r, \hat{M}, \hat{P}, \hat{P}^*, \hat{I}, \hat{A}, g$ & z.
The exogenous variables are ε_z & ε_g.
The parameters are $\alpha, \delta, \beta, a, n, \mu, \gamma, \psi, \lambda, \rho_z, \sigma_z, \rho_g, \sigma_g, D$ & h_0.

The model as written above cannot be solved because it includes variables that are a function of *i*. These need to be substituted out before we can numerically solve and simulate.

Eliminating $\hat{M}$ & $\hat{H}$ and equations (2.3) & (2.17).

$$g' = \rho_g g + \varepsilon_g'; \; \varepsilon_g' \sim N(0, \sigma_g^2) \tag{3.1}$$

$$z' = \rho_z z + \varepsilon_z'; \quad \varepsilon_z' \sim N(0, \sigma_z^2) \tag{3.2}$$

$$\hat{c} = \frac{e^{\mu+g}\hat{M}}{\hat{P}} \tag{3.3}$$

$$h = \frac{\hat{K}' e^{a+n}}{\hat{w}} + \frac{M_0 e^{\mu+g'}}{\hat{w}\hat{P}} - (1 + r - \delta)\frac{\hat{K}}{\hat{w}} \tag{3.4}$$

$$1 = \beta E\{e^{-\gamma a} \frac{\hat{w}}{\hat{w}'}(1 - \delta + r')\} \tag{3.5}$$

$$-\widetilde{D} = \beta E\{(\hat{c}' e^{a})^{-\gamma} \frac{\hat{w}\hat{P}}{\hat{P}' e^{\mu-a-n+g'}}\} \tag{3.6}$$

$$[\hat{P}' e^{\mu-a-n+g'}]^{1-\psi} = \lambda \hat{P}^{1-\psi} + (1-\lambda)[\hat{P}^{*}{}' e^{\mu-a-n+g'}]^{1-\psi} \tag{3.7}$$

$$\hat{P}^{*} = \frac{\psi}{1-\psi} \frac{E\left\{\sum_{s=0}^{\infty} (\beta\lambda e^{\mu})^{s} e^{G_{t+s}} \hat{Y}_{t+s}(i) \hat{P}_{t+s} \left[e^{(\alpha-1)z_{t+s}} [\frac{r_{t+s}(1-\alpha}{\hat{w}_{t+s}\alpha}]^{\alpha} \frac{\hat{w}_{t+s}}{1-\alpha}\right]\right\}}{E\left\{\sum_{s=0}^{\infty} (\beta\lambda e^{\mu})^{s} e^{G_{t+s}} \hat{Y}_{t+s}(i)\right\}} \tag{3.8}$$

$$\hat{Y}(i) = \hat{K}(i)^{\alpha}[e^{z}\hat{H}(i)]^{1-\alpha} \tag{3.9}$$

$$\hat{K}(i) = \hat{Y}(i) e^{(\alpha-1)z}[\frac{r(1-\alpha)}{\hat{w}\alpha}]^{\alpha-1} \tag{3.10}$$

$$\hat{H}(i) = \hat{Y}(i) e^{\alpha z}[\frac{r(1-\alpha)}{w\alpha}]^{\alpha} \tag{3.11}$$

$$\hat{K} = \int_0^1 \hat{K}(i)\, di \tag{3.12}$$

$$h = \int_0^1 \hat{H}(i)\, di \tag{3.13}$$

$$\hat{I} = e^{a+n}\hat{K}' - (1-\delta)K \tag{3.14}$$

14 equations in 14 unknowns

$\hat{c}, \hat{K}, h, \hat{Y}, \hat{Y}(i), \hat{K}(i), \hat{H}(i), \hat{w}, r, \hat{P}, \hat{P}^{*}, \hat{I}, g$ & z .

We need to eliminate $\hat{Y}(i), \hat{K}(i), \hat{H}(i)$ from the system.

Define $\phi(i) \equiv \frac{\hat{Y}(i)}{\hat{Y}}$ and note that by (3.10) & (3.11) we get $\phi(i) = \frac{\hat{K}(i)}{\hat{K}} = \frac{\hat{H}(i)}{h}$.

Using the intermediate goods production function:

$$\hat{Y}(i) = e^{(1-\alpha)z}\hat{K}(i)^{\alpha}\hat{H}(i)^{1-\alpha}$$

$$\phi(i)\hat{Y} = e^{(1-\alpha)z}\left[\phi(i)\hat{K}\right]^{\alpha}\left[\phi(i)h\right]^{1-\alpha}$$

$$\phi(i)\hat{Y} = e^{(1-\alpha)z}\phi(i)\hat{K}^{\alpha}h^{1-\alpha}$$

$$\hat{Y} = e^{(1-\alpha)z}\hat{K}^{\alpha}h^{1-\alpha}$$

Use the above in place of (3.9) with (3.4) becoming redundant

Also note that $\int_0^1 \phi(i)\, di = 1$, making (3.12) & (3.13) redundant.

(3.10) and (3.11) become:

$$\hat{K} = \hat{Y}e^{(\alpha-1)z}\left[\tfrac{r(1-\alpha)}{\hat{w}\alpha}\right]^{\alpha-1}$$

$$h = \hat{Y}e^{\alpha z}\left[\tfrac{r(1-\alpha)}{\hat{w}\alpha}\right]^{\alpha}$$

Which with a little algebra reduce to:

$$r\hat{K} = \alpha\hat{Y}$$

$$\hat{w}h = (1-\alpha)\hat{Y}$$

Lastly, we need to remove $\hat{Y}(i)$ from (3.8):

Let's start with the nonstationary version, noting that $P_t^*(i) = P_t^* \ \forall i$.

$$P_t^* = \frac{\psi}{1-\psi}\frac{E\left\{\sum_{s=0}^{\infty}(\beta\lambda)^s\left(\frac{P_{t+s}}{P_t^*}\right)^{\psi}Y_{t+s}P_{t+s}\left[e^{(\alpha-1)[a(t+s)+z_{t+s}]}\left[\frac{r_{t+s}(1-\alpha)}{w_{t+s}\alpha}\right]^{\alpha}\frac{w_{t+s}}{1-\alpha}\right]\right\}}{E\left\{\sum_{s=0}^{\infty}(\beta\lambda)^s\left(\frac{P_{t+s}}{P_t^*}\right)^{\psi}Y_{t+s}\right\}}$$

Stationarizing:

$$\frac{P_t^* e^{(\mu-a-n)t+G_t}}{e^{(\mu-a-n)t+G_t}} = \frac{\psi}{1-\psi}\frac{E\left\{\sum_{s=0}^{\infty}(\beta\lambda)^s\left(\frac{P_{t+s}}{P_t^*}\right)^{\psi}\frac{Y_{t+s}e^{(a+n)(t+s)}}{e^{(a+n)(t+s)}}\frac{P_{t+s}e^{(\mu-a-n)(t+s)+G_{t+s}}}{e^{(\mu-a-n)(t+s)+G_{t+s}}}\left[e^{(\alpha-1)z_{t+s}}\left[\frac{r_{t+s}(1-\alpha)}{\hat{w}_{t+s}\alpha}\right]^{\alpha}\frac{\hat{w}_{t+s}}{1-\alpha}\right]\right\}}{E\left\{\sum_{s=0}^{\infty}(\beta\lambda)^s\left(\frac{P_{t+s}}{P_t^*}\right)^{\psi}\frac{Y_{t+s}e^{(a+n)(t+s)}}{e^{(a+n)(t+s)}}\right\}}$$

$$\left(\frac{P_{t+s}}{P_t^*}\right)^{\psi} = \left(\frac{P_{t+s}e^{(\mu-a-n)(t+s)+G_{t+s}}e^{(\mu-a-n)t+G_t}}{P_t^* e^{(\mu-a-n)t+G_t}e^{(\mu-a-n)(t+s)+G_{t+s}}}\right)^{\psi} = \left(\frac{\hat{P}_{t+s}e^{(\mu-a-n)(t+s)+G_{t+s}}}{\hat{P}_t^* e^{(\mu-a-n)t+G_t}}\right)^{\psi} = \left(\frac{\hat{P}_{t+s}}{\hat{P}_t^*}\right)^{\psi}e^{\psi[(\mu-a-n)s+G_{t+s}-G_t]}$$

$$\hat{P}_t^* e^{(\mu-a-n)t+G_t} = \frac{\psi}{1-\psi} \frac{e^{\mu t+G_t} E\left\{\sum_{s=0}^{\infty} (\beta\lambda e^{\mu+\psi(\mu-a-n)})^s e^{(1+\psi)(G_{t+s}-G_t)} \left(\frac{\hat{P}_{t+s}}{\hat{P}_t^*}\right)^{\psi} \hat{Y}_{t+s} \hat{P}_{t+s} \left[e^{(\alpha-1)z_{t+s}} \left[\frac{r_{t+s}(1-\alpha)}{\hat{w}_{t+s}\alpha}\right]^{\alpha} \frac{\hat{w}_{t+s}}{1-\alpha}\right]\right\}}{e^{(a+n)t} E\left\{\sum_{s=0}^{\infty} (\beta\lambda e^{a+n+\psi(\mu-a-n)})^s e^{\psi(G_{t+s}-G_t)} \left(\frac{\hat{P}_{t+s}}{\hat{P}_t^*}\right)^{\psi} \hat{Y}_{t+s}\right\}}$$

$$\hat{P}_t^* = \frac{\psi}{1-\psi} \frac{\hat{P}_t^{*-\psi} E\left\{\sum_{s=0}^{\infty} (\beta\lambda e^{\mu+\psi(\mu-a-n)})^s e^{(1+\psi)(G_{t+s}-G_t)} \hat{P}_{t+s}{}^{\psi} \hat{Y}_{t+s} \hat{P}_{t+s} \left[e^{(\alpha-1)z_{t+s}} \left[\frac{r_{t+s}(1-\alpha)}{\hat{w}_{t+s}\alpha}\right]^{\alpha} \frac{\hat{w}_{t+s}}{1-\alpha}\right]\right\}}{\hat{P}_t^{*-\psi} E\left\{\sum_{s=0}^{\infty} (\beta\lambda e^{a+n+\psi(\mu-a-n)})^s e^{\psi(G_{t+s}-G_t)} \hat{P}_{t+s}{}^{\psi} \hat{Y}_{t+s}\right\}}$$

$$\hat{P}_t^* = \frac{\psi}{1-\psi} \frac{E\left\{\sum_{s=0}^{\infty} (\beta\lambda e^{(1+\psi)\mu-\psi(a+n)})^s e^{(1+\psi)(G_{t+s}-G_t)} \hat{P}_{t+s}{}^{1+\psi} \hat{Y}_{t+s} \left[e^{(\alpha-1)z_{t+s}} r_{t+s}{}^{\alpha} \hat{w}_{t+s}{}^{1-\alpha} \alpha^{-\alpha} (1-\alpha)^{\alpha-1}\right]\right\}}{E\left\{\sum_{s=0}^{\infty} (\beta\lambda e^{(1-\psi)(a+n)+\psi\mu})^s e^{\psi(G_{t+s}-G_t)} \hat{P}_{t+s}{}^{\psi} \hat{Y}_{t+s}\right\}}$$

Now transform sums in equation above into state variables.

Define $S_D = \sum_{s=0}^{\infty} (\beta\lambda e^{(1-\psi)(a+n)+\psi\mu})^s e^{\psi(G_{t+s}-G_t)} \hat{P}_{t+s}{}^{\psi} \hat{Y}_{t+s}$ and

$$S_N = \sum_{s=0}^{\infty} (\beta\lambda e^{(1+\psi)\mu-\psi(a+n)})^s e^{(1+\psi)(G_{t+s}-G_t)} \hat{P}_{t+s}{}^{1+\psi} \hat{Y}_{t+s} \left[e^{(\alpha-1)z_{t+s}} r_{t+s}{}^{\alpha} \hat{w}_{t+s}{}^{1-\alpha} \alpha^{-\alpha} (1-\alpha)^{\alpha-1}\right]$$

Rewrite these as:

$S_D' = \dfrac{S_D - \hat{P}^{\psi}\hat{Y}}{\beta\lambda e^{(1-\psi)(a+n)+\psi\mu} e^{\psi g}}$ and

$$S_N' = \frac{S_N - \hat{P}^{1+\psi}\hat{Y}\left[e^{(\alpha-1)z} r^{\alpha} \hat{w}^{1-\alpha} \alpha^{-\alpha} (1-\alpha)^{\alpha-1}\right]}{\beta\lambda e^{(1+\psi)\mu-\psi(a+n)} e^{(1+\psi)g}}$$

$$g' = \rho_g g + \varepsilon_g'; \; \varepsilon_g' \sim N(0, \sigma_g^2) \tag{4.1}$$

$$z' = \rho_z z + \varepsilon_z'; \quad \varepsilon_z' \sim N(0, \sigma_z^2) \tag{4.2}$$

$$\hat{c} = \frac{e^{\mu+g}\hat{M}}{\hat{P}} \tag{4.3}$$

$$1 = \beta E\{e^{-\gamma a}\frac{\hat{w}}{\hat{w}'}(1-\delta+r')\} \tag{4.4}$$

$$-\widetilde{D} = \beta E\{(\hat{c}' e^{a})^{-\gamma}\frac{\hat{w}\hat{P}}{\hat{P}' e^{\mu-a-n+g'}}\} \tag{4.5}$$

$$[\hat{P}' e^{\mu-a-n+g'}]^{1-\psi} = \lambda\hat{P}^{1-\psi} + (1-\lambda)[\hat{P}^{*}{}' e^{\mu-a-n+g'}]^{1-\psi} \tag{4.6}$$

$$\hat{P}^{*} = \frac{\psi}{1-\psi}\frac{E\{S_N\}}{E\{S_D\}} \tag{4.7}$$

$$S_D' = \frac{S_D - \hat{P}^{\psi}\hat{Y}}{\beta\lambda e^{(1-\psi)(a+n)+\psi(\mu+g)}} \tag{4.8}$$

$$S_N' = \frac{S_N - \hat{P}^{1+\psi}\hat{Y}\left[e^{(\alpha-1)z} r^{\alpha}\hat{w}^{1-\alpha}\alpha^{-\alpha}(1-\alpha)^{\alpha-1}\right]}{\beta\lambda e^{(1+\psi)(\mu+g)-\psi(a+n)}} \tag{4.9}$$

$$\hat{Y} = e^{(1-\alpha)z}\hat{K}^{\alpha}h^{1-\alpha} \tag{4.10}$$

$$r\hat{K} = \alpha\hat{Y} \tag{4.11}$$

$$\hat{w}h = (1-\alpha)\hat{Y} \tag{4.12}$$

$$\hat{I} = e^{a+n}\hat{K}' - (1-\delta)K \tag{4.13}$$

13 equations in 13 unknowns:
$\hat{c}, \hat{K}, h, \hat{Y}, \hat{w}, r, \hat{P}, \hat{P}^{*}, \hat{I}, S_N, S_D, g \;\&\; z$.

Steady State

$$\bar{g} = 0 \tag{5.1}$$
$$\bar{z} = 0 \tag{5.2}$$
$\overline{M}$ is an arbitrary constant pinned down by its initial value of 1. (5.3)
$$\bar{c} = \tfrac{(1+\mu)\overline{M}}{\overline{P}} \tag{5.4}$$
$$\bar{h} = \tfrac{(a+n-r+\delta)}{\overline{w}} + \tfrac{\overline{M}(1+\mu)}{\overline{w}\overline{P}} \tag{5.5}$$
$$1 = \beta e^{-\gamma a}(1-\delta+\bar{r}) \tag{5.6}$$
$$-\widetilde{D} = \beta\bar{c}^{-\gamma}\tfrac{\overline{w}}{(1+\mu-a-n)} \tag{5.7}$$
$$[\overline{P}(1+\mu-a-n)]^{1-\psi} = \lambda\overline{P}^{1-\psi} + (1-\lambda)[\overline{P}^{*}(1+\mu-a-n)]^{1-\psi} \tag{5.9}$$
$$\overline{Y}(i) = \overline{K}(i)^{\alpha}\,\overline{H}(i)^{1-\alpha} \tag{5.10}$$
$$\overline{P}^{*} = \frac{\psi}{1-\psi}\frac{\overline{Y}(i)\overline{P}[\frac{\bar{r}(1-\alpha}{\overline{w}\alpha}]^{\alpha}\frac{\overline{w}}{1-\alpha}}{\overline{Y}(i)} \tag{5.11}$$
$$\overline{K}(i) = \overline{Y}(i)[\tfrac{\bar{r}(1-\alpha)}{\overline{w}\alpha}]^{\alpha-1} \tag{5.12}$$
$$\overline{H}(i) = \overline{Y}(i)[\tfrac{\bar{r}(1-\alpha)}{\overline{w}\alpha}]^{\alpha} \tag{5.13}$$
$$\overline{K} = \int_0^1 \overline{K}(i)\,di \tag{5.14}$$
$$\bar{h} = \int_0^1 \overline{H}(i)\,di \tag{5.15}$$
$$\bar{I} = (a+n+\delta)\overline{K} \tag{5.16}$$
$$\overline{A} = 1 \tag{5.17}$$

Manipulating (5.9) gives:

$$\overline{P}^{1-\psi}(1+\mu-a-n)^{1-\psi} = \lambda\overline{P}^{1-\psi}\tfrac{(1+\mu-a-n)^{1-\psi}}{(1+\mu-a-n)^{1-\psi}} + (1-\lambda)\overline{P}^{*1-\psi}(1+\mu-a-n)^{1-\psi}$$
$$(1-\lambda\tfrac{1}{(1+\mu-a-n)^{1-\psi}})\overline{P}^{1-\psi}(1+\mu-a-n)^{1-\psi} = (1-\lambda)\overline{P}^{*1-\psi}(1+\mu-a-n)^{1-\psi}$$
$$(1-\lambda\tfrac{1}{(1+\mu-a-n)^{1-\psi}})\overline{P}^{1-\psi} = (1-\lambda)\overline{P}^{*1-\psi}$$
$$(\tfrac{(1+\mu-a-n)^{1-\psi}-\lambda}{(1+\mu-a-n)^{1-\psi}})\overline{P}^{1-\psi} = (1-\lambda)\overline{P}^{*1-\psi}$$
$$\overline{P}^{*1-\psi} = \tfrac{(1+\mu-a-n)^{1-\psi}-\lambda}{(1+\mu-a-n)^{1-\psi}(1-\lambda)}$$
$$\overline{P}^{*} = \Phi\overline{P};\;\; \Phi \equiv \left(\tfrac{(1+\mu-a-n)^{1-\psi}-\lambda}{(1+\mu-a-n)^{1-\psi}(1-\lambda)}\right)^{\frac{1}{1-\psi}} \tag{5.9'}$$

Putting this into the SS version of (1.7) gives:

$$\overline{Y}(i) = \overline{Y}$$

(5.12) - (5.15) then give:

$$\overline{K} = \overline{K}(i) \tag{5.14'}$$
$$\bar{h} = \overline{H}(i) \tag{5.15'}$$
$$\overline{K} = \overline{Y}[\tfrac{\bar{r}(1-\alpha)}{\overline{w}\alpha}]^{\alpha-1} \tag{5.12'}$$

$$\overline{h} = \overline{Y}[\tfrac{\overline{r}(1-\alpha)}{\overline{w}\alpha}]^{\alpha} \tag{5.13'}$$

(5.10) gives:

$$\overline{Y} = \overline{K}^{\alpha}\overline{h}^{1-\alpha} \tag{5.10'}$$

The SS is found by noting that (5.6) implies:

$$\bar{r} = \tfrac{1}{\beta}e^{\gamma a} + \delta - 1 \qquad (5.6")$$

Using (5.11):

$$\bar{w} = [\tfrac{1-\psi}{\psi}\bar{r}^{-\alpha}(1-\alpha)^{1-\alpha}\alpha^{\alpha}]^{\frac{1}{1-\alpha}} \qquad (5.11")$$

(5.7) gives:

$$\bar{c} = [\beta \tfrac{\bar{w}}{-\tilde{D}(1+\mu-a-n)}]^{\frac{1}{\gamma}} \qquad (5.7")$$

$\overline{M}$ is an arbitrary constant pinned down by its initial value.

(5.4) gives:

$$\overline{P} = \tfrac{(1+\mu)\overline{M}}{\bar{c}} \qquad (5.4")$$

Then (5.9') gives:

$$\overline{P}^{*} = \Phi\overline{P} \qquad (5.9")$$

Use (5.5) as is with (5.15'):

$$\bar{h} = \overline{H}(i) = \tfrac{(a+n-r+\delta)}{\bar{w}} + \tfrac{\overline{M}(1+\mu)}{\bar{w}\overline{P}} \qquad (5.5)$$

(5.12') & (5.13') together with (5.14') give:

$$\overline{K} = \overline{K}(i) = \tfrac{\bar{w}\alpha}{\bar{r}(1-\alpha)}\bar{h} \qquad (5.12")$$

(5.10') gives:

$$\overline{Y} = \overline{Y}(i) = \overline{K}^{\alpha}\bar{h}^{1-\alpha} \qquad (5.10')$$

Finally, use (5.16) as is.

$$\bar{I} = (a+n+\delta)\overline{K} \qquad (5.16)$$

Log-Linearization Approach

The equations (2.1) – (2.7) & (2.9) – (2.17) need to be log linearized.
(2.1) & (2.2) are already log-linearized.

Examine (2.9):

$$(\overline{P}e^{\widetilde{P}'}e^{\mu-a-n})^{1-\psi} = \lambda(\overline{P}e^{\widetilde{P}})^{1-\psi} + (1-\lambda)(\overline{P}^{*}e^{\widetilde{P}^{*}{}'}e^{\mu-a-n})^{1-\psi}$$

Rearranging:

$$(\overline{P}e^{\mu-a-n})^{1-\psi}e^{\widetilde{P}'(1-\psi)} = \lambda\overline{P}^{1-\psi}e^{\widetilde{P}(1-\psi)} + (1-\lambda)(\overline{P}^{*}e^{\mu-a-n})^{1-\psi}e^{\widetilde{P}^{*}{}'(1-\psi)}$$

$$(\overline{P}e^{\mu-a-n})^{1-\psi}[1+\widetilde{P}'(1-\psi)] = \lambda\overline{P}^{1-\psi}[1+\widetilde{P}(1-\psi)] + (1-\lambda)(\overline{P}^{*}e^{\mu-a-n})^{1-\psi}[1+\widetilde{P}^{*}{}'(1-\psi)]$$

$$(\overline{P}e^{\mu-a-n})^{1-\psi} + (\overline{P}e^{\mu-a-n})^{1-\psi}\widetilde{P}'(1-\psi) =$$

$$\lambda\overline{P}^{1-\psi} + \lambda\overline{P}^{1-\psi}\widetilde{P}(1-\psi) + (1-\lambda)(\overline{P}^{*}e^{\mu-a-n})^{1-\psi} + (1-\lambda)(\overline{P}^{*}e^{\mu-a-n})^{1-\psi}\widetilde{P}^{*}{}'(1-\psi)$$

Using (5.9) & (5.9'):

$$(\overline{P}e^{\mu-a-n})^{1-\psi}\widetilde{P}'(1-\psi) = \lambda\overline{P}^{1-\psi}\widetilde{P}(1-\psi) + (1-\lambda)(\Phi\overline{P}e^{\mu-a-n})^{1-\psi}\widetilde{P}^{*}{}'(1-\psi)$$

$$\widetilde{P}' = \tfrac{\lambda}{e^{(\mu-a-n)(1-\psi)}}\widetilde{P} + (1-\lambda)\Phi^{1-\psi}\widetilde{P}^{*}{}'$$

$$\widetilde{P}' = \tfrac{\lambda}{e^{(\mu-a-n)(1-\psi)}}\widetilde{P} + (1-\lambda)\tfrac{e^{(\mu-a-n)(1-\psi)}-\lambda}{e^{(\mu-a-n)(1-\psi)}(1-\lambda)}\widetilde{P}^{*}{}'$$

$$\widetilde{P}' = \tfrac{\lambda}{e^{(\mu-a-n)(1-\psi)}}\widetilde{P} + \left(\tfrac{e^{(\mu-a-n)(1-\psi)}-\lambda}{e^{(\mu-a-n)(1-\psi)}}\right)\widetilde{P}^{*}{}' \tag{6.9}$$

The most challenging equation is (2.11):

$$\hat{P}_t^{*}E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}\hat{Y}_{t+s}(i)\right\} = \frac{\psi}{1-\psi}E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}\hat{Y}_{t+s}(i)\hat{P}_{t+s}\left[e^{(\alpha-1)z_{t+s}}\left[\tfrac{r_{t+s}(1-\alpha)}{\hat{w}_{t+s}\alpha}\right]^{\alpha}\tfrac{\hat{w}_{t+s}}{1-\alpha}\right]\right\}$$

Linearize LHS first:

$$E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}\hat{P}_s^{*}\hat{Y}_{t+s}(i)\right\} = E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}\overline{P}\,\overline{Y}e^{\widetilde{P}_t^{*}+\widetilde{Y}_{t+s}}\right\}$$

$$= \overline{P}\,\overline{Y}E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}(1+\widetilde{P}_t^{*}+\widetilde{Y}_{t+s})\right\}$$

$$= \overline{P}\,\overline{Y}(1+\widetilde{P}^{*})\frac{1}{1-\beta\lambda e^{\mu}} + \overline{P}\,\overline{Y}E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}\widetilde{Y}_{t+s}\right\} \tag{6.11 LHS}$$

Next linearize RHS:

$$\frac{\psi}{1-\psi}E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}\overline{P}\,\overline{Y}e^{\widetilde{P}_{t+s}+\widetilde{Y}_{t+s}}e^{(\alpha-1)z_{t+s}}(\overline{r}e^{\widetilde{r}_{t+s}})^{\alpha}(\overline{w}e^{\widetilde{w}_{t+s}})^{1-\alpha}(1-\alpha)^{\alpha-1}\alpha^{\alpha}\right\}$$

$$\frac{\psi}{1-\psi}E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}\overline{P}\,\overline{Y}\overline{r}^{\alpha}\overline{w}^{1-\alpha}(1-\alpha)^{\alpha-1}\alpha^{\alpha}e^{\widetilde{P}_{t+s}+\widetilde{Y}_{t+s}+\alpha\widetilde{r}_{t+s}+(1-\alpha)\widetilde{w}_{t+s}-(1-\alpha)z_{t+s}}\right\}$$

$$\frac{\psi}{1-\psi}\overline{P}\,\overline{Y}\overline{r}^{\alpha}\overline{w}^{1-\alpha}(1-\alpha)^{\alpha-1}\alpha^{\alpha}E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}[1+\widetilde{P}_{t+s}+\widetilde{Y}_{t+s}+\alpha\widetilde{r}_{t+s}+(1-\alpha)\widetilde{w}_{t+s}-(1-\alpha)z_{t+s}]\right\}$$

Noting from (5.11) that $\frac{\psi}{1-\psi}\overline{r}^{\alpha}\overline{w}^{1-\alpha}(1-\alpha)^{\alpha-1}\alpha^{\alpha} = 1$

$$\bar{P}\bar{Y}E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}[1+\widetilde{P}_{t+s}+\widetilde{Y}_{t+s}+\alpha\widetilde{r}_{t+s}+(1-\alpha)\widetilde{w}_{t+s}-(1-\alpha)z_{t+s}]\right\} \quad (6.11\ \text{RHS})$$

Combining the two sides again:

$$(1+\widetilde{P}_t^*)\frac{1}{1-\beta\lambda e^{\mu}}+E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}\widetilde{Y}_{t+s}\right\}$$

$$=E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}[1+\widetilde{P}_{t+s}+\widetilde{Y}_{t+s}+\alpha\widetilde{r}_{t+s}+(1-\alpha)\widetilde{w}_{t+s}-(1-\alpha)z_{t+s}]\right\}$$

$$\widetilde{P}_t^*=(1-\beta\lambda e^{\mu})E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}[\widetilde{P}_{t+s}+\alpha\widetilde{r}_{t+s}+(1-\alpha)\widetilde{w}_{t+s}-(1-\alpha)z_{t+s}]\right\} \quad (6.11)$$

Substitute this into lagged version of (6.9) and add current period subscripts back to notation:

$$\widetilde{P}_t=\tfrac{\lambda}{e^{(\mu-a-n)(1-\psi)}}\widetilde{P}_{t-1}+\left(\tfrac{e^{(\mu-a-n)(1-\psi)}-\lambda}{e^{(\mu-a-n)(1-\psi)}}\right)(1-\beta\lambda e^{\mu})E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}[\widetilde{P}_{t+s}+\alpha\widetilde{r}_{t+s}+(1-\alpha)\widetilde{w}_{t+s}-(1-\alpha)z_{t+s}]\right\}$$

Quasi-difference by $1-\beta\lambda e^{\mu}L^{-1}$:

$$\widetilde{P}_t-\beta\lambda e^{\mu}\widetilde{P}_{t+1}=\tfrac{\lambda}{e^{(\mu-a-n)(1-\psi)}}\widetilde{P}_{t-1}-\tfrac{\beta\lambda^2 e^{\mu}}{e^{(\mu-a-n)(1-\psi)}}\widetilde{P}_t$$

$$+\tfrac{e^{(\mu-a-n)(1-\psi)}-\lambda}{e^{(\mu-a-n)(1-\psi)}}(1-\beta\lambda e^{\mu})E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}[\widetilde{P}_t+\alpha\widetilde{r}_{t+s}+(1-\alpha)\widetilde{w}_{t+s}-(1-\alpha)z_{t+s}]\right\}$$

$$-\tfrac{e^{(\mu-a-n)(1-\psi)}-\lambda}{e^{(\mu-a-n)(1-\psi)}}(1-\beta\lambda e^{\mu})E\left\{\sum_{s=0}^{\infty}(\beta\lambda e^{\mu})^{s}[\widetilde{P}_{t+1}+\alpha\widetilde{r}_{t+s+1}+(1-\alpha)\widetilde{w}_{t+s+1}-(1-\alpha)z_{t+s+1}]\right\}$$

Cancelling:

$$\widetilde{P}_t-\beta\lambda e^{\mu}\widetilde{P}_{t+1}=\tfrac{\lambda}{e^{(\mu-a-n)(1-\psi)}}\widetilde{P}_{t-1}-\tfrac{\beta\lambda^2 e^{\mu}}{e^{(\mu-a-n)(1-\psi)}}\widetilde{P}_t+\tfrac{e^{(\mu-a-n)(1-\psi)}-\lambda}{e^{(\mu-a-n)(1-\psi)}}(1-\beta\lambda e^{\mu})[\widetilde{P}_t+\alpha\widetilde{r}_t+(1-\alpha)\widetilde{w}_t-(1-\alpha)z_t]$$

Collecting terms:

$$-\beta\lambda e^{\mu}\widetilde{P}_{t+1}+\left[1+\tfrac{\beta\lambda^2 e^{\mu}}{e^{(\mu-a-n)(1-\psi)}}-\tfrac{(e^{(\mu-a-n)(1-\psi)}-\lambda)(1-\beta\lambda e^{\mu})}{e^{(\mu-a-n)(1-\psi)}}\right]\widetilde{P}_t-\left[\tfrac{\lambda}{e^{(\mu-a-n)(1-\psi)}}\right]\widetilde{P}_{t-1}$$

$$=\tfrac{(e^{(\mu-a-n)(1-\psi)}-\lambda)(1-\beta\lambda e^{\mu})}{e^{(\mu-a-n)(1-\psi)}}[\alpha\widetilde{r}_t+(1-\alpha)\widetilde{w}_t-(1-\alpha)z_t]$$

Defining $RMC_t\equiv\alpha\widetilde{r}_t+(1-\alpha)\widetilde{w}_t-(1-\alpha)z_t$

$$-\beta\lambda e^{\mu}\widetilde{P}_{t+1}+\left[1+\tfrac{\beta\lambda^2 e^{\mu}}{e^{(\mu-a-n)(1-\psi)}}-\tfrac{e^{(\mu-a-n)(1-\psi)}-\lambda}{e^{(\mu-a-n)(1-\psi)}}\right]\widetilde{P}_t-\left[\tfrac{\lambda}{e^{(\mu-a-n)(1-\psi)}}\right]\widetilde{P}_{t-1}=\tfrac{(e^{(\mu-a-n)(1-\psi)}-\lambda)(1-\beta\lambda e^{\mu})}{e^{(\mu-a-n)(1-\psi)}}RMC_t$$

Working on the coefficient on $\widetilde{P}_t$:

$$\tfrac{e^{(\mu-a-n)(1-\psi)}}{e^{(\mu-a-n)(1-\psi)}}+\tfrac{\beta\lambda^2 e^{\mu}}{e^{(\mu-a-n)(1-\psi)}}-\tfrac{(e^{(\mu-a-n)(1-\psi)}-\lambda)(1-\beta\lambda e^{\mu})}{e^{(\mu-a-n)(1-\psi)}}$$

$$\tfrac{e^{(\mu-a-n)(1-\psi)}}{e^{(\mu-a-n)(1-\psi)}}+\tfrac{\beta\lambda^2 e^{\mu}}{e^{(\mu-a-n)(1-\psi)}}-\tfrac{(e^{(\mu-a-n)(1-\psi)}-\lambda-\beta\lambda e^{\mu}e^{(\mu-a-n)(1-\psi)}+\beta\lambda^2 e^{\mu})}{e^{(\mu-a-n)(1-\psi)}}$$

$$\tfrac{\lambda+\beta\lambda e^{\mu}e^{(\mu-a-n)(1-\psi)})}{e^{(\mu-a-n)(1-\psi)}}$$

$$-\beta\lambda e^{\mu}\widetilde{P}_{t+1}+\left[\tfrac{\lambda+\beta\lambda e^{\mu}e^{(\mu-a-n)(1-\psi)})}{e^{(\mu-a-n)(1-\psi)}}\right]\widetilde{P}_t-\left[\tfrac{\lambda}{e^{(\mu-a-n)(1-\psi)}}\right]\widetilde{P}_{t-1}=\tfrac{(e^{(\mu-a-n)(1-\psi)}-\lambda)(1-\beta\lambda e^{\mu})}{e^{(\mu-a-n)(1-\psi)}}RMC_t$$

$$\tfrac{\lambda}{e^{(\mu-a-n)(1-\psi)}}(\widetilde{P}_t-\widetilde{P}_{t-1})=\beta\lambda e^{\mu}(\widetilde{P}_{t+1}-\widetilde{P}_t)-\tfrac{(e^{(\mu-a-n)(1-\psi)}-\lambda)(1-\beta\lambda e^{\mu})}{e^{(\mu-a-n)(1-\psi)}}RMC_t$$

$$(\widetilde{P}_t - \widetilde{P}_{t-1}) = e^{(\mu-a-n)(1-\psi)}\beta e^{\mu}(\widetilde{P}_{t+1} - \widetilde{P}_t) - \tfrac{1}{\lambda}(e^{(\mu-a-n)(1-\psi)} - \lambda)(1-\beta\lambda e^{\mu})RMC_t \tag{6.9'}$$

Or defining inflation as $\widetilde{\pi}_t \equiv \widetilde{P}_t - \widetilde{P}_{t-1}$

$$\widetilde{\pi}_t = \beta e^{(\mu-a-n)(1-\psi)+\mu}\widetilde{\pi}_{t+1} - \tfrac{1}{\lambda}(e^{(\mu-a-n)(1-\psi)} - \lambda)(1-\beta\lambda e^{\mu})[\alpha\widetilde{r}_t + (1-\alpha)\widetilde{w}_t - (1-\alpha)z_t]$$

$$\widetilde{M}_{t+1} = g_t + \widetilde{M}_t \tag{6.3}$$

$$\widetilde{c}_t = g + \widetilde{M}_t - \widetilde{P}_t \tag{6.4}$$

$$\overline{h}\widetilde{h}_t = \tfrac{\overline{K}e^{a+n}}{\overline{w}}(\widetilde{K}_{t+1} - \widetilde{w}_t) + \tfrac{\overline{M}e^{\mu}}{\overline{w}\overline{P}}(\widetilde{M}_{t+1} - \widetilde{w}_t - \widetilde{P}_t) + \tfrac{\overline{K}}{\overline{w}}\overline{r}\widetilde{r}_t \tag{6.5}$$

$$\widetilde{w}_t - \widetilde{w}_{t+1} + \overline{r}\widetilde{r}_{t+1} = 0 \tag{6.6}$$

$$-\gamma\widetilde{c}_{t+1} + \widetilde{w}_t + \widetilde{P}_t - \widetilde{P}_{t+1} = 0 \tag{6.7}$$

$$\widetilde{Y}_t = \int_0^1 \overline{Y}\widetilde{Y}_t(i)\,di \tag{6.8}$$

$$\widetilde{Y}_t(i) = \alpha\widetilde{K}_t(i) + (1-\alpha)[z_t + \widetilde{H}_t(i)] \tag{6.10}$$

$$\widetilde{K}_t(i) = \widetilde{Y}_t(i) + (1-\alpha)(\widetilde{w}_t - \widetilde{r}_t - z_t) \tag{6.12}$$

$$\widetilde{H}_t(i) = \widetilde{Y}_t(i) + \alpha(\widetilde{r}_t + z_t - \widetilde{w}_t) \tag{6.13}$$

$$\widetilde{K}_t = \int_0^1 \overline{K}\widetilde{K}_t(i)\,di \tag{6.14}$$

$$\widetilde{h}_t = \int_0^1 \overline{h}\widetilde{H}_t(i)\,di \tag{6.15}$$

$$\overline{I}\widetilde{I}_t = e^{a+n}\overline{K}\widetilde{K}_{t+1} - (1-\delta)\overline{K}\widetilde{K}_t \tag{6.16}$$

$$\widetilde{A}_t = z_t \tag{6.17}$$

We can remove $\widetilde{Y}_t(i), \widetilde{K}_t(i)$ & $\widetilde{H}_t(i)$ from the system as follows.

Linearizing stationary versions of (1.7) and (1.8):

$$\widetilde{Y}_t(i) = \psi\widetilde{P}_t - \psi\widetilde{P}_t(i) + \widetilde{Y}_t \tag{6.18}$$

$$\widetilde{P}_t = \int_0^1 \overline{P}\widetilde{P}_t(i)\,di \tag{6.19}$$

Using (6.12) with (6.18):

$$\widetilde{K}_t(i) = \psi\widetilde{P}_t - \psi\widetilde{P}_t(i) + \widetilde{Y}_t + (1-\alpha)(\widetilde{w}_t - \widetilde{r}_t - z_t)$$

Substituting this into (6.14):

$$\widetilde{K}_t = \int_0^1 \overline{K}\Big[\psi\widetilde{P}_t - \psi\widetilde{P}_t(i) + \widetilde{Y}_t + (1-\alpha)(\widetilde{w}_t - \widetilde{r}_t - z_t)\Big]di$$

$$\widetilde{K}_t = \overline{K}[\widetilde{Y}_t + (1-\alpha)(\widetilde{w}_t - \widetilde{r}_t - z_t)] \tag{6.14'}$$

Similarly:

$$\widetilde{h}_t = \widetilde{Y}_t + \alpha(\widetilde{r}_t + z_t - \widetilde{w}_t) \tag{6.15'}$$

Combining (6.8) & (6.10):

$$\widetilde{Y}_t = \int_0^1 \overline{Y}\{\alpha\widetilde{K}_t(i) + (1-\alpha)[z_t + \widetilde{H}_t(i)]\}\,di$$

$$\widetilde{Y}_t = \overline{Y}[\alpha \widetilde{K}_t + (1-\alpha) z_t + (1-\alpha)\widetilde{h}_t] \tag{6.8'}$$

Since we have used (6.8) which was our redundant equation by Walras Law, we need to drop (6.5) instead.

The system is defined by the 11 equations: (2.1), (2.2), (6.3), (6.4), (6.6), (6.7), (6.8'), (6.9'), (6.14'), (6.15') & (6.16).
The 11 variables are $z_t, g_t, \widetilde{Y}_t, \widetilde{K}_t, \widetilde{h}_t, \widetilde{r}_t, \widetilde{w}_t, \widetilde{P}_t, \widetilde{c}_t, \widetilde{M}_t, \widetilde{I}_t$.