```
Clear["`*"]
```

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Physics 321

Homework Due 9-14-12

P5.9

```
m = 2; α = 2000; g[t_] :=36 Cos[ω t];
```

The question didn't say anything about a resistive force, so the equation of motion can be written in the form of equation (5.14), without the 2mK:

$$m\ddot{x} + 2000\,x = 36\cos\omega t$$

We can let *Mathematica* solve this for us:

```
eqn = m x''[t] + 2000 x[t] == g[t];
DSolve[eqn, x[t], t]//FullSimplify
```

$$\left\{\left\{x(t) \to c_2 \sin\left(10\sqrt{10}\ t\right) + c_1 \cos\left(10\sqrt{10}\ t\right) - \frac{18\cos(t\,\omega)}{\omega^2 - 1000}\right\}\right\}$$

We know that the general solution of this equation will be the homogeous solution plus the driven response (x(t) = $x_h + x_d$). The homogenous solution (x_h) is equal to $c_2 \sin\left(10\sqrt{10}\ t\right) + c_1 \cos\left(10\sqrt{10}\ t\right)$ so that means $x_d = -\frac{18\cos(\omega t)}{\omega^2-1000} = \frac{18\cos(\omega t)}{1000-\omega^2}$.

We care more about the amplitude of the soloution and that is equal to $\bar{x} = \frac{18}{|1000-\omega^2|}$.

Becuase the mass is hanging, the total extension of the spring will be the sum of the displacement due to gravity acting on the block and the displacement caused by the driving force. With a spring constant $\alpha = 2000$, the extension of the spring in equilibirium is $\frac{20}{2000} = \frac{1}{100}\,m$. If we add that to the amplitude of the driven response we get the total amplitude of the extension:

$$\bar{x} = \frac{1}{100}\,m + \frac{18}{|\,1000 - \omega^2\,|}\,m$$

The description of the problem said the spring will be OK if $\bar{x} \leq 0.04\,m$. If we put that in we get the following:

$$0.04 \geq 0.01 + \frac{18}{|\,1000 - \omega^2\,|} \to 0.03 \geq \frac{18}{|\,1000 - \omega^2\,|}$$

Because of the absolute value we have to look at two cases: one where $\omega^2 < 1000$ and one where it is > 1000.

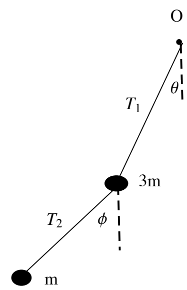
With $\omega^2 > 1000$ we have $0.03 \geq \frac{18}{\omega^2-1000} \to 18 \leq 0.03\,(\omega^2 - 1000) \to 600 \leq \omega^2 - 1000 \to 1600 \leq \omega^2 \to \boldsymbol{\omega \geq 40\ \frac{\textbf{rad}}{\textbf{sec}}}$

With $\omega^2 > 1000$ we have $0.03 \geq \frac{18}{1000-\omega^2} \to 18 \leq 0.03\,(1000 - \omega^2) \to 600 \leq 1000 - \omega^2 \to 400 \leq \omega^2 \to \boldsymbol{\omega \leq 20\ \frac{\textbf{rad}}{\textbf{sec}}}$

If either of the two conditions in bold are met, the spring is safe. □

P5.18

This is a double pendlum that can be represented as follows:



Deriving the linear approximation.

We can let x be the horizontal displacement from the line directly below O, adn y can be the vertical component of the forces.

We then use Newton's 2nd law to get the following equations.

$$m_1 \ddot{x}_1 = 3 m \ddot{x_1} = -T_1 \sin\theta + T_2 \sin\phi$$

$$m_2 \ddot{x}_2 = m \ddot{x_2} = -T_2 \sin\phi$$

$$m_1 \ddot{y}_1 = 3 m \ddot{y}_1 = T_1 \cos\theta - T_2 \cos\phi - 3 m g$$

$$m_1 \ddot{y}_2 = m \ddot{y}_2 = T_2 \cos\phi - m g$$

Using a linear approximation, we can ignore any terms of order 2 or more. The small angle approximation allows us to say that sin x = x and cos x = 1. We also know that $x_1 = a\theta$ and $x_2 = a(\theta + \phi)$, because of the small angle approximation and the length of the strings both being a. We can solve for equilibirum values of T_1 and T_2 by setting the equaitons in y equal to zero and solving for T_1 and T_2. We do this below

$$T_1 \cos\theta - T_2 \cos\phi - 3 m g = 0 \to T_1 - T_2 - 3 m g = 0$$

$$T_2 \cos\phi - m g \to T_2 - \text{mg} = 0$$

We can then solve for T_1 and T_2 under these approximations and get $T_2 = \text{mg}$ and $T_1 = 4\,\text{mg}$.

Using this and eventually substituting $n^2 = g/a$ we get the following:

$$3 m a \ddot{\theta} = -T_1 \theta + T_2 \phi \to 3\ddot{\theta} + 4\frac{g}{a}\theta - \frac{g}{a}\phi = 0 \to \boldsymbol{3\ddot{\theta} + 4 n^2 \theta - n^2 \phi = 0}$$

$$m a\left(\ddot{\theta} + \ddot{\phi}\right) = -T_2 \phi \to \ddot{\theta} + \ddot{\phi} + \frac{g}{a}'\phi = 0 \to \boldsymbol{\ddot{\theta} + \ddot{\phi} + n^2 \phi = 0}$$

Solving for the normal modes

Equation (5.28) tells us that these functions have a general solution of the form $x_1 = A\cos(\omega t - \gamma)$ and $x_2 = B\sin(\omega t - \gamma)$. The second derivatives would then $\ddot{x}_1 = -\omega^2 A\cos(\omega t - \gamma)$ and $\ddot{x}_2 = -\omega^2 B\sin(\omega t - \gamma)$. Using this the general equations are:

$$-3\omega^2 A\cos(\omega\text{t} - \gamma) + 4 n^2 A\cos(\omega\text{t} - \gamma) - n^2 B\sin(\omega\text{t} - \gamma) = 0$$

$$-\omega^2 B\sin(\omega\text{t} - \gamma) - -\omega^2 A\cos(\omega\text{t} - \gamma) + n^2 B\sin(\omega\text{t} - \gamma) = 0$$

We can again make the small angle approximations so that cos x = 1 and sin x = x. The above equations then simplify to

$$\left(4 n^2 - 3\omega^2\right) A - n^2 B = 0 \tag{0.1}$$

$$-\omega^2 A + \left(n^2 - \omega^2\right) B = 0 \tag{0.2}$$

Following the process on pages 122-123 we get a non trivial solution when $\det\begin{pmatrix} (4 n^2 - 3\omega^2) & -n^2 \\ -\omega^2 & (n^2 - \omega^2) \end{pmatrix} = 0$

Mathematica can simplify the algebra for us:

```
sol = Det[( (4n^2 - 3ω^2)    - n^2
              -ω^2        (n^2 - ω^2) ) ]
```

$4 n^4 - 8 n^2 \omega^2 + 3 \omega^4$

We can solve this for ω and get :

```
ans = ω/.Solve[sol ==0 , ω];
normalModes = {ans[[2]], ans[[4]]};
Print["The two normal modes are:", normalModes]
```

The two normal modes are:$\left\{\sqrt{\frac{2}{3}}\ n, \sqrt{2}\ n\right\}$

To solve for the forms at these modes we look at the one at a time. The fast modes oscilate with frequency $\omega^2 = 2\, n^2$.

Substituting this into the equations (0.1, 0.2) we get:

$$\left(4 n^2 - 3\left(2 n^2\right)\right) A - n^2 B = 0 \to -2 n^2 A - n^2 B = 0 \to A = -B/2$$

This corresponds to coefficients of A = ξ and B = -2 ξ so that the equations become (note we use the form in the first line under the section “Solving for the normal modes”):

$$x_1 = \xi \cos\left(\sqrt{2}\ n t - \gamma\right)$$

$$x_2 = -2 \xi \sin\left(\sqrt{2}\ n t - \gamma\right)$$

We can do the same for the slow modes and get:

$$\left(4 n^2 - 3\left(\frac{2}{3} n^2\right)\right) A - n^2 B = 0 \to 2 n^2 A - n^2 B = 0 \to A = B/2$$

This corresponds to coefficients of ξ and 2 ξ. The solution is then:

$$x_1 = \xi \cos\left(\sqrt{\frac{2}{3}}\ n t - \gamma\right)$$

$$x_2 = B \xi \sin\left(\sqrt{\frac{2}{3}}\ n t - \gamma\right)$$

P6.1

```
m = 4 (* kg *); F = {4, 12 t^2,0}; v0 = {2, 1, 2};
```

We are given the force acting on our particle and the initial velocity. We can use Newton’s second law to solve for the velocity.

$$4 \frac{\partial \boldsymbol{v}}{\partial t} = 4\, i + 12 t^2 j$$

$$\partial \boldsymbol{v} = \left(i + 3 t^2 j\right) \partial t$$

$$\boldsymbol{v} = t\, i + t^3 j + \boldsymbol{C}$$

$$\boldsymbol{v}(0) = \{2, 1, 2\} = \{0, 0, 0\} + \boldsymbol{C} \to \boldsymbol{C} = \{2, 1, 2\}$$

$\boldsymbol{v} = (t + 2)\, i + \left(t^3 + 1\right) j + 2\, \boldsymbol{k}$

We can now let mathematica know about the velocity function:

```
v = {2 + t, t^3 + 1, 2};
```

Equation (6.4) tells us that W = $\int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v}\, dt$.

```
Integrate[Dot[F, v], {t, 0, 1}]
```

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We also know that the kinetic energy at any given time *t* is given by T = $\frac{1}{2} m \left| \boldsymbol{v} \right|^2$.

The velocity at t = 0 is {2, 1, 2}. So T(0) = $\frac{1}{2} 4 (2^2 + 1^1 + 2^2) = 2 * (9) = 18$.

The velocity at *t* = 1 is {3, 2, 2}. So T(1) = $\frac{1}{2} 4 (3^2 + 2^2 + 2^2) = 2 * (17) = 34$.

The change in kinetic energy it T(1) - T(0) = 34 - 18 = 16.

This is the same as the work done, thus illustrating the princple of conservation of energy.