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Physics 441: Assignment #4 - Potentials

Due on Friday, May 31, 2013

May 24, 2013

Problem 3.3

Find the general solution to Laplace's equation in spherical coordinates, for the case where V depends only on r. Do the same for cylindrical coordinates, assuming V depends only on s

Laplace's equation in spherical coordinates takes the following form:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2} = 0$$

The problem says that V only depends on r, so we can simplify this to parts that only contain derivatives in r. In this case, Laplace's equation becomes:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

We can cancel out the leading $\frac{1}{r^2}$ (because the left hand side is 0) and get

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

which simplifies down to

$$\left(r^2 \frac{\partial V}{\partial r}\right) = c \rightarrow \frac{\partial V}{\partial r} = \frac{c}{r^2} \rightarrow V = -\frac{c}{r} + k$$

Laplace's equation in cylindrical coordinates is

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Doing the same and keeping only terms with derivatives in s we simplify to

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) = 0$$

I again multiply both sides by s to arrive at the following expression, which I simplify to get the final result:

$$\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) = 0 \rightarrow s\frac{\partial V}{\partial s} = c \rightarrow \frac{\partial V}{\partial s} = \frac{c}{s} \rightarrow V = c\ln s + k$$

□

Problem 3.10

A uniform line charge λ is placed on an infinite straight wire, a distance d above a grounded conducting plane. (Let's say the wire runs parallel to the x-axis and directly above it, and the conducting plane is the xy plane.)

1. Find the potential in the region above the plane [HINT: refer to problem 2.52]
2. Find the charge density σ induced on the conducting plane

1. We know that energy (potential in parenthesis) is a function of $\hat{\boldsymbol{r}} r^2$ $(-\hat{\boldsymbol{r}} r)$ in 3d and $\hat{\boldsymbol{r}}/r$ $(\ln r \hat{\boldsymbol{r}})$ in 3d. Our problem is a 2-d problem (we are dealing with a line). Using that, and remembering we pick up factor of $2\pi\varepsilon_0$ from integrating $E \rightarrow V$ we know that the general formula for potential in 2d is the following.

$$\begin{aligned} V(\boldsymbol{s}) &= \frac{\lambda}{2\pi\varepsilon_0} ln(\boldsymbol{a}/\boldsymbol{s}) \\ &= \frac{\lambda}{4\pi\varepsilon_0} ln(\boldsymbol{a}^2/\boldsymbol{s}^2) \end{aligned}$$

   In this case we say that $\boldsymbol{a} = -\boldsymbol{s}$ (it is an image problem!). We also say that $\boldsymbol{a}$ is the distance away from the x-axis in the xy plane. An expression for this is $\boldsymbol{a} = y + (z + d)$. We can now apply this to our expression for the potential energy to get the answer:

$$V(y,z) = \frac{\lambda}{4\pi\varepsilon_0} ln\left(\frac{y^2 + (z+d)^2}{y^2 + (z-d)^2}\right)$$

2. Now we need to find σ on the plane $(z = 0)$. We will use the equation at the top of page 126:

$$\sigma = -\varepsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0}$$

   We apply this equation and simplify to get $\sigma(y)$ (Note I let the computer do the algebra for me and I have included the code below)

$$\begin{aligned}
\sigma(y) &= -\varepsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0} \\
&= \left.\frac{\lambda\left(y^2+(-d+z)^2\right)\left(\frac{(2d-2z)\left(y^2+(d+z)^2\right)}{\left(y^2+(-d+z)^2\right)^2}+\frac{2d+2z}{y^2+(-d+z)^2}\right)}{4e_0\pi\left(y^2+(d+z)^2\right)}\right|_{z=0} \\
&= -\frac{d\lambda}{\pi\left(d^2+y^2\right)}
\end{aligned}$$

```
import sympy as sym

y, z, d, lamb, pi, e0 = sym.symbols('y, z, d, lamb, pi, e0 ')

expr = lamb / (4 * pi * e0) * sym.log((y**2 + (z + d) **2) / (y**2 + (z-d)**2))

print('\\ve_0 * dv/dz: ')
sym.pprint(-e0 * expr.diff(z))
print(sym.latex(-e0 * expr.diff(z)))

print('\n\n\n\n \\ve_0 * dv/dz at z=0: ')
sym.pprint(-e0 * expr.diff(z).subs({z:0}))
print(sym.latex(-e0 * expr.diff(z).subs({z:0})))
```

□

Problem 3.14

For the infinite slot (see example 3.3), detemrine the charge density $\sigma(y)$ on teh strip at $x = 0$, assuming it is a conductor at constant potential V_0

Problem 3.19

The potential at the surface of a sphere (radius R) is given by

$$V_0 = k\cos 3\theta$$

where k is a constant. Find the potential inside and outside the sphere, as well as the surface charge density $\sigma(\theta)$ on the sphere. (assume there is not charge inside or outside the sphere.)

Problem 3.24

Solve Laplace's equation by separation of variables in cylindrical coordinates, assuming there is no depndence on z (cylindrical symmetry). [Make sure you find all solutions to the radial equation; in particular, your results must accommodate the case of an infinite line charge, for which (of course) we already know the answer.]

Problem 3.24 continued on next page…

Problem 3.29

For particles (one of charge q, one of charge $3q$, and two of charge $-2q$) are placed at the following points:

- $q \rightarrow (0,0,-1)$
- $3q \rightarrow (0,0,1)$
- $-2q \rightarrow (0,-1,0)$
- $-2q \rightarrow (0,1,0)$

Find a simple approximate formula for the potential, valid at points far from the origin (Express the answer in spherical coordinates)

Problem 3.31

For the dipole in example 3.10, expand $1/r_{\pm}$ to order $(d/r)^3$, and use this to determine the quadropole and octopole terms in the potential.