A 3rd Real Business Cycle Model

Major Features of the Model

Add a labor-leisure decision with indivisible labor hours to model 1
One source of uncertainty: z
Stochastic technology growth about a deterministic trend

We first define the important terms that will be used throughout:
Endogenous variables that change over time:

z	productivity (temporary or permanent)
K	capital stock owned by households
H	labor supplied by households
C	consumption
w	wage rate
r	interest rate
Y	output of final goods

Parameters:

α	capital share in output from a Cobb-Douglas production function
δ	rate of depreciation
β	time discount factor; $\beta<1$
a	trend in z
γ	elasticity of substitution, $\gamma>0$
B	leisure weight in utility
ρ	autocorrelation parameter for z; $0<\rho<1$
σ	standard deviations of the shocks to z; $0<\sigma$
H_0	hours worked by household that have a job

Nonstationary Model

Households *ex ante* enter a labor lottery with probability π of working H_0 hours. Consumption is fully insured, so all households have the same consumption. Households that "win" the lottery work, those that lose do not work.

Using the same utility function as before,
$u(C,1-H)=\frac{1}{1-\gamma}(C^{1-\gamma}-1)+B\frac{1}{1-\gamma}\{[e^{at}(1-H)]^{1-\gamma}-1\}$. The *ex ante* expected utility is

$$u(C,1-H)=\frac{1}{1-\gamma}(C^{1-\gamma}-1)+\pi B\frac{1}{1-\gamma}\{[e^{at}(1-H_0)]^{1-\gamma}-1\}+(1-\pi)B\frac{1}{1-\gamma}\{[e^{at}(1-0)]^{1-\gamma}-1\}$$

$$u(C,1-H)=\frac{1}{1-\gamma}(C^{1-\gamma}-1)+B\frac{1}{1-\gamma}\{e^{(1-\gamma)at}[\pi(1-H_0)^{1-\gamma}+(1-\pi)]-1\}$$

$$u(C,1-H)=\frac{1}{1-\gamma}(C^{1-\gamma}-1)+\pi e^{(1-\gamma)at}B[(1-H_0)^{1-\gamma}-1]+B\frac{1}{1-\gamma}[e^{(1-\gamma)at}-1]$$

In equilibrium all households enter the lottery, so $H=\pi H_0$. Households choose the level of consumption and whether to enter the lottery or not. The expected utility from entering the lottery must be greater than or equal to the utility from not entering. Since all households are identical, they must all enter the lottery in order for any labor to be supplied at all. If expected utility from entering the lottery is strictly greater than that from not entering, firms can reduce the wage and get the same amount of labor. In equilibrium the households will be indifferent between entering the lottery or not. The effect of this in the aggregate is that households collectively behave as if they were choosing the probability of winning the lottery. Since $\pi = \frac{H}{H_0}$ this is the same as choosing the aggregate hours worked, H.

Given information on prices and shocks, $\mathbf{\Omega} = \{w, r, z\}$, the household solves the following non-linear program when the factor markets clear.

$$V(K, \mathbf{\Omega}) = \underset{K', H}{Max} \frac{1}{1-\gamma}(C^{1-\gamma} - 1) + He^{(1-\gamma)at}\widetilde{D} + \widetilde{F}e^{(1-\gamma)at} - \widetilde{F} + \beta E\{V(K', \mathbf{\Omega}')\}$$

where $\widetilde{D} \equiv \frac{1}{H_0} D[(1-H_0)^{1-\gamma} - 1] < 0$, $\widetilde{F} \equiv D\frac{1}{1-\gamma}$, and

$$C = wH + (1-\delta + r)K - K' \tag{1.1}$$

The first-order conditions are:

$$C^{-\gamma}(-1) + \beta E\{V_K(K', \Omega')\} = 0$$

$$C^{-\gamma}w + e^{(1-\gamma)at}\widetilde{D} = 0$$

The envelope condition from this problem is as follows.

$$V_K(K, \Omega) = C^{-\gamma}(1-\delta + r)$$

The Euler equations are:

$$C^{-\gamma} = \beta E\{C'^{-\gamma}(1-\delta + r')\} \tag{1.2}$$

$$C^{-\gamma}w = -e^{(1-\gamma)at}\widetilde{D} \tag{1.3}$$

Additional Behavioral Equations

The law of motion for z is:

$$z' = \rho z + \varepsilon'; \text{ where } \varepsilon' \text{ is distributed normal with a mean of 0 and a variance of } \sigma^2 \tag{1.4}$$

An assumption of competition in the goods market along with a Cobb-Douglas production function gives the following shares in output for labor & capital. Market clearing conditions have already been imposed.

$$Y = K^{\alpha}(e^{at+z}H)^{1-\alpha} \tag{1.5}$$

$$wH = (1-\alpha)Y \tag{1.6}$$

$$rK = \alpha Y \tag{1.7}$$

Definitions for Later Use

$$I \equiv K' - (1-\delta)K \tag{1.8}$$

$$A \equiv e^{at+z} \tag{1.9}$$

Eqs (1.1)-(1.9) are the system.

Transformation & Simplifications

If z is stationary ($\rho < 1$):

Transform the problem by dividing all growing variables by $A \equiv e^{at}$, denoting with a carat.

$z' = \rho z + \varepsilon'$; where ε' is distributed normal with a mean of 0 and a variance of σ^2 (2.1)

$$\hat{C} = \hat{w}H + (1-\delta+r)\hat{K} - (1+a)\hat{K}' \tag{2.2}$$

$$1 = \beta E\left\{\left(\frac{\hat{C}}{(1+a)\hat{C}'}\right)^{\gamma}(1-\delta+r')\right\} \tag{2.3}$$

$$\hat{C}^{-\gamma}\hat{w} = \widetilde{D} \tag{2.4}$$

$$\hat{Y} = \hat{K}^{\alpha}(e^{z}H)^{1-\alpha} \tag{2.5}$$

$$\hat{w}H = (1-\alpha)\hat{Y} \tag{2.6}$$

$$r\hat{K} = \alpha\hat{Y} \tag{2.7}$$

$$\hat{I} = (1+a)\hat{K}' - (1-\delta)K \tag{2.8}$$

$$\hat{A} \equiv e^{z} \tag{2.9}$$

These are the equations we will use in Dynare.

The endogenous variables are $\hat{C}, \hat{K}, H, \hat{Y}, \hat{w}, r, \hat{I}, \hat{A}$ & z .

The exogenous variable is ε.

The parameters are $\alpha, \delta, \beta, a, \gamma, \rho, \sigma, D$ & H_0 .