

Laplace's Equation in 2-d using Complex Variable

We start by identifying a 2-d physical space with a complex plane and taking a one-to-one correspondence between vectors (x, y) and complex numbers $x + i y$.

A complex function $w = w(z)$ of a complex variable z maps the complex z -plane onto the complex w -plane. Separating z and w into real and imaginary parts:

$$z = x + i y \tag{1}$$

$$w = u + i v \tag{2}$$

gives u and v as functions of x and y :

$$u = u(x, y) \tag{3}$$

$$v = v(x, y) \tag{4}$$

The very existence of the derivative dw/dz imposes strict relations between the partial derivatives of u and v with respect to x and y , called the Cauchy-Riemann relations. These, in turn, imply a very important property of the real and imaginary parts of w : they are both solutions of Laplace's equation in two dimensions.

Lemma: The existence of a well-defined complex derivative dw/dz implies the Cauchy-Riemann conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{5}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{6}$$

The proof follows from the fact that the derivative must have the same value for any direction in the complex plane as we take the limit in the derivative. So let us write the complex derivative as:

$$\frac{dw}{dz} = \frac{du + i dv}{dx + i dy} = \frac{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx} + i \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \frac{dy}{dx} \right)}{1 + i \frac{dy}{dx}}. \tag{7}$$

Equating dw/dz for the case $dy/dx = 0$ and for the case $dx/dy = 0$:

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}, \quad (8)$$

from which we obtain the Cauchy-Riemann conditions.

Theorem: The functions $u = u(x, y)$ and $v = v(x, y)$ obeying Cauchy-Riemann relations are solutions to Laplace's equation in two dimensions.

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) - \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial x} \right) = 0, \quad (9)$$

with a similar derivation for $v(x, y)$.

One more theorem proves to be important for the physical interpretation of the contours corresponding to $u = \text{const.}$ and $v = \text{const.}$, namely,

Theorem: $\nabla u \cdot \nabla v = 0$.

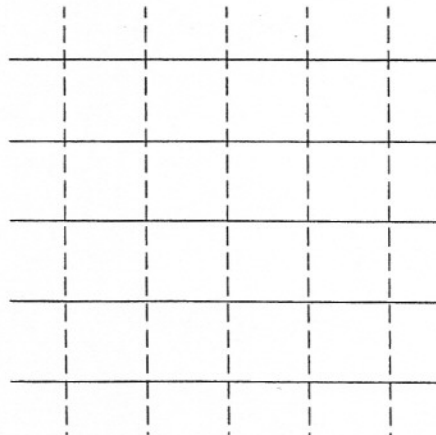
This follows again from the Cauchy-Riemann conditions, as can be easily verified.

EXAMPLES

1.- Ideal Capacitor.

Choose $w(z) = z$.

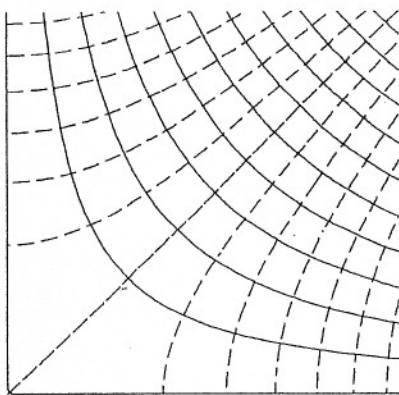
We can take $V(x, y) = \text{Im } w = y$ as a solution to Laplace's equation. The contours corresponding to equipotential lines are vertical lines in the $x - y$ plane equally spaced. We could now place a conductor along any equipotential line with a fixed value of the potential V_0 . Or two of them to form a 2-d ideal capacitor. The contours $u = \text{Re } w = \text{const.}$ are orthogonal to the equipotential contours and hence follow the electric field lines.



2.- Two Perpendicular Conducting Lines.

Take $w(z) = z^2$.

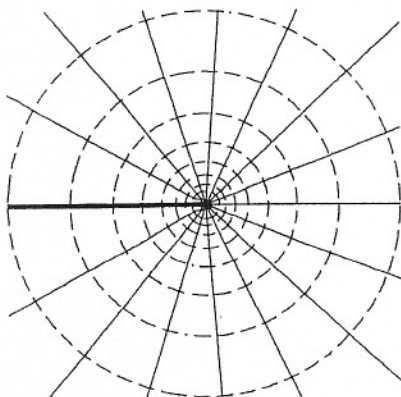
It is convenient to choose $V(x, y) = \text{Im } w = 2xy$ as the solution to Laplace's equation. The equipotential contours are hyperbolas in the x - y plane. In particular, their asymptotes, the semiaxes $x > 0$ and $y > 0$ correspond to the equipotential $V = 0$. We can place a conductor along these two perpendicular lines with vanishing potential. The hyperbolas corresponding to $u = \text{Re } w = \text{const.}$ are orthogonal to the equipotential contours and follow the electric field lines.



3.- Point Charge.

A singularity at the origin corresponding to a point charge can be accomplished by choosing $w(z) = \ln z$.

In this case we look for the real part of w , so that $V(x, y) = \text{Re } w = \ln s$ (in polar coordinates) is the solution to Laplace's equation. The contours corresponding to equipotential lines are concentric circles centered at the origin. The field lines $v = \text{Im } w = \text{const.}$ are radial lines with $\theta = \text{const.}$

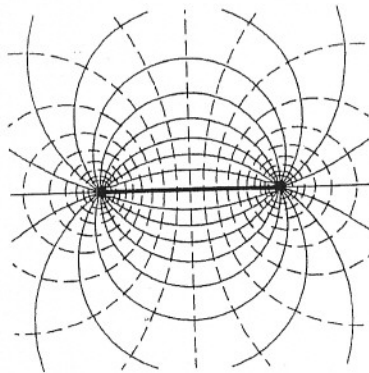


4.- Electric Dipole.

Using Example 3, we combine two point particles with opposite charges along the x -axis at $x = \pm 1/2$,

$$w(z) = \ln\left(\frac{z + 1/2}{z - 1/2}\right), \quad (10)$$

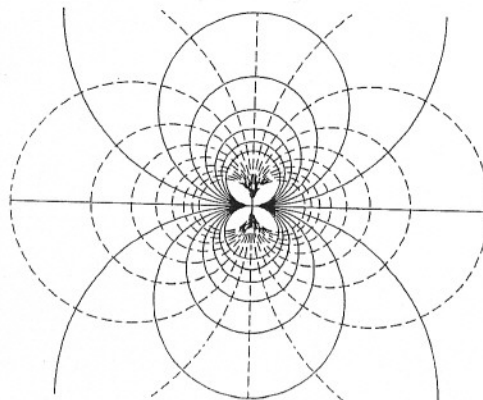
to obtain a finite dipole.



5.- Point Dipole.

Let us now look at a negative power: $w(z) = 1/z = e^{-i\phi}/s$.

We choose $V(x, y) = \operatorname{Re} w = \cos \phi / s$ as the appropriate solution to Laplace's equation. The equipotential contours are pairs of vertical circles passing through the origin. This solution can be thought of as the limit of a (finite) dipole as the two charges approach each other, keeping the dipole moment at a fixed value. The horizontal circles corresponding to $u = \operatorname{Im} w = \text{const.}$ follow the electric field lines.



6.- Expulsion of Electric field from a Circular Conductor.

Choose $w(z) = (z + 1/z)$.

We can take $V(x, y) = \text{Im}w = (s - 1/s) \sin \phi$ as the solution to Laplace's equation. The equipotential contour $V = 0$ (and the corresponding grounded conductor) is a unit circle centered at the origin. The electric field approaches a constant far away from the unit circle, but vanishes in the interior.

