GEOMETRIC ALGEBRAS IN E.M.

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Physical Applications:

- Electro-magnetostatics
- Dispersion and diffraction E.M.

Examples of geometric algebras

- Complex Numbers $\mathcal{G}_{0,1} = \mathrm{C}$
- Hamilton Quaternions $\mathcal{G}_{0,2} = \mathrm{H}$
- Pauli Algebra $\mathcal{G}_{3}$
- Dirac Algebra $\mathcal{G}_{1,3}$

Extension of the vector space

- inverse of a vector: $\mathbf{A}^{-1}, \quad \nabla^{-1}$
- reflections, rotations, Lorentz transform.
- integrals: Cauchy (≥ 2 d), Stokes' theorem

Based on the idea of **MULTIVECTORS**:

$$\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3 \ldots$$

and including lengths and angles: $\mathbf{a}_1 \cdot \mathbf{a}_2$

Algebraic Properties of $\mathbb{R}$

$+$

- closure
- commutativity
- associativity
- zero
- negative

$\cdot$

- closure
- commutativity
- associativity
- unit
- reciprocal
- and $+$ distributivity

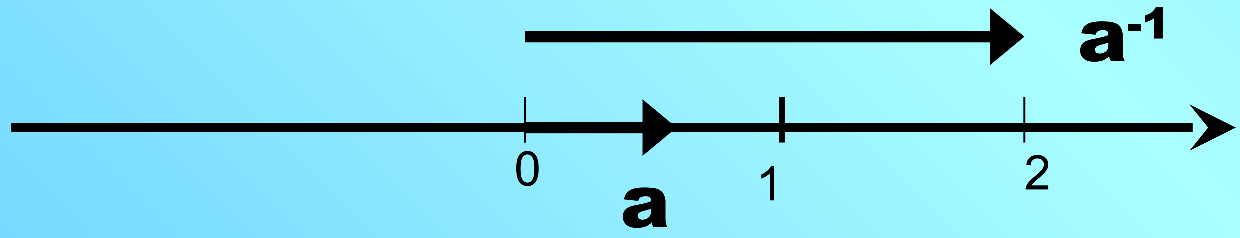
Vector spaces : linear combinations

Algebras

- include metrics: $\mathbf{a} \cdot \mathbf{b}$
- define geometric product

FIRST STEP: Inverse of a vector a ≠ 0

$$\mathbf{a}^{-1} = \frac{\hat{\mathbf{a}}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|^2} = \frac{\mathbf{a}}{a^2}, \qquad a^2 = \mathbf{a} \cdot \mathbf{a}$$



Orthonormal basis

3d: $\hat{\mathbf{e}}_1\ \hat{\mathbf{e}}_2\ \hat{\mathbf{e}}_3$

$\hat{\mathbf{e}}_i \bullet \hat{\mathbf{e}}_k = \delta_{ik}$

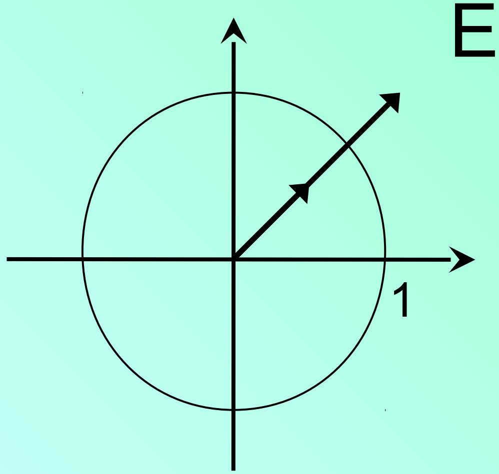
$\hat{\mathbf{e}}_i^{-1} = \hat{\mathbf{e}}_i$

Euclidian

4d: $\boldsymbol{\gamma}_0\ \boldsymbol{\gamma}_1\ \boldsymbol{\gamma}_2\ \boldsymbol{\gamma}_3$

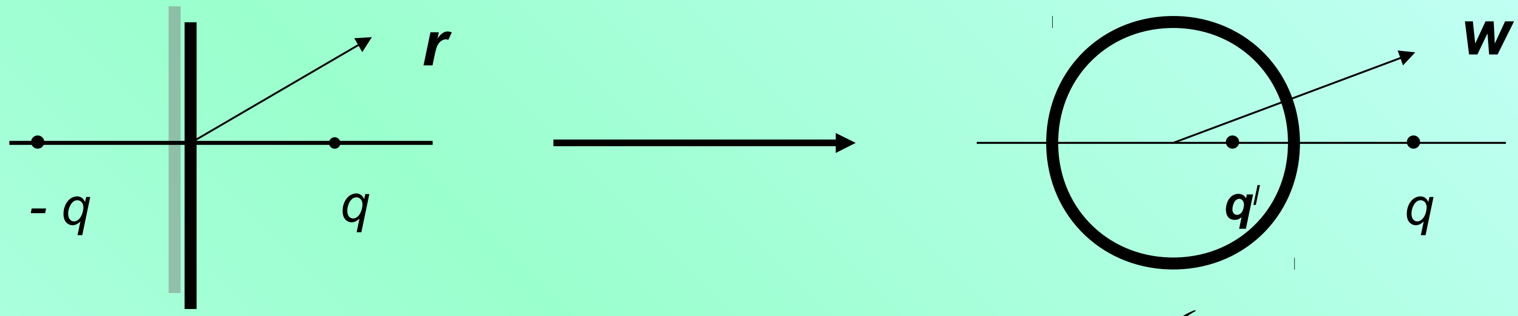
$\boldsymbol{\gamma}_\mu \bullet \boldsymbol{\gamma}_\nu = g_{\mu\nu} \Rightarrow (1,-1,-1,-1)$

Minkowski



Example: $(\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2)^{-1} = (\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2)/2$

Electrostatics: method of images



$$\mathbf{r} \to \mathbf{r} + \hat{\mathbf{e}}_1 \to \frac{1}{\mathbf{r} + \hat{\mathbf{e}}_1} \to \frac{1}{\mathbf{r} + \hat{\mathbf{e}}_1} - \frac{\hat{\mathbf{e}}_1}{2} \to 2\left(\frac{1}{\mathbf{r} + \hat{\mathbf{e}}_1} - \frac{\hat{\mathbf{e}}_1}{2} \right)$$

Plane: charge $-q$ at $c\hat{\mathbf{e}}_1$, image $(-q)$ at $-c\hat{\mathbf{e}}_1$

Sphere radius a, charge q at $b\hat{\mathbf{e}}_1$, find image (?)

r	**w**	charge
0	a	$V = 0$
c	b	q
$-c$	?	$q' = ?$

Choose scales for **r** and **w**:

$$\mathbf{w} = 2a\left(\frac{1}{\beta\mathbf{r}/c + \hat{\mathbf{e}}_1} - \frac{\hat{\mathbf{e}}_1}{2}\right)$$

$$V(\mathbf{w}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\left|\mathbf{w} - b\hat{\mathbf{e}}_1\right|} - \frac{a}{b}\frac{q}{\left|\mathbf{w} - a^2\hat{\mathbf{e}}_1/b\right|}\right]$$

SECOND STEP: EULER FORMULAS

$$e^{i\mathbf{a}} = \cos a + i\hat{\mathbf{a}} \sin a \qquad (i\hat{\mathbf{a}})^2 = -1$$

$$e^{\mathbf{b}} = \cosh b + \hat{\mathbf{b}} \sinh b \qquad \hat{\mathbf{b}}^2 = 1$$

In 3-d: $$e^{\theta \hat{\mathbf{k}}\times} \mathbf{A}_\perp = (\cos\theta + \sin\theta\ \hat{\mathbf{k}}\times)\mathbf{A}_\perp$$

given that $$\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{A}_\perp) = -\mathbf{A}_\perp$$

In general, rotating **A** about θ **k:**

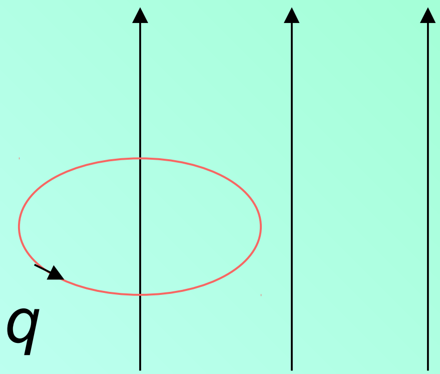
$$\boxed{e^{\theta \hat{\mathbf{k}}\times} \mathbf{A} = \mathbf{A}'}$$

Example: Lorentz Equation

$$\theta = -\omega t \qquad \mathbf{v}(t) = e^{-\omega t \hat{\mathbf{k}}\times}\mathbf{v}_0$$

$$\text{Derivative:} \quad \dot{\mathbf{v}}(t) = -\omega \hat{\mathbf{k}} \times \mathbf{v}(t)$$

$$\text{solves:} \quad m\dot{\mathbf{v}} = -qB\,\hat{\mathbf{k}} \times \mathbf{v}$$

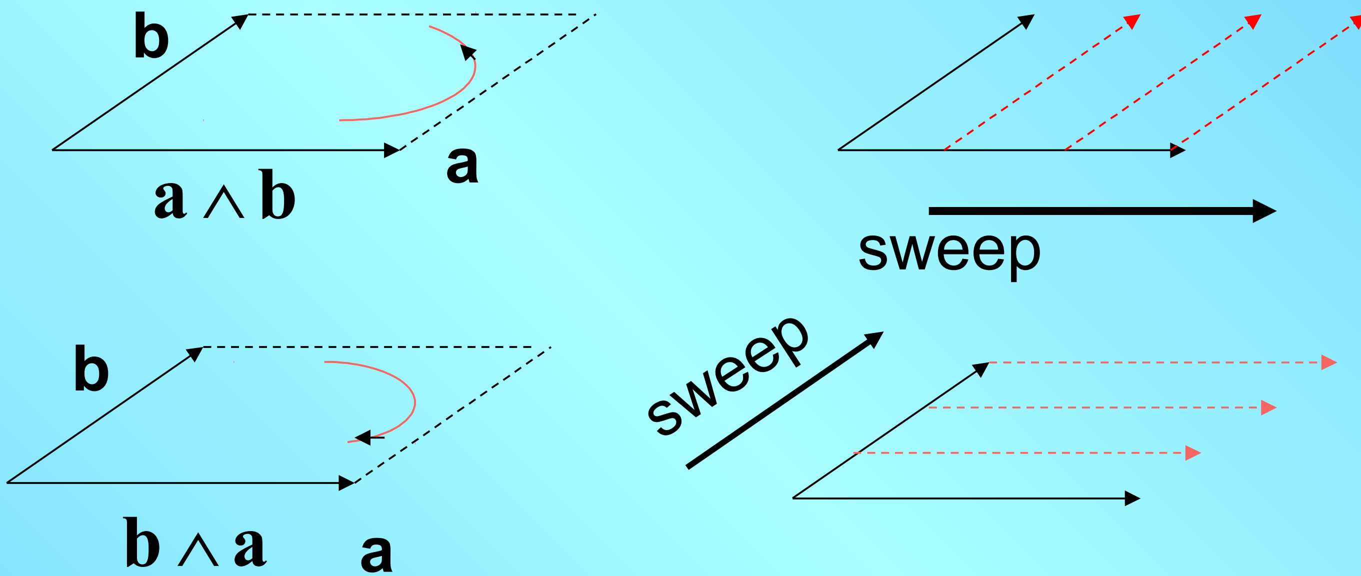


$$\hat{\mathbf{k}} = \frac{\mathbf{B}}{B}, \qquad \omega = \frac{qB}{m}$$

B = B **k**

$$\boxed{\mathbf{v}(t) = \mathbf{v}_{0=} + \cos(\omega t)\mathbf{v}_{0\perp} - \sin(\omega t)\,\hat{\mathbf{k}} \times \mathbf{v}_{0\perp}}$$

THIRD STEP: ANTISYM. PRODUCT



- anticommutative
- associative
- distributive
- absolute value => area

Geometric or matrix product

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$$

$$\mathbf{ba} = \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \wedge \mathbf{b}$$

$$\mathbf{aa} = \mathbf{a} \cdot \mathbf{a} = a^2$$

$$\mathbf{a}^{-1} = \frac{\mathbf{a}}{a^2}$$

- non commutative
- associative
- distributive
- closure:
  extend vector space
- unit = 1
- inverse (conditional)

Examples of Clifford algebras

Notation	Geometry	Dim.
$\mathcal{G}_2$	plane	4
$\mathcal{G}_{0,1}$	complex	2
$\mathcal{G}_{0,2}$	quaternions	4
$\mathcal{G}_3$	Pauli	8
$\mathcal{G}_{1,3}$	Dirac	16
$\mathcal{G}_{m,n}$	signature (m,n)	2^{m+n}

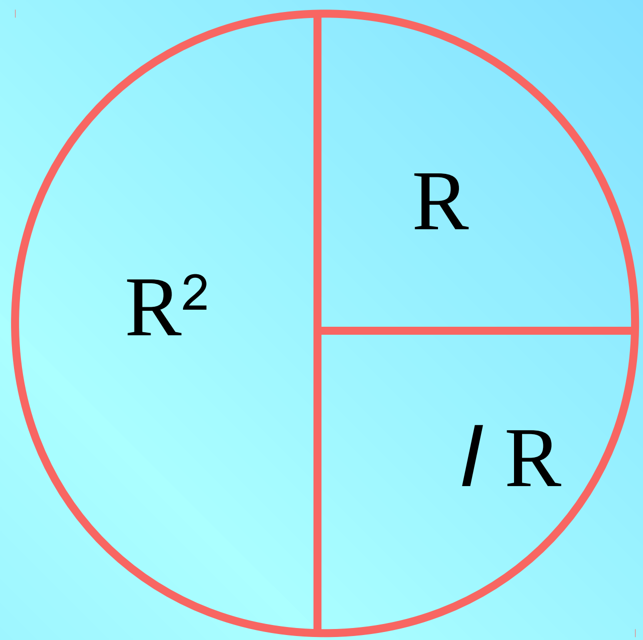
$\mathcal{G}_2$:

1		scalar
$\hat{\mathbf{e}}_1$	$\hat{\mathbf{e}}_2$	vector
$\hat{\mathbf{e}}_1 \wedge \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2 = I$		bivector

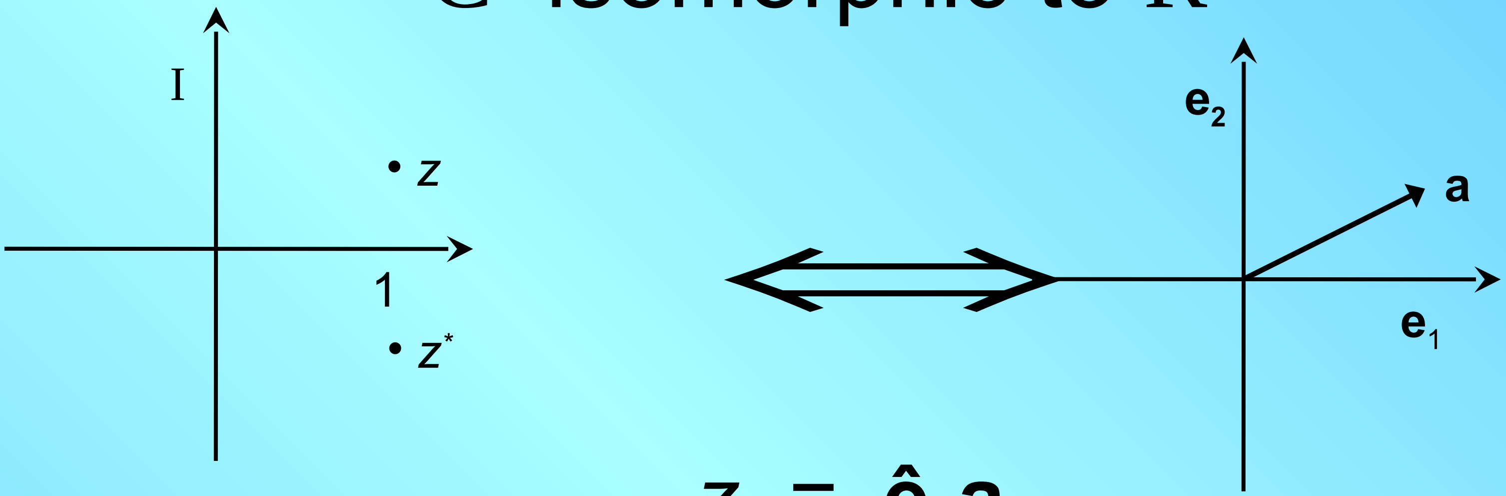
$$\{\hat{\mathbf{e}}_i, \hat{\mathbf{e}}_k\} = \hat{\mathbf{e}}_i \hat{\mathbf{e}}_k + \hat{\mathbf{e}}_k \hat{\mathbf{e}}_i = 2\delta_{ik}$$

$$I^2 = -1 \qquad\qquad I\,\mathbf{a} = -\mathbf{a} I$$

C = even algebra = spinors



$\mathbb{C}$ isomorphic to $\mathbb{R}^2$



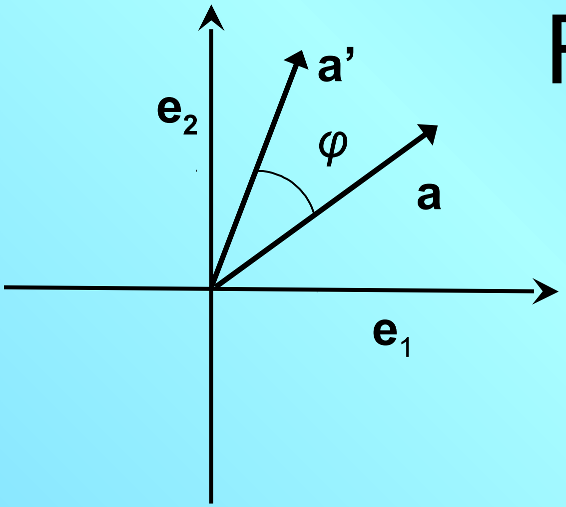
$$z = \hat{\mathbf{e}}_1 \mathbf{a}$$

$$\hat{\mathbf{e}}_1 z = \mathbf{a}$$

Reflection: $z \rightarrow z^*$

$$\mathbf{a} \rightarrow \mathbf{a}' = \hat{\mathbf{e}}_1 z^* = \hat{\mathbf{e}}_1 \mathbf{a}\, \hat{\mathbf{e}}_1$$

In general, $\mathbf{a}' = \mathbf{n}\, \mathbf{a}\, \mathbf{n}$, with $\mathbf{n}^2 = 1$



Rotations in R^2

$$\mathbf{a}' = \mathbf{a}\, e^{I\varphi} = e^{-I\varphi}\mathbf{a}$$

$$\mathbf{a}' = e^{-I\varphi/2}\mathbf{a}e^{I\varphi/2}$$

Euler:

$$\boxed{e^{(\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2)\varphi} = \cos\varphi + \hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 \sin\varphi}$$

Inverse of multivector in $\mathcal{G}_2$

Conjugate:

$$A = \alpha + \mathbf{a} + \beta I \quad \Leftrightarrow \quad \widetilde{A} = \alpha - \mathbf{a} - \beta I$$

$$A\widetilde{A} = \widetilde{A}A \qquad \text{scalar}$$

$$A^{-1} = \frac{\widetilde{A}}{\widetilde{A}A} = \frac{\widetilde{A}}{A\widetilde{A}}$$

Generalizing $\quad \mathbf{a}^{-1} = \dfrac{\mathbf{a}}{a^2} \quad$ **and** $\quad z^{-1} = \dfrac{z^*}{z\,z^*}$

$\mathcal{G}_3$

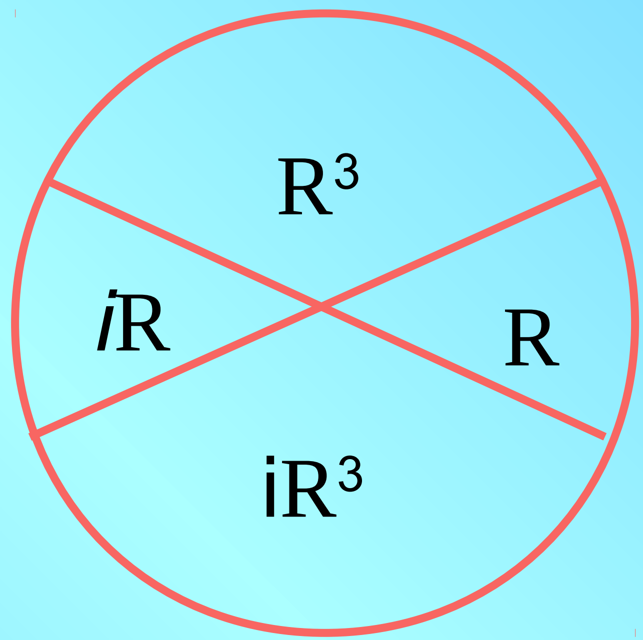
1			1	scalar
$\hat{\mathbf{e}}_1$	$\hat{\mathbf{e}}_2$	$\hat{\mathbf{e}}_3$	3	vector
$\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 = i\hat{\mathbf{e}}_3$	$\hat{\mathbf{e}}_3\hat{\mathbf{e}}_1 = i\hat{\mathbf{e}}_2$	$\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3 = i\hat{\mathbf{e}}_1$	3	bivector
$\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3 = i$			1	pseudoscalar

$$\{\hat{\mathbf{e}}_i, \hat{\mathbf{e}}_k\} = \hat{\mathbf{e}}_i\hat{\mathbf{e}}_k + \hat{\mathbf{e}}_k\hat{\mathbf{e}}_i = 2\delta_{ik}$$

$$i^2 = -1 \qquad\qquad i\mathbf{a} = \mathbf{a}i$$

$$\mathrm{H} = \mathrm{R} + i\,\mathrm{R}^3 =$$

= even algebra = spinors



Geometric product in $\mathcal{G}_3$:

$$\boxed{\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + i\,\mathbf{a}\times\mathbf{b}}$$

$$\mathbf{a}\wedge\mathbf{b} = i\,\mathbf{a}\times\mathbf{b}, \qquad \mathbf{a}\times\mathbf{b} = -i\mathbf{a}\wedge\mathbf{b}$$

Rotations:

$$\mathbf{a} = \hat{\mathbf{n}}(\hat{\mathbf{n}}\mathbf{a}) = \mathbf{a}_{=} + \mathbf{a}_{\perp}$$

$$\mathbf{a}'_{\perp} = \mathbf{a}_{\perp}\, e^{i\hat{\mathbf{n}}\varphi} = e^{-i\hat{\mathbf{n}}\varphi}\mathbf{a}_{\perp}$$

$$\mathbf{a}' = e^{-\frac{1}{2}i\hat{\mathbf{n}}\varphi}\,\mathbf{a}\, e^{\frac{1}{2}i\hat{\mathbf{n}}\varphi}$$

$e^{P/2}$ generates rotations with respect to plane defined by bivector P

Inverse of multivector in $\mathcal{G}_3$
defining Clifford conjugate:

$$A = \alpha + \beta i + \mathbf{a} + \mathbf{b}i \quad \Leftrightarrow \quad \widetilde{A} = \alpha + \beta i - \mathbf{a} - \mathbf{b}i$$

$$A\widetilde{A} = \widetilde{A}A \qquad \text{scalar}$$

$$A^{-1} = \frac{\widetilde{A}}{\widetilde{A}A} = \frac{\widetilde{A}}{A\widetilde{A}}$$

- Generalizing: $\mathbf{a}^{-1} = \dfrac{\mathbf{a}}{a^2}$

Geometric Calculus (3d)

∇ is a <u>vector</u> differential operator acting on:

$\varphi(\mathbf{r})$ – scalar field

$\mathbf{E}(\mathbf{r})$ – vector field

$$\nabla(\varphi + \mathbf{E}) = \nabla\varphi + \nabla\cdot\mathbf{E} + \nabla\wedge\mathbf{E} =$$

$$= \nabla\varphi + \nabla\cdot\mathbf{E} + i\nabla\times\mathbf{E}$$

First order Green Functions

$$\nabla^{-1} = \frac{\nabla}{\nabla^2} \qquad \nabla^2\left(-\frac{1}{4\pi r}\right) = \delta(\mathbf{r})$$

Euclidian spaces :

$$g(\mathbf{x},\mathbf{y}) = \frac{1}{\Omega}\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^n}$$ is solution of

$$\nabla g(\mathbf{x},\mathbf{y}) = \delta^{(n)}(\mathbf{x}-\mathbf{y})$$

Maxwell's Equation

- Maxwell's multivector $F = \mathbf{E} + i\,c\,\mathbf{B}$
- current density paravector $J = (\varepsilon_0 c)^{-1}(c\rho + \mathbf{j})$
- Maxwell's equation:

$$\boxed{\tilde{\partial}\,F = \tilde{J}}$$

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \nabla\right)(\mathbf{E} + ic\mathbf{B}) = \frac{1}{\varepsilon_0 c}(c\rho - \mathbf{j})$$

Electrostatics

Gauss's law. Solution using first order Green's function:

$$\nabla \mathbf{E} = \frac{1}{\varepsilon_0}\rho \qquad\qquad \mathbf{E} = \nabla^{-1}\left(\frac{1}{\varepsilon_0}\rho\right) = \nabla\frac{1}{\nabla^2}\left(\frac{1}{\varepsilon_0}\rho\right)$$

Explicitly:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\mathbf{r}}{r^3}\rho(\mathbf{x}')d\tau' \quad \text{with} \quad \mathbf{r} = \mathbf{x} - \mathbf{x}'$$

Magnetostatics

Ampère's law. Solution using first order Green's function:

$$\nabla \mathbf{B} = i\mu_0 \mathbf{j} \qquad \mathbf{B} = i\mu_0 \frac{1}{\nabla^2} \nabla \, \mathbf{j} = -\mu_0 \nabla \times \frac{1}{\nabla^2} \mathbf{j}$$

Explicitly:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{x}') \times \mathbf{r}}{r^3} d\tau' \quad \text{with} \quad \mathbf{r} = \mathbf{x} - \mathbf{x}'$$

Fundamental Theorem of Calculus

- Euclidian case :

  $d^k x$ = oriented volume, *e.g.* $\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_3 = i$

  $d^{k\text{-}1}x$ = oriented surface, *e.g.* $\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2 = i\hat{\mathbf{e}}_3$

$F(x)$ – multivector field, ∂V – boundary of V

$$\int_V d^k x \nabla F = (-1)^{k-1} \oint_{\partial V} d^{k-1} x\, F$$

$$\int_V \overset{S}{G} d^k x \overset{S}{\nabla} = (-1)^{k-1} \oint_{\partial V} G\, d^{k-1} x$$

Example: divergence theorem

- $d^3x = i\, d\tau$, where $d\tau = |d^3x|$
- $d^2x = i\, \mathbf{n} da$, where $da = |d^2x|$
- $F = \mathbf{v}(\mathbf{r})$

$$\int_V \nabla \mathbf{v}\, i d\tau = \oint_{\partial V} \hat{\mathbf{n}}\, \mathbf{v} i da$$

$$\text{scalar part:} \quad \int_V \nabla \cdot \mathbf{v}\, d\tau = \oint_{\partial V} \hat{\mathbf{n}} \cdot \mathbf{v}\, da$$

Green’s Theorem (Euclidian case)

$$\mathsf{F}(\mathbf{x}) = -\frac{(-)^n}{I(\mathbf{x})}\int_V g(\mathbf{x},\mathbf{x}')\, d^n x' \nabla' \mathsf{F}(\mathbf{x}') +$$

$$+\frac{(-)^n}{I(\mathbf{x})}\oint_{\partial V} g(\mathbf{x},\mathbf{x}') d^{n-1} x' \mathsf{F}(\mathbf{x}')$$

- where the first order Green function is:

$$g(\mathbf{x},\mathbf{x}') = \frac{1}{\Omega_n}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^n} \text{ with } \Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

Cauchy's theorem in n dimensions

- particular case : F = f y ∇ f = 0

$$f(\mathbf{x}) = \frac{(-)^n}{I(\mathbf{x})} \oint_{\partial V} \frac{1}{\Omega_n} \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^n} d^{n-1}\mathbf{x}' f(\mathbf{x}')$$

where f (**x**) is a monogenic multivectorial field: ∇ f = 0

Inverse of differential paravectors:

$$\tilde{\partial} = \frac{\partial}{\partial t} + \nabla, \qquad \tilde{\partial}^{-1} = \frac{\frac{\partial}{\partial t} - \nabla}{\frac{\partial^2}{\partial t^2} - \nabla^2}$$

Helmholtz:

$$(\nabla - jk)^{-1} = \frac{\nabla + jk}{\nabla^2 + k^2}, \qquad G(\mathbf{x} - \mathbf{x}') = \frac{-1}{4\pi} \frac{e^{\pm jkr}}{r},$$

$$j^2 = -1$$

Spherical wave (outgoing or incoming)

Electromagnetic diffraction

- First order Helmholtz equation:

$$\nabla \mathrm{F} = \tilde{\mathrm{J}} + jk\mathrm{F}$$

- Exact Huygens' principle:

$$\mathrm{F}(\mathbf{x}) = -\oint_S (\nabla + jk) G(\mathbf{x} - \mathbf{x}')\, \hat{\mathbf{n}}'\, \mathrm{F}(\mathbf{x}')\, da'$$

$$\textbf{for} \qquad \mathrm{F} = \mathbf{E} + ic\mathbf{B}$$

In the homogeneous case J = 0

Special relativity and paravectors in $\mathcal{G}_3$

- Paravector $p = p_0 + \mathbf{p}$

Examples of paravectors:

x	$ct + \mathbf{r}$
u	$\gamma(1 + \mathbf{v}/c)$
p	$E/c + \mathbf{p}$
Φ	$\varphi + c\mathbf{A}$
j	$c\rho + \mathbf{j}$

$$\gamma = \frac{1}{\sqrt{1 - v^2 / c^2}}$$

Lorentz transformations

$$B = e^{\frac{1}{2}\mathbf{w}} = \cosh(\tfrac{1}{2}w) + \hat{\mathbf{w}}\sinh(\tfrac{1}{2}w)$$

$$R = e^{-\frac{1}{2}i\boldsymbol{\theta}} = \cos(\tfrac{1}{2}\theta) - i\hat{\boldsymbol{\theta}}\sin(\tfrac{1}{2}\theta)$$

Transform the paravector

$p = p_0 + \mathbf{p} = p_0 + \mathbf{p}_= + \mathbf{p}_\perp$ into:

$\mathrm{B}\, p\, \mathrm{B} = \mathrm{B}^2 (p_0 + \mathbf{p}_=) + \mathbf{p}_\perp$

$\mathrm{R}\, p\, \mathrm{R}^\dagger = p_0 + \mathbf{p}_= + \mathrm{R}^2 \mathbf{p}_\perp$