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Physics 321

Homework Due 9-12-12

P4.17

Because the resistance is proportional to the square of the speed, we can write an equation of motion for the object when it is travelling down at its terminal velocity.

$$\text{mg} - m K V^2 = 0 \to V^2 = \frac{g}{K}$$

In this case the total acceleration is 0, because we are traveling at the terminal velocity.

Now we look at the case where the object is moving upwards.

$$m \frac{\partial^2 x}{\partial t^2} = -m g - m K v^2 \to \frac{\partial^2 x}{\partial t^2} = -g\left(1 + \frac{v^2}{V^2}\right) \tag{0.1}$$

Now we make the substitution that $\frac{\partial x}{\partial t} = v$.

$$\frac{\partial v}{\partial t} = -g\left(1 + \frac{v^2}{V^2}\right) \tag{0.2}$$

Mathematica can solve this for us like so (note that we put the first oder condition that v[0] = u):

```
vel = v[t]/.DSolve[{v'[t] == -g(1 + v[t]^2/V^2), v[0] == u}, v[t],t]//FullSimplify//Quiet
```

$$\left\{-V \tan\left(\frac{g t}{V} - \tan^{-1}\left(\frac{u}{V}\right)\right)\right\}$$

We can easily get the time associated with the highest point by setting vel = 0 and solving for time.

```
Solve[vel ==0 ,t]//Quiet
```

$$\left\{\left\{t \to \frac{V \tan^{-1}\left(\frac{u}{V}\right)}{g}\right\}\right\}$$

Now we will start at equation (0.2) and make a substitution of variables. It is written as $\frac{\partial v}{\partial t} =$ stuff, but we can write $\frac{\partial v}{\partial t} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial t} = v \frac{\partial v}{\partial x}$. This yields the following:

$$v \frac{\partial v}{\partial x} = -g\left(1 + \frac{v^2}{V^2}\right) \tag{0.3}$$

Mathematica can also solve this one for us (again first order coniditon says v=u when x=0):

```
velx = v[x]/.DSolve[{v[x] v'[x] == -g * (1 + v[x]^2/V^2),  v[0]==u},v[x],x]//FullSimplify//Qui
```

$$\left\{-\sqrt{\left(u^2 + V^2\right) e^{-\frac{2 g x}{V^2}} - V^2}, \sqrt{\left(u^2 + V^2\right) e^{-\frac{2 g x}{V^2}} - V^2}\right\}$$

Just like before, the maximum occurs when velx =0. We let mathematica solve this for us

```
xpos = Solve[velx == 0, x]//Quiet
```

$$\left\{\left\{x \to \frac{V^2 \log\left(\frac{u^2+V^2}{V^2}\right)}{2\, g}\right\}\right\}$$

Now we turn to the 3rd part of this question (it's really long). We can redefine our coordinate system setting the highest point we just characterized as the origin. Borrowing from (0.1), the equation of motion becomes:

$$m \frac{\partial v}{\partial t} = m\, g - m\, K\, v^2 \tag{0.4}$$

Note that the differnence is a sign change on the (mg) term.

We now write the equivalent form of (0.2):

$$\frac{\partial v}{\partial t} = g\left(1 - \frac{v^2}{V^2}\right) \tag{0.5}$$

Now we write the equivalent form of (0.3):

$$v \frac{\partial v}{\partial x} = g\left(1 - \frac{v^2}{V^2}\right) \tag{0.6}$$

Mathematica can solve this for us:

```
velx2 = v[x]/.DSolve[{v[x] v'[x] == g(1 - v[x]^2/V^2), v[0]==0},v[x],x][[2]]//FullSimplify//Qu
```

$$\sqrt{V^2\left(1 - e^{-\frac{2 g x}{V^2}}\right)}$$

We now use the answer to part 2 of this question and realize that the ball is at is origin position when it reaches the height found in that part, in the new coordinate system. We solve for 'x' in the equation above, set it equal to the x just mentioned and solve for v.

```
x1 = x/.xpos[[1]]//Quiet;
x2 = x/.Solve[velx2== v, x][[1]]//Quiet;
Solve[x1== x2, v][[2]]
```

$$\left\{v \to \frac{u\, V}{\sqrt{u^2 + V^2}}\right\}$$

P4.26

Proof in the beginning

Thankfully, the answer to the differential equation associated with this problem is given to us in Example 4.7 of the text. It tells us that:

$$x = \frac{u \cos\alpha}{K}\left(1 - e^{-K t}\right),\; z = \frac{K\, u \sin\alpha + g}{K^2}\left(1 - e^{-K t}\right) - \frac{g}{K}\, t \tag{0.7}$$

It is very simple to write the position vector **r** now:

$$\boldsymbol{r} = x\,\boldsymbol{i} + z\,\boldsymbol{k} = \frac{u \cos\alpha}{K}\left(1 - e^{-K t}\right)\boldsymbol{i} + \left(\frac{K\, u \sin\alpha + g}{K^2}\left(1 - e^{-K t}\right) - \frac{g}{K}\, t\right)\boldsymbol{k} \tag{0.8}$$

We can simplify this equation by noticing that both the **i** and **k** components are multiplied by the constant $\frac{1}{K}\,(1 - e^{-K t})$

$$\boldsymbol{r} = \frac{1}{K}\left(1 - e^{-K t}\right)(u \cos\alpha\, \boldsymbol{i} + u \sin\alpha\, \boldsymbol{k}) - \left(\mathrm{Kt} - \left(1 - e^{-\mathrm{Kt}}\right)\right)\frac{g}{K^2}\, \boldsymbol{k} = \beta(t)\, \boldsymbol{u} - \alpha(t)\, \boldsymbol{k},\ \text{where}$$

$$\beta(t) = \frac{1}{K}\left(1 - e^{-\mathrm{Kt}}\right),\ \alpha(t) = e^{-\mathrm{Kt}} - t + \mathrm{Kt},\ \boldsymbol{u} = u\cos\alpha\,\boldsymbol{i} + u\sin\alpha\,\boldsymbol{k}$$

Proving (i)

This is a pretty simple mental exercise. If we label the starting position of the particles as being O, we can define the position of any particle as being the magnitude of the vector OP, where P is the location of a particle after a certain amount of time. If all the particles start moving with the same speed $u = |\,\boldsymbol{u}\,|$, then the distance is just equal to $u\beta$(t). This is true for all particles, which means that the particles lie on a sphere centered at O, with a radius equal to $u\beta$(t).

Proving (ii)

Again, we define the starting point of the particles to be O. If all the particles are shot in the same direction, we can say that their velocity vectors point in the same direction. If that is the case then at any time t, the position of each of the particles will be a point $u\beta$(t) away from the origin. If the velocities are all in the same direction, then the motion is in the same direction, and you could draw a line from the particle furthest from O, and all other particles would also be on that line.

I know that isn't a very clear proof, but it is true. To summarize, if particles are released from a common point, moving the same direction, they will stay on a line that points from their place of origin along the velocity vector.

Proving (iii)

Let us define the three particles as x, y, z. We can express their position using the general formula we proved when we started this problem:

$$\boldsymbol{x} = -\alpha(t)\,\boldsymbol{k} + \beta(t)\,\boldsymbol{u}^x$$

$$\boldsymbol{y} = -\alpha(t)\,\boldsymbol{k} + \beta(t)\,\boldsymbol{u}^y$$

$$\boldsymbol{z} = -\alpha(t)\,\boldsymbol{k} + \beta(t)\,\boldsymbol{u}^z$$

In order to define a plane, one only needs two vectors in the plane as well as a vector normal to both of them. The vector **(x - y)** and **(z - y)** are the vecotrs pointing from **y** to both **x** and **z** and are therefore in the plane created by the three particles. To get the normal vector I just need to take the cross product of those two:

$$(\boldsymbol{x} - \boldsymbol{y}) \times (\boldsymbol{z} - \boldsymbol{y}) = \beta(t)^2\,(\boldsymbol{u}^x - \boldsymbol{u}^y) \times (\boldsymbol{u}^z - \boldsymbol{u}^y)$$

The above vector has a constant direction, which means that the plane isn't chaning its orientation or that it is always parallell to some fixed plane.

Pre-class assignment for 9-12-12

Like problem 5.6 in Gregory says, critical dampening occurs when K = Ω.

We can write the general equation of motion for a SHO as:

$$\ddot{x} + 2K\dot{x} + K^2 x = 0 \rightarrow \ddot{x} + 2\Omega\dot{x} + \Omega^2 x = 0 \tag{0.9}$$

This is a simple, linear, 2nd order ODE. We can write the characteristic equation as $\lambda^2 + 2\Omega\lambda + \Omega^2 = 0$, which has the repeated root λ = - Ω.

From a differential equations class, I know that the general form of a 2nd order ODE with repeated roots is:

$$x = \left\{e^{-\Omega t},\ t\,e^{-\Omega t}\right\} = C_1 e^{-\Omega t} + C_2\,t\,e^{-\Omega t} = \boldsymbol{e^{-\Omega t}(C_1 + t\,C_2)}$$