What Does it Mean to Integrate?

Ross L. Spencer

Department of Physics and Astronomy

Brigham Young University

In elementary physics classes we do not expect you to perform difficult integrations, but we do expect you to know what integration means. Most of you know that $\int x dx = x^2/2$, but you may not know when to use $\int x dx$, or $\int \sin x dx$, or some other integral. The following examples are offered to help you understand what it means to integrate, and especially to introduce you to the geometries used in the surface and volume integrations that occur in our study of electricity and magnetism.

We will hardly ever use integration to find the area under a curve. Instead, we use integration to find the total amount of something by adding up the contributions to it from billions and billions of tiny parts. Of course, integrating to find the area under a curve is an example of this process: the total area under the curve is the sum of the areas of infinitely many infinitesimally small rectangles, as suggested by Figure 1. In a similar way, we could find the length of a curve by cutting it up into many tiny line segments and adding together the lengths of all the segments; or we could find the mass of an object by breaking it into many pieces and adding up the masses of all the pieces; or we could find the value of the electric field at a certain point in space by adding together the contributions to the electric field from many small bits of charge distributed in some way through space. Hence, whenever we use the symbol $\int$ we shall mean that we are adding up many small bits of something to find the whole; once you have this concept firmly in mind, integration will not seem so mysterious.

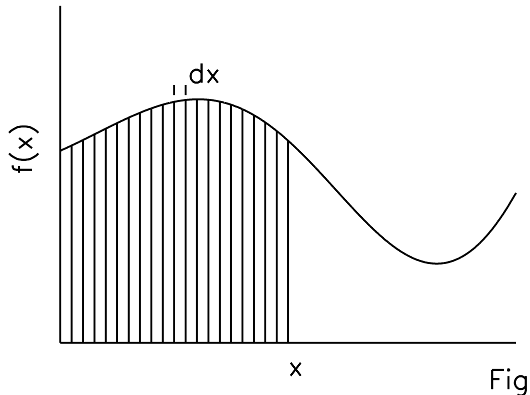


Figure 1

As a first example, let us find the length of a meter stick by integrating. Before we can express its length as an integral, we need a coordinate system. (You will notice that we almost always begin integration problems by setting up a coordinate system.) For instance, we could lay the meter stick along the x-axis with the zero mark at $x = 0.2$ m and with the 1 m mark at $x = 1.2$ m, as shown in Figure 2. To find the length of the stick, we now imagine that we slice it up along

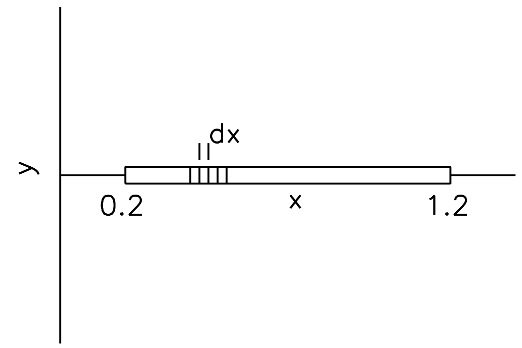


Figure 2

its length into a billion tiny pieces. The slices are made across the width of the stick, one million of them per millimeter. Each tiny piece is located at some value of x between $x = 0.2$ m and $x = 1.2$ m. The coordinate x thus serves as a way of giving each tiny piece a unique label. This is very important. The integration variable is always the variable that uniquely labels the pieces that are being added together. The tiny length of each piece we denote by the symbol, dx, meaning "an infinitesimally small length along x." Note that this means that dx is not just an abstract mathematical symbol. We think of dx as the very small, but very real, length of a tiny slice of the meter stick. And like any length, dx has units of meters. We are now ready to find the total length of the stick by adding up all of the little dx's between $x = 0.2$ m and $x = 1.2$ m. This operation is expressed mathematically by writing, for the total length L,

$$L = \int_{0.2m}^{1.2m} dx \ .$$

The symbol $\int$ means, "add up all the pieces," the lower limit tells us at what value of x the first piece is located, and the upper limit tells us the location of the last piece. Now that we have translated the physical process of adding up all of the pieces into the mathematical process of integration, the rest is easy: $\int dx = x$, so we may write

$$L = \int_{0.2m}^{1.2m} dx = x \mid_{0.2m}^{1.2m} = 1.2 \ m - 0.2 \ m = 1.0 \ m \ ,$$

as expected for a meter stick. (In the integrals that follow, 0.2 and 1.2 will be written without units to avoid clutter.)

Now let's try something a bit more difficult. We will find the mass of our meter stick by integrating. Once again we slice it along its length into many small pieces. The mass of each piece we denote by the symbol dm, having units of mass and representing the small mass of the slice of length dx. The total mass, m, is simply expressed by writing

$$m = \int dm \ .$$

But now we have a problem. The coordinate x labels each piece, so we want to use x-values for the upper and lower limits of the integral, but this means that we need dx in the integrand, and we have dm instead. So we need to express dm in terms of dx before we can proceed with the integration. Now, dm and dx must be rather simply related, because if we double the length of the slice its mass must also double. Hence, we must be able to write

$$dm = \lambda dx \ ,$$

where λ is a constant of proportionality. To understand the meaning of λ, consider its units. For $dm = \lambda dx$ to make sense, λ must have units of mass/length. It is, in fact, the "linear mass density," or the "mass per unit length" of the stick.

When you hear the word density, most of you think of mass per unit volume, but we will often use density in a more general sense to mean some quantity per unit length, area, or volume. We will usually use the symbol λ to denote densities per unit length, the symbol σ to denote densities per unit area, and the symbol ρ to denote densities per unit volume. In the examples that follow, each kind of of density will be used.

In general, the linear mass density λ might be different for every value of x along the stick. To find it at each value of x we would need to measure the length of the slice dx, weigh it to find its mass dm, then take the ratio of the mass to the length to find λ: $\lambda = dm/dx$, or equivalently, $dm = \lambda dx$. If we happen to know that the density is uniform, i.e., that λ is the same at each point along the stick, then we may find λ by taking the ratio of the entire mass to the entire length, $\lambda = m/L$. Once we know λ we may write

$$m = \int_{0.2}^{1.2} \lambda dx \ .$$

If λ is uniform, then it is a constant and can come outside of the integral sign, so

$$m = \lambda \int_{0.2}^{1.2} dx = \lambda \times 1.0 \ m \ .$$

i.e., the total mass is simply the mass per unit length times the length.

Let's try a case now where the density is not uniform. Suppose that we make careful measurements on the stick and discover that the linear mass density is well approximated by the formula $\lambda = 0.5x$ kg/m^2. Note that this formula says that at $x = 0.2$ m the density has the value $\lambda = 0.1$ kg/m and that at $x = 1.2$ m it has the value $\lambda = 0.6$ kg/m; between these two extremes, the density varies linearly with x. (Note that λ has the proper units of kg/m since x has units of meters.) The total mass is given by the same formula as before, except that now the factor of x in λ must stay inside the integral:

$$m = \int dm = \int_{0.2}^{1.2} \lambda dx = 0.5 \text{ kg/m}^2 \int_{0.2}^{1.2} x dx \ .$$

Performing the integral gives

$$m = 0.5 \text{ kg/m}^2 \frac{1}{2}(1.2^2 - 0.2^2) m^2 = 0.35 \text{ kg} \ .$$

It is interesting to note that if we try to estimate the answer by taking the average of the densities at each end and multiplying by the total length we find $m = (0.1+0.6)/2$ kg/m$\times$ 1.0 m $= 0.35$ kg. Averaging will not always give the same answer as integration, but a well-chosen average will always give approximately the same answer as integration. It is a good idea to use an average to check for big mistakes.

For the next example, we will find the circumference of a circle of radius a. As usual, we first need a coordinate system. For convenience, let's place the center of the circle at the origin of the coordinate system, as shown in Figure 3. Now imagine that the circle is chopped up into tiny line segments, each of length $d\ell$. To write the length as an integral, we need to find a coordinate that

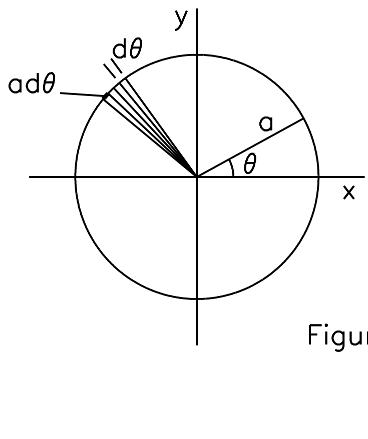


Figure 3

uniquely labels each little segment. It is possible to do this problem in (x, y) coordinates, but it is a mess. It is much simpler to use a coordinate system that fits the circle, namely polar, or (r,θ) coordinates, as shown in Figure 3. This coordinate system is nice for this problem because r has the value a at each little segment, and the single coordinate θ uniquely labels each segment. To go all the way around the circle, we start with the segment labeled by $\theta = 0$ and add segments up until we arrive at the one labeled by $\theta = 2\pi$. But before we can proceed, we need a relation between each little bit of length, $d\ell$, and each little piece of angle, $d\theta$. The desired relation is particularly simple when θ is measured in radians: $d\ell = rd\theta$ In our case, $r = a$, and a is constant, so it can come outside the integral giving

$$L = \int d\ell = a\int_0^{2\pi} d\theta = a\theta \mid_0^{2\pi} = 2\pi a \ ,$$

as expected.

Now let's find the area of the same circle. It is possible to find areas by performing multiple integrals, but since many physics students have not seen these before, we will find areas by cleverly cutting the shape whose area is to be found into strips of small area. We will then add up the small areas, by integrating, and find the total. There are many ways of cutting a figure into strips, but usually only one or two of these ways makes the integration easy. In the case of the circle, one of the simple ways is to cut it up into concentric circular strips, as shown in Figure 4. Each strip is at a different radius, so r is the coordinate that labels each small piece, and we will integrate with respect to r. To find the total area of the circle, we will start with the tiny strip at $r = 0$ and add up the areas of all the strips out to the one with radius $r = a$. The total area is simply given by $A = \int dA$, where dA is the area of each strip. Since the variable of integration is r, we can't perform this integral unless we have a relation between dA and dr. To see what this relation is, imagine that you are so small that you can stand on one of the circular strips. The strip will appear to you as a very long, slightly curved, path, and if you wanted to estimate the area of the path, you would simply take the width of the path and multiply it by its length. This won't be quite right because the path curves, but as the width becomes smaller and smaller (and as you shrink along with it) the path appears to become straighter and straighter until in the limit of zero width, the width-times-length formula is exactly correct. The length of the path is simply the circumference of a circle at that radius, $2\pi r$, and the width of the path is dr, so its area, dA, is given by $dA = 2\pi rdr$. Hence, we may write

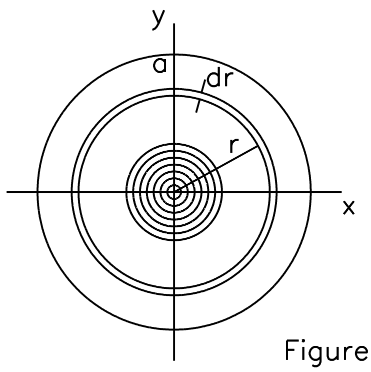


Figure 4

$$A = \int dA = \int_0^a 2\pi rdr = 2\pi\int_0^a rdr = 2\pi\frac{1}{2}(a^2 - 0^2) = \pi a^2,$$

a familiar result.

Suppose now that we are interested in finding the mass of a thin circular disk. Since the disk only has area, the kind of density that we use is an area density, i.e., the mass per unit area, σ. The mass per unit area of some part of the disk could be measured by cutting out a small square piece of the disk, measuring the side-length of the square and computing its area, dA, then finally weighing the piece to find its mass, dm. We would then take the ratio to find σ: $\sigma = dm/dA$. To find the relation between dm and dA, we simply solve this equation for dm: $dm = \sigma dA$. Suppose now that σ for the disk is given by the formula

$$\sigma = 2.0 \text{ kg/m}^2 (r/a).$$

(I just made this up as an example.) This means that the density in the center is zero, and that the density at the outer edge of the disk is 2.0 kg/m^2. To find the total mass of the disk we cut the disk up into concentric strips of width dr, as before. Each strip has area $dA = 2\pi r dr$ and mass dm given by

$$dm = \sigma dA = 2.0 \text{ kg/m}^2 (r/a) 2\pi r dr \ .$$

The total mass of the disk can then be calculated by writing

$$m = \int dm = \int_0^a (2.0 \text{ kg/m}^2) \frac{2\pi}{a} r^2 dr = (\frac{4\pi}{a} \text{ kg/m}^2) \frac{a^3}{3} = (2 \text{ kg/m}^2)(\frac{2}{3})\pi a^2.$$

Note that this shows that for this case, the total mass is the same as that calculated by assuming a uniform density equal to 2/3 of the outer edge value, so if we made our estimate of the mass by using the density halfway out, we would be a little low. But the estimate would still be reasonably close.

For the next example, we will move into three dimensions and consider a circular cylinder of length L and radius a, as shown in Figure 5. First, let's find the area of the outer surface of the cylinder. This area comes in three pieces: two circular end pieces and the curved cylindrical part (shaped like the cardboard center from a roll of toilet paper). The circular end pieces we have already done; each one contributes area πa^2 to the total. The other part is easy too, but just for practice, let's do it by integrating. Choose for the coordinate system cylindrical coordinates, (r,θ,z), with the cylinder centered as shown in Figure 5. This time there are two simple ways of cutting the area into strips. The first is to cut the surface into thin strips along the z-axis. Each strip has length L and width $ad\theta$, and the strips are labeled by the angle θ, as shown in Figure 5. The area of each strip is just the length times the width: $dA = Lad\theta$. The area is given by

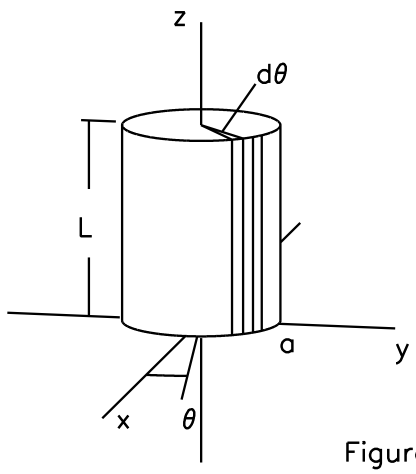


Figure 5

$$A = \int_0^{2\pi} Lad\theta = 2\pi a L \ ,$$

or simply the circumference of the circle times the length.

The other simple way to compute the area is to cut the surface into thin circular strips of length $2\pi a$ and width dz, as shown in Figure 6. This time the strips are labeled by the coordinate z, so we integrate with respect to z. The area is given by

$$A = \int_0^L 2\pi a dz = 2\pi a L \ ,$$

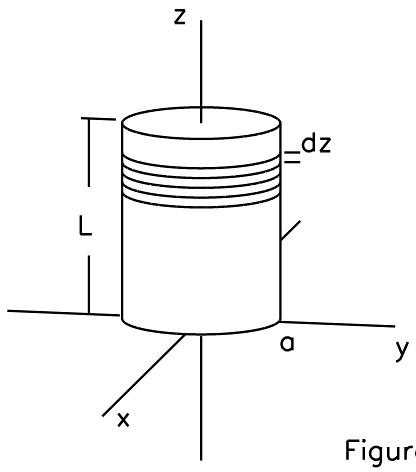


Figure 6

the same answer as from Figure 5. Hence, the total area of the cylinder is given by

$$A_T = 2(\pi a^2) + 2\pi a L = 2\pi a(a + L) \ .$$

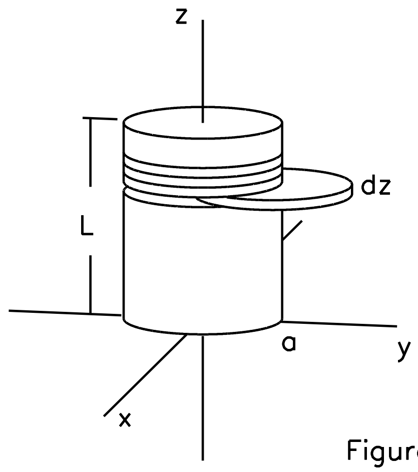


Figure 7

Now let's find the volume of the cylinder. One simple method is to slice the cylinder up as if it were a cylindrical loaf of bread, as shown in Figure 7. Each slice would then have volume $dV = \pi a^2 dz$ (area times thickness), each slice is labeled by the coordinate z, and the total volume is given by

$$V = \int dV = \int_0^L \pi a^2 dz = \pi a^2 L \ .$$

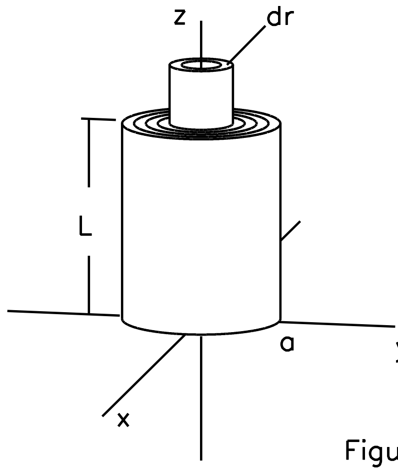


Figure 8

Another simple way of slicing the cylinder up is to imagine that the cylinder is the trunk of a tree. The separate slices of the cylinder are now the growth rings of the tree, as shown in Figure 8. The radial coordinate, r, now labels each of the little volumes, and each slice has length L, circumference $2\pi r$, and thickness dr, giving a small volume of $dV = 2\pi r L dr$. This time we integrate with respect to r to find the total volume:

$$V = \int dV = \int_0^a 2\pi L r dr = \pi a^2 L \ ,$$

as before. Note that both methods give the result that the volume is the base area multiplied by the length.

Sometimes it is confusing to know which slicing method to use. If everything is uniform, then it really doesn't matter, but you might be given a volume mass density, ρ, that depends on z, or on r. If it depends on z, then you need to use the bread slices because each slice is at a different value of z, and will hence have a different small mass, dm. If it depends on r, then the tree rings must be used because each cylindrical shell will have a different density, and hence a different small mass, dm.

To see how to work such a problem, suppose that you are given a plastic cylinder that has been built up by laying down thin cylindrical shells of charged plastic. The shells near the center have very little charge, but the outer ones have quite a bit. Suppose that the volume charge density ρ has been found by experimenting to be approximately given by the formula $\rho = 5\mu\mathrm{C/m^3}(r/a)^2$. (Note that we use the symbol ρ for any volume density. We use it for mass per unit volume, charge per unit volume, etc..) If you wanted to measure the density experimentally to see if this formula is really is a good description, you would need to remove a small bit of plastic from inside the cylinder, measure its charge, dq, and its volume, dV, and take the ratio: $\rho = dq/dV$. This would give the density at the radius from which you removed the sample. We can use this ratio to find the relation between the small charge and the small volume: $dq = \rho dV$. To find the total charge of the cylinder, we use the tree-rings-method since the density is different at each radial position. The total charge is given by

$$q = \int dq = \int \rho dV = \int_0^a (5\mu\mathrm{C/m^3})\frac{2\pi L}{a^2} r^3 dr \ .$$

Performing this integral gives

$$q = \frac{1}{2}(5\mu\mathrm{C/m^3})\pi a^2 L \ ,$$

so the total charge is the same as if the density were uniform with a value equal to 1/2 the value of the density at the outer edge of the cylinder. Since the density is small inside the cylinder and has its maximum value at $r = a$, we know that the proper average density will be less than the outer edge value, but there is no simple way to guess the factor of one-half. But if we had come out with an average density larger than the outer edge value, negative, or ridiculously small, we would know we had made a mistake.

Now we will study the sphere. We will begin with the surface area of a sphere of radius a. Let's use spherical coordinates with the origin of the coordinate system at the center of the sphere. For the slicing method we use lines of latitude spaced $d\theta$ radians apart, as shown in Figure 9. The coordinate that labels each strip is the polar angle, θ, so we will integrate with respect to θ from the north pole, $\theta = 0$, to the south pole, $\theta = \pi$. Each little strip has the same width, $ad\theta$, but the length of each strip varies with θ. At the north or south pole, the strips are quite short; if you stood at the pole you would see these little strips surrounding you. But at the equator, they are very long, passing all the way around the sphere. If you look carefully at Figure 9, you should be able to convince yourself that the

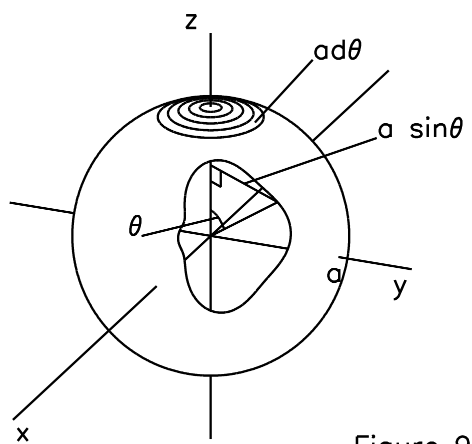


Figure 9

radius of each strip is given by $r = a \sin\theta$, so that the length of each strip is $2\pi a \sin\theta$, and the area of the strip is given by

$$dA = (2\pi a \sin\theta)(a d\theta) = 2\pi a^2 \sin\theta d\theta \ .$$

The total area is then given by

$$A = \int dA = \int_0^\pi 2\pi a^2 \sin\theta d\theta = 2\pi a^2 (-\cos\theta \mid_0^\pi) = 4\pi a^2 \ .$$

Now let's find the volume of the sphere. We use the same coordinate system, and imagine the sphere to be built up by laying down thin spherical shells, like the layers of a pearl. Each shell is labeled by its radius r, and the thickness of each shell is dr. If the layers are very thin, the volume of each shell will simply be the area of the shell multiplied by the thickness of the shell, i.e., $dV = 4\pi r^2 dr$. The total volume is then given by

$$V = \int dV = 4\pi \int_0^a r^2 dr = \frac{4}{3}\pi a^3 \ .$$

Now suppose that we have a plastic sphere made up of thin layers of charged plastic, similar to the charged cylinder we did before. In fact, let the charge density be given by the same formula, $\rho = 5\mu\mathrm{C/m}^3 (r/a)^2$, but here r is a spherical radius instead of a cylindrical radius. We divide the sphere into concentric thin spherical shells, and write for the charge of each thin shell, $dq = \rho dV = \rho 4\pi r^2 dr$. The sphere's total charge is then given by

$$q = \int dq = \int_0^a \rho dV = \int_0^a (5\mu\mathrm{C/m}^3)\frac{4\pi}{a^2} r^4 dr \ .$$

Performing this integral gives

$$q = \frac{4\pi}{5}(5\mu\mathrm{C/m}^3)a^3 \ = \ \frac{3}{5}(5\mu\mathrm{C/m}^3)\frac{4}{3}\pi a^3 \ .$$

so the total is the same as if the density were uniform with a value equal to $3/5$ of the value of the density at the outer edge of the sphere. Note that as in the case of the cylinder, the average density is less than the peak value, but the factor has changed from $1/2$ to $3/5$. The proper factor can only be gotten by integrating.

Now that we have a feel for how integrating is adding up little bits of something to find the total, let's find the electric field at a point in space by adding up the contributions to it from many small bits of charge. Suppose that a thin glass rod of length b is charged positively with constant linear charge density λ; the rod is laid along the y-axis, as shown in Figure 10. We need to find the electric field, both magnitude and direction, on the x-axis a distance a from the origin. It is impossible to do this problem without using integration because each little bit of the charged rod produces an electric field of different magnitude and direction at $x = a$. To find the total electric field, we must add up all of the electric field vectors produced by the charges on the rod. We proceed, as usual, by slicing the rod up into tiny bits. The coordinate y labels each slice, so y is our variable of integration. The small amount of charge contained in each slice is just the linear charge density multiplied by the small length: $dq = \lambda dy$. Now we need to find the electric field produced at $x = a$ by this little bit of charge at position y on the y-axis. Be careful not to confuse the observation point ($x = a$, the place where we are to find the electric field) with the source point (y, the location of the little bit of charge that contributes to the electric field at the observation point). Since each slice of the rod is small, we may find the electric field it produces by using the point charge formula:

$$\mathbf{E} = \frac{k_e q \mathbf{r}}{r^3} \ .$$

This is the best form of the point charge formula to use because the addition we will be doing as we integrate is a vector addition, and the easiest way to add vectors is to split them into components and add the components together. The vector $\mathbf{r}$ in the numerator of this formula is the vector that points from the position of the charge, q, to the point where the field is to be evaluated. In our problem the charge, q, is the little bit of charge, $dq = \lambda dy$, and the vector, $\mathbf{r}$, is the one drawn in Figure 10. As you can see, $\mathbf{r} = a\mathbf{i} - y\mathbf{j}$, and hence the magnitude of $\mathbf{r}$ is given by $r = (y^2 + a^2)^{1/2}$. The little bit of electric field at $x = a$ due to the little bit of charge on the y-axis can then be written as

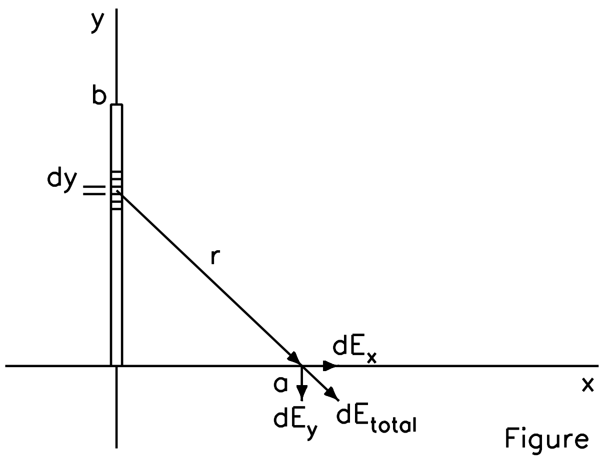


Figure 10

$$d\mathbf{E} = \frac{k_e \lambda dy (a\mathbf{i} - y\mathbf{j})}{(y^2 + a^2)^{3/2}} \ .$$

The x- and y-components of this expression are the x- and y-components of $d\mathbf{E}$, as shown in Figure 10. With the writing of this expression, all of the hard work is over. We now find the total electric field by adding up all of the contributions from $y = 0$ to $y = b$.

$$\mathbf{E} = \int d\mathbf{E} = \int_0^b \frac{k_e \lambda (a\mathbf{i} - y\mathbf{j}) dy}{(y^2 + a^2)^{3/2}} \ .$$

Notice that the unit vectors, $\mathbf{i}$ and $\mathbf{j}$, do not change as y is varied during the integration. The vector $\mathbf{i}$ always points in the x-direction and has magnitude 1, and the vector $\mathbf{j}$ always points in the y-direction and has magnitude 1. Hence, they can come outside the integral:

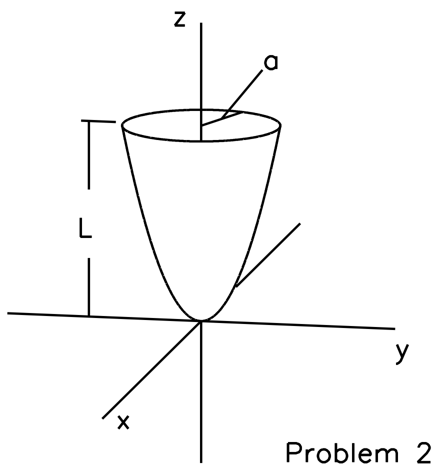
$$\mathbf{E} = k_e \lambda a \mathbf{i} \int_0^b \frac{dy}{(y^2 + a^2)^{3/2}} - k_e \lambda \mathbf{j} \int_0^b \frac{y dy}{(y^2 + a^2)^{3/2}} \ .$$

In the $\mathbf{i}$ integral, we are able to pull out $a\mathbf{i}$ because a is constant, but in the $\mathbf{j}$ integral, y must stay inside since it varies during the integration. We now have two integrations to perform; the one multiplied by $\mathbf{i}$ gives the x-component of the electric field while the one multiplied by $\mathbf{j}$ gives its y-component. We could work these integrals out using the techniques taught in Math 113, or we could get an integral table and simply look them up. With given values for λ, a, and b, we could even use a programmable calculator or a computer to find the answer. Whichever method is used, the problem is now reduced to standard mathematical operations. The hard part is arriving at the integral that is to be performed, and with some practice, even that part won't seem so hard.

Integration Problems

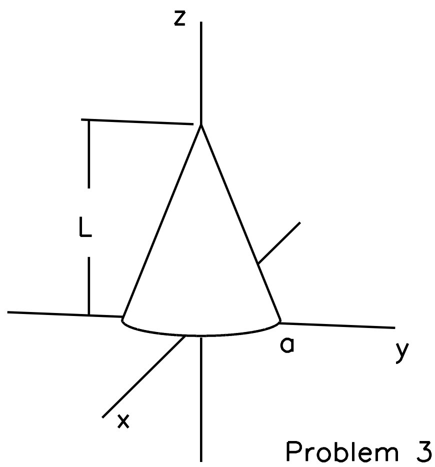
1. Consider the region in the x-y plane bounded below by the x-axis and bounded above by the curve $y = 1 - x^2$, for $-1 \le x \le 1$. Find the area of this region.

2. Consider a paraboloid "drinking cup," as shown. The height of the cup is L, and the radius of the cup at the top is a. The radius of the side of the cup as a function of height, z, is given by $r = a\sqrt{z/L}$. Find a formula for the volume of the cup.



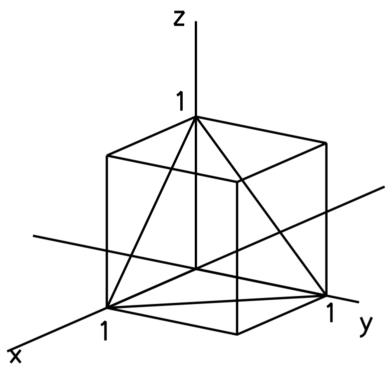
Problem 2

3. Consider a cone of height L and of base radius a. (a) Find its volume. (b) Find its surface area.



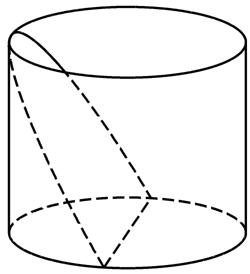
Problem 3

4. Consider the cube shown at the right. What fraction of its volume is contained in the pyramidal shape with vertices at (0,0,0), (1,0,0), (0,1,0), and (0,0,1)?



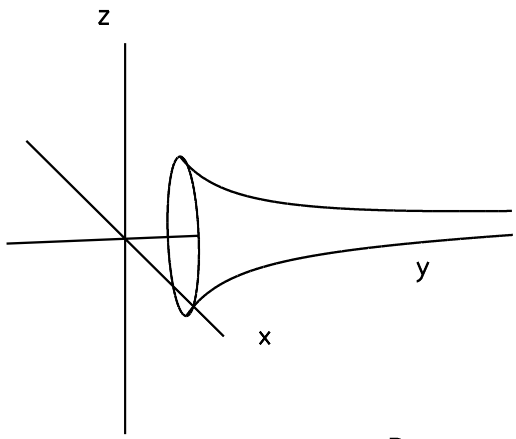
Problem 4

5. A cylindrical measuring cup that holds 2 cups is being used to measure water. It has been tipped so that the surface of the water just touches one lip of the cup and lies along a diameter of the circular bottom of the cup, as shown (the cup has been drawn upright for convenience in seeing how to set up the integration.) How many cups of water are in the cup?



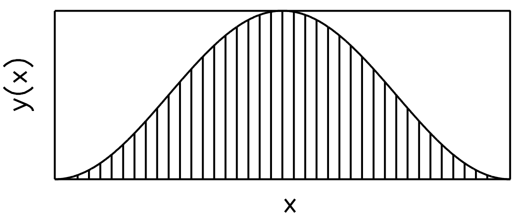
Problem 5

6. An infinitely-long funnel-shaped vessel has been formed by rotating about the y-axis the function $z = 1/y$ from $y = 1$ to infinity. Find the volume of this vessel. This vessel illustrates one of the oddities of infinite shapes: this one has a finite volume, but has infinite surface area(you might try proving this fact.)



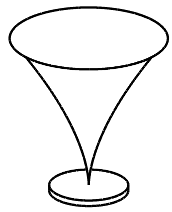
Problem 6

7. Consider the rectangle of length π and width 1, as shown. This rectangle is divided into two parts by the function $y = \sin^2 x$. What fraction of the area of the rectangle is in the shaded portion? Hint: $\sin^2 x = (1 - \cos(2x))/2$.



Problem 7

8. A very sharply-pointed martini glass has a shape given by $z = r^{3/5}$. The height of the martini glass is 1. Find the volume of the martini glass.



Problem 8

9. A long pipe oriented vertically along the z-axis contains a fluid whose density is given by $\rho = 1.0\ \text{kg/m}^3 e^{-0.1z}$, where z is in meters. If the radius of the pipe is 20 cm, find the mass of the fluid contained in the section of the pipe from $z = 0$ to $z = 5$ m.

Answers

1. $4/3$.
2. $\frac{\pi}{2}a^2L$.
3. $\frac{\pi}{3}a^2L$.
4. $1/6$.
5. $\frac{4}{3\pi}$ cups.
6. π .
7. $1/2$.
8. $\frac{3\pi}{13}$.
9. $\frac{2\pi}{5}(1 - e^{-.5})$.