Infinite Horizon Models of Entry and Exit

Introduction

Two approaches to industrial organization:

Game theory

- Emphasizes strategic interaction
- Results often depend on solution concepts, etc.
- Folk theorems suggest anything can happen
- Testable implications are rare (though this may reflect reality)

Dynamic competitive theory

- Ignores strategic interaction
- Emphasizes sunk costs and uncertainty
- Testable implications are common

Goals of the lectures

- Description of various models
- Exploration of technical tools
- Discussion of empirical relevance

A Taxonomy of Dynamic Competitive Models

Taxonomy according to the nature of the uncertainty

Internal (or idiosyncratic) shocks models, also called learning models, can be further divided into passive and active learning models. They emphasize randomness, such as uncertainty concerning productivity or outcomes for research and development, that is internal to the firm.

External (or common) shocks models emphasize randomness, such as uncertainty concerning demand conditions or factor prices, that is external to the firm.

Hybrids are also possible.

Taxonomy according to the treatment of time.

Discrete time models.

Continuous time models.

A Passive Learning Model in Discrete Time (Based on Jovanonic 1982)

Each firm is infinitesimal.

Each entrant independently draws a cost parameter $\theta \sim N(\overline{\theta}, \sigma_\theta^2)$. Entrants do not observe their draws. High θ means high cost.

After a firm produces in period t, it observes $\eta_t = \theta + \varepsilon_t$, where $\varepsilon_t \sim N(0, \sigma^2)$ iid.

Production decisions:

A firm's costs in period t are $c(q_t)x_t$, where $x_t = x(\eta_t)$.

> x is positive, continuous, and increasing with $\lim_{\eta_t \to -\infty} x(\eta_t) = \alpha_1 > 0$ and $\lim_{\eta_t \to \infty} x(\eta_t) = \alpha_2$.
> c satisfies $c(0) = 0$, $c'(0) = 0$, $c''(q) > 0$, and $\lim_{q \to \infty} c'(q) = \infty$.

An active firm in period t chooses output to maximize $p_t q_t - c(q_t)x_t^*$, where x_t^* is the expectation of x_t given past experience.

Entry and exit decisions:

Let ξ be the entry cost.

Let χ be the scrap value.

Let $p = \{p_t\}_{t=1}^{\infty}$ be a (deterministic) price path.

Let $\pi(p_t, x)$ be the maximized expected value of profit.

Let $V(x, n, t; p)$ be the value of a firm of age n at time t with expected cost parameter x, that remains active in period t and then behaves optimally.

Let $D_t(Q_t)$ be the (inverse) demand curve in period t.

Informal description of equilibrium:

Potential entrants enter in period t until $V(\overline{\theta}, 0, t; p) = \xi$, given the price sequence p.

Firms of age n and expected cost parameter x in period t exit if $V(x, n, t; p) \leq \chi$, given the price sequence p.

Active firms maximize current expected profit given current price. Markets clear.

A digression on dynamic programming

Let S be the set of states (e.g. expected efficiency).

Let A be the set of feasible actions.

Let $\rho(s'|s,a)$ be the probability that the new state is s' if the current state is s and the current action is a. (Note first-order Markov structure.)

Let $\pi(s,a)$ be the current return function.

Let $\beta \in (0,1)$ be the discount factor.

Let $d : S \to A$ be a decision rule and let D be the set of decision rules.

Consider the problem of choosing a decision rule to maximize the expected value of $\Sigma_{t=1}^{\infty}\beta^{t-1}\pi(s_t,a_t)$.

Let M be the set of bounded functions mapping S into $\Re$. Consider the mapping $U : M \to M$ defined by

$$U(m(s)) = \sup_{a\in A}\left\{\pi(s,a) + \beta\int_S m(s')d\rho(s'|s,a)\right\}.$$

Under technical continuity and boundedness conditions, U has a unique fixed point, say $V : S \to R$, called a value function. Thus

$$V(s) = \sup_{a \in A} \left\{ \pi(s,a) + \beta \int_S V(s')d\rho(s'|s,a) \right\}.$$

Under additional technical boundedness conditions, V can be used to find a decision rule that maximizes the expected value of $\Sigma_{t=1}^{\infty} \beta^{t-1} \pi(s_t, a_t)$.

Specifically, $d(s)$ maximizes $\pi(s,a) + \beta \int_S V(s')d\rho(s'|s,a)$ for each $s \in S$.

One approach to finding d(x)

For any initial $m \in M$, $V(s) = \lim_{n\to\infty} U(m(s))$. Use this to find V, and then use V to find d. (Can be done numerically for finite S.)

Another approach to finding d(x)

Guess the value function V.

End of digression on dynamic programming

A passive learning model, continued

Recall $V(x,n,t;p)$, the expected value of a firm of age n in time t with expected cost parameter x given the price sequence p. Using dynamic programing:

$$V(x,n,t;p) = \pi(x,n,t;p) + \beta \int \max\{\chi, V(z,n+1,t+1;p)dP(z|x,n)\}$$

where P reflects the expected efficiency in period $t+1$ conditioned on current efficiency and age of firm.

Theorem: $V(x,n,t;p)$ is decreasing in x.

Corollary: There is a unique value of x, say $\gamma(n,t;p)$, such that $V(\gamma,n,t;p) = \chi$. Thus an active firm's optimal exit rule induces exit iff $x_t \geq \gamma(n,t;p)$.

Some observations from the model for an exogenously given price sequence.

1) Since output depends positively on efficiency, exiting firms are smaller than average in a given cohort.

2)Since firms are ex ante identical, their efficiency diverges over time and production becomes increasingly concentrated in the most efficient firms (though not necessarily monotonically).

3) If prices are roughly constant over time, average profits in the industry rise for each cohort.

4) High profit leads to high growth: high efficiency today leads firms to expect high efficiency tomorrow and thus produce more. This effect is dampened over time for each cohort. This contrasts with the standard "liquidity constraint" explanation.

5) Simultaneous entry and exit occur, but not in the limit.

Endogenizing the price sequence

Recall the definition of the firms' optimal exit rule $\gamma(n, t; p)$.

Let $\Psi(x|t, \tau; p)$ be the probability that a firm entering at τ is still active at t with $x^* \leq x$.

Let $q(p_t|x)$ be profit maximizing output.

Let $\phi(t, \tau; p) = \int q(p_t, x) d\Psi(x|t, \tau; p)$ be output by cohort τ.

Let $Q_t(p, y) = \Sigma_{\tau=0}^{t} y_\tau \phi(t, \tau; p)$ be aggregate output, where y is an entry sequence.

Equilibrium: The pair of functions $q(p_t|x)$ and $\gamma(n, t; p)$, and a pair of nonnegative sequences (p, y) such that, for all t,

$$p_t = D_t(Q_t)$$

$$V(\overline{\theta}, 0, t; p) = \xi \text{ if } y_t > 0, \text{ and}$$

$$V(\overline{\theta}, 0, t; p) \leq \xi \text{ if } y_t = 0.$$

Note that the model is closed with rational expectations.

Equilibrium solves a social planner's problem.

This problem could be solved using dynamic programming, but the state space – which is the space of measures of (x_t^*, n), is unwieldy.

A digression on direct constrained optimization

Kuhn-Tucker Theorem: Let X be a vector space, Z a normed vector space, $\Omega \subseteq X$ a convex set, and P the positive cone in Z. Let $f : \Omega \rightarrow \Re$ and $G : \Omega \rightarrow Z$ be convex functions. Assume there exists $x^1 \in X$ such that $-G(x^1)$ is in the interior of P. Then there exists a linear functional $z^* \in X^*$ such that if x^o minimizes $f(x)$ subject to $G(\theta) \leq \theta$ then it minimizes $f(x) + z^*(G(x))$ and $z^*(G(x^o)) = 0.$

End of digression on direct constrained optimization

Think of a social planner choosing an entry sequence y, an exit rule γ, and an output function q. Call this triple a "plan." The cost of a plan is

$$\sum_{t=0}^{\infty} \beta^t \sum_{\tau=0}^{t} y_\tau \int_{\alpha_1}^{\alpha_2} [c(q_t(x))x] d\Psi_\gamma(x|t,\tau)$$

$$-\sum_{t=0}^{\infty} \beta^t \sum_{\tau=0}^{t} y_\tau [\Psi_\gamma(\alpha_2|t-1,\tau) - \Psi_\gamma(\alpha_2|t,\tau)]\chi$$

$$+\sum_{t=0}^{\infty} \beta^t y_t \xi$$

Under technical continuity and boundedness conditions, it can be shown that, for a given output sequence Q, there exists a plan that minimizes the cost of producing Q. (The Kuhn-Tucker theorem establishes first order conditions.)

Let $K(Q)$ be the cost of the cost-minimizing plan for producing Q.

Define social surplus by

$$S(Q) = \sum_{t=0}^{\infty} \beta^t \int_0^{Q_t} D_t(z)dz - K(Q)$$

It can be shown that $S(Q)$ is strictly concave. So there exists a unique optimal production sequence. Comparing first order conditions to equilibrium conditions verifies their equivalence. In particular:

$$\begin{aligned} \frac{dS(Q)}{dQ_t} &= \beta^t D_t(Q_t) - \frac{dK(Q)}{dQ_t} \\ &= \beta^t D_t(Q_t) - \lambda_t^* = 0 \end{aligned}$$

pins down the price sequence. Plugging this price sequence into the first order conditions for cost minimization returns the equilibrium conditions.

Theorem: Assume D_t is nondecreasing in t and that $q(p_t|x)$ is concave in x. Then the equilibrium price sequence is constant, $y_t > 0$ for all t, and $\Sigma_{t=0}^{\infty} y_t < \infty$.

An active learning model in discrete time (based on Ericson and Pakes 1995)

The number of firms is an integer.

Potential entrants make entry decision, entrants draw an efficiency parameter $\omega_t \in \Omega^e$, observe ω_t, and become incumbents in the next period. ω_t is an integer.

Incumbents decide whether to exit. Those that remain choose how much to invest in R&D, x_t. Their next efficiency parameter is a stochastic function of their current ω_t and their current x_t; $\rho(\omega_{t+1}|\omega_t, x_t)$ is the transition function.

Let s_t denote a state. It is a counting measure over the values of ω_t, that is, a list of how many incumbents exhibit each ω_t.

An incumbent's current payoff is $\pi(x_t; \omega_t, s_t) = A(\omega_t, s_t) - c(\omega_t)x_t$. A is a reduced form profit function, $c(\omega_t)$ is the (constant) marginal cost of R&D.

ξ is the entry cost; χ is the scrap value of the firms.

An exit rule is a stochastic sequence a interpreted as follows: $a_t = 1$ if the firm is active and $a_t = 0$ otherwise. For any realization, $a_t = 0$ implies $a_\tau = 0$ for all $t < t$.

An investment rule is a stochastic sequence x. Note $a_t = 0$ implies $x_t = 0$.

Incumbent's problem: Choose a and x at time t to maximize

$$\max\left\{ E_t\left[\sum_{\tau=t}^{\infty} \beta^{\tau-t}\pi(x_\tau;\omega_\tau,s_\tau)a_\tau + (a_{\tau-1} - a_\tau)\chi \right], \chi \right\}$$

Using dynamic programming, the incumbent's value function satisfies

$$V(\omega_t, s_t) =$$

$$\max\left\{ \max_{x_t}[\pi(x_t;\omega_t,s_t) + \beta E V(\omega_{t+1},s_{t+1})], \chi \right\}$$

Given the stochastic process that determines the expectation, an incumbent's optimal policy can be derived from the above equation.

Entrant's problem: Enter at time t if and only if, given the number of other entrants,

$$\beta EV(\omega_{t+1}, s_{t+1}) \geq \xi$$

Equilibrium is a list of stochastic functions:

$x(\omega, s)$, $a(\omega, s)$, $Q(s'|s)$, $m(s)$

such that

1) the investment function x and exit rule a solve the incumbent's problem given the stochastic process Q;

2) the number of entrants is the maximum number of entrants that solves the entrant's problem; and

3) given the entry process m, adoption of x and a by all incumbents generates the stochastic process Q.

Existence of equilibrium

Due to imperfect competition, equilibrium may not solve a social planner's problem.

Fortunately, under boundedness assumptions, it can be shown that if a firm becomes very efficient or very inefficient it stops investing.

Furthermore, the number of incumbents is bounded.

Thus the number of attainable states is finite.

This allows proof using Brouwer's fixed point theorem.

Digression on Brouwer's fixed point theorem

Definition: ℓ is a fixed point of Γ if $\ell = \Gamma(\ell)$.

Theorem: If L is convex and compact and if $\Gamma : L \to L$ is continuous, then Γ has a fixed point.

End of digression on Brouwer's fixed point theorem

Let $(\Delta^S)^S$ be the set of transition processes on S, where S is the finite set of attainable states.

Let $\bar{x}$ and $\bar{V}$ be upper bounds on investment and firm value. Under technical boundedness assumptions, these exist.

Define

$$\zeta : (\Delta^S)^S \to [0,\bar{x}]^{\Omega\times S} \times [\chi,\bar{V}]^{\Omega\times S} \times [\chi,\bar{V}]^{\Omega\times S}$$

as the function that associates optimal investment – and its associated entrant and incumbent value – with each transition process.

Define

$$\psi : [0,\bar{x}]^{\Omega\times S} \times [\chi,\bar{V}]^{\Omega\times S} \times [\chi,\bar{V}]^{\Omega\times S} \to (\Delta^S)^S$$

as the function that associates the induced transition process with an investment ruler and its associated entrant and incumbent values.

It can be shown that ζ and ψ are continuous; hence $\Gamma = \zeta \bullet \psi$ is continuous.

With S finite, $(\Delta^S)^S$ is compact and convex. So Γ has a fixed point and equilibrium exists.

Observations from the model (under reasonable assumptions):

1) Each firm dies in finite time.

2) Extremely efficient and extremely inefficient firms "coast" by doing no R&D.

3) Simultaneous entry and exit can occur.

4) Profits have no intertemporal pattern.

5) No relationship exists between profitability and growth.

6) Exiting firms are smaller then average.

In **passive learning models**, by contrast,

1) Efficient firms never die.

2) No R&D.

3) Entry and exit disappear in the limit.

4) Profits rise over time.

5) Profitable firms grow faster.

6) Exiting firms are smaller than average here too.

Digression on finite ergodicity

The active learning model exhibited ergodicity. Intuitively, current values of variable **do not** depend on the distant past.

The passive learning model did not. Intuitively, current values of variables **do** depend on the distant past.

Definition: A Markov process $Q(s'|s)$ is ergodic if, for all $s', s \in S$,

(1) $\eta_{s'} = \lim_{t\to\infty} Q^t(s'|s)$ exists and is independent of t;

(2) $\eta_s > 0$; and

(3) $\Sigma_s \eta_s = 1$.

Theorem: If S is finite and there exists τ such that

$$\min_{ss'} Q^{\tau}(s'|s) > 0$$

then $Q(s'|s)$ is ergodic. Furthermore, for all $s' \in S$,

$$\eta_{s'} = \Sigma_s Q(s'|s)$$

The distribution η is called the limiting distribution of Q and can be used to calculate long run averages.

End of digression on ergodicity.

Pakes and Ericson (1998) empirically investigated some of these distinctions and found that passive learning is more consistent with retailing while active learning is more consistent with manufacturing.

An external shocks model in discrete time (based on Lambson 1992)

Each firm is infinitesimal.

The market condition $m \in M$ describes all relevant exogenous variables.

The t-period history $h \in H_t$ lists the first t realized market conditions.

For $\tau \geq t$, $\rho(g|h)$ is the probability that $g \in H_\tau$ is realized if $h \in H_t$ is realized. Note that a Markov structure is not required.

All firms are identical, that is, they all have the same cost function, $\phi_m + C(q,m)$.

The inverse demand curve is market condition m is $p = d(yq,m)$.

ξ_m is the entry cost.

χ_m is the scrap value.

p and q are derived from y (given m) if $p = d(yq,m)$ and q maximizes $pq - C(q,m)$.

$\pi(y,m) = pq - C(q,m) - \phi_m$ where p and q are derived from y given m.

An exit rule $\gamma \subset \bigcup_{t=1}^{\infty} H_t \equiv H$ is a set of histories that induce exit.

Define $\theta(\gamma, s)$ as the set of s-period histories that do not induce exit.

Define $\gamma_s = \gamma \cap H_s$ as the set of s-period histories that induce exit.

Let Y be a stochastic sequence of y's. Define a firm's value in period t by

$$V(Y,h) = \pi(y(h), m(h))$$

$$+ \sup_{\gamma}\{\sum_{s=t+1}^{\infty} \beta^{s-1} \sum_{g\in\theta(\gamma,s)} \rho(g|h)\pi(y(g), m(g))$$

$$+ \sum_{s=t+1}^{\infty} \beta^{s-1} \sum_{g\in\gamma_s} \rho(g|h) x_{m(g)}\}$$

Given an initial mass of plants y^o, an equilibrium is a stochastic sequence $\{Y, Q, P\}$ such that Q and P are derived from Y and such that, for all $h \in H$,

$$V(Y,h) \leq \xi_{m(h)}$$

$$V(Y,h) = \xi_{m(h)} \text{ if } y(h) > y(h^{-1})$$

$$V(Y,h) \geq \chi_{m(h)} \text{ if } y(h) > 0$$

$$V(Y,h) = \chi_{m(h)} \text{ if } y(h^{-1}) > y(h) > 0$$

$$V(Y,h) \leq \chi_{m(h)} \text{ if } y(h^{-1}) > y(h) = 0$$

Digression on limit approach

Truncate to T-period model

Solve for intervals $[N^T(h), X^T(h)]$ by backward induction.

Note that intervals are bounded and monotonic in T, and hence have limits.

Take limits and construct
$y(h) = \min\{X(h), \max[N(h), y(h^{-1})]\}$.

End of digression on limit approach

Observations from the model

1) If m follows a Markov process then $s = (y, m)$ follows a Markov process, but y need not. (Hysteresis)

2) If m is ergodic then $s = (y, m)$ is ergodic and anything that depends on (y, m) – e.g. π – is ergodic.

3) Long run average profits do not equalize across industries. For example, if every change in market condition induces a change in y, and if ξ and χ are constants, then $\Sigma_s \eta_s \pi(s)/\xi$ is decreasing in ξ and increasing in χ. More generally, nothing can be said.

4) The range of firm value, $\xi - \chi$, equal sunk cost and thus is increasing in sunk cost.

5) The range in the mass of firms is decreasing in ξ and increasing in χ.

Digression on countable ergodicity

Let $\rho_{ss'}$ be the probability that s' occurs immediately after s.

Let $f^t_{ss'}$ be the probability that s' first occurrence after s occurs after t periods.

Let $f_{ss'}$ be the probability that s' occurs some time after s.

A state s is aperiodic if one is the largest integer value of d such that $\rho^t_{ss} > 0$ only if t is a multiple of d.

A state s is recurrent if $f_{ss} = 1$.

A recurrent state s is positive if $\Sigma_{t=1}^{\infty} t f^t_{ss} < \infty$, that is, its average time for recurrence is finite.

A set of states E is indecomposable if all states in E "communicate" with each other but not with states outside E.

Theorem: A Markov process exhibits a limiting distribution independent of initial conditions iff there is exactly one aperiodic, positive, recurrent class of states C such that $f_{ss'} = 1$ for all $s \in S$ and for all $s' \in C$.

If $\sup_m N_m > \inf_m X_m$, and if the conditions of the theorem hold for the stochastic process on market conditions, then the conditions of the theorem hold for the stochastic process on states.

Extension to heterogeneous firms

Same as before except entering firms must install a technology $\alpha \in A$. This extension allows for additional results:

1) Simultaneous entry and exit can occur when factor prices change.

2) Simultaneous entry and exit can occur when demand alone changes.

3) The vintage assumption is suspect.

Technical note: The limit approach fails in the extension because the interval for each technology depends on the number of firms adopting the others. The social planner's approach works at the cost of additional boundedness assumptions.

Another external shocks model in discrete time (based on Amir and Lambson 1999)

The framework is the same as before except there are only countably many firms.

Each firm i has a strategy $\sigma^i : S \to \{0,1\}$, where S is the set of histories. (A history includes past and current market conditions and past actions by the firms.)

An induced strategy σ_s^i is σ^i with domain restricted to histories following s.

σ and σ_s list all the firms' strategies and induced strategies, respectively.

Firm i's value given the induced strategy profile σ_s and the history s is

$$\begin{aligned} V_s^i(\sigma_s) = \sum_{\tau=t}^{\infty} \delta^{\tau-t} \sum_{g \geq h} \rho(g|h)\{ & a_g^i \pi(y_g, m_g) \\ & - \max(0, a_g^i - a_{g-1}^i)\xi_g \\ & + \max(0, a_{g-1}^i - a_g^i)\chi_g\} \end{aligned}$$

A **subgame perfect Nash equilibrium** is a list of strategies $\tilde{\sigma}$ such that $V_s^i(\tilde{\sigma}_s) \geq V_s^i(\sigma_s^i, \tilde{\sigma}_s^{-i})$ for all i, all S, and all σ_s^i.

Proof that equilibrium exists is more difficult than in continuum case because (1) analogous intervals are not monotonic in T and (2) each firm's strategy must be specified.

To handle (1), a "diagonalization argument" is used.

To handle (2), strategies can be defined by a LIFO rule:

If $a_{h-1}^i = 0$ then $\tilde{a}_h^i = 1$ iff $h = h(i)$ and $N_h - y_{h-1} > \eta^i$.

If $a_{h-1}^i = 1$ then $\tilde{a}_h^i = 1$ iff $X_h \geq \iota_h(i)$.

Application 1: Higher entry barriers can cause lower concentration.

Consider a Cournot market with two market conditions, H and L. Thus

$$\pi(y,m) = \left(\frac{a_m}{y+1}\right)^2, m \in \{H,L\}.$$

H occurs first; thereafter $\rho_H = \rho_L = .5$.

$a_H = 2; a_L = 1; \delta = .9; \chi = 1.35.$

If $\xi = 1.65$ **then**

Three firms enter immediately and never exit. No other firms enters unless there is exit. Thus $y_H = y_L = 3$.

If $\xi = 1.39$ **then**

One firm enters and never exits. Other firms enter – three at a time – when $m = H$ and exit when $m = L$. Thus $y_H = 4$ and $y_L = 1$.

In the first case, that is, $\xi = 1.65$,

$$V_H(3,3) \approx 1.656 > \xi > 1.469 \approx V_L(3,3) > \chi$$

In the second case, that is, $\xi = 1.39$

$$V_H^x(4,1) \approx 1.395 > \xi > 1.31 \approx V_H^x(5,1)$$

$$V_L(4,1) \approx 2.095 > \chi > 1.33 \approx V_L(4,2)$$

Other possible deviations can also be ruled out.

Note: Average number of firms over time is higher when ξ is higher.

Intuition: Although higher entry barriers reduce entry when times are good, they also reduce exit when times are bad. The net effect on the long-run average is ambiguous.

Application 2: Preemptive entry with negative current profits

Consider a Cournot model with three market conditions: H, M, L.

$$\pi(y,m) = \left(\frac{a_m}{y+1}\right)^2 - \phi$$

If $a_H = 15$, $a_M = 8$, $a_L = 3$, $\rho_H = .6$, $\rho_M = .3$, $\rho_L = .1$, $\xi = 1$, $\chi = 0$, $\phi = 2$

Then $y_H = 8$, $y_{MH} = 6$, $y_{ML} = 5$, $y_L = 2$

Furthermore: $\pi(8,H) \approx .78$ and $\pi(5,M) \approx -.22$

Thus entry in M (following L) exhibits negative current profits.

Note: This is inefficient; there is excessive entry.

Application 3: Product cycles

Consider a Cournot model such that, in period t,

$$\pi(y,t) = \left(\frac{a_t}{y+1}\right)^2$$

where $a_1 = 1$, $a_2 = 4 - \varepsilon$, and $a_t = 6$ for $t \geq 3$.

Furthermore, $\xi = 1$, $\chi = 0$, and $\delta = .5$

Consider the first 3 truncated models: $T = 1,2,3.$

$$y_1^1 = 0$$

$$y_1^2 = 1 \quad y_2^2 = 2$$

$$y_1^3 = 0 \quad y_2^3 = 4 \quad y_3^3 = 5$$

Note the nonmonotonicity. (Thus N_h^T and X_h^T might not converge.)

Application 4: Tendency for excessive numbers of firms

Let y_h^e be the equilibrium number of firms.

Let y_h^* be the optimal number of firms given post-entry imperfect competition.

Assume:

$Q_{y_h} > Q_{y_h'}$ for all h and for all $y_h > y_h'$

$q_{y_h} < q_{y_h'}$ for all h and for all $y_h > y_h'$

$p(Q_{y_h}, h) - c'(q_{y_h}, h) \geq 0$ for all h and for all y_h.

The following theorem extends Mankiw's and Whinston's result.

Theorem: For all h, $y_h^e \geq y_h^* - 1/$

An external shocks model in continuous time (based on Dixit 1989)

A partial equilibrium model, focusing on a single firm.

The firm is either inactive or active.

If inactive, it can pay an entry fee ξ and become active.

If active, it can accept a scrap value χ and become inactive.

While active it pays a variable production cost of w and produces one unit of output, which it sells at price p.

r is the discount rate.

ξ, χ, and w are constants.

p follows a geometric Brownian motion.

Digression on Brownian motions and geometric Brownian motions

Given an initial state $x_o \in \Re$, a stochastic process x follows a Brownian motion if, for all $t \geq 0$, x_t is normally distributed with mean $x_o + vt$ and variance $\eta^2 t$, where $v \in \Re$ and $\eta \in \Re_+$.

A Wiener process is a Brownian motion with $v = 0$ and $\eta^2 = 1$.

Thus for small changes, dx has mean vdt and variance $\eta^2 dt$ for a Brownian motion.

A Brownian motion can be decomposed as follows:

$$dx = vdt + \eta dz$$

where z follows a Wiener process.

A Brownian motion is the continuous-time limit of a discrete-time random walk.

Brownian motions are inappropriate for nonnegative price sequences. So let $p = e^x$, where x follows a Brownian motion. Then p follows a geometric Brownian motion.

A geometric Brownian motion may be written as follows:

$$\frac{dp}{p} = \mu dt + \sigma dz$$

where $\mu \in \Re$ and $\sigma \in \Re_+$.

Recall that if x follows a Brownian motion with parameters v and η, then x_t is normally distributed with mean $x_o + vt$ and variance $\eta^2 t$, where $v \in \Re$ and $\eta \in \Re_+$.

If $p = e^x$ follows the associated geometric Brownian motion, it has parameters $\mu = v + \frac{1}{2}\sigma^2$ and $\sigma = \eta$. Furthermore, p_t is distributed lognormally with mean $p_o e^{\mu t}$ and variance $p_o^2 e^{2\mu t}(e^{\sigma^2 t} - 1)$.

End of digression on Brownian motions

If p_t where constant then a firm would enter if

$$\frac{p_t - w}{r} \geq \xi$$

and would exit if

$$\frac{p_t - w}{r} \leq \chi$$

Let p^H and p^L solve the respective equations.

Dixit shows that entry (exit) is only provoked by prices strictly higher (lower) than p^H (p^L). His method, borrowed from finance, is informal but instructive.

Digression on (one form of) Ito's lemma

Suppose p follows a geometric Brownian motion:

$$dp = p\mu dt + p\sigma dz$$

Let $V(p)$ be a smooth function of p. Surprisingly,

$$dV \neq V'(p)dp$$

To see this intuitively, look at the approximate Taylor series expansion, where *h.o.t.* stands for higher order terms in dt:

$$
\begin{aligned}
dV &= V'(p)dp + \frac{1}{2}V''(d)(dp)^2 + h.o.t. \\
&= V'(p)(p\mu dt + p\sigma dz) + \frac{1}{2}V''(d)(p\mu dt + p\sigma dz)^2 \\
&\quad + h.o.t. \\
&= V'(p)(p\mu dt + p\sigma dz) \\
&\quad + \frac{1}{2}V''(p)(p^2\mu^2(dt)^2 + 2p^2\mu\sigma dtdz + p^2\sigma^2(dz)^2) \\
&\quad + h.o.t. \\
&= V'(p)(p\mu dt + p\sigma dz) \\
&\quad + \frac{1}{2}V''(p)(2p^2\mu\sigma dtdz + p^2\sigma^2(dz)^2) + h.o.t.
\end{aligned}
$$

Note that dV equals $V'(p)dp$ (as usual) **plus** a term that depends on V''. (Other terms a higher order.) Taking expectations, recalling that $E(dz) = 0$ and $E(dz^2) = dt$ yields:

$$E(dV) = V'(p)p\mu dt + \frac{1}{2}V''(p)p^2\sigma^2 dt$$

End of digression on Ito's lemma.

Let $V_O(p)$ be the value of being out of the market when the price is p, given an optimal entry strategy.

Let $V_I(p)$ be the value of being in the market when the price is p, given an optimal exit strategy.

Note that both being in and being out have option values.

Consider a firm that is out and intends to remain out for a little while. Its expected present value V_O can be broken into two parts:

$$V_O(p) = 0 + e^{-rdt}E[V_O(p + dp)]$$

Since $e^{-rdt} = \Sigma_{n=0}^{\infty}(-rdt)^n/n!$, $e^{-rdt} \approx 1 - rdt$. So, suppressing higher order terms in dt,

$$V_O(p) = (1 - rdt)[V_O(p) + E[V_O(p + dp)] - V_O(p)]$$
$$= (1 - rdt)V_O(p) + E(dV_O)$$

and thus

$$rV_O(p)dt = E(dV_O)$$

(This has an economic interpretation.)

Rewriting $E(dV_O)$ using Ito's lemma yields

$$rV_O(p)dt = V_O'(p)p\mu dt + \frac{1}{2}V_O''(p)p^2\sigma^2 dt \tag{*}$$

Now consider a firm that is in and intends to remain out for a little while. Its expected present value V_I can be broken into two parts:

$$V_O(p) = (p-w)dt + e^{-rdt}E[V_O(p+dp)]$$

The same argument as before establishes that this can be written

$$rV_I(p)dt = (p-w)dt + E(dV_I)$$

(which has a similar appealing economic interpretation). Rewriting $E(dV_I)$ using Ito's lemma yields

$$rV_I(p)dt = (p-w)dt + V_I'(p)p\mu dt + \frac{1}{2}V_I''(p)p^2\sigma^2 dt \tag{**}$$

Thus V_O and V_I, respectively, solve the two differential equations given by (*) and (**). These have solutions:

$$V_O(p) = A_O p^{-\alpha} + B_O p^{\beta}$$

$$V_I(p) = A_I p^{-\alpha} + B_I p^{\beta} + \left(\frac{p}{\rho - \mu} - \frac{w}{\rho}\right)$$

where $\alpha > 0$ and $\beta > 0$ are known functions of μ and σ and where A_O, B_O, A_I, and B_I are to be determined. They reflect option values.

A_O reflects the option value of being able to enter when prices are very low. Thus $A_O = 0$.

B_1 reflects the option value of being able to exit when prices are very high. Thus $B_I = 0$.

Determining $B_O \geq 0$ and $A_I \geq 0$ requires application of the value-matching and smooth-pasting conditions.

Value-matching conditions

Let p^h be the price at which an inactive firm would enter. Then

$$V_O(p^h) = V_I(p^h) - \xi$$

Let p^ℓ be the price at which an active firm would exit. Then

$$V_I(p^\ell) = V_O(p^\ell) + \chi$$

Smooth-pasting conditions

$$V_O'(p^h) = V_I'(p^h)$$

$$V_I'(p^\ell) = V_O'(p^\ell)$$

The previous six equations allow simultaneous solutions for B_O, A_I, p^h and p^ℓ.

Main result: $p^h > P^H$ and $p^\ell < P^L$ if $\xi > 0$ or if $\chi < 0$.

Some numerical results:

If $w = 1; \xi = 4; r = .025; \chi = 0; \sigma = .01; \mu = 0.$

Then

$p^h = 1.4667; p^H = 1.1; p^\ell = 0.7657; p^L = 1.$

Drawback: Aggregate entry and exit behavior is ignored. This is partially overcome by Caballero and Pindyck 1996.

Some empirical results (based on Lambson and Jensen 1995, 1998)

Let i index firms.

Let I be the set of firms in industry I.

Let t index time.

An implication of learning models:

$$\begin{aligned} Range_{i \in I} V_{it} &= \max_i V_{it} - \min_i V_{it} \\ &= (\max_i V_{it} - \bar{\xi}) + (\bar{\xi} - \bar{\chi}) - (\min_i V_{it} - \bar{\chi}) \end{aligned}$$

where bars denote industry averages.

An implication of external shocks models:

$$\begin{aligned} Range_t V_{it} &= \max_t V_{it} - \min_t V_{it} \\ &= (\max_t V_{it} - \xi_i) + (\xi_i - \chi_i) - (\min_t V_{it} - \chi_i) \end{aligned}$$

Table 1. Additional Sunk Costs[a] for Orchards Compared to Cropland

Location	Crop	Year	Amount per Acre[b]
Northern San Joaquin Valley, California	Apricots	1991	$1,797
Southern San Joaquin Valley, California	Pistachios	1990	$2,155
Sierra Nevada	Plums	1988	$2,300-$5,400
Foothills, California	Pears	1988	$3,100-$5,300
	Nectarines	1988	$2,150-$4,700
	Cherries	1988	$2,600-$5,200
Glenn and Tehama	Olives	1989	$1,200
Ventura County, California	Avocados	1987	$3,716
South Florida Flatwoods	Citrus	1990-91	$2,154
North Florida	Citrus	1990-91	$2,454

[a] Additional sunk costs include fumigation, tree costs, survey and planting costs, training, pruning, thinning, cultivation after planting, irrigation costs after planting, fertilization, pesticides and labor costs up to year of first harvest. Not included are land costs, land preparation, level levelling, canals, ditches, drainage systems, dikes, reservoirs, irrigation systems, property taxes and overhead costs.

[b] Information was obtained a number of different university and cooperative extension service bulletins.

Table 2. Farm Real Estate Value as a Percent of Total Farm Assets

Year	California	Florida
1960	86.5	85.5
1970	85.0	83.3
1980	86.5	87.9
1990	85.5	85.3
1991	85.1	84.0

Source: U.S. Department of Agriculture. Farm Business Balance Sheet, 1960-91. USDA Statistical Bulletin No. 856, Washington, D.C. May 1993.

Table 3. Average Real Normalized Ranges for Orchard and Nonorchard Land by Region, California 1981-90

Region	Orchard	Nonorchard
1	2,090	586
2	2,641	609
3	2,939	738
4	3,269	813
5	--	362
Mean*	2,735	686

* Excluding Region 5 because of missing values.

Table 4. Average Real Normalized Ranges for Orchard and Nonorchard Land by County, Florida 1984-91.

County	Orchard	Nonorchard
Brevard	1169	921
Broward	8649	9222
Charlotte	8582	1843
Citrus	–	494
Collier	2775	1118
Dade	–	1776
DeSoto	6355	623
Glades	4375	536
Hardee	3646	328
Hendry	3627	324
Hernando	125	965
Highlands	6147	1218
Hillsborough	4249	1776
Indian River	4524	2257
Lake	4340	882
Lee	7048	1852
Manatee	1775	1329
Marian	4884	760
Martin	2248	1566
Okeechobee	2755	437
Orange	2620	1621
Osceola	1863	623
Palm Beach	–	2552
Pasco	9968	981
Polk	3879	725
Lucie	2110	534
Sarasota	7366	1650
Seminole	2782	905
Sumter	–	588
Volusia	1468	1683
Washington	3353	1361
Mean*	4173	1409

*Excluding Citrus, Dade, Palm Beach, and Sumter counties because of missing values.

Table 5. Results of Regression Analysis

Dependent Variable[a]	State	Intercept	Orchard Binary Variable	Adjusted R^2
Range	California	612	2,057 (6.56)	.40
Range	Florida	1,399	2,714 (7.86)	.26

t values are included in parentheses.

a. Dependent variable is range of real normalized land value.

Table 1: Sunk Cost Regressions

Model	Variable	Coefficient	S.E.	R^2	Observations
Learning	Intercept	1958.37	211.76	.34	6266
	K	9.71	.17		
External Shocks	Intercept	1625.21	138.56	.04	4209
	K	1.21	.09		

For the learning model an observation corresponds to an industry in a given year, the dependent variable is the intra-industry range of real firm value, and K is the intra-industry mean of firm-level capital costs. For the external shocks model an observation corresponds to a firm, the dependent variable is the intertemporal range of real firm value, and K is the intra-industry mean of the intertemporal mean of firm-level capital costs.

Table 2: Size Regressions

Model	Variable	Coefficient	S.E.	R^2	Observations
Learning	Intercept	1120.42	226.12	.31	6064
	Range(L)	140.99	2.68		
External Shocks	Intercept	731.88	115.30	.35	3927
	L	168.08	3.70		

For the learning model an observation corresponds to an industry in a given year, the dependent variable is the intra-industry range of real firm value, and Range(L) is the intra-industry range of firm-level employment. For the external shocks model an observation corresponds to a firm, the dependent variable is the intertemporal range of real firm value, and L is the intertemporal mean of firm-level employment.

Table 3: Normalized Sunk Cost and Size Regressions

Model	Variable	Coefficient	S.E.	R^2	Observations
Learning	Intercept	-2648.91	1213.04	.15	6042
	K/L	94.28	2.86		
	Range(L)	-9.95	14.12		
External Shocks	Intercept	-685.75	679.23	.07	3876
	K/L	11.56	.69		
	L	-.26	20.94		

For the learning model an observation corresponds to an industry in a given year, the dependent variable is the intra-industry range of real firm value per worker, K/L is the intra-industry average of firm-level capital costs per worker, and Range(L) is the intra-industry range of firm-level employment. For the external shocks model an observation corresponds to a firm, the dependent variable is the intertemporal range of real firm value per worker, K/L is the intra-industry mean of the intertemporal mean of firm-level capital costs per worker, and L is the intertemporal mean of firm-level employment.