# A Simple Real Business Cycle Model

**Major Features of the Model**
One source of uncertainty: $z$
Stochastic technology growth about a deterministic trend

We first define the important terms that will be used throughout:
Endogenous variables that change over time:

| | |
|---|---|
| $z$ | productivity (temporary or permanent) |
| $K$ | capital stock owned by household |
| $C$ | consumption |
| $w$ | wage rate |
| $r$ | interest rate |
| $Y$ | output of final goods |

Parameters:

| | |
|---|---|
| $\alpha$ | capital share in output from a Cobb-Douglas production function |
| $\delta$ | rate of depreciation |
| $\beta$ | time discount factor; $\beta<1$ |
| $a$ | trend in $z$ |
| $\gamma$ | elasticity of substitution, $\gamma>0$ |
| $\rho$ | autocorrelation parameter for $z$; $0<\rho<1$ |
| $\sigma$ | standard deviations of the shocks to $z$; $0<\sigma$ |
| $H$ | fixed labor endowment |

We next setup the timing of the model, where a prime on a variable indicates its value next period.

1) Beginning of period –$z$ known
2) Factor markets open & clear
   $K$ is loaned out to production firms and $r$ is determined
   $H$ is hired out to production firms and $w$ is determined
3) Production of goods occurs
4) Factor payments made ($wH$, $rK$)
5) $K'$ is chosen
   Consumption, $C$, occurs
6) Temporary shocks, $z'$ revealed
   End of period

**Nonstationary Model**
**Households**
Given information on prices and shocks, $\mathbf{\Omega} = \{w, r, z\}$, the household solves the following non-linear program when the factor markets clear.

$$V(K, \mathbf{\Omega}) = \underset{K'}{Max}\, u(C) + \beta E\{V(K', \mathbf{\Omega}')\}$$

where:

$$C = wH + (1-\delta + r)K - K' \tag{1.1}$$

The first-order condition is:
One condition for *K*':

$$u'(C)(-1) + \beta E\{V_K^i(K', \Omega')\} = 0$$

The envelope condition from this problem is as follows.
One condition for *K*:

$$V_K(K, \Omega) = u'(C)(1-\delta + r)$$

Picking functional form of $u(C) = \frac{1}{1-\gamma}(C^{1-\gamma} - 1)$

$$u'(C) = C^{-\gamma}$$

The Euler equation from combining the first-order condition with next period's version of the envelope condition is:

$$1 = \beta E\{(\tfrac{C}{C'})^{\gamma}(1-\delta + r')\} \tag{1.2}$$

**Additional Behavioral Equations**
The law of motion for *z* is:

$$z' = \rho z + \varepsilon'\text{; where } \varepsilon' \text{ is distributed normal with a mean of 0 and a variance of } \sigma^2 \tag{1.3}$$

An assumption of competition in the goods market along with a Cobb-Douglas production function gives the following shares in output for labor & capital. Market clearing conditions have already been imposed.

$$Y = K^{\alpha}(e^{at+z}H)^{1-\alpha} \tag{1.4}$$

$$wH = (1-\alpha)Y \tag{1.5}$$

$$rK = \alpha Y \tag{1.6}$$

**Definitions for Later Use**

$$I \equiv K' - (1-\delta)K \tag{1.7}$$

$$A \equiv e^{at+z} \tag{1.8}$$

Eqs (1.1)-(1.8) are the system.

**Transformation & Simplifications**

1) If $z$ is stationary ( $\rho < 1$ ):

Transform the problem by dividing all growing variables by $e^{at}$ , denoting with a carat.

$z' = \rho z + \varepsilon'$ ; where $\varepsilon'$ is distributed normal with a mean of 0 and a variance of $\sigma^2$ (2.1)

$$\hat{C} = \hat{w}H + (1-\delta+r)\hat{K} - (1+a)\hat{K}' \tag{2.2}$$

$$1 = \beta E\left\{\left(\frac{\hat{C}}{(1+a)\hat{C}'}\right)^{\gamma}(1-\delta+r')\right\} \tag{2.3}$$

$$\hat{Y} = \hat{K}^{\alpha}(e^{z}H)^{1-\alpha} \tag{2.4}$$

$$\hat{w}H = (1-\alpha)\hat{Y} \tag{2.5}$$

$$r\hat{K} = \alpha\hat{Y} \tag{2.6}$$

$$\hat{I} = (1+a)\hat{K}' - (1-\delta)\hat{K} \tag{2.7}$$

$$\hat{A} = e^{z} \tag{2.8}$$

2) If $z$ is non-stationary ( $\rho = 1$ ):

Transform the problem by dividing all growing variables by $e^{at+z}$ , denoting with a carat.

$\Delta z' = \varepsilon'$ ; where $\varepsilon'$ is distributed normal with a mean of 0 and a variance of $\sigma^2$ (2.1)

$$\hat{C} = \hat{w}H + (1-\delta+r)\hat{K} - (1+a+\Delta z')\hat{K}' \tag{2.2}$$

$$1 = \beta E\left\{\left(\frac{\hat{C}}{(1+a+\Delta z')\hat{C}'}\right)^{\sigma}(1-\delta+r')\right\} \tag{2.3}$$

$$\hat{Y} = \hat{K}^{\alpha}H^{1-\alpha} \tag{2.4}$$

$$\hat{w}H = (1-\alpha)\hat{Y} \tag{2.5}$$

$$r\hat{K} = \alpha\hat{Y} \tag{2.6}$$

$$\hat{I} = (1+a+\Delta z')\hat{K}' - (1-\delta)\hat{K} \tag{2.7}$$

$$\hat{A} = 1 \tag{2.8}$$

These are the equations we will use in Dynare.
The endogenous variables are $\hat{C}, \hat{K}, \hat{Y}, \hat{w}, r, \hat{I}, \hat{A},$ & $\Delta z$ or $z$ .
The exogenous variable is $\varepsilon$.
The parameters are $\alpha, \delta, \beta, a, \gamma, \rho, \sigma$ & $H$ .

**Steady State**

System of 6 equations in 6 unknowns, $\gamma, \bar{z}$ or $\Delta\bar{z}, \overline{C}, \overline{K}, \overline{w}, \overline{Y}$

Parameters are $\bar{r}, H, \alpha, \beta, \delta, a$

Note that we could switch $\bar{r}$, $\gamma$ and $a$, but by (3.3) any two determine the remaining one.

$$\bar{z} = 0 \text{ or } \Delta\bar{z} = 0 \tag{3.1}$$

$$\overline{C} = \overline{w}H + (\bar{r} - \delta - g)\overline{K} \tag{3.2}$$

$$(1+a)^{\gamma} = \beta(1 - \delta + \bar{r}) \tag{3.3}$$

$$\overline{Y} = \overline{K}^{\alpha} H^{1-\alpha} \tag{3.4}$$

$$\overline{w}H = (1-\alpha)\overline{Y} \tag{3.5}$$

$$\bar{r}\overline{K} = \alpha\overline{Y} \tag{3.6}$$

(3.1) eliminates $\bar{z}$ or $\Delta\bar{z}$

(3.3) eliminates $\sigma$

$$\gamma = \frac{\ln\beta + \ln(1-\delta+\bar{r})}{\ln(1+a)} \cong \frac{\ln\beta + \bar{r} - \delta}{a} \tag{3.3'}$$

(3.4) into (3.5) & (3.6) and eliminate $\overline{Y}$

$$\overline{w} = (1-\alpha)\overline{K}^{\alpha} H^{-\alpha} \tag{3.5'}$$

$$\bar{r} = \alpha\overline{K}^{\alpha-1} H^{1-\alpha} \tag{3.6'}$$

(3.6') yields a solution for $\overline{K}$

$$\overline{K} = \left(\frac{\alpha}{\bar{r}}\right)^{\frac{1}{1-\alpha}} H \tag{3.7}$$

**Parameterization**

We need to choose parameters that are reasonable given evidence from micro studies or from other sources. Picking parameters so that the model matches empirical evidence is the simulation version of data mining.

| | |
|---|---|
| $\alpha$ | .3 – average observed US capital share |
| $\delta$ | .02 – quarterly rate of depreciation |
| $\beta$ | .995 – quarterly time discount factor |
| $a$ | .008341 – average quarterly growth of GDP in post-war USA |
| $\bar{r}$ | .026214 – implies a user cost of capital of 2.5% APR |
| $H$ | normalize to one |

Using equations (3.3') & (3.7) these give:

$\gamma = 0.1423$

$\overline{K} = 32.5288$

Parameterizing the law of motion

| | |
|---|---|
| $\rho$ | .9 |
| $\sigma$ | .02 |