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Physics 441: Assignment #2 - Integral Calculus

Due on Friday, May 17, 2013

May 15, 2013

Problem 1.25

Using vectors

$$\boldsymbol{A} = (x)\hat{\boldsymbol{x}} + (2y)\hat{\boldsymbol{y}} + (3z)\hat{\boldsymbol{z}} \qquad \boldsymbol{B} = (3y)\hat{\boldsymbol{x}} + (-2x)\hat{\boldsymbol{y}} + (0)\hat{\boldsymbol{z}}$$

Check the following identities:

1. $\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) + \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})$
2. $\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A}) + (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A}$
3. $\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A})$

For this problem we will use the following definitions:

- $\nabla \times \boldsymbol{A} = (0)\hat{\boldsymbol{x}} + (0)\hat{\boldsymbol{y}} + (0)\hat{\boldsymbol{z}}$
- $\nabla \times \boldsymbol{B} = (0)\hat{\boldsymbol{x}} + (0)\hat{\boldsymbol{y}} + (-5)\hat{\boldsymbol{z}}$
- $\nabla \times \boldsymbol{A} = 6$
- $\nabla \times \boldsymbol{B} = 0$
- $\boldsymbol{A} \times \boldsymbol{B} = (6xz)\hat{\boldsymbol{x}} + (9yz)\hat{\boldsymbol{y}} + (-2x^2 - 6y^2)\hat{\boldsymbol{z}}$
- $\boldsymbol{A} \cdot \boldsymbol{B} = -xy$

We will now verify each of the identities:

1.

$$\begin{aligned}15z = \nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) &= \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})\\ &= \boldsymbol{B}\cdot\big((0)\hat{\boldsymbol{x}}+(0)\hat{\boldsymbol{y}}+(0)\hat{\boldsymbol{z}}\big) - \boldsymbol{A}\cdot\big((0)\hat{\boldsymbol{x}}+(0)\hat{\boldsymbol{y}}+(-5)\hat{\boldsymbol{z}}\big)\\ &= 0-(-15z)\\ &= 15z \quad \square\end{aligned}$$

2. Below I make the following substitutions:

   - $\boldsymbol{A}\times[(0)\hat{\boldsymbol{x}}+(0)\hat{\boldsymbol{y}}+(-5)\hat{\boldsymbol{z}}] = (-10y)\hat{\boldsymbol{x}}+(5x)\hat{\boldsymbol{y}}+(0)\hat{\boldsymbol{z}}$
   - $\boldsymbol{B}\times[(0)\hat{\boldsymbol{x}}+(0)\hat{\boldsymbol{y}}+(0)\hat{\boldsymbol{z}}] = 0$
   - $(\boldsymbol{A}\cdot\nabla)\boldsymbol{B} = (6y)\hat{\boldsymbol{x}}+(-2x)\hat{\boldsymbol{y}}+(0)\hat{\boldsymbol{z}}$
   - $(\boldsymbol{B}\cdot\nabla)\boldsymbol{A} = (3y)\hat{\boldsymbol{x}}+(-4x)\hat{\boldsymbol{y}}+(0)\hat{\boldsymbol{z}}$

$$\begin{aligned}(-y)\hat{\boldsymbol{x}}+(-x)\hat{\boldsymbol{y}}+(0)\hat{\boldsymbol{z}} = \nabla(\boldsymbol{A}\cdot\boldsymbol{B}) &= \boldsymbol{A}\times(\nabla\times\boldsymbol{B}) + \boldsymbol{B}\times(\nabla\times\boldsymbol{A}) + (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A}\\ &= (-10y)\hat{\boldsymbol{x}}+(5x)\hat{\boldsymbol{y}}+(0)\hat{\boldsymbol{z}}+0+(6y)\hat{\boldsymbol{x}}+(-2x)\hat{\boldsymbol{y}}+(0)\hat{\boldsymbol{z}}+(3y)\hat{\boldsymbol{x}}+(-4x)\hat{\boldsymbol{y}}+(0)\hat{\boldsymbol{z}}\\ &= (-y)\hat{\boldsymbol{x}}+(-x)\hat{\boldsymbol{y}}+(0)\hat{\boldsymbol{z}} \quad \square\end{aligned}$$

3. Below I make the following substitution (as well as some from the previous part): $\nabla\times(\boldsymbol{A}\times\boldsymbol{B}) = (-21y)\hat{\boldsymbol{x}}+(10x)\hat{\boldsymbol{y}}+(0)\hat{\boldsymbol{z}}$.

$$\begin{aligned}(-21y)\hat{\boldsymbol{x}}+(10x)\hat{\boldsymbol{y}}+(0)\hat{\boldsymbol{z}} = \nabla\times(\boldsymbol{A}\times\boldsymbol{B}) &= (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + \boldsymbol{A}(\nabla\cdot\boldsymbol{B}) - \boldsymbol{B}(\nabla\cdot\boldsymbol{A})\\ &= (3y)\hat{\boldsymbol{x}}+(-4x)\hat{\boldsymbol{y}}+(0)\hat{\boldsymbol{z}} - (6y)\hat{\boldsymbol{x}}+(-2x)\hat{\boldsymbol{y}}+(0)\hat{\boldsymbol{z}}+\boldsymbol{A}(0)+\boldsymbol{B}(6)\\ &= (3y)\hat{\boldsymbol{x}}+(-4x)\hat{\boldsymbol{y}}+(0)\hat{\boldsymbol{z}} - (6y)\hat{\boldsymbol{x}}+(-2x)\hat{\boldsymbol{y}}+(0)\hat{\boldsymbol{z}}+(18y)\hat{\boldsymbol{x}}+(-12x)\hat{\boldsymbol{y}}+(0)\hat{\boldsymbol{z}}\\ &= (-21y)\hat{\boldsymbol{x}}+(10x)\hat{\boldsymbol{y}}+(0)\hat{\boldsymbol{z}} \quad \square\end{aligned}$$

Problem 1.27

Prove that the divergence of a curl is always zero. Check it for

$$\boldsymbol{v} = (x^2)\hat{\boldsymbol{x}}+(3xz^2)\hat{\boldsymbol{y}}+(-2xz)\hat{\boldsymbol{z}}$$

The general form for the curl of a vector function is

$$\nabla\times\boldsymbol{F}(x,y,z) = \left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\hat{\boldsymbol{x}} - \left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial F_x}{\partial y}-\frac{\partial F_y}{\partial x}\right)\hat{\boldsymbol{z}}$$

Problem 1.27 continued on next page...

Taking the divergence of this we get:

$$\begin{aligned}\nabla \cdot \left(\nabla \times \boldsymbol{F}(x,y,z)\right) &= \nabla \cdot \left[\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\boldsymbol{x}} - \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x}\right)\hat{\boldsymbol{z}}\right] \\ &= \frac{\partial}{\partial x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x}\right) \\ &= \left[\frac{\partial^2 F_z}{\partial x \partial y} - \frac{\partial^2 F_z}{\partial y \partial x}\right] + \left[\frac{\partial^2 F_y}{\partial x \partial z} - \frac{\partial^2 F_y}{\partial z \partial x}\right] + \left[\frac{\partial^2 F_x}{\partial z \partial y} - \frac{\partial^2 F_x}{\partial y \partial z}\right] \qquad = 0 + 0 + 0 = 0 \quad \square\end{aligned}$$

Now I will verify using the vector $\boldsymbol{v}$

$$\begin{aligned}\nabla \cdot (\nabla \times \boldsymbol{v}) &= \nabla \cdot \left((-6xz)\hat{\boldsymbol{x}} + (2z)\hat{\boldsymbol{y}} + (3z^2)\hat{\boldsymbol{z}}\right) \\ &= \frac{\partial}{\partial x}[-6xz] + \frac{\partial}{\partial y}[2z] + \frac{\partial}{\partial z}\left[3z^2\right] \\ &= -6z + (6z) = 0 \quad \square\end{aligned}$$

Problem 1.32

Check the fundamental theorem for gradients, using $T = x^2 + 4xy + 2yz^3$, the points $\boldsymbol{a} = (0,0,0), \boldsymbol{b} = (1,1,1)$, and the three paths in Figure 1. The paths are:

1. $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$
2. $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$
3. The parabolic path $z = x^2, y = x$

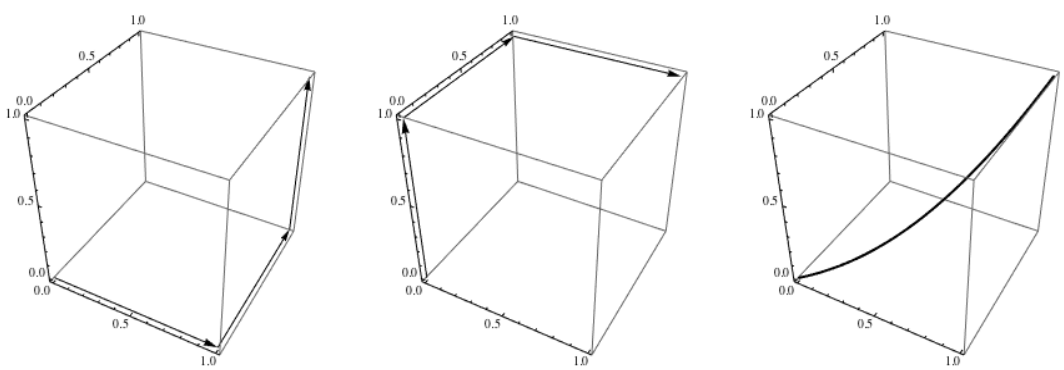

Figure 1: The three paths for problem 1.32

We first start with

$$\nabla T = (2x + 4y)\hat{\boldsymbol{x}} + (4x + 2z^3)\hat{\boldsymbol{y}} + (6yz^2)\hat{\boldsymbol{z}}$$

We will be integrating

$$\nabla T \cdot dl = (2x + 4y)\partial x + (4x + 2z^3)\partial y + (6yz^2)\partial z$$

1. The three integrals below represent the three segments.
   - $y = z = \partial y = \partial z = 0$ x from 0 to 1: $\int_0^1 (2x + 4[y = 0])\,dx = x^2|_0^1 = 1$

- $x = 1$, $\partial x = \partial z = z = 0$, and we integrate y from 0 to 1: $\int_0^1 (4[x=1] + 2[z=0]^3)dy = 4y|_0^1 = 4$
- $x = y = 1$, $dx = dy = 0$ and we integrate z from 0 to 1: $\int_0^1 (6[y=1]z^2)dz = 2z^3|_0^1 = 2$

Putting them together we get that $\int_a^b \nabla T \cdot dl = 1 + 4 + 2 = 7$

2. - $y = x = dy = dx = 0$ z from 0 to 1: $\int_0^1 (6[y=0]z^2)dz = 0$
   - $z = 1$, $dx = dz = x = 0$, and we integrate y from 0 to 1: $\int_0^1 (4[x=0] + 2[z=1]^3)dy = 2y|_0^1 = 2$
   - $z = y = 1$, $dz = dy = 0$ and we integrate x from 0 to 1: $\int_0^1 (2x + 4[y=1])dx = x^2|_0^1 + 4 = 5$

   Putting them together we get that $\int_a^b \nabla T \cdot dl = 0 + 2 + 5 = 7$

3. We do this one in a single shot be replacing all y and z terms in $\nabla T \cdot dl$ with terms of only x. We need to remember that $y = x$ and $z = x^2$. That makes $dy = dx$ and $dz = 2dx$.

$$\begin{aligned}\nabla T \cdot dl &= (2x + 4x)dx + (4x + 2x^6)dx + (3x^4)2dx \\ &= (10x + 14x^6)dx\end{aligned}$$

Now I can integrate x from 0 to 1:

$$\begin{aligned}\int_0^1 (10x + 14x^6)dx &= \left[5x^2 + 2^7\right]\Big|_0^1 \\ &= 2 + 5 = 7\end{aligned}$$

These are all the same so we have verified the fundamental theorem for gradients for this function along the given paths. □

Problem 1.34

Test Stokes' theorem for the function $\boldsymbol{v} = (xy)\hat{\boldsymbol{x}} + (2yz)\hat{\boldsymbol{y}} + (3zx)\hat{\boldsymbol{z}}$ using the triangular region defined by the points $(0,0,0), (0,0,2),$ and $(0,2,0)$ that moves in the counter clockwise direction..

Stokes' theorem gives the following:

$$\int_S (\nabla \times \boldsymbol{v}) \cdot d\boldsymbol{a} = \oint_{\mathbb{P}} \boldsymbol{v} \cdot d\boldsymbol{l}$$

We will need to know that $\nabla \times \boldsymbol{v} = (-2y)\hat{\boldsymbol{x}} + (-3z)\hat{\boldsymbol{y}} + (-x)\hat{\boldsymbol{z}}$.

Also, from the rotation of the curve, we can say that $d\boldsymbol{a}$ is oriented in the $\hat{\boldsymbol{x}}$ direction. With that we can say that $d\boldsymbol{a} = (dydz)\hat{\boldsymbol{x}}$.

We can finally say that $(\nabla \times \boldsymbol{v}) \cdot d\boldsymbol{a} = -2ydydz$

We need to evaluate the integral $\int_S (\nabla \times \boldsymbol{v}) \cdot d\boldsymbol{a}$, but before we can to that we need to define the bounds of integration for the integrals in y and z. A simple way to do this is to write y as a function of z using the expression

$$y = 2 - z$$

. The bounds of integration then become 0 to $z - 2$ for y and 0 to 2 for z.

$$
\begin{aligned}
\int_S (\nabla \times \boldsymbol{v}) \cdot d\boldsymbol{a} &= \int_0^2 \left(\int_0^{z-2} -2y\,dy \right) dz \\
&= \int_0^2 \left(-y^2|_0^{z-2} \right) dz \\
&= \int_0^2 -(2-z)^2 dz \\
&= -\int_0^2 (z^2 - 4z + 4) dz \\
&= -\left[\frac{1}{3} z^3 - 2z^2 + 4z \right] |_0^2 \\
&= -\left(\frac{8}{3} - 8 + 8 \right) = \frac{8}{3}
\end{aligned}
$$

To verify this we need to check that $\oint_{\mathbb{P}} \boldsymbol{v} \cdot d\boldsymbol{l} = \frac{8}{3}$. To do this we need to know that $\boldsymbol{v} \cdot d\boldsymbol{l} = (xy)dx + (2yz)dy + (3zx)dz$ I will break the integral into three sections, each representing one leg of the triangle.

1. Bottom leg: $x = z = dx = dz = 0$ and y goes from 0 to 2. Plugging zeros in we see that $\boldsymbol{v} \cdot d\boldsymbol{l} = 0$, therefore the integral for this section in 0.

2. Hypotenuse: $x = dx = 0$, $z = 2 - y$, $dz = -dy$, and y goes from 2 to 0. Plugging things in we see that $\boldsymbol{v} \cdot d\boldsymbol{l} = (2yz)dy = (2y(2-y))dy$

$$
\begin{aligned}
\int_2^0 (2y(2-y))dy &= \int_2^0 4y - y^2 dy \\
&= -\int_0^2 4y - y^2 dy \\
&= -\left(2y^2 - \frac{2}{3} y^3 \right)\Big|_0^2 \\
&= -(8 - \frac{16}{3}) = -\frac{8}{3}
\end{aligned}
$$

3. Vertical leg: $x = dx = y = dy = 0$. z goes from 2 to 0. Plugging zeros in we see that $\boldsymbol{v} \cdot d\boldsymbol{l} = 0$, therefore the integral for this section in 0.

Putting all three pieces together we see that the line integral way gave us $0 - \frac{8}{3} + 0 = -\frac{8}{3}$, which is what we got above. □

Problem 1.43

1. find the divergence of the function

$$\boldsymbol{v} = (s(2 + \sin^2(\phi)))\hat{\boldsymbol{s}} + s \sin\phi \cos\phi \hat{\boldsymbol{\phi}} + (3z)\hat{\boldsymbol{z}}$$

2. Test the divergence theorem for this function, using the quarter of radius 2, height 5 defined in the first quadrant of the x-y plane.

3. Find the curl of $\boldsymbol{v}$.

1. From the book we can get that the divergence in spherical coordinates is equal to:

$$\frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Applying that to our function we get

$$\begin{aligned}
\nabla \cdot \boldsymbol{v} &= \frac{1}{s}\frac{\partial}{\partial s}(ss(2+\sin^2(\phi))) + \frac{1}{s}\frac{\partial(s\sin\phi\cos\phi)}{\partial\phi} + \frac{\partial 3z}{\partial z} \\
&= \frac{1}{s}2s\left(2+\sin^2\phi\right) + \frac{1}{s}s\left(\cos^2\phi - \sin^2\phi\right) + 3 \\
&= 4 + 2\sin^2\phi + \cos^2\phi - \sin^2\phi + 3 \\
&= 4 + \cos^2\phi + \sin^2\phi + 3 \\
&= 4 + 1 + 3 = 8
\end{aligned}$$

2. We need to show that the two sides of the integral are equal. The first is easy.

$$\begin{aligned}
\int \nabla \cdot \boldsymbol{v} d\tau &= \int_0^2 \int_0^{\pi/2} \int_0^5 8 s ds d\phi dz \\
&= 8\left(\frac{s^2}{2}\right)\Big|_0^2 (\phi)|_0^{\pi/2} (z)|_0^5 \\
&= 40\pi
\end{aligned}$$

Now for the hard part... We need to integrate over all 5 pieces of the surface (top, bottom, left, right, curved).

(a) Top: $z=5$, where $\phi : 0 \to \pi/2$ and $s : 0 \to 2$. We also need to know that $\boldsymbol{v}\cdot d\boldsymbol{a} = 3zsdsd\phi$

$$\begin{aligned}
\int \boldsymbol{v}\cdot d\boldsymbol{a} &= \int_0^2 \int_0^{\pi/2} 3(z=5) s ds d\phi \\
&= 15\left(\frac{s^2}{2}\right)\Big|_0^2 (\phi)|_0^{\pi/2} = 15\pi
\end{aligned}$$

(b) Left: $\phi = \pi/2$, where $z : 0 \to 5$ and $s : 0 \to 2$. We also need to know that $\boldsymbol{v}\cdot d\boldsymbol{a} = s\sin\phi\cos\phi dsdz = 0$. That makes the integral for this part equal to 0.

(c) Bottom: $z = 0$, where $\phi : 0 \to \pi/2$ and $s : 0 \to 2$. We also need to know that $\boldsymbol{v}\cdot d\boldsymbol{a} = -3(z=0)sdsd\phi = 0$. This makes the integral for this part equal to 0.

(d) Right: Left: $\phi = 0$, where $z : 0 \to 5$ and $s : 0 \to 2$. We also need to know that $\boldsymbol{v}\cdot d\boldsymbol{a} = -s\sin\phi\cos\phi dsdz = 0$. That makes the integral for this part equal to 0.

(e) Curved part: $s = 2$, where $\phi : 0 \to \pi/2$ and $z : 0 \to 5$. We also need to know that $\boldsymbol{v}\cdot d\boldsymbol{a} = (s = 2)(2+\sin^2\phi)(s=2)dzd\phi = 4(2+\sin^2\phi)dzd\phi$

$$\begin{aligned}
\int \boldsymbol{v}\cdot d\boldsymbol{a} &= \int_0^5 \int_0^{\pi/2} 4(2+\sin^2\phi) dz d\phi \\
&= \left(\frac{5x}{2} - \frac{1}{4}\sin(2x)\right)\Big|_0^{\pi/2} (z)|_0^5 = 25\pi
\end{aligned}$$

Putting all these pieces together we get that the integral is $15\pi + 0 + 0 + 0 + 25\pi = 40\pi$, which is the same thing we got doing the integral the other way.

3. We now find the curl of $\boldsymbol{v}$ using the expression for the curl in spherical coordinates that we find in the book.

$$\begin{aligned}
\nabla \times \boldsymbol{v} &= \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\boldsymbol{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi - \frac{\partial v_s}{\partial \phi}\right]\hat{\boldsymbol{z}} \\
&= (0-0)\hat{\boldsymbol{s}} + (0-0)\hat{\boldsymbol{y}} + \frac{1}{s}\left[2s\sin\phi\cos\phi - 2s\sin\phi\cos\phi\right]\hat{\boldsymbol{z}} = 0 \quad \square
\end{aligned}$$

Problem 1.45

Evaluate the following integrals

1. $\int_{-2}^{2}(2x+3)\delta(3x)dx$
2. $\int_{0}^{2}(x^3+3x+2)\delta(1-x)dx$
3. $\int_{-1}^{1}9x^2\delta(3x+1)dx$
4. $\int_{-\infty}^{a}\delta(x-b)dx$

For this problem we simply use various properties of the delta distribution that we can find the in book.

1. For this part we use: $\int_{-\infty}^{\infty} f(x)\delta(kx) = \int_{\infty}^{\infty}\left[\frac{1}{|k|}\delta(x)\right]$

$$\begin{aligned}\int_{-2}^{2}(2x+3)\delta(3x)dx &= \int_{-2}^{2}(2x+3)\frac{1}{3}\delta(x)dx\\ &= \frac{1}{3}[2(x=0)+3]\\ &= \frac{1}{3}(0+3)=1\end{aligned}$$

2. For this part we use the above property and: $f(x)\delta(x-b) = f(b)$

$$\begin{aligned}\int_{0}^{2}(x^3+3x+2)\delta(1-x)dx &= \int_{0}^{2}(x^3+3x+2)\delta(-1[x-1])dx\\ &= \int_{0}^{2}(x^3+3x+2)\delta(x-1)dx\\ &= [x=1]^2+3[x=1]+2=6\end{aligned}$$

3. For this part we use the same properties as the previous part

$$\begin{aligned}\int_{-1}^{1}9x^2\delta(3x+1)dx &= \int_{-1}^{1}9x^2\delta(3[x+1/3])dx\\ &= \int_{-1}^{1}9x^2\frac{1}{3}\delta(x+1/3)dx\\ &= 9(x=-1/3)^2(1/3)=1/3\end{aligned}$$

4. This one is a bit tricky, but it equal to the following:

$$\int_{-\infty}^{a}\delta(x-b)dx = \begin{cases}1 & \text{if } a>b\\ 0 & \text{otherwise}\end{cases} \qquad \square$$

Problem 1.49

Evaluate the integral

$$J = \int_{\mathbb{V}} e^{-r}\cdot\left(\nabla\cdot\frac{\hat{\boldsymbol{r}}}{r^2}\right)d\tau$$

Problem 1.49 continued on next page...

(where $\mathbb{V}$ is a sphere of radius R , centered at the origin) by two different methods as in Ex. 1.16

We will follow the same course that Example 1.16 did:

1. We begin by re-writing the divergence:
$$\nabla \cdot \frac{\hat{\boldsymbol{r}}}{r^2} = 4\pi\delta^3(\boldsymbol{r})$$
We can now evaluate the integral
$$\begin{aligned} J &= \int_{\mathbb{V}} e^{-r} \cdot \left(\nabla \cdot \frac{\hat{\boldsymbol{r}}}{r^2}\right) d\tau \\ &= \int_{\mathbb{V}} e^{-r} 4\pi\delta^3(\boldsymbol{r}) d\tau \\ &= 4\pi e^{-0} = 4\pi \end{aligned}$$

2. To do this part we move the derivative term over to the e^{-r} and then finish it out. Note that we use $d\boldsymbol{a} = r^2 sin\theta d\theta d\phi \hat{\boldsymbol{r}}$ and $d\tau = 4\pi r^2 dr$
$$\begin{aligned} J &= \int_{\mathbb{V}} e^{-r} \cdot \left(\nabla \cdot \frac{\hat{\boldsymbol{r}}}{r^2}\right) d\tau \\ &= -\int_{\mathbb{V}} \frac{\hat{\boldsymbol{r}}}{r^2} \cdot \left[\nabla e^{-r}\right] d\tau + \oint_{\mathbb{S}} e^{-r} \frac{\hat{\boldsymbol{r}}}{r^2} \cdot d\boldsymbol{a} \\ &= \int \frac{1}{r^2} e^{-r} 4\pi r^2 dr + \int e^{-r} \frac{\hat{\boldsymbol{r}}}{r^2} \cdot r^2 sin\theta d\theta d\phi \hat{\boldsymbol{r}} \\ &= 4\pi \int_0^\infty e^{-r} dr + e^{-R} \int sin\theta d\theta d\phi \\ &= 4\pi \left(-e^{-r}\right)\Big|_0^\infty + 4\pi e^{-R} = 4\pi \end{aligned}$$

Those two integrals gave us the same answer so we are done. □

Problem 1.63

1. Find the divergence of the function
$$\boldsymbol{v} = \frac{\hat{\boldsymbol{r}}}{r}.$$
First compute it directly, as in Eq 1.84. Test your result using the divergence theorem as in Eq 1.85. Is there a delta function at the origin, as there was for $\frac{\hat{\boldsymbol{r}}}{r^2}$? What is the general formula for the divergence of $r^n \hat{\boldsymbol{r}}$? [*Answer*: $\nabla \cdot (r^n \hat{\boldsymbol{r}}) = (n+2) r^{n-1}$, unless $n = -2$, in which case it is $4\pi\delta^3(\boldsymbol{r})$; for $n < -2$, the divergence is ill-defined at the origin.]

2. Find the *curl* of $r^n \hat{\boldsymbol{r}}$. Test your colclusion using problem 1.61b [*Answer:* $\nabla \times (r^n \hat{\boldsymbol{r}}) = 0$]

1. We now find the divergence using the expression for divergence in spherical coordinates.
$$\begin{aligned} \nabla \cdot \boldsymbol{v} &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \left[v_r = \frac{1}{r} \right] \right) + 0\hat{\boldsymbol{\theta}} + 0\hat{\boldsymbol{\phi}} \\ &= \frac{1}{r^2} \frac{\partial r}{\partial r} \\ &= \frac{1}{r^2} \end{aligned}$$

Recall that the divergence theorem states that

$$\int (\nabla \cdot \boldsymbol{A}) d\tau = \oint \boldsymbol{A} \cdot d\boldsymbol{a}$$

We now use the divergence theorem to verify our result. Note that we follow equation 1.85 in using $d\boldsymbol{a} = R^2 sin\theta d\theta d\phi \hat{\boldsymbol{r}}$ and $d\tau = r^2 \sin\theta dr d\theta d\phi$

$$\begin{aligned}
\int (\nabla \cdot \boldsymbol{v}) d\tau &= \int \frac{1}{r^2} r^2 \sin\theta dr d\theta d\phi \\
&= \int_0^R \int_0^\pi \int_0^{2\pi} dr \sin\theta d\theta d\phi \\
&= 4\pi R
\end{aligned}$$

We now verify that this is the same as the other half of the divergence theorem

$$\begin{aligned}
\oint \boldsymbol{v} \cdot d\boldsymbol{a} &= \oint (\frac{1}{R} \hat{\boldsymbol{r}} \cdot (R^2 \sin\theta d\theta d\phi \hat{\boldsymbol{r}}) \\
&= R \int_0^\pi \int_0^{2\pi} \sin\theta d\theta d\phi \\
&= 4\pi R
\end{aligned}$$

These two expressions are equal so we did calculate the divergence correctly.

It doesn't look like there is a delta function at the origin because the function is linear in R.

We now need an expression for the divergence of $r^n \hat{\boldsymbol{r}}$. We will use the general form of the divergence in spherical coordinates from the book.

$$\begin{aligned}
\nabla \cdot r^n \hat{\boldsymbol{r}} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 [r^n]) + 0 + 0 \\
&= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 r^n) \\
&= \frac{1}{r^2} (n+2) r^{n+1} \\
&= (n+2) r^{n-1}
\end{aligned}$$

2. We now seek to find the curl of $r^n \hat{\boldsymbol{r}}$. We again use the expression for the curl in spherical coordinates:

$$\nabla \times \boldsymbol{v} = \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial \theta} (\sin\theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{\boldsymbol{r}} + \frac{1}{r} \left[\frac{1}{\sin\theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\boldsymbol{\theta}} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r v_\theta - \frac{\partial v_r}{\partial \theta} \right] \hat{\boldsymbol{\phi}}$$

Applying that definition to our function we get:

$$\begin{aligned}
\nabla \times (r^n \hat{\boldsymbol{r}}) &= 0 \hat{\boldsymbol{r}} + 0 \hat{\boldsymbol{\theta}} + 0 \hat{\phi} \\
&= 0
\end{aligned}$$

□