A Simple Overlapping Generations Model

Major Features of the Model

One source of uncertainty: z
Stochastic technology growth about a deterministic trend

We first define the important terms that will be used throughout:
Endogenous variables that change over time:

z	productivity (temporary or permanent)
k_s	capital stock owned by household of age s
c_s	consumption
w	wage rate by household of age s
r	interest rate
Y	output of final goods

Parameters:

α	capital share in output from a Cobb-Douglas production function
δ	rate of depreciation
β	time discount factor; $\beta<1$
a	trend in z
γ	elasticity of substitution, $\gamma>0$
ρ	autocorrelation parameter for z; $0<\rho<1$
σ	standard deviations of the shocks to z; $0<\sigma$
h_s	fixed labor endowment for household of age s

Nonstationary Model

Households

Every a new cohort of households is born that will live for S periods. Hence, each period there are S types of households, one for each age.

<u>Age S household</u>

Given information on prices and shocks, $\mathbf{\Omega}=\{w,r,z\}$, the household solves the following non-linear program when the factor markets clear.

$$V^S(k_S,\mathbf{\Omega})=\underset{k_{S+1}'}{Max}\,u(c_S)+\beta E\{V^{S+1}(k_{S+1}',\mathbf{\Omega}')\}$$

where:

$$c_S=wh_S+(1-\delta+r)k_S \tag{1.1a}$$

This problem is trivial. Since the household will be dead next period, the value function next period is zero.

Picking functional form of $u(c)=\frac{1}{1-\gamma}(c^{1-\gamma}-1)$:

$$V^S(k_S,\mathbf{\Omega})=\tfrac{1}{1-\gamma}[(wh_S+(1-\delta+r)k_S]^{1-\gamma}-1]$$

Age *S*-1 household

The household solves the following non-linear program when the factor markets clear.

$$V^{S-1}(k_{S-1},\mathbf{\Omega}) = \underset{k_S'}{Max}\, u(c_{S-1}) + \beta E\{V^S(k_S',\mathbf{\Omega}')\}$$

where:

$$c_{S-1} = wh_{S-1} + (1-\delta+r)k_{S-1} - k_S \qquad (1.1b)$$

The first-order condition is:

$$u_c(c_{S-1})(-1) + \beta E\{V_k^S(k_S',\Omega')\} = 0$$

From the S-age household we know next periods value function and derivative. This gives us the following Euler equation:

$$c_{S-1}{}^{-\gamma} = \beta E\{c_S{}'^{-\gamma}(1-\delta+r')\} \qquad (1.2a)$$

Generic age *s* household

The household solves the following non-linear program when the factor markets clear.

$$V^s(k_s,\mathbf{\Omega}) = \underset{k_{s+1}'}{Max}\, u(c_s) + \beta E\{V^{s+1}(k_{s+1}',\mathbf{\Omega}')\}$$

where:

$$c_s = wh_s + (1-\delta+r)k_s - k_{s+1} \qquad (1.1c)$$

The first-order condition is:

$$u_c(c_s)(-1) + \beta E\{V_k^{s+1}(k_{s+1}',\Omega')\} = 0$$

From the S-age household we know next periods value function and derivative. This gives us the following Euler equation:

$$c_s{}^{-\gamma} = \beta E\{c_{s+1}{}'^{-\gamma}(1-\delta+r')\} \qquad (1.2b)$$

Age 1 household

The age 1 household is born with no capital, but otherwise looks like any other household aged less than *S*.

Generalizing various versions of (1.1) and (1.2) gives:

$$c_s = wh_s + (1-\delta+r)k_s - k_{s+1} \quad \text{for } s \in \{1,2,..S\} \qquad (1.1)$$

$$c_s{}^{-\gamma} = \beta E\{c_{s+1}{}'^{-\gamma}(1-\delta+r')\} \quad \text{for } s \in \{1,2,..S-1\} \qquad (1.2)$$

$$k_{S+1} = 0 \qquad (1.3)$$

$$k_1 = 0 \qquad (1.4)$$

Additional Behavioral Equations

The law of motion for *z* is:

$z' = \rho z + \varepsilon'$; where ε' is distributed normal with a mean of 0 and a variance of σ^2 (1.5)

An assumption of competition in the goods market along with a Cobb-Douglas production function gives the following shares in output for labor & capital. Market clearing conditions have already been imposed.

$$Y = K^\alpha(e^{at+z}H)^{1-\alpha} \qquad (1.6)$$

$$wH = (1-\alpha)Y \tag{1.7}$$
$$rK = \alpha Y \tag{1.8}$$

Market clearing conditions give:

$$K = \sum_{s=2}^{S} k_s \tag{1.9}$$
$$H = \sum_{s=1}^{S} h_s$$

Definitions for Later Use

$$C = \sum_{s=1}^{S} c_s \tag{1.10}$$
$$I \equiv K' - (1-\delta)K \tag{1.11}$$
$$A \equiv e^{at+z} \tag{1.12}$$

Eqs (1.1)-(1.12) are the system.

Transformation & Simplifications

Transform the problem by dividing all growing variables by e^{at}, denoting with a carat.

$z' = \rho z + \varepsilon'$; where ε' is distributed normal with a mean of 0 and a variance of σ^2 (2.1)

$$\hat{c}_s = \hat{w}h_s + (1-\delta+r)\hat{k}_s - k_{s+1}(1+a) \quad \text{for } s \in \{1,2,..S\},\ \hat{k}_{S+1} = 0,\ \hat{k}_1 = 0 \tag{2.2}$$

$$\hat{c}_s^{\ -\gamma} = \beta E\{[\hat{c}_{s+1}'(1+a)]^{-\gamma}(1-\delta+r')\} \quad \text{for } s \in \{1,2,..S-1\} \tag{2.3}$$

$$\hat{Y} = \hat{K}^{\alpha}(e^z H)^{1-\alpha} \tag{2.4}$$

$$\hat{w}H = (1-\alpha)\hat{Y} \tag{2.5}$$

$$r\hat{K} = \alpha\hat{Y} \tag{2.6}$$

$$\hat{K} = \sum_{s=2}^{S}\hat{k}_s \tag{2.7}$$

$$\hat{C} = \sum_{s=1}^{S}\hat{c}_s \tag{2.8}$$

$$\hat{I} = (1+a)\hat{K}' - (1-\delta)\hat{K} \tag{2.9}$$

$$\hat{A} = e^z \tag{2.10}$$

These are the equations we will use in Dynare.

The endogenous variables are $\{\hat{c}_s\}_{s=1}^{S}, \{\hat{k}_s\}_{s=2}^{S}, \hat{Y}, \hat{w}, r, \hat{K}, \hat{C}, \hat{I}, \hat{A}$ & z.

The exogenous variable is ε.

The parameters are $\alpha, \delta, \beta, a, \gamma, \rho, \sigma$ & $\{h_s\}_{s=1}^{S}$, with $H \equiv \sum_{s=1}^{S} h_s$