A 6th Real Business Cycle Model

Major Features of the Model

Add money and utility from money to model 4
Add stochastic money growth about a deterministic trend
Two sources of uncertainty: z and g
Stochastic technology growth about a deterministic trend
Labor-leisure decision with indivisible labor hours
Population growth follows a deterministic trend

We first define the important terms that will be used throughout:
Endogenous variables that change over time:

z	productivity
g	money growth
K	capital stock owned by households
h	labor supplied by a single individual
c	consumption by a single individual
w	wage rate
r	interest rate
Y	output of final goods
N	number of persons per household
m	money balances per household
M	aggregate money supply
P	price of good in terms of money

Parameters:

α	capital share in output from a Cobb-Douglas production function
δ	rate of depreciation
β	time discount factor; $\beta<1$
a	trend in z
μ	trend in g
n	trend in N
γ	elasticity of substitution, $\gamma>0$
D	leisure weight in utility
J	money weight in utility
ρ_i	autocorrelation parameter for $i=z,g$; $0<\rho_i<1$
σ_z	standard deviations of the shocks to $i=z,g$; $0<\sigma_i$

Nonstationary Model

Households have increasing numbers of members, denoted N.
The law of motion for N is:

$$N' = e^n N \text{ or } N = e^{nt} N_0 \tag{1.1}$$

Households face only a budget constraint:

$wh + (1-\delta + r)\frac{K}{N} + \frac{m}{PN} + \frac{(g+\mu)M}{PN} = c + \frac{K'}{N} + \frac{m'}{PN}$

Given information on prices and shocks, $\mathbf{\Omega} = \{w, r, z, g\}$, the household solves the following non-linear program when the factor markets clear.

$$V(K, m, \mathbf{\Omega}) = \underset{K', m', h}{Max}\left[\tfrac{1}{1-\gamma}(c^{1-\gamma} - 1) + he^{(1-\gamma)at}\widetilde{D} + \widetilde{F}e^{(1-\gamma)at} - \widetilde{F} + J\tfrac{1}{1-\gamma}[(\tfrac{m}{PN})^{1-\gamma} - 1]\right]N$$
$$+ \beta E\{V(K', m', \mathbf{\Omega}')\}$$

$\widetilde{D} \equiv \frac{1}{H_0} D[(1-h_0)^{1-\gamma} - 1] < 0$, $\widetilde{F} \equiv D\frac{1}{1-\gamma}$

$$c = wh + (1-\delta + r)\tfrac{K}{N} + \tfrac{m}{PN} + \tfrac{(\mu+g)M}{PN} - \tfrac{K'}{N} - \tfrac{m'}{PN} \quad (1.2)$$

The first-order conditions are:

$c^{-\gamma}(-\frac{1}{N})N + \beta E\{V_K(K', \Omega')\} = 0$

$c^{-\gamma}(-\frac{1}{PN})N + \beta E\{V_m(K', \Omega')\} = 0$

$c^{-\gamma}wN + e^{(1-\gamma)at}\widetilde{D}N = 0$

The envelope conditions are:

$V_K(K, \Omega) = c^{-\gamma}(-\frac{1}{N})(1-\delta + r)N$

$V_m(K, \Omega) = c^{-\gamma}\frac{1}{PN}N + F(\frac{m}{PN})^{-\gamma}\frac{1}{PN}N$

The Euler equations are:

$$c^{-\gamma} = \beta E\{c'^{-\gamma}(1-\delta + r')\} \quad (1.3)$$

$$c^{-\gamma} = \beta E\{[c'^{-\gamma} + F(\tfrac{m'}{P'N'})^{-\gamma}]\tfrac{P}{P'}\} \quad (1.4)$$

$$c^{-\gamma}w = -e^{(1-\gamma)at}\widetilde{D} \quad (1.5)$$

Additional Behavioral Equations

Money changes over time according to:

$$M' = e^{\mu+g}M = M_0\prod_{s=1}^{t+1} e^{\mu+g_s} \quad (1.6)$$

The law of motion for g is:

$g' = \rho_g g + \varepsilon_g'$; where ε_g' is distributed normal with a mean of 0 and a variance of σ_g^2 (1.7)

The law of motion for z is:

$z' = \rho_z z + \varepsilon_z'$; where ε_z' is distributed normal with a mean of 0 and a variance of σ_z^2 (1.8)

An assumption of competition in the goods market along with a Cobb-Douglas production function gives the following shares in output for labor & capital. Market clearing conditions have already been imposed.

$$Y = K^{\alpha}(e^{at+z}H)^{1-\alpha} \quad (1.9)$$

$$wH = (1-\alpha)Y \quad (1.10)$$

$$rK = \alpha Y \quad (1.11)$$

Aggregating over household members gives:

$$H = Nh \quad (1.12)$$

Money market clearing gives:

$$M = m \tag{1.13}$$

Definitions:

$$I \equiv K' - (1-\delta)K \tag{1.14}$$

$$A \equiv e^{at+z} \tag{1.15}$$

Eqs (1.1)-(1.15) are the system.

Transformation & Simplifications

Without loss of generalization set $\hat{N} = N_0 = 1$, and eliminate it from the system.
Use (1.12) to eliminate *H* from the system.
Use (1.13) to eliminate *m* from the system.

Transform the problem by dividing:
c, w, A by e^{at}
K, Y, I by $e^{(a+n)t}$
M by $e^{\mu t + G_t}$; $G_t \equiv \sum_{s=1}^{t} g_s$ *M* has a unit root.
P by $e^{(\mu - a - n)t + G_t}$
r & *h* do not need to be transformed.

$$g' = \rho_g g + \varepsilon_g' \tag{2.1}$$

$$z' = \rho_z z + \varepsilon_z' \tag{2.2}$$

$$\hat{M}' = \hat{M} = M_0 \tag{2.3}$$

$$\hat{c} = \hat{w}h + (1 - \delta + r)\hat{K} + \tfrac{(1+\mu+g)\hat{M}}{\hat{P}} - \hat{K}(1 + a + n) - \tfrac{\hat{M}'(1+\mu+g')}{\hat{P}} \tag{2.4}$$

$$1 = \beta E\{(\tfrac{\hat{c}}{\hat{c}' e^{a}})^{-\gamma}(1 - \delta + r')\} \tag{2.5}$$

$$\hat{c}^{-\gamma} = \beta E\{e^{-\gamma a}[\hat{c}'^{-\gamma} + F(\tfrac{\hat{M}'}{\hat{P}'})^{-\gamma}]\tfrac{\hat{P}}{\hat{P}'(1+\mu-a-n+g')}\} \tag{2.6}$$

$$\hat{c}^{-\gamma}\hat{w} = -\widetilde{D} \tag{2.7}$$

$$\hat{Y} = \hat{K}^{\alpha}(e^{z}h)^{1-\alpha} \tag{2.8}$$

$$\hat{w}h = (1 - \alpha)\hat{Y} \tag{2.9}$$

$$r\hat{K} = \alpha\hat{Y} \tag{2.10}$$

$$\hat{I} = (1 + a + n)\hat{K}' - (1 - \delta)K \tag{2.11}$$

$$\hat{A} \equiv e^{z} \tag{2.12}$$

These are the equations we will use in Dynare.
The endogenous variables are $\hat{c}, \hat{K}, h, \hat{Y}, \hat{w}, r, \hat{M}, \hat{P}, \hat{I}, \hat{A}, g$ & z.
The exogenous variables are ε_z & ε_g.
The parameters are $\alpha, \delta, \beta, a, \mu, \gamma, \rho_z, \sigma_z, \rho_g, \sigma_g, D, F$ & h_0.