Physics 441

Electro-Magneto-Statics

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1. Introduction

- Electricity and Magnetism as a single **field**
  (even in static case, where they decouple)
  Maxwell: * vector fields
  * sources (and sinks)
- Linear coupled PDE's
  * first order (grad, div, curl)
  * inhomogeneous (charge & current distrib.)

$$\boxed{\nabla \mathcal{F} = \tilde{J} \quad \Rightarrow \quad \mathcal{F} = \nabla^{-1} \tilde{J}}$$

Tools

Math

- trigonometry
- vectors (linear combination)
  dot, cross, Clifford
- vector derivative operators
  $\partial / \partial x_i, \quad \nabla, \quad \nabla \cdot, \quad \nabla \times, \quad \nabla \circ$
- Dirac delta function
- **DISCRETE TO CONTINUUM**
- INTEGRAL THEOREMS:
  * Gauss, Stokes, FundThCalc
- cylindrical, spherical coord.
- LINEARITY

Physics

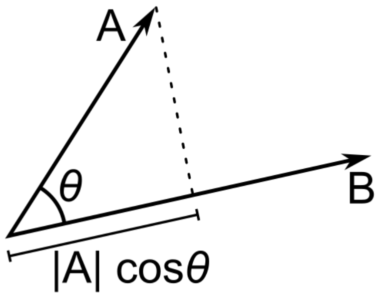
- trajectories: $\mathbf{r}(t)$
- **FIELDS**: * scalar, vector
  * static, t-dependent
- **SOURCES**: charge, current
- superposition of sources =>
- superposition of fields
- unit **point** sources
- Maxwell equation
- field lines
- potentials
- charge conservation

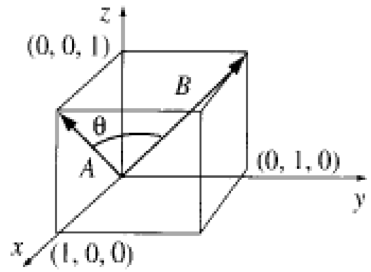
interpretation of equations and their solutions

2. Math. Review

- sum of vectors: $\mathbf{A}$, $\mathbf{B}$ => $\mathbf{A} + \mathbf{B}$
- dilation: c , $\mathbf{A}$ => $c\,\mathbf{A}$
- linear combinations: $c_1\,\mathbf{A} + c_2\,\mathbf{B}$
- scalar (dot) product:
  $\mathbf{A}$, $\mathbf{B}$ => $\mathbf{A} \cdot \mathbf{B} = A\,B \cos(\theta)$, **a scalar**
  where $A^2 = \mathbf{A} \cdot \mathbf{A}$ (magnitude square)
- cross product:
  $\mathbf{A}$, $\mathbf{B}$ => $\mathbf{A} \times \mathbf{B} = \mathbf{n}\, A\, B\, |\sin(\theta)|$, **a new vector**,
  with $\mathbf{n} \perp \mathbf{A}$ and $\mathbf{B}$, and $\mathbf{n} \cdot \mathbf{n} = n^2 = 1$
- orthonormal basis: $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$

Geometric Interpretation: Dot product





Vector products (components)

- Dot product (a scalar):

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{A}_x\mathbf{B}_x + \mathbf{A}_y\mathbf{B}_y + \mathbf{A}_z\mathbf{B}_z$$

- Cross product:

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \mathbf{A}_x & \mathbf{A}_y & \mathbf{A}_z \\ \mathbf{B}_x & \mathbf{B}_y & \mathbf{B}_z \end{vmatrix}$$

a vector. Its magnitude corresponds to the area of the parallelogram {**A**, **B**}

Law of cosines, law of sines

- cosines (use dot product)

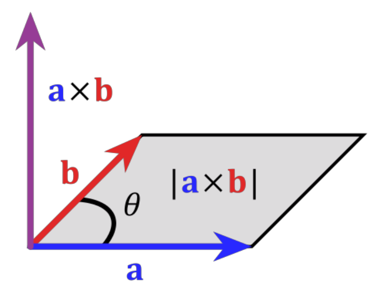
$$\mathbf{c} = \mathbf{a} - \mathbf{b}$$
$$c^2 = a^2 + b^2 - 2\mathbf{a}\cdot\mathbf{b}$$

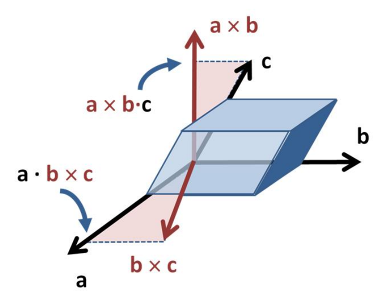
sines (use magnitude of cross product):

$$\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0} \quad \Rightarrow \quad |\mathbf{a}\times\mathbf{b}| = |\mathbf{a}\times\mathbf{c}|$$

$$ab\sin(C) = ac\sin(B) \quad \Rightarrow \quad \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

Cross Product and triple dot product





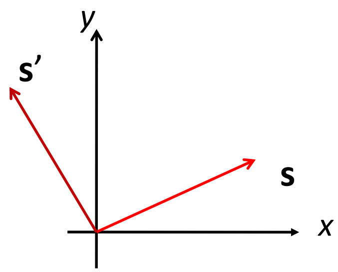
Rotation of a vector (plane)

- Assume **s** in the *x-y* plane
- Vector **s**‘ = **k** x **s**
- Operation **k** x rotates **s** by 90 degrees

  $$\mathbf{s} = x\mathbf{i} + y\mathbf{j}$$

  $$\mathbf{s}' = \mathbf{k} \times (x\mathbf{i} + y\mathbf{j}) = -y\mathbf{i} + x\mathbf{j}$$

- **k** x followed by **k** x again equivalent to multiplying by -1 in this case!!



Rotation of a vector (3-d)

n unit vector: $n^2 = 1$ defines rotation axis

ϕ = rotation angle

vector **r** ➔ **r**’

$$\mathbf{r}' = e^{\phi\,\mathbf{n}\times}\mathbf{r}$$

$$\mathbf{r}' = e^{\phi\,\mathbf{n}\times}(\mathbf{r}_{//} + \mathbf{r}_{\perp}) = \mathbf{r}_{//} + e^{\phi\,\mathbf{n}\times}\mathbf{r}_{\perp}$$

$$\boxed{\mathbf{r}' = \mathbf{r}_{//} + \cos\phi\,\mathbf{r}_{\perp} + \sin\phi\,\mathbf{n}\times\mathbf{r}}$$

where

$$\mathbf{r}_{=} = \mathbf{n}(\mathbf{n}\cdot\mathbf{r})$$

$$\mathbf{r}_{\perp} = \mathbf{r} - \mathbf{r}_{//} = -\mathbf{n}\times(\mathbf{n}\times\mathbf{r})$$

Triple dot product

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) =$$

$$= \mathbf{A}\cdot\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \mathbf{B}_x & \mathbf{B}_y & \mathbf{B}_z \\ \mathbf{C}_x & \mathbf{C}_y & \mathbf{C}_z \end{vmatrix} = \begin{vmatrix} \mathbf{A}_x & \mathbf{A}_y & \mathbf{A}_z \\ \mathbf{B}_x & \mathbf{B}_y & \mathbf{B}_z \\ \mathbf{C}_x & \mathbf{C}_y & \mathbf{C}_z \end{vmatrix}$$

is a scalar and corresponds to the (oriented) volume of the parallelepiped {**A**, **B**, **C**}

Triple cross product

- The cross product is **not** associative!
- Jacobi identity:

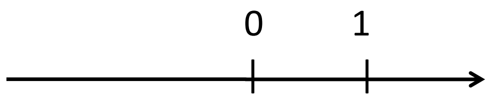
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) = 0$$

- BAC-CAB rule:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

is a vector linear combination of **B** and **C**

Inverse of a Vector



$\mathbf{n}^{-1} = \mathbf{n}$ as a unit

vector
$$\boxed{\mathbf{A}^{-1} = (A\mathbf{n})^{-1} = \frac{\mathbf{n}}{A} = \frac{\mathbf{A}}{A^2}}$$

along any direction in $\mathbf{R}^2$ or $\mathbf{R}^3$

Clifford Algebra Cl_3 (product)

- starting from $\mathbf{R}^3$ basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, generate all possible l.i. products ➔
- $8 = 2^3$ basis elements of the algebra Cl_3
- Define the product as:

$$\boxed{\mathbf{AB} = \mathbf{A}\cdot\mathbf{B} + i\mathbf{A}\times\mathbf{B}}$$

- with $i = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$

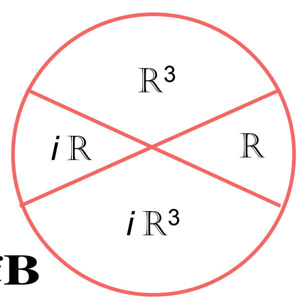
1			0	scalar
$\hat{\mathbf{e}}_1$	$\hat{\mathbf{e}}_2$	$\hat{\mathbf{e}}_3$	1	vector
$i\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2\hat{\mathbf{e}}_3$	$i\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3\hat{\mathbf{e}}_1$	$i\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1\hat{\mathbf{e}}_2$	2	bivector
$i = \hat{\mathbf{e}}_1\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3$			3	pseudoscalar

$$\mathbf{e}_1^{\ 2} = \mathbf{e}_2^{\ 2} = \mathbf{e}_3^{\ 2} = 1,$$

$$\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1 \qquad (\mathbf{e}_1\mathbf{e}_2)^2 = -1,$$

$$\{\hat{\mathbf{e}}_i, \hat{\mathbf{e}}_k\} = \hat{\mathbf{e}}_i\hat{\mathbf{e}}_k + \hat{\mathbf{e}}_k\hat{\mathbf{e}}_i = 2\delta_{ik}$$

$$\mathbf{C} = \alpha + i\beta + \mathbf{A} + i\mathbf{B}$$



Clifford Algebra Cl_3

- Non-commutative product w/ $\mathbf{A}\,\mathbf{A} = A^2$
- Associative: $(\mathbf{A}\,\mathbf{B})\,\mathbf{C} = \mathbf{A}\,(\mathbf{B}\,\mathbf{C})$
- Distributive w.r. to sum of vectors
- *symmetric part ➔ dot product

  *antisym. part ➔ proportional cross product
- Closure: extend the vector space until every product is a linear combination of elements of the algebra

Subalgebras

- $\mathbb{R}$ Real numbers
- $\mathbb{C} = \mathbb{R} + i\mathbb{R}$ Complex numbers
- $\mathbb{Q} = \mathbb{R} + i\mathbb{R}^3$ Quaternions

Product of two vectors is a ***quaternion:***

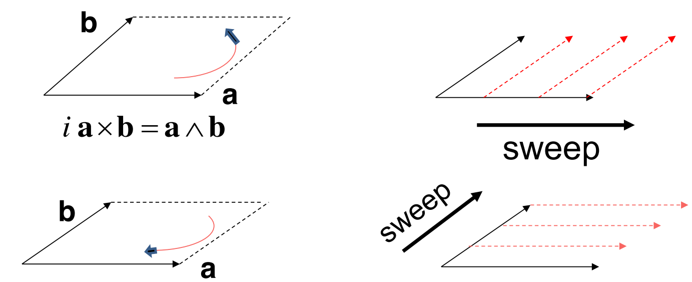
$$\mathbf{AB} = \mathbf{A}\cdot\mathbf{B} + i\mathbf{A}\times\mathbf{B}$$

$$\langle \mathbf{AB}\rangle_{scalar} = \mathbf{A}\cdot\mathbf{B} = (\mathbf{AB}+\mathbf{BA})/2$$

$$\langle \mathbf{AB}\rangle_{bivector} = i\mathbf{A}\times\mathbf{B} = (\mathbf{AB}-\mathbf{BA})/2 = \mathbf{A}\wedge\mathbf{B}$$

represents the oriented surface (plane) orthogonal to **A** x **B.**

Bivector: oriented surface



$i\,\mathbf{a}\times\mathbf{b} = \mathbf{a}\wedge\mathbf{b}$

$i\,\mathbf{b}\times\mathbf{a} = \mathbf{b}\wedge\mathbf{a} = -\mathbf{a}\wedge\mathbf{b}$

- antisymmetric, associative
- absolute value ➔ area

- BAC-CAB rule:

$$\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} = \frac{1}{2}(\mathbf{BAC}+\mathbf{BCA}-\mathbf{ABC}-\mathbf{BAC}) =$$

$$= \frac{1}{2}[(\mathbf{BC})\mathbf{A} - \mathbf{A}(\mathbf{BC})] = \frac{i}{2}[(\mathbf{B}\times\mathbf{C})\mathbf{A} - \mathbf{A}(\mathbf{B}\times\mathbf{C})] =$$

$$= \frac{2i^2}{2}(\mathbf{B}\times\mathbf{C})\times\mathbf{A} = \mathbf{A}\times(\mathbf{B}\times\mathbf{C})$$

- Decomposing **A** in // and $\perp$ w.r.t. to $\hat{n}$

$$\mathbf{A} = \hat{n}^2\mathbf{A} = \hat{n}(\hat{n}\cdot\mathbf{A}) + i\hat{n}(\hat{n}\times\mathbf{A}) =$$

$$= \hat{n}(\hat{n}\cdot\mathbf{A}) - \hat{n}\times(\hat{n}\times\mathbf{A})$$

so that

$$\mathbf{A}_{//} = \hat{n}(\hat{n}\cdot\mathbf{A}) \quad \text{and} \quad \mathbf{A}_{\perp} = -\hat{n}\times(\hat{n}\times\mathbf{A})$$

Differential Calculus

- Chain rule: $$df = \sum_i \frac{\partial f}{\partial x_i} dx_i$$

In 2-d: $$f = f(x,y) \qquad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

- Is $\begin{cases} xdx + 2ydy \\ 2ydx + xdy \\ ydx + xdy \end{cases}$ an "exact differential"?

- given $A(x,y)dx + B(x,y)dy$, $\dfrac{\partial A}{\partial y} = \dfrac{\partial B}{\partial x}$

- In 3-d:

$$1.- \quad df = \nabla f \cdot d\mathbf{r} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

$$2.- \quad \oint df = 0$$

$$3.- \quad \int_a^b df = F(b) - F(a) \qquad \text{independent of path}$$

- Geometric interpretation:

$$dT = |\nabla T||d\mathbf{r}|\cos\theta = \begin{cases} 0 & \text{when } \nabla T \perp d\mathbf{r} \\ \max & \text{when} \nabla T \,//\, d\mathbf{r} \end{cases}$$

∇T points in direction of steepest ascent

$$\nabla \doteq \hat{\mathbf{e}}_1 \frac{\partial}{\partial x} + \hat{\mathbf{e}}_2 \frac{\partial}{\partial y} + \hat{\mathbf{e}}_3 \frac{\partial}{\partial z}$$ **del** operator

Examples: $$\nabla z = \hat{\mathbf{k}}, \qquad \nabla g(x) = \frac{dg}{dx}\hat{\mathbf{i}}$$

$$\nabla r = \hat{\mathbf{r}}$$

$$\nabla f(r) = \frac{df}{dr}\hat{\mathbf{r}}$$

$$\boxed{\nabla \frac{1}{r} = -\frac{1}{r^2}\hat{\mathbf{r}} = -\frac{\mathbf{r}}{r^3}}$$

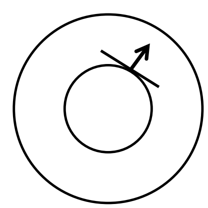
Gradient of *r*

- Contour surfaces: spheres
- ➔ gradient is *radial*

$f(\mathbf{r}) = r$ and $\nabla r \,//\, \hat{\mathbf{r}}$

$df = dr = (\nabla r)\cdot d\mathbf{r} = |\nabla r| dr$

$\Rightarrow |\nabla r| = 1$



$$\boxed{\nabla r = \hat{\mathbf{r}}}$$

- Algebraically:

$$\nabla r^2 = 2r\nabla r = \nabla(x^2 + y^2 + z^2) = 2\mathbf{r} \quad \Rightarrow \nabla r = \hat{\mathbf{r}}$$

and, in general, $$\nabla f(r) = f'(r)\hat{\mathbf{r}} = \frac{df}{dr}\hat{\mathbf{r}}$$

Divergence of a Vector Field

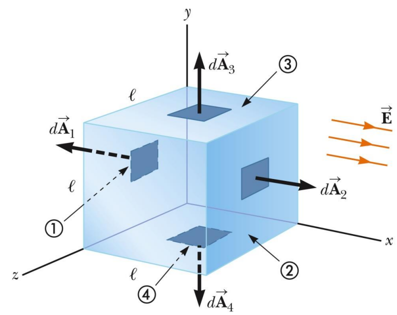
- **E** (**r**) ➔ scalar field (w/ dot product)

$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$= \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

- It is a measure of how much the field lines diverge (or converge) from a point (a line, a plane,…)

- Divergence as FLUX:



$$\frac{E_x(x+dx)A - E_x(x)A}{Adx} \approx \frac{\partial E_x}{\partial x}$$

Examples:

$$\nabla \cdot \hat{\mathbf{k}} = 0 \qquad \nabla \cdot (z\hat{\mathbf{k}}) = 1$$

$$\nabla \cdot \mathbf{r} = 3$$

$$\nabla \cdot (\hat{\mathbf{r}}\, F(r)) = \frac{1}{r^2}\frac{d}{dr}(r^2 F(r))$$

$$\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 0 \qquad \text{for} \quad r \neq 0$$

Curl of a Vector Field

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\varphi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ E_r & rE_\theta & r\sin\theta\, E_\phi \end{vmatrix}$$

The curl measures **circulation** about an axis.

Examples:

$$\mathbf{E}(\mathbf{r}) = x\hat{\mathbf{j}} \qquad \nabla \times \mathbf{E} = \hat{\mathbf{k}}$$

$$\mathbf{E}(\mathbf{r}) = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}} \qquad \nabla \times \mathbf{E} = 2\hat{\mathbf{k}}$$

Clifford product **del** w/ a cliffor

- For a scalar field $T = T(\mathbf{r})$,

$$\boxed{\nabla T = \mathrm{grad} T}$$

- For a vector field **E** = **E**(**r**),

$$\boxed{\nabla \mathbf{E} = \nabla \cdot \mathbf{E} + i\nabla \times \mathbf{E}}$$

- For a bivector field i **B** = i **B**(**r**),

$$\boxed{\nabla(i\mathbf{B}) = i\nabla \cdot \mathbf{B} - \nabla \times \mathbf{B}}$$

Products

- Gradient: $\nabla(fg) = f\nabla g + g\nabla f$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

- Divergence:

$$\nabla\cdot(f\mathbf{E}) = f\nabla\cdot\mathbf{E} + \mathbf{E}\cdot\nabla f$$

- Curl:

$$\nabla\times(f\mathbf{E}) = f\nabla\times\mathbf{E} + \mathbf{E}\times\nabla f$$

Second order derivatives

- For a scalar:

$$\nabla^2 T = \nabla(\nabla T) = \nabla\cdot(\nabla T) + i\nabla\times(\nabla T)$$
$$\Rightarrow \nabla^2 T = \nabla\cdot(\nabla T),$$
$$\Rightarrow \nabla\times(\nabla T) = 0$$

- For a vector:

$$\nabla^2\mathbf{E} = \nabla(\nabla\mathbf{E}) = \nabla(\nabla\cdot\mathbf{E}) + i\nabla(\nabla\times\mathbf{E})$$
$$\Rightarrow \nabla^2\mathbf{E} = \nabla(\nabla\cdot\mathbf{E}) - \nabla\times(\nabla\times\mathbf{E}),$$
$$\Rightarrow \nabla\cdot(\nabla\times\mathbf{E}) = 0$$

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

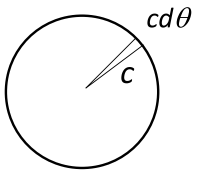
(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

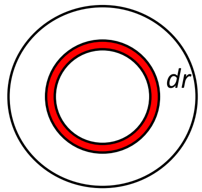
Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

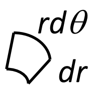
What do we mean by "integration"?

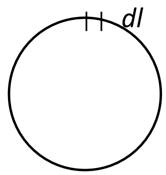




$dw = c(cd\theta)/2$

$dg = (2\pi r)\, dr$





$da = (rd\theta)dr$

$$dl = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Line, Surface, Volume integrals

- Vector field **F**: $W = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l}$ (work)

in general is path dependent

- Surface integral: $\Phi = \int_S \mathbf{E} \cdot d\mathbf{a} = \int_S \mathbf{E} \cdot \hat{\mathrm{n}}\, da =$

(electric flux) $(= \int_S \mathbf{E} \cdot \hat{\mathrm{k}}\, dxdy = \int_S E_z\, dxdy)$

- Volume integral:

$$\int_V T d\tau = (\int_V T\, dxdydz)$$

Cliffor differentials

- $d^k\alpha$ is a cliffor representing the "volume" element in *k* dimensions
- $k = 1$ ➔ $d\mathbf{l}$ is a vector ➔ $\mathbf{e}_1\, dx$ (path integral)
- $k = 2$ ➔ $i\, \mathbf{n}da$ bivector ➔ $\mathbf{e}_1\, dx\, \mathbf{e}_2\, dy$ (surface integral)
- $k = 3$ ➔ $i\, d\tau$ ps-scalar ➔ $\mathbf{e}_1\, dx\, \mathbf{e}_2\, dy\, \mathbf{e}_3\, dz$ (volume integral)

Fundamental Theorem of Calculus

$$\boxed{\int_V d^k\alpha\, \nabla F(\alpha) = (-)^{k-1} \oint_{\partial V} d^{k-1}\alpha\, F(\alpha)}$$

Particular cases: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a})$

$\int_V (\nabla \cdot \mathbf{E}) d\tau = \oint_{\partial V} \mathbf{E} \cdot d\mathbf{a}$ Gauss's theorem

$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \oint_{\partial S} \mathbf{B} \cdot d\mathbf{l}$ Stokes' theorem

Divergence theorem

- Choose $\mathbf{E}(x,y,z) = y^2\hat{i} + (2xy + z^2)\hat{j} + 2yz\hat{k}$

$$\nabla \cdot \mathbf{E} = 2(x+y) \qquad \int_V (\nabla \cdot \mathbf{E})\, d\tau = 2$$

V = unit cube

$$\int_V (\nabla \cdot \mathbf{E})\, d\tau \;?=?\; \oint_{\partial V} \mathbf{E} \cdot d\mathbf{a}$$

$$\oint_{\partial V} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{3} - \frac{1}{3} + \frac{4}{3} - \frac{1}{3} + 1 + 0 = 2$$

Curl theorem

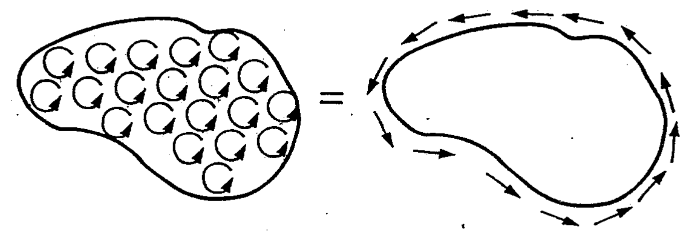


Figure 1.31

Integration by parts

- Scalar field $f(\mathbf{r})$, vector field $\mathbf{A}(\mathbf{r})$:

$$\int_V f(\nabla \cdot \mathbf{A})\, d\tau = \oint_{\partial V} f\mathbf{A} \cdot d\mathbf{a} - \int_V \mathbf{A} \cdot (\nabla f)\, d\tau$$

Delta “function” (distribution)

- 1-d:

$$\delta(x) = 0 \quad \text{for } x \neq 0, \qquad \int_{-\infty}^{\infty} \delta(x')dx' = \int_{-\varepsilon}^{\varepsilon} \delta(x')dx' = 1, \quad \text{unit area}$$

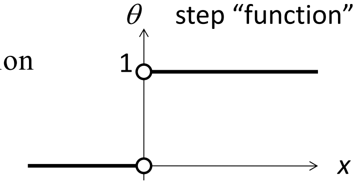
$$\int_{-\infty}^{\infty} \delta(x') f(x')dx' = f(0), \qquad \int_{-\infty}^{\infty} \delta(x-x') f(x')dx' = f(x)$$

$$[\delta * f](x) = f(x) \qquad \text{unit convolution}$$

$$\delta(x) = \frac{d\theta(x)}{dx} \qquad \text{as a distribution}$$

$$\delta(ax) = \frac{1}{|a|}\delta(x) \qquad \text{scaling}$$

step “function”



Convolutions

- Definition: given two fields $A(\mathbf{r})$ and $B(\mathbf{r})$

$$C(\mathbf{r}) = A * B = \int_{-\infty}^{\infty} A(\mathbf{r}-\mathbf{r}')B(\mathbf{r}')d\tau'$$

- Commutativity: $A * B = B * A$
- Associativity: $(A * B) * C = A * (B * C)$
- Unit element: $\delta * A = A * \delta = A$
- Derivative: $\partial_k (A * B) = (\partial_k A) * B$

- Divergence theorem and unit point source

apply $\int_V (\nabla \cdot \mathbf{E}) d\tau = \oint_{\partial V} \mathbf{E} \cdot d\mathbf{a}$ to $\mathbf{E}(\mathbf{r}) = \frac{\hat{\mathbf{r}}}{r^2}$

for a sphere of radius ε

$$\oint_{\partial V} \mathbf{E} \cdot d\mathbf{a} = \oint_{\partial V} \frac{\hat{\mathbf{r}}}{r^2} \cdot d\mathbf{a} = \oint_{\partial V} \frac{\hat{\mathbf{r}}}{r^2} \cdot \hat{\mathbf{r}} r^2 d\Omega = \oint_{\partial V} d\Omega = 4\pi$$

➔

$$\int_O (\nabla \cdot \mathbf{E}) d\tau = \int_O \left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} \right) d\tau = 4\pi$$

➔ $\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) = 4\pi \delta^{(3)}(\mathbf{r})$ and

$$\boxed{\nabla^2 \left(-\frac{1}{4\pi\, r} \right) = \delta^{(3)}(\mathbf{r})} \qquad g(\mathbf{r}) = -\frac{1}{4\pi r}$$

Displacing the vector **r** by **r'**:

$$\nabla^2 \left(-\frac{1}{4\pi |\mathbf{r}-\mathbf{r}'|} \right) = \delta^{(3)}(\mathbf{r}-\mathbf{r}').$$

Inverse of Laplacian

- To solve $\nabla^2 A(\mathbf{r}) = B(\mathbf{r})$

$(\nabla^2 g) * B = \nabla^2 (g * B) = \delta * B \quad \Rightarrow \quad A = g * B$

where $\nabla^2 g(\mathbf{r}) = \delta^{(3)}(\mathbf{r}) \quad \Rightarrow \quad g(\mathbf{r}) = -\frac{1}{4\pi r}$

$$A(\mathbf{r}) = \frac{1}{\nabla^2} B(\mathbf{r}) = \left(-\frac{1}{4\pi r}\right) * B(\mathbf{r}) \qquad \boxed{\frac{1}{\nabla^2} \doteq g(\mathbf{r}) *}$$

- In short-hand notation:

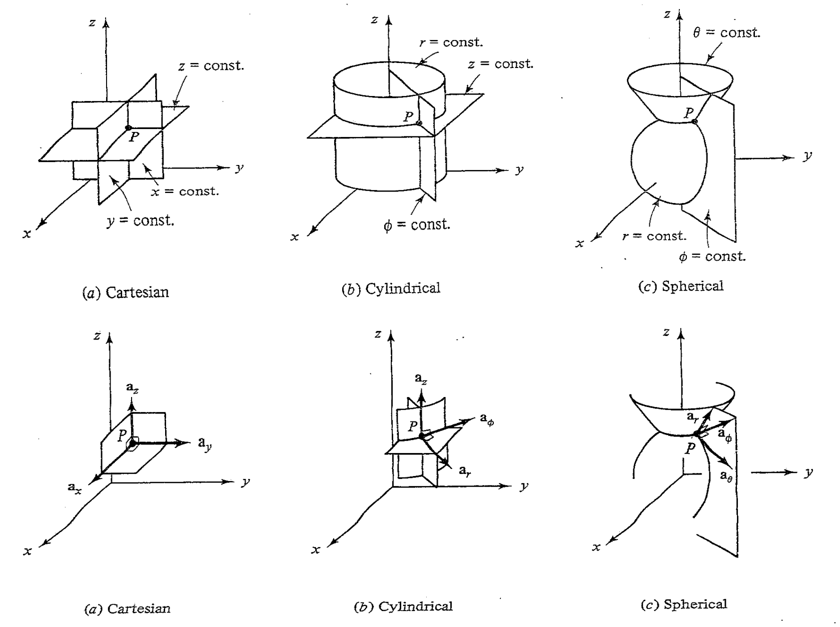
$$A(\mathbf{r}) = -\frac{1}{4\pi} \int \frac{1}{\mathbf{\mathscr{r}}} B(\mathbf{r}') d\tau', \qquad \text{where } \mathbf{\mathscr{r}} = |\mathbf{r}-\mathbf{r}'|$$

Orthogonal systems of coordinates

- coordinates: (u_1, u_2, u_3)
- orthogonal basis: $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$
- scale factors: (h_1, h_2, h_3)
- volume: $d(\text{vol}) = d\tau = h_1 h_2 h_3 \, du_1 du_2 du_3$
- area $\perp u_3$: $d\mathbf{a}_3 = h_1 h_2 \, du_1 du_2 \, \mathbf{e}_3$

- displacement vector:

$$d\mathbf{l} = h_1 du_1 \mathbf{e}_1 + h_2 du_2 \mathbf{e}_2 + h_3 du_3 \mathbf{e}_3$$

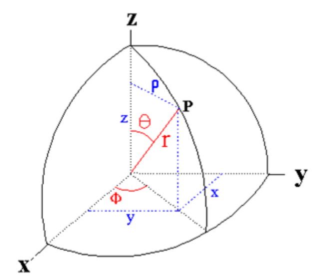


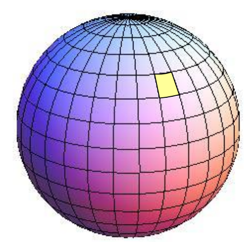
Scale Factors

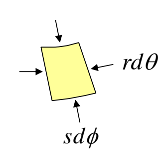
	du_1	du_2	du_3	h_1	h_2	h_3
Cartesian	dx	dy	dz	1	1	1
Cylindrical	ds	$d\phi$	dz	1	s	1
Spherical	dr	$d\theta$	$d\phi$	1	r	$r\sin\theta$

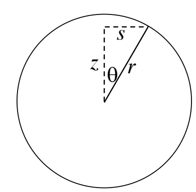
- polar (s, ϕ): $\mathbf{s} = s\,\hat{\mathbf{s}}$
- cylindrical (s, ϕ, z): $\mathbf{r} = s\,\hat{\mathbf{s}} + z\,\hat{\mathbf{e}}_3$
- spherical (r, θ, ϕ): $\mathbf{r} = r\,\hat{\mathbf{r}}$

Spherical Area









$s = r\sin(\theta)$

$z = r\cos(\theta)$

$$dA = r d\theta \, s d\phi = r^2 \sin\theta \, d\theta \, d\phi$$

- Grad: $$\nabla T = \frac{\hat{\mathbf{u}}_1}{h_1}\frac{\partial T}{\partial u_1} + \frac{\hat{\mathbf{u}}_2}{h_2}\frac{\partial T}{\partial u_2} + \frac{\hat{\mathbf{u}}_3}{h_3}\frac{\partial T}{\partial u_3}$$
- Div: $$\nabla \cdot \mathbf{E} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(E_1 h_2 h_3)}{\partial u_1} + \frac{\partial\left(h_1 E_2 h_3\right)}{\partial u_2} + \frac{\partial(h_1 h_2 E_3)}{\partial u_3}\right)$$
- Curl: $$\nabla \times \mathbf{E} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{\mathbf{u}}_1 & h_2 \hat{\mathbf{u}}_2 & h_3 \hat{\mathbf{u}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 E_1 & h_2 E_2 & h_3 E_3 \end{vmatrix}$$
- Laplacian:

$$\nabla^2 T = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial T}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial T}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial T}{\partial u_3}\right)\right)$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\,\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $d\tau = r^2 \sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

Maxwell's Equations

- Electro-statics: $$\nabla\mathbf{E}(\mathbf{r}) = \frac{1}{\varepsilon_0}\rho(\mathbf{r})$$
- Magneto-statics: $$\nabla(i\mathbf{B}(\mathbf{r})) = -\mu_0\mathbf{J}(\mathbf{r})$$
- Maxwell: $$\boxed{\nabla\mathcal{F} = \nabla(E + icB) = \frac{1}{\varepsilon_0 c}(c\rho - \mathbf{J})}$$

$$\frac{1}{\varepsilon_0 c} = \mu_0 c \approx 377\,\Omega$$

Formal solution

$$\nabla \mathcal{F}(\mathbf{r}) = \widetilde{\mathcal{J}}(\mathbf{r}) \qquad \Rightarrow \qquad \mathcal{F} = \nabla^{-1}\widetilde{\mathcal{J}}$$

separates into:

$$\nabla \mathbf{E}(\mathbf{r}) = \frac{1}{\varepsilon_0}\rho(\mathbf{r}) \quad \Rightarrow \quad \mathbf{E} = \nabla^{-1}\left(\frac{1}{\varepsilon_0}\rho\right) = \nabla \frac{1}{\nabla^2}\left(\frac{1}{\varepsilon_0}\rho\right)$$

and

$$i\nabla \mathbf{B}(\mathbf{r}) = -\mu_0 \mathbf{J}(\mathbf{r}) \quad \Rightarrow \quad i\mathbf{B} = \nabla \frac{1}{\nabla^2}\left(-\mu_0 \mathbf{J}\right)$$

Electro-statics

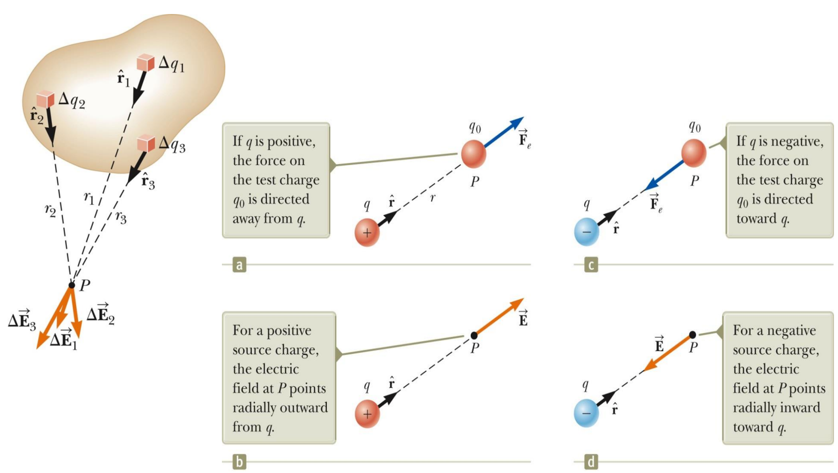
Convolution: $\mathbf{E}(\mathbf{r}) = -\frac{1}{4\pi\varepsilon_0}\nabla\left(\frac{1}{r} * \rho(\mathbf{r})\right) = \mathbf{E}_0(\mathbf{r}) * \rho(\mathbf{r})$

where $\mathbf{E}_0(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2}$ for point charge.

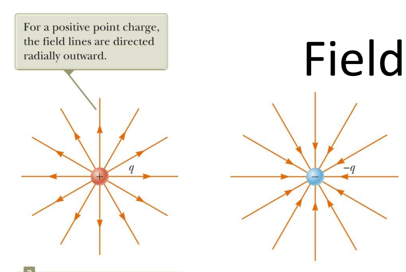
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\rho(\mathbf{r}')d\tau' = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{\boldsymbol{\imath}}}{\imath^2}\rho(\mathbf{r}')d\tau'$$

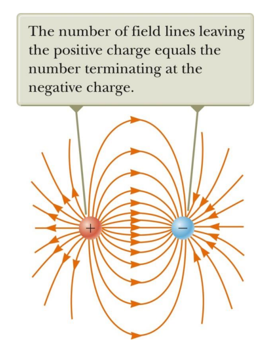
$\boxed{\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})}$ where $V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')d\tau'$

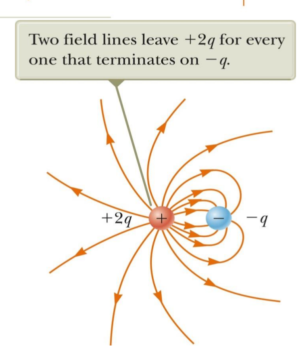
Electric Field from a Point Charge

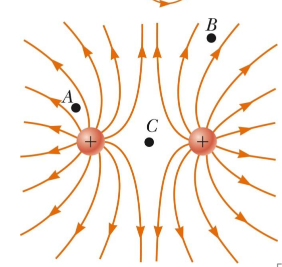


Field Lines

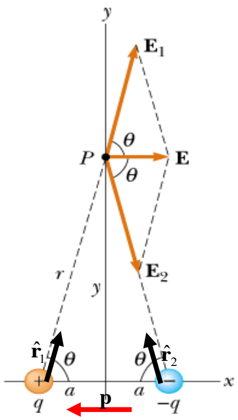


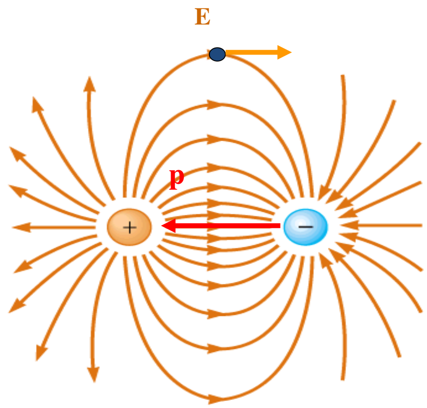




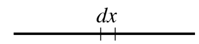


Dipole Field



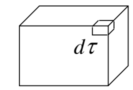


Non-uniform Density



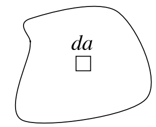
$$dq = \lambda(x)\,dx$$

$$q = \int dq = \int \lambda(x)\,dx$$



$$dq = \rho(\mathbf{r})\,d\tau$$

$$q = \int dq = \int \rho(\mathbf{r})\,d\tau$$



$$dq = \sigma(\mathbf{r})\,da$$

$$q = \int dq = \int \sigma(\mathbf{r})\,da$$

Superposition of charges

- For *n* charges $\{dq_1, dq_2, \ldots, dq_n\}$

$$V(\mathbf{r}_i) = \frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{n}\frac{dq_j}{r_{ij}} = \frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{n}\frac{dq_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

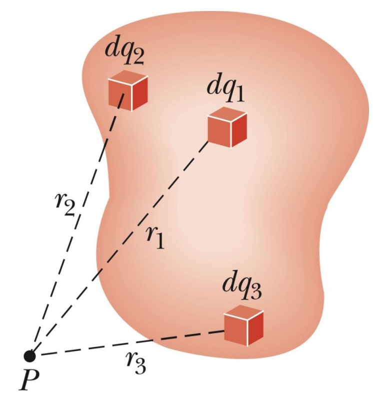
- continuum limit:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{dq'}{|\mathbf{r}-\mathbf{r}'|} \qquad \mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$$

- where

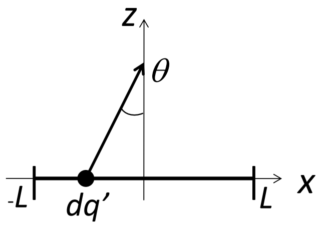
$$dq' = \begin{cases} \lambda(\mathbf{r}')\,dl' \\ \sigma(\mathbf{r}')\,da' \\ \rho(\mathbf{r}')\,d\tau' \end{cases}$$

Superposition Principle for Potential



$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{dq'}{|\mathbf{r}-\mathbf{r}'|} \qquad \mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$$

- linear uniform charge density $\lambda\,(x')$ from $-L$ to L
- field point @ $x = 0$, z variable



$$dq' = \lambda\, dx'$$
$$\boldsymbol{\mathscr{r}} = z\,\mathbf{k} - x'\,\mathbf{i}$$
$$\mathscr{r} = \sqrt{x'^2 + z^2} = z\sec\theta$$
$$x' = z\tan\theta$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{-L}^{L}\frac{\lambda\,\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}\,dx'$$

with $dx' = z\sec^2\theta\, d\theta,$

$$\mathbf{E}(\mathbf{r}) = \frac{2\lambda\hat{\mathbf{k}}}{4\pi\varepsilon_0}\int_0^{\theta_0}\frac{z^2\sec^2\theta}{z^3\sec^3\theta}\,d\theta = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda\hat{\mathbf{k}}}{z}\sin\theta_0$$

so
$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{z}\frac{L}{\sqrt{L^2+z^2}}\hat{\mathbf{k}}.}$$

- Limits:

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \dfrac{1}{2\pi\varepsilon_0}\dfrac{\lambda}{s}\hat{\mathbf{s}} & L \to \infty \\ \dfrac{q}{4\pi\varepsilon_0}\dfrac{1}{r^2}\hat{\mathbf{r}} & z \to \infty,\quad L \text{ fixed} \end{cases}$$

Gauss's law

- Flux of **E** through a surface S: $\Phi_E = \int_S \mathbf{E}\cdot d\mathbf{a}$
  volume V enclosed by surface S.
- Flux through the closed surface:

$$\boxed{\oint_S \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\varepsilon_0}Q_{enc} = \frac{1}{\varepsilon_0}\int_V \rho\, d\tau}$$

- choose a "Gaussian surface" (symmetric case)

$$E\,\text{Area}(G.S.) = \frac{1}{\varepsilon_0}Q_{enc} \qquad \hat{\mathbf{E}} \text{ determined by symmetry}$$

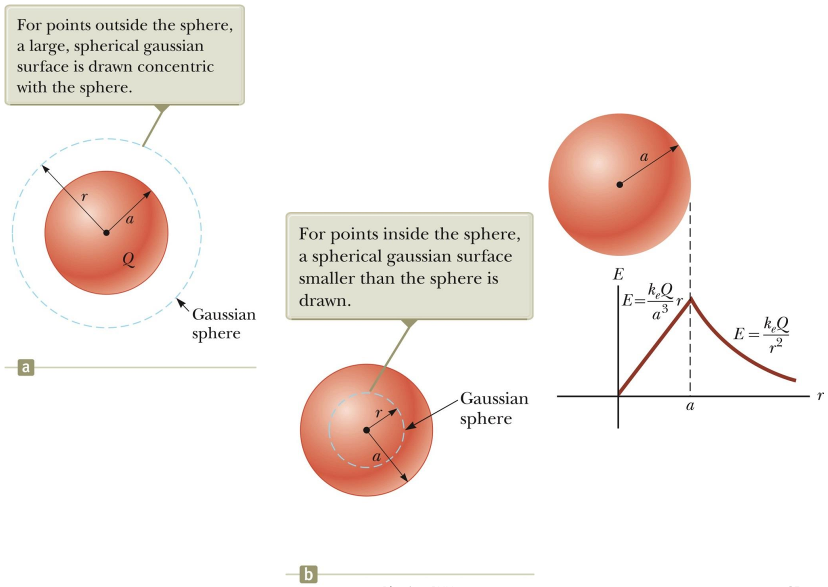
Examples:

- **Charged sphere** (uniform density) radius R
  Gaussian surface: $A = 4\pi r^2$

a) $r < R$:
$$Q_{enc} = \int_{Vol(r)}\rho(\mathbf{r}')\,d\tau' = Q\frac{r^3}{R^3}$$
$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0}\frac{r}{R^3}\hat{\mathbf{r}} \qquad \text{for } r < R$$

b) $r > R$: $Q_{enc} = Q \;\Rightarrow\; \mathbf{E}(\mathbf{r}) = \dfrac{Q}{4\pi\varepsilon_0}\dfrac{\hat{\mathbf{r}}}{r^2} \qquad \text{for } r > R$

as if **all** Q is concentrated @ origin



- **Thin wire**: linear uniform density (C/m)

  Gaussian surface: cylinder

$$A = 2\pi\, sl \qquad Q_{enc} = \lambda\, l$$

$$E(2\pi\, sl) = \frac{\lambda l}{\varepsilon_0} \quad \Rightarrow \quad \mathbf{E(s)} = \frac{\lambda}{2\pi\varepsilon_0}\frac{\hat{\mathbf{s}}}{s}$$

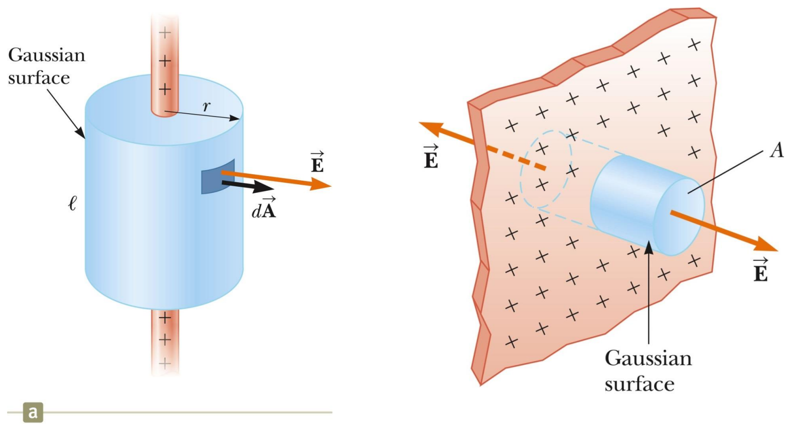
- **Plane**: surface uniform density (C/m^2)

  Gaussian surface: "pill box" straddling plane

$$E\,(2A) = \frac{\sigma\, A}{\varepsilon_0} \quad \Rightarrow \quad \mathbf{E} = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}$$

  CONSTANT, pointing AWAY from surface (both sides)

Gauss's Law II



Boundary conditions for **E**

- Gaussian box w/ small area ΔA // surface w/ charge density σ

  $\varepsilon\, \hat{\mathbf{n}}\mathbf{E}\Big|_2^1 = \sigma$ with **n** pointing away from 1

- Equivalently: $\boxed{\mathbf{E}\big|_1 - \mathbf{E}\big|_2 = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}}$

- Component parallel to surface is continuous
- Discontinuity for perp. component $= \sigma/\varepsilon_0$

Electric Potential (V = J/C)

- Voltage

$$V(\mathbf{r}) = V_0(\mathbf{r}) * \rho(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')d\tau' = \frac{1}{4\pi\varepsilon_0}\int \frac{dq'}{\mathscr{r}}$$

solves Poisson's equation: $\nabla^2 V(\mathbf{r}) = -\frac{1}{\varepsilon_0}\rho(\mathbf{r})$

- point charge Q at the origin $\rho(\mathbf{r}) = Q\delta(\mathbf{r})$

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}}$$

Potential Difference (voltage)

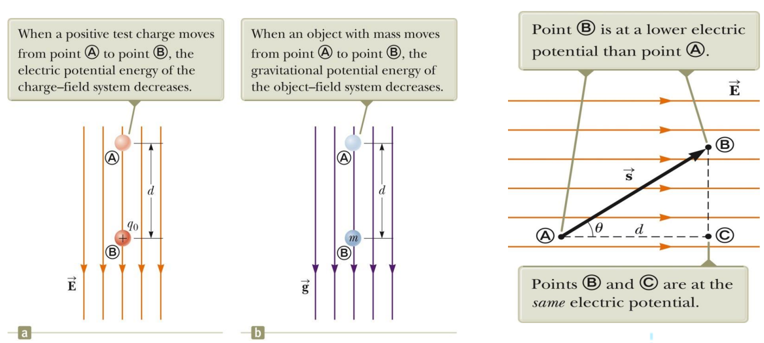
- in terms of **E**: $V(\mathbf{r}) - V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{l}$ and

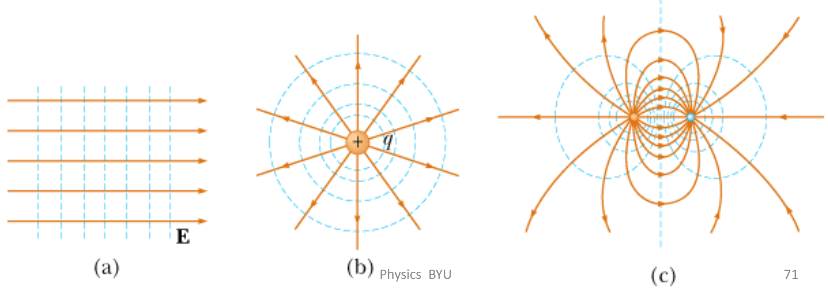
$$\oint \mathbf{E}\cdot d\mathbf{l} = 0$$

- Spherical symmetry: $V = V(r)$

$$\mathbf{E}(\mathbf{r}) = -\frac{dV(r)}{dr}\hat{\mathbf{r}} \quad \text{where } r = |\mathbf{r}|$$

- Potential energy: $U = q\,V$ (joules)
- Equi-potential surfaces: perpendicular to field lines





- Example: spherical shell

radius R, uniform surface charge density σ

Gauss's law ➔ **E**(r) = 0 inside ($r < R$)

For $r > R$:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \quad \text{where } q = 4\pi R^2\sigma \quad \text{and}$$

$$V(r) = -\int_{\infty}^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{l} = -\frac{1}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{q}{r^2}dr = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

- Example: infinite straight wire, uniform line charge density λ Gauss's law:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{s}\hat{\mathbf{s}} \qquad \text{and}$$

$$V(r) = -\int_{\mathbf{a}}^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{l} = -\frac{1}{2\pi\varepsilon_0}\int_a^s \frac{\lambda}{s}ds = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{a}{s}$$

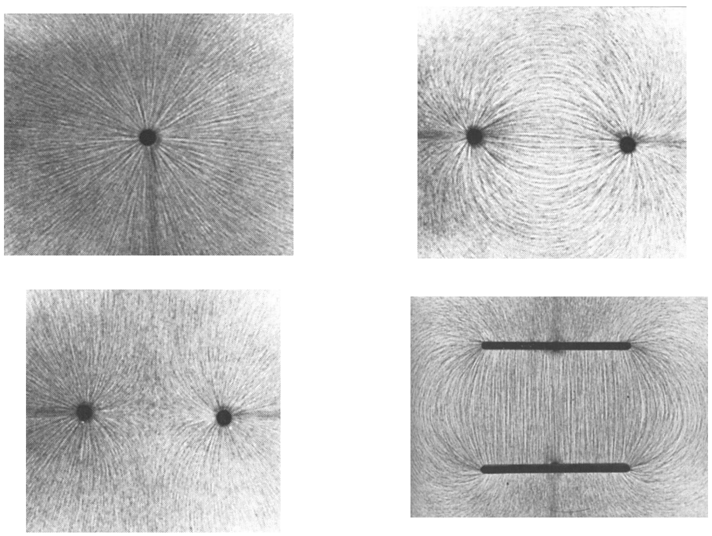
Electric-Magnetic materials

- conductors
  - surface charge
  - boundary conditions
  - 2^{nd} order PDE for V (Laplace)
- dielectrics
  - auxiliary field **D** (electric displacement field)
- non-linear electric media
- magnets
  - auxiliary field **H**
- ferromagnets
  - non-linear magnetic media

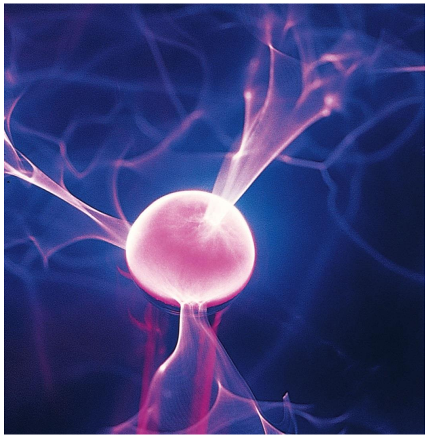
Perfect conductors

- Charge *free* to move with no resistance
- **E** = 0 INSIDE the conductor
- ρ = 0 inside the conductor
- **NET** charge resides entirely on the surface
- The conductor surface is an EQUIPOTENTIAL
- **E** *perpendicular* to surface just OUTSIDE
- E = σ/ε_0 **locally**
- Irregularly shaped ➔ σ is GREATEST where the curvature radius is SMALLEST

E-field perpendicular to conducting surfaces



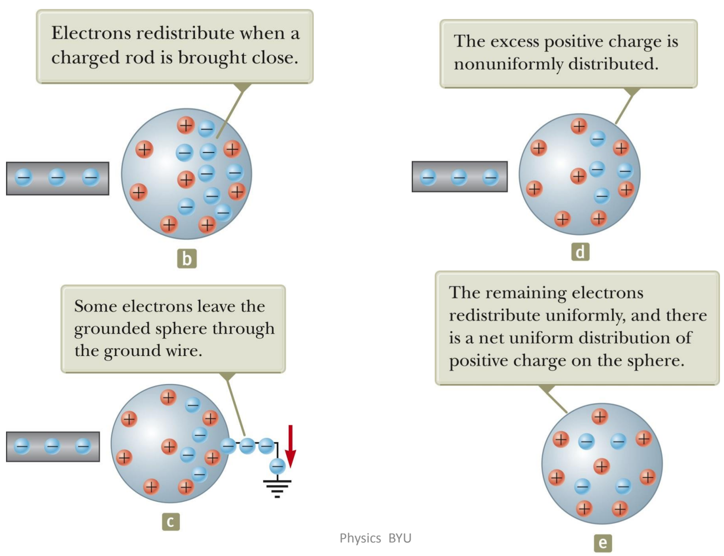
Corona Discharge



Induced charge

- Charge held close to a conductor will induce charge displacement due to the attraction (repulsion) of the inducing charge and the mobile charges in the conductor

Induced Charge II



Work and Energy

- Work: $W = -\int_{\mathbf{a}}^{\mathbf{b}} Q\mathbf{E}\cdot d\mathbf{l} = Q[V(\mathbf{b}) - V(\mathbf{a})] = Q\Delta V$
- for n interacting charges:

$$W = \frac{1}{4\pi\varepsilon_0}\frac{1}{2}\sum_{i\neq j}^{n}\frac{q_i q_j}{r_{ij}} = \frac{1}{2}\sum_{i}^{n} q_i V(\mathbf{r}_i)$$

- continuous distribution ρ

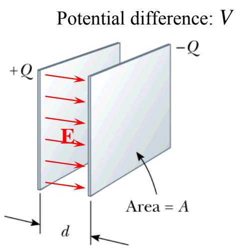
$$W = \frac{1}{2}\int \rho V d\tau = \frac{\varepsilon_0}{2}\int (\nabla\cdot\mathbf{E}) V d\tau = \frac{\varepsilon_0}{2}\left(\int E^2 d\tau + \oint V\mathbf{E}\cdot d\mathbf{a}\right)$$

- Energy density: $\boxed{u(\mathbf{r}) = \frac{\varepsilon_0}{2}E^2(\mathbf{r})}$

  also electric PRESSURE

Capacitors (vacuum)

- Capacitor: charge $+Q$ and $-Q$ on each plate



$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A\varepsilon_0},$$

$$\text{so } \Delta V = E\,d = Q\frac{d}{A\varepsilon_0}$$

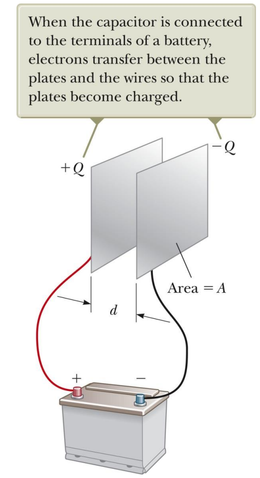
define capacitance C : $Q = C\,\Delta V$

with units $1F = 1C/1V$

$$\boxed{C = \frac{\varepsilon_0 A}{d}}$$

- charging a capacitor C: move dq at a given time from one plate to the other adding to the q already accumulated

$$dW = \frac{q}{C}dq \quad \Rightarrow \quad W = \frac{1}{2}\frac{Q^2}{C} = \frac{CV^2}{2}$$



- in terms of the field **E**

$$W = \frac{1}{2}\varepsilon_0\frac{A}{d}(Ed)^2 = \frac{1}{2}\varepsilon_0 E^2(\text{Vol})$$

➔ $u(\mathbf{r}) = \frac{\varepsilon_0}{2}E^2(\mathbf{r})$

energy density

Laplace's equation

- V at a boundary (instead of σ)

$$\boxed{\nabla^2 V = 0}$$

one dimension: $\frac{d^2V}{dx^2} = 0 \quad \Rightarrow \quad V(x) = mx + b$

capacitor V_0 → V_1 (0 to d)

$$V(x) = \frac{V_1 - V_0}{d}x + V_0 \quad \text{from} \quad 0 \to d$$

$V(x)$ is the **AVERAGE VALUE** of $V(x\text{-}a)$ and $V(x\text{+}a)$

- Harmonic function has NO maxima or minima inside. All extrema at the BOUNDARY
- Average value (for a sphere):

$$V_0 = V(r=0) = \frac{1}{\text{Area}}\oint V(\mathbf{r})da$$

Proof: $\int \nabla^2 V d\tau = \int \nabla\cdot(\nabla V)d\tau = \oint(\nabla V)\cdot d\mathbf{a} =$

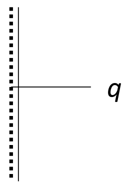
$$= R^2\oint\frac{\partial V}{\partial r}d\Omega = R^2\frac{d}{dR}\oint V d\Omega = 0$$

➔ $\oint V d\Omega = V_0 4\pi \quad \text{for} \quad R \to 0$

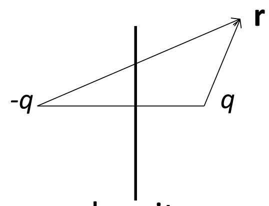
$\oint V da = V_0 4\pi R^2 \quad \text{for finite } R$

Method of images

- Problem: find *V* for a point charge *q* facing an infinite conducting plane



equivalent to



potential:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_1} - \frac{q}{r_2}\right)$$

charge density:

$$\sigma = -\varepsilon_0 \left.\frac{\partial V}{\partial n}\right|_{\text{plane}}$$

Laplace's equation in 2-d

Complex variable $z = x + iy$ ➔ $w'(z)$ well defined

$$\frac{dw(z)}{dz} = \frac{\frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy}{dx + idy} = \begin{cases} \frac{\partial w}{\partial x} & \text{for } dy = 0 \\ -i\frac{\partial w}{\partial y} & \text{for } dx = 0 \end{cases}$$

$$\Rightarrow \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w(x+iy) = 0$$

both real and imaginary parts of *w* fulfill Laplace's equation

Examples and equipotentials in 2-d

- $w(z) = z$ $\quad V(x, y) = y$
  equipotential lines: $\quad y = a$ (const)
- $w(z) = z^2$ $\quad V(x, y) = 2xy$, or $x^2 - y^2$
  e. l.: $\quad xy = a$ (hyperbolae)
- $w(z) = \ln z$ $\quad V(s, \phi) = \ln s$, $\arctan(y/x)$
  e. l.: $\quad s = \exp(a)$
- $w(z) = 1/z = \exp(-i\,\phi)/s$ $\quad V(s, \phi) = \cos\phi/s$
  e. l.: $\quad s = \cos\phi/a$ (circles O)

- $w(z) = \ln\left(\frac{z + 1/2}{z - 1/2}\right)$ $\quad V(x, y)$ = finite dipole
- $w(z) = z + 1/z$ $\quad V(s, \phi) = (s - 1/s)\sin\phi$
  e. l.: $\quad s = 1$ for $a = 0$
- w(x) = exp(z) $\quad V(x, y) = \exp(x)\cos y$

$V_k(x, y) = \exp(k\,x)\cos(k\,y)$ & $\exp(k\,x)\sin(k\,y)$

Separation of variables (polar)

- Powers of *z*: $z^n = s^n\, e^{in\phi}$

$$V_n(s,\phi) = S_n(s)\Phi_n(\phi) \qquad \text{where}$$

$$S_n(s) = s^n \qquad \text{and} \qquad \Phi_n(\phi) = \begin{cases} \cos(n\phi) \\ \sin(n\phi) \end{cases}$$

- Solution of $\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} = 0$

$$\boxed{V = V_0 + a_0 \ln s + \sum_{n=1} (a_n s^n + b_n s^{-n})[c_n \cos(n\phi) + d_n \sin(n\phi)]}$$

Separation of variables (*x*, *y*)

- k^2 = separation constant $V(x,y) = X(x)Y(y)$

$$\frac{1}{V}\nabla^2 V = \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = k^2 - k^2 = 0$$

- Eigenvalue equation for *X* and *Y*

$$\frac{d^2X(x)}{dx^2} = k^2 X(x) \qquad \& \qquad \frac{d^2Y(y)}{dy^2} = -k^2 Y(y)$$

- Construct linear combination, determine coefficients w/ B.C. using orthogonality

- Example: i) $V(x=0) = V_0$ ii) $V(x \to \infty) \to 0$
  iii) $V(y=0) = 0$ iv) $V(y=\pi) = 0$

solution:

$$X_k(x) = A\,e^{kx} + B\,e^{-kx}$$
$$Y_k(y) = C\sin(ky) + D\cos(ky)$$

B.C.: ii) $A = 0$; iii) $D = 0$;
iv) quantization $k = 1, 2, 3, \ldots$

➔ $$V(x,y) = \sum_{n=1} C_n\, e^{-nx} \sin(ny)$$

- orthonormality:

$$\int_0^\pi \sin(ny)\sin(my)\,dy = \frac{\pi}{2}\delta_{nm}$$

i) $$C_m = \frac{2}{\pi}\int_0^\pi V_0 \sin(my)\,dy = \begin{cases} 0 & m \text{ even} \\ \frac{4V_0}{m\pi} & m \text{ odd} \end{cases}$$

solution: $$V(x,y) = \frac{4V_0}{\pi}\sum_{n=1}^{odd} \frac{1}{n} e^{-nx} \sin(ny)$$

Separation of variables - spherical

- Assume axial symmetry, *i.e.* no ϕ dependence

$$\nabla^2 \to \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)$$

separation constant $l(l+1)$

$$V(r,\theta)=R(r)\Theta(\theta)$$

$$\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)=l(l+1)R(r)$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\,\Theta(\theta)}{d\theta}\right)=-l(l+1)\Theta(\theta)$$

Legendre polynomials

- change of variable

$$\Theta(\theta)=P_l(\zeta)=P_l(\cos\theta)$$

$$\frac{d}{d\zeta}\left[(1-\zeta^2)\frac{d\,P_l(\zeta)}{d\zeta}\right]+l(l+1)P_l(\zeta)=0$$

- orthogonality

$$\int_{-1}^{1}P_l(\zeta)P_m(\zeta)\,d\zeta=\frac{2}{2l+1}\delta_{lm}$$

- “normalization” $P_l(1) = 1$

- Radial function $R(r) = r^l$ or r^{-l-1}

$$\boxed{V(r,\theta)=\sum_{l=0}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)}$$

$$\Rightarrow\begin{cases} A_0+A_1 r\cos\theta+\ldots=A_0+A_1 z+\ldots \\ \dfrac{B_0}{r}+\dfrac{B_1}{r^2}\cos\theta+\ldots \end{cases}$$

- Given $V_0(\theta)$ on the surface of a hollow sphere (radius R), find $V(r)$
- Soln: $B_l = 0$ for $r < R$, and $A_l = 0$ for $r > R$

B.C. @ $r = R$:

$$V_0(\theta)=\begin{cases}\sum_{l=0}^{\infty}A_l R^l P_l(\cos\theta)\\ \sum_{l=0}^{\infty}B_l R^{-l-1}P_l(\cos\theta)\end{cases}$$

using orthogonality:

$$A_l=\frac{2l+1}{2R^l}\int_0^{\pi}V_0(\theta)P_l(\cos\theta)\sin\theta\,d\theta \qquad r<R$$

$$B_l=\frac{2l+1}{2}R^{l+1}\int_0^{\pi}V_0(\theta)P_l(\cos\theta)\sin\theta\,d\theta \qquad r>R$$

- Uncharged conducting sphere in uniform electric field $\mathbf{E} = E_0\,\mathbf{k}$

$$V = (A_0 + \frac{B_0}{r}) + (A_1 r + \frac{B_1}{r^2})\cos\theta + \ldots$$

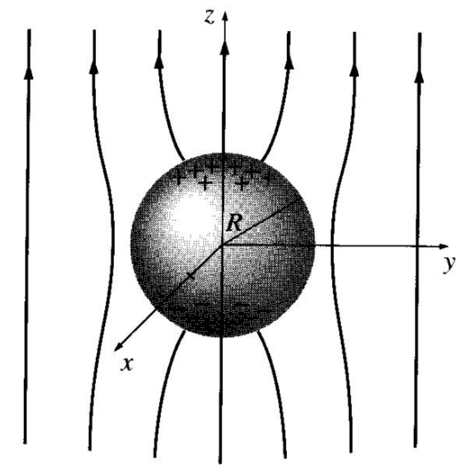
$V(r = R) = 0$: $\quad A_l R^l + \dfrac{B_l}{R^{l+1}} = 0 \quad \Rightarrow \quad B_l = -A_l R^{2l+1}$

$V \rightarrow -E_0 r \cos\theta \quad r >> R$:

$$A_0 = B_0 = 0 \qquad B_1 = -A_1 R^3 \qquad A_1 = -E_0$$

$$\Rightarrow \quad V(\mathbf{r}) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta$$

$$\sigma(\theta) = -\varepsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=R} = 3\varepsilon_0 E_0 \cos\theta$$



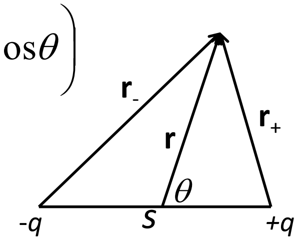
Electric Dipole

- finite dipole $\quad V(r) = \dfrac{q}{4\pi\varepsilon_0}\left(\dfrac{1}{r_+} - \dfrac{1}{r_-}\right)$

as $r \rightarrow \infty \qquad \dfrac{1}{r_\pm} \rightarrow \dfrac{1}{r}\left(1 \pm \dfrac{s}{2r}\cos\theta\right)$

and $\qquad \dfrac{1}{r_+} - \dfrac{1}{r_-} \rightarrow \dfrac{s}{r^2}\cos\theta$

$$V(\mathbf{r}) \rightarrow \frac{qs}{4\pi\varepsilon_0}\frac{1}{r^2}\cos\theta$$



$p = qs$ is the dipole moment for a finite dipole

Multipole expansion

- Legendre polynomials: coefficients of $1/\mathscr{r}$ expansion in powers of $\alpha = r'/r < 1$

$$\mathscr{r} = |\mathbf{r} - \mathbf{r}'| = \sqrt{r^2 + r'^2 - 2\mathbf{r}\cdot\mathbf{r}'} = r\sqrt{1+\varepsilon}$$

where $\quad \varepsilon = \alpha^2 - 2\alpha\cos\theta \qquad$ and

$$(1+\varepsilon)^{-1/2} \approx 1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 + \ldots$$

$$\approx 1 + \alpha\cos\theta + \alpha^2\left(\frac{3\cos^2\theta - 1}{2}\right) + \ldots$$

- Multipole expansion of V: using

$$\frac{1}{\mathscr{r}} = \sum_{l=0} \frac{1}{r}\left(\frac{r'}{r}\right)^l P_l(\cos\theta') \approx \frac{1}{r} + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{r^2} + \ldots$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}\frac{1}{r^{l+1}}\int r'^{\,l}\, P_l(\cos\theta')\, dq'$$

$$V(\mathbf{r}) \approx \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} + \ldots\right)$$ with

$$Q = \int dq' \qquad \text{and} \qquad \mathbf{p} = \int \mathbf{r}'\, dq'$$

Electric field of a dipole

- Choosing $\mathbf{p}$ along the z axis

$$V_{dip}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{pz}{r^3}$$

and $\mathbf{E}_{dip}(\mathbf{r}) = -\nabla V_{dip}(\mathbf{r}) = -\frac{1}{4\pi\varepsilon_0}p\nabla\left(\frac{z}{r^3}\right)$

$$\boxed{\mathbf{E}_{dip}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]}$$

Clifford form: $\mathbf{E}_{dip}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{2r^3}(3\hat{\mathbf{r}}\mathbf{p}\hat{\mathbf{r}} + \mathbf{p})$

Polarization

- **P**(**r**) polarization, vector field =

  = dipole moment/volume.

Compare: $$V_{ch\,dist}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathscr{r}}d\tau'$$

w/ potential due to dipole distribution:

$$V_{pol}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{P}(\mathbf{r}')}{\mathscr{r}^2}d\tau'$$

A useful integral over the sphere

$$\mathbf{I}_0(\mathbf{r}) = \int_{sph}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}d\tau'$$ corresponds to electric field w/ constant density: $\rho = 4\pi\varepsilon_0$

using Gauss's law:

$$\mathbf{I}_0(\mathbf{r}) = \frac{4\pi}{3}\begin{cases}\mathbf{r} & \text{inside}\\ \frac{R^3}{r^2}\hat{\mathbf{r}} & \text{outside}\end{cases}$$

Applications: constant charge density, polarization, and magnetization

- Example: Find the electric field **E** produced by a uniformly polarized sphere of radius *R*

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\mathbf{P}\cdot\int_{sph}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}d\tau'$$

where $\displaystyle\int_{sph}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}d\tau' = \frac{4\pi}{3}\begin{cases}\mathbf{r} \\ R^3\hat{\mathbf{r}}/r^2\end{cases}$

and

$$\mathbf{E} = -\nabla V = \begin{cases}-\dfrac{1}{3\varepsilon_0}\mathbf{P} & r<R \\ \text{dipole field} & r>R\end{cases}$$

Equivalent "bound charges"

- Integration by parts:

$$\nabla'\cdot\left(\frac{\mathbf{P}(\mathbf{r}')}{\mathscr{r}}\right) = \mathbf{P}(\mathbf{r}')\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right) + \frac{1}{\mathscr{r}}\nabla'\cdot\mathbf{P}(\mathbf{r}') =$$

$$= \frac{\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{P}(\mathbf{r}')}{\mathscr{r}^2} - \frac{1}{\mathscr{r}}\nabla\cdot\mathbf{P}$$

so that:

$$\boxed{V_{pol}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{1}{\mathscr{r}}\rho_b(\mathbf{r}')d\tau' + \frac{1}{4\pi\varepsilon_0}\oint\frac{1}{\mathscr{r}}\sigma_b(\mathbf{r}')da'}$$

where $\rho_b = -\nabla\cdot\mathbf{P}$ and $\sigma_b = \hat{\mathbf{n}}\cdot\mathbf{P}$

and $\nabla\cdot\mathbf{P}\neq 0$ only if $\mathbf{P}$ is not uniform

- surface charge:

$\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow$ ➔ -- $\longrightarrow$ +

- In a dielectric **charge** does not migrate. It gets **polarized**.

Electric Displacement field D(r)

- Total charge density: free plus bound
  $\rho(\mathbf{r}) = \rho_f(\mathbf{r}) + \rho_b(\mathbf{r}) = \rho_f(\mathbf{r}) - \nabla\cdot\mathbf{P}$
- Defining $\mathbf{D}(\mathbf{r}) = \varepsilon_0\mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})$

we get: $\boxed{\nabla\cdot\mathbf{D} = \rho_f}$ and $\boxed{\nabla\times\mathbf{E} = 0}$

- Gauss's law for **D**: $\oint\mathbf{D}\cdot da = Q_{f\ enc}$

Linear Dielectrics

- **D**, E, and **P** are proportional to each other for **linear materials**

ε_0	$-$	permittivity space
ε	$-$	permittivity material
κ	$\varepsilon / \varepsilon_0$	dielectric constant
χ_e	$\kappa - 1$	susceptibility

$$\mathbf{D} = \varepsilon \mathbf{E} = \kappa \varepsilon_0 \mathbf{E} \qquad C = \kappa C_0$$

$$\mathbf{P} = \mathbf{D} - \varepsilon_0 \mathbf{E} = (\varepsilon - \varepsilon_0)\mathbf{E} = \varepsilon_0 \chi_e \mathbf{E}$$

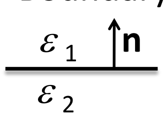
- **Theorem**:

In a **linear** dielectric material with constant susceptibility the *bound* charge distribution is **proportional** to the *free* charge distribution.

$$\rho_b = -\nabla \cdot \mathbf{P} = -\nabla \cdot \left(\varepsilon_0 \chi_e \frac{\mathbf{D}}{\varepsilon} \right) = -\frac{\chi_e}{1+\chi_e} \rho_f$$

Boundary conditions

- Boundary between two linear dielectrics



Boundary conditions:

$$\mathbf{D}_1^{\perp} - \mathbf{D}_2^{\perp} = \sigma_f \hat{\mathbf{n}}$$

$$\varepsilon_1 \left.\frac{\partial V}{\partial n}\right|_1 - \varepsilon_2 \left.\frac{\partial V}{\partial n}\right|_2 = -\sigma_f$$

- Example: Homogeneous dielectric sphere, radius R in uniform external electric field $\mathbf{E}_0$
- Solution:

$$\rho_b \propto \rho_f = 0 \quad \Rightarrow \quad \mathbf{E}_{sph} = -\frac{1}{3\varepsilon_0}\mathbf{P}$$

in terms of the unknown **P**.

Total field: $\mathbf{E} = \mathbf{E}_{sph} + \mathbf{E}_0$ and polarization:

eliminating **P**:

Energy in dielectrics

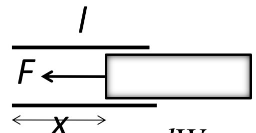
- vacuum energy density:
- dielectric: $u(\mathbf{r}) = \frac{1}{2}\mathbf{D}\cdot\mathbf{E}$
- Energy stored in dielectric system:

$$W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\, d\tau$$

- energy stored in capacitor:

$$W = \frac{1}{2}CV^2 = \frac{\kappa}{2}C_0 V^2$$

Fringing field in capacitors



for fixed charge Q:

$$F = -\frac{dW}{dx} = -\frac{1}{2}Q^2\frac{d}{dx}\left(\frac{1}{C(x)}\right) = \frac{1}{2}V^2\frac{dC}{dx}$$

where $C(x) = \frac{w}{d}[\varepsilon_0 x + \varepsilon(l-x)] = \frac{\varepsilon_0 w}{d}(\kappa l - \chi_e x)$

so $$\boxed{F = -\frac{\varepsilon_0 w \chi_e}{2d}V^2 < 0}$$

Magnetostatics duality in Clifford space

- Electric
- Field lines: from + to -
- **E**(**r**): vector field
- pos. &neg. charge
- ρ (**r**): scalar source
- *V* (**r**): scalar potential
- units [E] = V/m

$$\nabla\mathbf{E} = \frac{1}{\varepsilon_0}\rho$$

- Magnetic
- Field lines: closed
- *i* **B**: bivector field
- **no** magnetic monopoles
- **J**(**r**): vector source
- **A** (**r**): vector potential
- units [B] = Vs/m^2

$$i\nabla\mathbf{B} = -\mu_0\mathbf{J}$$

Lorentz force

- Point charge *q* in an external magnetic field **B**

$$\mathbf{F} = q\mathbf{v}\times\mathbf{B}$$

- velocity dependent (but **no**-friction)
- perpendicular to motion: **no** work done

$$\mathbf{F}\cdot d\,\mathbf{l} = q(\mathbf{v}\times\mathbf{B})\cdot\mathbf{v}\,dt = 0$$

Example: uniform magnetic field **B**

define cyclotron (angular) frequency: $\omega = \frac{qB}{m}$

equation of motion:

$$\dot{\mathbf{v}} = -\omega \hat{\mathbf{n}} \times \mathbf{v} \quad \text{where} \quad \hat{\mathbf{n}} = \frac{\mathbf{B}}{B}$$

solution:

$$\mathbf{v}(t) = \exp(-\omega t\, \hat{\mathbf{n}}\times)\, \mathbf{v}(0)$$
$$= \mathbf{v}_{//}(0) + \cos(\omega t)\mathbf{v}_{\perp}(0) - \sin(\omega t)\hat{\mathbf{n}} \times \mathbf{v}_{\perp}(0)$$

with $\mathbf{v}_{\perp}(t)^2 = \mathbf{v}_{\perp}(0)^2 = \text{const.}$

and radius: $qv_{\perp}B = m\dfrac{v_{\perp}^{\,2}}{R}$ so $R = \dfrac{v_{\perp}}{\omega}$

Add uniform electric field **E** perpendicular to **B**:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad \dot{\mathbf{v}} + \omega \hat{\mathbf{n}} \times \mathbf{v} = \frac{q\mathbf{E}}{m}$$

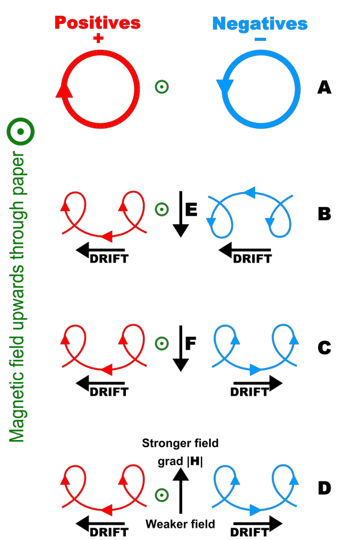
a particular solution is given by:

$$\mathbf{v}(t) = \frac{\mathbf{E} \times \hat{\mathbf{n}}}{B} \quad \text{given that} \quad \mathbf{E} \cdot \hat{\mathbf{n}} = 0$$

and
$$\boxed{\mathbf{v}(t) = \exp(-\omega t \hat{\mathbf{n}}\times)\mathbf{v}_0 + \frac{\mathbf{E} \times \mathbf{B}}{B^2}}$$

Applications:

- cyclotron motion
- velocity selector



Charge current densities

- current $I = \dfrac{dq}{dt}$ [I] = Amp = C/s ➔
  flow of charge per unit time
- current density
$$q\mathbf{v} \quad \Rightarrow \quad \begin{cases} \mathbf{J} = \rho \mathbf{v} & \text{volume} \quad (\text{A/m}^2) \\ \mathbf{K} = \sigma \mathbf{v} & \text{surface} \quad (\text{A/m}) \\ \mathrm{I} = \lambda \mathbf{v} & \text{wire} \quad (\text{A}) \end{cases}$$
- magnetic force
$$\mathbf{F} = \int \mathbf{v} \times \mathbf{B}\, dq = \begin{cases} \int \mathbf{J} \times \mathbf{B}\, d\tau \\ \int \mathbf{K} \times \mathbf{B}\, da \\ \int \mathbf{I} \times \mathbf{B}\, dl \end{cases}$$

- Relation between I and J

- a) consider cylinder, radius R, with uniform steady current I

$$J = \frac{I}{\pi R^2}$$

- b) for $\mathbf{J} = k\mathbf{s}$

$$I = \int_{\perp} \mathbf{J} \cdot d\mathbf{a} = \int (ks) s ds d\phi = 2\pi \int_0^R ks^2 ds = \frac{2\pi}{3} kR^3$$

Charge conservation

- current across *any* closed surface

$$I = \oint \mathbf{J} \cdot d\mathbf{a} = \int \nabla \cdot \mathbf{J} d\tau = -\frac{d}{dt} \int \rho d\tau$$

- local (differential) form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

- steady currents: $\nabla \cdot \mathbf{J} = 0$

Solving Magnetostatic Equation

- Two Maxwell equations rewritten as

$$\nabla \mathbf{B} = i\mu_0 \mathbf{J}$$

formally solved as:

$$\mathbf{B} = i\mu_0 \nabla \frac{1}{\nabla^2} \mathbf{J} = -\mu_0 \nabla \times \left(\frac{1}{\nabla^2} \mathbf{J} \right) = -\frac{\mu_0}{4\pi} \nabla \times \left(\frac{1}{r} * \mathbf{J} \right)$$

$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$ where $\mathbf{A} = -\mu_0 \frac{1}{\nabla^2} \mathbf{J}$

Explicitly:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \left(\frac{1}{r} * \mathbf{J}(\mathbf{r}) \right) = \frac{\mu_0}{4\pi} \int \frac{1}{\mathscr{r}} \mathbf{J}(\mathbf{r}') d\tau'$$

Using $\nabla \times \left(\frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} \right) = -\mathbf{J}(\mathbf{r}') \times \nabla \left(\frac{1}{\mathscr{r}} \right) = \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2}$

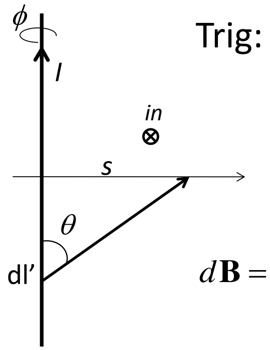
the magnetic field is given in terms of the vector source as:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau' = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d\tau'$$

For current on a wire (Biot-Savart expression):

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}(\mathbf{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} dl' = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

Calculate the magnetic field due to a uniform steady current *I*



Trig: $z' = -s\cot\theta \qquad \frac{1}{\mathscr{r}^2} = \frac{\sin^2\theta}{s^2} \qquad dz' = \frac{s\,d\theta}{\sin^2\theta}$

Biot-Savart:

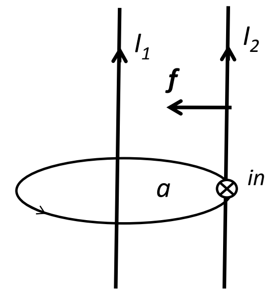
$$d\mathbf{l}'\times\hat{\mathscr{r}} = dz'\sin\theta\,\hat{\phi}$$

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi}\frac{d\mathbf{l}'\times\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0 I}{4\pi}\left(\frac{\sin^2\theta}{s^2}\right)\left(\frac{s\,d\theta}{\sin^2\theta}\sin\theta\,\hat{\phi}\right)$$

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi}\frac{\sin\theta}{s}\,d\theta\,\hat{\phi}$$

Integration: $\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi s}\hat{\phi}\int_0^{\pi}\sin\theta\,d\theta = \boxed{\frac{\mu_0 I}{2\pi s}\hat{\phi}}$

Force between two // currents



at wire 2: $B_1 = \frac{\mu_0 I_1}{2\pi a}$

$\mathbf{I}_2 \perp \mathbf{B}_1 \qquad F_{1\to 2} = I_2 B_1 l$

attractive force per unit length:

$$\mathbf{f} = \frac{\mathbf{F}_{1\to 2}}{l} = \frac{\mu_0 I_1 I_2}{2\pi a}(-\hat{s}_2)$$

Ampere's Law

- using Stokes' theorem

$$\int_S (\nabla\times\mathbf{B})\cdot d\mathbf{a} = \oint_{\partial S}\mathbf{B}\cdot d\mathbf{l}$$

we get

$$\boxed{\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{encl}}$$

over "Amperian loop".

Symmetric case:

$$\mathbf{B}L = \mu_0 I_{enc} = \mu_0\int_{surf}\mathbf{J}\cdot d\mathbf{a}$$

Applications of Ampere's law

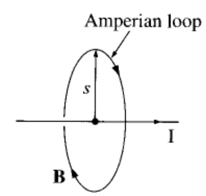
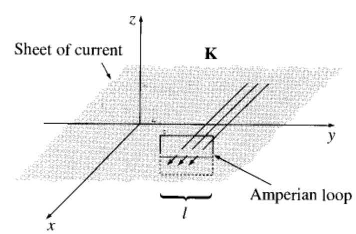


Figure 5.32

B tangent to Amperian loop

$BL = \mu_0 I \qquad \text{where} \qquad L = 2\pi s$

$$\boxed{\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}}$$

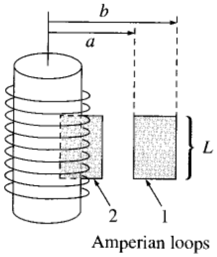


Current in *x* direction

$\mathbf{K} = K\hat{i} \quad \text{so} \quad B(2l) + 0 = \mu_0 K l$

$$\boxed{\mathbf{B} = -(\operatorname{sgn} z)\frac{\mu_0}{2}K\hat{j}}$$

- Solenoid: *n* turns/length



current *I*, $\mathbf{K} = nI\hat{\phi}$

magnetic field in *z* direction

Outer loop: $\mathbf{B}(a) = \mathbf{B}(b) = 0$

Straddling loop: $BL = \mu_0 nIL$

$$\boxed{\mathbf{B} = \mu_0 nI\hat{k} \ \text{ inside;} \quad \mathbf{B} = 0 \ \text{ outside}}$$

- Torus: Amperian loop inside the torus

N = TOTAL number of turns

$$B(2\pi s) = \mu_0 I \qquad \boxed{\mathbf{B} = \frac{\mu_0 NI}{2\pi s}\hat{\phi} \ \text{ inside;} \quad \mathbf{B} = 0 \ \text{ outside}}$$

Vector Potential **A**

- Poisson equation: $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$
- Solution:

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} d\tau'$$

- Disadvantages:
  integral diverges
  it is a vector field (not scalar)
- Advantage: $d\mathbf{A} \propto d\mathbf{J}$ in the same direction

Examples:

- Long thin wire with current *I*

$$A = \frac{\mu_0 I}{4\pi}\int_{-L}^{L}\frac{dz'}{\sqrt{z'^2 + s^2}} = \frac{\mu_0 I}{4\pi} 2\ln(z + \sqrt{z^2 + s^2})\Big|_0^L$$

$$\to \frac{\mu_0 I}{4\pi} 2\ln\left(\frac{2L}{s}\right)$$

so

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 I}{2\pi}\ln\left(\frac{s}{s_0}\right)\hat{\mathbf{k}} \quad \text{and} \quad \mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

- Example: Find the vector field corresponding to a uniform magnetic field.
- Using Stokes' theorem:

$$\oint \mathbf{A}\cdot d\mathbf{l} = \int (\nabla\times\mathbf{A})\cdot d\mathbf{a} = \int \mathbf{B}\cdot d\mathbf{a} = \text{magnetic flux}$$

- Amperian loop FOR **A** $\quad A(2\pi s) = B(\pi s^2)$

$$\oint A$$

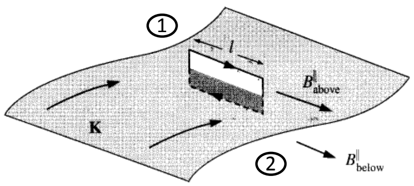
$$\mathbf{A}(\mathbf{r}) = \frac{1}{2}Bs\hat{\phi} = \frac{1}{2}B(\hat{\mathbf{k}}\times\mathbf{s}) = \boxed{\frac{1}{2}\mathbf{B}\times\mathbf{r}}$$

- For a solenoid $B = \mu_0 nI$ inside, and

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2s}B\hat{\phi}\begin{cases} s^2 \\ a^2 \end{cases} = \frac{\mu_0 nI}{2}\hat{\phi}\begin{cases} s \\ a^2/s \end{cases}$$

Boundary conditions for **B**

- Integrating Maxwell's equation for **B**



$$\nabla \mathbf{B} = i\mu_0 \mathbf{J}$$

$$\hat{\mathbf{n}}\mathbf{B}\Big|_2^1 = i\mu_0 \mathbf{K}$$

or, equivalently $\quad \mathbf{B}_1 - \mathbf{B}_2 = i\mu_0 \hat{\mathbf{n}}\mathbf{K} = \mu_0 \mathbf{K}\times\hat{\mathbf{n}}$

- $\mathbf{B}_\perp$ is CONTINUOUS

while $\quad \mathbf{B}_1^{//} - \mathbf{B}_2^{//} = \mu_0 \mathbf{K}$

Multipole expansion: magnetic moment

- Applying the expansion

$$\frac{1}{\mathscr{r}} = \sum_{l=0} \frac{1}{r}\left(\frac{r'}{r}\right)^l P_l(\cos\theta') \approx \frac{1}{r} + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{r^2} + \ldots$$

to the vector potential for a CLOSED loop:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\oint \frac{1}{\mathscr{r}} d\mathbf{l}' \approx$$

$$\approx \frac{\mu_0 I}{4\pi}\left(\frac{1}{r}\oint d\mathbf{l}' + \frac{1}{r^2}\oint (\hat{\mathbf{r}}\cdot\mathbf{r}')d\mathbf{l}' + \ldots\right)$$

- The dipole term is:

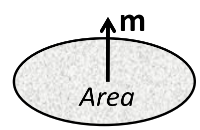
$$\mathbf{A}_{dipole}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint (\hat{\mathbf{r}}\cdot\mathbf{r}')d\mathbf{l}'$$

- Using the fundamental theorem in 2-*d*

$$i\int_S d\mathbf{a}\,\nabla\Phi = -\oint_{\partial S}\Phi\, d\mathbf{l}, \quad \Phi(\mathbf{r}) = \mathbf{C}\cdot\mathbf{r} \text{ a scalar field}$$

$$\nabla\Phi = \nabla(\hat{\mathbf{k}}\cdot\mathbf{r}) = \hat{\mathbf{k}} \quad \text{and} \quad \nabla'\Phi(\mathbf{r}') = \nabla'(\hat{\mathbf{r}}\cdot\mathbf{r}') = \hat{\mathbf{r}}$$

- Finally: $\oint (\hat{\mathbf{r}}\cdot\mathbf{r}')d\mathbf{l}' = -i\int d\mathbf{a}'\hat{\mathbf{r}} = \left(\int d\mathbf{a}'\right)\times\hat{\mathbf{r}}$



$$\boxed{\mathbf{A}_{dipole}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}}$$

where the **magnetic dipole moment** is:

$$\boxed{\mathbf{m} = I\int d\mathbf{a} = I \text{ Area}}$$

Curl: $\quad \nabla\times\left(\frac{\mathbf{m}\times\mathbf{r}}{r^3}\right) = \frac{1}{r^3}\nabla\times(\mathbf{m}\times\mathbf{r}) - \frac{3}{r^4}\hat{\mathbf{r}}\times(\mathbf{m}\times\mathbf{r})$

Magnetic field for a dipole

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]$$

compared to the electric dipole field:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]$$

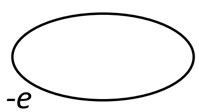
Magnetism in Matter

- Paramagnets **M // B**
- Diamagnets **M // -B**
- Ferromagnets nonlinear - permanent **M**

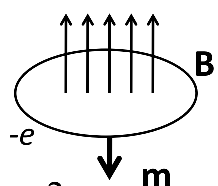
Dipoles

$\mathbf{p} = q\,\mathbf{s}$	$\mathbf{m} = I\,\mathbf{a}$
$\mathbf{F} = 0$ uniform $\mathbf{E}$	$\mathbf{F} = 0$ uniform $\mathbf{B}$
$\mathbf{N} = \mathbf{p}\times\mathbf{E}$ torque	$\mathbf{N} = \mathbf{m}\times\mathbf{B}$ torque
$W = -\mathbf{p}\cdot\mathbf{E}$	$W = -\mathbf{m}\cdot\mathbf{B}$
$\mathbf{F} = \nabla(\mathbf{p}\cdot\mathbf{E})$ point dipole	$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$ point dipole

Atomic diamagnetism



add **B** ➔



$$I = -\frac{ev}{2\pi a}$$

$$\mathbf{m} = \frac{-ev}{2\pi a}\pi a^2\hat{\mathbf{k}}$$

$$F_{cent} = \frac{m_0 v^2}{a}$$

$$\Delta F = \Delta\frac{m_0 v^2}{a} \approx \frac{2m_0}{a} v\,\Delta v = evB$$

$$\Delta v = \frac{eBa}{2m_0}$$

$$\Delta\mathbf{m} = -\frac{e}{2}a\hat{\mathbf{k}}\,\Delta v = -\frac{e^2a^2}{4m_0}\mathbf{B}$$

Energy: $$W = -\mathbf{m}\cdot\mathbf{B} = \frac{eva}{2}\hat{\mathbf{m}}\cdot\mathbf{B} = +\frac{e}{2m_0}\mathbf{L}\cdot\mathbf{B}$$

Magnetization

- Magnetization = (dipole moment)/volume
- Vector potential: $d\mathbf{A}' = \frac{\mu_0}{4\pi}\frac{d\mathbf{m}(\mathbf{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\mathbf{M}(\mathbf{r}')\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau' = \frac{\mu_0}{4\pi}\int\mathbf{M}(\mathbf{r}')\times\left(\nabla'\frac{1}{\mathscr{r}}\right)d\tau'$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\nabla'\times\mathbf{M}(\mathbf{r}')}{\mathscr{r}}d\tau' + \frac{\mu_0}{4\pi}\oint\frac{\mathbf{M}(\mathbf{r}')\times d\mathbf{a}'}{\mathscr{r}} =$$

$$= \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}d\tau' + \frac{\mu_0}{4\pi}\oint\frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}da'$$

Sphere - constant magnetization

- sphere radius R - uniform **M**

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{M}\times\int\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau' = \frac{\mu_0}{3}\mathbf{M}\times\begin{cases}\mathbf{r} & \text{inside}\\ \dfrac{R^3}{r^2}\hat{\mathbf{r}} & \text{outside}\end{cases}$$

$$\mathbf{B} = \nabla\times\mathbf{A} = \begin{cases}\dfrac{2}{3}\mu_0\mathbf{M} & \text{inside}\\ \text{(dipole @ center)} & \text{outside}\end{cases}$$

with magnetic moment $\mathbf{m} = \frac{4}{3}\pi R^3\mathbf{M}$

- Bound current densities:

$$\mathbf{J}_b(\mathbf{r}) = \nabla\times\mathbf{M}(\mathbf{r}) \quad \text{and} \quad \mathbf{K}_b(\mathbf{r}) = \mathbf{M}(\mathbf{r})\times\hat{\mathbf{n}}$$

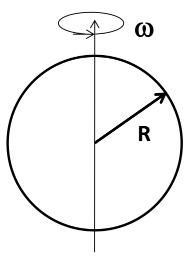
- Auxiliary field **H**

$$\frac{1}{\mu_0}\nabla\times\mathbf{B} = \mathbf{J} = \mathbf{J}_f + \mathbf{J}_b = \mathbf{J}_f + \nabla\times\mathbf{M}$$

- Defining

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \quad \text{we get} \quad \boxed{\nabla\times\mathbf{H} = \mathbf{J}_f}$$

- An equivalent problem: sphere with uniform surface charge σ spinning with constant angular velocity $\boldsymbol{\omega}$.



Tangential velocity & surface current

$$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{s} = \boldsymbol{\omega}\times\mathbf{R}$$

$$\mathbf{K} = \sigma\,\mathbf{v} = \sigma\,\boldsymbol{\omega}\times\mathbf{R}$$

equivalent to uniform **M** w/bound $\mathbf{K}_b$

$$\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{r}}, \quad \text{so} \quad M \Leftrightarrow \sigma\omega R$$

$\mathbf{B} = \frac{2}{3}\mu_0\sigma R\boldsymbol{\omega}$ inside spinning sphere

- Ampere's law for magnetic materials

$$\boxed{\oint\mathbf{H}\cdot d\mathbf{l} = I_{f\,encl}}$$

- Experimentally: *I* & *V* ➔ **H** & **E**

Example: copper rod of radius R carrying a uniform free current I

Amperian loop radius s:
$$H(2\pi s) = I\begin{cases}\pi s^2/\pi R^2 & \text{inside}\\ 1 & \text{outside}\end{cases}$$

outside **M** = 0 and $\mathbf{B} = \mu_0\mathbf{H} = \frac{\mu_0 I}{2\pi s}\hat{\phi} \quad (s \ge R)$

Boundary conditions for **B** & **H**

- Integrating Maxwell's equations:

$$\nabla\times\mathbf{H}=\mathbf{J}_f \qquad \nabla\cdot\mathbf{B}=0$$

$$\hat{\mathbf{n}}\times\mathbf{H}\Big|_2^1=\mathbf{K}_f \qquad \hat{\mathbf{n}}\cdot\mathbf{B}\Big|_2^1=0$$

or, equivalently

$$\mathbf{H}_1^{//}-\mathbf{H}_2^{//}=\mathbf{K}_f\times\hat{\mathbf{n}} \qquad \text{and} \qquad \mathbf{B}_1^{\perp}=\mathbf{B}_2^{\perp}$$

Linear Magnetic Materials

- Susceptibility and permeability:

$$\mathbf{M}=\chi_m\mathbf{H} \qquad \text{and} \qquad \mathbf{B}=\mu_0(\mathbf{H}+\mathbf{M})=\mu\mathbf{H}$$

$$\mu=\mu_0\left(1+\chi_m\right) \qquad \text{magnetic permeability}$$

$$\mathbf{B}=\mu\mathbf{H}$$

- diamagnetic ~ -10^{-5}
  paramagnetic ~ $+10^{-5}$
  Gadolinium ~ +0.5

- **Theorem**:

  In a **linear** homogeneous magnetic material (constant susceptibility) the volume *bound* current density is **proportional** to the *free* current density :

$$\mathbf{J}_b=\nabla\times\mathbf{M}=\nabla\times(\chi_m\mathbf{H})=\chi_m\mathbf{J}_f$$

  Bound surface current:

$$\mathbf{K}_b=\mathbf{M}\times\hat{\mathbf{n}}=\chi_m\mathbf{H}\times\hat{\mathbf{n}}$$

  example: solenoid $\mathbf{K}_b=\chi_m nI\hat{\phi}$

Ferromagnetism

- Permanent magnetization (**no** external field necessary)
- non-linear relation between *M* and *I*
- hysteresis loop
- dipole orientation in DOMAINS
- magnetic domain walls disappear with increasing magnetization
- phase transition (Curie point): iron *T* ~ 770 C