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Physics 441: Assignment #6 - Magnetic Fields in Matter

Due on Wednesday, June 19, 2013

June 19, 2013

Problem 6.3

Find the force of attraction between two magnetic dipoles, $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ oriented as shown in figure 6.7, a distance r apart:

1. Using equation 6.2
2. Using equation 6.3

1. Equation 6.2 says

$$F = 2\pi IRB\cos\theta$$

To evaluate this I need an expression for $B\cos\theta = \boldsymbol{B}\cdot\hat{\boldsymbol{y}}$. I did problem 5.34 and showed that

$$\boldsymbol{B}_{\text{dipole}} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\boldsymbol{m}_1\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{m}_1\right]$$

Applying that I can get the following expression:

$$\begin{aligned}\boldsymbol{B}\cdot\hat{\boldsymbol{y}} &= \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\boldsymbol{m}_1\cdot\hat{\boldsymbol{r}})(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{y}}) - (\boldsymbol{m}_1\cdot\hat{\boldsymbol{y}}\right]\\ &= \frac{\mu_0}{4\pi}\frac{1}{r^3}3m_1\sin\phi\cos\phi\end{aligned}$$

The above is true because $m_1\cdot\hat{\boldsymbol{y}} = 0$, $m_1\hat{\boldsymbol{r}} = m1cos\phi$, and $\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{y}} = \sin\phi$. I now plug this in to equaiton 6.2 to get

$$F = 2\pi IR\frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3m_1\sin\phi\cos\phi\right]$$

I can simplify the trig functions here and get a final answer (note that I apply the identity that $m_2 = IR^2\pi$ and realize that $r >> R$ to simplify a square root).

$$\begin{aligned}
F &= 2\pi IR\frac{\mu_0}{4\pi}\frac{1}{r^3}3m_1 \sin\phi\cos\phi \\
&= 2\pi IR\frac{\mu_0}{4\pi}\frac{1}{r^3}3m_1 \left(\frac{R}{r}\right)\phi\cos\phi \\
&= 2\pi IR\frac{\mu_0}{4\pi}\frac{1}{r^3}3m_1 \left(\frac{R}{r}\right)\left(\frac{\sqrt{r^2-R^2}}{r}\right) \\
&= 2\pi IR^2\frac{\mu_0}{4\pi}\frac{1}{r^5}3m_1 \sqrt{r^2-R^2} \\
&= m_2\frac{\mu_0}{2\pi}\frac{1}{r^5}3m_1 \sqrt{r^2-R^2} \\
&= \frac{\mu_0}{2\pi}\frac{1}{r^4}3m_1 m_2
\end{aligned}$$

2. Now I will use equation 6.3:

$$\begin{aligned}
\boldsymbol{F} &= \nabla(\boldsymbol{m}\cdot\boldsymbol{B}) \\
&= (m_2\cdot\nabla)\boldsymbol{B} \\
&= \left(m_2\frac{\partial}{\partial z}\right)\frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\boldsymbol{m}_1\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}}-\boldsymbol{m}_1\right] \\
&= \left(m_2\frac{\partial}{\partial z}\right)\frac{\mu_0}{4\pi}\frac{1}{z^3}\left[3(\boldsymbol{m}_1\cdot\hat{\boldsymbol{z}})\hat{\boldsymbol{z}}-\boldsymbol{m}_1\right] \\
&= \left(m_2\frac{\partial}{\partial z}\right)\frac{\mu_0}{4\pi}\frac{1}{z^3}\left[2\boldsymbol{m}_1\right] \\
&= m_2\frac{\partial}{\partial z}\frac{1}{z^3}\frac{\mu_0}{4\pi}\left[2\boldsymbol{m}_1\right] \\
&= \frac{\partial}{\partial z}\frac{1}{z^3}\frac{\mu_0}{4\pi}\left[2\boldsymbol{m}_1 m_2\right] \\
&= \left(\frac{-3}{z^4}\right)\frac{\mu_0}{2\pi}m_1 m_2
\end{aligned}$$

Those are the same, so I am done. □

Problem 6.6

Of the following materials which would you expect to be paramagnetic and which diamagnetic:

- aluminium
- copper
- copper chloride ($CuCl_2$)
- carbon
- lead
- nitrogen(N_2)

Molecule	# of electrons	Magnetism
Al	13	paramagnetic
Cu	The book gave the answer	diamagnetic
$CuCl_2$	29 + (17 * 2) = 63	paramagnetic
C	6	diamagnetic
Pb	82	diamagnetic
N_2	14	diamagnetic
NaCl	11 + 17 = 28	diamagnetic
S	16	diamagnetic
H_2O	(1 *2) + 8 = 10	diamagnetic

Table 1: Table describing magnetism of different molecules

- salt (NaCl)
- sulfur
- water

The key to this problem is determining if each of the molecules listed has an even of odd number of electrons. If there is an even number I expect the molecule to be diagmagnetic, if there is an odd number I would expect paramagnetism. See Table 1 for the answer.

□

Problem 6.12

An infinitely long cylinder, of radius R, carries a "frozen-in" magnetization, parallel to the axis,

$$\boldsymbol{M} = kx\hat{\boldsymbol{z}}$$

where k is a constant and s is the distance from the axis; there is no free current anywhere. FInd the magnetic field inside and outside the cylinder by two different methods:

1. As in Section 6.2, locate all the bound currents, and calculate the field they produce
2. Use Ampere's law (in the form of equation 6.20) to find $\boldsymbol{H}$, and then get $\boldsymbol{B}$ from equation 6.18 (Notice that the second method is much faster, and avoids any explicit reference to the bound currents.)

1. To do this I will need to apply equations 6.13 and 6.14 to get expressions for $\boldsymbol{J}_b$ and $\boldsymbol{K}_b$, respectively.

$$\boldsymbol{J}_b = \nabla \times \boldsymbol{M} = -k\hat{\boldsymbol{\phi}}$$
$$\boldsymbol{K}_b = \boldsymbol{M} \times \hat{\boldsymbol{n}} = kR\hat{\boldsymbol{\phi}}$$

It is easy to show that $B = 0$ outside the surface. All you need to know is that B is in the $\hat{\boldsymbol{z}}$ direction. To find the value of B inside the surface I will use equation 5.44:

$$\oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I_{\text{enc}}$$

I do this below

$$\begin{aligned}
\oint \boldsymbol{B} \cdot d\boldsymbol{l} = Bl &= \mu_0 I_{\text{enc}} \\
&= \mu_0 \left[\int \boldsymbol{J}_b da + \boldsymbol{K}_b l \right] \\
&= \mu_0 \left[\int (-k\hat{\boldsymbol{\phi}}) da + (kR\hat{\boldsymbol{\phi}}) l \right] \\
&= \mu_0 kls \\
B &= \mu_0 ks\hat{\boldsymbol{z}}
\end{aligned}$$

2. Now I will do it the easy way using Ampere's law. I can say that $\boldsymbol{H}$ points in the $\hat{\boldsymbol{z}}$ direction. I will use the equation 6.20 and integrate over the same loop I just used before:

$$\begin{aligned}
\oint \boldsymbol{H} \cdot d\boldsymbol{l} &= I_{f_{\text{enc}}} \\
Hl &= 0 \\
\boldsymbol{H} &= 0
\end{aligned}$$

I can then use equation 6.18 ($\boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}$) to say that

$$B = \mu_0 \boldsymbol{M}$$

. I can use the same arguments as before to say that outside the surface I have $\boldsymbol{M} = 0$ and inside $\boldsymbol{M} = kx\hat{\boldsymbol{z}}$ so I get the final answer that

$$\begin{cases} B = 0 & \text{outside} \\ B = \mu_0 kx\hat{\boldsymbol{z}} & \text{inside} \end{cases}$$

□

Problem 6.23

A familiar toy consists of donut-shaped permanent magnets (magnetization parallel to the axis), which slide frictionlessly on a vertical rod (Figure 6.31). Treat the magnets as dipoles, which mass m_d and dipole moment $\boldsymbol{m}$.

1. If you put two back-to-back magnets on teh rod, the upper one will :float: – the magnetic force upward balancing the gravitational force downward. At what height (z) does it float?
2. If you now add a third magnet (parallel to the bottom one), what is the ratio of the two heights? (Determine the actual number to 3 significant digits)

1. I start this one using our friend in equation 5.88: the expression for the magnetic field of a dipole. I do this with $\theta = 0$ to get
$$\boldsymbol{B}_1 = \frac{\mu_0}{4\pi}\frac{2m}{z^3}\hat{\boldsymbol{z}}$$
. I now compute this in the direction of $\boldsymbol{m}_2$:
$$\boldsymbol{m}_2 \cdot \boldsymbol{b}_1 = -\frac{\mu_0}{2\pi}\frac{m^2}{z^3}$$
Now I need to use equation 6.3 to get an expression for $\boldsymbol{F}$:
$$\begin{aligned} \boldsymbol{F} &= \nabla(\boldsymbol{m} \cdot \boldsymbol{B}) \\ &= \nabla(\boldsymbol{m}_2 \cdot \boldsymbol{B}_1) \\ &= \frac{\partial}{\partial z}\left[-\frac{\mu_0}{2\pi}\frac{m^2}{z^3}\right]\hat{\boldsymbol{z}} \\ &= \frac{3\mu_0}{2\pi}\frac{m^2}{z^4}\hat{\boldsymbol{z}} \end{aligned}$$
I now need this to balance the gravitational force down (mg) and solve for z:
$$\begin{aligned} \boldsymbol{F} &= \boldsymbol{G} \\ \frac{3\mu_0}{2\pi}\frac{m^2}{z^4}\hat{\boldsymbol{z}} &= m_d g - \hat{\boldsymbol{z}} \\ z &= \left(\frac{3\mu_0 m^2}{2\pi m_d g}\right)^{1/4} \end{aligned}$$

2. I can use the same expression for the force that I just derived. The only exception is that I need to replace z with z_t and z_m for the distance between middle and top and the distance between middle and bottom, respectively. Doing this I can get expressions for the net force acting in the z direction on the middle and top magnets:
$$\begin{cases} F_{\text{net}} = 0 = \frac{3\mu_0}{2\pi}\frac{m^2}{z_t^4}\hat{\boldsymbol{z}} - \frac{3\mu_0}{2\pi}\frac{m^2}{z_m^4}\hat{\boldsymbol{z}} - m_d g\hat{\boldsymbol{z}} & \text{middle magnet} \\ F_{\text{net}} = 0 = \frac{3\mu_0}{2\pi}\frac{m^2}{z_m^4}\hat{\boldsymbol{z}} - \frac{3\mu_0}{2\pi}\frac{m^2}{(z_m+z_t)^4}\hat{\boldsymbol{z}} - m_d g\hat{\boldsymbol{z}} & \text{top magnet} \end{cases}$$
I can subtract these two expressions and simplify to get that $\frac{1}{z_t^4} - \frac{2}{z_m^4} + \frac{1}{(z_t+z_m)^4} = 0$. I use this to get the expression $\frac{1}{(z_t/z_m)^4} + \frac{1}{(z_t/z_m+1)^4} = 2$. I let the computer solve this for me and I got an answer of $z_t/z_m = 0.8501$

□

Problem 6.25

Notice the following parallel:

$$\begin{cases} \nabla \cdot \boldsymbol{D} = 0, \quad \nabla \times \boldsymbol{E} = 0, \quad \epsilon_0 \boldsymbol{E} = \boldsymbol{D} - \boldsymbol{P} & \text{(no free charge)} \\ \nabla \cdot \boldsymbol{B} = 0, \quad \nabla \times \boldsymbol{H} = 0, \quad \mu_0 \boldsymbol{H} = \boldsymbol{B} - \mu_0 \boldsymbol{M} & \text{(no free charge)} \end{cases}$$

Thus, the transcription $\boldsymbol{D} \to \boldsymbol{B}, \boldsymbol{E} \to \boldsymbol{H}, \boldsymbol{P} \to \mu_0 \boldsymbol{M}, \epsilon_0 \to -\mu_0$ turn an electrostatic problem into an analogous magnetostatic one. Use this, together with your knowledge of the electro-static results to re-derive:

1. The magnetic field inside a uniformly magnetized sphere
2. The magnetic field inside a sphere of linear magnetic material in an otherwise uniform magnetic field (problem 6.18)
3. The average magnetic field over a sphere, due to steady currents within the sphere (equation 5.93)

1. I will use equation 4.14 ($\boldsymbol{E} = -\frac{1}{3\epsilon_0}\boldsymbol{P}$) to say that

$$\boldsymbol{H} = -\frac{1}{3\mu_0}(\mu_0 M) = -\frac{1}{3}M$$

   I then use equation 6.18 ($\boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}$) to say that

$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}) = \mu_0(-\frac{1}{3}\boldsymbol{M} + \boldsymbol{M}) = \frac{2}{3}\mu_0 \boldsymbol{M}$$

2. I will use equation 4.49 $\left(\boldsymbol{E} = \frac{3}{\epsilon_r + 2}\boldsymbol{E}_0 \to \boldsymbol{E} = \frac{1}{1+\chi_e/3}\boldsymbol{E}_0\right)$ to say that

$$\boldsymbol{H} = \frac{1}{1+\chi_m/3}\boldsymbol{H}_0$$

   I then use equation 6.30 $\left(\boldsymbol{B} = \mu_0(1+\chi_m)\boldsymbol{H}\right)$ and equation 6.31 $\left(\boldsymbol{B}_0 = \mu_0 \boldsymbol{H}_0\right)$ to set up the following expression

$$\frac{\boldsymbol{B}}{\mu_0(1+\chi_m)} = \frac{1}{(1+\chi_m/3)}\frac{\boldsymbol{B}_0}{\mu_0} \to \boldsymbol{B} = \left(\frac{1+\chi_m}{1+\chi_m/3}\right)\boldsymbol{B}_0$$

   .

3. I begin this part with the average electric field over a sphere: $\boldsymbol{E}_{\text{ave}} = -\frac{1}{4\pi\epsilon_0}\frac{P}{R^3}$ I will use this in addition to equation 4.39 ($-\rho = \nabla \cdot \boldsymbol{P}$). I can use this as well as an understanding that there are no free charges to re-write the expression for the average electric field:

$$\boldsymbol{E}_{\text{ave}} = -\frac{1}{4\pi\epsilon_0}\frac{1}{R^3}\int \boldsymbol{P} d\tau$$

   I am now in a position to make the substitutions indicated in the problem description to obtain the following:

$$\boldsymbol{H}_{\text{ave}} = \frac{1}{4\pi\mu_0}\frac{1}{R^3}\int \mu_0 \boldsymbol{M} d\tau = -\frac{1}{4\pi R^3}\boldsymbol{m}$$

   I again return to using equation 6.18 (given above) to say that

$$
\begin{aligned}
\boldsymbol{B}_{\text{ave}} &= -\frac{\mu_0 m}{4\pi R^3} + \mu_0 \boldsymbol{M}_{\text{ave}} \\
&= -\frac{\mu_0 m}{4\pi R^3} + \mu_0 \frac{m}{4/3\pi R^3} \\
&= \frac{2\mu_0 m}{4\pi R^3}
\end{aligned}
$$

This is the same as equation 5.93, so I am done.

□