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Physics 441: Assignment #5 - Electric Fields in Matter

Due on Monday, June 10, 2013

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Problem 4.2

According to quantum mechanics, the electron could for a hydrogen atom in the ground state has a charge density

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a},$$

where q is the charge of the electron and a is the Bohr radius. Find the atomic polarizability of such an atom [Hint: First calculate the electric field for the electron could, $E_e(r)$; then expand the exponential assuming that $r >> a$]

I will use Gauss' Law to find an expression for E. Recall that Gauss' Law is $\int \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{Q}{/ve_0}$. We need to find Q, which we can do by integrating the expression for charge density.

$$\begin{aligned} Q &= \int_0^r \rho d\tau \\ &= \frac{4\pi q}{\pi a^3} \int_0^r e^{-2r/a} r^2 dr \\ &= -\frac{q\left(-a^2 e^{2\frac{r}{a}} + a^2 + 2ar + 2r^2\right) e^{-2\frac{r}{a}}}{a^2} \end{aligned}$$

Now that we have Q we just need to find E from Gauss' law.

$$
\begin{aligned}
E &= \frac{1}{4\pi\varepsilon_0 r^2} Q \\
&= -\frac{q\left(-a^2 e^{2\frac{r}{a}} + a^2 + 2ar + 2r^2\right) e^{-2\frac{r}{a}}}{4\pi a^2 \epsilon_0 r^2} \\
&= \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \left[1 - e^{-2r/a}\left(1 + 2\frac{r}{a} + 2\left(\frac{d}{a}\right)^2\right)\right]
\end{aligned}
$$

We now need to expand the exponential term in E. I do this below

$$
e^{-2r/a} = -\frac{4}{3}\frac{r^3}{a^3} + 2\frac{r^2}{a^2} - 2\frac{r}{a} + 1 + \mathcal{O}\left(\frac{r^4}{a^4}\right)
$$

If we plug this into the solution for E, we get the following:

$$
\begin{aligned}
E &= \frac{q}{4\pi\varepsilon_0 r^2}\left[1 - e^{-2r/a}\left(1 + 2\frac{r}{a} + 2\left(\frac{r}{a}\right)^2\right)\right] \\
&= \frac{q}{4\pi\varepsilon_0 r^2}\left[1 - 1 - 2\frac{r}{a} - 2\frac{r^2}{a^2} + 2\frac{r}{a} + 4\frac{r^2}{a^2} + 4\frac{r^3}{a^3} - 2\frac{r^2}{a^2} - 4\frac{r^3}{a^3} - \frac{4}{3}\frac{r^3}{a^3}\right] \\
&= \frac{q}{4\pi\varepsilon_0 r^2}\left[\frac{4}{3}\frac{r^3}{a^3}\right] \\
&= \frac{1}{3\pi\varepsilon_0 a^3} qr \\
&= \alpha p
\end{aligned}
$$

where $\alpha = 3\pi\varepsilon_0 a^3$. □

Problem 4.5

In Figure 4.6, $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are (perfect) dipoles at a distance r apart. What is the torque on $\boldsymbol{p}_1$ due to $\boldsymbol{p}_2$? What is the torque on $\boldsymbol{p}_2$ due to $\boldsymbol{p}_1$? [In each case, I want the torque on the dipole about its own center]

For this problem we will use equation 3.103: $\boldsymbol{E}_{\text{dipole}} = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}})$ and equation 4.4: $\boldsymbol{N} = \boldsymbol{p} \times \boldsymbol{E}$
We can find the torque of $\boldsymbol{p}_1$ on $\boldsymbol{p}_2$ by finding $\boldsymbol{E}_1$, which is what we get when $\theta = \pi/2$ in equation 3.103 and plugging the result into equation 4.4

$$
\begin{aligned}
\boldsymbol{E}_1 &= \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}) \\
&= \frac{p_1}{4\pi\varepsilon_0 r^3}(2\cos\pi/2\hat{\boldsymbol{r}} + \sin\pi/2\hat{\boldsymbol{\theta}}) \\
&= \frac{p_1}{4\pi\varepsilon_0 r^3}\hat{\boldsymbol{\theta}} \\
\boldsymbol{N}_2 &= \boldsymbol{p}_2 \times \boldsymbol{E}_1 \\
&= p_2 E_1 \\
&= \frac{p_1 p_2}{4\pi\varepsilon_0 r^3}
\end{aligned}
$$

We now repeat the analysis above using $\theta = \pi$ for $\boldsymbol{p}_2$:

$$
\begin{aligned}
\boldsymbol{E}_2 &= \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}) \\
&= \frac{p_2}{4\pi\varepsilon_0 r^3}(2\cos\pi\hat{\boldsymbol{r}} + \sin\pi\hat{\boldsymbol{\theta}}) \\
&= \frac{p_2}{4\pi\varepsilon_0 r^3} - 2\hat{\boldsymbol{r}} \\
\boldsymbol{N}_1 &= \boldsymbol{p}_1 \times \boldsymbol{E}_2 \\
&= p_1 E_2 \\
&= \frac{2p_1 p_2}{4\pi\varepsilon_0 r^3}
\end{aligned}
$$

□

Problem 4.10

A sphere of radius R carries a polarization

$$\boldsymbol{P}(\boldsymbol{r}) = k\boldsymbol{r}$$

where k is a constant and $\boldsymbol{r}$ is the vector from the center.

1. Calculate the bound of charges σ_b and ρ_b
2. Find the field inside and outside the sphere

1.
   - σ_b is found using equation 4.11: $\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}} = k\hat{r} \cdot \hat{\boldsymbol{n}} = kR$
   - ρ_b is found using equation 4.12 (Note I use the expression for the gradient in spherical coordinates as found in the front cover of the book): $\rho_b = -\nabla \cdot \boldsymbol{P} = -\nabla \cdot k\boldsymbol{r} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 kr\right) = -\frac{1}{r^2}3kr^2 = -3k$
2.
   - For inside the sphere ($r < R$) we will use Gauss' law to find an expression for E in terms of ρ.

$$
\begin{aligned}
\oint \boldsymbol{E} \cdot d\boldsymbol{a} = Er\pi r^2 &= \frac{1}{\varepsilon_0}Q = \frac{1}{\varepsilon_0}\frac{4}{3}\pi r^3 \rho \\
\boldsymbol{E} &= \frac{1}{3\varepsilon_0}\rho r\hat{\boldsymbol{r}}
\end{aligned}
$$

   We simply plug our ρ in to get:

$$\boldsymbol{E} = \frac{1}{3\varepsilon_0} - 3kr\boldsymbol{r} = -(kr/\varepsilon_0)\hat{\boldsymbol{r}}$$

   - Outside the sphere ($r > R$) we can treat it as if all the charge were at the center. This makes $Q = (kR)(4\pi R^2) + (-3k)(\frac{4}{3}\pi R^3 = 0$ so $\boldsymbol{E} = 0$. Gauss' law can help is verify this intuitively.

□

Problem 4.15

A thick spherical shell (inner radius a, outer radius b) is made of dielectric material with "frozen-in" polarization

$$\boldsymbol{P} = \frac{k}{r}\hat{\boldsymbol{r}}$$

Problem 4.15 continued on next page...

where k is a constant and r is the distance from the center. (There is no free charge in this problem.) Find the electric field in all three regions by two different methods:

1. Locate all the bound charge, and use Gauss' law (Equation 2.13) to calculate the field it produces
2. Use equation 4.23 to find $\boldsymbol{D}$, and then get $\boldsymbol{E}$ from equation 4.21. [Notice that the second method is much faster and avoids any reference to bound charges.]

1. We start by finding σ_b and ρ_b like we did in the previous problem. $\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}} = \begin{cases} \boldsymbol{P} \cdot \hat{\boldsymbol{r}} = k/b & \text{r=b} \\ -\boldsymbol{P} \cdot \hat{\boldsymbol{r}} = -k/a & \text{r=a} \end{cases}$
   and $\rho_b = -\nabla \cdot \boldsymbol{P} = -\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 (k/r) \right) = -\frac{k}{r^2}$.
   We will now apply Gauss' law ($\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2}$) to three different regions
   (a) $r < a$. Here $Q = 0$ so $\boldsymbol{E} = 0$
   (b) $a < r < b$: Here we need to calculate $Q = \sigma_b A + \int \rho_b dv$:
   $$Q = \left(\frac{-k}{a} \right) (4\pi a^2) + \int_a^r \left(\frac{-k}{r^2} \right) 4\pi r^2 dr = -4\pi k a - 4\pi k (r-1) = -4\pi k r$$
   We plug this in to get that $\boldsymbol{E} = -(k/\varepsilon_0 r)\hat{\boldsymbol{r}}$
   (c) $r > b$: Here $Q = 0$ so $\boldsymbol{E} = 0$
2. Equation 4.23 says that $\oint D \cdot d\boldsymbol{a} = Q_{f_{enc}}$. In our case there are not free charges so $Q_{f_{enc}} = 0 \rightarrow \boldsymbol{D} = 0$. We now say that
   $$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} \rightarrow \boldsymbol{E} = (-1/\varepsilon_0)\boldsymbol{P}$$
   We get the same answer as before because inside a and outside b, $\boldsymbol{P} = 0$ and plugging $\boldsymbol{P}$ into our expression above yields:
   $$\boldsymbol{E} = (-1/\varepsilon_0) k/r \hat{\boldsymbol{r}} = -(k/\varepsilon_0 r)\hat{\boldsymbol{r}}$$

Problem 4.19

Suppose you have enough linear dielectric material, of dielectric constant ε_r, to half-fill a parallel-plat capacitor. By what fraction is the capacitance increased when you distribute the material as shown in figure 4.25(a)? How about figure 4.25(b)? For a given potential difference V between the plates, find $\boldsymbol{E}$, $\boldsymbol{D}$, and $\boldsymbol{P}$ in each region and the free and bound charge on all surfaces, for both cases.

Problem 4.26

A spherical conductor of radius a, carries a charge Q. It is surrounded by linear dielectric material of susceptibility χ_e, out to radius b. FInd the energy of this configuration.

Problem 4.30

An electric diopoly $\boldsymbol{p}$, pointing in the y direction, is placed midway between two large conducting plates, as shown in figure 4.33. Each plate makes a small angle θ with respect to the x axis, and they are maintained at potentials $\pm V$. What is the direction of the net force on $\boldsymbol{p}$? (There's nothing to calculate, but explain your answer qualitatively.)

Problem 4.36

At the interface between one linear dielectric and another, the electric field lines bend(see figure 4.34). Show that

$$\tan_{\theta_2} / \tan_{\theta_1} = \varepsilon_2 \varepsilon_1$$

assuming there is no free charge at the boundary.