Asset Allocation Models

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v2.56, 18 February 2017

Modern Portfolio Theory

Introduction, Background, Focus, and Motivation

1. Scope of Modern Portfolio Theory: Modern Portfolio (MPT), or Mean Variance Analysis, is a mathematical framework for assembling assets such that the expected return is maximized for a given level of risk, defined as variance (Modern Portfolio Theory (Wiki)).
2. Asset Risk and Return Assessment: MPT's key insight is that an asset's risk and return should not be assessed by itself, but by how it contributes to the portfolio's overall risk and return (Markowitz (1952)).

Mathematical Model

1. Investor Risk and Return Preferences: MPT assumes that investors are risk averse, meaning that, given two portfolios that offer the same expected returns, investors will prefer the less risky one. Thus an investor will take on increased risks only if compensated with higher expected returns. Conversely, an investor who wants higher expected returns must accept more risk.
2. Investor Risky Portfolio Choice Assessment: The exact trade-off curve will be the same for all the investors, but different investors will evaluate the trade-off differently based on the individual risk aversion characteristics. The implication is that a rational investor will not invest in a portfolio if a second portfolio exists with a more favorable risk-to-expected return

profile – i.e., if for that level of risk an alternate portfolio exists that has better expected returns.

3. <u>Portfolio Properties Used by MPT</u>: Under the MPT, the portfolio return is the proportion-weighted combination of the constituent assets' returns. Portfolio volatility is a function of the correlations ρ_{ij} of the component assets for all asset pairs (i, j).
4. <u>Multi Asset Portfolio Expected Return</u>: The expected portfolio return is

$$\mathbb{E}[R_p] = \sum_i w_i \mathbb{E}[R_i]$$

where R_p is the return on the portfolio, R_i is the return on the asset i, and w_i is the weight of the asset i. The portfolio variance is

$$\sigma_p{}^2 = \sum_i w_i{}^2 \sigma_i{}^2 + \sum_i \sum_{j \neq i} w_i w_j \sigma_i \sigma_j \rho_{ij}$$

where ρ_{ij} is the correlation coefficient between the returns on assets i and j. Alternatively the above expression can be re-cast as

$$\sigma_p{}^2 = \sum_i \sum_j w_i w_j \sigma_i \sigma_j \rho_{ij}$$

where

$$\rho_{ij} = 1$$

for

$$i = j$$

The corresponding portfolio return volatility (standard deviation) is

$$\sigma_p = \sqrt{{\sigma_p}^2}$$

5. Two Asset Portfolio Returns/Variance: Portfolio Returns is

$$\mathbb{E}[R_p] = w_A\mathbb{E}[R_A] + w_B\mathbb{E}[R_B] = w_A\mathbb{E}[R_A] + (1 - w_A)\mathbb{E}[R_B]$$

The Portfolio Variance is

$${\sigma_p}^2 = {w_A}^2{\sigma_A}^2 + {w_B}^2{\sigma_B}^2 + 2w_Aw_B\rho_{AB}\sigma_A\sigma_B$$

6. Three Asset Portfolio Returns/Variance: Portfolio Returns is

$$\mathbb{E}[R_p] = w_A\mathbb{E}[R_A] + w_B\mathbb{E}[R_B] + w_C\mathbb{E}[R_C]$$

The Portfolio Variance is

$$\begin{aligned}{\sigma_p}^2 = {w_A}^2{\sigma_A}^2 + {w_B}^2{\sigma_B}^2 + {w_C}^2{\sigma_C}^2 + 2w_Aw_B\rho_{AB}\sigma_A\sigma_B + 2w_Bw_C\rho_{BC}\sigma_B\sigma_C \\ + 2w_Cw_A\rho_{CA}\sigma_C\sigma_A\end{aligned}$$

7. Portfolio Diversification - Inversely Correlated Assets: An investor can reduce the portfolio risk simply by holding combinations of instruments that are not perfectly positively correlated, i.e., correlation coefficient

$$-1 \leq \rho_{ij} < 1$$

In other words, investors can reduce their exposure to individual asset risk by holding a diversified portfolio of assets. Diversification may allow for the same portfolio expected return with reduced risk.

8. Special Case - Completely Uncorrelated Assets: If all asset pairs have correlations of 0, i.e., they are perfectly uncorrelated, the portfolio's return variance is the sum over all assets of the square of the fraction held in the asset times the asset's return variance.
9. Frontier Without Risk-Free Asset: Every possible combination of the risk assets, without including any holdings of the risk-free assets, can be plotted in the risk-expected returns space, and a collection of all such possible portfolios defines a region in this space. The left boundary of this region is a hyperbola (Merton (1972)), and the upper edge of this region is the efficient frontier in the absence of a risk-free asset (sometimes called that Markowitz bullet).
10. Portfolio at the Efficient Frontier: Combinations along this upper edge represent portfolios (including no holdings of the risk-free asset) for which there is lowest risk for a given level of expected return. Equivalently, a portfolio lying on the efficient frontier represents the combination offering the best possible expected return for a given risk level. The tangent to the hyperbola at the tangency point indicates the best possible Capital Allocation Line (CAL).
11. Determination of the Efficient Frontier: For a given risk tolerance

$$q \in [0, \infty)$$

the efficient frontier is found by minimizing $w^T \Xi w - q R^T w$ where w is the vector of portfolio weights, and

$$\sum_i w_i = 1$$

(the weights can be negative, which means that the investors can short that security), Ξ is the covariance matrix for the returns on the assets in the portfolio, R is the vector of expected returns, $w^T \Xi w$ is the variance of the portfolio returns, and $R^T w$ is the expected return on the portfolio.
12. The Risk Tolerance Parameter Interpretation:

$$q \geq 0$$

is a "risk tolerance" factor, where 0 results in a portfolio with minimum risk, and ∞ results in the portfolio being infinitely far out in the frontier with both the expected return and the risk unbounded. The above optimization finds the point on the frontier at which the inverse of the slope of the frontier would be q if the portfolio return variance instead of the standard deviation were plotted horizontally. The frontier in its entirety is parametric on q. Many software packages, including MATLAB, Microsoft Excel, Mathematica, and R provide optimization routines suitable for the above problem.

13. <u>Alternate Efficient Frontier Approach Specification</u>: An alternate approach to specifying the efficient frontier is to do so parametrically on the expected portfolio return $R^T w$. This version of the problem requires that we minimize $w^T \Xi w$ subject to

$$R^T w = \mu$$

for a parameter μ. This problem is easily solved using a Lagrange multiplier.

14. <u>Two Mutual Fund Theorem - Concept</u>: One key result of the above analysis is the two mutual fund theorem (Merton (1972)). The theorem states that any portfolio on the efficient frontier can be generated by any two give portfolios on the frontier; the latter two portfolios are the mutual funds in the theorem's name. So in the absence of a risk-free asset, the investor can achieve any desired efficient portfolio even if all that is accessible is a pair of efficient mutual funds.
15. <u>Two Mutual Fund Theorem - Allocation</u>: If the location of the desired portfolio on the frontier is between the location of the two mutual funds, both mutual funds will be held in positive quantities. If the desired portfolio is outside the range spanned by the two mutual funds, then one of the mutual funds must be sold short (held in negative quantity) while the size of the investment in the other mutual fund must be greater than the amount available for investment – the excess being funded by borrowing from the other mutual fund.
16. <u>Characteristics of the Risk-Free Asset</u>: The risk free asset is a hypothetical asset that pays a risk free rate. In practice, short term government securities such as UST bills are used as risk-free asset because they pay a fixed coupon and have an exceptionally low default risk. The

risk-free rate has zero variance in its returns (hence risk-free); it is also uncorrelated with any other asset (by definition, since its variance is zero).As a result, when combined with any other asset or portfolio of assets, the change in return is linearly related to the change in risk as the proportions in the combination vary.

17. Risk Free Asset Efficient Frontier: When a risk-free asset is introduced the corresponding half-line becomes the new efficient frontier – this line is the tangent to the hyperbola at the pure risky portfolio with the highest Sharpe ratio.
18. Corresponding Efficient Frontier Portfolio Weights: As demonstrated in Modern Portfolio Theory (Wiki), the vertical intercept represents a portfolio with 100% of holdings in the risk-free asset; the tangency with the hyperbola represents a portfolio with no risk-free holdings and 100% of the assets held in the portfolio occurring at the tangency point; and points on the half-line beyond the tangency point are leveraged portfolios involving negative holdings of the risk free asset (the latter has been sold short – in other words the investor has borrowed at the risk-free rate) and an amount invested in the tangency portfolio equal to more than 100% of the investor's initial capital.
19. The Capital Allocation Line - Definition: The efficient half-line is called the Capital Allocation Line (CAL) and its formula can be shown to be

$$\mathbb{E}[R_C] = R_F + \frac{\sigma_C}{\sigma_P}(\mathbb{E}[R_P] - R_F)$$

Here P is the sub-portfolio of the risky assets at the tangency point with the Markowitz bullet, F is the risk-free asset, and C is a combination of portfolios P and F.

20. Improved Risk Adjusted Returns Profile: Thus the introduction of the risk-free asset as a possible component of the portfolio has improved the range of the risk-expected returns available, because everywhere except at the tangency portfolio the half-line gives a higher expected returns than the hyperbola does at every possible risk level.
21. The One Mutual Fund Theorem: The fact that all points on the linear efficient locus can be achieved by a combination of holdings of the risk-free asset and the tangency portfolio is known as the One Mutual Fund Theorem (Merton (1972)), where the mutual fund referred to is the tangency portfolio.

Asset Pricing

1. MPT Basis For Asset Pricing: The analysis above describes the optimal behavior of an individual investor. Asset pricing theory builds on this analysis as follows. Since everyone holds risky assets in identical proportions to each other – namely in proportions given by the tangency portfolio – in market equilibrium, the risky assets' prices, and therefore their expected returns, will adjust so that the ratios in the tangency portfolio are the same as in the ratios in which the risky assets are supplied to the market.
2. Asset Supply/Demand Re-adjustment: Thus the relative supplies will equal relative demands. MPT derives the required expected return for a correctly priced asset in this context.
3. The Diversifiable Idiosyncratic/Specific Risk: Specific risk is the risk associated with the individual assets – within a portfolio these risks can be reduced through diversification (i.e., specific risks cancel out). Thus specific risk is also called diversifiable, unique, unsystematic, or idiosyncratic risk.
4. The Non-Diversifiable Systematic Risk: Systematic risk (a. k. a portfolio risk or market risk) refers to the risk common to all securities – except for selling short, systematic risk cannot be diversified away in one market. Within the market portfolio, asset specific risk will be diversifiable away to the extent possible. Systematic risk is therefore equated with the risk (standard deviation) of the market portfolio.
5. Rational Approach to Security Purchase: Since a security will be purchased only if it improves the risk-expected return characteristics of the market portfolio, the relevant measure of the risk of a security is the risk it adds to the market portfolio, and not its risk in isolation.
6. Conditional Asset Pricing Model Approaches: In this context, the volatility of the asset, and its correlation with the market portfolio, are historically observed, and are therefore given. There are several approaches to asset pricing that attempt to price assets by modeling the

stochastic properties of the moments of the assets' returns – these are broadly referred to as conditional asset pricing models.

7. Long/Short Market Model Portfolios: Systematic risks within one market can be managed through a strategy of using both long and short portfolios within one portfolio, creating a market neutral portfolio. Market neutral portfolios will therefore have correlations of zero.
8. Capital Asset Pricing Model - Definition: The asset returns depend on the amount paid for the asset today. The price paid must ensure that the market portfolio's risk/return characteristics improve when the asset is added to it. The CAPM is a model that derives the theoretical required expected return (i.e., discount rate) for an asset in the market, given the risk-free rate available to investors and the risk of the market as a whole.
9. CAPM - Mathematical Framework Specification: The CAPM is usually expressed as

$$\mathbb{E}[R_i] = R_i + \beta_i\left(\mathbb{E}[R_m] - R_f\right)$$

Here β_i is the measure of the asset sensitivity to a movement in the overall market; β_i is usually estimated through regression on historical data. β_i exceeding 1 signifies more than average riskiness in the sense of the asset's contribution to the overall portfolio risk; β_i below one indicate a lower than average risk combination. $\mathbb{E}[R_m] - R_f$ is the market premium, the expected excess return of the market portfolio's expected return over the risk-free rate.

10. CAPM Derivation Market Portfolio Risk: The incremental impact on risk and expected return when an additional risky asset a is added to the market portfolio m follows the formula for a two-asset portfolio. The results are used to derive the asset appropriate discount rate. Market portfolio's risk is ${w_m}^2{\sigma_m}^2 + ({w_a}^2{\sigma_a}^2 + 2w_a w_m \rho_{am}\sigma_a\sigma_m)$ Hence the risk added to the portfolio is ${w_a}^2{\sigma_a}^2 + 2w_a w_m \rho_{am}\sigma_a\sigma_m$, but since the weight of the asset is relatively low

$${w_a}^2 \approx 0$$

i.e., the additional risk is $2w_a w_m \rho_{am}\sigma_a\sigma_m$.

11. CAPM Derivation Market Portfolio Return: The market portfolio's expected return is $w_m\mathbb{E}[R_m] + (w_a\mathbb{E}[R_a])$ Hence the additional expected return is $w_a\mathbb{E}[R_a]$.

12. CAPM Derivation - Risk/Return Portfolio: If an asset a is correctly priced, the improvement in the risk-to-expected return ratio achieved by adding it to a market portfolio m will at least match the gains of spending that amount on an increased stake in the market portfolio. The assumption is that the investor will purchase the asset with funds borrowed at the risk free rate R_f; this is rational if

$$\mathbb{E}[R_a] > R_f$$

13. CAPM Derivation Excess Returns/Risk: Thus

$$\frac{w_a\left(\mathbb{E}[R_a] - R_f\right)}{2w_a w_m \rho_{am} \sigma_a \sigma_m} = \frac{w_m\left(\mathbb{E}[R_m] - R_f\right)}{2w_a w_m \sigma_m \sigma_m}$$

i.e.

$$\mathbb{E}[R_a] = R_f + \frac{\rho_{am}\sigma_a\sigma_m}{\sigma_m\sigma_m}\left(\mathbb{E}[R_m] - R_f\right)$$

i.e.

$$\mathbb{E}[R_a] = R_f + \frac{\sigma_{am}}{\sigma_{mm}}\left(\mathbb{E}[R_m] - R_f\right)$$

14. CAPM Derivation Asset-to-Market: $\frac{\sigma_{am}}{\sigma_{mm}}$ is the beta, - the covariance between the asset return and the market return divided by the variance of the market return – i.e., the sensitivity of the asset price to movement in the market's portfolio value.
15. The Security Characteristic Line (SCL): The excess asset returns can be estimated statistically using the following regression equation:

$$R_i(t) - R_f = \alpha_i + \beta_i\left(R_M(t) - R_f\right) + \epsilon_i(t)$$

where α_i is called the asset's alpha, and β_i is the asset's beta coefficient.

16. Expected Returns as Asset Numeraire: Once an asset's expected return $\mathbb{E}[R_a]$ is calculated using CAPM the future cash flows of the asset can be discounted to their present values using this rate to establish the correct price for the asset. A riskier stock will have a higher beta and be discounted at a higher rate, and a less sensitive stock will have a lower beta and be discounted at a lower rate. In theory an asset is correctly priced when its observed price is the same as its value calculated using the CAPM derived discount rate. If the observed price is higher than the valuation, the asset is over-valued; it is under-valued for too low a price.

Criticisms

1. Historical vs Future Statistical Inputs: The risk, the return, and the correlation measures used by MPT are based on expected, i.e., they are mathematical values about the future. In practice, investors must substitute predictions based on historical measurements of asset returns and volatility for these values. Very often such expected values fail to take into account new circumstances that did not exist when the historical data were first generated (Damghani (2013)).
2. Structural Approach to Risk Management: More fundamentally investors are stuck with estimating key parameters from past market data because MPT attempts to model risk in terms of likelihood of losses, but says nothing about why those losses might occur. The risk measurements used are probabilistic in nature, not structural. This is a major difference compared to many engineering approaches to risk management (Hubbard (2009)).
3. Better Measures for Investors' Preferences: MPT uses the mathematical concept of variance to quantify risk, and this may be justified under elliptically distributed returns such as normal returns, but for general return distributions, other risk measures (e.g. coherent risk measures) might reflect investors' true preferences.
4. Fundamentally Asymmetric Nature of Risk: In particular, variance is a symmetric measure that counts abnormally high returns as just as risky as abnormally low returns. In reality,

many investors are only concerned about losses, and do not care about the dispersion or tightness of above-average returns. According to this view, the intuitive concept of risk is fundamentally asymmetric in nature.

5. Gaussian Nature of Returns Distribution: MPT has also been criticized because it assumes that returns follow a Gaussian distribution (Taleb (2007)). Researchers have built on previous work that proposed using stable distributions instead, and have presented strategies for deriving optimal portfolios in such settings (Rachev and Mittnik (2000), Risk Manager Journal (2006), Doganoglu, Hartz, and Mittnik (2007)).
6. Extensions by Black and Litterman: The Black-Litterman model is an extension of the unconstrained Markowitz optimization that incorporates relative and absolute "views" on inputs of risk and returns.

Other Applications

1. MPT Applied in Regional Science: In a series of seminal works, Michael Conroy modeled the labor force in economy using portfolio-theoretic methods to examine growth and variability in the labor force. This was followed by a long literature on the relationship between economic growth and volatility (Chandra (2003)).
2. Modeling of the Concept of "Self": MPT has been used to model the concept of "self" in social psychology. When the self attributes comprising the self-concept constitute a well-diversified portfolio, the psychological outcomes at the level of an individual such as mood and self-esteem should be more stable than when the self-concept is undiversified (Chandra and Shadel (2007)). This prediction has been confirmed in studies involving human subjects.
3. Correlation/Uncertainty in Information Retrieval: Recently MPT has been applied to modeling the uncertainty and correlation between documents in information retrieval. Given a query, the aim is to maximize the overall relevance of a ranked list of documents and at the same time minimize the overall uncertainty of the ranked list (Wang (2009)).

4. <u>Project Portfolios/Non-Financial Assets</u>: When MPT is applied outside of traditional financial portfolios, some differences may need to be considered, e.g., the discrete nature of projects (i.e., projects may be all/nothing, have inseparable logical units, etc.), launch/closure limitations (e.g., no recovery/salvage of a half complete project) (Hubbard (2007), Sabbadini (2010)).
5. <u>Re-defining the Investment Boundaries</u>: Some of the simplest elements of MPT are applicable to virtually any kind of portfolio. As an example, for major projects the MPT investment boundary can be defined in more general terms like "choice of an ROI less than the cost of capital" or "chance of losing more than half of the investment" as opposed to the well-defined historical variance used in MPT. When risk is specified in terms of the uncertainty of forecasts and possible losses, the concept is then transferrable to various types of investments (Hubbard (2007)).

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The Black-Litterman Model

Overview

1. Black Litterman Chronology and Taxonomy: This section surveys the literature on the Black-Litterman model. The survey, drawn heavily from Walters (2014), is provided as both a chronology and as a taxonomy, as there are many claims on the model in the literature.
2. The Canonical Model and Derivations: A complete description of the canonical model including full derivations from the underlying principles using both Theil's Mixed Estimation Model and the Bayes' Theory is provided.
3. Model Parameters Calibration and Computation: The various parameters of the model are considered, along with the information on the calibration and the computation. Further treatment is given to several of the key papers, including worked samples illustrating the concept.

Introduction

1. Black Litterman Model and Evolution: The Black-Litterman model was first published by Fischer Black and Robert Litterman in 1990. During the 20+ years since the original paper, many authors have published research referring to their model as Black-Litterman. This has led to a variety of models being labeled as Back-Litterman even though they may be very different from the original model.

2. Variants in Literature and Taxonomy: In the chronological survey of the literature, Walters (2014) introduces several papers that make significant contributions to the literature on the Black-Litterman model. He also documents the taxonomy of models that have been described as Black-Litterman, and further refers to the Black-Litterman Model as described in the original papers as the Canonical Black-Litterman Model.
3. Equilibrium Market Portfolio as Prior: The Canonical Model of Black-Litterman makes two significant contributions to the problem of asset allocation. First it provides an intuitive prior – the equilibrium market portfolio – as a starting point for the estimation of the asset returns.
4. Alternative Priors in Earlier Work: Previous similar work started with the uninformative uniform prior distribution or with the global minimum variance portfolio. The latter method, studied by Frost and Savarino (1986) and Jorion (1986), took a shrinkage approach to describe the final asset allocation. Neither of these methods have an intuitive connection back to the market.
5. Shrinkage Based on "Reverse Optimization": The idea one could use "reverse optimization" to generate a stable distribution of returns from the equilibrium market portfolio as a starting point for shrinkage is a significant improvement to the process of returns estimation.
6. Incorporating Investors' Views on Returns: Second, the Black-Litterman model provides a clear way for the investors to specify views on returns and to blend those views with prior information. The investors' views' are allowed to be partial or complete, and the views can span arbitrary and overlapping sets of assets. The model outputs estimates of excess returns and the corresponding precision.
7. Blending Prior with the Investors' Views: Prior to Black and Litterman (1991a) nothing similar had been proposed for the portfolio selection problem. Theil's mixing model had been developed, but nobody had applied it to the problem of estimating asset returns. No research linked the process of specifying the views to the blending of the prior and the investors' views. The Black-Litterman model provides a quantitative framework for specifying the investors' views, and a clear way to combine those investors' views with an intuitive prior to arrive at a new combined distribution.
8. Enhancements to the Original Model: The state of the art in the portfolio selection problem has significantly changed during the time since the publication of the original Black-

Litterman model. Because of its rich theoretical basis it can be applied alongside several modern portfolio selection approaches as can be seen in the literature.

Historical Taxonomy and Literature Survey

1. Black-Litterman Model Classification Traits: The primary dimensions used by Walters (2014) to classify the models will be: does it specify the parameters as distributions or as point estimates, and does it include the parameter τ. In order for an author's model to be considered as the Canonical Black-Litterman it would need to match both these criteria.
2. Canonical Black-Litterman Reference Model: Walters (2014) collects the efforts into three different Reference Models based on the dimensions provided above. The situation where both of the above conditions are met is called the Canonical Reference Model, i.e., the model described by the original authors.
3. Alternative Black-Litterman Reference Model: The Alternate Reference Models describe models that use point estimates, but for some reason include τ, which now becomes a scaling factor with no quantitative basis. These authors typically treat the model as just a shrinkage/mixing estimator, thereby losing the explicit connection to Bayesian statistics. As Walters (2014) shows, using the Bayes' law formula – but substituting in different variables – is not theoretically tenable.
4. "Beyond Black Litterman" Reference Model: Finally the "Beyond Black Litterman" Reference Model uses point estimates and no longer includes τ at all. This makes it a pure shrinkage/mixing model.
5. Black-Litterman Model First Version: The Black-Litterman Model was first published by Fischer Black and Robert Litterman of Goldman Sachs in an internal Goldman Sachs Fixed Income Research Note (Black and Litterman (1990)). This research note was extended into a paper (Black and Litterman (1991a)). The paper provides a good overview of the features of then model, but does not provide all of the formulas used in the model.

6. Black-Litterman Model Second Version: A second internal Goldman Sachs Fixed Income research Note was published in the same year (Black and Litterman (1991b)). This paper was later extended and published in the Financial Analysis Journal (Black and Litterman (1992)). It provides the rationale for the methodology, and some steps involved in the derivation, but does not show all the formulas or a full derivation. It also includes a rather complex worked example based on global equilibrium – see Litterman (2003) on the details of the methods required to address this problem.
7. Black- Litterman Model Third Version: Unfortunately, because of the merging of the two problems, the second version of the Black-Litterman Model produces results that are difficult to reproduce. He and Litterman (1999) provide more details on the workings of the model, but not a complete set of formulas. They do provide, however, a much simpler to reproduce working model.
8. Goldman Sachs Asset Allocation Process: Bevan and Winkelmann provide details on how they use the Black-Litterman Model as part of their broader asset allocation process at Goldman Sachs, including some calibrations of the model which they perform. This is useful to anyone planning on building a Black-Litterman approach onto an ongoing asset allocation process.
9. The Satchell and Scowcroft Model: Satchell and Scowcroft (2000) attempt to de-mystify the Black-Litterman Model, but instead introduce a new non-Bayesian expression of the model. Their model uses point estimates for the prior and the views, and use τ and Ω only to control the shrinkage of the views on the prior. Because they use point estimates instead of distributions, their model does not include any information on the precision of the estimate. This allows them to recommend setting

$$\tau = 1$$

10. Use of a Stochastic τ: Satchell and Scowcroft (2000) also introduce a stochastic τ, but because they use point estimates, this really becomes a model with stochastic covariance of returns. The model was occasionally used in the literature after this point, but was largely replaced by Meucci's models in the mid-2000's.

11. Black Litterman Model Confidence Estimation: Drobetz (2001) provides a further description of the Black-Litterman Model including a good description of how to interpret the confidence in the estimates including a diagram – he also works out an example.
12. The Fusai and Meucci Model: Fusai and Meucci (2003) introduced yet another non-Bayesian variant of the model that removed the parameter τ altogether. Meucci (2005) followed up on this paper and coined the phrase "Beyond Black Litterman". Once the model is viewed as only a shrinkage model, Ω provides enough degrees of freedom to the shrinkage and the parameter τ becomes superfluous and confusing. Since the mid-2000's, there has been a mixture of the "Canonical" and the "Beyond Black Litterman" Models used in the literature (a more comprehensive survey can be found at http://www.blacklitterman.org/methods.html). However these hybrid models were later debunked by Michaud et al (2013).
13. The Firoozye and Blamont Model: Firoozye and Blamont (2003) provide a good overview of the Canonical Reference Model and present an intuition into the parameter τ. They also illustrate the reduction in the variance of the posterior estimate as a result of the mixing.
14. The Herold Model - Active Portfolios: Herold (2003) provides an alternate view of the problem where he examines optimizing alpha generation for active portfolio management, essentially specifying that the sample distribution has zero mean. He uses the Alternate Reference Model with point estimates and tracking error to determine how much shrinkage to allow. The two significant contributions by his paper are: a) application of the model to active portfolio management, and b) designing some additional measures that can be used to validate that the views are reasonable.
15. The Koch Model - 100% Certainty: Koch (2008) includes derivations of the "master formula" and the alternative form under 100% certainty. He does not mention posterior variance, or show the alternate form of master formula under uncertainty (the general case). He does include a slide on the sensitivity of the posterior estimate of τ, but he uses the Alternate Reference Model, so this treatment is not valid for the Canonical Model.
16. The Idzorek Model - Shrinkage Impact: Idzorek (2005) introduced a technique for specifying Ω such that the impact of the shrinkage was specified in terms of percentage of change in the weights between 0 and the change caused by 100% confidence. His paper uses the Alternative Reference Model but his techniques can be applied to the Canonical Black-Litterman Model because it is sensitive to the value of τ specified by the investor.

17. The Krishnan and Mains Model: Krishnan and Mains (2005) provide an extension to the Black-Litterman model to account for an additional factor that is uncorrelated with the market. They call this the two-factor Black-Litterman Model and they demonstrate an example of extending the Black-Litterman model with a recession factor. They show how it intuitively impacts the expected returns computed from the model.
18. The Mankert Model - Sampling Theory: Mankert (2006) provides a detailed walk-through of the model and works the elaborate transformation between the two specifications of the Black-Litterman "master formula" for the estimated asset returns. She approaches the problem from the point of view of Sampling Theory and provides additional intuition on the value of τ in the context of the Alternate Reference Model.
19. The Meucci Model - Non-Normality: Meucci (2006) provides a method for using non-normal views in the Black-Litterman model. Meucci (2008) extends this model to a wider range of parameters, and allows for the usage of the full distribution as well as for scenario analysis.
20. The Beach and Orlov Model: Beach and Orlov (2007) illustrate using GARCH models to generate the views. They use a model for international equities across 20 countries. They show how the results change as the investor uses different values for τ. They do not provide exact details for the uncertainty of the views and appear to be using an Alternative Reference Model even though their techniques can be applied to the Canonical Reference Model.
21. The Braga and Natale Model: Braga and Natale (2007) describe a model for calibrating the uncertainty in the views using the Tracking Error Volatility (TEV). They provide sensitivities for the posterior estimates to the various views. The TEV metric is well-known for its use in active portfolio management. They use the Alternative Reference Model, but their work could be used for the Canonical Reference Model as well.
22. The Martellini and Ziemann Model: Martellini and Ziemann (2007) describe an approach to the active management of a fund of hedge funds. They use VaR as their objective function for the reverse optimization, and they use a variant of the CAPM model extended to include skewness and kurtosis in determining the market portfolio. They use a factor model to generate the rankings, and then convert the rankings into their confidence in the views. They do not use Bayesian features of the model, but rather use point estimates, and thus do not have information on the precision of their estimates.

23. Stable Paretian Distributions for Measures: Giacometti, Bertocci, Rachev, and Fabozzi (2007) provide an approach to computing the neutral portfolio using stable Paretian distributions rather than the normal distribution described in the original Black and Litterman model. They also use multiple different measures of risk for their portfolio selection model – variance VaR and CVaR.
24. The Cheung Model: Cheung (2009) introduces the concept of an augmented model. This version of the model integrates a factor model and does a joint estimation of the factor returns. Cheung uses an Alternate Reference Model, his variant of the model can easily be used with the Canonical Reference Model.
25. Bertsimas, Gupta, and Paschalidis Model: Bertsimas, Gupta, and Paschalidis (2012) introduce a way to measure the alignment of views with prior estimate by comparing the view portfolio weights with the eigenvalues of the prior covariance matrix.
26. Michaud, Esch, and Michaud Model: Michaud, Esch, and Michaud (2012) provide a blistering critique of the Alternative Reference Model. Owing to their focus on the Alternative Reference Model, a significant part is devoted to problems with point estimates, and much of it is not relevant to the Canonical Reference Model. Further, the basic arguments are on the elementary statistical properties of time series, essentially ignoring non-stationary and auto-regressive models, as well as the richer, state-of-the-art econometric models.
27. Model Interoperability: Much of the work presented with any of the reference models can be used with the Canonical Reference Model, even if it has been initially formulated for the Alternative Reference Model (for additional notes, see Christodoulakis (2002)).

Canonical Black-Litterman Reference Model

1. The Black-Litterman Reference Model: The reference model for the returns is the base upon which the rest of the Black-Litterman Model is built. It includes assumptions on which variables are random, and which ones are not. It also defines which parameters are modeled,

and which ones are not modeled. Most importantly, some authors of papers on the Black-Litterman model use reference models, e.g., Alternate Reference Models, or Beyond Black-Litterman Model, and these do not have the same theoretical basis as the Canonical one which was initially specified in Black and Litterman (1992).

2. Normal Distribution for Expected Returns: The normally distributed expected returns can be expressed as

$$r \sim \mathcal{N}(\mu, \Psi)$$

The fundamental goal of the Black-Litterman model is to model the expected returns, which are assumed to be normally distributed with a mean μ and variance Ψ. Note that we will need at least these values – the expected returns and the covariance matrix later as inputs into a portfolio selection model.

3. The Random Unknown Mean Return: We define μ the unknown men return, as a random variable itself distributed as

$$\mu \sim \mathcal{N}(\varkappa, \Psi_{\varkappa})$$

$\varkappa$ is our estimate of the mean and $\Psi_{\varkappa}$ is the variance of the unknown mean μ about the estimate. Another way to view this simple relationship is shown in the formula below:

$$\mu = \varkappa + \epsilon_{\varkappa}$$

i.e., the prior returns are normally distributed around $\varkappa$ with a disturbance value $\epsilon_{\varkappa}$.

4. Relation Between $\epsilon_{\varkappa}$ and ϵ: $\epsilon_{\varkappa}$ is normally distributed with zero mean and $\Psi_{\varkappa}$ variance, and is assumed to be uncorrelated with ϵ, where ϵ is defined as

$$r = \mu + \epsilon$$

5. Relationship between all the Variances: The Reference Model can be completed by defining Ψ_r as the variance of the returns about our estimate $\varkappa$. From

$$\mu = \varkappa + \epsilon_{\varkappa}$$

and the assumption above that $\varkappa$ and μ are not correlated, the formula to compute Ψ_r is

$$\Psi_r = \Psi + \Psi_{\varkappa}$$

The formula indicates that the proper relationship between the variances is

$$\Psi_r \geq \max(\Psi, \Psi_{\varkappa})$$

6. Boundary Conditions Check for Ψ_r: A check at the boundaries ensures that the Reference Model is correct. In the absence of estimation error, e.g.

$$\epsilon = 0$$

then

$$\Psi_r = \Psi$$

As the estimate gets worse, i.e., as $\Psi_{\varkappa}$ increases, Ψ_r increases as well.

7. Alternate vs. Canonical μ Estimate: The Canonical Reference Model for the Black-Litterman Model expected return is

$$r \sim \mathcal{N}(\varkappa, \Psi_r)$$

A common misconception about the Canonical Black-Litterman Reference Model is that

$$r \sim \mathcal{N}(\mu, \Psi)$$

is a Reference Model, and that μ is a point estimate. This approach is what is labeled the Alternate Reference Model.

8. Impact of the Reference Model: Several authors approach the model from this point of view so it cannot be neglected, but it is a fundamentally different model. When considering the results from the various Black-Litterman implementations it is important to understand how the various parameters will impact the results.

Computing the Equilibrium Returns

1. Using the General Equilibrium Theory: The Black-Litterman Model starts with a market neutral equilibrium portfolio for the prior estimate of returns. The model relies on the General Equilibrium Theory to state of the aggregate portfolio is at equilibrium, each of the sub-portfolio must be at equilibrium.
2. Quadratic Utility for CAPM Portfolio: The Black-Litterman Model can be used with any utility function, which makes it very flexible. In practice most practitioners use the Quadratic Utility Function and assume a risk-free asset, and thus the equilibrium model reduces to the Capital Asset Pricing Model (CAPM). The neutral portfolio in this situation is the CAPM Market Portfolio.
3. Consistency across the Utility Functions: Some authors have used other utility functions – most notably Bertsimas, Gupta, and Paschalidis (2012) – but others have used CVaR and the other measures of portfolio risk without applying the same theoretical basis. In order to preserve the symmetry of the model the practitioners should use the same utility function to both identify the neutral portfolio as well as in the portfolio selection area.
4. The Standard Formulation Setting Choices: In his paper Walters (2014) uses the Quadratic Utility Function, the CAPM, and the Unconstrained Mean-Variance Optimizer because it is a well-understood model.

5. <u>CAPM Market Excess Returns Prior</u>: Given the previous assumption, the prior distribution for the Black-Litterman portfolio is the estimated mean excess return from the CAPM market portfolio. The process of computing the CAPM equilibrium excess returns is straightforward.
6. <u>The Basic CAPM Market Model</u>: CAPM is based on the concept that there is a linear relationship between risk (as measured by the standard deviation of returns) and expected returns. Further it requires returns to be normally distributed. The model is of the form

$$\mathbb{E}[r] = r_f + \beta r_m + \alpha$$

where r_f is the risk free rate, r_m is the excess returns of the market portfolio, β is a regression coefficient computed as

$$\beta = \rho \frac{\sigma_p}{\sigma_m}$$

and α is the residual – or asset specific (idiosyncratic) – excess return.
7. <u>Idiosyncratic Risk Reduction through Diversification</u>: Under CAPM the idiosyncratic risk associated with an asset is uncorrelated with that from the other assets, and this risk can be reduced through diversification. Thus the investor is rewarded for taking systematic risk measured by β but is not rewarded for the idiosyncratic risk associated with α.
8. <u>CAPM Market Portfolio Special Features</u>: In the CAPM world all investors should hold the same risky portfolio – the CAPM market portfolio. Because all investors hold risky assets only in their market portfolio, at equilibrium the market capitalization of the various assets will determine their weights in the market portfolio. Defining the Sharpe ratio as the excess returns divided by the excess risk, i.e. $\frac{r-r_f}{\sigma}$, it can be seen that the CAPM market portfolio has the maximum Sharpe Ration of any portfolio on the efficient frontier.
9. <u>Challenges Computing the Optimized Market Portfolio</u>: Note that the CAPM Market Portfolio contains all investable assets, which makes it very hard to actually specify. Because we are in equilibrium all sub markets should also be in equilibrium so that any sub-market we choose is part of the global equilibrium. While this allows us to reverse optimize the excess returns from the market capitalization and the covariance matrix, forward

optimization from this point to identify the investors' optimal portfolio within CAPM is problematic as we do not have information for the entire market portfolio.

10. Optimal Allocation among the Assets: In general, however, the investor usually selects an investible universe and the desired optimal asset allocation within that universe, so the theoretical problem with the market portfolio can be ignored.
11. The Capital Market Line Construction: The Capital Market Line is the line through the risk-free rate and the CAPM market portfolio. The Two-Fund Separation Theorem, closely related to the CAPM, states that all investors should hold portfolios on the Capital Market Line.
12. Significance of the Capital Market Line: Any portfolio on the Capital Market Line dominates all the portfolios on the Efficient Frontier, the CAPM being the only point on both the Efficient Frontier and the Capital Market Line. Depending on their risk aversion the investor will hold arbitrary fractions of wealth in the risk-free asset and/or the CAPM market portfolio.
13. Risk-Free Asset in the Portfolio: Starting with the market portfolio, the starting point will be a set of weights which are all greater than zero and naturally sum to one. The market portfolio only includes risk assets, because by definition, the investor is rewarded only for taking systematic risk. Thus, in CAPM, the risk-free asset with

$$\beta = 0$$

will not be included in the market portfolio. Walters (2014) examines the case where the Bayesian investor may invest in the risk-free asset based on their confidence on their returns estimates.

14. Historical Returns Covariance Matrix Estimation: One assumption here is that the covariance matrix of the returns Ψ is known. In practice, this covariance matrix is estimated from historical returns data. It is often computed from higher frequency data and scaled up to the time frame required for the asset allocation problem. By computing it from the actual historical data, we can ensure that the covariance matrix is positive definite.
15. Other Returns Covariance Estimation Techniques: Without basing the estimation process on actual data, there are significant issues involved in ensuring that the covariance matrix is

positive definite. One could, however, apply shrinkage or modern random matrix theory filters to make it more robust.

16. Note on the Notation Used: This section follows Walters (2014) where the notation used is similar to that used in He and Litterman (1999) for all the terms in the formulas. The notation is a little different from (and conflicts with) the notation used later in the section on Bayesian Theory.
17. Equation for the “Reverse Optimization”: One can derive the equation for reverse optimization starting from the quadratic utility function

$$U = w^T \Pi - \frac{\delta}{2} w^T \Psi w$$

where represents the investors’ utility – this is the objective function used for the Mean-Variance Optimization, w is the vector of weights invested in each asset, Π is the vector or equilibrium excess returns for each asset, δ is the risk aversion parameters, and Ψ is the covariance matrix of the excess returns of the assets.

18. Maxima for the Utility Function: U is a convex function, so it will have a single maxima. If we maximize the utility with no constraints, there is a closed form solution. The exact solution is found by taking the first derivative with respect to the weights w and setting it to zero.

$$\frac{\partial U}{\partial w} = \Pi - \delta \Psi w = 0$$

19. Estimation of the Excess Returns Vector: Solving this for Π – the vector of excess returns – yields

$$\Pi = \delta \Psi w$$

20. Estimating the Risk Aversion Factor: In order to use the above formula to solve for the CAPM Market Portfolio, there needs to be a value for δ - the risk aversion coefficient of the

market. One way of finding δ is by multiplying both sides by w^T and replacing the vector terms with scalar terms

$$r - r_f = \delta\sigma^2$$

21. <u>δ Estimate for Equilibrium Portfolio</u>: The expression at equilibrium above states that the excess returns on the portfolio is equal to the risk aversion parameter scaled by the variance of the portfolio. Conversely

$$\delta = \frac{r - r_f}{\sigma^2}$$

where r is the total return on the Market Portfolio, i.e.

$$r = w^T \Pi + r_f$$

r_f is the risk-free rate, and σ^2 is the variance of the Market Portfolio, i.e.

$$\sigma^2 = w^T \Psi w$$

22. <u>δ Values used in Literature</u>: Most authors specify that value of δ that they used. Bevan and Winkelmann (1998) describe their process of calibrating their returns to an average Sharpe Ratio based on their experience. For global fixed income – their area of expertise – they use a Sharpe Ratio of 1.0. Black and Litterman (1992) use a Sharpe Ratio closer to 0.5 in the example in their paper.
23. <u>Risk Aversion from Sharpe Ratio</u>: Given the Sharpe Ratio one can write the formula for δ in terms of Sharpe Ratio as

$$\delta = \frac{SharpeRatio}{\sigma_m}$$

Thus one can calibrate the returns in terms of

$$\delta = \frac{r - r_f}{{\sigma_m}^2}$$

or

$$\delta = \frac{SharpeRatio}{\sigma_m}$$

24. Sharpe Ratio of Market Portfolio: In order to use

$$\delta = \frac{r - r_f}{\sigma^2}$$

one needs to have an implied return for the market portfolio which may be harder to estimate than the Sharpe Ratio of the market portfolio.

25. No-Constraint Reverse Portfolio Optimization: Once there is a value for

$$\Pi = \delta \Psi w$$

plugging in w, δ, and Ψ into

$$\Pi = \delta \Psi w$$

generates the set of equilibrium returns.

$$\Pi = \delta \Psi w$$

is the closed form solution to the reverse optimization problem for computing asset returns given an mean-variance portfolio in the absence of constraints. Re-arranging

$$\Pi = \delta\Psi w$$

produce a formula for the closed form calculation of the optimal portfolio weights in the absence of constraints:

$$w = (\delta\Psi)^{-1}\Pi$$

26. Reverse Optimization under Simple Constraints: Herold (2005) provides insights into how implied returns can be computed in the presence of simple equality constraints such as the budget constraint of the full investment

$$\sum w = 1$$

constraint. He illustrates how errors can be introduced during the reverse optimization if the constraints are assumed to be non-binding when they are actually binding for a given portfolio. Note that since one is dealing with the market portfolio which has only positive weights that sum to 1, one can safely assume that there are no binding constraints on the reverse optimization.

27. Incorporating the Prior Distribution Variance: The only piece missing is the variance of our estimate of the mean. Looking back at the Reference Model what is needed is $\Psi_{\varkappa}$. Black and Litterman made the simplifying assumption that the structure of the covariance matrix of the estimate is proportional to the covariance of the returns Ψ. They created a parameter τ as the constant of proportionality. Given the assumption

$$\Psi_{\varkappa} = \tau\Psi$$

the prior distribution is

$$Prior(A) \sim \mathcal{N}(\varkappa, \tau\Psi)$$

or

$$r_A \sim \mathcal{N}(Prior(A), \Psi)$$

This is the prior distribution of the Black Litterman Model. It represents the estimate of the mean, which is expressed as a distribution of the actual unknown mean about the estimate.

28. Incorporating Prior and Conditional Variance: Using

$$r \sim \mathcal{N}(\varkappa, \Psi_r)$$

one can re-write the above formula in terms of $\varkappa$ as

$$r_A \sim \mathcal{N}(\varkappa, [1+\tau]\Psi)$$

29. No Views Unconstrained Portfolio Weights: It is often written that an investor with no views and using an unconstrained mean-variance portfolio selection model will invest 100% in the neutral portfolio, but this is only true if they apply a budget constraint. Because of the uncertainty in the estimates they will invest $\frac{1}{1+\tau}$ in the neutral portfolio an $\frac{\tau}{1+\tau}$ in the risk-free asset. This can be seen from the following:

$$\Pi = \delta\Psi w$$

$$w = (\delta\Psi)^{-1}\Pi$$

$$\hat{w} = ([1+\tau]\delta\Psi)^{-1}\Pi$$

$$\hat{w} = \frac{1}{1+\tau}(\delta\Psi)^{-1}\Pi$$

$$\hat{w} = \frac{1}{1+\tau} w$$

Walters (2014) illustrates the concept graphically.

30. Confidence Shift on the Efficient Frontier: One can alternatively view the Bayesian Efficient Frontier as a shift to the right if they plot the generated Efficient Frontier with an increased covariance matrix and a budget constraint. In this case the uncertainty just pushes each point further to the right in the risk/return space.

Specifying the Views

1. Investors Views as Conditional Distribution: This section describes the process of specifying the investors' views on the estimated mean excess returns. The combination of the investors' views is defined as the conditional distribution.
2. Zero Correlation among the Views: First, by construction, each view will be unique and uncorrelated with the other views. This gives the conditional distribution the property that the covariance matrix will be diagonal, with off-diagonal entries equaling zero. This constraint on the problem simplifies it and improves the stability of the result. Estimating co-variances between views is more error prone and complicated than estimating the view variances.
3. Fully Invested Absolute/Relative Views: The views are also further constrained to be fully invested, i.e., the sum of the weights in the view is either zero (a relative view) or one (an absolute view). Views are not required on any or all assets. In addition, the views can conflict – the mixing process will merge the views based on the confidence on them, and confidence on the prior.
4. View Representation Matrix - Asset Weights: The investors' k views on n assets is represented as follows. First, $\mathbb{P}$ refers to the $k \times n$ matrix of the asset weights within each view. For a relative view the sum of the weights will be zero, and for an absolute view the sum will be 1. Different authors compute the weights within the views differently – He and

Litterman (1999) and Idzorek (2005) use a market capitalization weighted scheme, whereas Satchell and Scowcroft (2000) use an equi-weighted scheme in their examples. In practice, the weights will be a mixture depending upon the process used to estimate the view returns.

5. View Representation Matrix - Excess Returns: Second, $\mathbb{Q}$ is a $k \times 1$ vector of weights for each view.
6. View Representation Matrix - Returns Variance: Finally Ω is a $k \times k$ matrix of the covariance of the views. Ω is diagonal, as the views are required to be independent and uncorrelated. Ω^{-1} is known as the confidence in the investors' views. The i^{th} diagonal element of Ω is represented w_i.
7. Invertibility of the $\mathbb{P}$ Matrix: $\mathbb{P}$ is not required to be invertible. Meucci (2006) describes a method of augmenting the matrices to make the $\mathbb{P}$ matrix invertible, while not changing the net results.
8. Invertibility of the Ω Matrix: Ω is symmetric and zero on all non-diagonal elements, but may also be zero on a diagonal if the investor is certain of the given view. This means that Ω may or may not be invertible. At a practical level one may require

$$w_i > 0$$

so that Ω is made invertible, but from a formulation point of view Ω is not required to be invertible.

9. Relative and Absolute Views Specification: As an example of how these matrices would be populated, Walters (2014) examines some investors' views, using four assets and two views. The first is a relative view in which the investor believes that asset #1 will outperform asset #2 by 2% with a confidence of w_1. The second is an absolute view in which the investor believes that asset #2 will return 3% with a confidence w_2.
10. Sample Absolute/Relative View Matrices: Note that the sample above does not express a view on asset #4, and thus its returns should not be directly adjusted. The views are specified as follows:

$$\mathbb{P} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathbb{Q} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Omega = \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix}$$

11. <u>Conditional Distribution in the View Space</u>: Given the above specification of the views, the conditional distribution mean and variance can be formulated as

$$\mathcal{P}(\mathbb{B}|\mathbb{A}) \sim \mathcal{N}(\mathbb{Q}, \Omega)$$

In general this cannot be converted into a useful expression in the asset space because of the mixture of relative and absolute views, and because the $\mathbb{P}$ matrix is not required to be of full rank.

12. <u>View Representation in Asset Space</u>: In the asset space the conditional distribution can be represented as

$$\mathcal{P}(\mathbb{B}|\mathbb{A}) \sim \mathcal{N}(\mathbb{P}^{-1}\mathbb{Q}, [\mathbb{P}^T \Omega^{-1} \mathbb{P}])$$

This representation, however, is not of much practical use. Incomplete views and relative views make the variance non-invertible, and relative views also impact the mean term. Also $\mathbb{P}$ may not be invertible, and even if it is $[\mathbb{P}^T \Omega^{-1} \mathbb{P}]$ may not be, making the above expression impossible to evaluate in practice. Luckily to work with the Black-Litterman Model, one does not need to explicitly evaluate $\mathcal{P}(\mathbb{B}|\mathbb{A})$ above. It is, however, of interest to see how the views are projected onto the asset space.

View Distribution in the Asset Space – Derivation

1. Expected View Return Conditional Distribution: The starting point is the definition of the views

$$\hat{Q} = Q + \epsilon$$

where $\hat{Q}$ is the $k \times 1$ vector of the unknown mean returns to the views, Q is the $k \times 1$ vector of the estimated mean returns to the views, and ϵ is a $n \times 1$ matrix of the residuals from the regression where

$$\mathbb{E}[\epsilon] = 0$$

$$\mathbb{V}[\epsilon] = \mathbb{E}[\epsilon\epsilon^T] = \Omega$$

and Ω is non-singular. The expression for $\hat{Q}$ above can be re-written as an explicit distribution of $\hat{Q}$ as

$$\hat{Q} \sim \mathcal{N}(Q, \Omega)$$

2. Projections onto the Asset Space: Using the definition above the unknown mean returns of the views based on the unknown mean returns of the assets and the portfolio pick matrix P may be re-written as

$$P\hat{\Pi} = \hat{Q}$$

where P is the $k \times n$ matrix of weights for the view portfolios and $\hat{\Pi}$ is the $n \times 1$ vector of the unknown returns of the assets. Substitution of

$$\hat{Q} = Q + \epsilon$$

onto $\hat{Q}$ above yields

$$P\widehat{\Pi} = Q + \epsilon$$

3. View Estimated Mean in Asset Space: Assuming P is invertible, which requires it to be of full rank, both sides above may be multiplied by P^{-1}. This is the projection of the view-estimated means onto the asset space representing the Black-Litterman conditional distribution. If P is not invertible a slightly different formulation is needed, adding another term on the right hand side

$$\widehat{\Pi} = P^{-1}Q + P^{-1}\epsilon$$

4. View Covariance in Asset Space: The next task is to represent the above formula as a distribution. For this purpose the covariance of the random term needs to be computed. The variance of the unknown asset means about the estimated view means projected onto the asset space is calculated as follows:

$$Variance = \mathbb{E}[P^{-1}\epsilon\epsilon^T(P^{-1})^T] = P^{-1}\mathbb{E}[\epsilon\epsilon^T](P^{-1})^T = P^{-1}\Omega(P^{-1})^T$$

Using

$$(P^T)^{-1} = (P^{-1})^T$$

and

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$Variance = P^{-1}\Omega(P^{-1})^T = P^{-1}\Omega(P^T)^{-1} = (P^T\Omega^{-1}P)^{-1}$$

5. Consolidated Asset Space View Projection: Thus one arrives at the projection of the views onto the asset space as

$$\widehat{\Pi} \sim \mathcal{N}(\mathbb{P}^{-1}\mathbb{Q}, [\mathbb{P}^T\Omega^{-1}\mathbb{P}])$$

The covariance term here is the covariance of the unknown mean returns about the estimated returns from the views, not the covariance of the expected returns.

Specifying Ω

1. Benchmark of Investor Confidence: Ω, the variance of the views, is inversely related to the investor confidence in the view. However, the basic Black-Litterman Model does not provide an intuitive way of quantifying this relationship. It is up to the investor to compute this variance of the views.
2. Typical Methods to calculate Ω: The common ways to calculate Ω are:
   a. Choose an Ω proportional to the variance of the prior
   b. Specify Ω from a confidence interval
   c. Use the variance of residuals in a factor model
   d. Use Idzorek's method to specify confidence along the weight dimension

Ω Proportional to the Variance of the Prior

1. He and Litterman (1999) Ω Specification: Here the assumption is that the variance of the views is just proportional to the variance of the asset returns, just as the variance of the prior distribution is. Both He and Litterman (1999) and Meucci (2006) use this method, though they use it differently. He and Litterman (1999) set the variance of the views as

$$w_{ij} = P(\tau\Psi)P^T \ \forall\ i = j$$

$$w_{ij} = 0 \; \forall \; i \neq j$$

or

$$\Omega = Diagonal[P(\tau\Psi)P^T]$$

2. Implication of the Above Specification: The above specification of the variance or the uncertainty of the views essentially equally weights investors' views and the market equilibrium weights. By adding τ in the expression the posterior estimate of the returns becomes independent of τ as well. This seems to be the most common method used in the literature.
3. Meucci's Specification – Non-diagonal Ω: Meucci (2006) doesn't bother with diagonalization at all – he just sets

$$\Omega = \frac{1}{c} P\Psi P^T$$

He further sets

$$c > 1$$

and one obvious choice for c is τ^{-1}. As will be seen later this form of the variance of views lends itself to some simplifications of the Black-Litterman formulas.

Using Confidence Interval for Ω

1. Ω as One Standard Deviation: The investor can specify the variance using a confidence interval around an estimated mean return, e.g., an asset that has an estimated mean return of

3% with an expectation that it is 68% likely to be within the interval $(2\%, 4\%)$. Knowing that 68% of the outcomes falls within one standard deviation of the mean allows one to translate this into a variance for the view as $(1\%)^2$

2. Ω as a Confidence Metric: Note that Ω is the estimate of the uncertainty around the mean, and not the variance of the returns around the mean. This formulation of the variance of the view is consistent with the Canonical Reference Model.

Ω as the Variance of Residuals from a Factor Model

1. Generalized Factor Model for Returns: If the investor is using a factor model to compute the views they can use the variance of the residuals from the model to drive the variance of the return estimates. The general expression for the factor model of returns is

$$r = \sum_{i=1}^{n} \beta_i f_i + \epsilon$$

where r is the return on the asset, β_i is the factor loading for the factor i, f_i is the return due to factor i, and ϵ is an independent normally distributed residual.

2. Variance of Factor Model Returns: The general expression for the variance of the returns from a factor model is

$$\mathbb{V}[r] = B\mathbb{V}[f]B^T + \mathbb{V}[\epsilon]$$

where B is the factor loading matrix, and F is the vector of returns due to the various factors. Given

$$r = \sum_{i=1}^{n} \beta_i f_i + \epsilon$$

and the assumption that ϵ is independent and normally distributed, one can compute the variance of ϵ directly as part of the regression. While the regression may yield a full covariance matrix, the mixing model will be more robust if only the diagonal elements are used.

3. GARCH Style View Factor Models: Beach and Orlov (2006) describe their work using GARCH style factor models to generate their views for the Black-Litterman model. They generate their precision of views using GARCH models.

Using Idzorek's Method for Ω

1. Asset Weights as Confidence Metric: Idzorek (2005) describes a method for specifying the confidence in the views in terms of a percentage move of the weights on the interval from 0% confidence to 100% confidence. This algorithm is examined in the section on extensions.

The Estimation Model

1. Theil and Bayesian Estimation Approaches: The original Black-Litterman Model reference Theil's Mixed Estimation Model rather than a Bayesian Estimation Model, though similar results are obtained from both methodologies. Theil's method serves as a starting point because it is simpler and cleaner. The Bayesian method and its derivation are also worked out for completeness.

2. Mean and Estimated Returns Distribution: With either approach the Canonical Black Litterman Reference Model will be used. In this reference model the estimation model is used to compute the distribution of returns about the mean return. This distinction is important in understanding the values used for τ and Ω, and for computing the variance of the prior and the posterior distributions of returns.
3. Posterior Estimate of the Means: The posterior estimate of the mean generated by the estimation model is more precise than either the prior estimate or the investors' views. Note that one would not expected changes in the variance of the distribution of returns because the estimate of the mean is more precise.
4. Posterior Means Estimate - Sample Illustration: A prototypical illustration of the above would be to blend the distributions

$$\mathbb{P}(A) \sim \mathcal{N}(10\%, 20\%)$$

and

$$\mathbb{P}(B|A) \sim \mathcal{N}(12\%, 20\%)$$

If one applies either estimation model in a straightforward fashion

$$\mathbb{P}(A|B) \sim \mathcal{N}(11\%, 10\%)$$

Clearly with the data the variance of the returns distribution did not get cut in half just because one has a slightly better estimate of the mean. In this case the mean is a random variable, and the variance of the posterior corresponds to the variance of the estimated mean around the mean return, not the variance of the distribution of returns around the mean return. In this case the posterior

$$\mathbb{P}(A) \sim \mathcal{N}(11\%, 10\%)$$

makes sense.

5. Shrinkage from Blending of Distributions: By blending these two estimates of the mean, one has an estimate of the mean with much less uncertainty (less variance) than either of the estimates, even though there is no improvement in the estimate of the actual distribution of returns around the mean.

Theil's Mixed Estimation Model

1. Theil's Estimation Model – Setup: Theil's mixed estimation model was created for the purpose of estimating parameters from a mixture of complete prior data and partial conditional data. This is a good problem fit in the current case as it allows one to express views on all of them. The views can be expressed on a single asset, or on a combination of assets. The views do not even to be consistent, the estimation model will take each into account based on the investors' confidence.
2. Linear Models for Parameters' Estimation: Theil's Mixed Estimation Model starts from a linear model for the parameters to be estimated. One can use

$$\mu = \varkappa + \epsilon_{\varkappa}$$

   from the reference model as a starting point.
3. Linear Models for Prior Returns: A simple linear factor model can be expressed as

$$x\beta = \varkappa + u$$

   where $\varkappa$ is the $n \times 1$ vector of equilibrium returns for the assets, x is the $n \times n$ matrix of the factor loadings I_n for the model, β is the $n \times 1$ vector of unknown means for the asset return process, and u is an $n \times 1$ matrix of the residuals from the regression where

$$\mathbb{E}[u] = 0$$

$$\mathbb{V}[u] = \mathbb{E}[u^T u] = \Phi$$

and Φ is non-singular.

4. Linear Model Returns Variance Estimator: The Black-Litterman Model uses a very simple linear model where the expected return for each asset is modeled by a single factor which has a coefficient of 1. Thus x becomes the identity matrix. Given that β and u are independent, and that x is constant, one can model the variance of $\varkappa$ as

$$\mathbb{V}[\varkappa] = x^T \mathbb{V}[\beta] x + \mathbb{V}[u]$$

which can be simplified to

$$\mathbb{V}[\varkappa] = \Psi + \Phi$$

where Ψ is the historical covariance matrix of the returns as used earlier, and Φ is the covariance of the residuals or of the estimate about the actual mean.

5. Total Variance of Estimated Return: This ties back to the formula

$$\Psi_r = \Psi + \Psi_\varkappa$$

used in the reference model. The total variance of the estimated return is the form of the variance of the actual return process plus the variance of the estimate of the mean. This relation will be re-visited later.

6. Conditional Linear Models for Returns: The next consideration is the additional information that needs to be combined with the prior. This information can be subjective views, or it can be derived from statistical data. This is also allowed to be incomplete, meaning that there may not be an estimate for each asset. As before, a simple linear factor model can be expressed as

$$p\beta = q + v$$

where q is the $k \times 1$ vector of equilibrium returns for the assets, p is the $k \times n$ matrix of the factor loadings I_n for the model, β is the $n \times 1$ vector of unknown means for the asset return process, and v is an $k \times 1$ matrix of the residuals from the regression where

$$\mathbb{E}[v] = 0$$

$$\mathbb{V}[v] = \mathbb{E}[v^T v] = \Omega$$

and Ω is non-singular.

7. <u>Combining Prior and Conditional Models</u>: One can combine prior and conditional models as

$$\begin{bmatrix} x \\ p \end{bmatrix} \hat{\beta} = \begin{bmatrix} \varkappa \\ q \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix}$$

where the expected value of the residual is 0, and the expected value of the variance of the residual is

$$\mathbb{V}\begin{bmatrix} v \\ u \end{bmatrix} = \mathbb{E}\left[\begin{bmatrix} v \\ u \end{bmatrix} \begin{bmatrix} u^T & v^T \end{bmatrix}\right] = \begin{bmatrix} \Phi & 0 \\ 0 & \Omega \end{bmatrix}$$

8. <u>Estimation Using Generalized Lease Squares</u>: One can then apply the generalized least squares procedure, which leads to estimating $\hat{\beta}$ as

$$\hat{\beta} = \left[\begin{bmatrix} x^T & p^T \end{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & \Omega \end{bmatrix}^{-1} \begin{bmatrix} x \\ p \end{bmatrix}\right]^{-1} \left[\begin{bmatrix} x^T & p^T \end{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & \Omega \end{bmatrix}^{-1} \begin{bmatrix} \varkappa \\ q \end{bmatrix}\right]$$

This can be re-written without the matrix notation as

$$\hat{\beta} = [x^T \Phi^{-1} x + p^T \Omega^{-1} p]^{-1} [x^T \Phi^{-1} \varkappa + p^T \Omega^{-1} q]$$

9. Simplification to the Black-Litterman Case: For the Black-Litterman Model – which is a single factor per asset - x can be dropped as it is an identity matrix. If one wanted to use a multi-factor model for the equilibrium then x would be the equilibrium factor loading matrix

$$\hat{\beta}_{BL} = [\Phi^{-1} + p^T \Omega^{-1} p]^{-1} [\Phi^{-1} \varkappa + p^T \Omega^{-1} q]$$

10. Interpretation of the $\hat{\beta}$ Estimate: This new $\hat{\beta}$ is the weighted average of the estimates. The precision is inverse of the variance. The posterior estimate $\hat{\beta}$ is also the best linear unbiased estimate given the data, and has the property that it minimizes the variance of the residual. Note that given a new $\hat{\beta}$ an updated expectation for the variance of the residual should also be available.
11. Multi-Factor Prior Returns Model: If one were using a factor model for the prior, one would retain x – the factor weightings – in the formulas. This results in a multi-factor model where all the factors will be priced into the equilibrium.
12. Variation of the Estimation Residual: One can reformulate the combined relationship in terms of the estimate of $\hat{\beta}$ and a new residual $\tilde{u}$ as

$$\begin{bmatrix} x \\ p \end{bmatrix} \hat{\beta} = \begin{bmatrix} \varkappa \\ q \end{bmatrix} + \tilde{u}$$

Once again

$$\mathbb{E}[\tilde{u}] = 0$$

leading to the expression for the variance of the new residual as

$$\mathbb{V}[\tilde{u}] = \mathbb{E}[\tilde{u}^T \tilde{u}] = [\Phi^{-1} + p^T \Omega^{-1} p]^{-1}$$

and the total variance is

$$\mathbb{V}[y \quad \varkappa] = \mathbb{V}\left[\hat{\beta}\right] + \mathbb{V}[\tilde{u}]$$

13. Simplification of the Residual Variance: This section began with the assertion that the variance of the returns process is a known quantity. Improved estimation of the quantity $\hat{\beta}$ does not alter the estimate of the variance of the returns distribution Ψ. Because of the improved estimate one expects that the variance of the residual estimate has decreased, thereby the total variance has changed. This simplifies the variance formula

$$\mathbb{V}[\varkappa] = \Psi + \Phi$$

to

$$\mathbb{V}[y \quad \varkappa] = \Psi + \mathbb{V}[\tilde{u}]$$

14. Interpretation of the Posterior Variance: This is a clearly intuitive result consistent with the realities of financial time series. The two estimates of the mean have been combined to arrive at a better estimate of the mean. The variance of this estimate has been reduced, but the actual variance of the underlying process has not changed. Given the uncertain estimate of the process, the total variance of the estimated process has also improved incrementally, but it has the asymptotic limit that it cannot be less than the variance of the underlying process.
15. Consistency with Canonical Reference Model: The above is the convention for computing the covariance of the posterior distribution of the Canonical reference Model as shown in He and Litterman (1999).
16. Simplification under Absence of Views: In the absence of views the above formula simplifies to

$$\mathbb{V}[y \quad \varkappa] = \Psi + \Phi$$

which is the variance of the prior distribution of returns.

A Quick Introduction to Bayes' Theorem

1. Application of Bayes' Theorem/Nomenclature: This section provides a quick overview of the relevant portions of the Bayes' Theory in order to create a common vocabulary that can be used in analyzing the Black-Litterman model from a Bayesian point of view.
2. Bayes' Theory – Basic Statement: Bayes' theory states

$$P[A|B] = \frac{P[B|A]P[A]}{P[B]}$$

3. LHS - The Conditional Posterior Distribution: $P[A|B]$ is the conditional probability of A given B. This is also referred to as the posterior distribution.
4. RHS - The Conditional Sampling Distribution: $P[B|A]$ is the conditional probability of B given A. This is also known as the sampling distribution, thus will be referred to as the conditional distribution from here on.
5. RHS - The Unconditional Prior Distribution: $P[A]$ is the unconditional probability of A – also known as the prior distribution. This will be referred to as the prior distribution from here on.
6. RHS - The Unconditional Normalizing Distribution: $P[B]$ is the unconditional probability of B, and is known as the normalizing constant. When applying the Bayes' formula and solving for the posterior distribution the normalizing constant typically disappears inside the constants of the integration.
7. Use of Normal Distributions throughout: A general problem in applying Bayes' Theory is the identification of an intuitive and tractable prior distribution. As seen earlier one of the core assumptions of the Black-Litterman model and the Mean Variance Optimization is that the asset returns are normally distributed. For that reason the treatment here is confined to the case of normally distributed prior and conditional distributions. Give that these inputs are normal distributions it then follows that the posterior will also be normally distributed.
8. Using non-Normal Probability Distributions: When the prior and the posterior distributions have the same functional form the prior distribution is referred to as the conjugate prior.

Given interest there is nothing that prevents the construction of the Black-Litterman model using different distributions, however the normal distribution is generally the most straightforward.

9. Unknown Mean and Known Variance: Another core assumption of the Black-Litterman model is that the variances of the prior and the conditional distributions about the actual mean are known, but the actual mean itself is unknown. This scenario corresponds to the *Unknown Mean and Known Variance* case and is well-documented in the Bayesian literature. This matches the model used by Theil, where the estimate of the mean is uncertain, but the variance is known.
10. Specification of the Prior Distribution: The prior distribution is given as

$$P[A] \sim \mathcal{N}\left(x, \frac{S}{n}\right)$$

    where S is the sample variance about the mean of the distribution with n samples and $\frac{S}{n}$ is the variance of x about the mean.
11. Specification of the Conditional Distribution: The conditional distribution is specified as

$$P[B|A] \sim \mathcal{N}(\mu, \Omega)$$

    Here is Ω is the uncertainty of the estimate of μ in the mean, and is *not* the variance of the full distribution about its mean.
12. Specification of the Posterior Distribution: From the above the posterior can be specified as

$$P[A|B] \sim \mathcal{N}([\Omega^{-1} + nS^{-1}]^{-1}[\Omega^{-1}\mu + nS^{-1}x]^{T}, [\Omega^{-1} + nS^{-1}]^{-1})$$

    The variance term above is the variance of the estimated mean about the actual mean.
13. Precision in Bayesian Distributions: In Bayesian statistics the inverse of the variance is known as the precision. The posterior mean can be described as the weighted mean of the prior and conditional means, where the weighting factor is the corresponding precision. The expression for the posterior distribution $P[A|B]$ above requires that the precisions of both the

prior and the conditional be non-infinite and that the sum be non-zero. Infinite precision corresponds to a variance of 0 or absolute confidence. Zero precision corresponds to infinite variance or complete uncertainty.

The PDF Based Approach

1. PDF Based Black-Litterman Derivation: This section contains a derivation of the Black-Litterman master formula using the standard Bayesian approach for modeling the posterior of two normal distributions. An alternate derivation is presented in Mankert (2006) where the author derives the Black-Litterman *Master Formula* from Sampling Theory, and also shows the detailed transformation between the two forms of this formula.

2. PDF Based Posterior Bayesian Estimation*:* This approach follows a Bayesian approach to computing the PDF of the posterior distribution when the prior and the conditional distributions are both normal distributions. The derivation here is based on the proof shown in De Groot (1970). A similar approach is taken by Scowcroft and Satchell (2000).

3. PDF Based Bayesian Approach Outline: The method of this proof is to examine all the terms in the PDF of each distribution that depend on $\mathbb{E}[r]$ neglecting the other terms as they have no dependence on $\mathbb{E}[r]$ and are thus constant with respect to $\mathbb{E}[r]$.

4. PDF for the Prior Distribution*:* Starting from the prior distribution one can derive an expression proportional to the value of the PDF

$$P[A] \sim \mathcal{N}\left(x, \frac{S}{n}\right)$$

with n samples from the population. Thus the PDF $\xi(x)$ of $P[A]$ satisfies

$$\xi(x) \propto e^{-\left[\frac{S}{n}\right]^{-1}(\mathbb{E}[r]-x)^2}$$

5. PDF for the Conditional Distribution: One next considers the PDF of the conditional distribution

$$P[B|A] \sim \mathcal{N}(\mu, \Omega)$$

The PDF $\xi(\mu|x)$ of $P[B|A]$ satisfies

$$\xi(\mu|x) \propto e^{-\Omega^{-1}(\mathbb{E}[r]-\mu)^2}$$

6. Combining Prior and Conditional Distributions: Combining the distributions $\xi(x)$ and $\xi(\mu|x)$ from above the expression for the PDF of the posterior distribution satisfies

$$\xi(x|\mu) \propto e^{-\Omega^{-1}(\mathbb{E}[r]-\mu)^2-\left[\frac{S}{n}\right]^{-1}(\mathbb{E}[r]-x)^2}$$

or

$$\xi(x|\mu) \propto e^{-\Phi}$$

7. Simplification of the Combined Exponent: Looking at the quantity in the exponent Φ and simplifying it yields

$$\Phi = \Omega^{-1}(\mathbb{E}[r] - \mu)^2 + \left[\frac{S}{n}\right]^{-1}(\mathbb{E}[r] - x)^2$$

$$\Phi = \Omega^{-1}\{(\mathbb{E}[r])^2 - 2\mu\mathbb{E}[r] + \mu^2\} + \left[\frac{S}{n}\right]^{-1}\{(\mathbb{E}[r])^2 - 2x\mathbb{E}[r] + x^2\}$$

$$\Phi = (\mathbb{E}[r])^2\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\} - 2\mathbb{E}[r]\left\{\mu\Omega^{-1} + x\left[\frac{S}{n}\right]^{-1}\right\} + \mu^2\Omega^{-1} + x^2\left[\frac{S}{n}\right]^{-1}$$

8. Introducing the Precision Weighted Mean: On introducing

$$y=\frac{\mu\Omega^{-1}+x\left[\frac{S}{n}\right]^{-1}}{\Omega^{-1}+\left[\frac{S}{n}\right]^{-1}}$$

and substituting it in the second term

$$\Phi=(\mathbb{E}[r])^2\left\{\Omega^{-1}+\left[\frac{S}{n}\right]^{-1}\right\}-2y\mathbb{E}[r]\left\{\Omega^{-1}+\left[\frac{S}{n}\right]^{-1}\right\}+\mu^2\Omega^{-1}+x^2\left[\frac{S}{n}\right]^{-1}$$

9. Simplification Using the Posterior Mean: On adding

$$0=y^2\left\{\Omega^{-1}+\left[\frac{S}{n}\right]^{-1}\right\}-\left\{\mu\Omega^{-1}+x\left[\frac{S}{n}\right]^{-1}\right\}^2\left\{\Omega^{-1}+\left[\frac{S}{n}\right]^{-1}\right\}^{-1}$$

$$\Phi=(\mathbb{E}[r])^2\left\{\Omega^{-1}+\left[\frac{S}{n}\right]^{-1}\right\}-2y\mathbb{E}[r]\left\{\Omega^{-1}+\left[\frac{S}{n}\right]^{-1}\right\}+\mu^2\Omega^{-1}+x^2\left[\frac{S}{n}\right]^{-1}$$
$$+y^2\left\{\Omega^{-1}+\left[\frac{S}{n}\right]^{-1}\right\}+\left\{\mu\Omega^{-1}+x\left[\frac{S}{n}\right]^{-1}\right\}^2\left\{\Omega^{-1}+\left[\frac{S}{n}\right]^{-1}\right\}$$

$$\Phi=(\mathbb{E}[r])^2\left\{\Omega^{-1}+\left[\frac{S}{n}\right]^{-1}\right\}-2y\mathbb{E}[r]\left\{\Omega^{-1}+\left[\frac{S}{n}\right]^{-1}\right\}+y^2\left\{\Omega^{-1}+\left[\frac{S}{n}\right]^{-1}\right\}+\mu^2\Omega^{-1}$$
$$+x^2\left[\frac{S}{n}\right]^{-1}-\left\{\mu\Omega^{-1}+x\left[\frac{S}{n}\right]^{-1}\right\}^2\left\{\Omega^{-1}+\left[\frac{S}{n}\right]^{-1}\right\}^{-1}$$

$$\Phi=\left\{\Omega^{-1}+\left[\frac{S}{n}\right]^{-1}\right\}\{(\mathbb{E}[r])^2-2y\mathbb{E}[r]+y^2\}+\mu^2\Omega^{-1}+x^2\left[\frac{S}{n}\right]^{-1}$$
$$-\left\{\mu\Omega^{-1}+x\left[\frac{S}{n}\right]^{-1}\right\}^2\left\{\Omega^{-1}+\left[\frac{S}{n}\right]^{-1}\right\}^{-1}$$

$$\Phi = \left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}\{(\mathbb{E}[r])^2 - 2y\mathbb{E}[r] + y^2\} - \left\{\mu\Omega^{-1} + x\left[\frac{S}{n}\right]^{-1}\right\}^2 \left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}^{-1}$$
$$+ \left\{\mu^2\Omega^{-1} + x^2\left[\frac{S}{n}\right]^{-1}\right\}\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}^{-1}$$

$$\Phi = \left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}\{(\mathbb{E}[r])^2 - 2y\mathbb{E}[r] + y^2\}$$
$$- \left\{\mu^2\Omega^{-2} + 2\mu x\Omega^{-1}\left[\frac{S}{n}\right]^{-1} + x^2\left[\frac{S}{n}\right]^{-2}\right\}\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}^{-1}$$
$$+ \left\{\Omega^{-2}\mu^2 + x^2\left[\frac{S}{n}\right]^{-1}\Omega^{-1} + \mu^2\Omega^{-1}\left[\frac{S}{n}\right]^{-1} + x^2\left[\frac{S}{n}\right]^{-2}\right\}\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}^{-1}$$

$$\Phi = \left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}\{(\mathbb{E}[r])^2 - 2y\mathbb{E}[r] + y^2\}$$
$$+ \left\{\left[\frac{S}{n}\right]^{-1}\Omega^{-1}x^2 + \mu^2\Omega^{-1}\left[\frac{S}{n}\right]^{-1} - 2\mu x\Omega^{-1}\left[\frac{S}{n}\right]^{-1}\right\}\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}^{-1}$$

$$\Phi = \left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}\{(\mathbb{E}[r])^2 - 2y\mathbb{E}[r] + y^2\} + \left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}^{-1}(x-\mu)\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}$$

10. <u>The Posterior Mean and Variance</u>: The second term has no dependency on $\mathbb{E}[r]$ and thus can be included in the proportionality factor. One is left with

$$\xi(x|\mu) \propto e^{-\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}^{-1}(\mathbb{E}[r]-y)^2}$$

Thus the posterior mean is y as defined in

$$y = \frac{\mu\Omega^{-1} + x\left[\frac{S}{n}\right]^{-1}}{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}}$$

and the corresponding variance is $\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}^{-1}$

Using Bayes' Theorem for the Estimation Model

1. Specification of the Prior Distribution: In the Black-Litterman model the prior distribution is based on the equilibrium implied excess returns. One of the major assumption made by the Black-Litterman model is that the covariance of the prior estimate is proportional to the covariance of the actual returns, but the two quantities are independent. The parameter τ will serve as the constant of proportionality. The prior distribution for the Black-Litterman model was specified using

$$P[A] \sim \mathcal{N}(\Pi, \tau\mathcal{H})$$

and

$$r_A \sim \mathcal{N}(P[A], \mathcal{H})$$

2. Specifying Views Conditional on Prior: The conditional distribution is based on the investors' views. The investors' views are specified as returns to portfolio of assets, and each view has an uncertainty that will impact the overall mixing process. The conditional distribution from the investors' views was specified in

$$P[B|A] \sim \mathcal{N}(PQ, [P^T\Omega^{-1}P]^{-1})$$

3. Blending Conditional and Prior Views: The posterior distribution from the Bayes' Theorem is the precision weighted mean of the prior estimate and the conditional estimate. Therefore

one can readily apply Bayes' Theory to the problem of blending the prior and the conditional distributions to create new posterior distribution of asset returns.

4. Posterior Distribution of Asset Returns: Given

$$P[A] \sim \mathcal{N}(\Pi, \tau\mathcal{H})$$

$$r_A \sim \mathcal{N}(P[A], \mathcal{H})$$

and

$$P[B|A] \sim \mathcal{N}(P^{-1}Q, [P^T\Omega^{-1}P]^{-1})$$

for the prior and the conditional distributions respectively, one can apply Bayes' Theorem and derive the following expression for the posterior distributions of the asset returns.

$$P[B|A] \sim \mathcal{N}(\{[\tau\mathcal{H}]^{-1}\Pi + P^T\Omega^{-1}Q\}\{[\tau\mathcal{H}]^{-1} + P^T\Omega^{-1}P\}^{-1}, \{[\tau\mathcal{H}]^{-1} + P^T\Omega^{-1}P\}^{-1})$$

This is sometimes referred to as the Black-Litterman master formula.

5. Alternate Representation of the Posterior Returns: An alternate representation of the same expression for the mean returns $\widehat{\Pi}$ and covariance M is

$$\widehat{\Pi} = \Pi + \tau\Psi P^T[P\tau\Psi P^T + \Omega]^{-1}[Q - P\Pi]$$

and

$$M = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}$$

To re-iterate the posterior M is the variance of the posterior mean about the actual mean. It is the uncertainty in the posterior mean estimate, and not the gross variance of the returns.

6. Derivation of the Alternate Representation: This part contains the derivation of the alternate representation of the Black-Litterman master formula for the posterior expected returns. Starting from

$$P[A|B] \sim \mathcal{N}([(\tau\Psi)^{-1}\Pi + P^T\Omega^{-1}Q][(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}, [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1})$$

one derives

$$\mathbb{E}[r] = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Psi)^{-1}\Pi + P^T\Omega^{-1}Q]$$

7. Separating the Covariance Term: Separating parts of the second term

$$\mathbb{E}[r] = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}(\tau\Psi)^{-1}\Pi + [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}P^T\Omega^{-1}Q$$

8. Replacing Precision in the First Term: Replacing the precision term in the first term with the alternate form one gets

$$\mathbb{E}[r] = \{(\tau\Psi - \tau\Psi P^T[P\tau P^T + \Omega]^{-1}P\tau\Psi)(\tau\Psi)^{-1}\Pi\} + \{[(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}P^T\Omega^{-1}Q\}$$

$$\mathbb{E}[r] = \{\Pi - (\tau\Psi P^T[P\tau\Psi P^T + \Omega]^{-1}P\Pi)\} + \{[(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}P^T\Omega^{-1}Q\}$$

$$\mathbb{E}[r] = \{\Pi - (\tau\Psi P^T[P\tau\Psi P^T + \Omega]^{-1}P\Pi)\} + \{(\tau\Psi)(\tau\Psi)^{-1}[(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}P^T\Omega^{-1}Q\}$$

$$\mathbb{E}[r] = \{\Pi - (\tau\Psi P^T[P\tau\Psi P^T + \Omega]^{-1}P\Pi)\} + \{(\tau\Psi)[I_n + P^T\Omega^{-1}P\tau\Psi]^{-1}P^T\Omega^{-1}Q\}$$

$$\mathbb{E}[r] = \{\Pi - (\tau\Psi P^T[P\tau\Psi P^T + \Omega]^{-1}P\Pi)\} + \{\tau\Psi[I_n + P^T\Omega^{-1}P\tau\Psi]^{-1}[\Omega(P^T)^{-1}]^{-1}Q\}$$

$$\mathbb{E}[r] = \{\Pi - (\tau\Psi P^T[P\tau\Psi P^T + \Omega]^{-1}P\Pi)\} + \{\tau\Psi[\Omega(P^T)^{-1} + P\tau\Psi]^{-1}Q\}$$

$$\mathbb{E}[r] = \{\Pi - (\tau\Psi P^T[P\tau\Psi P^T + \Omega]^{-1}P\Pi)\} + \{\tau\Psi P^T(P^T)^{-1}[\Omega(P^T)^{-1} + P\tau\Psi]^{-1}Q\}$$

$$\mathbb{E}[r] = \{\Pi - (\tau\Psi P^T[P\tau\Psi P^T + \Omega]^{-1}P\Pi)\} + \{\tau\Psi P^T[\Omega + P\tau\Psi P^T]^{-1}Q\}$$

9. <u>Alternate Black Litterman Formula Form</u>: Thus the alternate form of the Black-Litterman expression expected returns is

$$\mathbb{E}[r] = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Psi)^{-1}\Pi + P^T\Omega^{-1}Q]$$

10. <u>Posterior Covariance of Returns</u>: Computing the posterior covariance of the returns requires adding the variance of the estimate about the mean to the variance of the distribution about the estimate as seen in

$$V([y \quad \Pi]) = \Psi + V[\tilde{u}]$$

This is mentioned in He and Litterman (1999) but not in the other papers.

$$\Psi_P = \Psi + M$$

Substituting the posterior variance from

$$M = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}$$

results in

$$\Psi_P = \Psi + [(\tau\Omega)^{-1} + P^T\Omega^{-1}P]^{-1}$$

11. <u>Posterior Covariance without Views</u>: In the absence of views this reduces to

$$\Psi_P = \Psi + \tau\Psi$$

or

$$\Psi_P = (1 + \tau)\Psi$$

Thus when applying the Black-Litterman model in the absence of views the variance of the estimated returns will be greater than the prior distribution variance. The impact of this formula can be seen in the results shown in He and Litterman (1999) where the investors' weights sum to less than 1 if they have no views. Idzorek (2005) and most other authors do not compute a new posterior variance but instead use a known input variance of the returns about the mean.

12. Views on Asset Subset: In the event that the investor only has partial views – that is views on a subset of assets – usage of the posterior estimate of the variance tilts the posterior weights towards assets with lower variance and higher precision of the estimated mean, and away from higher variance – lower precision of the estimated mean. Thus the existence of views and of the updated covariance will tilt the optimizer towards using or not using those assets. Thus tilt may not be very large if τ is small, but will still be measurable.
13. Estimation of the Prior Covariance: Since one often builds the unknown covariance matrix of returns Ψ from historical data, basic methods of statistics may be used to compute τ. τ may also be estimated from the confidence of the prior distribution. Both of these provide an intuition for selecting a value of τ that is closer to 0 than to 1.
14. Choice of τ used in the Literature: Black and Litterman (1992), He and Litterman (1999), and Idzorek (2005) all indicate that they used small values of τ in their calculations, on the order of $0.025 - 0.050$ Satchell and Scowcroft (2000) state that many investors use a τ around 1, which has no intuitive connection to data, and in fact demonstrates that their paper uses the Alternate Reference Model.
15. Boundary Conditions 100% View Uncertainty: The results can be checked to see if they correspond to the intuition around the boundary conditions. Given

$$\hat{\Pi} = \Pi + \tau\Psi P^T[P\tau\Psi P^T + \Omega]^{-1}[Q - P\Pi]$$

it is easy to see that letting

$$\Omega \to 0$$

corresponding to 100% certainty of views results in

$$\hat{\Pi} = \Pi + \Psi P^T [P\Psi P^T]^{-1}[Q - P\Pi]$$

Thus under 100% certainty of views the estimated return is insensitive to the value of τ used. Further if P is invertible – which means that a view has been offered on every asset – then

$$\hat{\Pi} = P^{-1}Q$$

16. <u>Boundary Conditions 100% View Uncertainty</u>: If the investor is not sure about the views than

$$\Omega \to \infty$$

and

$$\hat{\Pi} = \Pi + \tau\Psi P^T [P\tau\Psi P^T + \Omega]^{-1}[Q - P\Pi]$$

reduces to

$$\hat{\Pi} = \Pi$$

17. <u>Boundary Conditions - Posterior Covariance</u>: An alternate form of the posterior covariance

$$M = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}$$

derived using the Woodbury Identity Matrix is

$$M = \tau\Psi - \tau\Psi P^T [\Omega + P\tau\Psi P^T]^{-1} P\tau\Psi$$

If

$$\Omega \to 0$$

– total confidence in views, plus every asset is in at least one view, the above reduces to

$$M = 0$$

If on the other hand the investor is not confident on the views

$$\Omega \to \infty$$

and the above reduces to

$$M = \tau\Psi$$

The Alternate Reference Model

1. The Satchell and Scowcroft Model: The most common Alternate Reference Model is the one used in Satchell and Scowcroft (2000) and in the work of Meucci prior to his introduction of *Beyond Black Litterman*.

$$\mathbb{E}[r] = \sim\mathcal{N}(\mu, \Psi)$$

In this model $\mathbb{E}[r]$ is normally distributed with a variance Ψ While μ is estimated, it is not considered a random variable. This is commonly referred to as having

$$\tau = 1$$

but the estimate is a point estimate, thus the parameter τ gets eliminated.

2. Interpretation of the Model Ω: In this model Ω becomes the covariance of returns to the views around the unknown mean return, just as Ψ is the covariance of the prior around its mean. Given that point estimates are now being used, the posterior now is a point estimate, and the posterior covariance is no longer being estimated. Corresponding there is no posterior precision to use downstream in the portfolio selection model.

3. Scaling the Covariance by τ: On re-writing

$$M = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}$$

   and moving τ around one gets

$$\hat{\Pi} = \Pi + \Psi P^T \left[P\tau\Psi P^T + \frac{\Omega}{\tau}\right]^{-1} [Q - P\Pi]$$

   Here τ appears on only term in the formula. Because the Alternate Reference Model does not include updating the covariance of the estimates this is the only formula.

4. Recast of the Posterior Mean: Given that the investor selects both Ω and τ to control the blending of the prior and the views, one of these terms can be eliminated. Since τ is a single term for all the views and Ω has a separate term for each view Ω will be retained. The posterior estimate of the mean can be re-written as

$$\hat{\Pi} = \Pi + \tau\Psi P^T[P\tau\Psi P^T + \Omega]^{-1}[Q - P\Pi]$$

5. Elimination of the Posterior Variance: As a note none of the authors prior to Meucci (2008) except for Black and Litterman (1992) and He and Litterman (1999) make any mention of the details of the Canonical Reference Model or of the fact that different authors actually use quiet different reference models.

6. Updated Posterior Covariance Estimate: In the Canonical Reference Model the updated posterior covariance of the unknown mean about the estimate will be smaller than the

covariance of either the prior or the conditional estimates indicating that the addition of more estimates will reduce the uncertainty of the model. The posterior variance of the returns

$$\Psi_p = \Psi + M$$

will never be lesser than the prior variance of the returns.

7. Improved Estimate of the Returns Variance: This matches intuition as adding more information reduces the uncertainty of the estimates. Given that there is some uncertainty in the value M

$$\Psi_p = \Psi + M$$

provides a better estimate of the variance of returns than the prior variance of the returns.

The Impact of τ

1. Canonical/Alternate Reference Models Usage: The meaning and impact of the parameter τ causes a great deal of confusion for many users of the Black-Litterman Model. It is obvious that investors using the Canonical Reference Model use τ and that it has a very precise meaning in that model. An author that selects essentially a random value for τ is not using the Canonical Reference Model, but is instead using the Alternate Reference Model.
2. τ in the Canonical Reference Model: Given the Canonical Reference Model one can still perform an exercise to understand the impact of τ on the results. Starting with an expression for Ω similar to the one used by He and Litterman (1999), rather than using only the diagonal, the entire structure of the covariance matrix is retained.
3. He and Litterman Posterior Returns: Substituting

$$\Omega = P\tau\Psi P^T$$

into

$$\hat{\Pi} = \Pi + \tau \Psi P^T [P \tau \Psi P^T + \Omega]^{-1} [Q - P\Pi]$$

one gets

$$\begin{aligned}\hat{\Pi} = \Pi + \tau \Psi P^T [P \tau \Psi P^T + \Omega]^{-1} [Q - P\Pi] &= \Pi + \tau \Psi P^T [P \tau \Psi P^T + P \tau \Psi P^T]^{-1} [Q - P\Pi] \\ &= \Pi + \tau \Psi P^T [2 P \tau \Psi P^T]^{-1} [Q - P\Pi] = \Pi + \frac{1}{2} \tau \Psi P^T (P^T)^{-1} [P \tau \Psi]^{-1} [Q - P\Pi] \\ &= \Pi + \frac{1}{2} \tau \Psi [\tau \Psi]^{-1} P^{-1} [Q - P\Pi] = \Pi + \frac{1}{2} P^{-1} [Q - P\Pi]\end{aligned}$$

i.e.

$$\hat{\Pi} = \Pi + \frac{1}{2} [P^{-1} Q - \Pi]$$

4. He and Litterman Proportional τ: Clearly using

$$\Omega = P[\tau \Psi] P^T$$

is a simplification, but shows that setting Ω proportional to τ eliminates it from the final formula for $\hat{\Pi}$. However in the Canonical Reference Model it does not eliminate τ from the posterior covariance expression

$$\Psi_p = \Psi + M$$

5. Proportional τ-Generated Posterior Returns: In general if Ω is formulated as

$$\Omega = P[\alpha \tau \Psi] P^T$$

then

$$\hat{\Pi} = \Pi + \frac{1}{2}[P^{-1}Q - \Pi]$$

can be written as

$$\hat{\Pi} = \Pi + \frac{1}{1+\alpha}[P^{-1}Q - \Pi]$$

6. Proportional τ Posterior Covariance: Similarly substituting

$$\Omega = P[\tau\Psi]P^T$$

into

$$M = \tau\Psi - \tau\Psi P^T[\Omega + P\tau\Psi P^T]^{-1}P\tau\Psi$$

produces a similar result

$$M = \tau\Psi - \tau\Psi P^T[P\tau\Psi P^T + P\tau\Psi P^T]^{-1}P\tau\Psi = \tau\Psi - \tau\Psi P^T[2P\tau\Psi P^T]^{-1}P\tau\Psi$$
$$= \tau\Psi - \frac{1}{2}\tau\Psi P^T[P^T]^{-1}[\tau\Psi]^{-1}P^{-1}P\tau\Psi = \tau\Psi - \frac{1}{2}\tau\Psi = \frac{1}{2}\tau\Psi$$

Thus

$$M = \frac{1}{2}\tau\Psi$$

in this case.

7. τ Impact on the Posterior Covariance: Note that τ is not eliminated from M. It may also be observed that if τ is of the order of 1 and one were to use

$$\Psi_p = \Psi + M$$

the uncertainty on the estimate of means would be a significant part of the variance of returns. With Alternate Reference Model no posterior variance calculations are performed and the mixing is weighted by the variance of the returns.

8. <u>He and Litterman Symmetric τ</u>: In both cases the choice for τ has evenly weighted prior and conditional distributions in the estimate of the posterior distribution. This matches the intuition considering that two impacts have been blended and both have the same level of uncertainty. The posterior will be an average of the two distributions.
9. <u>Meuccis's Scaling of Prior τ</u>: If one solves for the more general case of

$$\Omega = P[\alpha\tau\Psi]P^T$$

where

$$\alpha \geq 0$$

substitution into

$$\hat{\Pi} = \Pi + \tau\Psi P^T[P\tau\Psi P^T + \Omega]^{-1}[Q - P\Pi]$$

and following the same logic used to derive

$$M = \frac{1}{2}\tau\Psi$$

results in

$$\hat{\Pi} = \Pi + \frac{1}{1+\alpha}[P^{-1}Q - \Pi]$$

10. Impact on Stability of Results: This parametrization of the uncertainty is specified in Meucci (2005) and offers an option between using the same uncertainty for the prior and the views versus having to specify a separate and unique uncertainty for each view. Given that the prior covariance matrix is essentially being multiplied by a constant, this parametrization of the uncertainty of views does not have a negative impact on the stability of the results.
11. Impact on the View Covariance: Note that this specification of the uncertainty in the views changes the assumption from the views being uncorrelated to one where the views have the same correlations as the prior returns.
12. Cross Model τ Usage – Summary: In summary if the investor uses the Alternate Reference Model and makes Ω proportional to Ψ then they only need to calibrate the constant of proportionality α that indicates their relative confidence in their views versus the equilibrium. If they use the Canonical Reference Model and set Ω proportional to $\tau\Psi$ then the returns estimate does not depend on τ but the posterior covariance does.

Calibration of τ

1. τ from Daily Returns Series: The first method for calibrating τ relies on falling back to basic statistics. When estimating the mean of a distribution the uncertainty of the mean estimate will be proportional to the number of samples. Given that the covariance matrix is estimated from historical data

$$\tau = \frac{1}{T}$$

results from the maximum likelihood estimator;

$$\tau = \frac{1}{T - k}$$

results from the best quadratic unbiased estimator. Here T is the number of samples and k is the number of assets.

2. Typical Estimates for τ Values: There are a number of other estimators, but usually the maximum likelihood estimator above is the one used. Given that one usually aims for a sample number of around 60 - 5 years of monthly samples - τ is on the order of 0.02. This is consistent with several papers that indicate they used τ values in the range of $0.025 - 0.050$.
3. Using Intuition in τ Calibration: The easiest and the most intuitive way to calibrate τ is as part of a confidence interval for the prior mean estimates. This concept can be illustrated using a simple example. The scenario used is one where

$$\tau = 0.05$$

with a single asset having a prior estimate of 7.0% for the excess returns and 15.0% is the known standard deviation of the returns around the mean. The confidence interval used is $(1.0\%, 5.0\%)$ with 68.0% confidence.
4. Prior Distribution Impact - First Scenario: Two scenarios respectively with

$$\tau = 0.05$$

and

$$\tau = 1.00$$

are considered, with the ratio of View Precision to Prior Precision held constant across them. Even though the first scenario uses a seemingly small

$$\tau = 0.05$$

the prior estimate has a relatively low precision based on the width of the confidence interval, and the posterior estimate will be heavily weighted towards the view.

5. Prior Distribution Impact - Second Scenario: In the second scenario

$$\tau = 1$$

the prior confidence interval is so worthless as to make the prior estimate worthless. In order to keep the posterior estimate the same across the scenarios the view estimate also has a wide confidence interval indicating that the investor is not really confident in any of their estimates.

6. Table Depicting Prior/View Confidence:

$\boldsymbol{\tau}$	**Prior at 68% Confidence**	**Prior Precision**	**View $\boldsymbol{\sigma}$**	**View Precision**	**View at 68% Confidence**	**View/Prior Precision**
0.05	$(4.60\%, 9.40\%)$	888	2.00%	2500	$(1.00\%, 5.00\%)$	2.81
1.00	$(-8.00\%, 22,00\%)$	44.40	8.90%	125	$(-5.90\%, 11.90\%)$	2.81

7. View/τ Variance Interplay: Given such wide intervals for the

$$\tau = 1$$

scenario - 16% chance that the asset has mean returns less than -8.00% - it is hard to imagine having much convection in using the final asset allocation. Thus the interplay between the selection of τ and the specification of the variance of the views is critical.

8. Model Specific τ Usage Impact: Further this example illustrates the difference between using the Alternate Reference Model and the Canonical Reference Model. Specifying

$$\tau = 1$$

is the Alternate Reference Model, but results in very uncertain outputs in the Canonical Reference Model.

9. Calibration from the CAPM Market Weights: Finally τ could be calibrated from the amount invested in the risk free asset given the prior distribution. It can be seen here that he portfolio invested in risky assets given the prior weights will be

$$w = \Pi[\delta(1+\tau)\Psi]^{-1}$$

Thus the weights allocated to the assets are smaller by $\frac{1}{1+\tau}$ than the CAPM market weights. This is because the Bayesian investor is uncertain in the estimate of the prior and does not want to be 100% invested in the risky assets.

Black Litterman Model Implementation Steps

1. CAPM Asset Class Equilibrium Weights: w_{eq} is the Equilibrium Weight for each asset class. It is derived from the capitalization weighted CAPM market portfolio.
2. Historical Asset Returns Covariance: Ψ is the matrix of covariance between the asset classes, and is computed from the historical data.
3. Risk Free Asset Class Yield: r_f is the risk-free rate for the base currency.
4. Market Portfolio Risk Aversion Coefficient: δ is the risk aversion coefficient of the market portfolio. This can be computed using the returns and the standard deviation of the market portfolio, or can be extraneously supplied.
5. Prior Returns Distribution Covariance: τ represents the measure of uncertainty of the equilibrium variance. This is usually set to a small number of the order of $0.025 - 0.050$
6. Setting P, Ω, and Q: First the vector of equilibrium returns Π is computed using reverse optimization from

$$\Pi = \delta\Psi w$$

Then the investors views are formulated, and P, Ω, and Q are specified. Given k views and n assets, P is a $k \times n$ matrix where each row sums to 0 for a relative view and to 1 for an absolute view. Q is a $k \times 1$ vector of excess returns for each view. Ω is a diagonal $k \times k$ matrix of the variance of the views, i.e., the confidence in the views. As a starting point most authors call for the values of w_i to be set equal to $p^T \tau \Psi_i p$ where p is the row from P for the specific view.

7. Applying the Black Litterman Master Formula: Assuming uncertainty in all the views the Black Litterman master formula may be applied to compute the joint returns:

$$\hat{\Pi} = \Pi + \tau \Psi P^T [P \tau \Psi P^T + \Omega]^{-1} [Q - P\Pi]$$

8. Estimation of the Joint Returns Covariance: The joint covariance is computed from

$$M = \tau \Psi - \tau \Psi P^T \left[\frac{\Omega}{\tau} + P \tau \Psi P^T\right]^{-1} P \Psi$$

9. Computation of the Posterior Covariance: After the above step the sample posterior covariance is computed from

$$\Psi_p = \Psi + M$$

10. Computing the Optimal Portfolio Weights: Finally the weights are computed for the optimal portfolio on the unconstrained (or with constraints, as the case may be) efficient frontier using

$$\hat{w} = \left[\delta \Psi_p\right]^{-1} \hat{\Pi}$$

Extensions to the Black-Litterman Model

1. <u>Principal Black Litterman Model Extensions</u>: The principal extensions to the Black Litterman approach covered over a couple of sections and chapters are the ones proposed by Qian and Gorman (2001), Fusai and Meucci (2003), Idzorek (2005), and Krishnan and Mains (2006).
2. <u>Idzorek's Method for Variance Calibration</u>: Idzorek (2005) presents a means to calibrate the confidence or the variance of the investors' views in a simple and straightforward method. This is treated in detail in a later chapter.
3. <u>Estimation of Quality of Views</u>: Next is a section on measures of extremity or of quality of views. Fusai and Meucci (2003) propose a way to measure how consistent a posterior estimate of the mean is with regards to the prior, or some other estimate. Braga and Natale (2007) describe how to use tracking error to measure the distance from the equilibrium to the posterior portfolio. Walters (2014) provides additional original work on using relative entropy to measure quality of views.
4. <u>Multi-Factor Black Litterman Returns</u>: Finally larger extensions to the model such as the one by Krishnan and mains (2006) demonstrate a method to incorporate additional factors into the model. Qian and Gorman (2001) present a method to integrate views on the covariance matrix as well as on the returns.

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The Intuition behind Black Litterman Model Portfolios

Purpose, Drivers, and Primary Motivation

1. Principle Intuition behind Black Litterman: He and Litterman (1999) demonstrate that the optimal portfolios generated by Black-Litterman asset allocation model have a simple and intuitive property. The unconstrained optimal portfolio in the Black-Litterman model is the scaled market equilibrium portfolio – reflecting the uncertainty in the equilibrium expected returns – plus a weighted sum of portfolios representing the investors' views.
2. Projection mode Bullish than the Unadjusted Portfolio: The weight on a portfolio representing a view is positive when the view is more bullish than the one implied by the equilibrium and the other views.
3. More Confidence in the Projection: The weight increases as the investor becomes more bullish in the view, and the magnitude of the weight also increases as the investor becomes more bullish about the view.

Analyzing the Unconstrained Optimal Portfolio

1. Fundamental Black Litterman Posterior Expressions: In practice it is quiet straightforward to apply

$$\bar{\mu} = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Psi)^{-1}\Pi + P^T\Omega^{-1}Q]$$

for calculating the mean of the expected returns. However it is often difficult to find the original economic intuitions of the views from these numbers, particularly when the number of assets is very large.

2. Cross Asset Covariance Dependencies: The trouble with trying to make sense of these expected returns of different asset classes is that they are related to each other through their relative volatilities and correlations.
3. Impact of German Outperformance: For example it is easy to conjecture that the German equity market will outperform the rest of the European markets, but it is much less to imagine what the implication of that conjecture is for the relative expected returns of the German and the other markets.
4. Cross Asset Correlation/Volatilities Impact: To increase the expected returns on the German equities and hold all the other expected returns does not, in the mathematics of the optimizer, suggest an overweight on German equities – rather it suggests to the optimizer that by using the relative volatilities and the correlations of different markets it can create a much more complicated portfolio with higher expected return and lower risk than would be available by simply creating an overweight of German equities.
5. Projection Representation under Black Litterman: In Black Litterman approach a view that the German equities will outperform the rest of Europe is expressed as an expectation of positive return on a portfolio consisting of a long position in German equities and market capitalization weighted short positions in the rest of the European markets.
6. Adjustments to the Expected Returns Vector: This view projection is then translated using

$$\bar{\mu} = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Psi)^{-1}\Pi + P^T\Omega^{-1}Q]$$

   which appropriately takes volatilities and correlations into account into the expected returns in all of the markets.
7. Components of the Expected Returns Adjustment: As shown by He and Litterman (1999) these adjustments to the expected returns are exactly those needed to suggest to the optimizer that the best opportunity is a simple overweighting of the German equity market, financed by the underweighting of the rest of European markets.

8. <u>Weights in Place of Returns</u>: These types of complex transformations, from views on portfolios to the expected returns vector, and from the expected returns vector to the optimal portfolio, are generally difficult to understand. Thus instead of looking at the expected returns directly, He and Litterman (1999) examine Black-Litterman optimal portfolio weights.
9. <u>Black Litterman Asset Allocation Insights</u>: They begin with the case of an unconstrained investor with representative risk aversion parameter equal to δ. The optimal portfolio then provides some very interesting insights into the Black-Litterman asset allocation model.
10. <u>Prior Distribution of the Returns Vector</u>: Because the expected returns themselves are random variables in the Black Litterman Model the distribution of returns is no longer simply $\mathcal{N}(\bar{\mu}, \Psi)$. Using

$$r \sim \mathcal{N}(\mu, \Psi)$$

$$\bar{\mu} = [(\tau\Psi)^{-1} + P^T \Omega^{-1} P]^{-1} [(\tau\Psi)^{-1} \Pi + P^T \Omega^{-1} Q]$$

and

$$\bar{M}^{-1} = [(\tau\Psi)^{-1} + P^T \Omega^{-1} P]^{-1}$$

the posterior distribution of the returns becomes

$$r \sim \mathcal{N}(\bar{\mu}, \bar{\Psi})$$

where

$$\bar{\Psi} = \Psi + \bar{M}^{-1}$$

11. <u>Risk Aversion Based Optimal Portfolio</u>: Given the mean $\bar{\mu}$ and the covariance $\bar{\Psi}$ the optimal portfolio can be constructed using the standard mean-variance optimization method. For an investor with a risk aversion parameter δ the maximization problem can be written as

$$\max\left[w^T\bar{\mu} - \frac{\delta}{2}w^T\overline{\Psi}w\right]$$

12. Expression for Posterior Optimal Weights: The first order condition yields that

$$\bar{\mu} = \delta\overline{\Psi}w^*$$

or equivalently

$$w^* = \frac{1}{\delta}\overline{\Psi}^{-1}\bar{\mu}$$

where w^* is the vector of the optimal portfolio weights. Using

$$\bar{\mu} = [(\tau\mathit{\Psi})^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\mathit{\Psi})^{-1}\mathit{\Pi} + P^T\Omega^{-1}Q]$$

the optimal portfolio weights can be written as

$$w^* = \frac{1}{\delta}\overline{\Psi}^{-1}\overline{M}^{-1}[(\tau\mathit{\Psi})^{-1}\mathit{\Pi} + P^T\Omega^{-1}Q]$$

13. Recast for Posterior Optimal Weights: From the fact that

$$\overline{\Psi}^{-1} = [\Psi + \overline{M}^{-1}]^{-1} = \overline{M} - [\overline{M} + \Psi^{-1}]^{-1}\overline{M}$$

the term $\overline{\Psi}^{-1}\overline{M}^{-1}$ can be simplified as

$$\overline{\Psi}^{-1}\overline{M}^{-1} = \frac{\tau}{1+\tau}\left[1 - P^TA^{-1}P\frac{\Psi}{1+\tau}\right]$$

where the matrix

$$A = \frac{\Omega}{\tau} + P\frac{\Psi}{1+\tau}P^T$$

14. Posterior Portfolio Factor Attribution Components: The optimal portfolio weights

$$w^* = \frac{1}{\delta}\bar{\Psi}^{-1}\bar{\mu}$$

now can be written as

$$w^* = \frac{1}{1+\tau}\left[w_{eq} + P^T\Lambda\right]$$

where

$$w_{eq} = [\delta\Psi]^{-1}\Pi$$

is the market equilibrium portfolio and Λ is a vector defined as

$$\Lambda = \frac{\tau\Omega^{-1}Q}{\delta} - A^{-1}P\frac{\Psi}{1+\tau}w_{eq} - A^{-1}P\frac{\Psi}{1+\tau}P^T\frac{\tau\Omega^{-1}Q}{\delta}$$

15. Projection Portfolio Pick Vector Impact: Since each column of the matrix P^T is a new portfolio

$$w^* = \frac{1}{1+\tau}\left[w_{eq} + P^T\Lambda\right]$$

shows that the investor's optimal portfolio is the market equilibrium portfolio w_{eq} plus a weighted sum of the portfolios forming the views, then scaled by a factor of $\frac{1}{1+\tau}$. The weight for each portfolio is given by the corresponding element in the vector Λ.

16. First Component Impact on Projection: There are intuitive interpretations of the weights in

$$\Lambda = \frac{\tau\Omega^{-1}Q}{\delta} - A^{-1}P\frac{\Psi}{1+\tau}w_{eq} - A^{-1}P\frac{\Psi}{1+\tau}P^T\frac{\tau\Omega^{-1}Q}{\delta}$$

The first term shows that the stronger the view is, i.e., either with a higher expected return q_k or with a lower level of uncertainty and a higher value of the precision $\frac{w_k}{\tau}$ the more the weight it carries in the final optimal portfolio.

17. Second Component Market Projection Impact: The second term shows that the weight of a view is penalized for the covariance between the view portfolio and the market equilibrium portfolio. Since the market equilibrium information is already presented by the prior, the covariance of the view portfolio with the market equilibrium portfolio indicates that the view carries less new information and the penalty makes sense.

18. Third Component Cross Projection Impact: Similarly the last term shows that the weight is penalized for the covariance of a view portfolio with other view portfolios. Again since the covariance between these view portfolios indicates in a sense that the information is being double counted, it is intuitive that there should be a penalty on the weight associated with an increased covariance with other view portfolios.

19. Risk Aversion Posterior Weights Scaler: For an investor with a different risk tolerance the optimal portfolio weight $\widehat{w}^*$ can be obtained by scaling the portfolio w^*

$$\widehat{w}^* = \frac{\delta}{\hat{\delta}}w^*$$

where $\hat{\delta}$ is the risk aversion parameter for the investor.

20. Total Limit Risk Posterior Scaler: For an investor with a fixed risk, i.e., standard deviation, limit σ the portfolio optimization is formulated as

$$\max_{w^T\Psi w \le \sigma^2} w^T\bar{\mu}$$

The optimal portfolio $\widetilde{w}^*$ can be obtained by scaling the portfolio w^*

$$\widetilde{w}^* = \frac{\sigma\delta}{\sqrt{\bar{\mu}^T \bar{\Psi} \bar{\mu}}} w^*$$

21. Posterior Distribution Based Constrained Allocation: For an investor with other constraints on the portfolio the optimal portfolio can be obtained by using the usual portfolio optimization package with $\bar{\mu}$ and $\bar{\Psi}$ as inputs.
22. Projection Impact on Posterior Portfolio: Since the weight Λ plays a very important role in the portfolio construction process, one needs to know when the weight is positive and how the weight changes. He and Litterman (1999) illustrate the impact using the two properties discussed below.

Impact of an Incremental Projection

1. Parameters of an Incremental Projection: Let P, Q, Ω and represent the K views held by the investor initially, $\bar{\mu}$ be the mean of the expected returns by using these views in the Black-Litterman model, Λ be the weight vector defined by

$$\Lambda = \frac{\tau\Omega^{-1}Q}{\delta} - A^{-1}P\frac{\Psi}{1+\tau}w_{eq} - A^{-1}P\frac{\Psi}{1+\tau}P^T\frac{\tau\Omega^{-1}Q}{\delta}$$

   The investor is now assumed to use an incremental view represented by p, q, and ω.

2. Update to the Posterior Weights: For the case of $K+1$ views the new weight vector $\widehat{\Lambda}$ is given by

$$\widehat{\Lambda} = \begin{pmatrix} \Lambda - \hat{\lambda}_{K+1}A^{-1}b \\ \hat{\lambda}_{K+1} \end{pmatrix}$$

where

$$b = P\frac{\Psi}{1+\tau}p$$

$$c = \frac{\omega}{\tau} + p^T\frac{\Psi}{1+\tau}p$$

and $\hat{\lambda}_{K+1}$ the weight on the addition view is

$$\hat{\lambda}_{K+1} = \frac{q - p^T\Psi\bar{\Psi}^{-1}\bar{\mu}}{[c - b^TA^{-1}b]\delta}$$

3. <u>Base Views Implied Expected Returns</u>: Letting

$$\tilde{\mu} = \Psi\bar{\Psi}^{-1}\bar{\mu}$$

since

$$w^* = [\delta\bar{\Psi}]^{-1}\bar{\mu}$$

is the unconstrained optimal portfolio of the Black Litterman model with the first K views

$$\tilde{\mu} = \delta\Psi w$$

can be seen as the implied expected returns of the first K views, even though $\tilde{\mu}$ is usually different from $\bar{\mu}$ due to the difference between Ψ and $\bar{\Psi}$. In the limit as τ and Ω go to zero while $\frac{\Omega}{\tau}$ stays finite, $\tilde{\mu}$ and $\bar{\mu}$ become identical.

4. <u>Bullish vs. Bearish Projection Contributions</u>: Since

$$c - b^TA^{-1}b > 0$$

$$\hat{\lambda}_{K+1} = \frac{q - p^T \Psi \bar{\Psi}^{-1} \bar{\mu}}{[c - b^T A^{-1} b]\delta}$$

shows that $\hat{\lambda}_{K+1}$, the weight on the additional view, will have the same sign as the expression $q - p^T \tilde{\mu}$. This means that the weight $\hat{\lambda}_{K+1}$ is positive (negative) when the strength of the new view on the portfolio p is more bullish (bearish) than the portfolio's implied expected returns $p^T \tilde{\mu}$.

5. Neutral Projection Contribution to Posterior: The addition view will have zero weight if

$$q = p^T \tilde{\mu}$$

   In this case the weights on the first K views are identical to the weights generated by the Black Litterman model with the first K views.

6. The Initial Projection Contribution Scaler: In the case where the investor holds the initial K views the weight vector can be written as

$$\Lambda = \frac{\tau \Omega^{-1} Q}{\delta} - A^{-1} P \frac{\Psi}{1+\tau} w_{eq} - A^{-1} P \frac{\Psi}{1+\tau} P^T \frac{\tau \Omega^{-1} Q}{\delta}$$

7. The Updated Projection Contribution Scaler: Similarly in the case where the investor holds an additional view the weight vector $\hat{\Lambda}$ is

$$\hat{\Lambda} = \frac{\tau \hat{\Omega}^{-1} \hat{Q}}{\delta} - \hat{A}^{-1} \hat{P} \frac{\Psi}{1+\tau} w_{eq} - \hat{A}^{-1} \hat{P} \frac{\Psi}{1+\tau} \hat{P}^T \frac{\tau \hat{\Omega}^{-1} \hat{Q}}{\delta}$$

   where $\hat{P}$, $\hat{Q}$, $\hat{\Omega}$, and $\hat{A}$ are counterparts of P, Q, Ω, and A in the case of $K+1$ views.

8. Components of the Incremental Projection Scaler: They can be written as

$$\hat{P} = \begin{pmatrix} P \\ p \end{pmatrix}$$

$$\hat{Q} = \begin{pmatrix} Q \\ q \end{pmatrix}$$

$$\widehat{\Omega} = \begin{pmatrix} \Omega & 0 \\ 0 & \omega \end{pmatrix}$$

and

$$\widehat{\mathrm{A}} = \begin{pmatrix} \mathrm{A} & b \\ b^T & c \end{pmatrix}$$

The assumption that the matrix $\widehat{\Omega}$ be diagonal is not needed here. It only needs to be block diagonal so that

$$\widehat{\Omega} = \begin{pmatrix} \Omega & 0 \\ 0 & \omega \end{pmatrix}$$

holds.

9. Updated Projection Contribution Scaler Decomposition: Using

$$\bar{\mu} = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Psi)^{-1}\Pi + P^T\Omega^{-1}Q]$$

after a few steps of algebra

$$\widehat{\Lambda} = \frac{\tau\widehat{\Omega}^{-1}\hat{Q}}{\delta} - \hat{A}^{-1}\hat{P}\frac{\Psi}{1+\tau}w_{eq} - \hat{A}^{-1}\hat{P}\frac{\Psi}{1+\tau}\hat{P}^T\frac{\tau\widehat{\Omega}^{-1}\hat{Q}}{\delta}$$

can be written as

$$\widehat{\Lambda} = \frac{1}{\delta}\left[\begin{pmatrix} \tau\Omega^{-1}Q \\ \tau\omega^{-1}q \end{pmatrix} - \frac{\hat{A}^{-1}}{1+\tau}\begin{pmatrix} P\bar{\mu} + P\Psi P^T\tau\Omega^{-1}\bar{\mu} + P\Psi p\omega^{-1}q \\ p^T\bar{\mu} + p^T\Psi P^T\tau\Omega^{-1}\bar{\mu} + p^T\Psi p\omega^{-1}q \end{pmatrix}\right]$$

10. Inverse of the $\hat{A}$ Matrix: The inverse of the $\hat{A}$ matrix is

$$\hat{A}^{-1} = \begin{bmatrix} A^{-1} + \dfrac{A^{-1}bb^TA^{-1}}{d} & -\dfrac{A^{-1}b}{d} \\ -\dfrac{b^TA^{-1}}{d} & \dfrac{1}{d} \end{bmatrix}$$

where

$$d = c - b^TA^{-1}b$$

11. Estimation of the Incremental Projection Contribution: Applying the last row of $\hat{A}^{-1}$ into

$$\widehat{\Lambda} = \frac{1}{\delta}\left[\begin{pmatrix} \tau\Omega^{-1}Q \\ \tau\omega^{-1}q \end{pmatrix} - \frac{\hat{A}^{-1}}{1+\tau}\begin{pmatrix} P\bar{\mu} + P\Psi P^T\tau\Omega^{-1}\bar{\mu} + P\Psi p\omega^{-1}q \\ p^T\bar{\mu} + p^T\Psi P^T\tau\Omega^{-1}\bar{\mu} + p^T\Psi p\omega^{-1}q \end{pmatrix}\right]$$

gives the last element of $\widehat{\Lambda}$ as

$$\hat{\lambda}_{K+1} = \frac{\tau\omega^{-1}q}{\delta} + \frac{b^TA^{-1} - 1}{(1+\tau)d\delta}\begin{pmatrix} P\bar{\mu} + P\Psi P^T\tau\Omega^{-1}\bar{\mu} + P\Psi p\omega^{-1}q \\ p^T\bar{\mu} + p^T\Psi P^T\tau\Omega^{-1}\bar{\mu} + p^T\Psi p\omega^{-1}q \end{pmatrix}$$

12. Simplification of Incremental Projection Contribution: After expanding the matrix multiplication and after a few steps of algebra the above equation directly leads to

$$\hat{\lambda}_{K+1} = \frac{q - p^T\Psi\bar{\Psi}^{-1}\bar{\mu}}{[c - b^TA^{-1}b]\delta}$$

which is what was seen earlier.

13. Estimation of the Updated Projection Contribution: Likewise applying the first K rows of $\hat{A}^{-1}$ in

$$\widehat{\Lambda} = \frac{1}{\delta}\left[\begin{pmatrix} \tau\Omega^{-1}Q \\ \tau\omega^{-1}q \end{pmatrix} - \frac{\hat{A}^{-1}}{1+\tau}\begin{pmatrix} P\bar{\mu} + P\Psi P^T\tau\Omega^{-1}\bar{\mu} + P\Psi p\omega^{-1}q \\ p^T\bar{\mu} + p^T\Psi P^T\tau\Omega^{-1}\bar{\mu} + p^T\Psi p\omega^{-1}q \end{pmatrix}\right]$$

gives the first K elements of $\widehat{\Lambda}$ as

$$\widetilde{\Lambda} = \tau\Omega^{-1}Q$$
$$-\left(A^{-1} + \frac{A^{-1}bb^TA^{-1}}{c - b^TA^{-1}b} \quad - \frac{A^{-1}b}{c - b^TA^{-1}b}\right)\begin{pmatrix} P \\ p^T \end{pmatrix}\frac{\Psi}{1+\tau}\left[\delta w_{eq} + P^T\tau\Omega^{-1}Q\right.$$
$$\left. + p\tau\omega^{-1}q\right]$$

14. Recast of the Projection Contribution Update: After some simple but lengthy algebra $\widetilde{\Lambda}$ can be simplified into the following form:

$$\widetilde{\Lambda} = \Lambda - \hat{\lambda}_{K+1}A^{-1}b$$

Combining

$$\hat{\lambda}_{K+1} = \frac{q - p^T\Psi\overline{\Psi}^{-1}\bar{\mu}}{[c - b^TA^{-1}b]\delta}$$

and the expression for $\widetilde{\Lambda}$ above it is easy to see that

$$\widehat{\Lambda} = \begin{pmatrix} \Lambda - \hat{\lambda}_{K+1}A^{-1}b \\ \hat{\lambda}_{K+1} \end{pmatrix}$$

15. Positive Definite Nature of $\hat{A}^{-1}$: Since $\widehat{\Omega}$ is positive definite and $\hat{P}\Psi\hat{P}^T$ is positive semi-definite, the sum of the two - matrix $\hat{A}$ - is also positive definite, and therefore so is $\hat{A}^{-1}$. And since $c - b^TA^{-1}b$ is a diagonal element of $\hat{A}^{-1}$, it must be positive.
16. Sign of Incremental Projection Update: Therefore $\hat{\lambda}_{K+1}$ - the weight on the last view - has the same sign as the numerator in

$$\hat{\lambda}_{K+1} = \frac{q - p^T\Psi\overline{\Psi}^{-1}\bar{\mu}}{[c - b^TA^{-1}b]\delta}$$

In the limit when τ and $\widehat{\Omega}$ go to zero and $\frac{\widehat{\Omega}}{\tau}$ stays constant the numerator becomes $q - p^T \bar{\mu}$.

Projection Distribution Dependence on Parameters

1. Projection Contribution Returns/Variance Dependence: For a particular view k its weight λ_k is an increasing function of its expected return q_k. The absolute value of λ_k is an increasing function of its confidence level ${w_k}^{-1}$
2. Posterior Portfolio Component Attribution Recast: Notice that

$$\Lambda = \frac{\tau \Omega^{-1} Q}{\delta} - A^{-1} P \frac{\Psi}{1+\tau} w_{eq} - A^{-1} P \frac{\Psi}{1+\tau} P^T \frac{\tau \Omega^{-1} Q}{\delta}$$

can be re-arranged as

$$\Lambda = \frac{\mathrm{A}^{-1} Q}{\delta} - A^{-1} P \frac{\Psi}{1+\tau} w_{eq}$$

3. Posterior Portfolio Component Attribution Sensitivity: It is clear that

$$\frac{\partial \lambda_i}{\partial q_k} = \frac{1}{\delta} [A^{-1}]_{ik}$$

in particular

$$\frac{\partial \lambda_k}{\partial q_k} = \frac{1}{\delta} [A^{-1}]_{kk}$$

Because $[A^{-1}]_{kk}$ is a diagonal element of a positive definite matrix it must be positive. Therefore λ_k is an increasing function of q_k.

4. Projection Contribution Sensitivity to Variance: Using

$$\Lambda = \frac{\mathrm{A}^{-1}Q}{\delta} - A^{-1}P\frac{\Psi}{1+\tau}w_{eq}$$

the partial derivative $\frac{\partial\Lambda}{\partial {w_k}^{-1}}$ can be written as

$$\frac{\partial\Lambda}{\partial {w_k}^{-1}} = \frac{\partial\mathrm{A}^{-1}}{\partial {w_k}^{-1}}\left[\frac{Q}{\delta} - P\frac{\Psi}{1+\tau}w_{eq}\right]$$

5. Projection Contribution Variance Sensitivity Simplification: Because

$$\frac{\partial\mathrm{A}^{-1}}{\partial {w_k}^{-1}} = -\mathrm{A}^{-1}\frac{\partial A}{\partial {w_k}^{-1}}\mathrm{A}^{-1}$$

and

$$\frac{\partial A}{\partial {w_k}^{-1}} = -\frac{{w_k}^2 i_{kk}}{\tau}$$

the above expression for $\frac{\partial\Lambda}{\partial {w_k}^{-1}}$ becomes

$$\frac{\partial\Lambda}{\partial {w_k}^{-1}} = \frac{{w_k}^2\mathrm{A}^{-1}i_{kk}\Lambda}{\tau}$$

where i_{kk} is a matrix with 1 as the k^{th} diagonal element and 0 everywhere else.

6. Sign of Projection Confidence Sensitivity: The k^{th} element of the above partial derivative is

$$\frac{\partial \lambda_k}{\partial {w_k}^{-1}} = \frac{{w_k}^2}{\tau}[A^{-1}]_{kk}\lambda_k$$

This has the same sign as λ_k. For a positive (negative) λ_k increasing ${w_k}^{-1}$ would cause λ_k to increase (decrease). In other words the absolute value of λ_k is an increasing function of ${w_k}^{-1}$.

Black Litterman Intuition Numerical Examples

1. Sample Portfolio – G7 Equity Index: He and Litterman (1999) present several numerical examples to illustrate the results. In all these examples the market consists of the equity indexes of seven major industrial countries. The tables below contain the index returns, the index return volatilities, the market capitalization weights, and the equilibrium risk premia. The δ parameter representing the world average risk aversion is assumed to be 2.5.
2. Equity Market Index Returns Correlations: Correlations among the Equity Market Index Returns of the seven countries

	AUD	**CAD**	**FRA**	**GER**	**JPN**	**UK**	**USA**
AUD	1.000	0.488	0.478	0.515	0.439	0.512	0.491
CAD	0.488	1.000	0.664	0.655	0.310	0.608	0.779
FRA	0.478	0.664	1.000	0.861	0.355	0.783	0.668
GER	0.515	0.655	0.861	1.000	0.354	0.777	0.653
JPN	0.439	0.310	0.355	0.354	1.000	0.652	0.306
UK	0.512	0.608	0.783	0.777	0.405	1.000	0.405
USA	0.491	0.779	0.668	0.653	0.306	0.652	1.000

3. Volatility, Market Weights, Equilibrium Premia:

	Annualized Volatility	Market Capitalization Weights	Equilibrium Risk Premia
AUD	16.0%	1.6%	3.9%
CAD	20.3%	2.2%	6.9%
FRA	24.8%	5.2%	8.4%
GER	27.1%	5.5%	9.0%
JPN	21.0%	11.6%	4.3%
UK	20.0%	12.4%	6.8%
USA	18.7%	61.5%	7.6%

4. Initial Weights - CAPM Market Portfolio: They first examine the example of a view on the German equity market seen before using the traditional mean-variance approach. Assuming that the investor uses the CAPM as a starting point and that they do not have any view they presume that the expected returns equal the equilibrium risk premia. As a result they hold the market portfolio.
5. Views under Mean-Variance Optimization: With the view that the German Equity market will outperform the rest of Europe by 5% the investor needs to modify the expected returns. One way to proceed is as follows. To be precise in expressing the view investor sets the expected returns for Germany 5% higher than the market capitalization weighted average of the expected returns of France and the United Kingdom.
6. Impact on the Resulting Returns: The investor keeps the market weighted average expected returns for the European countries and the spread between France and the United Kingdom unchanged from their equilibrium values. The investor also keeps the expected returns for non-European countries unchanged from their equilibrium values. Since the equilibrium already implies that Germany will outperform the rest of Europe the changes to the expected returns are quiet small.
7. Results from the Traditional Mean Variance Approach: This table shows the expected returns, the optimal portfolio weights, and the changes from their equilibrium values in the

traditional mean-variance approach. The expected returns are adjusted to reflect the view that the German equities will outperform the rest of the European equities by 5%.

	$\boldsymbol{\mu}$	$\boldsymbol{w_{opt}}$	$\boldsymbol{\mu - \Pi}$	$\boldsymbol{w_{opt} - w_{eq}}$
AUD	3.9%	-5.1%	0.0%	-6.7%
CAD	6.9%	-2.3%	0.0%	-4.5%
FRA	7.6%	-50.1%	-0.8%	-55.1%
GER	11.5%	83.6%	2.4%	78.1%
JPN	4.3%	14.9%	0.0%	3.3%
UK	6.0%	-22.8%	-0.8%	-35.2%
USA	7.6%	66.6%	0.0%	5.1%

8. Impact on the Resulting Weights: However the optimal portfolio is quite different than what one would have expected. Although the increased weights in the Germany market and the decreased weights in the United Kingdom and the France markets are expected, the size of the changes are quite dramatic. Even more puzzling are the reduced weights on Australia and Canada as well as the increased weights on Japan and the United States.
9. Black Litterman Sample Views Specification: With the Black Litterman approach the same view is expressed as an expectation of a 5% return on a portfolio consisting of a long position in Germany equities and a market capitalization weighted short position in the rest of the European markets. He and Litterman (1999) assume the value of τ to be 0.05 which corresponds to the confidence level of the CAPM prior mean if it was estimated using 20 years of data.
10. Posterior Returns and Optimal Portfolio: The Black Litterman expected returns are calculated using

$$\bar{\mu} = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Psi)^{-1}\Pi + P^T\Omega^{-1}Q]$$

and the optimal portfolio weights are calculated by $[\delta\bar{\Psi}]^{-1}\bar{\mu}$. The view portfolio, the expected returns, and the deviation from the initial portfolio weights are given in the table below.

11. Black Litterman Single View Results: The first column is the portfolio used by the investor to express the view on Germany versus the rest of Europe in the Black Litterman approach. The expected return on this portfolio is 5%. The relative uncertainty of the view represented by the value of $\frac{\omega}{\tau}$ which equals 0.021. The rest of the table shows the expected returns calculated using

$$\bar{\mu} = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Psi)^{-1}\Pi + P^T\Omega^{-1}Q]$$

the optimal portfolio $[\delta\bar{\Psi}]^{-1}\bar{\mu}$, and the deviation of the final portfolio from the initial portfolio $\frac{w_{eq}}{1+\tau}$.

	$\boldsymbol{p}$	$\boldsymbol{\bar{\mu}}$	$\boldsymbol{w^*}$	$\boldsymbol{w^* - \frac{w_{eq}}{1+\tau}}$
AUD	0.0%	4.3%	1.5%	0.0%
CAD	0.0%	7.6%	2.1%	0.0%
FRA	-29.5%	9.3%	-4.0%	-8.9%
GER	100.0%	11.0%	35.4%	30.2%
JPN	0.0%	4.5%	11.0%	0.0%
UK	-70.5%	7.0%	-9.5%	-21.3%
USA	0.0%	8.1%	58.6%	0.0%

12. CAPM Bayesian Scaled Market Portfolio: The optimal weight vector is very intuitive in this case. Being a Bayesian the investor has started with a scaled market portfolio of $\frac{w_{eq}}{1+\tau}$ reflecting their uncertainty on the CAPM. In case of no additional views the quantities $\bar{\mu}$ and $\bar{\Psi}$ are reduced to Π and $(1+\tau)\Psi$.

13. Size of the Projection Portfolio Exposure: Since the investor also expressed a view on the portfolio of Germany versus the rest of Europe they just add an exposure to the view portfolio in addition to their initial portfolio – the scaled market equilibrium portfolio. The size of this exposure is given by $\frac{\lambda}{1+\tau}$.
14. Magnitude of the Optimal Deviation: The weight λ is calculated using the Black Litterman model through

$$\Lambda = \frac{\tau\Omega^{-1}Q}{\delta} - A^{-1}P\frac{\Psi}{1+\tau}w_{eq} - A^{-1}P\frac{\Psi}{1+\tau}P^T\frac{\tau\Omega^{-1}Q}{\delta}$$

In this instance the value of λ is 0.302. This can easily be verified in the Table above – the optimal deviation is exactly the new portfolio multiplied by λ.
15. Black Litterman Two Views Specification: As a second instance, He and Litterman (1999) consider the case where the investor has more than one view. In addition to the first view of Germany versus the rest of Europe, the investor also believes the Canadian equities will outperform the US equities by 3% a year. The view is expressed as an expected 3% annualized returns for the portfolio of long Canadian equities and short US equities.
16. Black Litterman Two Views Results: The first two columns are the two views held by the investor. The rest of the table shows the expected returns calculated using

$$\bar{\mu} = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Psi)^{-1}\Pi + P^T\Omega^{-1}Q]$$

the optimal portfolio $[\delta\overline{\Psi}]^{-1}\bar{\mu}$, and the deviation of the optimal portfolio from the scaled equilibrium weights.

	$\boldsymbol{p}$		$\bar{\boldsymbol{\mu}}$	$\boldsymbol{w}^*$	$\boldsymbol{w}^* - \frac{\boldsymbol{w_{eq}}}{\boldsymbol{1+\tau}}$
AUD	0.0%	0.0%	4.4%	1.5%	0.0%
CAD	0.0%	100.0%	8.7%	41.9%	39.8%
FRA	-29.5%	0.0%	9.5%	-3.4%	-8.4%
GER	100.0%	0.0%	11.2%	33.6%	29.4%

JPN	0.0%	0.0%	4.6%	11.0%	0.0%
UK	-70.5%	0.0%	7.0%	-8.2%	-20.0%
USA	0.0%	-100.0%	7.5%	18.8%	39.8%
$\boldsymbol{q}$	5.00%	3.00%			
$\boldsymbol{\frac{\omega}{\tau}}$	0.021	0.017			
$\boldsymbol{\lambda}$	0.298	0.418			

17. Optimal Deviation as a Weighted Sum: The table above shows the views, the expected returns, and the optimal portfolio. Again the results clearly show that the optimal deviation is a weighted sum of the view portfolios.
18. Exposures to the Individual Views: The corresponding weights $\frac{\lambda_1}{1+\tau}$ and $\frac{\lambda_2}{1+\tau}$ are equal to 0.284 and 0.398 respectively. Again the result is very easy to understand – the investor has two weights, so they add exposures to the portfolios. The Black Litterman model gives optimal weights.
19. Impact of Increased View Bullishness: Assume now that the investor becomes more bullish on the Canada/US view. Instead of 3% outperformance the investor now believes that the Canadian equities will outperform the US equities by 4%. According to the second property seen earlier the weight λ_2 on the second portfolio should increase. As demonstrated in the table below the value of λ_2 increased from 0.418 in the previous table to 0.538.
20. Black Litterman Two Views Results: The Canada/US view is more bullish than in the previous case. The expected returns on the view is 4% instead of the 3% shown in the previous table.

	$\boldsymbol{p}$		$\boldsymbol{\bar{\mu}}$	$\boldsymbol{w^*}$	$\boldsymbol{w^* - \frac{w_{eq}}{1+\tau}}$
AUD	0.0%	0.0%	4.4%	1.5%	0.0%
CAD	0.0%	100.0%	9.1%	53.3%	51.3%
FRA	-29.5%	0.0%	9.5%	-3.3%	-8.2%

GER	100.0%	0.0%	11.3%	33.1%	27.8%
JPN	0.0%	0.0%	4.6%	11.0%	0.0%
UK	-70.5%	0.0%	7.0%	-7.8%	-19.6%
USA	0.0%	-100.0%	7.3%	7.3%	-51.3%
$\boldsymbol{q}$	5.00%	4.00%			
$\boldsymbol{\frac{\omega}{\tau}}$	0.021	0.017			
$\boldsymbol{\lambda}$	0.292	0.538			

21. Impact of Decreased View Confidence: Assume now the investor becomes less confident in the view of Germany versus rest of Europe, say only 50% as confident in the view now. This is reflected by a change of $\frac{\omega}{\tau}$ from 0.021 to 0.043. The view on Canada/US is unchanged at 4.0%. According to the second property above the absolute weight on the first portfolio λ_1 should decrease. The table below shows that the value decreased from 0.292 to 0.193.
22. Less Certain on Germany/Europe: In this case the investor becomes less certain in the view of Germany versus the rest of Europe. It is represented by the value of $\frac{\omega}{\tau}$ being double the value in the previous table.

	$\boldsymbol{p}$		$\boldsymbol{\bar{\mu}}$	$\boldsymbol{w}^*$	$\boldsymbol{w}^* - \frac{\boldsymbol{w_{eq}}}{\boldsymbol{1+\tau}}$
AUD	0.0%	0.0%	4.3%	1.5%	0.0%
CAD	0.0%	100.0%	8.9%	53.9%	51.8%
FRA	-29.5%	0.0%	9.3%	-0.5%	-5.4%
GER	100.0%	0.0%	10.6%	23.6%	18.4%
JPN	0.0%	0.0%	4.6%	11.0%	0.0%
UK	-70.5%	0.0%	6.9%	-1.1%	-13.0%
USA	0.0%	-100.0%	7.2%	6.8%	51.8%
$\boldsymbol{q}$	5.00%	4.00%			

$\frac{\omega}{\tau}$	0.043	0.017
λ	0.193	0.544

23. Replication of Equilibrium in Projection: The final example from He and Litterman (1999) illustrates the first property. In addition to the two views in the previous table the investor has a third view. The third view is that the Canadian equities will outperform the Japanese equities by 4.12%. But using

$$\tilde{\mu} = \Psi\overline{\Psi}^{-1}\bar{\mu}$$

the implied expected return of the portfolio of long Canada and short Japan is exactly 4.12% when the investor had two views.

24. Identical Equilibrium and Posterior Allocation: In this case the investor has three views. However since the third view is already implied by the equilibrium and the first two views as in the previous table the weights on the first two views are the same as in the previous case. The optimal portfolio in this case is identical to the values in the previous case.

	$\boldsymbol{p}$			$\overline{\boldsymbol{\mu}}$	$\boldsymbol{w}^*$	$\boldsymbol{w}^* - \frac{\boldsymbol{w}_{eq}}{\mathbf{1}+\boldsymbol{\tau}}$
AUD	0.0%	0.0%	0.0%	4.3%	1.5%	0.0%
CAD	0.0%	100.0%	100.0%	8.9%	53.9%	51.8%
FRA	-29.5%	0.0%	0.0%	9.3%	-0.5%	-5.4%
GER	100.0%	0.0%	0.0%	10.6%	23.6%	18.4%
JPN	0.0%	0.0%	-100.0%	4.6%	11.0%	0.0%
UK	-70.5%	0.0%	0.0%	6.9%	-1.1%	-13.0%
USA	0.0%	-100.0%	0.0%	7.2%	6.8%	51.8%
$\boldsymbol{q}$	5.00%	4.00%	4.12%			
$\frac{\omega}{\tau}$	0.043	0.017	0.059			

$\boldsymbol{\lambda}$	0.193	0.544	0.000

25. Lack of Change in the Weights: The expected returns of the third portfolio matches exactly the expected returns implied by the Black Litterman Model using only two views. According to the first property the weights on the first two views should not be changed and the weight on the third view should be zero.
26. Corresponding View Portfolio Exposure Loading: The table above shows that the weight λ_3 on the third view is zero and the expected returns and the optimal portfolio weights are exactly the same as in the previous table.

References

- He, G., and R. Litterman (1999): The Intuition behind the Black-Litterman Model Portfolios **Goldman Sachs Asset Management**.

Incorporating User-Specified Confidence Levels

Overview

1. The Black-Litterman Model Motivation: As seen before the Black Litterman model enables users to combine their unique views regarding the performance of the various assets with the market equilibrium in a manner the results in intuitive diversified portfolios.
2. Tilt Based User Confidence Specification: Idzorek (2005) introduces a new method for controlling the tilts and the final portfolio weights caused by views. The method asserts that the magnitude of the tilt should be controlled by the user-specified confidence level based on an intuitive 0% to 100% confidence level. This is an intuitive method for specifying one of the most abstract mathematical parameters of the Black Litterman model.

Introduction

1. Problem with the Markowitz Paradigm: The Black Litterman asset allocation model is a highly sophisticated portfolio construction method that overcomes the problems of unintuitive highly concentrated portfolios, high input sensitivity, and high estimation error. These three related and well-documented problems with the mean-variance optimization are the most likely reasons that practitioners do not use the Markowitz paradigm, in which the returns are maximized for a given level of risk.
2. Basis behind the Black Litterman Model: The Black Litterman model uses a Bayesian approach to combine the subjective views of an investor with respect to one or more assets

with the market equilibrium vector of expected returns – the prior distribution – to form a new, mixed estimate of the expected returns.

3. Basic Black Litterman Model Literature: The model was introduced by Black and Litterman (1990), expanded in Black and Litterman (1991, 1992), and treated in greater detail by Bevan and Winkelmann (1998), He and Litterman (1999), and Litterman (2003). Other key works on the model were done by Lee (2000), Satchell and Scowcroft (2000), and for the mathematically more inclined, by Christodoulakis (2002).
4. Conceptual Components behind the Model: The Black Litterman Model combines CAPM (Sharpe (1964)), reverse optimization (Sharpe (1974)), mixed estimation (Theil (1971, 1978)), universal hedge ratio/Black's global CAPM (Black (1989a, 1989b), Litterman (2003)), and mean-variance optimization (Markowitz (1952)).

Estimating the Excess Returns Distribution

1. Advantages of the Black Litterman Model: The Black Litterman Model creates stable, mean-variance efficient portfolios based on the investors' unique insights, which overcome the problem of input sensitivity. According to Lee (2000) the Black Litterman model also largely mitigates the problem of estimation error-maximization (Michaud (1989)) by spreading the errors throughout the vector of expected returns.
2. Small Changes in Expected Returns: The most important input in mean-variance optimization is the vector of expected returns. However best and Grauer (1991) show that a small change in the expected returns of one of the portfolio's assets can force half of the assets from the portfolio.
3. Starting Point for Expected Returns: In a search for a reasonable starting point for expected returns Black and Litterman (1992), He and Litterman (1999), and Litterman (2003) all explore several alternative forecasts, historical returns, equal "mean" returns for all assets, and risk-adjusted equal mean returns.

4. Persistence of the Extreme Portfolios: They demonstrate that these alternative forecasts lead to extreme portfolios – when unconstrained portfolios with large long and short positions, and when subject a long-only constraint, portfolios that are concentrated in only a relatively small number of assets.

Reverse Optimization of Expected Returns

1. Basic Idea behind Reverse Optimization: The Black Litterman Model uses the market equilibrium returns as a neutral starting point. Equilibrium returns are the set of returns that clear the market. The equilibrium returns are derived using a reverse optimization method in which the vector of implied excess equilibrium returns is extracted from known information using reverse optimization.
2. Reverse Optimization Equilibrium Returns Expression:

$$\Pi = \lambda \Psi w_{MKT}$$

   where Π is the Implied Equilibrium Return Vector - $N \times 1$ column vector, λ is the risk-aversion coefficient; Ψ is the covariance matrix of excess returns - $N \times N$ matrix, and w_{MKT} is the market capitalization weights of the assets - $N \times 1$ column vector. Possible alternatives to the market capitalization weights include a presumed benchmark and a set of float adjusted capitalization weights.
3. The Risk-Return Tradeoff: The risk aversion coefficient λ characterizes the expected risk-return tradeoff. It is the rate at which an investor will forego expected excess return for less variance.
4. Excess Return Reverse Optimization Estimate: In the reverse optimization process the risk aversion coefficient acts as a scaling factor for the reverse optimization estimate of the excess returns; the weighted reverse optimization excess returns equals the specified market risk premium.

5. The Implied Risk Aversion Coefficient: More excess returns per unit risk – larger λ – increases the estimated excess returns. The *implied* risk aversion coefficient λ for a portfolio can be estimated by dividing the expected excess returns by the variance of the portfolio (Grinold and Kahn (1999))

$$\lambda = \frac{\mathbb{E}[R] - r_f}{\sigma^2}$$

where $\mathbb{E}[R]$ is the expected market or the benchmark total returns, r_f is the risk-free rate, and

$$\sigma^2 = {w_{MKT}}^T \Psi w_{MKT}$$

is the total variance of the market – or benchmark – excess returns.

6. Currency Returns Incorporation in the Model: To illustrate the model and keep the scope manageable Idzorek (2005) presents an eight asset example in addition to the general model, and avoids discussing currencies. Currency returns in the model are treated in detail by Black (1989a, 1989b), Black and Litterman (1991, 1992), Grinold (1996), Meese and Crownover (1999), Grinold and Meese (2000), and Litterman (2003).
7. Expected Excess Returns Vector Table

Asset Class	**Historical** $\boldsymbol{\mu_{HIST}}$	**CAPM GSMI** $\boldsymbol{\mu_{GSMI}}$	**CAPM Portfolio** $\boldsymbol{\mu_P}$	**Implied Equilibrium Returns Vector** $\boldsymbol{\Pi}$
US Bonds	3.15%	0.02%	0.08%	0.08%
International Bonds	1.75%	0.18%	0.67%	0.67%
US Large Growth	-6.39%	5.57%	6.41%	6.41%
US Large Value	-2.86%	3.39%	4.08%	4.08%
US Small Growth	-6.75%	6.59%	7.43%	7.43%
US Small Value	-0.54%	3.16%	3.70%	3.70%
International Developed Equity	-6.75%	3.92%	4.80%	4.80%

International Emerging Equity	-5.26%	5.60%	6.60%	6.60%

All four estimates are based on 60 months of excess returns over the risk-free rate. The two CAPM estimates are based on a risk premium of 3. Dividing the risk premium by the variance of the market – or benchmark – excess returns σ^2 results in a risk aversion coefficient λ of approximately 3.07.

8. Asset Set Expected Excess Returns: The table presents four estimates of expected excess returns for the eight assets – US Bonds, International Bonds, US Large Growth, US Large Value, US Small Growth, US Small Value, International Developed Equity, and International Emerging Equity.
9. UBS Global Securities Market Index: The GSMI CAPM excess returns vector is calculated relative to the UBS Global Securities Market Index GSMI – a global index and a good proxy for the world market portfolio.
10. CAPM Implied Equilibrium Vector: The second CAPM excess-returns vector is calculated relative to the market capitalization weighted portfolio using *implied bets* and is identical to the Implied Equilibrium Returns Vector Π.
11. CAPM Returns Based Implied Betas: Literature on the Black Litterman Model often refers to the reverse optimized Implied Returns Equilibrium Vector Π as the CAPM returns, which can be confusing. CAPM returns based on regression betas can be significantly different from the CAPM based on *implied betas*. Idzorek (2005) uses the procedure in Grinold and Kahn (1999) to calculate the implied betas.
12. Market Capitalization Weighted Portfolio Beta: Just as one is able to use the market capitalization weights and the covariance matrix to infer the Implied Equilibrium Returns Vector, one can extract a vector of implied betas. These implied betas are the betas of the N assets relative to the market capitalization weighted portfolios. As one would expect the market capitalization weighted beta of the market portfolio is 1.
13. Implied Beta Formulation and Components:

$$\beta = \frac{\Psi w_{MKT}}{{w_{MKT}}^{T} \Psi w_{MKT}} = \frac{\Psi w_{MKT}}{\sigma^2}$$

where β is the vector of implied betas, Ψ is the covariance matrix of excess returns, w_{MKT} is the vector of market capitalization weights, and

$$\sigma^2 = {w_{MKT}}^T \Psi w_{MKT} = \frac{1}{\beta^T \Psi^{-1} \beta}$$

is the variance of the market – or the benchmark – excess returns. The vector of the CAPM returns is the same as the vector of the reverse optimized returns when the CAPM returns are based on implied betas relative to the market capitalization weighted portfolio.

14. Historical vs. GSMI Capital Returns: The Historical Returns Vector has a larger standard deviation and range compared to the other vectors. The GSMI CAPM Returns Vector is quiet similar to the Implied Equilibrium Returns vector Π - the correlation is 98%.
15. Solution to the Unconstrained Variance Minimization: Rearranging

$$\Pi = \lambda \Psi w_{MKT}$$

and substituting μ – representing any vector of excess returns – for Π – the vector of Implied Equilibrium Excess Returns – leads to the solution to the unconstrained minimization problem

$$\underset{w}{max}\ w^T \mu - \frac{\lambda w^T w}{2}$$

results in

$$w = [\lambda \Psi]^{-1} \mu$$

If μ does not equal Π w will not equal w_{MKT}.

16. The Corresponding Optimal Portfolio Weights: The Table on Recommended Portfolio Weights uses

$$w = [\lambda\Psi]^{-1}\mu$$

to find the optimum weights for these portfolios based on the returns vectors from the first Table. The market capitalization weights are presented in the last column.

17. Table of Reverse Optimization Weights:

Asset Class	**Weights Based on Historical μ_{HIST}**	**Weights Based on CAPM GSMI μ_{GSMI}**	**Weights Based on CAPM Portfolio μ_P**	**Weights Based on Implied Equilibrium Returns Vector Π**
US Bonds	1144.32%	21.33%	19.34%	19.34%
International Bonds	-104.59%	5.19%	26.13%	26.13%
US Large Growth	54.99%	10.80%	12.09%	12.09%
US Large Value	-5.29%	10.82%	12.09%	12.09%
US Small Growth	-60.52%	3.73%	1.34%	1.34%
US Small Value	81.47%	-0.49%	1.34%	1.34%
International Developed Equity	-104.36%	17.10%	24.18%	24.18%
International Emerging Equity	14.59%	14.59%	3.49%	3.49%
High	1144.32%	21.33%	26.13%	26.13%
Low	-104.59%	-0.49%	1.34%	1.34%

18. Historical Return Vector Optimal Portfolio: Not surprisingly the Historical Return Vector produces an extreme portfolio.
19. GSMI vs. Equilibrium Returns Comparison: Those not familiar with mean-variance optimization may expected two highly correlated Returns Vectors to lead to similarly correlated vectors of portfolio holdings. Nevertheless despite the similarity between the CAPM GSMI Returns vector and the Implied Equilibrium Returns Vector Π the returns vectors produce two rather distinct weight vectors – the correlation coefficient is 66%.

20. GSMI vs. Equilibrium Weights Comparison: Most of the weights of the CAPM GSMI based portfolios are different significantly than the benchmark market capitalization weighted portfolio, especially the allocation to International Bonds. As one would expect, since the process of extracting Implied Equilibrium Returns from using the Market Capitalization is reversed, the Implied Equilibrium Returns Vector Π leads back to the Market Capitalization weighted portfolio.
21. Black-Litterman Model Starting Point: In the absence of views that differ from the Implied Equilibrium Returns, investors should hold the market portfolio. The Implied Equilibrium Vector Π is the market-neutral starting point for the Black Litterman Model.

The Black Litterman Model

1. The Fundamental Black-Litterman Expression: K is used to represent the number of views, and N is used to express the number of assets in the formula. The expression for the New Combined Returns Vector $\mathbb{E}[R]$ is

$$\mathbb{E}[R] = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Psi)^{-1}\Pi + P^T\Omega^{-1}Q]$$

2. The Black Litterman Expression Parameters: Here $\mathbb{E}[R]$ is the Posterior Combined Return Vector - $N \times 1$ column vector; τ is a scalar; Ψ is the covariance matrix of excess returns - $N \times N$ matrix; P is a matrix that identifies the assets involved in the views - $K \times N$ matrix, or $1 \times N$ in the special case of one view; Ω is a diagonal covariance matrix of error terms from the expressed view representing the uncertainty in each view - $K \times K$ matrix; Π is the Implied Equilibrium Returns Vector - $N \times 1$ column vector; and Q is the View Returns Vector - $K \times 1$ column vector.
3. First Idzorek Projection - Absolute View: International Developed Equity will have an absolute return of 5.25% - Confidence in this view is 5.25%. This is an example of the

absolute view. From the final column of the First Table the Implied Equilibrium Returns for International Developed Equity is 4.80%, 45 bp lower than this view of 5.25%.

4. Second Idzorek Projection - Relative View: International Bonds will outperform US Bonds by 25 bp – Confidence of the View is 50%.
5. Assessing the Second View Impact: The second view says that the return of International Bonds will be 0.25% greater than that of the US Bonds. In order to gauge whether the second view will have a positive or a negative effect on the International Bonds relative to the US Bonds it is necessary to evaluate the respective Implied Equilibrium Returns of the two assets in the View.
6. View Based Cross Asset Tilt: The first Table indicates that the Implied Equilibrium Returns for the International and the US Bonds is 0.67% and 0.08% respectively, a difference of 0.59%. The second view of 0.25% is less than the 0.59% by which the returns of the International Bonds exceeds that of the US Bonds; thus one would expect the model to tilt the portfolio away from the International Bonds in favor of the US Bonds.
7. View Returns Lesser than Implied: In general – and in the absence of constraints and additional views – if the view is lesser than the difference of the two Implied Equilibrium returns, the model tilts the portfolio towards the underperforming asset.
8. View Returns Greater than Implied: Likewise if the view is greater than the two implied equilibrium returns, the model tilts the portfolio toward the underperforming assets.
9. Relative Under/Over Performing Assets: The third view demonstrates a scenario involving multiple assets and that the terms *outperforming* and *underperforming* are relative. The number of outperforming assets need not match the number of assets underperforming.
10. Long/Short Sub-portfolio Components: The results from views that involve multiple assets with a range of different Implied Equilibrium Returns can be less intuitive. The assets in a relative view form two separate mini sub-portfolios – a long portfolio and a short portfolio.
11. Relative Weighting of Each Asset: The relative weighting of each nominally outperforming asset is proportional to the asset's market capitalization divided by the sum of the market capitalizations of the other nominally outperforming assets of that view. Likewise the relative weighting of each nominally underperforming asset is proportional to that asset's market capitalization divided by the market capitalizations of the other underperforming assets. The net long position less the net short positions equal 0.

12. Equilibrium View Returns Differential Impact: The mini-portfolio that actually receives the positive view may not be the nominally outperforming assets from the expressed view. In general if the view return is greater than the weighted average Implied Equilibrium differential return the model will tend to overweight the “outperforming” assets.
13. View 3 Table of Nominally Outperforming Assets:

Asset Class	**Market Capitalization (Billions)**	**Relative Weight**	**Implied Equilibrium Returns Vector Π**	**Weighted Excess Return**
US Large Growth	$5174	90.00%	6.41%	5.77%
US Small Growth	$575	10.00%	7.43%	0.74%
TOTAL	$5749	100.00%	-	6.52%

14. US Large and Small Growth: Idzorek’s View 3 shows that the nominally outperforming assets are the US Large Growth and the US Small Growth and the nominally underperforming assets are the US Large Value and the US Small Value. The Table above demonstrates that the weighted average Implied Equilibrium Returns of the mini-portfolio formed from US Large Growth and the US Small Growth is 6.52%.
15. View 3 Table of Nominally Underperforming Assets:

Asset Class	**Market Capitalization (Billions)**	**Relative Weight**	**Implied Equilibrium Returns Vector Π**	**Weighted Excess Return**
US Large Value	$5174	90.00%	4.08%	3.67%
US Small Value	$575	10.00%	3.70%	0.37%
TOTAL	$5749	100.00%	-	4.04%

16. US Large and Small Value: Using the Table above the weighted average Implied Equilibrium Return of the mini-portfolio formed from the US Large Value and the US Small Value is 4.04%. The weighted average Implied Equilibrium Return differential is 2.47%.
17. Impact of the Return Differentials: Because View 3% states that the US Large Growth and the US Small Growth will outperform the US Large Value and the US Small Value by only 2% - a reduction from the current weighted average Implied Equilibrium Return differential of 2.47% - the view appears to actually represent a reduction in the performance of US Large Growth and US Small Growth relative to US Large Value and US Small Value.
18. Corresponding Increments in the Allocation: The effect is numerically demonstrated later, where the nominally outperforming assets of the third View – US Large Growth and US Small Growth – receive reductions in their allocations, and the nominally underperforming assets – US Large Value and US Small Value – receive increases in their allocations.

Building the Inputs

1. *P* Matrix Specification Litterman's Approach: Methods for specifying the *P* matrix vary. Litterman (2003) assigns a percentage value to the assets in question.
2. Satchell and Scowcroft *P* Matrix: Satchell and Scowcroft (2000) use an equal weighting scheme. Under this system the weightings are proportional to 1 divided by the number of respective assets underperforming or outperforming. View 3 has 2 nominally underperforming assets, each of which receives a -0.5 weighting. View 3 also contains 2 nominally outperforming assets, each of which receives a +0.5 weighting. This weighting scheme ignores the market capitalization of the assets involved in the view.
3. Satchell and Scowcroft Approach Impact: The market capitalizations of the US Large Growth and the US Large Value asset classes are 9 times the market capitalizations of the US Small Growth and the US Small Value asset classes. Yet Satchell and Scowcroft affects the relative weights equally causing large changes in the two smaller asset classes.

4. <u>P Matrix Specification Idzorek Approach</u>: In contrast to Satchell and Scowcroft (2000) equal weighting scheme Idzorek prefers to use a market capitalization weighting scheme. More specifically the relative weighting of each asset is proportional to the asset's market capitalization divided by the total capitalization of either the outperforming or the underperforming assets of that particular view.
5. <u>Idzorek's P Matrix Scoping Loadings</u>: As shown in the previous tables the relative market capitalization weights of the nominally outperforming assets are 0.9 for US Large Growth and 0.1 for US Small Growth while the relative market capitalization weights for the nominally underperforming assets are -0.9 for US Large Value and -0.1 for US Small Value. These figures are used to create a new P matrix that is used for all of the subsequent calculations.
6. <u>Market Capitalization Method P Matrix</u>:

$$P = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ -1.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.9 & -0.9 & 0.1 & -0.1 & 0.0 & 0.0 \end{bmatrix}$$

7. <u>Computation of the View Variances</u>: Once the matrix P is defined one can calculate the variance of each individual view portfolio. The variance of an individual view portfolio is ${p_k}^T \Psi p_k$ where p_k is a single $1 \times N$ row vector from matrix P that corresponds to the k^{th} view and Ψ is the covariance of matrix of excess returns.
8. <u>Covariance of the Projection Uncertainty</u>: The respective variance of each individual view portfolio is an important source of information regarding the certainty – or lack thereof – of the level of confidence that should be placed on a view. This information is shortly used to re-visit the error variances ω that form the diagonal elements of Ω.
9. <u>Variance of the View Portfolios</u>:

View	**Expression**	**Value**
1	${p_1}^T \Psi p_1$	2.836%
2	${p_2}^T \Psi p_2$	0.563%
3	${p_3}^T \Psi p_3$	3.462%

10. Specifying Scoping and Projection Uncertainties: Conceptually the Black-Litterman Model is a complex weighted average of the Implied Equilibrium Returns Vector Π and the View Returns Vector Q in which the relative weightings are a function of the scalar τ and the uncertainty Ω. Unfortunately the scalar and the uncertainty in the views are the most abstract and difficult to specify parameters of the model.
11. Enhanced Scoping/Projection Uncertainty Impact: The greater the level of confidence – certainty – in the expressed views, the closer the new return vector will be to the views. If the investor is less confident on the expressed views, the new returns vector should be closer to the Implied Equilibrium Returns Vector Π.
12. Estimating the Scoping Uncertainty Parameter: The scalar τ is more or less inversely proportional to the relative weight given to the Implied Equilibrium Returns Vector Π. Unfortunately guidance in the literature for setting the scalar's value is scarce.
13. Estimate of Means vs. Variance: Both Black and Litterman (1992) and Lee (2000) address this issue; since the uncertainty in the mean is less than the uncertainty in the returns the scalar τ is close to zero. One would expect Equilibrium Returns to less volatile than Historical Returns.
14. Lee (2000) Estimation of the Scalar τ: Lee, who has considerable experience working with a variant of the Black Litterman model typically sets the value of the scalar τ between 0.01 and 0.05, and then calibrates the model based upon a target level of the tracking error.
15. Satchell and Scowcroft τ Estimate: Conversely Satchell and Scowcroft (2000) indicate that the value of τ is often set to 1. They also include an advanced mathematical discussion of one method for establishing a conditional value for τ.
16. Blamont and Firoozye τ Estimate: Finally Blamont and Firoozye (2003) interpret $\tau\Psi$ as the standard error of the estimate of the Implied Equilibrium Vector Π; thus τ is approximately 1 divided by the number of observations.
17. Departure from the Asset's Prior Weight: In the absence of constraints the Black Litterman model only recommends departure from the market capitalization weight only if the asset is a subject of a view. For assets that are the subject of a view, the magnitude of their departure

from the market capitalization weight is controlled by the ratio of τ to the variance of the error term ω of the view in question.

18. Impact of the Additional Views: The magnitude of the departure from the market capitalization weights is also affected by the other views. Additional views lead to a different Combined Returns Vector $\mathbb{E}[R]$, which in turn leads to a new vector of recommended weights.
19. He and Litterman τ Estimate: The easiest way to calibrate the Black Litterman Model is to make an assumption on the value of τ. He and Litterman (1999) calibrate the confidence of a view so that the ratio $\frac{\omega}{\tau}$ is equal to the variance of the view portfolio ${p_k}^T \Psi p_k$.
20. Explicit Calculation of the Ω Elements: Assuming

$$\tau = 0.025$$

and using the individual variances of the view portfolios ${p_k}^T \Psi p_k$ from the previous Table the covariance matrix of the error term has the following form. The general case is

$$\Omega = \begin{pmatrix} ({p_1}^T \Psi p_1)\tau & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & ({p_k}^T \Psi p_k)\tau \end{pmatrix}$$

and in the case of Idzorek's example

$$\Omega = \begin{bmatrix} 0.000709 & 0.000000 & 0.000000 \\ 0.000000 & 0.000141 & 0.000000 \\ 0.000000 & 0.000000 & 0.000866 \end{bmatrix}$$

21. Impact of $\frac{\omega}{\tau}$ on Returns: When the covariance matrix of the error term is calculated using this method the actual value of the scalar τ becomes irrelevant, as only the ratio $\frac{\omega}{\tau}$ enters the model. For example changing the assumed value of the scalar τ from 0.025 to 15 dramatically changes the value of the diagonal elements Ω but the new Combined Returns Vector $\mathbb{E}[R]$ is unaffected.
22. Returns Vectors and Portfolio Weights:

Asset Class	New Combined Returns Vector $\mathbb{E}[\boldsymbol{R}]$	Implied Equilibrium Returns Vector $\boldsymbol{\Pi}$	Difference $\mathbb{E}[\boldsymbol{R}] - \boldsymbol{\Pi}$	New Weight $\hat{\mathbf{w}}$	Market Capitalization Weight $\mathbf{w}_{\boldsymbol{MKT}}$	Difference $\hat{\mathbf{w}} - \mathbf{w}_{\boldsymbol{MKT}}$
US Bonds	0.07%	0.08%	-0.02%	29.88%	19.34%	10.54%
International Bonds	0.50%	0.67%	-0.17%	15.59%	26.13%	-10.54%
US Large Growth	6.50%	6.41%	0.08%	9.35%	12.09%	-2.73%
US Large Value	4.32%	4.08%	0.24%	14.82%	12.09%	2.73%
US Small Growth	7.59%	7.43%	0.16%	1.04%	1.34%	-0.30%
US Small Value	3.94%	3.70%	0.23%	1.65%	1.34%	0.30%
International Developed Equity	4.93%	4.80%	0.13%	27.81%	24.81%	3.63%
International Emerging Equity	6.84%	6.60%	0.24%	3.49%	3.49%	0.00%
TOTAL	-	-	-	103.63%	100.00%	3.63%

23. Deviation from the Asset's Equilibrium Weight: Even though the expressed views only involve 7 out of the 8 asset classes, the table above shows that the individual returns of all assets changed from their respective Implied Equilibrium Returns. A single view causes the return of every asset in the portfolio to change from its Implied Equilibrium Return since each individual return is linked to the other returns via the covariance matrix of excess returns Ψ.
24. Strength of Black Litterman Model: The new weight vector $\widehat{\omega}$ in the Table is based upon the Combined Excess Return Vector $\mathbb{E}[R]$. The Table also illustrates one of the strongest features of the Black Litterman model. Only the weights for the 7 assets for which the views were expressed changed from their original market capitalization weights and the directions of their changes are intuitive. No views were expressed on International Emerging Equity and its weights are unchanged.

25. Original Portfolio Plus Long/Short: From a macro perspective the new portfolio can be viewed as a sum of two portfolios where Portfolio 1 is the original market capitalization weighted portfolio and Portfolio 2 is a series of long and short positions based on the views.
26. Long/Short Estimations from Projections: Portfolio 2 can be sub-divided into mini-portfolios with offsetting long and short positions that sum to 0. View 1, the absolute view, increases the weight of the International Developed Equity without an offsetting position, resulting in portfolio positions that no longer sum to 1.
27. Black Litterman Model with Constraints: The intuitiveness of the Black Litterman model is less apparent when used investment constraints such as constraints on unity, risk, beta, and short selling. He and Litterman (1999) and Litterman (2003) suggest that, in the presence of constraints, the investor input the new Combined Returns Vector $\mathbb{E}[R]$ into a mean-variance optimizer.

Fine Tuning the Model

1. Principle behind Model Fine Tuning: One can fine tune the Black Litterman Model by studying the New Combined Return Vector $\mathbb{E}[R]$, calculating the anticipated risk-return characteristics of the new portfolio, and then adjusting the scalar τ and the individual variances of the error term ω that form the diagonal elements of the covariance matrix of the error term Ω.
2. Bevan and Winkelmann Fine Tuning Approach: Bevan and Winkelmann (1998) offer guidance in setting the weight given to the View Returns Vector Q. After deriving an initial Combined Return Vector $\mathbb{E}[R]$ and the subsequent portfolio weights, they calculate the anticipated Information Ratio of the new portfolio.
3. Maximum Anticipated Information Ratio Metric: They recommend a Maximum Anticipated Information Ratio of 2.0. If the Information Ratio is greater than 2.0 they recommend decreasing the weight given to the views, i.e., decreasing the value of τ and leaving the diagonal elements of Ω unchanged.

4. Table on Posterior Portfolio Statistics:

Measure	**Market Capitalization Weighted Portfolio $\mathbf{w}_{MKT}$**	**Black Litterman Portfolio $\hat{\mathbf{w}}$**
Excess Return	3.000%	3.101%
Variance	0.00979	0.01012
Standard Deviation	9.893%	10.058%
Beta	1	1.01256
Residual Return	-	0.063%
Residual Risk	-	0.904%
Active Return	-	0.101%
Active Risk	-	0.913%
Sharpe Ratio	0.3033	0.3083
Information Ratio	-	0.0699

5. Anticipated Risk-Return Characteristics Comparison: Idzorek (2005) compares the anticipated risk-return characteristics of the market capitalization weighted portfolio with the Black Litterman portfolio using the new weights produced by the New Combined Vector.
6. Idzorek's Calculations of Portfolio Statistics: The computations by Idzorek are based on the implied betas derived from the covariance matrix of historical excess returns and the mean-variance data of the market capitalization weighted benchmark portfolio.
7. Grinold and Kahn Measure Expressions: From Grinold and Kahn (1999); The residual return is

$$\theta_P = \mathbb{E}[R_P] - \beta_P \mathbb{E}[R_B]$$

The Residual Risk is

$$\omega_P = \sqrt{{\sigma_P}^2 - {\beta_P}^2 {\sigma_B}^2}$$

The Active Return is

$$\mathbb{E}[R_{PA}] = \mathbb{E}[R_P] - \mathbb{E}[R_B]$$

The Active Risk is

$$\psi_P = \sqrt{{\omega_P}^2 + {\beta_{PA}}^2 {\sigma_B}^2}$$

The Active Portfolio Beta is

$$\beta_{PA} = \beta_P - 1$$

8. Grinold and Kahn Measure Dictionary: $\mathbb{E}[R_P]$ is the expected returns of the posterior portfolio, $\mathbb{E}[R_B]$ is the expected returns of the benchmark market capitalization weighted portfolio based on the New Combined Expected Returns Vector $\mathbb{E}[R]$, ${\sigma_B}^2$ is the variance of the benchmark portfolio, and ${\sigma_P}^2$ is the variance of the posterior portfolio.
9. Posterior Sharpe/Information Ratio Estimates: Overall the views have very little effect on the expected risk-return characteristics of the new portfolio. However both the Sharpe Ratio and the Information Ratio have increased slightly. The ex-ante information ratio is well below the recommended maximum of 2.0.
10. Verifying Absence of Unintended Results: Next the outputs from the views should be evaluated to confirm that there are no unintended results. For instance investors confined to unity may want to remove absolute views, such as View 1.
11. Sources for Asset Projection Views: Investors should evaluate their ex-post Information Ratios for guidance when setting their confidence on various views. An investment manager could the set the confidence on a particular view based in part on the corresponding analyst's Information Coefficient.

12. The Investment Manager's Information Coefficient: According to Grinold and Kahn (1999) a manager's Information Coefficient is the correlation of his forecasts with the actual results. This gives greater relative importance to the more skillful analysts.
13. Accurate Estimation of the Asset Variance: Most of the examples in the literature, including the eight asset sample presented by Idzorek (2005), use a simple covariance matrix of historical returns. However investors should use the best possible estimate of the covariance matrix of the excess returns. Litterman and Winkelmann (1998) and Litterman (2003) outline the methods they prefer for estimating the covariance matrix of returns, as well as several alternate methods of estimation.
14. The Qian and Gorman Extension: Qian and Gorman (2001) extend the Black-Litterman Model, enabling investors to express views on volatilities and correlations in order to derive a conditional estimate of the covariance matrix of returns. They assert that the conditional covariance matrix stabilizes the results of mean-variance optimization.

Method for Incorporating User-Specified Confidence Levels

1. Difficulty in the Specification of Ω: As the discussion above illustrates Ω is one of the most abstract mathematical parameters of the Black-Litterman Model. Unfortunately, according to Litterman (2003), how to specify the diagonal elements of Ω, representing the uncertainty of the views, is a common question without an universal answer.
2. View Specific Probability Density Function: Regarding Ω Herold (2003) indicates that the major difficulty of the Black-Litterman model is that it forces the user to specify a probability density function for each view, which makes Black-Litterman model suitable only for quantitative managers.
3. Idzorek's Implied Confidence Estimation Method: Idzorek presents a new method for determining the implied confidence levels in the views and how an implied confidence level framework can be coupled with an intuitive 0% to 100% user specified confidence level in

each view to determine the values of Ω which simultaneously removes the need for specifying a value for τ.

Implied Confidence Levels

1. Specifying an Intuitive Confidence Level: It was seen earlier that the individual variances of the error terms ω that form the diagonal elements of the covariance matrix Ω of the error term ω were based on the view portfolios' variances ${p_k}^T \Psi p_k$ multiplied by the scalar τ. However Idzorek (2005) contends that there are other sources of variances in addition ${p_k}^T \Psi p_k$ that affect the investor's confidence in a view. When each view was stated an intuitive level of confidence from 0% to 100% was assigned to each view.
2. Factors affecting Confidence in the View: Presumably additional factors can affect an investor's confidence in a view, such as historical accuracy or *score* of the model, the screen, or the analyst the produced the view, as well as the difference between the view and the implied market equilibrium.
3. Accommodation of the Multiple Factors: These factors, and possibly others, should then be combined with the variance of the view portfolio ${p_k}^T \Psi p_k$ to produce the best possible estimate of the confidence levels on the views. Doing so would enable the Black-Litterman model maximize across much of the investors' information.
4. 100% Confidence in a View: Setting all of the diagonal elements of Ω equal to zero is equivalent to specifying 100% confidence in each of the K views. *Ceteris paribus* doing so would produce the greatest departure from the benchmark capitalization weights for the assets named in the views.
5. Expected Returns Under 100% Confidence: When 100% confidence is specified for all of the views the Black Litterman expression for the New Combined Return Vector under 100% certainty $\mathbb{E}[R_{100\%}]$ is

$$\mathbb{E}[R_{100\%}] = \Pi + \tau \Psi P^T [P^T \Psi P]^{-1} [Q - P\Pi]$$

To distinguish the above expression from the Black Litterman expression

$$\mathbb{E}[R] = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Psi)^{-1}\Pi + P^T\Omega^{-1}Q]$$

the subscript 100% is added.

6. Allocation under Different Confidence Levels: Substituting $\mathbb{E}[R_{100\%}]$ for μ in

$$w = [\lambda\Psi]^{-1}\mu$$

leads to $w_{100\%}$ the weight vector based on 100% confidence on the views. Idzorek (2005) illustrates the relative weights w_{MKT}, $\widehat{w}$, and $w_{100\%}$ for each asset.

7. 100% Confidence for Single Asset: When an asset is named in only one view the vector of portfolio weights with 100% confidence $w_{100\%}$ enables one to calculate an intuitive 0% to 100% confidence for each view. In order to do so one must solve the unconstrained maximization problem twice – once using $\mathbb{E}[R]$ and once using $\mathbb{E}[R_{100\%}]$.
8. Weights under Different Confidence Levels: The New Combined Returns Vector $\mathbb{E}[R]$ based on the covariance matrix of the error term Ω leads to vector $\widehat{w}$ while the New Combined Return Vector $\mathbb{E}[R_{100\%}]$ based on 100% confidence levels leads to the vector $w_{100\%}$.
9. Implied Confidence Using Allocation Deviations: The departure of these new weight vectors from the vector of market capitalization weights w_{MKT} are $\widehat{w} - w_{MKT}$ and $w_{100\%} - w_{MKT}$ respectively. It is then possible to determine the *implied* level of confidence in these views by dividing each weight difference $\widehat{w} - w_{MKT}$ by the corresponding maximum weight difference $w_{100\%} - w_{MKT}$.
10. Implied Confidence Level of Views:

Asset Class	**Market Capitalization Weight $\mathbf{w}_{MKT}$**	**New Weight $\widehat{\mathbf{w}}$**	**Difference $\widehat{\mathbf{w}} - \mathbf{w}_{MKT}$**	**New Weights Based on 100%**	**Difference $\mathbf{w}_{100\%} - \mathbf{w}_{MKT}$**	**Implied Confidence Level $\frac{\widehat{\mathbf{w}} - \mathbf{w}_{MKT}}{\mathbf{w}_{100\%} - \mathbf{w}_{MKT}}$**

				Confidence $\mathbf{w}_{100\%}$		
US Bonds	19.34%	29.88%	10.54%	43.82%	24.48%	43.06%
International Bonds	26.13%	15.59%	-10.54%	1.65%	-24.48%	43.06%
US Large Growth	12.09%	9.35%	-2.73%	3.81%	-8.28%	33.02%
US Large Value	12.09%	14.82%	2.73%	20.37%	8.28%	33.02%
US Small Growth	1.34%	1.04%	-0.30%	0.42%	-0.92%	33.02%
US Small Value	1.34%	1.65%	0.30%	2.26%	0.92%	33.02%
International Developed Equity	24.81%	27.81%	3.63%	35.21%	11.03%	32.94%
International Emerging Equity	3.49%	3.49%	-	3.49%		

11. Idzorek's Empirical Implied Confidence Estimate: The implied level of confidence in a view based on the scaled variance of the individual view portfolios is shown in the Table above. The implied confidence levels in View 1, View 2, and View 3 in the instance above are 32.94%, 43.06%, and 33.02% respectively.
12. Principle Driver behind Idzorek's Approach: Given the discrepancy between the stated confidence levels and the implied confidence levels one could experiment with different ω's and recalculate the New Combined Return Vector $\mathbb{E}[R]$ and the new set of recommended portfolio weights. Idzorek's approach formalizes this algorithmically.

The Tilt Based Intuitive Approach

1. Tilt Induced by Specified Confidence: Idzorek (2005) proposes that the diagonal elements of be derived in a manner that is based on the user-specified confidence levels and that results in

portfolio tilts, which approximate $w_{100\%} - w_{MKT}$ multiplied by the user-specified confidence level C.

$$Tilt_k \approx (w_{100\%} - w_{MKT}) \times C_k$$

where $Tilt_k$ is the approximate tilt caused by the k^{th} view - $N \times 1$ column vector, and C_k is the confidence in the k^{th} view.

2. Incorporation of the Estimated Tilt: Furthermore in the absence of the other views the approximate recommended weight vector resulting from the view is

$$w_{k\%} \approx w_{MKT} + Tilt_k$$

where $w_{k\%}$ is the target weight vector based on the tilt caused by the k^{th} view - $N \times 1$ column vector.

3. Step #1 - 100% Confidence Returns: For each view k calculate the New Combined Return Vector $\mathbb{E}\left[R_{k,100\%}\right]$ using the Black Litterman formula under 100% certainty treating each view as if it were the only one.

$$\mathbb{E}\left[R_{k,100\%}\right] = \Pi + \tau\Psi {p_k}^T [{p_k}^T \Psi p_k]^{-1} [Q_k - p_k \Pi]$$

where $\mathbb{E}\left[R_{k,100\%}\right]$ is the Expected Return Vector based on 100% confidence on the k^{th} view - $N \times 1$ column vector, p_k identifies the assets involved in the k^{th} view - $1 \times N$ row vector, and Q_k is the k^{th} view returns.

4. Adjustment Applied to Absolute Views: If the view in question is an absolute view and the view is specified as a total return rather than as an excess return, subtract the risk-free rate from Q_k.

5. Step #2 - 100% Confidence Weights: Calculate $w_{k,100\%}$ the weight vector based on 100% confidence in the k^{th} view, using the unconstrained maximization formula

$$w_{k,100\%} = [\lambda\Psi]^{-1} \mathbb{E}\left[R_{k,100\%}\right]$$

6. Step #3 - Departure from Market: Calculate using pair-wise subtraction the maximum departures from the market capitalization weights caused by 100% confidence on the k^{th} view.

$$D_{k,100\%} = w_{k,100\%} - w_{MKT}$$

   where $D_{k,100\%}$ is the departure from the market capitalization weight based on 1005 confidence in k^{th} view - $N \times 1$ column vector.

7. Assets not in the View: The asset classes of $w_{k,100\%}$ that are not part of the k^{th} view retain their original weight leading to a value of 0 for the elements of $D_{k,100\%}$ that are not part of the k^{th} view.
8. Step #4 - 100% Confidence Tilt: Compute using pair-wise multiplication the N elements of $D_{k,100\%}$ by the unspecified confidence C_k in the k^{th} view to estimate the desired tilt caused by the k^{th} view.

$$Tilt_k = D_{k,100\%} \times C_k$$

   where $Tilt_k$ is the desired active weights caused by the k^{th} view - $N \times 1$ column vector – and C_k is an $N \times 1$ column vector where the assets that are part of the view receive the user-specified confidence level of the k^{th} view and the assets that are not part of the view are set to 0.

9. Step #5 - Tilt Based Weight: Estimate using pair-wise addition the target weight vector $w_{k,\%}$ based on the tilt:

$$w_{k,\%} = w_{MKT} + Tilt_k$$

10. Step #6 - Variance Error Minimization: Find the value of ω_k - the k^{th} diagonal element of Ω representing the uncertainty in the k^{th} view that minimizes the sum of the squared differences between $w_{k,\%}$ and w_k.

$$\min \sum \left(w_{k,\%} - w_k\right)^2$$

subject to

$$\omega_k > 0$$

where

$$w_k = [\lambda\Psi]^{-1}[(\tau\Psi)^{-1} + {p_k}^T {\omega_k}^{-1} p_k]^{-1}[(\tau\Psi)^{-1}\Pi + {p_k}^T {\omega_k}^{-1} Q_k]$$

11. Active Risk of the k^{th} View: Having just determined the weight vector associated with a specific view w_k it may be useful to calculate the active risk associated with the specific view in isolation as $\sqrt{{w_A}^T \Psi w_A}$
12. Active Risk Estimation Measure Dictionary: Here

$$w_A = w_k - w_{MKT}$$

is the active portfolio weights;

$$w_k = [\lambda\Psi]^{-1}[(\tau\Psi)^{-1} + {p_k}^T {\omega_k}^{-1} p_k]^{-1}[(\tau\Psi)^{-1}\Pi + {p_k}^T {\omega_k}^{-1} Q_k]$$

is the weight vector of the portfolio based on the k^{th} view and the user-specified confidence level, and Ψ is the covariance matrix of excess returns.
13. Constructing the Complete Ω Matrix: Repeat the above six steps for the k views, build a diagonal $K \times K$ matrix in which the diagonal elements of Ω are the ω_k values calculated in step #6, and solve for the New Combined Return Vector $\mathbb{E}[R]$ using

$$\mathbb{E}[R] = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Psi)^{-1}\mathit{\Pi} + P^T\Omega^{-1}Q]$$

14. Independence of the Prior Confidence Parameter: Throughout this process the value of the scalar τ is held constant and does not explicitly affect the New Combined Return Vector $\mathbb{E}[R]$ which eliminates the difficulties associated with specifying it.
15. Advantages of the Idzorek Approach: Despite the relative complexities of the steps for specifying the diagonal elements of Ω the key advantage of this new method is that it enables the user to determine the values of Ω based on an intuitive 0% to 100% confidence scale.
16. Alternate Methods for specifying Ω: Alternative methods for specifying the diagonal elements of Ω require one to specify these abstract values directly. Some of these approaches are explored in Zimmermann, Drobetz, and Oertmann (2002), Fusai and Meucci (2003), and Litterman (2003).

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Simplified Black Litterman Surplus Optimizer

Background

1. Investment Objectives of Individual Investors: Individual investors have vastly different investment objectives and constraints depending upon their expected future incomes, retirement pension benefits, and basic consumption needs, none of which appears in the standard single period MVO framework.
2. Infeasibility of the Inter-temporal Framework: Although such issues can be dealt with in a generalized inter-temporal optimization framework, formulating the problem and obtaining numerical solutions tend to be a complex, computationally intensive process.
3. Pension Fund Asset Liability Management: A much simpler approximate solution can be found, however, by extending the Surplus Optimization Framework, originally proposed by Leibowitz and Henriksson and explored by Sharpe and Tint (1990) in the context of pension fund asset-liability management.
4. Pension Fund ALM - Net Surplus: The basic insight is to apply the mean-variance approach to the "**surplus**' – as defined below – of the investor. In pension fund ALM "surplus" means "financial assets minus the PV of pension liabilities", but the definition used here is more general.
5. Components of Investors' Basic Consumption: One needs to recognize that the typical investor has substantial needs for "basic consumption" – consumption on basic goods and services – but also holds human capital (PV of future incomes) and can expect to receive retirement pension benefits.
6. Definition of Investors' Net Liability: Thus the natural definition of his/her **net liability** would be the PV of the basic consumption needs minus a conservative estimate of the human

capital, with the PV of the pension benefits to be further subtracted. If the net liability is negative one can ignore it and revert to the standard mean-variance approach.

7. <u>Estimating Profiles' Assets and Liabilities</u>: In practice one would need to estimate the values and the risk exposures of such liabilities based on the investors' responses to a few simple questions. Although estimation errors would be inevitably be significant, the belief is that it should still be significantly better to use rough estimates than to ignore all such information.
8. <u>Housing Assets and Mortgages Incorporation</u>: The plan is also to incorporate housing assets and mortgages, but this extension will be implemented at a later stage.
9. <u>Formulation of the Net Surplus</u>: Denote the return on financial assets by r_A and the returns on the net liability – the percentage changes in the PV of the net liability – by r_L.
10. <u>Net Surplus Optimization Objective Function</u>: Then the objective function is

$$\max\left(\mathbb{E}[r_S] - \frac{\gamma}{2}\mathbb{V}[r_S]\right)$$

where

$$r_S = r_A - q r_L$$

This is equivalent to

$$\max\left(\mathbb{E}[r_A] - \frac{\gamma}{2}\mathbb{V}[r_A] + \gamma q \mathbb{C}[r_A, r_L]\right)$$

where γ is the Investors' Risk Aversion Coefficient – not necessarily equal to the Global Risk Aversion Coefficient δ - and q is a non-negative constant that reflects the importance of the net liability – referred to here as **net liability weight**.

11. <u>Incorporating the Sharpe-Tint Formulation</u>: In the context of Pension Fund ALM, the Shape-Tint Formulation is

$$q = k\frac{L}{A}$$

which means that

$$r_S = r_A - k\frac{L}{A}r_L$$

where A is the amount of financial assets, L is the PV of the net liabilities, and k is the proportion of the net liability (between 0 and 1) that is taken into consideration.

12. <u>Full Pension Fund ALM Framework</u>: For a full pension fund ALM framework k would be set to 1 and q would therefore be equal to the inverse of the funding ratio.
13. <u>Estimating the Net Liability Weight</u>: In the current context one could calculate q if there is sufficient information on k, A, and L; otherwise values to q are simply assigned based on questionnaire's responses.
14. <u>Imposing the Black Litterman Approach</u>: Using the Black Litterman risk premia and the Black Litterman covariance matrix as inputs the first order condition for the optimum can be written as

$$\widehat{\Pi} - \gamma\Psi_p\widehat{w} + \gamma q\theta = 0$$

where

$$\theta = \begin{pmatrix} \mathbb{C}[r_1, r_L] \\ \vdots \\ \mathbb{C}[r_n, r_L] \end{pmatrix}$$

and r_i is the rate of returns on the class i.

15. <u>The Fund's Optimal Portfolio Weights</u>: The optimal portfolio weights are given by

$$\hat{w} = \frac{1}{\gamma}\Psi_{\mathrm{p}}{}^{-1}\hat{\Pi} + q\Psi_{p}{}^{-1}\theta$$

$$= Mean\ Variance\ Portfolio\ \left[\frac{1}{\gamma}\Psi_{\mathrm{p}}{}^{-1}\hat{\Pi}\right]$$

$$+ Liability\ Hedging\ Portfolio\ \left[q\Psi_{p}{}^{-1}\theta\right]$$

16. Components of the Optimal Portfolio: Thus the optimal portfolio would be the sum of a mean variance portfolio (MVP) and a liability hedging portfolio (LHP).
17. Extremes of Net Liability Weight: For a retiree with modest financial assets q would be relatively large, and the LHP would thus be an independent component of his portfolio. On the other hand for a wealthy person q would be small and therefore the optimal portfolio would be close to the standard MVP.
18. Estimating Asset Liability Covariance: In order to estimate $\mathbb{C}[r_i, r_L]$ one generates the time series of r_L based on the investors' projected net cash outflows – basic consumption expenditures minus conservative estimates of wages and pension incomes – and historical interest rate data, and then compute the historical covariance of r_L with the asset class i.

Black Litterman Surplus Optimizer Inputs

1. Time Series of Unhedged JPY Returns: Historical time series of ***unhedged*** JPY rates of returns on $\boldsymbol{n}$ asset classes based on the Topix, Nomura, MCSI, Citi, and/or Barclay's Capital Indices. "Unhedged JPY" means that the returns are not currency hedged with short term FX forward contracts.
2. Time Series of Hedged JPY Returns: Historical time series on ***hedged*** JPY returns on $\boldsymbol{m}$ $(< \boldsymbol{n})$ foreign asset classes based on the MSCI, Citi, and/or Barclays' Capital Indices. "Hedged JPY" means that the returns are currency hedged with short term FX forward contracts.

3. Current Asset Class Market Capitalization: Current market capitalization for the asset classes based on the Topix, Nomura, MSCI, Citi, and/or Barclays' Capital Indices.
4. Time Series of JPY LIBOR/Swap: Historical time series of JPY LIBOR and JPY Swap yields.
5. JGB Time Series: Historical time series of JGB Benchmark yields published by the Japanese Ministry of Finance.
6. Investor Total Financial Assets Input: The investor needs to provide his/her total financial asset W, which will be used in the final stage of the optimization – to calculate the actual amount of assets to buy – and the non-financial income Y_W which is used for the net cash flow projections as described later.
7. Questionnaire Based Exogenous Parameter Setting: In addition investor responses to the Questionnaire have to be translated into actual values for the following parameters. These parameters, however, can be treated as exogenous when developing the Optimizer.
8. The Risk Aversion Coefficient γ: This parameter can be expected to be somewhere between 1 and 10. The most typical value may be approximately 4.
9. The Net Liability Weight q: This parameter can be any non-negative number, but it makes sense to restrict this to be between 0 and 1 in order to reduce the likelihood of getting a solution involving a leveraged liability hedging portfolio.
10. Uncertainty of Equilibrium Variance τ: "Tau", the measure of the uncertainty of the equilibrium variance, is typically an administrator specified parameter. Walters (2014) contains a section on the various ways of estimating it in practice.

Cash Flow Projections and Liability Returns

1. In/out Cash Flow PV: In order to calculate the net liability one must first make a simple projection of the investor's future cash inflows and outflows, and then take the PV's of these cash flows.

2. Specification of the Investor Age: Denote the current age, the retirement age, and the maximum age of the investor by A_{NOW}, A_{RET}, and A_{MAX} respectively. Let T denote the time horizon in years with

$$T = 0$$

being the present.

3. Investors' Net Cash Out-flow: On expects the net cash outflow to be negative during the working life and positive in retirement. The typical value for A_{RET} would be 65 while A_{MAX} could either be set by the investor or be fixed at 85.

4. Expressions for In/Out Flows: For simplicity the following assumptions are made.

$$\mathbb{E}[Y(T)] = \delta(T)Y^* + [1 - \delta(T)]rY^*$$

$$\mathbb{E}[C(T)] = \delta(T)c_w Y^* + [1 - \delta(T)]c_r Y^*$$

$$T = 1, \cdots, A_{MAX} - A_{NOW}$$

where

$$\delta(T) = \begin{cases} 1 \ for \ T \leq A_{RET} - A_{NOW} \ (until \ retirement) \\ 0 \ for \ A_{RET} - A_{NOW} + 1 \leq T \leq A_{MAX} - A_{NOW} \ (after \ retirement) \end{cases}$$

5. Working Age and Retirement Incomes: Y^* is the working age income, which is assumed to be constant throughout the working life. At retirement the income is reduced to rY^* where r is the replacement rate of the expected pension benefits.

6. Working Age and Retirement Consumption: Both working age and retirement age basic consumption expenditures are assumed to be constant fractions - c_w and c_r respectively – of the working age income Y^*. A plausible set of parameters may be

$$c_w = 0.8$$

$$c_r = 0.6$$

and

$$r = 0$$

Once the parameters values are assumed as above all that needs to be known is the working age income Y^* which may be approximated by the current income.

7. PV of the Net Liability: The next step is to calculate the net liability, i.e., the present value of the investors' net cash outflows. Given the value of the continuously compounded T-year zero coupon rate $y^*(t,T)$ at time t the T-year discount factor $D_f(t,T)$ is calculated as

$$D_f(t,T,s) = e^{-[y^*(t,T)+s]} \; T = 1, \cdots, A_{MAX} - A_{NOW}$$

were s is the credit/liquidity spread applicable to the discounting of the investors' future cash flows.

8. Discounting Spread for Income/Consumption: The spread s is set to s_Y when discounting income cash flows, whereas it is set to s_C when discounting basic consumption cash flows. The spread for income may be somewhat higher than that for basic consumption, although there are reasons for them not to diverge too far from each other.

9. Flat 10Y JGB Yield Rate: For the sake of simplicity a flat term structure is assumed, treating all zero coupon rates as equal to the 10Y JGB yield $y(t)$

$$D_f(t,T,s) = e^{-[y(t,T)+s]} \; T = 1, \cdots, A_{MAX} - A_{NOW}$$

10. PV of the Net Liability: One can easily incorporate the non-flat yield curve by using, for example, splines, or parametric models such as the Nelson-Siegel model. Net liability can then be calculated as

$$L(t) = \sum_{T=1}^{A_{MAX}-A_{NOW}} D_f(t, T, s_C)\mathbb{E}[C(T)] - \sum_{T=1}^{A_{MAX}-A_{NOW}} D_f(t, T, s_Y)\mathbb{E}[Y(T)]$$

11. Net Liability Rate of Returns: Using the historical time series of the JGB yield $y(t)$ and the equation above for net liability one can calculate the time series of $L(t)$ from which the following time series can be obtained:

$$r_L(t) = \frac{L(t)}{L(t-1)} - 1$$

12. Asset Liability Covariance Estimation: By combining these data with the time series of returns on the asset classes one obtains an estimate of θ as

$$\theta = \begin{pmatrix} \mathbb{C}[r_1, r_L] \\ \vdots \\ \mathbb{C}[r_n, r_L] \end{pmatrix}$$

where r_i is the returns on the asset class i.

13. Handling Zero Net Liability Levels: Since one might encounter the problem of singularity when the net liability is sufficiently close to zero, one needs to have some procedures for dealing with such cases.

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Multiple Security Portfolios Optimal Trajectory

Trading Model

1. Nomenclature - Holdings and Trade List: With m securities (Almgren and Chriss (2000)), the position at each moment is a column vector

$$x_k = (x_{1k}, \cdots, x_{mk})^T$$

The initial values are

$$x_0 = X = (X_1, \cdots, X_m)^T$$

and the trade list is the column vector

$$n_k = x_{k-1} - x_k$$

If

$$x_{jk} > 0$$

then the security j is held short at time t_k; if then the security is being bought between t_{k-1} and t_k.

2. Multi-dimensional Arithmetic Random Walk: The model assumes that the column vector of the security price S_k follows a multi-dimensional arithmetic Brownian random walk with zero drift. Its dynamics are written as

$$S_k = S_{k-1} + \sigma\sqrt{\tau_k}\xi_k - \tau_k g\left(\frac{n_k}{\tau_k}\right)$$

where

$$\xi_k = (\xi_{1k}, \cdots, \xi_{rk})^T$$

is a vector of r independent Brownian increments with

$$r \le m$$

with σ an $m \times r$ additive volatility matrix.

$$C = \sigma\sigma^T$$

is the $m \times m$ symmetric positive definite variance-covariance matrix.

3. Per Asset Market Impact Function: The permanent impact $g(v)$ and the temporary impact $h(v)$ are vector functions of a vector. Only the following linear model is considered:

$$g(v) = \Gamma v$$

$$h(v) = \epsilon sgn(v) + \mathrm{H}v$$

where Γ and H are $m \times m$ matrices and ϵ is an $m \times 1$ column vector multiplied component-wise by $sgn(v)$. The ij element of Γ and of H represents the price depression on security i caused by selling security j at a unit rate.

4. Need for Positive Definite H: H is required to be positive definite, since if there were a non-zero v with

$$v^T \mathrm{H} v \le 0$$

then by selling at a rate v at a net benefit – or at least no loss – is obtained from the instantaneous market impact. Η and Γ are not assumed to be symmetric.

5. Portfolio Expected Loss and Variance: The market value of the initial position is $X^T S_0$. The loss in value is incurred by a liquidation profile $x_1, \cdots, x_N$ is calculated just as in

$$\sum_{k=1}^{N} n_k \tilde{S}_k = XS_0 + \sum_{k=1}^{N} \left[\sigma\sqrt{\tau_k}\xi_k - \tau_k g\left(\frac{n_k}{\tau_k}\right)\right] x_k$$

to obtain the matrix equivalent of

$$\mathbb{E}[x] = \sum_{k=1}^{N} \tau_k x_k g\left(\frac{n_k}{\tau_k}\right) + \sum_{k=1}^{N} n_k h\left(\frac{n_k}{\tau_k}\right)$$

$$\mathbb{V}[x] = \sigma^2 \sum_{k=1}^{N} \tau_k {x_k}^2$$

i.e.,

$$\mathbb{E}[x] = \epsilon^T |X| + \frac{1}{2} X^T \Gamma_S X + \sum_{k=1}^{N} \tau_k {v_k}^T \widetilde{\mathrm{H}} v_k + \sum_{k=1}^{N} \tau_k {x_k}^T \Gamma_A v_k$$

$$\mathbb{V}[x] = \sum_{k=1}^{N} \tau_k {x_k}^T C x_k$$

with

$$\widetilde{\mathrm{H}} = \mathrm{H}_S - \frac{1}{2}\tau\Gamma_S$$

6. <u>Decomposition of H and Γ</u>: The subscripts S and A are used to represent the symmetric and anti-symmetric parts respectively, so

$$\mathrm{H} = \mathrm{H}_S + \mathrm{H}_A$$

and

$$\Gamma = \Gamma_S + \Gamma_A$$

with

$$\mathrm{H}_S = \frac{1}{2}[\mathrm{H} + \mathrm{H}^T]$$

$$\Gamma_S = \frac{1}{2}[\Gamma + \Gamma^T]$$

$$\Gamma_S = \frac{1}{2}[\Gamma - \Gamma^T]$$

7. <u>Advantage of the $\mathrm{H}_S/\widetilde{\mathrm{H}}$ Formulation</u>: Note that H_S is positive definite as well as symmetric. It shall be assumed that τ is small enough so that $\widetilde{\mathrm{H}}$ is positive definite and hence invertible. The assumption is that each component of v has a consistent sign throughout the liquidation.
8. <u>The Scalar Market Impact Functionals</u>: Despite the multi-dimensional complexity of the problem, the set of all outcomes is completely defined by these two scalar functionals. The utility function and the value at risk objective functions are still given in terms of $\mathbb{E}[x]$ and $\mathbb{V}[x]$ by

$$\mathbb{U}[x] = \lambda_v \mathbb{V}[x] + \mathbb{E}[x]$$

and

$$\mathbb{V}ar_P[x] = \lambda_v \sqrt{\mathbb{V}[x]} + \mathbb{E}[x]$$

Optimal Trajectories

1. Multi-asset Holdings Difference Equations: Determination of the optimal trajectory for a portfolio is again a linear problem. It is readily found that stationarity of $\lambda_v \mathbb{V}[x] + \mathbb{E}[x]$ with respect to the variation of x_{jk} gives the multi-dimensional extension of

$$\frac{1}{\tau^2}\left(x_{j-1} - 2x_j + x_{j+1}\right) = \tilde{\kappa} x_j$$

$$\frac{x_{k-1} - 2x_k + x_{k+1}}{{\tau_k}^2} = \lambda \widetilde{H}^{-1} C x_k + \widetilde{H}^{-1} \Gamma_A \frac{x_{k-1} - x_{k+1}}{\tau_k}$$

for

$$k = 1, \cdots, N-1$$

2. Simplification under Change of Variables: Since $\widetilde{H}^{-1} C$ is not necessarily symmetric and $\widetilde{H}^{-1}$ is not necessarily anti-symmetric (despite the symmetry of $\widetilde{H}$) it is convenient to define a new solution variable using

$$y_k = \widetilde{H}^{\frac{1}{2}} x_k$$

3. Solution for Rebalanced Portfolio: One then obtains

$$\frac{y_{k-1} - 2y_k + y_{k+1}}{{\tau_k}^2} = \lambda A x_k + B \frac{x_{k-1} - x_{k+1}}{2\tau_k}$$

in which

$$A = \widetilde{H}^{\frac{1}{2}} C \widetilde{H}^{-\frac{1}{2}}$$

and

$$B = \widetilde{H}^{\frac{1}{2}} \Gamma_A \widetilde{H}^{-\frac{1}{2}}$$

are symmetric positive-definite and anti-symmetric, respectively. This is a linear system in $(\mathrm{N} - 1)m$ which can be solved numerically.

Explicit Solution for a Diagonal Model

1. Diagonal Γ and H Matrices: To write explicit solutions, the *diagonal* assumption that trading affects the price of that security only and no other prices is made. This corresponds to taking Γ and H to be diagonal matrices, with

   $$\Gamma_{jj} = \gamma_j$$

   and

   $$\mathrm{H}_{jj} = \eta_j$$

   It is required

   $$\gamma_j > 0$$

and

$$\eta_j > 0$$

2. <u>Parsimonization of the Estimation Coefficients</u>: With this assumption the number of coefficients needed in the model is equal to the number of securities, and their values can be plausibly estimated from the available data. For diagonal Γ and H, $\mathbb{E}[x]$ decomposes into a collection of sums over each security separately, but the co-variances still couple the whole system.
3. <u>Diagonalized Version of the $\tilde{\eta}$ Matrix</u>: In particular, since Γ is now symmetric

$$\Gamma_A = 0$$

and hence

$$B = 0$$

Further, $\tilde{H}$ is diagonal with

$$\tilde{H}_{jj} = \eta_j \left(1 - \frac{\tau_j \gamma_j}{2\eta_j}\right)$$

The diagonal elements are required to be positive, which will be the case if

$$\tau_j < \frac{2\eta_j}{\gamma_j}$$

or

$$\tau < \min_j \frac{2\eta_j}{\gamma_j}$$

Then the inverse of the square root is trivially computed.

4. Diagonalized Version of $\kappa/\tilde{\kappa}$: For

$$\lambda > 0$$

λA has a complete set of positive eigenvalues which we denote by ${\tilde{\kappa}_1}^2, \cdots, {\tilde{\kappa}_m}^2$ and a complete set of orthonormal eigenvectors Which form he columns of an orthogonal matrix U. The solution in the diagonal case is the combination of the exponentials $e^{\pm\kappa_j t}$ with

$$\frac{2}{\tau^2}\left(\cosh \kappa_j t - 1\right) = \tilde{\kappa}_j^{\;2}$$

5. Trajectory of the Holdings Portfolio: With

$$y_k = U_{z_k}$$

it may be written

$$z_{jk} = \frac{\sinh \kappa_j (T - t_k)}{\sinh \kappa_j t} z_{j0}$$

in which the column vector z_0 is given by

$$z_0 = U^T y_0 = U^T \widetilde{\mathrm{H}}^{\frac{1}{2}} X$$

Undoing all the changes of variables produces finally

$$x_k = \widetilde{\mathrm{H}}^{-\frac{1}{2}} U z_k$$

6. Caveat – Non-monotonic Trading Velocities: With multiple securities it is possible for some components of the velocities to be non-monotonic in time. For example, if the portfolio includes two securities whose fluctuations are highly correlated, with one much more liquid than the other, an optimal strategy could direct rapidly going short on the liquid one to reduce risk, whole slowly reducing the whole position to zero.
7. Consequence of Caveat - Estimation Inaccuracy: In this case, the above expressions are not exactly correct because of the changing sign of the cost associated with the bid-ask spread. Since this effect is probably very small, a reasonable approach in such a case is to set

$$\epsilon = 0$$

Almgren and Chriss (2000) Example

1. Sample Portfolio Different Liquidity Levels: Almgren and Chriss (2000) briefly consider an example with only two securities. For the first, the parameters used are the same as the ones seen earlier. The second security is chosen to be more liquid and less volatile, with a moderate amount of correlation. The parameters are summarized below. From the market data, the model parameters are determined exactly as seen earlier.
2. Parameters for the Two Security Example:

$$Share\ Price = \begin{pmatrix} \$50 \\ \$100 \end{pmatrix}$$

$$Daily\ Volume = \begin{pmatrix} 5\ million \\ 20\ million \end{pmatrix}$$

$$Annual\ Variance = \begin{pmatrix} 30\% & 10\% \\ 10\% & 15\% \end{pmatrix}$$

3. Efficient Frontier and the Corresponding Trajectories: The initial holdings are 10 $million$ shares in each security; the trading horizon is

$$T = 5\ days$$

and uses

$$N = 5\ periods$$

Almgren and Chriss (2000) graphically illustrate the efficient frontier in this case, and depict three trading trajectories corresponding to three points on the frontier.
4. Inter-dependence - Impact of Correlation: For these example parameters, Almgren and Chriss (2000) demonstrate that the trajectory of security 1 is almost to its trajectory in the absence of security 2. Increase in the correlations of the two securities increases the inter-dependence of their trajectories; the expectation is that relaxing the assumption of diagonal transaction costs would have the same effect.

References

- Almgren, R. F. and N. Chriss (2000): Optimal Execution of Portfolio Transactions *Journal of Risk* **3 (2)** 5-39.