Insights into Robust Portfolio Optimization: Decomposing Robust Portfolios into Mean-Variance and Risk-Based Portfolios

Romain Perchet

is head of Investment Solutions in the Financial Engineering team at

BNP Paribas Investment Partners

14 rue Bergère, 75009 Paris, France

and PhD student at 'École des Hautes Études en Sciences Sociales'

(EHESS-CAMS), Paris, France.

romain.perchet@bnpparibas.com, Tel. +33 (0)1 58 97 28 23

Xiao Lu

is head of Quantitative Research in the Financial Engineering team at

BNP Paribas Investment Partners

14 rue Bergère, 75009 Paris, France.

xiao.lu@bnpparibas.com, Tel. +33 (0)1 58 97 75 64

Raul Leote de Carvalho

is deputy head of Financial Engineering team at

BNP Paribas Investment Partners

14 rue Bergère, 75009 Paris, France.

raul.leotedecarvalho@bnpparibas.com, Tel. +33 (0)1 58 97 21 83

Thomas Heckel

is head of Financial Engineering team at

BNP Paribas Investment Partners

14 rue Bergère, 75009 Paris, France.

thomas.heckel@bnpparibas.com, Tel. +33 (0)1 58 97 76 13

September 7th 2015

Abstract:

For a number of different formulations of robust portfolio optimization, quadratic and absolute, we show that a) in the limit of low uncertainty in estimated asset mean returns the robust portfolio converges towards the mean-variance portfolio obtained with the same inputs; and b) in the limit of high uncertainty the robust portfolio converges towards a risk-based portfolio, which is a function of how the uncertainty in estimated asset mean returns is defined. We give examples in which the robust portfolio converges toward the minimum variance, the inverse variance, the equal-risk budget and the equally weighted portfolio in the limit of sufficiently large uncertainty in asset mean returns. At intermediate levels of uncertainty we find that a weighted average of the mean-variance portfolio and the respective limiting risk-based portfolio offer a good representation of the robust portfolio, in particular in the case of the quadratic formulation. The results remain valid even in the presence of portfolio constraints, in which case the limiting portfolios are the corresponding constrained mean-variance and constrained risk-based portfolios. We believe our results are important in particular for risk-based investors who wish to take into account expected returns to gently tilt away from their current allocations, e.g. risk parity or minimum variance.

Keywords: portfolio optimization, portfolio construction, robust optimization, risk-based portfolios, minimum variance, risk parity, equal-risk budget, equally-weighted, mean-variance, Markowitz

The idea that financial decisions and portfolio construction should be based on the trade-off between risk and return was introduced more than 60 years ago by Markowitz [1952]. This idea is the backbone of mean-variance optimization (MVO) which selects the optimal portfolio by trading-off portfolio return, estimated from the mean value of the random asset returns, against portfolio risk, estimated from asset return variances and covariances. In MVO, these inputs are assumed to be deterministic and the optimal portfolio can be found by solving a convex quadratic programming problem seeking either the maximization of portfolio returns for a maximum level of tolerated portfolio risk or minimizing the portfolio risk for a minimum level of acceptable portfolio returns.

As shown by Black and Litterman [1992], the optimal portfolio selected with MVO is *not* robust with respect to small changes in inputs, in particular in asset mean returns, since these can lead to large changes in portfolio allocation. Moreover, in MVO, assets with positive estimation error will be over-weighted while assets with negative estimation error will be under-weighted, which is why MVO is sometimes referred to as an error maximizer. Since estimates of market parameters are subject to significant statistical errors, the portfolios obtained from MVO are often not workable from a practical point of view. Kritzman [2006] discusses this at length.

Chopra [1993] and Frost and Savarino [1988] suggested adding portfolio constraints to the MVO problem so that the optimal portfolio is kept within feasible bounds. Unfortunately the optimal portfolio tends to be sensitive to the choice of portfolio constraints, which is often quite subjective. This solution is thus relatively arbitrary and of limited practical interest.

A more sophisticated approach is to use robust estimation approaches so as to reduce the statistical error in the mean returns. This is achieved by using some form of shrinkage of the mean returns towards a sensible value. Chopra and Ziemba [1993] proposed the use of the James-Stein estimator to shrink mean returns towards equal returns, while Black and Litterman [1990] proposed shrinking mean returns towards market-implied returns. While these approaches help in rendering MVO more robust, they also require additional shrinkage parameters that are not necessarily clear for estimation purposes and can have a significant impact on the optimal portfolio allocation.

More recently, we have seen an increase in interest in risk-based portfolio construction methods, as discussed by Leote De Carvalho, Lu and Moulin [2012]. These approaches increase the robustness of the optimal portfolio by avoiding the estimation of any or all of the mean returns, variance and covariances. Instead, these are set to what may be regarded as sensible values[1]. The minimum-variance portfolio is mean-variance efficient and has the highest Sharpe ratio if the asset mean returns are all equal. The equal risk budgeting portfolio proposed by Dalio [2005] (also known as risk parity) is mean-variance efficient with the highest Sharpe ratio portfolio if we assume that the asset mean returns are proportional to the asset volatilities and also that the return correlations are all equal. The

inverse variance (IV) portfolio is mean-variance efficient with the highest Sharpe ratio if all the assets have the same mean returns and the correlations between the returns of any pair of assets are equal. Finally, the equally-weighted portfolio is mean-variance efficient with the highest Sharpe ratio if all asset mean returns, variances and correlations are equal. Despite growing popularity, these approaches have been criticized for their implicit assumptions on asset mean returns. For example, the success of the equal risk budgeting approach (risk parity) to multi-asset portfolios has been attributed to its overweight in Treasuries when compared to more traditional multi-asset approaches. The future success of the approach for multi-asset portfolios is now in question when one takes into account the extremely low level of yields that makes it less likely for Treasuries to continue to perform as well in the future as they have in the past.

Michaud [1998] proposed the use of re-sampling methods to generate different samples, each with their own mean, variances and covariances. MVO can be applied to each sample and the final portfolio is an aggregation of the portfolios generated for each sample. Scenario-based stochastic programming methods also address the problem of uncertainty in the inputs by generating different scenarios in advance. Traditional stochastic linear programming selects the portfolio that produces the best objective function value over all scenarios. Dynamic programming methods could be used to deal with stochastic uncertainty over multiple periods, with the optimization problem solved backwards recursively, computing the optimal portfolio at each period. However, these approaches are not only computationally intensive; they become inefficient when the number of assets grows, as discussed by Scherer [2002], and they lack a strong theoretical underpinning.

Robust portfolio optimization was introduced by Lobo, Vandenberghe, Boyd and Lebret [1998] as a tractable alternative to stochastic programming. It is an extension of the robust optimization framework proposed by Ben-Tal and Nemirovski [1998], who study convex optimization while taking into account uncertainty in the data. This approach to portfolio selection considers the uncertainty associated with the inputs in the optimization problem. Instead of assuming mean returns, variances and covariances as deterministic, robust optimization seeks the optimal portfolio that remains optimal for all values of returns, variances and covariances that remain close to their estimated values. Relatively general assumptions on the distributions of the uncertainty of inputs are made in order to come up with formulations of the portfolio selection problem that are more tractable. These formulations are typically worst-case scenario optimizations of the original portfolio selection problem in terms of deviations of inputs from their normal values.

Tütüncü and König [2004] considered that neither the asset mean returns nor the covariance matrix is known deterministically. Instead, there are many possible sets of asset mean returns and many possible covariance matrices. Their early work introduces the idea of splitting the robust optimization

into a two-step max-min optimization problem: they propose a solution where they seek the vector of portfolio weights that maximizes the worst case utility for all possible combinations of covariance matrices and asset mean return vectors. They show that this problem can be simplified, in the case of a long only portfolio, to that of solving a traditional MVO problem for the worst variance covariance matrix and for the worst expected return vector.

Ceria and Stubbs [2006] reformulated the problem of robust optimization by proposing a quadratic form of the robust portfolio in which they create a distinction between the estimated covariance Σ of asset returns and the covariance matrix Ω of estimation errors in asset mean returns. Ceria and Stubbs [2006] claim this separation yields more robust portfolios but do not in our view provide sufficient details with regard to the choice of Ω or the impact of that choice on the robust optimization process. They also propose an additional constraint on how the uncertainty of returns is defined – the zero net adjustment – to render the robust optimization less conservative.

Scherer [2006] discusses the choice of Ω proportional to Σ. In this case, he demonstrates that robust optimization is equivalent to a non-normalized shrinkage approach. The robust portfolio is an exact-weighted average of the MVO and the minimum variance (MV) portfolio, at least in the absence of portfolio constraints other than the budget constraint, i.e. the sum of portfolio weights equals one. He also shows that if the estimation error is large then the robust portfolio converges towards the MV portfolio.

Fabozzi, Kolm, Pachamanova and Focardi [2007] introduced a form of robust optimization that is valid when the asset mean returns are not too far from the *true* expected returns. In this formulation, the absolute spread between the estimated and the *true* expected returns should be smaller than a given level of uncertainty, κ. However, the differences between this and other formulations were not discussed.

The goal of this paper is to take the understanding of robust optimization one step further. By looking at limiting portfolios in the cases of both large and small uncertainty in mean returns, we provide the intuition necessary to demystify portfolios selected by robust optimization. For this we extend the work of Ceria and Stubbs [2006], Scherer [2006] and Fabozzi et al. [2007] and consider both quadratic and absolute formulations of the robust optimization problem with different definitions of the uncertainty box about mean returns. In all the cases we considered, we find that the robust optimization problem can be reasonably well represented by a weighted average of the mean-variance portfolio and a risk-based portfolio. We also show that the robust portfolio converges towards the mean-variance portfolio when the uncertainty in the estimation of asset mean returns is low. In the limit of large uncertainty we show that the robust portfolio converges towards risk-based portfolios.

The risk-based portfolio is a function of how the uncertainty box about mean returns is defined. For the quadratic formulation of Ceria and Stubbs [2006], if $\boldsymbol{\Omega}$ is proportional to $\boldsymbol{\Sigma}$, we find that the robust portfolio converges towards MV. If $\boldsymbol{\Omega}$ is given by the identity matrix then we find the equally weighted (EW) portfolio in the limit of large uncertainty. If $\boldsymbol{\Omega}$ is a diagonal matrix only containing variances then we find that the robust portfolio converges towards the IV weighted portfolio. With this choice of $\boldsymbol{\Omega}$, if we add a zero-net adjustment constraint to the Sharpe ratio of assets we then find that the robust portfolio converges towards the equal-risk budgeting portfolio. For the absolute formulation we find that the robust portfolio converges towards the EW portfolio if the uncertainty is the same for all asset mean returns, and converges towards the equal risk-budget (ERB) portfolio if the uncertainty is the same for the Sharpe ratio of all assets.

We show that where there are small degrees of uncertainty, the robust portfolio deviates quickly from the mean-variance portfolio. If $\boldsymbol{\Omega}$ is proportional to $\boldsymbol{\Sigma}$ then the robust portfolio is always an exact-weighted average of the MVO portfolio and the MV portfolio, even when portfolio constraints are imposed. For other forms of the uncertainty box, the decomposition of the robust portfolio into a MVO and a risk-based portfolio is not exact but continues to be an approximate representation of the underlying robust portfolio allocation, even in the presence of portfolio constraints. This representation of the robust portfolio as a weighted average of the mean-variance portfolio and a risk-based portfolio is more accurate in the case of the quadratic formulation, as confirmed by numerical examples.

In the reminder of the paper we will overview some key aspects of mean-variance, risk-based and robust optimization that will support our claims. We introduce different formulations of robust optimization, quadratic and absolute, and derive the limiting risk-based portfolios found in the case of large estimation error. We conduct numerical exercises to investigate the claim that a weighted average of the MVO and a risk-based portfolio provides a good simplified representation of the robust portfolio and found this to be confirmed in the case of the quadratic formulation and also to some extent in the case of the absolute formulation. We also show that the analysis holds true even in the presence of portfolio constraints. Finally in this paper, we propose some applications of the results.

MEAN-VARIANCE PORTFOLIOS

The MVO portfolio selection problem can be written in terms of the following well-known quadratic problem for the asset weights in the portfolio for the explicit trade-off between portfolio returns and portfolio risk:

$$\mathbf{w}_{MVO} = argmax\left(\boldsymbol{\mu}^t\mathbf{w} - \frac{\lambda}{2}\mathbf{w}^t\boldsymbol{\Sigma}\mathbf{w}\right)$$
$$s.t.\,\mathbf{1}^t\mathbf{w} = 1$$

(1)

Here $\boldsymbol{\mu}$ is the vector of the estimated mean value of asset returns, $\boldsymbol{\Sigma}$ is the estimated covariance matrix of asset returns, λ the risk aversion of the investor and $\mathbf{1}$ is the vector of ones, $\mathbf{1}=(1, \ldots, 1)$. A closed solution to this optimization problem can be written in the form:

$$\mathbf{w}_{MVO} = \frac{\Sigma^{-1}\boldsymbol{\mu}}{\mathbf{1}^t\Sigma^{-1}\boldsymbol{\mu}} \tag{2}$$

Another well-known formulation of the closed solution to the MVO problem is:

$$\mathbf{w}_{MVO} = \frac{1}{\lambda}\Sigma^{-1}\left(\bar{\boldsymbol{\mu}} - \frac{\bar{\boldsymbol{\mu}}^t\Sigma^{-1}\mathbf{1}}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}}\mathbf{1}\right) + \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}} = \mathbf{w}_{speculative} + \mathbf{w}_{MV} \tag{3}$$

which separates the portfolio weights into the sum of a speculative portfolio $\mathbf{w}_{speculative}$, which is a function of the asset mean returns, and a MV portfolio $\mathbf{w}_{MV}$, which is not. This solution can be found by solving the optimization problem in equation (1) using its Lagrangian $L(\mathbf{w},\theta)$, where θ is the Lagrange multiplier associated with the budget constraint $\mathbf{1}^t\mathbf{w} = 1$:

$$L(\mathbf{w},\theta) = \bar{\boldsymbol{\mu}}^t\mathbf{w} - \frac{\lambda}{2}\mathbf{w}^t\Sigma\mathbf{w} - \theta(\mathbf{1}^t\mathbf{w} - 1) \tag{4}$$

Equation (3) can be obtained by differentiating the Lagrangian (4) with respect both to the Lagrange multiplier θ and to the portfolio weights $\mathbf{w}$, setting these partial derivatives to zero and then solving for the Lagrange multiplier and substituting it back into the condition for the derivative with respect to the portfolios weights:

$$\frac{\partial L}{\partial \mathbf{w}} = \bar{\boldsymbol{\mu}} - \lambda\Sigma\mathbf{w} - \theta\mathbf{1} = 0 \tag{5}$$

The MVO optimal portfolio often contains short positions and tends to be oversensitive to small changes in the mean value of asset returns $\boldsymbol{\mu}$ and also to changes in the covariance matrix $\boldsymbol{\Sigma}$. Small changes in the vector of mean returns are likely to lead to large changes in the portfolio, in particular in the presence of fairly close substitutes, i.e. strongly correlated assets. Because of estimation error in inputs, the actual efficient frontier estimated from MVO can be far from the true frontier and in general the estimated frontier is above the true frontier.

RISK-BASED PORTFOLIOS

Risk-based portfolios do not require an explicit estimation of asset mean value returns. These approaches for portfolio construction rely solely on the estimation of asset variances and covariances for managing portfolio risk and for increasing diversification. The approaches presented below are solutions to the MVO portfolio selection problem under certain simple assumptions for the asset mean returns, for the variances or for the covariances.

The EW portfolio is the simplest approach to portfolio construction based on the idea of diversification, allocating the same dollar amount to each asset. For N assets, the weight of each asset is the same and equal to N^{-1} and the vector of asset weights $\mathbf{w}_{EW}$ in the portfolio can be written as:

$$\mathbf{w}_{EW} = \frac{\mathbf{I1}}{\mathbf{1}^t\mathbf{I1}} = \frac{\mathbf{I}^{-1}\mathbf{1}}{\mathbf{1}^t\mathbf{I}^{-1}\mathbf{1}} \tag{6}$$

with $\mathbf{I}$ the identity matrix, i.e. a diagonal matrix of ones. This portfolio is mean-variance efficient, maximizing the Sharpe ratio if we assume that all assets have the same mean value return and the same volatility and if all pair-wise correlations are equal.

The ERB portfolio allocates the same risk budget to all assets and requires only the estimation of asset volatilities. The risk budget allocated to asset i is the product of the asset volatility σ_i with the asset weight w_i. Thus, the weight of asset i in the ERB portfolio is just the inverse of the volatility normalized by the sum of the volatility of all assets, $w_i = (1/\sigma_i)/(\sum_i 1/\sigma_i)$, and the vector of asset weights $\mathbf{w}_{ERB}$ in the portfolio is then:

$$\mathbf{w}_{ERB} = \frac{\mathbf{\Lambda}^{-1}\mathbf{1}}{\mathbf{1}^t\mathbf{\Lambda}^{-1}\mathbf{1}} \tag{7}$$

with $\mathbf{\Lambda}$ a diagonal matrix of asset volatilities. If the Sharpe ratio for each asset is the same and all pair-wise correlations are equal then the ERB portfolio is mean-variance efficient and has the highest possible Sharpe ratio.

In the IV portfolio the weight of each asset is inversely proportional to the variance σ_i^2 of asset i, $w_i = (1/\sigma_i^2)/(\sum_i 1/\sigma_i^2)$, and the vector of asset weights $\mathbf{w}_{IV}$ in the portfolio is then:

$$\mathbf{w}_{IV} = \frac{\mathbf{A}^{-1}\mathbf{1}}{\mathbf{1}^t\mathbf{A}^{-1}\mathbf{1}} \tag{8}$$

with $\mathbf{A}$ a diagonal matrix of asset variances. This portfolio is mean-variance efficient and maximizes the Sharpe ratio if all assets have the same mean value return and if all pair-wise correlations are equal to the same constant. This portfolio construction requires the estimation of asset variances only.

The MV portfolio has the lowest possible estimated variance and requires the estimation only of the asset covariance matrix $\mathbf{\Sigma}$. The asset weights in the MV portfolio can be estimated from:

$$\mathbf{w}_{MV} = \frac{\mathbf{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^t\mathbf{\Sigma}^{-1}\mathbf{1}} \tag{9}$$

If not constrained, the MV portfolio is likely to include short positions for the riskier assets. The MV portfolio is always optimal in a mean-variance sense. In the case where all asset returns are equal then the MV portfolio also maximizes the Sharpe ratio.

ROBUST OPTIMIZATION

Lobo, Vandenberghe, Boyd and Lebret [1998] introduced robust portfolio optimization, a process which takes into account the uncertainty in the inputs directly in the optimization problem. Although there is more than one formulation of the robust optimization problem, these formulations tend to result in a maximization of the worst-case scenario portfolio return for a given confidence level subject to risk-return considerations. In MVO, the portfolio selection problem simply maximizes portfolio return with risk considerations.

As shown by Merton [1980], the accuracy in the estimation of the covariance of asset returns increases when higher frequency data is available. This is much less the case for the estimation of mean returns. Moreover, although both the asset mean returns and the covariances change over time, the persistence of the covariance matrix is higher because variances and covariances change more slowly than mean returns. For this reason we focus our analysis on more tractable approaches that consider only the error in the estimation of returns. Other formulations exist that also take into account the error estimation in the covariance matrix. Unfortunately this renders the problem substantially more complex. Below, we discuss two formulations that have been put forward in the literature.

Quadratic form: definition

Let us consider a portfolio with N assets and $\boldsymbol{\mu}$ a vector of *true* mean returns, $\overline{\boldsymbol{\mu}}$ a vector of estimated mean returns and $\boldsymbol{\Omega}$ the covariance matrix of estimation errors for the vector of estimated returns[2]. $\boldsymbol{\Omega}$ is a symmetric positive definite matrix. It is then well known from statistics that the estimated mean returns lie inside the ellipsoidal confidence region:

$$(\boldsymbol{\mu} - \overline{\boldsymbol{\mu}})^t \boldsymbol{\Omega}^{-1} (\boldsymbol{\mu} - \overline{\boldsymbol{\mu}}) \leq \kappa^2 \qquad (10)$$

with a probability α, where $\kappa^2 = \chi_N^2 (1 - \alpha)$ and χ_N^2 is the inverse cumulative distribution function of the chi-squared distribution with N degrees of freedom. According to (10), the acceptable mean returns taking into account error estimation, will be such that the scaled sum of the square of the spreads between the *true* mean returns and the estimated returns will be smaller or equal to κ^2, with the scaling factor being the inverse of the covariance matrix of the estimation errors.

The robust optimization will seek the portfolio that remains optimal even when the mean returns take their worst possible values as defined in (10). Mathematically, this can be solved by modifying the original MVO problem (1) into a min-max problem, seeking the portfolio weights that maximize the returns of the worst scenario portfolio for a given level of risk, with the asset mean returns drawn for the confidence region in (10):

$$\mathbf{w}_{rob} = argmax\left(min(\boldsymbol{\mu}^t\mathbf{w}) - \frac{\lambda}{2}\mathbf{w}^t\boldsymbol{\Sigma}\mathbf{w}\right)$$
$$s.t.\,(\boldsymbol{\mu} - \overline{\boldsymbol{\mu}})^t\boldsymbol{\Omega}^{-1}(\boldsymbol{\mu} - \overline{\boldsymbol{\mu}}) \le \kappa^2 \qquad s.t.\,\mathbf{1}^t\mathbf{w} = 1 \qquad (11)$$

This equation can be simplified into a problem that closely resembles MVO. As proposed by Scherer [2006], the $min(\boldsymbol{\mu}^t\mathbf{w})$ in equation (11) under the constraint (10) can be found by assessing how large the spread between the expected and realized portfolio return can become, given a particular confidence level and a portfolio allocation. This is equivalent to maximizing the difference between the expected portfolio return $\overline{\boldsymbol{\mu}}^t\mathbf{w}$ calculated from the *true* asset mean returns and the worst statistically equivalent portfolio return $\boldsymbol{\mu}^t\mathbf{w}$ for those inputs that are along the ellipsoid defined in (10). This problem of maximizing the spread $\overline{\boldsymbol{\mu}}^t\mathbf{w} - \boldsymbol{\mu}^t\mathbf{w}$ for a given portfolio allocation can be solved using a standard Lagrangian multiplier approach. The Lagrangian associated with $max(\overline{\boldsymbol{\mu}}^t\mathbf{w} - \boldsymbol{\mu}^t\mathbf{w})$ is:

$$L(\boldsymbol{\mu}, \phi) = \mathbf{w}^t\overline{\boldsymbol{\mu}} - \mathbf{w}^t\boldsymbol{\mu} - \phi((\boldsymbol{\mu} - \overline{\boldsymbol{\mu}})^t\boldsymbol{\Omega}^{-1}(\boldsymbol{\mu} - \overline{\boldsymbol{\mu}}) - \kappa^2) \qquad (12)$$

with ϕ the Lagrange multiplier associated with the constraint on estimated returns defined in (10). Equating to zero both partial derivatives of $L(\boldsymbol{\mu}, \phi)$ relative to $\boldsymbol{\mu}$ and to ϕ and then solving for $\overline{\boldsymbol{\mu}}$ leads to the solution:

$$\boldsymbol{\mu} = \overline{\boldsymbol{\mu}} - \sqrt{\frac{\kappa^2}{\mathbf{w}^t\boldsymbol{\Omega}\mathbf{w}}}\boldsymbol{\Omega}\mathbf{w} \qquad (13)$$

If we multiply this result by $\mathbf{w}^t$ and use it in (11), we can rewrite equation (11) as:

$$\mathbf{w}_{rob} = argmax\left(\overline{\boldsymbol{\mu}}^t\mathbf{w} - \kappa\sqrt{\mathbf{w}^t\boldsymbol{\Omega}\mathbf{w}} - \frac{\lambda}{2}\mathbf{w}^t\boldsymbol{\Sigma}\mathbf{w}\right)$$
$$s.t.\,\mathbf{1}^t\mathbf{w} = 1 \qquad (14)$$

If κ is small then both the MVO and the robust portfolio will be similar as the term in the middle vanishes and equation (14) will be equivalent to equation (1). If κ is large then the optimal solution can deviate significantly from the solution proposed by MVO.

Equation (14) can no longer be solved using quadratic programming because of the square root in it. It must be solved using either second-order cone programming (SOCP) or an optimizer that can handle general convex expressions. These optimization algorithms are readily available in many standard statistical software packages and run almost as fast as traditional quadratic algorithms for practical portfolio optimization applications.

It is interesting to look in more detail at equation (14) and its properties. For this, we define a new Lagrangian $L(\mathbf{w}, \theta)$, now of equation (14), with θ the Lagrange multiplier associated with the budget constraint $\mathbf{1}^t\mathbf{w} = 1$:

$$L(\mathbf{w}, \theta) = \overline{\boldsymbol{\mu}}^t \mathbf{w} - \kappa\sqrt{\mathbf{w}^t \boldsymbol{\Omega} \mathbf{w}} - \frac{\lambda}{2}\mathbf{w}^t \boldsymbol{\Sigma} \mathbf{w} - \theta(\mathbf{1}^t \mathbf{w} - 1) \tag{15}$$

and differentiate the Lagrangian with respect to the portfolio weights to obtain the condition:

$$\frac{\partial L}{\partial \mathbf{w}} = \overline{\boldsymbol{\mu}} - \lambda\left(\frac{\kappa}{\lambda\sqrt{\mathbf{w}^t \boldsymbol{\Omega} \mathbf{w}}}\boldsymbol{\Omega} + \boldsymbol{\Sigma}\right)\mathbf{w} - \theta\mathbf{1} = 0 \tag{16}$$

Equation (16) is similar to equation (5) for MVO replacing $\boldsymbol{\Sigma}$ with a shrinkage non-normalized covariance matrix $\mathbf{Q}$ defined as:

$$\mathbf{Q} = \frac{\kappa}{\lambda\sqrt{\mathbf{w}^t \boldsymbol{\Omega} \mathbf{w}}}\boldsymbol{\Omega} + \boldsymbol{\Sigma} = \xi(n\boldsymbol{\Omega}) + \boldsymbol{\Sigma} \tag{17}$$

with the scalar defined as:

$$\xi = \frac{1}{\sqrt{n}} \times \frac{\kappa}{\lambda\sqrt{\mathbf{w}^t(n\boldsymbol{\Omega})\mathbf{w}}} \tag{18}$$

a function of the portfolio risk aversion λ, of the estimation error in the asset mean returns κ, of the number of observations n and of the standard deviation of expected returns $\sqrt{\mathbf{w}^t \boldsymbol{\Omega} \mathbf{w}}$. An increase in κ or a reduction in the number of observations n will increase the values of this new covariance matrix $\mathbf{Q}$. Since $\mathbf{Q}$ is a function of the portfolio weights, there is no closed solution for the optimal robust portfolio.

Differentiating equation (15), with respect to the Lagrange multiplier θ and to the vector of portfolio weights, setting the partial derivatives to zero and then solving for the Lagrange multiplier and substituting this back into the derivative (16) of the Lagrangian with respect of portfolio weights, leads to:

$$\mathbf{w}_{rob} = \frac{1}{\lambda}\mathbf{Q}^{-1}\left(\overline{\boldsymbol{\mu}} - \frac{\overline{\boldsymbol{\mu}}^t \mathbf{Q}^{-1}\mathbf{1}}{\mathbf{1}^t \mathbf{Q}^{-1}\mathbf{1}}\mathbf{1}\right) + \frac{\mathbf{Q}^{-1}\mathbf{1}}{\mathbf{1}^t \mathbf{Q}^{-1}\mathbf{1}} \tag{19}$$

which closely resembles equation (3) for MVO with $\mathbf{Q}$ replacing $\boldsymbol{\Sigma}$. Although there is no closed solution for the robust portfolio, because $\mathbf{Q}$ is a function of the portfolio weights, we can see that in the limit when the uncertainty in the asset mean return falls to zero or the number of observations raises to infinity, both leading to $\xi \to 0$, the matrix $\mathbf{Q}$ converges towards the estimated covariance matrix $\boldsymbol{\Sigma}$ and, from (19), it is clear that the robust optimization problem turns out to be equivalent to MVO, as should have been expected.

We can also investigate the limit of infinitely large uncertainty in the estimation of mean asset returns or number of observations falling to zero, both resulting in $\xi \to \infty$. In this case $\mathbf{Q}$ is a function of $\boldsymbol{\Omega}$ only as seen from (17) and:

11

$$lim_{\xi \to \infty}(\|\mathbf{Q}\|) \to \infty \tag{20}$$

and:

$$lim_{\xi \to \infty}(\mathbf{Q}) \to \mathbf{\Omega} \tag{21}$$

The impact of $\mathbf{\Sigma}$ on the selected robust portfolio vanishes in this limit of large uncertainty in the expected returns or falling number of observations. The structure of the covariance matrix of estimation errors for the vector of estimated returns dominates the problem of portfolio selection:

$$lim_{\xi \to \infty}(\mathbf{w}_{rob}) \to \frac{\mathbf{\Omega}^{-1}\mathbf{1}}{\mathbf{1}^t\mathbf{\Omega}^{-1}\mathbf{1}} \tag{22}$$

In this limit the only driver of the allocation is the choice of $\mathbf{\Omega}$.

Quadratic form: zero net adjustment

The formulation of the robust optimization problem as given in equation (11) is too conservative as it assumes the worst possible case scenario for estimation errors in asset mean returns. This pessimism can be softened by adding an additional constraint to render the problem less conservative. Indeed we add one more constraint to equation (11), to further limit the uncertainty box:

$$(\mathbf{\mu} - \overline{\mathbf{\mu}})^t \mathbf{D1} = 0 \tag{23}$$

If we use $\mathbf{D} = \mathbf{I}$, the identity matrix, then this constraint defines a zero net adjustment constraint imposed to the estimation error of asset mean returns, with as many realizations expected above the *true* mean returns as below. If we use $\mathbf{D} = \mathbf{\Lambda}$, a diagonal matrix of asset volatilities, then this constraint (23) instead defines a zero net adjustment constraint imposed to the estimation error of asset Sharpe ratios.

Despite this additional constraint, equation (11) can still be handled analytically as before. The Lagrangian defined in (12) now includes an additional Lagrange multiplier ψ to take into account the net zero adjustment constraint:

$$L(\mathbf{\mu}, \phi) = \mathbf{w}^t \overline{\mathbf{\mu}} - \mathbf{w}^t \mathbf{\mu} - \phi((\mathbf{\mu} - \overline{\mathbf{\mu}})^t \mathbf{\Omega}^{-1}(\mathbf{\mu} - \overline{\mathbf{\mu}}) - \kappa^2) - \psi((\mathbf{\mu} - \overline{\mathbf{\mu}})^t \mathbf{D1}) \tag{24}$$

This equation must be solved by setting to zero the partial derivatives of the Lagrangian relative to the two Lagrange multipliers and to the estimated asset mean returns. The algebra is tedious but straightforward and was given by Ceria and Stubbs [2006]. The equivalent of equation (13) turns out as:

$$\mu = \bar{\mu} - \sqrt{\dfrac{\kappa^2}{\left(\Omega w - \frac{w^t D\Omega 1}{1^t D\Omega D^t 1}\Omega D^t 1\right)^t \Omega^{-1}\left(\Omega w - \frac{w^t D\Omega 1}{1^t D\Omega D^t 1}\Omega D^t 1\right)}}\left(\Omega w - \frac{w^t D\Omega 1}{1^t D\Omega D^t 1}\Omega D^t 1\right) \tag{25}$$

And the robust optimization problem can be re-written as:

$$\mathbf{w}_{rob} = argmax\left(\bar{\mu}^t \mathbf{w} - \kappa\sqrt{\mathbf{w}^t \mathbf{\Phi} \mathbf{w}} - \frac{\lambda}{2}\mathbf{w}^t \Sigma \mathbf{w}\right)$$
$$s.t.\, \mathbf{1}^t \mathbf{w} = 1 \tag{26}$$

with the matrix $\mathbf{\Phi}$ defined as:

$$\mathbf{\Phi} = \Omega - \frac{1}{1^t D\Omega D^t 1}\Omega D^t 1 1^t D\Omega \tag{27}$$

The results in the previous section will continue to hold with $\mathbf{\Omega}$ now replaced by $\mathbf{\Phi}$. In particular we continue to find the MVO portfolio in the limit of low uncertainty. If $\mathbf{\Phi}$ is invertible then, in the limit of large uncertainty we still find a risk-based strategy as in equation (22), but now defined as:

$$lim_{\xi \to \infty}(\mathbf{w}_{rob}) \to \frac{\mathbf{\Phi}^{-1}\mathbf{1}}{\mathbf{1}^t \mathbf{\Phi}^{-1}\mathbf{1}} \tag{28}$$

If $\mathbf{\Phi}$ is not invertible then we need to solve for:

$$\lim_{\xi \to \infty}(\mathbf{w}_{rob}) \to argmin(\mathbf{w}^t \mathbf{\Phi} \mathbf{w})$$
$$s.t.\, \mathbf{1}^t \mathbf{w} = 1 \tag{29}$$

Below we consider a sensible choice of $\mathbf{D}$ and $\mathbf{\Omega}$ leading to a $\mathbf{\Phi}$ which cannot be inverted. However, in the example we can still find a risk-based portfolio when solving (29).

Quadratic form: examples

As suggested by Fabozzi *et al.* [2007], several approximate methods for estimating $\mathbf{\Omega}$ have been found to work well in practice. Simpler estimation approaches, such as computing the diagonal matrix of the variances of estimates, as opposed to the complete error covariance matrix, often provide most of the benefit in robust optimization.

We considered three possible choices[3] for $\mathbf{\Omega}$. In the first we assume that all errors in the estimation of the asset mean returns are of the same order of magnitude, i.e. they do not depend on the variance of the mean returns. Additionally, we assume that the error estimation of the mean return of an asset is fully independent of the error estimation of the mean returns of different asset, even if the returns appear correlated in the past. Under such assumptions, $\mathbf{\Omega}$ should be proportional to the identity matrix $\mathbf{I}$. If we use this in equation (22) we find that under such an assumption, in the limit of large uncertainty in the estimation of mean returns, the robust optimization converges towards the EW portfolio as defined in (6).

A second possible choice is to assume that the estimation error in the asset mean returns are proportional to the asset variance but that $\boldsymbol{\Omega}$ should still be diagonal, indicating that we still believe the estimation of error in the mean returns of an asset is independent from that for another asset. In this case $\boldsymbol{\Omega}$ is a diagonal matrix $\mathbf{A}$ with only asset variances, and the robust portfolio converges towards the inverse portfolio as given in (8).

Another simple choice is to assume that the covariance matrix of errors in estimated mean returns $\boldsymbol{\Omega}$ is proportional to the estimated covariance matrix of asset returns $\boldsymbol{\Sigma}$. In such a case the robust portfolio will converge towards the MV portfolio in (9) when the uncertainty is too large or the number of observations behind the estimated mean returns is too small. Moreover, as demonstrated by Scherer [2006], in this particular case the robust portfolio is nothing more than an exact-weighted average of MVO and MV for all κ. Indeed, from (19) we can show easily that with the choice $\boldsymbol{\Omega} = n^{-1}\boldsymbol{\Sigma}$ the robust portfolio is:

$$
\begin{aligned}
\mathbf{w}_{rob} &= \frac{1}{\lambda}\mathbf{Q}^{-1}\left(\overline{\boldsymbol{\mu}} - \frac{\overline{\boldsymbol{\mu}}^{t}\mathbf{Q}^{-1}\mathbf{1}}{\mathbf{1}^{t}\mathbf{Q}^{-1}\mathbf{1}}\mathbf{1}\right) + \frac{\mathbf{Q}^{-1}\mathbf{1}}{\mathbf{1}^{t}\mathbf{Q}^{-1}\mathbf{1}} \\
&= \frac{1}{\lambda} \times \frac{1}{1+\xi} \times \boldsymbol{\Sigma}^{-1}\left(\overline{\boldsymbol{\mu}} - \frac{\overline{\boldsymbol{\mu}}^{t}\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^{t}\boldsymbol{\Sigma}^{-1}\mathbf{1}}\mathbf{1}\right) + \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^{t}\boldsymbol{\Sigma}^{-1}\mathbf{1}} \\
&= \frac{1}{1+\xi} \times \mathbf{w}_{MVO} + \frac{\xi}{1+\xi} \times \mathbf{w}_{MV}
\end{aligned}
\tag{30}
$$

If we consider the quadratic formulation, which includes the zero net adjustment constraint as defined in (23), and choose $\boldsymbol{\Omega} = \mathbf{A}$, a diagonal matrix of asset variances, and $\mathbf{D} = \boldsymbol{\Lambda}$, zero net adjustment imposed to the Sharpe ratio estimation errors, then it can be shown from (29) that the robust portfolio converges towards the equal risk budgeting portfolio as defined in (7). We give more details in the appendix.

In Exhibit 1 we summarize these results. From a practical point of view, if the number of observations is small, then the error in the estimation of the variances and covariances is likely to be large. Should the number of observations be small, the second and third proposed forms for $\boldsymbol{\Omega}$ are not recommended.

Exhibit 1: Risk-based portfolios found as solutions to the quadratic formulation of the robust optimization portfolio selection problem in the limit of large uncertainty in expected returns (i.e. $\kappa \to \infty$) for three different forms for the covariance matrix $\boldsymbol{\Omega}$ of estimation errors in the asset mean returns.

Proposed covariance matrix for estimation error in mean returns	Limiting portfolio when uncertainty is large
Identity matrix	Equally weighted portfolio
Diagonal matrix with estimated variances	Inverse variance portfolio
Full estimated covariance matrix	Minimum variance portfolio
Diagonal matrix with estimated volatities zero net adjustment on Sharpe ratio	Equal risk budget portfolio

Quadratic form: numerical application

An important question for practitioners is the speed of convergence of the robust portfolio towards either the MVO solution in the case of small uncertainty or towards the risk-based portfolios in the case of large uncertainty. We shall investigate this using a numerical application of the quadratic form of the robust optimization.

All data was downloaded from Bloomberg. We used monthly time series of total returns in USD with dividends reinvested in the case of equities, coupons reinvested and interest accrued in the case of bonds. Data runs from January 1990 through November 2014.

All portfolios can invest in five assets and the inputs for robust optimization are estimated from historical returns in excess of the US 3-month T-bill yields (US0003M Index). The MSCI USA Index dividend reinvested measures the performance of largest capitalization-weighted large-cap US stocks (NDDUUS Index). The FTSE EPRA/NAREIT USA Index measures the performance of capitalization-weighted stocks in US real estate market (RUUS Index). The S&P Commodities Index measures the performance of commodities (SPGSCITR Index). The US Corporate Investment Grade Index (C0A0 Index) and US 10-year Government Bonds Index (SBUS10L Index) respectively measure the performance of US investment-grade corporate bonds and bonds issued by the US government.

Exhibit 2: Asset mean return, volatility and pair-wise correlations estimated from the historical monthly total returns in USD for the period January 1990 to November 2014. The number of observations n is 298.

Asset classes	Annualized excess returns	Annualized Volatility	Pairwise correlations				
			MSCI USA	US 10 years gov. bonds	US corp. inv. grade bonds	S&P Commodities	EPRA USA
MSCI USA	6.7%	14.9%	1.00	-0.10	0.30	0.17	0.66
US 10 years gov. bonds	4.5%	9.7%		1.00	0.66	-0.13	-0.01
US corp. inv. grade bonds	3.6%	5.3%			1.00	0.12	0.40
S&P Commodities	2.5%	21.2%				1.00	0.23
EPRA USA	5.5%	18.8%					1.00

In order to investigate the speed of convergence towards limiting cases we vary ξ which, as shown from its definition in (18), can change as a function of three parameters: the uncertainty in the expected returns κ, the number of observations n and the standard deviation of the errors estimation in expected returns $\sqrt{\mathbf{w}^t \mathbf{\Omega} \mathbf{w}}$. For simplicity we chose to vary ξ by changing the uncertainty in the expected returns κ only.

We first built the limiting portfolios for these five assets. In Exhibit 3 we show the MVO portfolio that maximizes the Sharpe ratio. This is found using the data in Exhibit 2 as input in equation (2). We also show the EW, ERB, IV and MV portfolios which can be found from equations (6), (7), (8) and (9), respectively, also using data from Exhibit 2 when required. All these portfolios show significantly different allocation to the available assets. For MVO and MV we also include the long-only constrained portfolios.

Exhibit 3: Limiting portfolios for quadratic formulation of robust optimization, based on the historical data in Exhibit 2 when required. Two of these portfolios, MVO and MV, include short positions. Thus, we also include the MVO and MV under long-only constraint.

	unconstrained					long-only constranied	
	MVO	EW	ERB	IV	MV	MVO	MV
MSCI USA	22.3%	20.0%	14.6%	8.1%	5.9%	16.7%	1.51%
US 10 year gov. bonds	14.3%	20.0%	22.5%	19.2%	-11.3%	16.8%	0.00%
US corp. inv. grade bonds	70.0%	20.0%	41.0%	63.6%	111.7%	65.0%	95.38%
S&P Commodities	1.9%	20.0%	10.3%	4.0%	2.9%	1.5%	3.11%
EPRA USA	-8.4%	20.0%	11.6%	5.1%	-9.2%	0.0%	0.00%

As shown in (30), if $\mathbf{\Omega}$ is proportional to $\mathbf{\Sigma}$ then the selected portfolio from robust optimization is a simple weighted average of the two limiting portfolios, the MVO and MV. In the other cases, the robust portfolio is no longer necessarily a weighted average of the MVO portfolio and the corresponding risk-based portfolio found in the limit of large uncertainty. Here we investigate to what extent the weighted average of the MVO portfolio and a risk-based portfolio is still a good representation of the robust portfolio at different levels of uncertainty for other two forms of $\mathbf{\Omega}$ in Exhibit 1. We generate the solution to the robust optimization problem using the inputs from Exhibit 2 and vary ξ by changing the error κ. The exercise is repeated for the three different choices of $\mathbf{\Omega}$, and also for the case with zero net adjustment imposed to the estimation error of Sharpe ratios. We then regress the robust portfolio asset weights on the asset weights of the two limiting portfolios from Exhibit 3: the unconstrained MVO and the corresponding unconstrained risk-based portfolio, as detailed in Exhibit 1. The asset weights in these portfolios are given in Exhibit 3. We only impose the budget constraint that the sum of asset weights in each portfolio adds to one. No other portfolio constraints are added. In the regression of the robust optimization portfolios on the two limiting

portfolios, we also impose the constraint that the sum of the two regression coefficients equals one. However, even when we did not impose this constraint we found that the results from the regression were still extremely close to those presented here. The results are shown in Exhibit 4. The exposure to the unconstrained risk-based portfolio is the coefficient obtained from the regression. The exposure to the unconstrained MVO portfolio is simply 100% minus the exposure to the unconstrained risk-based portfolio.

Exhibit 4: Exposure of the quadratic robust optimization portfolio to the risk-based portfolio as a function of the uncertainty in the estimated returns κ for three different forms of the covariance matrix of the estimation error in the mean returns, with one example also including a zero net adjustment imposed to the estimation error of Sharpe ratios. Unconstrained portfolios.
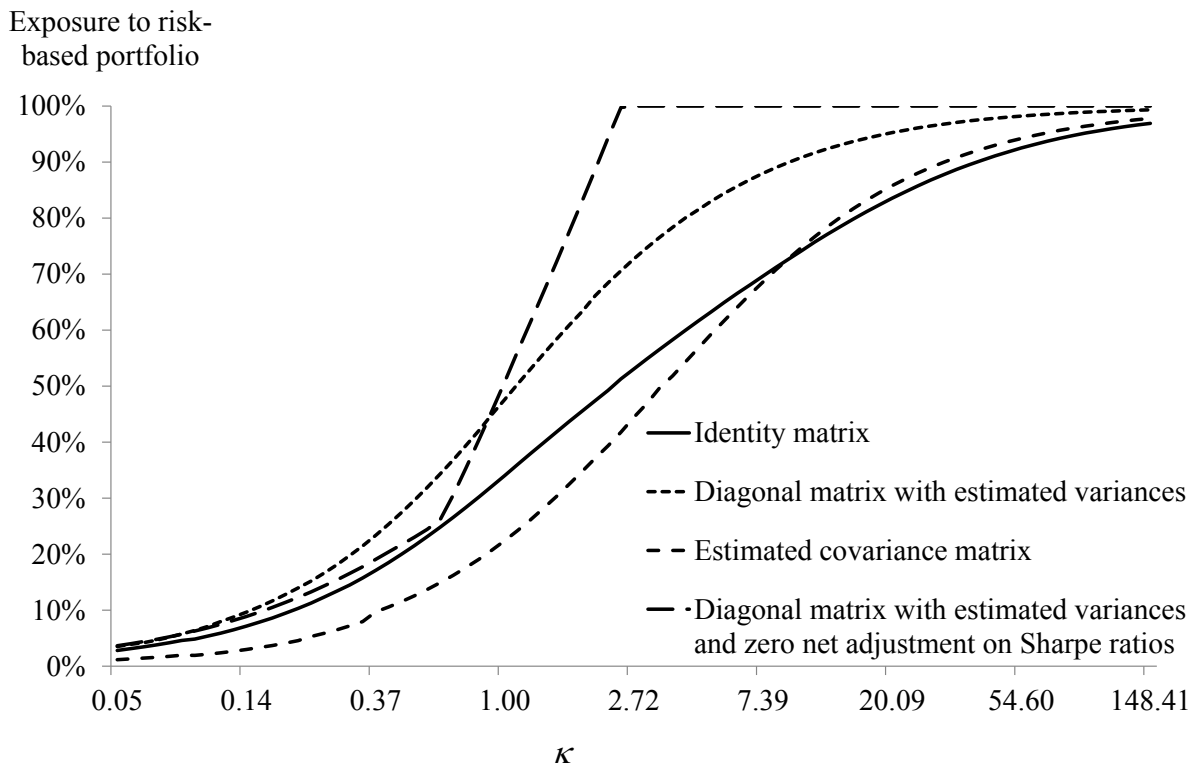
Without the zero net adjustment constraint, the exposure of the robust portfolio increases smoothly from zero in the limit when $\kappa, \xi \rightarrow 0$ to 100% in the limit when $\kappa, \xi \rightarrow \infty$. With the zero net adjustment constraint, the exposure to the risk-based portfolio no longer changes smoothly with κ. In this case, the maximum exposure to the risk-based portfolio is reached abruptly and the convergence is much faster than for the equivalent robust optimization problem without the zero net adjustment, i.e. with $\boldsymbol{\Omega} = \boldsymbol{A}$. It is interesting that although the zero net adjustment was introduced to make the robust optimization problem less conservative, the result is that the risk-based limit is reached faster. And the risk-based limit is perhaps even more conservative, i.e. ERB instead of IV.
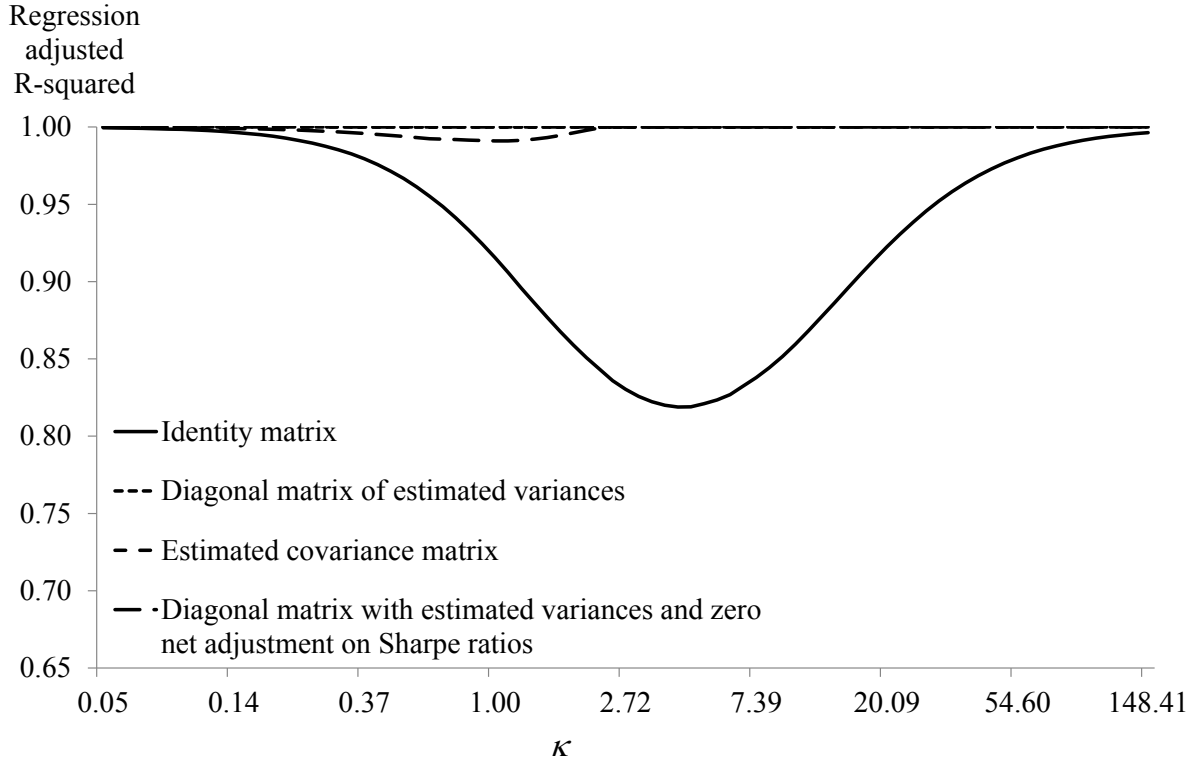
17

The percentage weight of risk-based portfolio in the robust portfolio increases quickly with confidence at lower κ values. We find that at $\alpha = 5\%$ confidence, which with five assets is found for $\kappa = 1.14$, the risk-based portfolios already represent a significant weight in the robust portfolio: 37% when the uncertainty in returns is defined by the identity matrix, 51% when the uncertainty is defined by the diagonal matrix with estimated variances and 25% when the uncertainty is defined by the estimated covariance matrix. The weight of the risk-based portfolio increases significantly at $\alpha = 50\%$ confidence level to 61%, 81% and 56%, respectively. This level of confidence is found with $\kappa = 4.35$. At larger κ values, the convergence towards the risk-based portfolios slows considerably. This is explained by the shape of the χ^2 distribution and is the reason we used an exponential scale in Exhibit 4. At $\alpha = 75\%$ confidence, with $\kappa = 6.69$, the robust portfolio is essentially dominated by the risk-based portfolio with 68%, 87% and 66%, respectively.

For the example with zero net adjustment we find that at $\alpha = 5\%$, $\kappa = 1.14$, the weight of the risk-based portfolio is already 58% and at $\alpha = 22\%$, with $\kappa = 2.46$, the robust portfolio is already undistinguishable from the risk-based portfolio, in this case the ERB.

We now investigate to what extent the simple weighted average of the MVO portfolio and the risk-based portfolio remains a valid representation of the selected portfolio from robust optimization, even beyond the choice $= n^{-1}\Sigma$, where we know from equation (30) that the result is exact.

In Exhibit 5 we show the R-squared from the regressions used in Exhibit 4 for the projection of the robust portfolio on the MVO and the respective risk-based portfolio. In the case where $\Omega = n^{-1}\Sigma$. the R-squared is exactly one for all κ as expected.

Exhibit 5: R-squared of the regression of the robust optimization portfolios against the MVO portfolios and the respective risk-based portfolios as a function of the uncertainty κ in asset mean returns. Unconstrained portfolios.

Regression adjusted R-squared

— Identity matrix

--- Diagonal matrix of estimated variances

– – Estimated covariance matrix

— · Diagonal matrix with estimated variances and zero net adjustment on Sharpe ratios

κ

In the case with $\boldsymbol{\Omega}$ equal to the diagonal matrix with the estimated variances, we find an R-squared extremely close to 1 for all κ. However, there is no longer an analytical reason for the robust portfolio to be the exact weighted average of the MVO portfolio and the IV portfolio. Clearly, this should be expected in the case where all estimated correlations are zero. However, there are significant differences in the estimated correlations as seen in exhibit 2.

Adding the zero net adjustment to the Sharpe ratio estimation errors decreases slightly the R-squared for intermediate levels of uncertainty κ. But the values of R-squared remain very high, suggesting that the decomposition into a mean-variance portfolio and risk-based portfolio is still a relatively accurate representation of the robust portfolio.

Finally, in the case where $\boldsymbol{\Omega}$ is the identity matrix $\mathbf{I}$, we find that the R-squared is no longer exactly one for all κ. The R-squared falls initially, as κ increases, reaching a minimum of 82% at $\kappa = 4.06$ as shown in Exhibit 5. Again, if the estimated correlations where zero and the estimated asset volatilities were the same, then we would expect an R-squared equal to one. But from Exhibit 2 we can see that the correlations are not zero and the volatilities are not comparable.

The absolute form: examples

Fabozzi, Kolm, Pachamanova and Focardi [2007] proposed another form of robust optimization applicable when the expected returns are not too far from the *true* expected returns. In this

formulation the absolute spread between the estimated and the *true* expected returns should be smaller than a given level of uncertainty κ. The robust portfolio can be found from:

$$\mathbf{w}_{rob} = argmax\left(min(\boldsymbol{\mu}^t\mathbf{w}) - \frac{\lambda}{2}\mathbf{w}^t\boldsymbol{\Sigma}\mathbf{w}\right)$$
$$s.t. \ \sum_i|\mu_i - \bar{\mu}_i| \leq \frac{\kappa\,\bar{\sigma}}{\sqrt{n}} \qquad s.t.\ \mathbf{1}^t\mathbf{w} = 1 \tag{31}$$

where $\bar{\sigma}$ is the average of the volatility of all assets and, as before, n is the number of observations in the estimation of mean returns. The constraint in equation (31) defines a set of estimated mean returns that are close to the *true* expected returns and κ can be chosen as a function of the confidence interval around each estimate μ_i.

In this formulation the optimization is less penalized by the estimation error in the vector of mean returns than it is in the quadratic formulation in (11), where the error in estimation is squared. In addition, no correlation between the errors in expected returns is considered.

In general, for the uncertainty set in estimated returns introduced in (31) we can write:

$$\boldsymbol{\mu}^t\mathbf{w} - \bar{\boldsymbol{\mu}}^t\mathbf{w} \geq -\sum_i|\mu_i - \bar{\mu}_i|\,max(|w_i|) \geq -\frac{\kappa\,\bar{\sigma}}{\sqrt{n}}max(|w_i|) \tag{32}$$

Using the same notation as for the quadratic formulation we can then rewrite the robust optimization problem as follows:

$$\mathbf{w}_{rob} = argmax\left(\bar{\boldsymbol{\mu}}^t\mathbf{w} - \frac{\kappa\,\bar{\sigma}}{\sqrt{n}}\,max(|w_i|) - \frac{\lambda}{2}\mathbf{w}^t\boldsymbol{\Sigma}\mathbf{w}\right)$$
$$s.t.\ \mathbf{1}^t\mathbf{w} = 1 \tag{33}$$

There are some important differences between this absolute formulation of the robust optimization problem and the quadratic formulation. Below we provide some intuition behind what to expect from solving equation (33) and highlight the key differences relative to the quadratic form.

Let us assume that asset p has the largest portfolio weight. We can then rewrite (33) by redefining the set of mean returns as follows:

$$If\ |w_p| \geq |w_i|\ \ \forall i$$
$$\mathbf{w}_{rob} = argmax\left(\bar{\boldsymbol{\mu}}^t\mathbf{w} - \frac{\kappa\,\bar{\sigma}}{\sqrt{n}}\,sgn(w_p) - \frac{\lambda}{2}\mathbf{w}^t\boldsymbol{\Sigma}\mathbf{w}\right)$$
$$\mathbf{w}_{rob} = argmax\left(\bar{\boldsymbol{\mu}}_{new}^t\mathbf{w} - \frac{\lambda}{2}\mathbf{w}^t\boldsymbol{\Sigma}\mathbf{w}\right) \tag{34}$$
$$\bar{\boldsymbol{\mu}}_{new}^t = \left(\bar{\mu}_1, \ldots \bar{\mu}_{p-1}, \bar{\mu}_p - \frac{\kappa\,\bar{\sigma}}{\sqrt{n}}\,sgn(w_p), \bar{\mu}_{p+1}, \ldots\right)$$

While we could define the Lagrangian multiplier of the formulation (32), we cannot easily differentiate it because of the function *max*. Nevertheless, we can still take the limit of the equation

(32) when $\kappa \to \infty$ thanks to the budget constraint of portfolio weights adding to one. When the uncertainty in the expected returns κ is sufficiently large, the expected return of the portfolio $\bar{\boldsymbol{\mu}}^t \mathbf{w}$ and the variance of the portfolio $\mathbf{w}^t \boldsymbol{\Sigma} \mathbf{w}$ can be neglected when compared to the term with the largest absolute asset weight in the portfolio. Thus, in the limit $\kappa \to \infty$ we have

$$\lim_{\kappa \to \infty} (\mathbf{w}_{rob}) \to -\frac{\kappa \, \bar{\sigma}}{\sqrt{n}} \times argmin(\max(|w_i|)) \qquad (35)$$
$$s.t. \, \mathbf{1}^t \mathbf{w} = 1$$

The only feasible solution to (35) is the EW portfolio. Thus, in this case the robust portfolio falls between the MVO portfolio in the deterministic limit of no uncertainty in the mean returns, $\kappa \to 0$, and the equally-weighted portfolio in the limit of large uncertainty, when $\kappa \to \infty$.

We can think of a different absolute formulation by defining the uncertainty set for the asset risk-adjusted mean returns, i.e. Sharpe ratio, instead of defining the asset mean returns as in (31). Indeed, the idea that the estimation error in the asset mean returns should not be the same for all assets, irrespective of their volatility, seems reasonable.

$$w_{rob} = argmax \left(min(\boldsymbol{\mu}^t \mathbf{w}) - \frac{\lambda}{2} \mathbf{w}^t \boldsymbol{\Sigma} \mathbf{w} \right)$$
$$s.t. \, \sum_i \frac{|\mu_i - \bar{\mu}_i|}{\sigma_i} \leq \frac{\kappa}{\sqrt{n}} \qquad s.t. \, \mathbf{1}^t \mathbf{w} = 1 \qquad (36)$$

Using the same arguments above, we can show that in the limit of large uncertainty in the estimation of the Sharpe ratio of different assets, the solution to this robust optimization problem converges towards:

$$\lim_{\kappa \to \infty} (\mathbf{w}_{rob}) \to -\frac{\kappa}{\sqrt{n}} \times argmin(\max(|\sigma_i w_i|)) \qquad (37)$$
$$s.t. \, \mathbf{1}^t \mathbf{w} = 1$$

In this case we need to maximize the risk budget allocated to each asset in the optimal portfolio instead of the weight of each asset. The only feasible solution to this problem is the ERB portfolio defined in (7). Thus, with this formulation, in the limit of large uncertainty in the Sharpe ratio of assets, the robust portfolio converges towards the ERB portfolio.

In Exhibit 6 we summarize the results found in the limit of large uncertainty when the absolute formulation of the robust optimization portfolio selection problem is used. The results are for unconstrained portfolios.

Exhibit 6: Risk-based portfolios found as solutions to the absolute formulation of the robust optimization portfolio selection problem in the limit of large uncertainty in expected returns (i.e.

$\kappa \to \infty$) for two different forms of estimation error in the vector of estimated returns. Unconstrained portfolios.

Proposed absolute estimation error	Limiting portfolio when uncertainty is large
Same error in asset mean returns	Equally weighted portfolio
Same error in asset Sharpe ratios	Equal risk budget portfolio

Exhibit 7: Exposure of the absolute robust optimization portfolios to the risk-based portfolio as a function of the uncertainty in the estimate returns κ for two different forms of the uncertainty box, one defined in terms of absolute error in asset mean returns and the other based on absolute error in asset Sharpe ratios. Unconstrained portfolios.
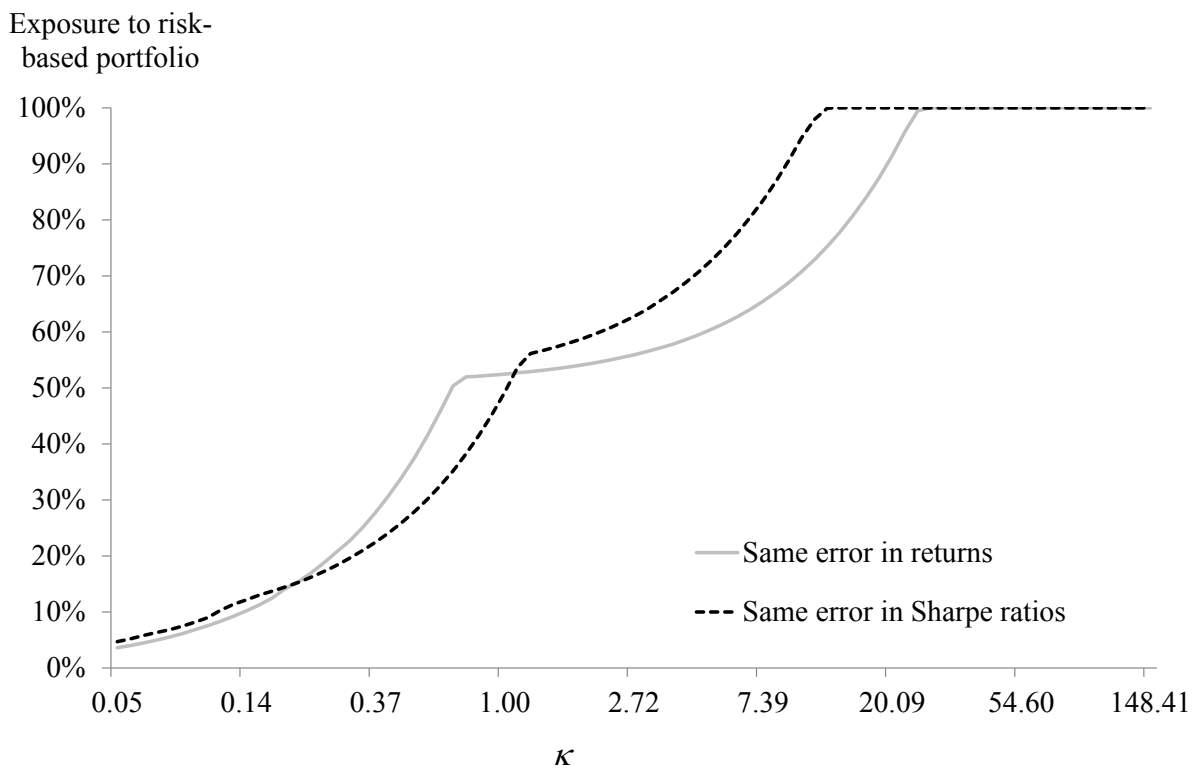

Absolute form: numerical application

As for the quadratic form, we investigated numerically the speed of convergence of the robust portfolios towards the MVO portfolio in the case of small uncertainty in mean returns and towards the risk-based portfolio in the case of large uncertainty in mean returns. We also projected the robust portfolio against the MVO and the respective risk-based limiting portfolio and investigated the

accuracy of this portfolio decomposition by looking at the R-squared of the regression. We used the same inputs as before for the quadratic form, which can be found in Exhibit 2. The limiting portfolio allocations derived from these inputs can be found in Exhibit 3.

Exhibit 8: R-squared of the regression of the absolute form robust optimization portfolios against the MVO portfolios and the respective risk-based portfolios as a function of the uncertainty κ in the mean returns. Unconstrained portfolios.
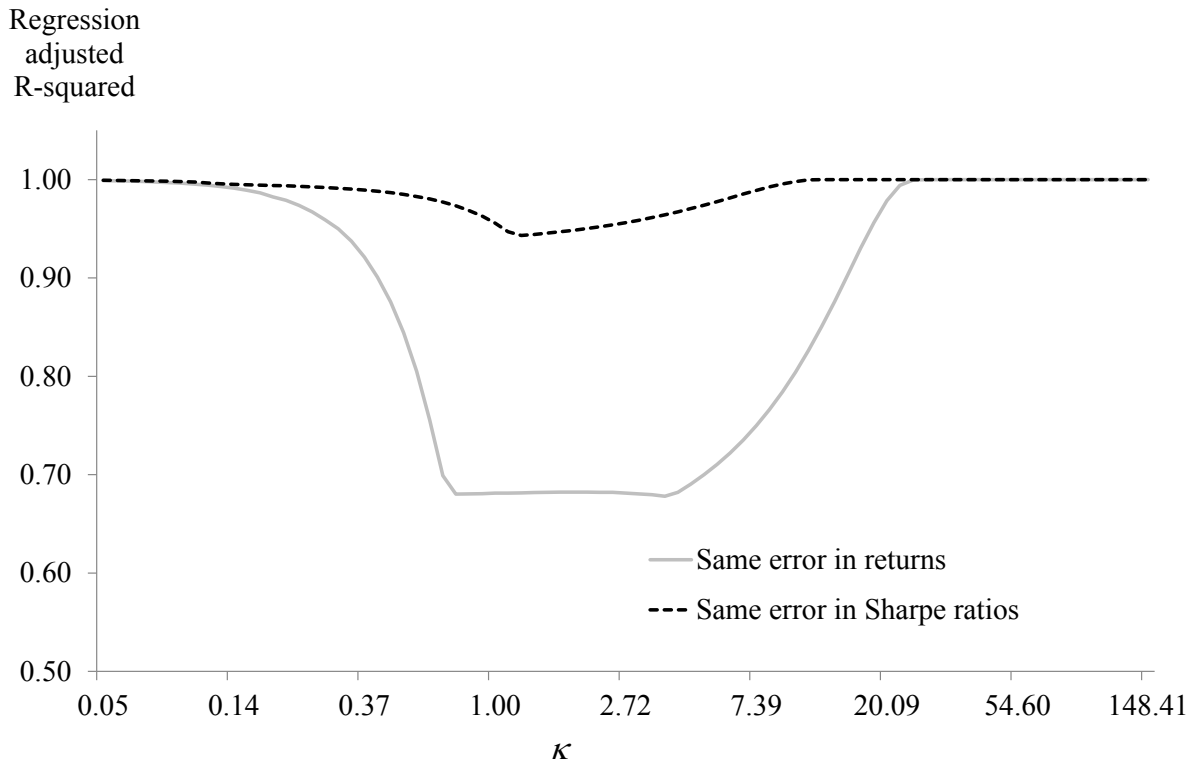

In Exhibit 7 we show the exposure of the robust portfolio to the respective risk-based limiting portfolio obtained from the regression of the robust portfolio against the two limiting portfolios, i.e. the MVO and the respective risk-based portfolio. As we also found in the quadratic formulation, the robust portfolio deviates quickly from the MVO portfolio as soon as some level of uncertainty in mean returns or Sharpe ratios is introduced. We also find that the convergence of the robust portfolios towards a risk-based portfolio seems faster with the absolute formulation than with the quadratic formulation. Finally, unlike what we had seen in the case of the quadratic formulation, the exposure to the limiting risk-based portfolio has discontinuities in its first derivative with respect to κ.

If we compare Exhibit 8 with Exhibit 5 we tend to find a lower R-squared for the absolute formulation of the robust optimization problem, in particular in the case where the estimation error is associated with the asset Sharpe ratios. The decomposition of the robust portfolio as a linear combination of an MVO and a risk-based portfolio seems to be more accurate in the case of the quadratic formulation.

Portfolio constraints

One important question is what if we add other portfolio constraints to the robust optimization problem, e.g. a long-only constraint. In fact, the results derived in this paper continue to hold even if subject to other portfolio constraints. The only difference is that the limiting portfolios in the case of small uncertainty and large uncertainty will be, respectively, the constrained MVO portfolio and the constrained risk-based portfolio obtained by solving equations (6) through (9) subject to those portfolio constraints.

Exhibit 9: Exposure of the quadratic robust optimization portfolio to the risk-based portfolio as a function of the uncertainty in the estimate returns κ for three different forms of the covariance matrix of the estimation error in the mean returns, with one example also including a zero net adjustment imposed to the estimation error of Sharpe ratios. Long-only constrained portfolios.
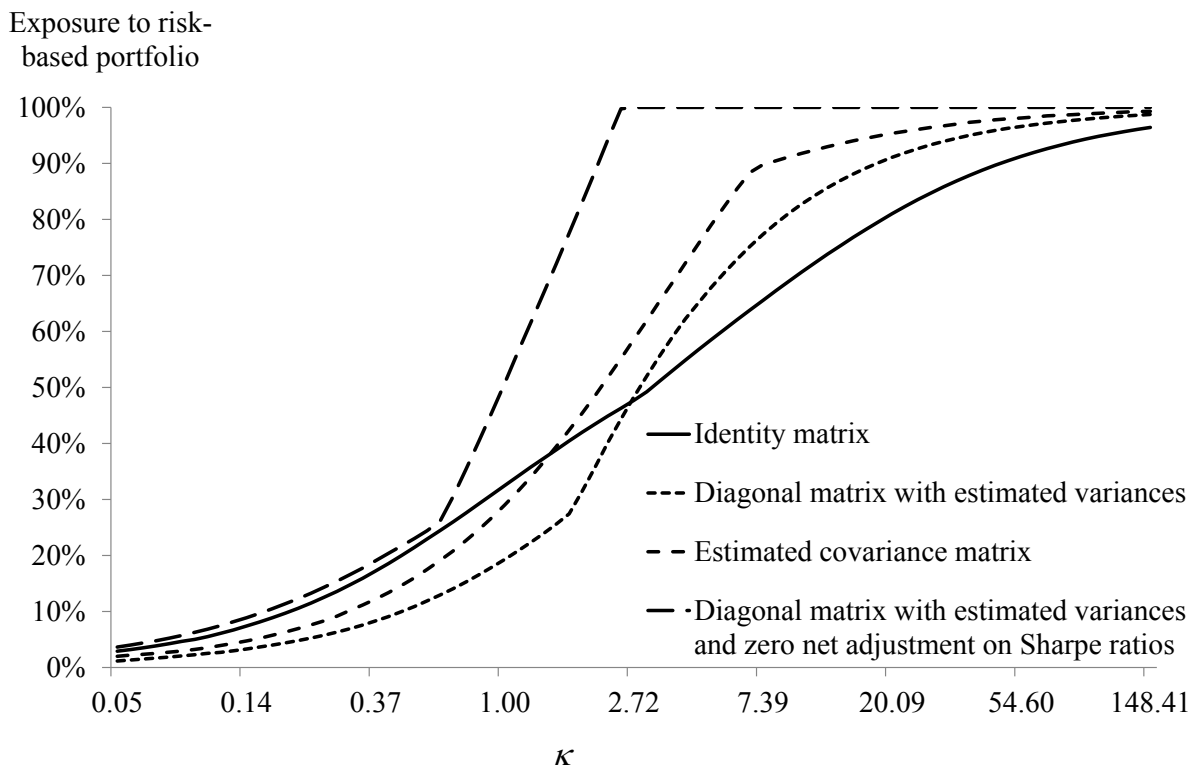

A second question is whether, at intermediate levels of uncertainty, the decomposition of robust portfolios into a weighted average of a constrained MVO portfolio and a constrained risk-based portfolio is less accurate. On this point, it can be shown that for the quadratic case with a choice $\mathbf{\Omega} = n^{-1}\mathbf{\Sigma}$, equation (23) still holds true, even in the presence of portfolio constraints. The decomposition into constrained MVO and constrained MV is still exact.

We repeated the analysis carried out earlier for the quadratic formulation and presented in Exhibits 4 and 5, but now also imposing a long-only constraint. The long-only constraint has an impact on the MVO portfolio and on the MV portfolio which can be found in Exhibit 3. The results can be found in Exhibits 9 and 10. The speed of convergence changes but the results do not change significantly and the decomposition of the robust portfolio still appears to be a reasonably good representation of the robust portfolio in most cases.

Exhibit 10: R-squared of the regression of the robust optimization portfolios against the MVO portfolios and the respective risk-based portfolios as a function of the uncertainty κ in asset mean returns. Long-only constrained portfolios.
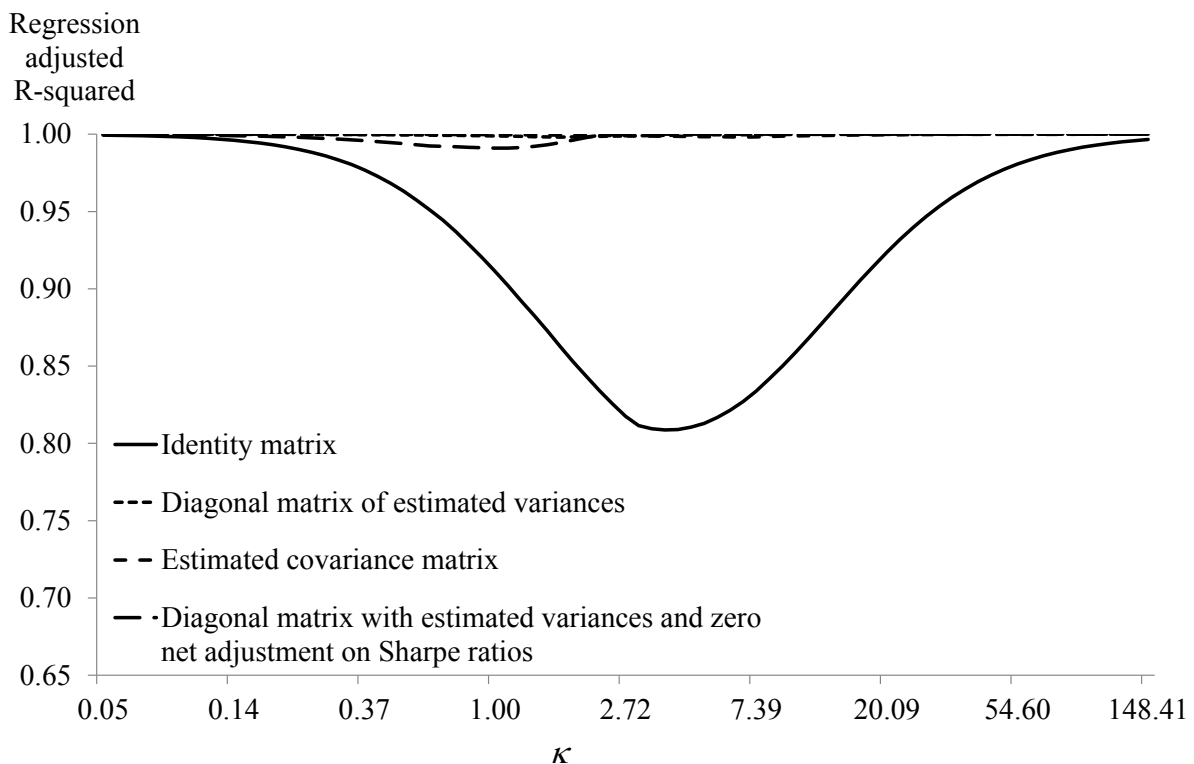
CONCLUSIONS

We show that robust portfolios converge towards the mean-variance portfolio when the uncertainty in the estimation of asset mean returns is small. When uncertainty is sufficiently large, the robust portfolio converges towards risk-based portfolio allocations. The type of risk-based portfolio found in this limit depends on the formulation of the robust optimization problem.

We considered two different types of formulation, quadratic and absolute, and different forms of uncertainty in the estimation of asset mean returns. The risk-based portfolio found in the limit of large uncertainty is intuitive. We could associate the MV, EW, IV and ERB portfolios as robust portfolio

allocations in the limit of large uncertainty, depending of the formulation chosen for the robust portfolio optimization problem.

We also show that a simple weighted average of the mean-variance portfolio and the risk-based limiting portfolio gives a good description of the underlying robust portfolio allocation in most cases and at different levels of uncertainty in the estimation of returns. For the quadratic formulation, assuming that the covariance of errors in the estimation of the asset mean returns is proportional to the estimated covariance of asset returns, this result is exact, even in the presence of portfolio constraints.

We have also investigated a less conservative form of the quadratic formulation in which we add a zero net adjustment constraint to render the problem less conservative. We still find that the limiting portfolios in the cases of low and high uncertainty are the mean-variance portfolio and a risk-based portfolio, respectively. And the robust portfolio is still relatively well represented by a weighted average of the mean-variance portfolio and the risk-based portfolio at intermediate levels of uncertainty.

Finally, we also show that the results hold even in the presence of portfolio constraints. In this case the limiting portfolios are the similarly constrained mean-variance portfolio and risk-based portfolios.

We believe that our results help to demystify robust optimization and give some guidance on how to interpret the final portfolio allocation even in the presence of constraints. We also give investors a tool to better understand the limitations of the approach and gauge its usefulness in practical investment problems.

In particular, robust optimization can be used to add expected returns in problems where investors had to some extent given up. We can think of risk parity multi-asset allocation as one example. Investors have been pouring assets into risk parity multi-asset portfolios. Such strategies risk delivering poorer investment returns in the future due to the current low level of Treasury yields. Results from Assness, Frazzini and Pedersen [2012] show that risk parity multi-asset strategies can underperform other well-known multi-asset strategies for decades in rising yield conditions. Robust optimization offers the possibility of tilting the portfolio away from the risk parity portfolio by taking into account expected returns views with a given level of uncertainty.

Recently we have also witness increasing flows into equity MV strategies. Again, robust optimization offers fund managers the means to include expected returns in their allocations by tilting gently away from the MV portfolio. For example, a view on stock returns generated from fundamental considerations could be added.

Other approaches competing with robust optimization have recently been proposed in the literature. In particular Heansen *et al.* [2014] and Jurczenko and Teiletche [2015] recently proposed Black-Litterman approaches in which the prior is no longer the market portfolio as in the original formulation of Black and Litterman [1992]. Instead they proposed the use of risk-based portfolios as a prior. The results from such approaches may not too different from those given by some robust optimization formulations that reach the same risk-based portfolio in the limit of large uncertainty, and provided they express their views using mean-variance portfolios. But even in such a case, the underlying allocations are not always necessarily comparable and robust optimization has a stronger theoretical underpinning, in our view. We believe that more research is needed in order to compare these different approaches and to investigate the added value from the departure of robust optimized portfolio from the simple weighted average of a mean-variance and a risk-based portfolio which is only exact in the case of the quadratic formulation with an covariance matrix of estimation errors proportional to the estimated covariance matrix of asset returns.

NOTES

1- Proponents of these approaches have used empirical evidence to defend their choices of parameters and approaches to risk-based portfolio construction. Indeed, there is empirical evidence that stock returns are relatively insensitive to risk and that the slope of the market line is practically zero, motivating the choice of equal means for all stocks and the use of the MV approach for stock portfolios. In turn there is empirical evidence that the Sharpe ratio of all asset classes is comparable (with the single exception of commodities). This has motivated the use of equal risk budgeting approaches (risk parity) for strategic asset allocation and could also justify the use of maximum diversification approaches for such problems.

2- In this paper we use the more conventional definition of $\mathbf{\Sigma}$ for the estimated covariance matrix and $\mathbf{\Omega}$ for the covariance matrix of estimated errors in asset mean returns. We note that Scherer [2006] uses exactly the opposite convention.

3- We could think of a fourth choice for the covariance matrix $\mathbf{\Omega}$ of estimation errors in the mean returns that in the limit of large uncertainty would lead to the maximum diversification portfolio proposed by Chouefaty and Coignard [2008]. The maximum diversification portfolio can also be obtained from MVO by assuming that the asset means returns are proportional to the estimated volatilities. If we defined two new matrices, one $\mathbf{C}$ with the estimated correlations of asset returns and a second $\mathbf{\sigma}$ a diagonal matrix with asset volatilities, then with the choice $\mathbf{\Omega} = \mathbf{C}\,\mathbf{\sigma}$, the robust portfolio will converge towards the maximum diversification portfolio in the limit of large uncertainty.

However, this $\boldsymbol{\Omega}$ is not necessarily symmetric and is difficult to interpret. For that reason we do not discuss this choice of quadratic robust optimization any further.

ANNEX

Here we investigate the limit of large uncertainty for the quadratic formulation of the robust portfolio optimization problem when a zero net adjustment constraint is imposed to the Sharpe ratio estimation errors. Equation (38) below closely resembles equation (11) if we choose $\boldsymbol{\Omega}$ = A, a diagonal matrix of asset variances. Here we added a zero net adjustment constraint:

$$\mathbf{w}_{rob} = argmax \left(min(\boldsymbol{\mu}^t \mathbf{w}) - \frac{\lambda}{2} \mathbf{w}^t \boldsymbol{\Sigma} \mathbf{w} \right)$$
$$s.t. \sum_i \left(\frac{\mu_i - \bar{\mu}_i}{\sigma_i} \right)^2 = \sum_i \Delta SR_i^2 \leq \frac{\kappa^2}{n} \quad s.t. \sum_i \left(\frac{\mu_i - \bar{\mu}_i}{\sigma_i} \right) = \sum_i \Delta SR_i = \mathbf{0} \quad s.t. \mathbf{1}^t \mathbf{w} = 1 \tag{38}$$

where ΔSR_i is the estimation error in the Sharpe ratio of asset i. From equation (24), i.e. the Lagrangian with all constraints, and taking into account the fact that $\boldsymbol{\Omega}$ = A, then:

$$\mathbf{w}_{rob} = argmax \left(\bar{\boldsymbol{\mu}}^t \mathbf{w} - \kappa \sqrt{\mathbf{w}^t \boldsymbol{\Phi} \mathbf{w}} - \frac{\lambda}{2} \mathbf{w}^t \boldsymbol{\Sigma} \mathbf{w} \right)$$
$$s.t. \mathbf{1}^t \mathbf{w} = 1 \tag{39}$$

with the matrix $\boldsymbol{\Phi}$ defined as in (27). However, since $\boldsymbol{\Omega} = $ A, $\mathbf{D} = \mathbf{A}^{-\frac{1}{2}}$, and we can write $\boldsymbol{\Phi}$ as:

$$\boldsymbol{\Phi} = \boldsymbol{\Omega} - \frac{1}{\mathbf{1}^t \mathbf{1}} (\mathbf{D}^{-1}) \mathbf{1} \mathbf{1}^t (\mathbf{D}^{-1}) \tag{40}$$

An increase in κ will increase the values of this covariance matrix $\boldsymbol{\Phi}$. The problem can then be written as follows when $\kappa \to \infty$:

$$\lim_{\kappa \to \infty} (\mathbf{w}_{rob}) \to argmax \left(-\kappa \sqrt{\mathbf{w}^t \boldsymbol{\Phi} \mathbf{w}} \right)$$
$$s.t. \mathbf{1}^t \mathbf{w} = 1 \tag{41}$$

With N assets, $\boldsymbol{\Omega}$ is a N x N matrix. Using the inequality of Frobenius (i.e. $rank(\mathbf{A}+\mathbf{B}) \leq rank(\mathbf{A}) + rank(\mathbf{B})$ where $\mathbf{A}$ and $\mathbf{B}$ have the same dimensions) and based on the fact that $rank(\boldsymbol{\Omega}) = N$ and $rank \left(\frac{1}{\mathbf{1}^t \mathbf{1}} (\mathbf{D}^{-1}) \mathbf{1} \mathbf{1}^t (\mathbf{D}^{-1}) \right) = 1$, then:

$$rank(\boldsymbol{\Phi}) \geq N - 1 \tag{42}$$

If $\mathbf{w}_{ERB}$ is the vector of weights of the ERB portfolio, then we can write that:

$$\boldsymbol{\Phi} \mathbf{w}_{ERB} = \boldsymbol{\Omega} \mathbf{w}_{ERB} - \frac{1}{\mathbf{1}^t \mathbf{1}} (\mathbf{D}^{-1}) \mathbf{1} \mathbf{1}^t (\mathbf{D}^{-1}) \mathbf{w}_{ERB}$$

$$= \mathbf{D}^{-1} \mathbf{1} - \frac{1}{\mathbf{1}^t \mathbf{1}} (\mathbf{D}^{-1}) \mathbf{1} (\mathbf{1}^t \mathbf{1})$$

$$= \mathbf{D}^{-1}\mathbf{1}(\mathbf{D}^{-1})\mathbf{1}$$

$$= 0 \tag{43}$$

We can use (43) show that $\mathbf{w}_{ERB}$ is the eigenvector according to the unique zero eigenvalue of $\boldsymbol{\Phi}$. Equation (44) then follows:

$$\mathbf{w}_{ERB} = \lim_{\kappa \to \infty}(\mathbf{w}_{rob}) \to argmax\left(-\kappa\sqrt{\mathbf{w}^t\boldsymbol{\Phi}\mathbf{w}}\right)$$
$$s.t.\,\mathbf{1}^t\mathbf{w} = 1 \tag{44}$$

And then, with a zero net adjustment constraint imposed to the Sharpe ratio estimation errors and $\boldsymbol{\Omega} = \mathbf{A}$, the robust portfolio converges towards the ERB portfolio.

REFERENCES

Asness, C.S., A. Frazzini, and L.H. Pedersen. "Leverage Aversion and Risk Parity." *Financial Analysts Journal*, Vol. 68, No. 1 (2012), pp. 47-59.

Ben-Tal, A., and A. Nemirovski. "Robust Convex Optimization." *Mathematics of Operations Research*, Vol. 23, No. 4 (1998), pp. 769-805.

Black, F., and R. Litterman. "Asset Allocation: Combining Investor Views with Market Equilibrium." Technical report, Goldman Sachs & Co., 1990.

Black, F., and R. Litterman. "Global Portfolio Optimization." *Financial Analysts Journal*, Vol. 48 (September-October 1992), pp. 28-43.

Ceria, S., and R.A. Stubbs. "Incorporating Estimation Errors into Portfolio Selection: Robust Portfolio Construction." *Journal of Asset Management*, Vol. 7, No. 2 (July 2006), pp. 109-127.

Chopra, V.K. "Improving Optimization." *Journal of Investing*, Vol. 2, No. 3 (Fall 1993), pp. 51-59.

Chopra, V.K., and W.T. Ziemba. "The effect of Errors in Means, Variances and Covariances on Optimal Portfolio Choice." *The Journal of Portfolio Management*, Vol. 19, No. 2 (Winter 1993), pp. 6-11.

Chouefaty, Y., and Y. Coignard. "Towards Maximum Diversification." *The Journal of Portfolio Management*, Vol. 35, No. 1 (Fall 2008), pp. 40-51.Dalio, R. "Engineering targeted returns and risks." Technical report, Bridgewater Associates LP, 2005.

Fabozzi, F.J., P.N. Kolm, D.A. Pachamanova, and S.M. Focardi. "Robust Portfolio Optimization." The Journal of Portfolio Management, Vol. 33, No. 3 (Spring 2007), pp. 40-48.

Frost, P.A., and J.E. Savarino. "For Better Performance: Constrain Portfolio Weights." *The Journal of Portfolio Management*, Vol. 15, No. 1 (Fall 1988), pp. 29-34.

Heasen, D., W.G. Hallerbach, T.D. Markwat, and R. Molenaar. "Risk Parity versus Mean-Variance: It's All in the Views." Working Paper, SSRN id=2478751, 2014.

Jurczenko, E., J. Teiletche. "Active Risk-Based Investing." Working Paper, SSRN id=2592904, 2015.

Kritzman, M. "Are Optimzers Error Maximizers?" The Journal of Portfolio Management, Vol. 32, No. 4 (Summer 2006), pp. 66-69.

Leote De Carvalho, R., X. Lu, and P. Moulin. "Demystifying Equity Risk–Based Strategies: A Simple Alpha plus Beta Description." *The Journal of Portfolio Management*, Vol. 38, No. 3 (Spring 2012), pp. 56-70.

Lobo, M.S., L. Vandenberghe, S. Boyd, and H. Lebret. "Applications of Second-order Cone Programming." *Linear Algebra and its Applications,* Vol. 284, No. 1-3 (1997), pp. 193-228.

Markowitz, H. "Portfolio Selection." *Journal of Finance*, Vol. 7, No. 1 (March 1952), pp. 77-91.

Merton, R. "On Estimating the Expected Return on the Market." *Journal of Financial Economics*, Vol. 8, No. 4 (1980), pp. 323-361.

Michaud, R.O. [1998] *Efficient Asset Management: A Practical Guide to Stock Portfolio Optimization and Asset Allocation.* Harvard Business School Press, Boston, MA, 1999

Scherer, B. "Portfolio Resampling: Review and Critique". *Financial Analysts Journal*, Vol. 58, No. 6 (2002), pp. 98-109

Scherer, B. "Can robust portfolio optimization help to build better portfolios?" *Journal of Asset Management*, Vol. 7, No. 6 (2006), pp. 374-387.

Tütüncü, R.H., and M. König "Robust Asset Allocation." *Annals of Operations Research*, Vol. 132, No. 1-4 (2004), pp. 157-187.