Letian Wang

SABR Model

Contents

- SABR Model
- Parameter Estimation
- Example SABR calibration

SABR Model

SABR model, or "Stochastic Alpha, Beta, Rho" model, is a stochastic volatility model for forward (LIBOR) rates. Consider the forward rate $f_t = F(t;S,T)$. **Under T-forward measure Q^T** with numeraire $P(t,T)$, this forward rate is a martingale. In addition, we assume its volatility is also a martingale under Q^T. The SDE is then specified as

$$df_t = \sigma_t (f_t)^\beta dz_t$$
$$d\sigma_t = v\sigma_t dw_t$$
$$E^{Q^T}[dz_t dw_t] = \rho dt$$
$$f_{t=0} = f_0, \sigma_{t=0} = \sigma_0$$

where the current forward price $f_{t=0} = f_0$ is observed in the market. So the model has four constant parameters: $\sigma_0 > 0, 0 \le \beta \le 1, -1 < \rho < 1, v \ge 0$. In terms of the model name, stochastic alpha stands for σ_t; beta and rho stand for their respective parameters.

If $\beta = 0$, the forward rate is normal; if $\beta = 1$, the forward rate is lognormal. If **the volatility of volatility parameter** $v = 0$, the model is reduced to the CEV (Constant Elasticity of Variance) model.

In SABR, it models directly the implied volatility curve which is then used to obtain European option prices via Black-76 model (Chapter 2, Chapter 7). The Black implied volatility is modeled as

$$\sigma^{Model}(K; f_0) = A \cdot \left(\frac{z}{\chi(z)}\right) \cdot B \qquad (1)$$

where

$$A = \frac{\sigma_0}{(f_0K)^{\frac{1-\beta}{2}}\left[1 + \frac{(1-\beta)^2}{24} ln^2 \frac{f_0}{K} + \frac{(1-\beta)^4}{1920} ln^4 \frac{f_0}{K} + \cdots\right]} \qquad (2)$$

$$B = \left[1 + \left(\frac{(1-\beta)^2}{24} \frac{\sigma_0^2}{(f_0K)^{1-\beta}} + \frac{\rho\beta v\sigma_0}{4(f_0K)^{\frac{1-\beta}{2}}} + \frac{2-3\rho^2}{24} v^2\right)T + \cdots\right] \qquad (3)$$

$$z = \frac{v}{\sigma_0}(f_0K)^{\frac{1-\beta}{2}} ln \frac{f_0}{K} \qquad (4)$$

$$\chi(z) = ln\left(\frac{\sqrt{1 - 2\rho z + z^2} + z - \rho}{1 - \rho}\right) \qquad (5)$$

Except for the special cases of β = 0 and β = 1, no closed form expression is known. But the approximation is very accurate as long as the option is not too out-of-money or T is not too large.

Equation (1) is implemented in function *QuantLib:: unsafeSabrVolatility().*

Parameter Estimation

The four model parameters influence differently the shape of implied volatility Curve.

Parameter	Curve Property	Direction
σ_0	Level	The curve shifts upward as it increases
β	Slope	The curve steepens as it decreases

ρ	Slope	The curve steepens as it decreases
v	Curvature	The curvature increases as it increases

To estimate the parameters, or in other words, to calibrate the model, it usually takes three steps.

1. Estimate β
2. Imply σ_0 from ρ and v
3. Calibrate ρ and v

According to the parameter table, β and ρ both control the slope of the volatility curve. Therefore to some degree the model is over-determined. A common industry practice is to skip step 1 by choosing directly $\beta = 0.5$.

Another way of calibration combines step 2 and 3 together and calibrate these three parameters directly.

Step 1 Estimate β

For at-the-money options, equation (1) can be re-written as

$$\sigma_{ATM}^{Mkt} \approx \sigma^{Model}(f_0; f_0) = \frac{\sigma_0}{f^{(1-\beta)}}\left[1 + \left(\frac{(1-\beta)^2}{24}\frac{\sigma_0^2}{(f_0)^{2-2\beta}} + \frac{\rho\beta v\sigma_0}{4(f_0)^{(1-\beta)}} + \frac{2-3\rho^2}{24}v^2\right)T\right] \quad (6)$$

Taking log (Hagan et al. 2002),

$$\ln\left(\sigma_{ATM}^{Mkt}\right) = ln\sigma_0 - (1-\beta)\,lnf_0$$

Therefore β can be estimated from a linear regression between log ATM volatilities and log forward rates time series.

Step 2 Imply σ_0 from ρ and v

Given current market ATM volatility σ_{ATM}^{Mkt}, we can invert equation (6) to get the following cubic equation in σ_0,

$$\left(\frac{(1-\beta)^2 T}{24(f_0)^{2-2\beta}}\right)\sigma_0^3 + \left(\frac{\rho\beta v T}{4(f_0)^{1-\beta}}\right)\sigma_0^2 + \left(1 + \frac{2-3\rho^2}{24}v^2 T\right)\sigma_0 - \sigma_{ATM}^{Mkt}(f_0)^{1-\beta} = 0$$

So σ_0, as the smallest positive real root to this equation, is then explicitly calibrated to at-the-money volatility σ_{ATM}^{Mkt}. It is expressed as a function of parameters ρ and v, which will be calibrated in the next step. The Tartaglia approach to cubic equation can be found in Flannery et al. (1992).

Step 3 Calibrate ρ and v

After step 2, there remains only two parameters to be calibrated: ρ and v. The calibration process is a rather standard one. We can choose the parameters that bring model volatilities to market (quoted implied) volatilities. That is,

$$(\rho, v) = \underset{\rho, v}{\operatorname{argmin}} \sum \left(\hat{\sigma}_i(\rho, v) - \sigma_i^{mkt}\right)^2$$

Following step 1, 2, and 3, SABR model in its primitive format can be relatively easy to calibrate. In general if one tries to calibrate a model to a volatility surface $\sigma(K,T)$ (or volatility cube $\sigma(K,T_\alpha,T_\beta)$ in case of swaptions), the process is usually complicated. So SABR fixes forward rate (and time) T and calibrates itself to volatility smile (skew) curve respect to strike K. The forward rate $f_t = F(t;S,T)$ is treated in its own T-forward measure and does not interact with other forward rates. Compare this with Cap vol calibration (Chapter 7) where At-the-Money volatility curve respect to time T is considered. In both cases, the curve is one dimensional. <u>In sum, as long as we don't consider the forward rate (and its volatility) dynamics under other T-forward measures, the calibration process demands much less effort</u>.

Example

This example fits on 2011-June-06 the Eurodollar futures options market. The option considered here is EDU1P, which matures on 2011-

September-19. Time-to-maturity is 109d. As discussed in Chapter 7, the Eurodollar futures put option is a call option on rates.

Current futures price is $F = 99.665$. The strike price can go beyond 100, implying a negative forward rate. To avoid the problem of negative strikes, here the Black Normal volatilities are used, which are then fitted to the SABR model with original strike price. For example, consider the market options quote

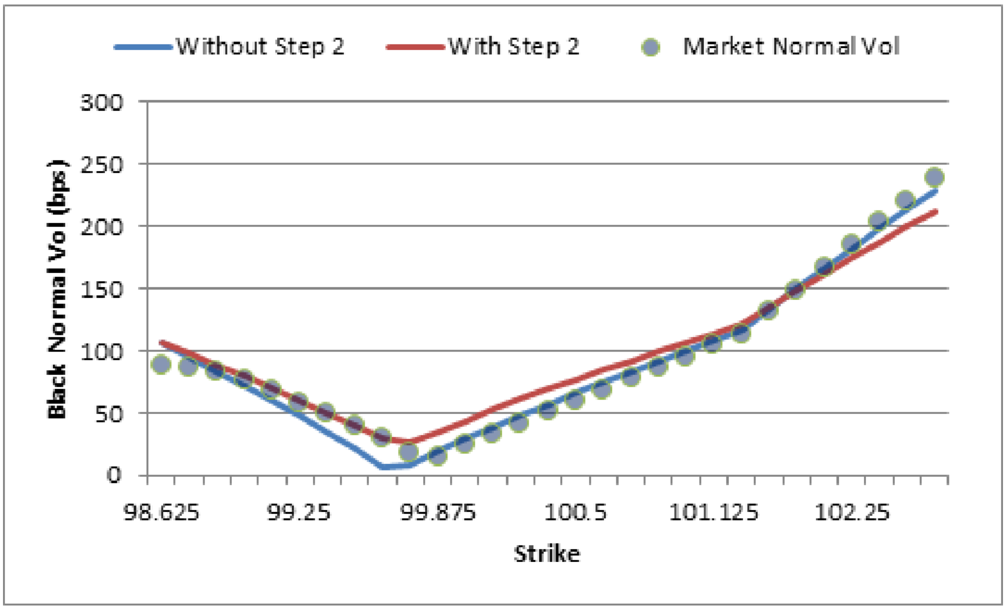
$$K = 101.125, Price = 1.4600$$

First use the Black Normal Volatility formula (Chapter 7)

$$1.46 \times 2500 = 0.99356 \times \$1M \times BN(f = 0.00335, k = -0.01125) \times (\frac{90}{360})$$

where $BN(f, k)$ stands for the Black Normal formula. It yields Black implied normal vol σ_N =104.527 (bps).

Then, we come back and use original F and K to fit the SABR model, the results are shown in the following figure.



In the figure, β is fixed to 0.5. “without step 2” means skip step 2 and calibrate the other three parameters simultaneously. “With step 2” follows the procedure discussed before. The dots represent market implied normal volatility. It seems that the first method fits in-the-money data better, while the second method fits more closely the out-the-money data.

See the C++ file for detail.

Reference

[1] Flannery, B.P. and Press, W.H. and Teukolsky, S.A. and Vetterling, W.

(1992). Numerical recipes in C. Press Syndicate of the University of Cambridge, New York.

[2] Hagan, PS and Kumar, D. and Lesniewski, AS (2002). Managing Smile Risk, Wilmott Magazine, pp 84—108.

[3] Rebonato, R, McKay, K, and White, R. (2009). The SABR/LIBOR market model: pricing, calibration, and hedging for complex interest-rate derivatives. John Wiley and Sons.

[4] West, G. (2005). Calibration of the SABR model in illiquid markets. *Applied Mathematical Finance*, 12(4), pp. 371-385.