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Hull-White Model

Contents

Hull-White Model

This section is adapted from Brigo and Mercurio (2006). As an extension of the Vasicek model, Hull-White model assumes that the short rate follows the mean-reverting SDE:

$$dr(t) = [\theta(t) - ar(t)]dt + \sigma dW(t)$$

where a and σ are positive constants; and $\theta(t)$ is time-dependent function that will be used to fit the current zero curve.

To solve this SDE, first apply Ito lemma to re^{at}

$$d(re^{at}) = e^{at}dr + are^{at}dt = (\theta dt + \sigma dW)\, e^{at}$$

Then integrate both sides over [s,t],

$$r(t)e^{at} - r(s)e^{as} = \int_s^t \theta(u)e^{au}du + \sigma\int_s^t e^{au}dW(u)$$
$$\Rightarrow r(t) = r(s)e^{-a(t-s)} + \int_s^t \theta(u)e^{-a(t-u)}du + \sigma\int_s^t e^{-a(t-u)}dW(u)$$

In order to fit the term structure of interest rates, the time-dependent $\theta(T)$ must satisfy

$$\theta(T) = \frac{\partial f^M(0,T)}{\partial T} + af^M(0,T) + \frac{\sigma^2}{2a}(1 - e^{-2aT})$$

where $f^M(0,T)$ is the market observed instantaneous forward rate at time 0 for the maturity T.

Then $r(t)$ can be re-written as

$$r(t) = r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW(u)$$

where

$$\alpha(t) = f^M(0,t) + \frac{\sigma^2}{2a^2}(1-e^{-at})^2$$

Therefore, $r(t)$ conditional on F_s is normally distributed with mean and variance given respectively by

$$E\{r(t)|F_s\} = r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)}$$
$$Var\{r(t)|F_s\} = \sigma^2 \int_s^t e^{-2a(t-u)} du = \frac{\sigma^2}{2a}\left[1-e^{-2a(t-s)}\right]$$

HW model is an **affine term structure model** where the continuously-compounded spot rate is an affine function in the short rate, i.e.,

$$R(t,T) = \alpha'(t,T) + \beta(t,T)r(t)$$

The zero coupon bond price is given by

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

where

$$A(t,T) = \frac{P^M(0,T)}{P^M(0,t)} exp\left\{B(t,T)f^M(0,t) - \frac{\sigma^2}{4a}(1-e^{-2at})B(t,T)^2\right\}$$
$$B(t,T) = \frac{1}{a}\left[1-e^{-a(T-t)}\right]$$

We can also find the close-form formulas for zero-coupon bond options, caps/floors, and swaptions. See Brigo and Mercurio (2006) for detail.

Hull-White Trinomial Tree

To construct HW tree, it is helpful to decompose the short rate into the following format:

$$r(t) = x(t) + \alpha(t)$$

Where

$$\alpha(t) = f^M(0,t) + \frac{\sigma^2}{2a^2}(1-e^{-at})^2 \quad (1)$$

$$dx(t) = -ax(t)dt + \sigma dW \qquad x(0) = 0 \quad (2)$$

$$x(t) = x(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW(u) \quad (3)$$

With this decomposition in hand, the tree construction task can be achieved in two steps. In the first step one constructs the trinomial tree for $x(t)$. Then in the second step, one shifts the tree by $\alpha(t)$ to bring it in line with the initial term structure.

Step One: Construct the Symmetric Trinomial Tree

From equations (2) and (3), it is known that

$$E\{x(t)|F_s\} = x(s)e^{-a(t-s)}$$
$$Var\{x(t)|F_s\} = \frac{\sigma^2}{2a}\left[1 - e^{-2a(t-s)}\right]$$

Denote the tree nodes by (i,j) where the time index i ranges from 0 to N and the space index j ranges from some $\underline{j}_i < 0$ to some $\bar{j}_i > 0$. Using the results in A.F (Brigo and Mercurio 2006), we have

$$E\{x(t_{i+1})|x(t_i) = x_{i,j}\} = x_{i,j}e^{-a\Delta t_i} =: M_{ij}$$
$$Var\{x(t_{i+1})|x(t_i) = x_{i,j}\} = \frac{\sigma^2}{2a}\left[1 - e^{-2a\Delta t_i}\right] =: V_i^2 \qquad (4)$$

where $\Delta t_i = t_{i+1} - t_i$.

Now, given the node $x_{i,j}$, we need to locate its subsequent nodes $x_{i+1,k+1}, x_{i+1,k}$ and $x_{i+1,k-1}$ with the respective transition probabilities p_u, p_m, p_d. This is done as follows. First find the space in the y direction as

$$\Delta x_{i+1} = V_i\sqrt{3} = \sigma\sqrt{\frac{3}{2a}\left[1 - e^{-2a\Delta t_i}\right]} \qquad (5)$$

Then locate the level k by

$$k = round\left(\frac{M_{i,j}}{\Delta x_{i+1}}\right) \qquad (6)$$

where round(x) function indicates the closest integer to the real number x. We then set

$$x_{i+1,k+1} = (k+1)\Delta x_{i+1}, x_{i+1,k} = k\Delta x_{i+1}, x_{i+1,k-1} = (k-1)\Delta x_{i+1} \qquad (7)$$

At last the transition probabilities are chosen in a way to match the conditional mean and variance,

$$p_u = \frac{1}{6} + \frac{\eta_{j,k}^2}{6V_i^2} + \frac{\eta_{j,k}}{2\sqrt{3}V_i},$$
$$p_m = \frac{2}{3} - \frac{\eta_{j,k}^2}{3V_i^2}, \qquad (8)$$
$$p_d = \frac{1}{6} + \frac{\eta_{j,k}^2}{6V_i^2} - \frac{\eta_{j,k}}{2\sqrt{3}V_i}$$

where

$$\eta_{j,k} = M_{i,j} - x_{i+1,k} = M_{i,j} - k\Delta x_{i+1} \qquad (9)$$

Step Two: Displace the Tree

The second step consists of displacing the tree nodes to obtain the corresponding tree for r. An easy way to do so is through equation (1), as in class *HullWhite::FittingParameter*, where the instantaneous forward rate is approximated by

$$f^M(0,t) \approx F^M(0;t,t+0.5bps)$$

This approach has to approximate continuously-compounded rate $R(0,t)$ with short rate r(0), therefore doesn't fit exactly the zero curve.

The other way uses helps from Arrow-Debrew prices. Denote α_i be the displacement at time t_i, and $Q_{i,j}$ be the Arrow-Debrew price of node (i,j). A **state price security** or an **Arrow-Debreu security** is defined as a contract which pays off $1 in a particular state at a particular time and pays zero in all other states. Its price (net present value) is referred to as **Arrow-Debreu price**.

The values of α_i and $Q_{i,j}$ are calculated recursively as follows.

1. Initialize

$$Q_{0,0} = 1$$

2. Find

$$\alpha_0 = -ln(P^M(0,t_1))/t_1$$

3. With $\alpha_i, (i = 0 \text{ } at \text{ } this \text{ } time)$ in hand, calculate

$$Q_{i+1,j} = \sum_h Q_{i,h} q(h,j) exp\{-(\alpha_i + h\Delta x_i)\Delta t_i\}$$

where $q(h,j)$ is the probability of moving from node (i,h) to node $(i+1,j)$.

4. With $Q_{i,j}$ $(i = 1 \text{ } at \text{ } this \text{ } time)$ in hand, find α_i by solving

$$P(0,t_{i+1}) = \sum_{j=\underline{j}_i}^{\bar{j}_i} Q_{i,j} exp\{-(\alpha_i + j\Delta x_i)\Delta t_i\}$$

that leads to

$$\alpha_i = \frac{1}{\Delta t_i} ln \frac{\sum_{j=\underline{j}_i}^{\bar{j}_i} Q_{i,j} exp\{-j\Delta x_i \Delta t_i\}}{P(0,t_{i+1})}$$

5. Loop step 3 and 4 to discover α_i and $Q_{i,j}$ for further steps (i++).

6. Short rate on each node

$$r_{i,j} = x_{i,j} + \alpha_i$$

Example: Hull-White Calibration

This section illustrates the Hull-White model calibration process with a real example. It calibrates the Hull-White tree to the LIBOR market on Monday, May 16, 2011. The settlement date is Wednesday, May 18, 2011.

Hull-White model has three parameters: a, σ, and $\theta(t)$. In this example, the first two will be calibrated to LIBOR swaptions; and the third one will be calibrated to LIBOR spot curve. To begin with, a and σ are initialized as

$$a = 0.1, \sigma = 0.01$$

Another input is LIBOR spot curve, which is given by

Table 1 -- LIBOR Curve

Tenor	Date	Time to Maturity	Year Fraction	Rate
3m	Thursday, August 18, 2011	92 days	0.255556	0.0026
6m	Friday, November 18, 2011	184 days	0.511111	0.00412
9m	Tuesday, February 21, 2012	279 days	0.775	0.00572

The rates are continuously compounded. Conventions are Act/360 and Modified Following.

The calibration process is carried out in four steps:

Step 1. construct trinomial tree for process $x(t)$;

Step 2. displace $x(t)$ to obtain the tree for $r(t)$;

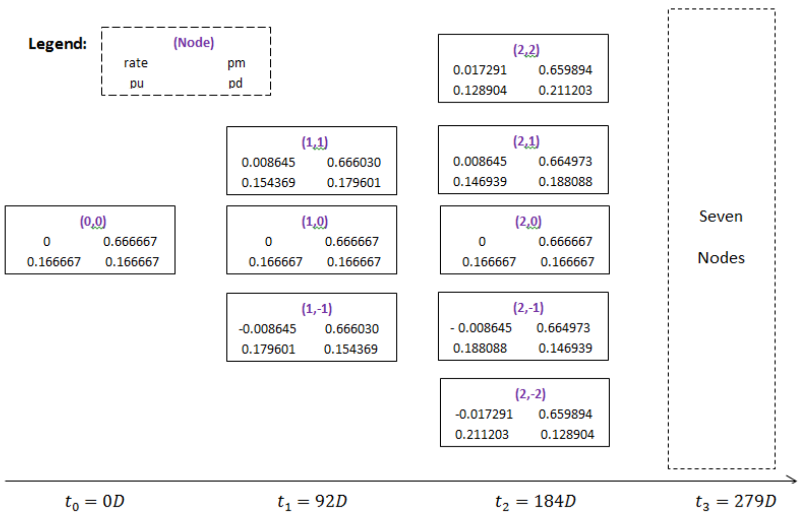
Step 3. price swaptions on this tree;

Step 4. calibrate the tree to swaptions market.

Step One: Construct Trinomial Tree for Process x(t)

The outcome of this step is shown in the following figure. This trinomial tree of $x(t)$ is symmetric.

Figure 1 – Trinomial Tree of x(t)



Now let's walk through the steps to create this figure.

1. From equation (2), $x(t_0 = 0) = 0$, or node (0,0) is 0.

2. **Consider node (0,0)**. From equation (4)

$$M_{0,0} = x_{0,0}e^{-(2)(0.1)(0.25556)} = 0$$
$$V_0 = \frac{(0.01)^2}{(2)(0.1)}\left[1 - e^{-(2)(0.1)(0.25556)}\right] = 0.000025$$

3. Go through equations (3) – (5) to locate its descendants: nodes (1,-1), (1,0) and (1,1).

$$\Delta x_1 = V_0\sqrt{3} = 0.008645$$
$$k = round\left(\frac{M_{0,0}}{\Delta x_1}\right) = 0$$
$$x_{1,-1} = -0.008645$$
$$x_{1,0} = 0$$
$$x_{1,1} = 0.008645$$

4. Using (8) and (9) to get transition probabilities from $x_{0,0}$ to its descendants.

$$\eta_{0,0} = M_{0,0} - x_{1,0} = 0$$
$$p_u = \frac{1}{6} + \frac{\eta_{0,0}^2}{6V_0^2} + \frac{\eta_{0,0}}{2\sqrt{3}V_0} = 0.166667$$
$$p_m = \frac{2}{3} - \frac{\eta_{0,0}^2}{3V_0^2} = 0.666667$$
$$p_d = \frac{1}{6} + \frac{\eta_{0,0}^2}{6V_0^2} - \frac{\eta_{0,0}}{2\sqrt{3}V_0} = 0.166667$$

By now node (0,0) is finished.

5. **Now consider node (1,-1)**. Follow step 2 to 4,

$$M_{1,-1} = x_{1,-1}e^{-(2)(0.1)(0.25556)} = -0.008427$$
$$V_1 = \frac{(0.01)^2}{(2)(0.1)}\left[1 - e^{-(2)(0.1)(0.25556)}\right] = 0.000025$$

and

$$\Delta x_2 = V_1\sqrt{3} = 0.008645$$
$$k = round\left(\frac{M_{1,-1}}{\Delta x_2}\right) = -1$$
$$x_{2,-2} = -0.017291$$
$$x_{2,-1} = -0.008645$$
$$x_{2,0} = 0$$

as well as

$$\eta_{-1,-1} = M_{1,-1} - x_{2,-1} = 0.000218$$
$$p_u = \frac{1}{6} + \frac{\eta_{-1,-1}^2}{6V_1^2} + \frac{\eta_{-1,-1}}{2\sqrt{3}V_1} = 0.179601$$
$$p_m = \frac{2}{3} - \frac{\eta_{-1,-1}^2}{3V_1^2} = 0.666030$$
$$p_d = \frac{1}{6} + \frac{\eta_{-1,-1}^2}{6V_1^2} + \frac{\eta_{-1,-1}}{2\sqrt{3}V_1} = 0.154369$$

6. Process similarly node (1,0) and node (1,1), then move on to the next tenor (6m). Then the figure will be created.

When the volatility is time-dependent, this trinomial tree is recombining.

Step Two: Displace x(t) to Obtain the Tree for r(t)

We can iteratively find state prices and $\alpha(t)$, then shift the symmetric tree from step one by $\alpha(t)$ to obtain the Hull-White tree. This has been explained in Section 2. Let's follow the procedure introduced in that section.

1. Initialize

$$Q_{0,0} = 1$$

2. Find displacement

$$\alpha_0 = -\ln(\exp(-0.255556 \times 0.0026))/0.0026 = 0.0026$$

3. **Move on to time step 1**, compute the state prices for the three nodes:

$$\begin{aligned} Q_{1,-1} &= Q_{0,0}q(0,-1)exp\{-(\alpha_0)\Delta t_0\} = 0.166556 \\ Q_{1,0} &= Q_{0,0}q(0,0)exp\{-(\alpha_0)\Delta t_0\} = 0.666224 \\ Q_{1,1} &= Q_{0,0}q(0,1)exp\{-(\alpha_0)\Delta t_0\} = 0.166556 \end{aligned}$$

4. The displacement for time step 2

$$\alpha_1 = \frac{1}{\Delta t_1} ln \frac{\sum_{j=-1}^{1} Q_{1,j} exp\{-x_{1,j}\Delta t_1\}}{\exp(-0.00412 \times 0.51111)} = 0.005643$$

5. **Move on to time step 2**, compute the state prices for the five nodes. For example, for the middle node (2,0), it has three incoming nodes: (1,-1), (1,0), and (1,1), then

$$\begin{aligned} Q_{2,0} = {} & Q_{1,-1}q(-1,0)exp\{-(\alpha_1 - x_{1,-1})\Delta t_1\} \\ & +Q_{1,0}q(0,0)exp\{-(\alpha_1 + x_{1,0})\Delta t_1\} \\ & +Q_{1,1}q(1,0)exp\{-(\alpha_1 + x_{1,1})\Delta t_1\} \\ & = 0.503250 \end{aligned}$$

6. Given the five state prices on time step 2, the displacement of this time step is

$$\alpha_2 = \frac{1}{\Delta t_2} ln \frac{\sum_{j=-2}^{2} Q_{2,j} exp\{-x_{2,j}\Delta t_2\}}{\exp(-0.00572 \times 0.775)} = 0.50325$$

The results are shown in the following table.

Table 2 – State Prices and Displacements

Time Step	0	1	2
Q			0.025617
		0.166556	0.221404
	1	0.666224	0.503250
		0.166556	0.221894
			0.025731
α	0.0026	0.005643	0.008832

By displacing the symmetric trinomial tree in step one with corresponding α_i, **the Hull-White tree is constructed** as in the following figure.

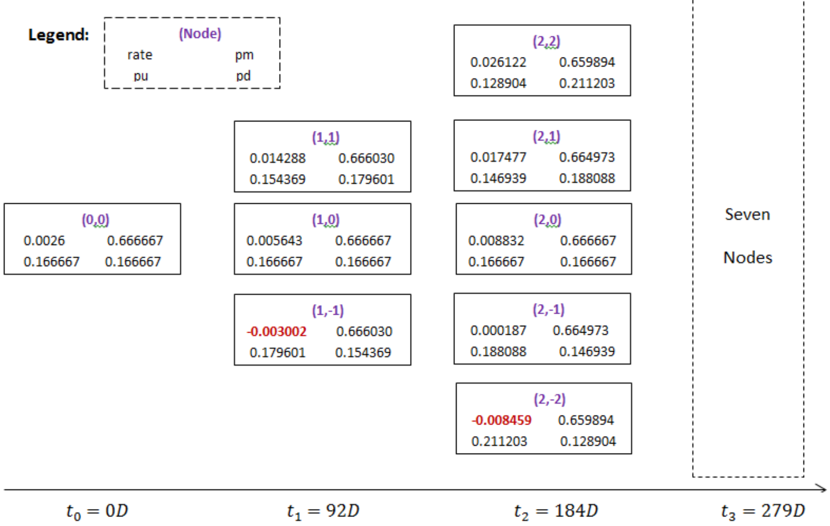


Figure 2 – Hull-White Trinomial Tree

Note that the rates on node (1,-1) and node (2,-2) are negative. Hull White model can produce negative rates due to normal distribution.

Step Three: Price Swaptions on the Tree

Before we can proceed to swaption pricing, it needs to point out **the way in calculating the discounting factor D(t,T) on the short rate tree**. To calculate the discount factor, for example $D(t_0,t_1)$, we need to retrieve the rate at the beginning of the period, in this case 0.0026 on time $t_0 = 0$. This contrasts with the general case when the term structure curve is used, where we usually retrieve the rate from the end of the period, in this case 0.0026 on time $t_1 = 3m$. This rule applies to other time steps as well.

Now we are ready to price on this tree a 3mx6m ATM European payer Swaption with notional \$1,000 and ATM rate 0.007427. The swaption can be exercised only on $t_1 = 3m$, giving the owner the right to enter into the long position (pay fixed) of a 3mx6m forward starting swap. Therefore the value of this swaption on each of the three nodes on $t_1 = 3m$, or nodes (1,j), j= -1, 0, 1, is simply

$$swaption\ value\ on\ node\ (1,j) = \max\{0, swap\ value\ on\ node\ (1,j)\}$$

So it comes down to price the underlying swap, which is priced by discounting its fixed leg cash flows and floating leg cash flows respectively, via the following formula,

$$NPV_{leg}(t) = E_t\left\{\sum D(t,T_i) \times CF_i\right\} = E_t\left\{\sum e^{-R(t,T_i)\tau(t,T_i)} \times CF_i\right\}$$

First consider the fixed leg. It pays on t_3 the amount

$$1000 \times 0.508333 \times 0.007427 = 3.775665$$

where the year fraction $0.508333 \approx (279-92)/360$ according to 30/360 day count. The results are shown in the following table.

Table 3 -- Fixed Leg Pricing (Backward Induction)

Node	Time step 1	Time step 2	Time step 3
3			3.775665
2		3.749728	3.775665
1	3.744813	3.758292	3.775665
0	3.761451	3.766876	3.775665
-1	3.778163	3.775480	3.775665
-2		3.784103	3.775665
-3			3.775665

Now we explain how to discount one step backward, from time step 3 to from step 2. Later we will use the same logic to discount backward the floating leg.

Consider the node (2,-2) for instance. It has three descendants: node (3,-3), (3, -2), and (3, -1). Denote $V_{3,j}$ the value of node (3, j). In this particular case

$$V_{3,j} = 3.775665, \qquad \text{for j} = -3 \text{ to } 3$$

To roll one step back,

$$\begin{aligned} V_{2,-2} &= \exp\left(-r_{2,-2} * t_2\right) \times \{q(-2,-3) \times V_{3,-3} \\ &\quad + q(-2,-2) \times V_{3,-2} \\ &\quad + q(-2,-1) \times V_{3,-1}\} \\ &= \exp(0.008459 \times 0.263889) \times \{0.128904 \times 3.775665 \\ &\quad + 0.659894 \times 3.775665 \\ &\quad + 0.211203 \times 3.775665\} \\ &= 3.784103 \end{aligned}$$

We continue to deal with other four nodes in step 2 and then move on to step 1. This leads to Table 3.

For the floating leg, we can do it similarly by first identifying the cash flows and then discounting them. An alternative and quicker way is through equivalent cash flows.

Unlike fixed leg, floating leg has tenor of 3 months (see chapter LIBOR Rates). Therefore, it contains two cash flows. The first one resets at time t_1 and pays at time t_2; the second one resets at time t_2 and pays at time t_3.

Look at the second cash flow. On node (2,-2), the rate is fixed at

$$L(t_2,t_3) = F(t_2;t_2,t_3) = \frac{1}{\tau(t_2,t_3)}\left(\frac{P(t_2,t_2)}{P(t_2,t_3)} - 1\right)$$

So the payment made at time t_3 is

$$N \times L(t_2,t_3) \times \tau(t_2,t_3)$$

which is equivalent to

$$N \times L(t_2,t_3) \times \tau(t_2,t_3) \times P(t_2,t_3) = N\{1 - P(t_2,t_3)\}$$

on time t_2 after discounting. Thus by starting with t_2 via equivalent cash flows, it saves us one step of backward induction. The results are shown in the following table.

Table 4 – Two floating leg cash flows (backward induction)

Node	Second cash flow		First cash flow	Total Value
	Time step 1	Time step 2	Time step 1	Time step 1
2		6.869698		
1	4.526675	4.601398	3.64483	8.171505
0	2.323698	2.327917	1.441107	3.764806
-1	0.106017	0.049244	-0.76749	-0.66147
-2		-2.23463		

In Table 4, the two floating cash flows are treated independently, and then added up together. Cash flows start with equivalent cash flows. For example, second cash flow pays at time step 3. Its equivalent cash flow on node (2, -2) at time step 2 is

$$1000 \times (1 - \exp(0.008459 \times 0.263889)) = -2.23463$$

After figuring out the other four nodes at time step 2, they are discounted back to time step 1 via the same procedure as has been seen in the fixed leg part. Total value at time step 1 is the sum of the first and second cash flows.

Now we have treated both floating and fixed legs, it is ready to price the swap and the 3mx6m swaption. The pricing procedure is shown in the following table.

Table 5 – 3mx6m Swaption Pricing

Node	Fixed Leg	Floating Leg	Swap	Swaption	State Prices	NPV
(1,1)	3.744813	8.171505	4.426692	4.426692	0.166556	0.737292
(1,0)	3.761451	3.764806	0.003355	0.003355	0.666224	0.002235
(1,-1)	3.778163	-0.66147	-4.439635	0.000000	0.166556	0.000000
Sum						**0.739527**

In Table 5, the fixed leg column and floating leg column are inherited from Table 3 and 4, respectively. Then a long swap position receives floating leg while pays fixed leg. The Swaption is only exercised when it is in the money, or underlying swap has positive value. The last column, NPV, is the product of the Swaption value column and state price column. Finally the NPV of swaption price is the sum of NPV column, or 0.739527.

This example is done in Excel. In comparison, the accompanying C++ code gives 0.739461.

Step Four: Calibrate the Tree to Swaptions Market

Step three calculates the 3mx6m swaption on the HW tree. It is known as the **model price**, which depends on the model parameters. In this case, it depends on the (initial) value of $\alpha = 0.1$ and $\sigma = 0.01$.

Market calibrates the Hull-White model to swaption volatility cube by minimizing

$$\min f(\alpha,\sigma) = \sum \left(\frac{P^{Model}(\alpha,\sigma) - P^{Market}}{P^{Market}} \right)^2$$

where model price $P^{Model}(\alpha,\sigma)$ is calculated by following step three; and P^{Market} is the **market price**, obtained by plugging the volatility quotes into Black model. Pay attention to whether the volatilities are quoted as log vol or normal vol (see Chapter LIBOR Volatility).

The optimization can be achieved by iterations. In that case, a new Hull-White tree will be constructed for each iteration (when α and σ change).

Bloomberg Commands: **SWPM, VCUB**.

Appendix

This appendix derives PDE for Interest Rate Derivatives (IRDs) in short rate model. Let the short rate SDE be

$$dr = a(r,t)dt + b(r,t)dW$$

An interest rate derivative (IRD) has payoff V at time T. Its value at time t is

$$V = V(r_t, t; T)$$

Using Ito lemma

$$\begin{aligned} dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial r}dr + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}d\langle r,r\rangle \\ &= \left(\frac{\partial V}{\partial t} + a\frac{\partial V}{\partial r} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial r^2}\right)dt + b\frac{\partial V}{\partial r}dW \\ &= \mu V dt + \sigma V dW \end{aligned}$$

by defining

$$\mu = \left(\frac{\partial V}{\partial t} + a\frac{\partial V}{\partial r} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial r^2}\right)/V$$
$$\sigma = b\frac{\partial V}{\partial r}/V$$

We construct a hedging portfolio with two instruments with two different maturities, T_1 and T_2

$$\Pi = \Delta_1 V_1 - \Delta_2 V_2$$
$$d\Pi = (\Delta_1\mu_1 V_1 - \Delta_2\mu_2 V_2)dt + (\Delta_1\sigma_1 V_1 - \Delta_2\sigma_2 V_2)dW$$

In order to be risk-free

$$\Delta_1\sigma_1 V_1 = \Delta_2\sigma_2 V_2$$

and no-arbitrage

$$d\Pi = r\Pi dt \Rightarrow \Delta_1\mu_1 V_1 - \Delta_2\mu_2 V_2 = r(\Delta_1 V_1 - \Delta_2 V_2)$$

$$\Rightarrow \Delta_1 V_1(\mu_1 - r) = \Delta_2 V_2(\mu_2 - r)$$
$$\Rightarrow \frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}$$

which shows that the market price of risk is independent of maturity T

$$\frac{\mu - r}{\sigma} = \lambda(r, t)$$

Then

$$\mu = r + \lambda\sigma = r + \lambda b \frac{\partial V}{\partial r}$$

Substitute it into the drift equation we obtain the PDE

$$rV + \lambda b V \frac{\partial V}{\partial r} = \frac{\partial V}{\partial t} + a \frac{\partial V}{\partial r} + \frac{1}{2} b^2 \frac{\partial^2 V}{\partial r^2}$$

A faster way to get the PDE under risk-neutral measure $\lambda = 0$ uses martingale property. Let the bank account numeraire be

$$B(t) = e^{\int_0^t r(s)ds}$$

We know that an instrument with payoff $V(r, t; T)$, $V(r, t; T)/B(t)$ is a martingale under risk-neutral measure Q. Therefore,

$$d\left(\frac{V(r,t)}{B(t)}\right) = d\left(V(r,t)e^{-\int_0^t r(s)ds}\right)$$
$$= \left(\frac{\partial V}{\partial r}dr + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}d\langle r, r\rangle + \frac{\partial V}{\partial t}dt\right)e^{-\int_0^t r(s)ds} + Ve^{-\int_0^t r(s)ds}(-r)dt$$

whose drift term should be 0. It leads to,

$$\frac{\partial V}{\partial t} + a \frac{\partial V}{\partial r} + \frac{1}{2} b^2 \frac{\partial^2 V}{\partial r^2} = rV$$

Reference

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