Fixed Income Analytics in DRIP

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Section I: Preface

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Terminology Background

Framework Glossary

1. Self-Jacobian: Self-Jacobian refers to the Jacobian of the Objective Function at any point in the variate to the Objective Function at the segment nodes, i.e., $\frac{\partial Y(t)}{\partial Y(t_K)}$.
2. Point-Measure State-Transform: Point-Measure transform refers to the one-to-one transform between a state measure at a predictor ordinate and its corresponding observation (e.g., discount factor from zero-coupon bond price observations).
3. Convolved-Measure State-Transform: Convolved-Measure transform refers to the many-to-one transform between a state metric/predictor ordinate combination to a given observation, i.e., a set of state metric/predictor ordinate pairs together imply an observation (e.g., zero rates from swap fair premium).
4. Discount-Curve Native Forward Curve: For discount curves built out of instruments dependent on forward rates, those rates and their discount curve usage ranges together constitute the discount curve's native forward curve range.

Document Layout

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Section II: Jurisdictions, Instruments, Trading, and Settlement Conventions

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Associations and Exchanges

Associations

1. ISDA: Many rules and standards are proposed or collected by financial associations. Chief among the, the ISDA, was founded in 1985. In particular ISDA publishes the ISDA definitions. Reference => http://www2.isda.org/
2. British Bankers' Association: The British Banker's Association (BBA) is the trade association for the UK banking and the financial services sector. Reference => http://www.bba.org.uk/
3. EURIBOR-EBF: The EURIBOR-EBF is international non-profit association founded in 1999 with the launch of the Euro. Its members are the national banking associations of the member nations of the European Union which are involved in the Euro-zone and the Euro system. Reference => http://www.euribor-ebf.eu/
4. Australian Financial Markets Association: AFMA was formed in 1986. Reference => http://www.afma.com.au
5. Danish Bankers' Association: The Danish Bankers' Association is an organization representing the banks in Denmark. It has the overall responsibility for the CIBOR indices. Reference => http://www.finansraadet.dk
6. Wholesale Markets' Brokers' Association: The WMBA is an association of the London brokers. Reference => http://www.wmba.org.uk
7. Japanese Bankers' Association: The Japanese Bankers' Association is a financial organization whose members consist of banks, bank holding companies, and bankers' associations in Japan. Reference => http://www.zenginko.or.jp/en/

Exchanges

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1. Introduction: There are many exchanges where financial instruments are traded throughout the world. This section includes the main ones where the interest rate derivatives are listed. Over the years a lot of mergers and acquisitions have taken place between the exchanges. The names and the organization structures have changed, and will certainly change again.
2. Australian Securities Exchange: In the interest rate landscape, the main products are the AUD bank bill futures and their options, and AUD bond futures. Reference => http://www.asx.com.au/
3. BM&FBovespa - Brazil: BM&FBovespa was created in 2008 through the integration between Sao Paolo Stock Exchange (Bolsa de Valores de Sao Paolo) and the Brazilian Mercantile and Futures Exchange (Bolsa de Mercadorias e Futuros). Reference => http://www.bmfbovespa.com.br
4. CME Group: The CME group is a result of mergers between the Chicago Mercantile Exchange (CME), the Chicago Board of Trade (CBOT), the New York Mercantile Exchange (NYMEX), and COMEX. In the interest rate landscape, the main products are the interest rate futures (on LIBOR) and their options listed on CME, the federal funds futures listed in CBOT, and the bond futures and their options listed in CBOT. CME also runs a swap clearing business. Reference => http://www.cmegroup.com
5. EUREX: Eurex is a derivatives exchange jointly operated by the Deutsche Borse AG and the SIX Swiss Exchange. It started its derivatives trading in 1998. In the interest rate landscape, the main products are the interest rate futures (on EURIBOR) and their options, and EUR bond futures. Reference => http://www.eurexchange.com/index.html
6. Intercontinental Exchange - ICE: ICE is a relatively recent exchange active mainly in commodity, energy, and credit. It is involved in interest rate derivatives mainly through its (as of November 2013) acquisition of NYSE EuroNext. Reference => http:///www.theice.com
7. LCH.ClearNet: LCH.ClearNet group is a clearing house serving major exchanges and platforms, as well as a range of OTC markets. ClearNet is owned 77.5% by its clients and 22.5% by the exchanges. Reference => http://www.lchclearnet.com
8. MEFF - Spain: MEFF is an official secondary market regulated by the Spanish laws and under the supervision of the Spanish National Securities Market Commission. Reference => http://www.meff.com

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Tel. 646-513-3300

9. Montreal Exchange: The Montreal Exchange is an electronic exchange dedicated to the development of the Canadian derivative markets. Reference => http://www.m-x.ca/
10. NASDAQ OMX: In the interest rate landscape the main products are the Nordic futures; CIBOR futures, STIBOR futures, and Swedish Bond Futures. They are also known for publishing the CIBOR and the SIOR rates. NASDAQ OMX also runs an exchange in London – the NLX (New London eXchange). References => http://www.nasdaqomx.com; http://www.nasdaqomx.com/transactions/markets/nlx
11. NYSE EuroNext: NYSE EuroNext resulted from the mergers/acquisitions between EuroNext, NYSE< LIFFE, and AMEX. The exchange was acquired by InterContinental Exchange in November 2013. In the interest rate landscape the main products are the interest rate future (on LIBOR and EURIBOR), and their listed options on LIFFE. Reference => http://www.euronext.com
12. Singapore Exchange - SGX: In the interest rate landscape, the products are Japanese and Singaporean Government Bond Futures, JPY (LIBOR and TIBOR), the Eurodollar STIR futures/options, and SGD futures. SGX also runs a swap clearing business. Reference => http://www.sgx.com
13. Tokyo Stock Exchange: In the interest rate landscape, the main products are JPY bond futures. Reference => http://www.tse.or.jp/english
14. South African Futures Exchange (SAFE): The Johannesburg Stock Exchange's Interest Rate Market offers bond futures and JIBAR 3M STIR futures. Reference => http://www.safex.co.za.

Date Conventions

Day Count Conventions

1. 1/1: The day count fraction is always 1.0; Definition 4.16 (a) in 2006 ISDA definitions (Open Gamma (2012)).
2. 30/360 Methods: Here the DCF is computed as

$$DCF = \frac{360(Y_2 - Y_1) + 30(M_2 - M_1) + (D_2 - D_1)}{30}$$

   The main differences are on how $Y_{1,2}$, $M_{1,2}$, and $D_{1,2}$ are calculated.
3. Generic 30/360: The generic 30.360 rules are:
   - If $D_1 == 31$, set $D_1 = 30$
   - If $D_2 == 31$ and $D_1 == 30/31$, set $D_2 = 30$

     This day count is also called 30/360 US, 30U/360, Bond Basis, 30/360, or 360/360. The last 3 are the ones used in the 2006 ISDA conventions. A variation of this uses an EOM Convention, which applies the following addition rule: If the EOM flag is turned on, and $D_{1,2}$ are the last days of February, the set $D_{1,2} = 30$. ISDA (as set out in 4.16(f)) does not use the EOM flag.
4. 30E/360: This is definition 4.16(g) in the 2006 ISDA definitions. The date adjustment rules are the following: If $D_1 == 31$, change it to $D_1 = 30$. Do the same for D_2 as well. This is also referred to as EUROBOND basis.
5. 30E/360 (ISDA): This is definition 4.16 (h) in ISDA 2006 definitions. The date adjustment rules are the following: a) If D_1 is the last day of the month, D_1 becomes 30. b) If D_2 is the last day of February, but not the termination date, or if $D_2 == 31$, then $D_2 = 30$.
6. 30E+/360 (ISDA): The date adjustment rules are the following: a) If $D_1 == 31$, the set $D_1 = 30$. b) If $D_2 == 31$, then change $D_2 = 1$ and $M_2 = M_2 + 1$. This convention is also called 30E+/360.
7. Act/360: This is definition 4.16 (e) in the 2006 ISDA definitions. The accrual factor is

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$$DCF = \frac{d_2 - d_1}{360}$$

where $d_2 - d_1$ is the number of days between the 2 dates. This is the most widely used convention for the money market instruments (maturity below one year). This day count is also called the MONEY MARKET BASIS, Actual 360, or French Money Market Basis.

8. Act/365 Fixed: This is definition 4.16 (d) in the 2006 ISDA definitions. The accrual factor is

$$DCF = \frac{d_2 - d_1}{365}$$

where $d_2 - d_1$ is the number of days between the 2 dates. The number 365 is used even in a leap year. This convention is also called the ENGLISH MONEY MARKET BASIS.

9. Act/365L: This convention, described in ICMA Rule 251.1 (i), is seldom used. It was originally designed for the Euro-Sterling FRNs. It is used to only compute the accrual factor of a coupon. The computation of the factor requires 3 dates – the accrual start date (d_1), the accrual factor date (d_2), and the accrual end date (d_3).

10. Application of 365L: For semi-annual coupons (the type of coupons for which it was originally designed for), the accrual factor is

$$DCF = \frac{d_2 - d_1}{Days\ in\ the\ End\ Year}$$

where $Days\ in\ the\ End\ Year$ is the number of days in the year contained by d_3. This convention is extended to annual coupons by

$$DCF = \frac{d_2 - d_1}{Denominator}$$

where $Denominator$ is 366 if 29 February is between d_1 (exclusive) and d_3 (inclusive), or 365 otherwise. The convention is also called Act/365 Leap Year.

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Tel. 646-513-3300

11. Act/365A: The accrual factor here is

$$DCF = \frac{d_2 - d_1}{Denominator}$$

where $Denominator$ is 366 if 29 February is between d_1 (exclusive) and d_2 (inclusive), or 365 otherwise. The convention is also called Act/365 Actual.

12. NL/365: The accrual factor is

$$DCF = \frac{Numerator}{365}$$

where $Numerator = d_2 - d_1 - 1$ if 29 February is between d_1 (exclusive) and d_2 (inclusive), or $d_2 - d_1$ otherwise. The convention is also called Act/365 No Leap Year.

13. Act/Act ISDA: This is definition 4.16(a) in 2006 ISDA definitions. The accrual factor is

$$DCF = \frac{Days\ in\ a\ Non-Leap\ Year}{365} + \frac{Days\ in\ a\ Leap\ Year}{366}$$

where the period first day is include and the period last day is excluded.

14. Act/Act ICMA – No Adjustment: This is taken from 4.16 (c) of the 2006 ISDA definitions. This convention is defined in Rule 251 of the ICMA Rule book. The accrual factor is

$$DCF = \frac{1}{Freq} \times Adjustment$$

where $Freq$ is the number of coupons per year and $Adjustment$ depends on the type of stub period. Where NO_ADJUSTMENT is set, $Adjustment = 1$, so the accrual factor becomes simply $\frac{1}{Freq}$.

15. Act/Act ICMA – Short at Start: Here the adjustment is computed as a ratio. The numerator is the number of days in the period, and the denominator is the number of days between the

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Tel. 646-513-3300

standardized start date computed as the coupon end date minus the number of month corresponding to the frequency (e.g., $\frac{12}{Freq}$) and the end date.

16. Act/Act ICMA – Long at Start: Two standardized start dates are computed as the coupon end date minus once and twice the number of months corresponding to the frequency. The numerator is the number of days between the start date and the denominator is the number of days between the first and the second standardized start date. The adjustment is the ratio of the numerator by the denominator plus 1.
17. Act/Act ICMA – Short at End: The adjustment is computed as a ratio. The numerator is the number of days between the start date and the standardized end date computed as the coupon start date plus the number of months corresponding to the frequency (i.e., $\frac{12}{Freq}$).
18. Act/Act ICMA – Long at End: Two standardized dates are computed as the coupon start dates plus once and twice the number of months corresponding to the frequency. The numerator is the number of months between the end date and the first standardized end date, and the denominator is the number of days between the second and the first standardized end date. The adjustment is the ratio of the numerator to the denominator plus 1.
19. Business/252: This day count is also called BUS/252. This day count is based on the business, not the calendar days. The accrual factor is

$$DCF = \frac{Business\ Days}{252}$$

where the numerator is the number of business days in a calendar year from and including the start date up to and excluding the end date. This day count convention is used in particular in the Brazilian market.

Business Day Conventions

1. Following: Business day convention is the convention for the adjustment of the dates when the specified date is not a good business day. The adjustment is applied with respect to a

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Tel. 646-513-3300

specified calendar. In the *following* convention, the adjusted day is the following business day.

2. Preceding: Here the adjusted day is the preceding good business day. This convention is often linked to loans and is a translation of the amount that should be paid on or before a specific date.
3. Modified Following: Here the adjusted date is the following good business day unless that date falls on the next calendar month, in which case it is taken to be the preceding good business day. This is the most used convention for interest rate derivatives.
4. Modified Following Bi-monthly: The adjusted date is the following good business day unless that adjusted date crosses mid-month (15^{th}) or the end of month, in which case the adjusted date is the preceding good business day.
5. End-of-Month: Where the tart of a period is on the final business day of a calendar month, the end date is also on the final business day of the end month (not necessarily the corresponding calendar date in the month).

References

- Open Gamma (2012): Interest Rate Instruments and Market Conventions Guide *Quantitative Research* **Open Gamma**.

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Tel. 646-513-3300

Overnight and IBOR-like Indexes

IBOR Indexes - Introduction

1. The Indexes: IBOR-like indexes are related to interbank lending for maturities ranging from 1D to 1Y. They are usually computed as a trimmed average between rates contributed by the participating banks. The rates are the banks' estimates, but usually do not refer to the actual transactions. The most common usage of these indexes in IRD is in IRS and caps/floors.
2. IBOR Indices for the main Currencies:

Currency	Name	Maturities	Convention	Spot Lag
CHF	LIBOR	ON-12M	Act/360	2
EUR	EURIBOR	1W-12M	Act/360	2
EUR	EURIBOR	1W-12M	Act/365	2
EUR	LIBOR	ON-12M	Act/360	2
EUR	LIBOR	ON-12M	Act/360	0
GBP	LIBOR	ON-12M	Act/365	0
JPY	LIBOR	ON-12M	Act/360	2
JPY	Japan TIBOR	1W-12M	Act/365	2
JPY	EuroYen TIBOR	1W-12M	Act/360	2
USD	LIBOR	ON-12M	Act/360	2

3. IBOR Indices for the Other Currencies:

Currency	Name	Maturities	Convention	Spot Lag
AUD	BBSW	1M-6M	Act/365 F	0
CAD	CDOR	1M-12M	Act/365 F	0
CZK	PRIBOR		Act/360	2

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Tel. 646-513-3300

DKK	CIBOR	1W-12M	Act/360	0
HKD	HIBOR	1M-12M	Act/365 F	0
HUF	BUBOR		Act/360	2
IDR	IDRFIX		Act/360	2
INR	MIFOR		Act/365 F	2
NOK	NIBOR		Act/360	2
NZD	BBR		Act/365	0
PLN	WIBOR		Act/365	2
RMB	SHIBOR	ON-12M	Act/360	0
SEK	STIBOR		Act/360	2
SKK	BRIBOR		Act/360	2
SGD	SIBOR		Act/365 F	2
SGD	SOR		Act/365 F	2
ZAR	JIBAR	1M-12M	Act/365 F	0

4. LIBOR: LIBOR is an acronym for London Interbank Offered Rate. It is calculated by Thomson Reuters on behalf of the British Bankers' Association. Major banks submit their cost of borrowing unsecured funds for several tenors and currencies.
5. LIBOR Administration: Up until 2012 there were 15 tenors in 10 currencies (AUD, CAD, DKK, EUR, JPY, NZD, GBP, SEK, CHF, and USD). Some have been phased out in the first half of 2013 – to 42 rates. Rates are now published for 6 currencies (EUR, EUR same day, JPY, GBP, CHF, USD) and seven tenors (ON/SN, !W, 1M, 2M, 3M, 6M, and 12M). NYSE Euronext will be in charge of administration through its subsidiary NYSE Euronext Rates Administration Limited, starting from 2014.
6. LIBOR Conventions: For all the currencies apart from EUR and GBP, the period between the fixing date and the value date will be 2 London business days after the Fixing date. However, if the day is not a business day at the corresponding financial center, it will be adjusted to the following day at both London and the location – this date will be the new value date. The business day convention is modified following, and the EOM rule applies. For all currencies except GBP, the day count convention is Act/360. References =>

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Tel. 646-513-3300

http://www.bbalibor.com/technical-aspects/fixing-value-and-maturity;
http://www.nyx.com/libor

Main IBOR-Indices

1. GBP LIBOR: The fixing date and the value date are the same (0 day spot lag). The day count convention is Act/365.
2. EUR LIBOR: The value date is 2 TARGET business days after the fixing date.
3. EURIBOR: The day count convention is Act/360 and the spot lag is 2 business days. The business day convention is *modified following* and the EOM rule applies. The related calendar is TARGET. There are 43 contributor banks, and the rates are published at 11:00 AM CET. Reference => http://www.euribor-ebf.eu/euribor-org/about-euribor.html
4. JPY TIBOR: TIBOR is the acronym for Tokyo InterBank Offered Rate. It is published by the Japanese Bankers Association. There are 2 types of TIBOR. The *Japanese Yen TIBOR* rates reflect the prevailing rates on the unsecured call market. The *EuroYen TIBOR* rates are the rates from offshore Japan market. The JBA TIBOR is calculated by the JBA as a prevailing market rate based on quotes for 13 different maturities (1W, 1M-12M) provided by banks as of 11:00 AM each business day. The day count convention is Act/365 for the domestic market and Act/360 for the EuroYen market. Reference => http://www.zenginkyo.or.jp/en/tibor/the-jba-tibor/

Other IBOR-Indices

1. AUD BBSW: The rate is Bank Bill Rates (BBSW) and is published by the Australian Financial Markets Association, and the maturities are 1M-6M. The day count convention is Act/365, and the spot lag is zero days. The business day convention is *modified following bi-monthly*. The rates are published at 10:00 AM. Reference => http://www.afma.com.au/data/bbsw.html

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Tel. 646-513-3300

2. CAD CDOR: CDOR is the acronym for Canadian Dealer Offered Rate. CDOR is determined daily from a survey of 9 market makers' in bank acceptances (BA). The survey is conducted at 10:00 AM each business day, with the results being quoted by 10:15 AM on the same day. The day count convention is Act/365. The fixing date and the value date are the same (0 spot lag). Reference => http://www.m-x.ca/marc_terme_bax_cdor_en.php
3. DKK CIBOR: CIBOR is the acronym for Copenhagen InterBank Offered Rate. It is the reference interest rate for liquidity offered in the inter-bank market (in Denmark) on an uncollateralized basis for maturities from 1W-12M. NASDAQ OMX publishes CIBOR on a daily basis at 11:00 AM. The Danish Bankers' Association has the overall responsibility for CIBOR. The day count convention is Act/360. References => http://www.finansraadet.dk; http://nasdaqomxnordic.com/obligationer/danmark.cibor
4. HKD HIBOR: Hungarian InterBank Offered Rate. Act/365 and *Modified Following*.
5. INR MIFOR: MIFOR is the acronym for Mumbai Interbank Forward Offered Rate. The day count convention is Act/365 and the spot lag is 2 days. It is published for 1M, 2M, 3M, 6M, and 12M tenors.
6. NOK NIBOR: NIBOR is the acronym for Norwegian InterBank Offered Rate. The day count convention is Act/360 and the business day convention is *modified following*.
7. RMB SHIBOR: SHIBOR is the acronym for SHanghai InterBank Offered Rate. The day count convention is Act/360 and the spot lag is zero days. It is published for ON, 1W, 2W, 1M, 2M, 3M, 6M, 9M, and 12M tenors. Reference => http://www.shibor.org/shibor/web/html/index_e.html
8. SEK STIBOR: STIBOR is the acronym for STockholm InterBank Offered Rate. The day count convention is Act/360. The business day convention is *modified following*.
9. SGD SIBOR: SIBOR is the acronym for Singapore InterBank Offered Rate. The day count convention is Act/365. An individual ABS SIBOR contributor bank contributes the rate at which it can borrow funds were it to do so by asking for and accepting inter-bank offers in a reasonable market size, just prior to 11:00 AM. The indexes are computed by the Association of Banks in Singapore. Reference => http://www.abs.org.sg
10. SGD SOR: SOR in the acronym for Swap Offered Rate. It is implied from the USD LIBOR and the forex forwards. The indexes are computed by the Association of Banks in Singapore.

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Tel. 646-513-3300

11. ZAR JIBAR: JIBAR is the acronym for Johannesburg InterBank Agreed Rate. The rate is calculated daily by SAFEX as the average prime lending rate quoted independently by a number of different banks. The rate is available in 1M, 3M, 6M, and 12M tenors.

Overnight Index Definitions

1. Setup: Overnight indexes are related to inter-bank lending on a one-day horizon. Most indexes are for overnight loans, and some are for tomorrow/next loans. The rates are computed as a weighted average of the actual transactions.
2. Main Currencies: To note: The publication lag is the number of days between the period start and the rate publication. A lag of 0 indicates that the rate is published on the start date, 1 indicates that the rate is published on the end date, and -1 indicates that the rate is published one day earlier.

Currency	Index Name	Reference	Convention	Lag
CHF	TOIS	TN	Act/360	-1
EUR	EONIA	ON	Act/360	0
GBP	SONIA	ON	Act/365	0
JPY	TONAR	ON	Act/365	1
USD	Fed Fund	ON	Act/360	1

3. Other Common Currencies:

Currency	Index Name	Reference	Convention	Lag
AUD	RBA ON/AONIA	ON	Act/365	0
CAD	CORRA	ON	Act/365	1
DKK	DNB TN	TN	Act/360	-1
CZK	CZEONIA		Act/360	
HKD	HONIX	ON	Act/365	0

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

HUF	HUFONIA	ON	Act/360	
INR	MIBOR ON	ON	Act/365	0
INR	MITOR	TN	Act/365	0
NZD	NZIONA	ON	Act/365	0
PLN	POLONIA	ON	Act/365	
SEK	TN SIBOR/STIBOR	TN	Act/360	-1
SGD	SONAR	ON	Act/365	0
ZAR	SAFEX ON Deposit Rate	ON	Act/365	
ZAR	SAONIA	ON	Act/365	

Overnight Index Committees and Meeting Dates

1. Meetings: Overnight rates are strongly influenced by central banks monetary policy decisions. The meeting dates of the main central banks can be found on the following sites:
    a. http://www.federalreserve.gov/monetarypolicy/fomccalendars.htm
    b. http://www.ecb.int/events/calendar/mgcgc/html/index.en.html
    c. http://www.bankofengland.co.uk/monetarypolicy/Pages/decisions.aspx
2. TN vs. Reference Lag: TN essentially refers to an overnight lending contract between $T+1B$ and $T+2B$, i.e., the TN contract. Therefore, these published rates correspond to a -1 publication lag; in fact, this is indeed the case for all the TN OIS contracts (CHF, DKK, SEK, etc.)
3. CHF TOIS: The reference rate used is the TOIS rate with TN inter-bank fixing. The index is calculated by Cosmorex AG, a division of Tullet Prebon.
4. EUR EONIA: EONIA is the acronym for Euro OverNight Index Average. It is computed as a weighted average of all the overnight unsecured lending transactions undertaken in the inter-bank market, initiated within the Euro area by the contributing banks (rounded to 3 decimal places). It is calculated by the European Central Bank. The rate is published in the evening

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

(CET 19:00) of the period start date. Day Count convention is Act/360. Reference => http://www.euribor-ebf.eu/euribor-eonia-org/about-eonia.html.

5. EUR EURONIA: This is the weighted average of all the unsecured Euro overnight cash transactions brokered in London by WMBA member firms between midnight and 16:15 CET with all the counterparts with a minimum deal size. Reference => http://www.wmba.org.uk/pages/index.cfm?page_id=32.
6. GBP SONIA: SONIA is an acronym for Sterling OverNight Index Average. It is the weighted average of all unsecured overnight sterling cash transactions brokered in London by WMBA member firms between midnight and 16:15 CET with all the counterparts with a minimum deal size of GBP 25 million (rounded to 4 decimal places). The rate is published in the evening (around 17:00 CET) of the period start date. The day count convention is Act/365. Reference => http://www.bba.org.uk/policy/article/sterling-overnight-index-average-sonia-a-guide/benchmarks
7. JPY TONAR Uncollateralized Overnight Call Rate: TONAR is an acronym for Tokyo OverNight Average Rate. It is the weighted average of all unsecured overnight cash transactions between financial institutions. The rate is published by Bank of Japan (BOJ), and the day count is Act/365. A provisional result is published on the evening (17:15 JST, except on the last business day of the month, when it is at 18:15 JST) of the period start. The final result is published on the morning (10:00 JST) of the end date. Reference => http://www.boj.or.jp/en/statistics/market/short/mutan
8. USD Effective Federal Funds Rate: The daily effective federal funds rate is a volume weighted average of the rates on trades arranged by the major brokers. The effective rate is calculated by the Federal Reserve Bank of New York using the data provided by the brokers and is subject to revision. The rate is published in the morning (between 7:00 AM and 8:30 AM EST) of the period end date. The day count is Act/365. Reference => http://www.newyorkfed.org/markets/omo/dmm/fedfundsdata.cfm
9. AUD RBA Interbank Overnight Cash Rate Survey AONIA: The rate is computed by the Reserve Bank of Australia (RBA). It is the weighted average of the rates at which a sample of banks transact in the domestic interbank market for overnight funds. The interbank overnight cash rate calculated from the survey is published on electronic media services at the conclusion of each trading day. The rate is published on the evening of the period start

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

date, and the day-count is Act/365. Reference => http://www.rba.gov.au/mkt-operations/tech-notes/interbank-survey.html

10. CAD CORRA: CORRA is the average for the Canadian Overnight Repo Rate Average. It is the weighted average of overnight general (non-specific) collateral repo trades that occur through designated inter-dealer brokers between 6:00 and 16:00 EDT on the specified date as reported by the Bank of Canada. The rate is published on the morning (9:00) of the end date, and the day count convention is Act/365. Reference => http://www.bankofcanada.ca/rates/interest_rates/money-market-yields
11. DKK Danmarks Nationalbank Tomorrow/Next: The TN money market rate is calculated and published by the Danmarks Nationalbank. The TN interest rate is an uncollateralized day-to-day interest rate for money market lending. Calculation of the TN interest rate is based on the daily reports from 11 banks. Each bank reports the day-to-day uncollateralized inter-bank lending and the average rates on these loans. The report is made with a time lag of one, e.g., Monday's lending is reported on Tuesday. The day-count convention is Act/360. Reference => http://www.nationalbanken.dk/dnuk/rates.nsf/side/reference_rates!opendocument
12. NZD NZIONA: The rate used is a reference rate equal to the official cash rate for that date set by RBNZ. It is publishes as of 10:00 AM Wellington time, and the day count is Act/365.
13. SEK SIOR/TN STIBOR: STIBOR (Stockholm Interbank Offered Rate) is a reference rate that shows an average of interest rates at which a number of banks active in the Swedish money market are willing to lend to one another without collateral at different maturities. The reference rate is SIOR or TN STIBOR. The rate is published by the OMX exchange. SIOR is a reference rate equal to the daily fixing for the Swedish krona tomorrow next deposits as published at 11:00 AM Stockholm time, on the day that is one Stockholm banking day prior to the start of the payment period. Reference => http://www.swedishbankers.se
14. SGD SONAR: The SONAR rate is published by the association of banks in Singapore. The rate is published at 11:00 AM Singapore time on the period start date, and the day count convention is Act/365.
15. ZAR - SFX ZAR OND: The rate SFX ZAR OND rate is published by SAFEX JIBAR. SAFEX publishes the rate which is the average rate that is receives on its deposits with the banks, weighted by the size of the investments placed at each bank.

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16. ZAR - SAONIA: The SAONIA rate is the weighted average rate paid on unsecured, interbank, overnight funding. For more details see *South African Financial Markets (G West, Financial Modeling Agency, 2009)*.

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Tel. 646-513-3300

Over-The-Counter Instruments

Forward Rate Agreement

1. Description: FRA's are OTC contracts linked to an IBOR-like index. At the trade date, a *Reference Rate* (R), a start period, and a *Reference Index* are all agreed to. The end period is equal to the start period plus the index tenor (i.e., a 6M start period and a 3M tenor result in a nominal 9M end period).
2. FRA Reference Period: The start of the reference period is computed from the spot date by adding the index spot lag, and then the spot tenor (using the business day convention and the calendar of the index). The reference period's end date is computed by adding the index spot lag and then the end period. The *Fixing Date* (or the exercise date) is the spot lag before the start date.
3. FRA Accrual DCF: The accrual factor between the start date and the end date (in the index day count) is denoted by δ. Occasionally the dates (and sometimes the accrual) described above are not calculated, but simply agreed upon arbitrarily by the counter-parties (usually changing the dates by 1-2 days for convenience and/or operational reasons).
4. FRA Settlement: The *FRA Settlement Date* is the start date (and NOT the end date). For the FRA buyer, the settlement day pay amount is

$$\delta \frac{L_\theta - R}{1 + \delta L_\theta}$$

   where L_θ is the value of the reference index on the fixing date, and R is the strike. The payoff for the FRA seller is obviously the same amount with the opposite sign.
5. FRA Accounting Treatment: In some accounting schemes, since the payment is always regarded as accruing between the start and the end dates, the FRA instrument is regarded as "being alive" even if it has already fully settled.
6. FRA Period Mismatch: The FRA's end date can be slightly different from the end date of the theoretical deposit underlying the IBOR rate. This potential mismatch comes from a

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Tel. 646-513-3300

mismatch comes from a difference in adjustment of the non-good business days between the different ways to compute the period.

7. IMM FRA's: FRAs can also be traded as IMM FRAs, i.e., FRAs with accrual dates equal to consecutive IMM dates (just as in STIR futures). The underlying IBOR rate has a tenor the one relevant to the IMM dates' frequency (3M IBOR for the quarterly dates and 1M IBOR for monthly dates).

Interest Rate Swaps

1. Interest Rate Swaps (Fixed for IBOR): Exchange of fixed payments for a floating stream of payments linked to an IBOR-like index. Typical payment/accrual periods and their generation rules are outlined below, and, since these are OTC, any variant is possible if agreed by the counterparts.
2. Forward Date Generation: The dates are computed from the start (or the settlement) date. The last date of the stream will be the start date plus the total length (maturity tenor) of the leg. The intermediate dates are regularly spaced, and the first one is the non-standard period. For example, a 15-month leg with a 6-month period can pay after 3 ($= 15 - 2 \times 6$), 9, and 15 months.
3. IRS - Date Adjustment: Dates adjustment is done by the business day convention and the EOM rule. All the dates are first computed without adjustment, and then all the dates are adjusted.
4. IRS Stubs: The non-standard period is referred to as the stub. It can be short (shorter than one period) or long (between one and two periods in length). The reason that the non-standard period is the first one is that once that period is finished, the instrument then has regular periods (similar to a standard one). If the stub was the last period, the swap could never become a standard one.
5. Start Date: The start (or the settlement) date of the swap is usually a certain lag called the spot lag after the trade date. The most common lag is 2 business days. The start date can also be a forward date. In that case, the start date is the forward period tenor plus the spot lag. The forward period tenor is specified W/M/Y.

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

6. Payer (Buyer) and Receiver (Seller) Swaps: The terms *Payer* and *Receiver* refer to the fixed leg of an IRS. The swap is a payer for one party if the swap pays the fixed leg and receives the floating leg. The payer swap for one party is the receiver swap for the other. Like FRA, the swap buyer buys the floating leg by paying the fixed.

Vanilla IRS

1. Notional and Coupon: In a vanilla IRS, all the coupons have the same notional, and the coupons on the fixed leg have the same rate.
2. Payments: The payments on the fixed leg are regularly placed, most with a 6M/12M gap. The payments on the floating leg are also regularly spaced, most with the 3M/6M gap. The fixing date for the floating payment is the index spot lag before the period start date. This lag is usually part of the index, and usually the same as the swap start spot lag.
3. Floater/index mismatch: The dates of the period corresponding to the deposit underlying the IBOR index can be slightly different from that of the coupon period. The difference is created by the adjustments due to non-good business days.
4. Main Currencies:

Currency	**Spot**	**Fixed**		**Floating**		
	Lag	**Period**	**Convention**	**Index**	**Period**	**Convention**
USD (NY)	2	6M	30/360	LIBOR	3M	Act/360
USD (London)	2	1Y	Act/360	LIBOR	3M	Act/360
EUR: 1Y	2	1Y	30/360	EURIBOR	3M	Act/360
EUR: > 1Y	2	1Y	30/360	EURIBOR	6M	Act/360
GBP: 1Y	0	1Y	Act/365	LIBOR	3M	Act/365
GBP: > 1Y	0	6M	Act/365	LIBOR	6M	Act/365
JPY	2	6M	Act/365	TIBOR	3M	Act/365
JPY	2	6M	Act/365	LIBOR	6M	Act/360
CHF: 1Y	2	1Y	30/360	LIBOR	3M	Act/360
CHF: > 1Y	2	1Y	30/360	LIBOR	6M	Act/360

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Tel. 646-513-3300

5. Other Currencies:

Currency	Spot	Fixed		Floating		
	Lag	Period	Convention	Index	Period	Convention
AUD: 1Y – 3Y	1	3M	Act/365	BBSW	3M	Act/365
AUD: >= 4Y	1	6M	Act/365	BBSW	6M	Act/365
DKK	2	1Y	30/360	CIBOR	6M	Act/360
INR: <= 1Y	2	1Y	Act/365	MIFOR	3M	Act/365
INR: > 1Y	2	6M	Act/365	MIFOR	6M	Act/365
HKD	0	3M	Act/365	HIBOR	3M	Act/365
NOK	2	1Y	30/360	NIBOR	6M	Act/360
HZD	0	6M	Act/365	BKBM	3M	Act/365
PLN	2	1Y	Act/Act ISDA	WIBOR	6M	Act/365
SEK	2	1Y	30/360	STIBOR	6M	Act/360
SGD	2	6M	Act/365	SIBOR	6M	Act/365
SGD	2	6M	Act/365	SOR	6M	Act/365
ZAR	0	3M	Act/365	JIBAR	3M	Act/365

6. Composition - Multi-reset Swaps: In some cases, the period between the payments is equal to the IBOR index, but a multiple thereof. The fixing rates are compounded over the sub-periods up to the payment at the end. The main currency for which this is a standard for vanilla swaps is CAD.
7. Reset Periods - Compounding: The description of the compounded coupon is as follows: The associated times are denoted $\{t_i\}_{i=0}^{n}$. The fixing for the period $[t_{i-1}, t_i]$ is denoted by $\{r_i\}_{i=1}^{n}$, and the corresponding accrual fraction in the index convention is δ_i. The fixing takes place at a date typically prior to the start of the accrual period, with the difference between the fixing date and the start date being referred to as the spot lag.
    a. Geometric Compounding => The coupon pays at t_n the amount (to be multiplied by the notional)

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

$$\prod_{i=1}^{n}(1+\delta_i r_i)-1$$

b. Compounded Spread => In case a spread is agreed to on the compounding leg, there are 3 standard ways to deal with the compounded spread: *COMPOUNDING*, *FLAT COMPOUNDING*, and *COMPOUNDING WITH SPREAD AS A SIMPLE INTEREST*. These methods are described in the ISDA document *Alternate compounding methods for the OTC derivative transaction (2009)*.

c. CAD Swap => Multi-reset Composition:

Currency	**Spot**	**Fixed**		**Floating**		
	Lag	**Period**	**Convention**	**Index**	**Period**	**Convention**
CAD: 1Y	0	1Y	Act/365	CDOR	1Y	Act/365
CAD: >= 1Y	0	6M	Act/365	CDOR	6M	Act/365
CNY	2	3M	Act/365	CNY-Repo	3M	Act/365

8. IMM Dated Swap: Like for FRA, there exists the IMM dates IRS. These swaps pay the fixed and the floating legs at the IMM dates. The most common are the quarterly IMM dates on the floating legs based on the IBOR 3M rates. It is also common that the fixed leg payment is every 2^{nd} (semi-annual) to 4^{th} (quarterly).
9. In-arrears Swap: Another type of IBOR swaps is a swap with fixing in-arrears. Here the reference period for the IBOR index and the accrual period for the coupon are disjoint, i.e., the accrual period precedes the IBOR period. Thus, the start date of the IBOR period is the payment date. The fixing date for the floating payment is the index spot lag before the accrual period end date.
10. Short and Long Tenors: For some swaps, the period between the payments is not equal to the index tenor. The payment period can be shorter than the index period (the short tenor swap), or longer (long tenor swap). Typically this type of swap has a 3M pay accrual period on 6M-12M IBOR index (short) or an annual pay accrual period on a 3M-6M IBOR index (long). The long/short tenor swap can also be of the (fixing) in-advance or the in-arrears type.

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

11. Step-up and step-down: The rate paid on the fixed leg coupons does not need to be the same for each coupon. The swap is then referred to as a step-up when the coupons increase and step-down when they decrease.
12. Amortized, Accruing, and Roller Coaster Swaps: The coupon notional does not need to be the same for all the coupons. In most cases, the coupons are the same for both the legs of the period. If the notional is decreasing over time, the swap is called *Amortizing Swap*, and if is increases it is called *Accruing*. If the notional first increases and then decreases up to maturity, it is called a *Roller Coaster*.

Interest Rate Swaps (Basis Swaps: IBOR for IBOR)

1. Description: In a basis swap, both legs are floating legs and depend on an IBOR index in the same currency. In most cases, the indexes have different tenors. A spread above the IBOR index is paid in one of the legs. The quote convention used quotes the spread the spread over IBOR on the shorter tenor leg, in such a way that the spread is positive.
2. EUR Basis Swap: In EUR alone, the basis swap is quoted as 2 different swaps. For e.g., the quote of EURIBOR 3M vs. EURIBOR 6M at 12 bp has the following meaning: In the first swap you receive a fixed rate and pay 3M EURIBOR. In the second swap you pay the same fixed rate plus a spread of 12 bp and receive 6m EURIBOR. Note that with this convention the spread is paid on an annual basis, like the standard fixed leg of a fixed-float IBOR swap. Even if the quote refers to the spread of a 3M vs. 6M swap, the actual spread is paid annually with a fixed leg convention.
3. Compounded Basis Swap: The multi-reset composition functionality described above is not restricted to fix-float IBOR swaps alone. Some basis swaps are also traded on a compounded basis to align the payments on both the legs. For example, a basis 1M LIBOR vs. 3M LIBOR swap can be quoted with the 1M LIBOR compounded over 3 periods and paid quarterly in line with the 3M period. Note that the exact convention for the spread convention needs to be indicated for the trade. The multi-reset composition of the shorter leg is currently the standard in USD.

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

Cross-Currency Swap, IBOR for IBOR

1. Description: Here the notional is not the same in both legs as they are in different currencies. The notional in one leg is usually the notional on the other translated to the other currency through an exchange rate. The rate is often an exchange rate at the moment of trade as agreed between the parties. The notional is paid out on both the legs, at the start and at the end of the swap.
2. Non-MTM FX Cross-Currency Swap: In some cases, the FX rates used are not in line with the market rates. Often this can be abused to disguise some debts from the accounting rules. These types of cross-currency swaps at non-market exchange rates were famously used by Greece to hide some of its debts when it entered the Euro. The swaps used for construction are at-the-money exchange rates (although initially set, therefore non-MTM).
3. MTM Cross-Currency Swap: There also exist cross-currency swaps with the FX rate reset, and this is called the *Cross-Currency Mark-to-Market Swap*. They are specified in article 10 of the 2006 ISDA definitions. For each period, the FX rate that is observed at the beginning of the period is used for the following period. The notional of one of the legs is unchanged, and the other is adapted according to the new exchange rate. At each payment date and MTM amount is paid. The amount is calculated as the new notional in the adapted leg minus the previous notional. This is equivalent, up to netting, to exchanging notionals at the start and at the end of each period. This feature was introduced to reduce the credit risk induced by the movement of the FX rates.
4. Cross-Currency Swap Stream Construction: Bothe legs of a cross-currency swap are linked to an IBOR like index. In the standard cross-currency swaps, the IBOR tenor on both the legs is the same. The payments are done on the same day for both the legs to reduce the credit risk. This means that the payment calendar is the joint calendar of both the swaps involved in the swap.
5. Typical Cross Currency Swaps: The most liquid cross-currency swaps exchange 3 month payments. Even if the index of one of the currency is 6M as its most commonly used one, the cross-currency swaps may still use 3M payments. This is in particular the case with

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Tel. 646-513-3300

USD/JPY and USD/EUR swaps, and these use 3M payments, even if 6M EUR EURIBOR and 6M JPY LIBOR are the standard floating references for these currencies.

6. Spreaded Cross-Currency Swaps: The cross-currency swaps also pay a spread on one of the legs. On which currency leg the spread is paid out depends on the currency pairs. When one of the currencies is USD, the convention is usually USD LIBOR flat vs. the other currency plus a spread. The 2 exceptions to this rule are – USD/MXN Peso Swaps and USD/CLP Chilean Peso Swaps.

Constant Maturity Swaps

1. Description: Constant Maturity Swaps are in some sense similar to the standard IRS, in that the swap is composed of 2 legs, and each leg has its own payment type. One leg is typically a fixed leg or an IBOR leg. The other is a floating leg, the rate of which is based on a swap index.
2. Key Features: The difference to that of a standard IBOR leg is that rate of the index period can be very different from the period on which it is paid. The CMS floating leg usually pays on a quarterly or on a semi-annual basis a swap rate. The most popular swap indexes are indices based on the 2Y, the 5Y, the 10Y, the 20Y, and the 30Y swaps.
3. CMS Fixing: The details of the fixing and the payment are similar to that of the IBOR coupons. The coupon fixing can be in-advance or in-arrears. For the fixing in advance, the fixing takes place at the start of the accrual period. For the fixing in-arrears, the fixing takes place at the end of the accrual period. The lag between the reference rate and the fixing is that spot lag of the swap index.
4. EUR CMS: In EUR, the most common CMS have quarterly payments on both legs. The non-CMS leg is 3M EURIBOR.
5. Swap Rate Fixings for the Main Currencies:

Currency	**Spot**	**Fixed**		**Floating**			
	Lag	**Period**	**Convention**	**Index**	**Period**	**Convention**	**Fixing Time**
EUR: 1Y	2	1Y	30/360	EURIBOR	3M	Act/360	11:00 CET

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

EUR: > 1Y	2	1Y	30/360	EURIBOR	6M	Act/360	11:00 CET
EUR: 1Y	2	1Y	30/360	EURIBOR	3M	Act/360	12:00 CET
EUR: > 1Y	2	1Y	30/360	EURIBOR	6M	Act/360	12:00 CET
EUR: 1Y	2	1Y	30/360	LIBOR	3M	Act/360	10:00 GMT
EUR: > 1Y	2	1Y	30/360	LIBOR	6M	Act/360	10:00 GMT
EUR: 1Y	2	1Y	30/360	LIBOR	3M	Act/360	11:00 GMT
EUR: > 1Y	2	1Y	30/360	LIBOR	6M	Act/360	11:00 GMT
USD	2	6M	30/360	LIBOR	3M	Act/360	11:00 EST
USD	2	6M	30/360	LIBOR	3M	Act/360	15:00 EST
GBP: 1Y	2	1Y	Act/365	EURIBOR	3M	Act/365	12:00 CET
GBP: > 1Y	2	6M	Act/365	EURIBOR	6M	Act/365	12:00 CET
CHF: 1Y	2	1Y	30/360	LIBOR	3M	Act/360	10:00 GMT
CHF: > 1Y	2	1Y	30/360	LIBOR	6M	Act/360	10:00 GMT

Swap Indexes

1. Introduction: The most common usage of these indexes is in CMS and CMS caps/floors. Swap rates for CHF, EUR, GBP, JPY, and USD are established by ISDA in co-operation with Reuters (now Thomson Reuters) and InterCapital Brokers (ICAP). The main fixing details are shown in the table above.
2. ISDA EUR: There are 4 daily fixings – 2 for swaps vs. LIBOR, and 2 for EURIBOR. The LIBOR fixings are at 10:00 GMT and 11:00 GMT. The EURIBOR fixings are at 11:00 CET and 12:00 CET. The maturities are 1Y-10Y, 12Y, 15Y, 20Y, 25Y, and 30Y. All are 6M tenor swaps, except for the 1Y maturity which is 3M.
3. ISDA USD: There are 2 fixings – at 11:00 EST and 15:00 EST. The maturities are 1Y-10Y, 15Y, 20Y, and 30Y. All swaps are vs. 3M LIBOR.
4. ISDA GBP: There is one fixing, at 11:00 GMT. The maturities are 1Y-10Y, 12Y, 15Y, 20Y, 25Y, and 30Y. All the swap fixings are quoted for 3M , except for the 1Y which is for 6M.
5. ISDA CHF: There is one fixing at 11:00 GMT. The maturities are 1Y-10Y. All swaps are vs. 6M except for the 1Y which is vs. 3M.

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

6. ISDA JPY: There are 2 fixings – at 10:00 and at 15:00 Tokyo time. The maturities are 1Y-10Y, 12Y, 15Y, 20Y, 25Y, 30Y, 35Y, and 40Y. All the swaps are vs. 6M. Note that for the JPY there is also an 18M fixing.

Overnight Indexed Swaps

1. Description: The overnight indexed swaps (OIS) exchange a leg of fixed payments for a leg of floating payments. The start (or the settlement) date of the swap is a certain lag (the spot lag) after the trade date. The most common lag is 2 business days.
2. Payments: The payments on the fixed leg are regularly spaced by the given period. Most of the OIS have a single payment if the maturity is shorter than 1Y, and a 12M period for longer swaps. The payments on the floating legs are also regularly spaced, usually on the same dates as the fixed leg. The amount paid on the floating leg is computed by compounding the rates.
3. Conventions for the Main Currencies:

Currency	**Spot**	**Fixed**		**Floating**		
	Lag	**Period**	**Convention**	**Index**	**Pay Lag**	**Convention**
USD <= 1Y	2	tenor	Act/360	Fed Fund	2	Act/360
USD > 1Y	2	1Y	Act/360	Fed Fund	2	Act/360
EUR: <= 1Y	2	tenor	Act/360	EONIA	2	Act/360
EUR: > 1Y	2	1Y	Act/360	EONIA	2	Act/360
GBP: <= 1Y	0	tenor	Act/365	SONIA	1	Act/365
GBP: > 1Y	0	1Y	Act/365	SONIA	1	Act/365
JPY: <= 1Y	2	tenor	Act/365	TONAR		Act/365
JPY: > 1Y	2	1Y	Act/365	TONAR		Act/365

4. Conventions for the Other Currencies: Pay Lag => The lag in days between the last fixing publication and the payment.

Currency	**Fixed**	**Floating**

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

	Spot Lag	Period	Convention	Index	Pay Lag	Convention
AUD <= 1Y	1	tenor	Act/365	RBA ON	1	Act/365
AUD > 1Y	1	1Y	Act/365	RBA ON	1	Act/365
CAD: <= 1Y	0	tenor	Act/365	CORRA	0	Act/365
CAD: > 1Y	0	1Y	Act/365	CORRA	0	Act/365
INR: <= 1Y	1	tenor	Act/365	ON MIBOR	1	Act/365
INR: > 1Y	1	6M	Act/365	ON MIBOR	1	Act/365
SGD: <= 1Y	2	tenor	Act/365	SONAR	1	Act/365

5. Payment Calculation: Let

$$0 < t_0 < t_1 < \cdots < t_n < t_{n+1}$$

be the relevant dates (all good business days) in the composite floating leg period. Let δ_i be the accrual factor between t_i and t_{i+1} $(1 \leq i \leq n)$ and δ the accrual factor for the period $[t_i, t_{n+1}]$. The overnight rates paid between t_i and t_{i+1} are given at t_i by ${F_i}^{ON}$. The paid amount is

$$\prod_{i=1}^{n} \left(1 + \delta_i {F_i}^{ON}\right) - 1$$

multiplied by the notional. The payment is usually not done at the end of the period t_{n+1}, but at a certain lag after the last fixing publication date. The reason for the lag is that the actual amount is known only at the very end of the period; the payment lag allows for a smooth settlement.

6. USD OIS: In USD the payment is 2D after the end of the fixing period. These two days are computed from the final publication date, which is at the end of the last period, plus 2 lag days.

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

7. EUR OIS: In EUR, the payment is 1D after the end of the fixing period. This one day is computed as the final publication date, which is the start of the previous period and 1D before the end of the previous period, plus 2 lag days.
8. OIS Committee Meetings: A somehow popular choice of start or end dates for the OIS Swaps are the dates of the relevant committee meetings, as shown earlier in the table.
9. Federal Funds Swaps: Federal Fund Swaps are a USD peculiarity. These are swaps exchanging quarterly USD LIBOR payments for the quarterly average of the effective USD Federal Funds Rate. They are often called as the Feds or the FED Swaps.
    a. Arithmetic Compounding => The particularity is that the rates paid is an arithmetic average of the Fed Funds Rates; the rates are not compounded as in the traditional OIS. The quarterly coupon payment is not equal to a 3M OIS.
    b. Fed Fund Coupon Calculation => Let

$$0 < t_0 < t_1 < \cdots < t_n < t_{n+1}$$

be the relevant dates (all good business days) in the composite floating leg period. Let δ_i be the accrual factor between t_i and t_{i+1} $(1 \leq i \leq n)$ and δ the accrual factor for the period $[t_i, t_{n+1}]$. The overnight rates paid between t_i and t_{i+1} are given at t_i by ${F_i}^{ON}$. The paid amount is

$$\sum_{i=1}^{n} \delta_i {F_i}^{ON}$$

multiplied by the notional.

    c. Fed Funds Final Fixing => The final Fed Funds effective fixing is applied to the last 2 fixing days. In terms of the above formula, it needs to be re-cast as

$$\sum_{i=1}^{n-1} \delta_i {F_i}^{ON} + \delta_n {F_{n-1}}^{ON}$$

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

It is possible to trade absent the rate cut-off, but this requires the counterparty to make payment on the same day the last fixing information is published.

d. Fed Funds vs. LIBOR Swaps => Here, the swaps are quoted with a spread over the Fed Funds ON leg. A quote of s often in bp means that the swap exchanges LIBOR for ON average plus a spread of s. The spread is usually positive. The computation of the interest on the ON floating leg is additive with simple compounding, and the spread is also additive with simple compounding. The multiple compounding alternatives as in IBOR compounding are not present here. The payment is (excluding the final day repeated fixing)

$$\sum_{i=1}^{n-1} \delta_i {F_i}^{ON} + \delta s$$

In some cases, the fed fund swaps are traded against 1M LIBOR. This type of swaps is less liquid.

10. OIS Indexes: The OIS Index Rates are reference rates to the standard OIS. As an example, the EONIA index is the average rate of rates provided by the prime banks to 3 decimal places that each panel bank believes is the mid-market rate of EONIA swap quotations between prime banks. It is quoted for spot value ($T + 2B$) and on an Act/360 basis with annual payments. The fixing time is 11:00 CET. The index covers swaps from 1W to 24M. The indexes are computed by the EURIBOR-EBF association, and were launched in 2005. Reference => http://www.euribor-ebf.eu/eoniaswap-org/about-eoniaswap.html

Swaption

1. Description: A swaption is an option on a swap. It is characterized by an exercise date and an underlying swap. The exercise date is on or before the swap start date. The option gives its holder the right (but not the obligation) to enter in the underlying swap. In theory an option can be written on any underlying swap. In practice, a large majority of swaptions are written on the vanilla interest rate swap.

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2. Strike: The strike of an option is the common fixed rate across all the fixed leg coupons. If the underlying swap has a different rate for each coupon (in a step-up or step-down swap, for example), the strike is ill-defined (at least as in the "common strike").
3. Payer/Receiver Swaptions: The term payer/receiver swaption refers to the payer/receiver of the underlying swap. A swaption is a payer/receiver swaption if the party long the option has the right to enter into a payer/receiver swap. Note that the payer/receiver indicator refers to the long party. Thus, if one is short a receiver option and the swaption is exercised, he enters into a payer swap (a receiver swap for the other party which is long the option). A payer swaption for one party is the receiver swaption for the other.
4. Swaption Dates: A swaption exercise date and its underlying swap start date are computed as follows for the standard swaptions. The swaption is described by an exercise tenor and an underlying swap tenor (like, as in, 6M by 10Y). The exercise date is computed as the spot date plus an exercise tenor, using the relevant calendar and the business day convention of the underlying swap. The swap settlement date is computed as the exercise date plus the underlying swap (or the underlying swap index) spot lag.
5. Swaption Settlement Conventions:

Currency	**Method**	**Sub-method**	**Expiry**
EUR	Cash Settled	IRR	11:00 CET
GBP	Cash Settled	IRR	11:00 GMT
CHF	Cash Settled	IRR	11:00 CET
DKK	Cash Settled	IRR	11:00 CET
NOK	Cash Settled	IRR	11:00 CET
SEK	Cash Settled	IRR	11:00 CET
USD	Cash Settled	Exact Curve	11:00 EST
JPY	Physical Delivery		17:00 Tokyo
AUD	Physical Delivery		

6. Physical Delivery Swaptions: When the swaption is settled with physical delivery, at the exercise date the parties enter into an actual swap (the underlying swap).

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Tel. 646-513-3300

7. Cash Settled Swaptions: When the swaption is cash-settled, a cash amount is paid (by the short party to the long party) at the exercise date (or, more precisely, spot lag after the exercise), and the actual swap is not entered into.
8. Yield Settled Swaptions: The cash amount to be paid to the long party is computed from a swap fixing rate using a conventional valuation formula of the theoretical underlying swap. The valuation is done using the swap fixing rate as the IRR for the swap. The cash-settled swaption can only be written on a vanilla swap with the standard convention. This is the standard convention for EUR and GBP.
    a. Yield Settlement – Computation => For a swaption with strike K and maturity M, the amount paid for a fixing s is $G(s)$ where m is the cash-annuity

$$G(s) = \sum_{i=1}^{mM} \frac{1/m}{\left(1 + \frac{1}{m}s\right)^i}$$

    where m is the number of payments per year.
9. Cash Settle Swaption – Exact Curve: The term cash-settled can also refer to another way to compute the cash amount. This approach is usually used for the USD cash-settled swaptions. The cash amount to be exchanged is explicitly calculated as the value of the underlying swap. To value the swap, a full yield curve (not just the fixing swap rate) has to be agreed to by the parties.
10. Upfront and Forward Premium: The standard for the options has been for a long time *Spot* payment. The premium relative to the option paid by the buyer to the seller was done at the spot date from the trade date. With the crisis that started in 2007, the credit risk awareness increased, and most of the dealers decided to change the standard to a forward premium. Since September 2010, in the main currencies, the premium is paid at the same date the swaption itself is settled. This is in general from the spot date from the exercise date.

Forex and Forward Swaps

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Tel. 646-513-3300

1. Description: A forex swap is essentially a contract on the interest-rate differences, and therefore similar to a cross-currency swap. The conventions on these transactions are similar to the conventions on interest-rate swaps.
2. Currency Pair Order: FX rates are usually quoted for the currency pairs in the conventional order. For the main currencies, the orders are: EUR/USD, GBP/USD, JPY/USD, and GBP/EUR. The first currency in these pairs is called the *Base Currency*, and the second one is called the *Quote Currency*.
3. Conventional Currency Strength Table:
   a. Strength #1 => EUR
   b. Strength #2 => GBP
   c. Strength #3 => AUD
   d. Strength #4 => NZD
   e. Strength #5 => USD
   f. Strength #6 => CAD
   g. Strength #7 => CHF
   h. Strength #8 => JPY
   i. Strength #9 => Other
4. FX Forward: An FX/Forex forward is simply another FX transaction taking place at a forward date. The payments are in one amount in one currency vs. another amount in another currency. The amount in the other currency is the base currency multiplied by the exchange rate agreed. The rate is often quoted in 2 parts – the spot rate and the *Forward Rate*.
5. FX Swap: The FX/Forex Swap is the exchange of an FX spot and an FX forward. An FX spot rate and an FX forward rate are first agreed to. The signs of the spot and the forward amount in the same currency are opposite. For e.g., in the following EUR/USD trade, the jargon used for FX swap trade would be something like: *I buy spot and sell forward 3M EUR vs. USD for 10m with 10 (forward) points and a spot of 1.25*. This means that on the spot date I receive € 10m and pay $ 12.5m, and at $T + 3M$ I pay € 10m and receive $(1.25 + 0.0010) \times \$\ 10m = \$\ 12.51m$. The Spot part is called the *Near Leg* and the Forward part is called the *Far Leg*.
6. FX Building Blocks: An FX spot is a pure currency trade. The FX swap is mainly an interest rate trade, it is a trade on the differences between interest rates in the 2 currencies. As the

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amounts in each currency are paid and received, there is almost no currency exposure, similar to the cross-currency swap with initial and final exchange of notional. In the Forex market, the trader sees an FX forward as a net between the FX spot and the FX swap, as opposed visualizing the FX Swap as a combination of the FX spot and the FX forward. The FX forward mixes up the currency exposure and the rate exposure, and is therefore not looked at as a building block.

7. Forward Point Quotation Factors: For most of the currencies, the forward point code is the same as the currency code. This is not the case, however, for non-deliverable currencies. The Bloomberg code is built from the prefix in the table below, the maturity (1W, 1M, 2M etc.), and the postfix _CRNCY.

Base Currency	**Other Currency**	**Factor**	**BBG Code Prefix**
AUD	EUR	10,000	AUDEUR
AUD	EUR	10,000	AUD
EUR	GBP	10,000	EURGBP
EUR	JPY	100	EURJPY
EUR	USD	10,000	EUR
GBP	JPY	100	GBPJPY
GBP	USD	10,000	GBP
USD	BRL	10,000	BCN
USD	CAD	10,000	CAD
USD	CHF	10,000	CHF
USD	CNY	1	CCN
USD	EGP	10,000	EPN
USD	HUF	100	HUF
USD	INR	100	INR
USD	JPY	100	JPY
USD	KRW	1	KRW
USD	MXN	10,000	MXN
USD	PLN	100	PLN
USD	TRY	100	TRY

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Tel. 646-513-3300

USD	TWD	1	TWD
USD	ZAR	10,000	ZAR

8. Forward Points: The *Forward Points* are quoted for the currency pairs in the conventional order. The mechanism of forward points is the same as that for FX forward and FX swap. The points are added to the FX spot rate to obtain the FX forward rate. For a spot rate s and points p, the forward rate is $s + p$. The points are usually quoted with a conventional factor (just as the interest rates, which are quoted in percentage). The factor is dependent upon the currency pair, as can be seen in the table above.

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

Exchange Traded Instruments

Introduction

1. <u>Exchange Month Codes</u>: The exchange traded instruments with a regular schedule (like futures) use the following code to refer to the corresponding months:
    a. January – F
    b. February – G
    c. March – H
    d. April – J
    e. May – K
    f. June – M
    g. July – N
    h. August – Q
    i. September – U
    j. October – V
    k. November – X
    l. December – Z
2. <u>Contract Event Dates</u>:
    a. Expiry Date.
    b. Last Trading Date.
    c. Final Settlement Date.

In addition, for contracts with *Physical Deliverables*, the following additional dates apply:

d. First Delivery Date.
e. Final Delivery Date.
f. Delivery Notice Date.

Typically all the dates are computed from a single date pivot (often the expiry date pivot).

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Overnight Futures

1. Overnight Index Linked Futures: The overnight index futures are linked to an average of overnight rates over a certain period (usually a calendar month). Here we look at in detail at the Fed Fund Futures, 1M EONIA Indexed Futures, and 1D Interbank Deposit Futures Contract – Brazil. All these are exchange traded.
2. Federal Funds Futures: The 30D Federal Funds Futures (called Fed Funds Futures) are based on the monthly average of the overnight Fed Funds rate for the contract month. The notional is $5m. The contract months are the first 36 calendar months. They are quoted on CBOT for USD.
    a. Pricing => Let

$$0 < t_0 < t_1 < \cdots < t_n < t_{n+1}$$

be the relevant dates for the Fed Funds Futures, with t_1 being the first business day of the month, t_{i+1} the business day following t_i, and t_{n+1} the first business day of the following month. Let δ_i be the accrual factor between t_i and t_{i+1} $(1 \leq i \leq n)$ and δ the day count fraction for the full period $[t_1, t_{n+1}]$. The day count convention for USD ON rate is Act/360. If the overnight rate between t_i and t_{i+1} is given at t_i as ${F_i}^{ON}$, the futures price on the final settlement date t_{n+1} is

$$\Phi_{t_{n+1}} = 1 - \frac{1}{\delta}\left[\sum_{i=1}^{n} \delta_i {F_i}^{ON}\right]$$

The margining is done on the price multiplied by the notional, and described by the monthly accrual fraction (i.e. $\frac{1}{12}$).

3. 1M EONIA Indexed Futures: The contract was introduced in 2008, and is traded on LIFFE. The notional is €3m and the underlying overnight rate is EONIA. The delivery month covers a single ECB Reserve Maintenance Period. The number of available delivery months will be

limited to the number of Reserve Maintenance Periods for which dates have been published by the ECB.

a. Pricing => The Exchange Delivery Settlement Price (EDSP) is one minus the EDSP Rate. The EDSP Rate is calculated as

$$\frac{1}{\delta}\left[\prod_{i=1}^{n}\left(1+\delta_i {F_i}^{ON}\right)\right]-1$$

Reference => https://globalderivatives.nyx.com/contract/content/29179/contract-specification

4. 1D InterBank Deposit Futures Contract – Brazil: They are also called 1D Futures, and are traded in BM&FBovespa. The underlying is the daily interest rate compounded until the contracts' expiration date. The rate is the Average 1D InterBank Deposit Rate (1D) as calculated by CETIP.

a. Contract Details => The *Expiration Date* is the first business day of the contract. The *Last Trading Day* is the business day preceding the expiration date. The quotations ar4 expressed as a rate per annum compounded daily based on a 252-day year to 3 decimal places. This the *Trading Price* is related to the *Quoted Rate* r as

$$\wp = \frac{100{,}000}{(1+r)^{\frac{n}{252}}}$$

where n is the number reserves between the trade date and the day preceding the expiration date.

b. Margining => On the trade date t, the margin is computed as (to be multiplied by the real value and the number of contracts)

$$\mathcal{M} = \mathbb{PA}_t - \mathbb{PO}$$

where $\mathbb{PA}_t$ is the contract settlement price on t, and $\mathbb{PO}$ is the initial price paid for the contract. The 1D margining increment can be computed as

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$$\Delta\mathcal{M} = \mathbb{PA}_t \times (\mathbb{PA}_{t-1} \times \mathbb{FC}_t)$$

where $\mathbb{FC}_t$ is the indexation factor estimates as

$$\mathbb{FC}_t = [1 + \mathbb{DI}_{t-1}]^{1/252}$$

where $\mathbb{DI}_{t-1}$ is the 1D rate corresponding to the period $[t-1, t]$.

Short-Term Interest Rate (STIR) Futures

1. IBOR Based STIR Futures: IBOR-based STIR Futures are also called Interest Rate Futures. The settlement mechanisms are common across all currencies, but the STIR contract details differ on the notional, the underlying rate index, and the exchange on which they are quoted. He dates relevant to the futures are based on the 3rd Wednesday of the month (and adjusted to the following day if the Wednesday is not a good business day) – this date is the *Start Date* of the IBOR rate underlying the future.
2. STIR Future Contract Dates: The rate is fixed at a *Spot Lag* prior to the *Start Date* (using the currency spot lag seen earlier). The fixing, thus, usually takes place on the Monday or on the Wednesday itself. The fixing date is also the *Last Trading Date* for the future. The *End Date* of the IBOR period is usually 1-3 months after the *Start Date* depending on the type of the future (using the conventions associated with the relevant IBOR index).
3. STIR Futures – Margining: The margining process works in the following way. For a given closing price (as published by the exchange), the daily margin paid is that price minus the *Reference Price* multiplied by the notional and the accrual factor of the future. Equivalently, it is the price difference multiplied by 100 *and* by the *Point Value* (*Point Value* being the margin that results from a 1% change in the price). The reference price is the trade price on the trade date and the previous closing price on subsequent dates.

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4. STIR Futures - Prices/Ticks: The futures price at t is denoted by Φ_t^j. On the fixing date, at the moment of publication of the underlying IBOR rate L_t^j, the future price is

$$\Phi_t^j = 1 - L_t^j$$

Before that moment, the price evolves according to the market dynamics including bid/offer. The *Tick Value* is the value of the smallest increment in price. The price usually changes in $\frac{1}{2}$ or 1 bp increments.

5. STIR Futures - Designation Codes: The futures are designated by character codes. The first part is dependent upon the data provider, and is usually 2-4 characters. The second part describes the month (from the month code table), and the year, with its last digit. As interest rate futures are quoted 10 years only, there is no ambiguity in using only one digit for the year. Note also that this means that when a future reaches its last trading date, a new one is created a couple of days later with the same name, but for a 10Y maturity in the future.

Currency Specific Futures

1. Main Currencies:

Currency	Tenor	Exchange	Underlying	Notional
CHF	3M	LIFFE	LIBOR	CHF 1m
EUR	3M	EUREX	EURIBOR	€ 1m
EUR	3M	LIFFE	EURIBOR	€ 1m
EUR	3M	NLX	EURIBOR	€ 1m
GBP	3M	LIFFE	LIBOR	£ ½ m
GBP	3M	NLX	LIBOR	£ ½ m
JPY	3M	SGX/CME	TIBOR	JPY 100 m
JPY	3M	SGX	LIBOR	JPY 100 m

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USD	3M	CME	LIBOR	$ 1m
USD	1M	CME	LIBOR	$ 3m
USD	3M	SGX	LIBOR	$ 1m

2. Other Currencies:

Currency	Tenor	Exchange	Underlying	Notional
CAD	3M	MX	CDOR	CAD 1m
DKK	3M	OMX	CIBOR	CAD 1m
ZAR	3M	SAFEX	JIBAR	ZAR 0.1m

3. USD: USD interest rate futures are traded on CME and on LIFFE. For the 3M futures, the notional is $ 1m and the accrual fraction is ¼. The fixing index is LIBOR 3M. For 1M Futures, then notional is $ 3m and the accrual factor is $\frac{1}{12}$. In both cases, the notional to multiply the accrual factor is 250,000.
4. EUR: The EUR 3M interest rate futures are traded on LIFFE, EUREX, and NLX. The notional is € 1m and the accrual factor is ¼. The fixing index is LIBOR.
5. GBP: The GBP 3M interest rate futures are traded on LIFFE and NLX. The notional is £ 1m and the accrual factor is ¼. The fixing index is LIBOR.
6. JPY: The JPY 3M interest rate futures are traded on CME and on SGX for TIBOR-based futures and on SGX for LIBOR-based futures. The notional is JPY 100m and the accrual factor is ¼.
7. CHF: CHF interest rate futures are traded on LIFFE. The fixing is LIBOR 3M.The notional is CHF 1m and the accrual factor is ¼.
8. AUD: Underlying Index: AUD BBSW 3M. Margin Based on

$$\frac{Price}{1 + X\ DT}$$

9. CAD: The CAD 3M interest rate futures (3M Canadian Banker's Acceptance Futures) are traded on MX. The notional is CAD 1m and the accrual factor is ¼. The fixing index is

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Tel. 646-513-3300

CDOR. The contract months are the quarterly March, June, September, and December months for up to 3 years, plus the 2 nearest non-quarterly months (serials). Reference => http://www.m-x.ca/produits_taux_int_bax_en.php

10. ZAR: The 3M ZAR interest rate futures are traded on SAFEX. The notional is ZAR 0.1 m and the accrual factor is ¼. The fixing index is JIBAR 3M, and the futures are traded 8 quarters ahead. Reference => http://www.jse.co.za/Libraries/Interest_Rate_Market_-_Products_Documentation/Jibar_FuturesContract_specifications.sf/b.ashx

Interest Rate Futures Option – Premium

1. Definition: An option on futures is described by the underlying future, an option expiration date θ, a strike K, and an option type (call/put). The expiration is before or on the futures trading date: $\theta < t_0$. The premium type options referred to here are the American type and pay premium upfront at the transaction date. The premium type is traded on CME and SGX, and there is no margining process for the option. On the CME, the options are Eurodollar futures (1M and 3M), on SGX the options are Eurodollar futures (1M), on JPY LIBOR futures, and JPY TIBOR futures.
2. Upfront Option Types: There are 3 types of options - the quarterly options, the mid-curve options, and the serial options. The quarterly options expire at the last trading day of the underlying future, i.e., $\theta = t_0$. The serial and the mid-curve options expire before the futures' last trading date. For the serial, the delay is 1-2 months (plus one weekend). For the mid-curve options, the delay is 1, 2, or 4 years. The quoted price for the option follows the same rule as the futures. For a quoted price, the amount paid is multiplied by the notional and the accrual factor of the underlying future.

Interest Rate Futures Option – Margin

1. Definition: An option on futures is described by the underlying future, the *Option Expiration Date* θ, a strike K, and an option type (call/put). The expiration is before or on the futures

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trading date: $\theta < t_0$. The option on futures referred to here are the American type and have a futures-like margining process. This type of option is traded on LIFFE for the EUR, GBP, CHF, and the USD futures (3M), and on EUREX for EUR 3M.

2. Options Margining and Quoted Price: Note that there are 2 margining processes involved in the instrument – the margining process on the underlying futures, and one on the quoted option itself. The quoted price for the option follows the same rule as that for the future. For a quoted price, the daily margin is paid on the current closing price minus the reference price multiplied by the notional and the accrual factor of the underlying future. The reference price is the trade price on the trade date, and the previous closing price on the subsequent dates.
3. The Standard Contracts List:

Currency	Tenor	Exchange	Underlying	Type
USD	3M	LIFFE	LIBOR	Option on Future
USD	3M	LIFFE	LIBOR	Mid-Curve Options
EUR	3M	LIFFE	EURIBOR	Option on Future
EUR	3M	LIFFE	EURIBOR	Mid-Curve Options
EUR	3M	LIFFE	EURIBOR	2Y Mid-Curve Options
EUR	3M	EUREX	EURIBOR	Option on Future
EUR	3M	EUREX	EURIBOR	1Y Mid-Curve Options
GBP	3M	LIFFE	LIBOR	Option on Future
GBP	3M	LIFFE	LIBOR	Mid-Curve Options
GBP	3M	LIFFE	LIBOR	2Y Mid-Curve Options
GBP	3M	LIFFE	LIBOR	Option on Future

4. Margined IRF Options: Trading Dates: For standard options (not mid-curve options), the last trading date is the same as the last trading date for the underlying future. For mid-curve options, the last trading date is 1BD before the last trading date of the future in the same month. For example, the EUR mid-curve options with expiry in March 2014 (OR14) on the March 2015 Future (ER15) have a last trading date on Friday 10 March 2014, while the March 2014 Futures (ER14) and their associated standard options (ER14) trade up to 13 March 2014.

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Bank Bill Futures – AUD Style

1. Definition: The AUD Bill Futures are traded on ASX. At expiry different bills can be delivered. The bills eligible for delivery are the bills having between 85 and 95 days to maturity on the settlement date. The bills issuers can be any banks in the approved banks' list (currently there are 4 banks – ANZ Banking Group, National Australia Bank Limited, Commonwealth Bank of Australia, and Westpac Banking Corporation).

Currency	Tenor	Exchange	Underlying	Notional
AUD	3M	ASX	Bank Bill	AUD 1m

2. Delivery: The party short on the futures chooses the bill it wants to deliver – for each contract, the short party can choose up to 10 different bills of AUD 0.1 m each. Thus, the short party has a delivery option – a situation very similar to bond futures in the main currencies.
3. Expiry: The expiry date θ (also called the announcement date) is the second Friday of the future month and the *Delivery Date* t_0 is the next business day (usually Monday). The futures are quoted with fixing upto 5 years.
4. Settlement: Let t_i $(1 \leq i \leq N)$ denote the possible maturity dates of the bills. At settlement, the price received for the bill will depend on the last quoted Future Index that we denote by F_θ. The yield associated with this index is

$$R_\theta = 1 - F_\theta$$

The paid price is

$$\frac{1}{1 + \delta_i R_\theta}$$

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Tel. 646-513-3300

where δ_i is the accrual factor between t_0 and t_i. For the AUD bill futures δ_i is Act/365. In exchange of the price the short party delivers the bills with the notional equivalent to that of the futures (remember, in practice, there can be at most 9 possible dates taking into account weekends).

Deliverable Swap (IRS) Futures (PV Quoted)

1. Definition: These futures are traded in CBOT/CME, and the notional is $ 0.1m per contract. The margining feature is the future-daily margin on the quoted price (note that the price is quoted in percentage points and $\frac{1}{32}^{\text{nd}}$ of a point, like the bond futures contract). The underlying swap has the standard convention for a USD swap – semi-annual bond-basis vs. 3M LIBOR. The futures are quoted swaps with tenors 2Y, 5Y, 10Y, and 30Y. The underlying swap has a fixed rate as decided by the exchange on the first trading date of the contract. The rate of change is in increments of 25 bp. The rate is NOT fixed at a pre-defined value, unlike the reference coupon of bond futures.
2. CBE/CBOT Deliverable Swap Futures in USD:

Contract	Notional	Coupon
2Y	$ 0.1m	0.50% as of March 2014
5Y	$ 0.1m	1.00% as of March 2014
10Y	$ 0.1m	2.50% as of March 2014
30Y	$ 0.1m	2.75% as of March 2014

3. CME/CBOT Deliverable Swap: Delivery/Trading Dates: The *Delivery Dates* follow the quarterly cycle standard to the interest rate futures. The *Delivery Date* is the 3$^{\text{rd}}$ Wednesday of the quarterly month (March, June, September, and December). The *Last Trading Date* or the *Expiry Date* is 2 trading days prior to that date, usually on the Monday.
4. Deliverable Swap – Delivery and Settlement: On the expiry date, the parties agree to enter into a swap where the party long the futures receives fixed on the swap and the party short

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Tel. 646-513-3300

the futures pays the fixed. The delivered swap is cleared on a CME clearing. The *Effective Date* of the Swap is also the *Delivery Date*. The swap has an upfront payment on the delivery date. The upfront payment is obtained from the futures settlement price on the last trading date, denoted by F_θ. The amount received by the long party is

$$N \times (1 - F_\theta)$$

(if the amount is negative, it is interpreted as the absolute value paid by the long party).

Bond Futures (non AUD/NZD)

1. Definition: These bond futures are exchange traded instruments. One of their peculiarities is that the underlying is not a single instrument, but a basket. For most of the instruments the short party has an option to deliver any of the instruments in the basket.
2. Basket: The basket is composed of government bonds from a unique issuer (country) with rules on initial maturity, remaining maturity, and size to be eligible. The bond futures are traded on different exchanges for different countries. In general, there are several maturity buckets for each underlying country.
3. Conversion Factor: The bonds in the basket are transformed to be comparable through a conversion factor mechanism. The factor is such that in a certain yield environment all the bonds have the same price. The reference yield acts in a way as the strike for a delivery process.
4. Main Contracts:

Country	Currency	Exchange	Number of Contracts
Canada	CAD	MSE	3
Germany	EUR	EUREX	4
Germany	EUR	NLX	3

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Tel. 646-513-3300

Italy	EUR	EUREX	2
Japan	JPY	TSE	3
Japan	JPY	LIFFE	1
Japan	JPY	SGX	1
Spain	EUR	MEFF	1
United Kingdom	GBP	LIFFE	3
United Kingdom	GBP	NLX	1
United States	USD	CBOT	5
Switzerland	CHF	EUREX	1

5. Embedded Options: Some of the other embedded options for certain currencies include:
    a. Timing Option => The delivery notice can be made inside of a period and not just on a single date. This provides some American Option flavor to the futures.
    b. Wild Card Option => The underlying bonds can be selected after the price of the future has been fixed. During the delivery period, there is a daily option between the end of the future trading at 14:00 and the end of the bond trading at 18:00. After the last trading date, there can be a period of up to 7 days where the future price is fixed, but the delivery notice has not been given yet.
6. Settlement: Suppose there are n bonds in the basket. Let $AccruedInterest_i(t)$ denote the accrued interest of bond i at the delivery date t. The conversion factor associated with bond i is K_i. The bond future delivery notice takes place at some date before the actual delivery, with this lag usually being around 1-2 days. If the *Futures Price* is denoted by F, at the delivery time the short party can choose the bond he wishes to deliver and receives at the delivery date the amount

$$F \times K_i + \ AccruedInterest_i(t_0)$$

    a. Settle Price Clarification => The term *Price* used above is the standard in the jargon for futures, however, it should be viewed as a *Number* or as a *Traded Reference Index.* The *Future Price* is never paid in itself. It only serves as an input for the eventual computation.

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

Country-specific Bond Futures - USD

1. Treasury Bond Futures: The futures on United States debt are traded on CBOT. The price is quoted in percentage points and 32nd of a point. Note that the last trading day and the last delivery date are not the same for all the basket underlyings.
2. Conversion Factor: The description of the price used in the delivery (using an explicit quote from the exchange) is: *The invoice price equals a future settlement price times a conversion factor, plus accrued interest. The conversion factor is the price of the delivered bond (USD 1 par value) to yield 6%*. The conversion factor is provided by the exchange and does not need to be computed by the users. Nevertheless, there are clear rules to compute them. The values do not change through the life of the future.
3. Long Futures: The Ultra T-Bond futures, the US Treasury Bond Futures, and the 10Y US Treasury Note Futures all have the same last trading day and the last delivery date. The last trading day is *the 7th business day preceding the last business day of the delivery month. Trading in expiring contracts closes at 12:01 PM on the last trading day*. Previously the US Treasury Long Bond Futures referred to all bonds with maturities greater than 15 years. That range has recently (since March 2011) been divided into 2 futures – the Ultra T-Bond Futures, and the US Treasury Bond Futures.
    a. Ultra T-Bond Futures => The underliers of the Ultra T-Bond Futures are the *US Treasury Bonds with remaining term to maturity of not less than 25 years from the first day of the futures contract delivery month*.
    b. US Treasury Bond Futures => Formerly called the 30 years futures, the deliverable grade for the Treasury Bond Futures are *bonds with remaining maturity of at least 15 years, but less than 25 years, from the first date of the delivery month*. These are also known as *Classic Bond Futures*. The Treasury Bond Futures are less liquid than the 5Y and the 10Y futures. To match the US Treasury Naming Convention, the futures would be better called *Note Futures*.
    c. Catalog => Volume refers to the Monthly volume, as on October 2013.

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

Contract	Maturity	Notional	Yield	Code	Volume
Ultra T-Bond	> 25Y	$ 0.1m	6.00 %	UB/UL/LBE	1,387,996
30Y Bond	15Y - 25Y	$ 0.1m	6.00 %	ZB/US	6,193,997
10Y Bond	6½Y – 10Y	$ 0.1m	6.00 %	ZN/TY	21,265,689
5Y Note	4Y2M – 5Y3M	$ 0.1m	6.00 %	ZF/FV	10,198,247
3Y Note	2Y9M - 3Y	$ 0.2m	6.00 %	Z3N/3YR	0
2Y Note	1Y9M – 2Y	$ 0.2m	6.00 %	ZT/TU	3,132,990

4. 10Y Treasure Note Futures: *US Treasury Notes with a remaining term to maturity of at least 6½Y, but no more than 10Y, from the first date of the delivery month*.
5. 5Y Treasury Note Futures: The last trading day is *the last business day of the calendar month*. The last delivery day is *the 3rd business day following the delivery day*. The eligible bonds are *US Treasury notes with the original term to maturity if note more than 5Y3M, and a remaining term to maturity of not less than 4Y2M as of the first day of the delivery month*.
6. 3Y Treasury Note Futures: The last trading day is *the last business day of the transaction month*. The notional is $ 0.2m. The eligible bonds are *US Treasury Notes that have an original maturity of 5Y3M and a remaining maturity of not less than 2Y9M from the first day of the delivery month, but not more than 3Y from the last day of the delivery month*.
7. 2Y Treasury Note Futures: The notional is $ 0.2m. The eligible bonds are *US Treasury Notes that have an original maturity of 5Y3M and a remaining maturity of not less than 1Y9M from the first day of the delivery month, but not more than23Y from the last day of the delivery month*.

Country-specific Bond Futures - Germany

1. German € Bond Futures Catalog: Volume is as of December 2011.

Contract	Maturity	Notional	Yield	Volume
EURO-BUXL	24Y – 35Y	€ 0.1m	6.00 %	222, 821

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

EURO-BOND	8½Y – 10½Y	€ 0.1m	6.00 %	11,778,488
EURO-BOBL	4½Y – 5½Y	€ 0.1m	6.00 %	7,252,498
EURO-SCHATZ	1¾Y – 2¼Y	€ 0.1m	6.00 %	8.659,722

2. The Contracts: All the futures are traded on EURX and NLX, except for EURO-BUXL which is only traded on EUREX. *The delivery option arising out of a short position may only be fulfilled by the delivery of certain securities issued by the Federal Republic of Germany with a remaining term on the delivery day within the remaining term of the underlying*. To be eligible, the debt securities must have a minimum issue of € 5bn.
3. Trading/Delivery Dates: The Delivery Date is *the 10th calendar day of the respective quarterly month, if this day is an exchange day; otherwise it is the exchange day immediately succeeding that day*. The last trading day is *2 exchange days prior to the Delivery Day of the relevant maturity month*.
4. Reference Yields: Note that the reference yield for the EURO-BUXL, which the most recent among the others, is 4% (and not 6% as for the majority of the others).

Country-specific Bond Futures - Spain

1. € Bond Futures: The BONO10 Futures Contract on the Spanish 10Y Government Bond was launched on 29 May 2012 by MEFF. The volumes are currently very low (quoted below for October 2013).

Contract	Maturity	Notional	Yield	Volume
BONO 10	> 8½Y	€ 0.1m	6.00 %	253

2. Underliers: The underlying asset is a national government bond with a 6.00% annual coupon and a maturity of 10Y. The contract face value is € 0.1m. The expiration day is the 10th day of the month of maturity (if holiday, the next business day). The last trading and the registration days are 2 business days prior to the expiration date. The bonds in the basket are Spanish government bonds with a remaining life of no less than 8½Y.

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

3. Settlement: The settlement price at the expiration date is calculated by dividing the CTD bond market price (ex-coupon) at the end of the session by the conversion factor of the bond. The market price of the CTD bond will be the closing price determined by SENAF.

Country-specific Bond Futures - £

1. UK £ Bond Futures Catalog: Volume is monthly as of December 2010. Note – the change from 6.00 % coupon to a lower coupon took place with the December 2011 contract.

Contract	Maturity	Notional	Yield	Volume
Long GILT Futures	8Y9M – 13Y	£ 0.1m	6.00 %/4.00 %	476,025
Medium GILT Futures	4Y – 6Y3M	£ 0.1m	6.00 %/4.00%	183
Short GILT Futures	1Y6M – 3Y3M	£ 0.1m	6.00 %/ 3.00%	1,131

2. The Contracts: All the futures are traded on LIFFE, and the Long GILTs are also traded on NLX. The first notice day in 2BD prior to the first day of the delivery month. The *Last Notice Day* is the first business day after the *Last Trading Day*. The *Last Trading Day* is 2BD prior to the last business day of the delivery month. The *Delivery Day* is any business day in the delivery month (at the sellers' choice). The deliverable bonds are subject to a coupon range of 3.00% around the reference yield.

Country-specific Bond Futures - ¥

1. Japan ¥ Bond Futures: These are traded on TSE. The notional is ¥ 100m. The *Final Settlement Day* is the 20th of each contract month. The *Last Trading Day* is the 7th business day prior to each delivery date. Trading for the new contract month begins on the business day following the last trading day. There also used to be a 20Y JGB futures, but its trading was halted in December 2002 due to lack of volume.

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

Contract	Maturity	Notional	Reference Yield	Volume
10Y JGB Futures	7Y - 10Y	¥ 100m	6.00 %	657,356
5Y JGB Futures	4Y – 5¼Y	¥ 100m	3.00 %	

Options On Bond Futures (non AUD/NZD) - Premium

1. Description: An option on futures is described by the underlying future, and expiration date θ, the strike K, and an option type (Call/Put). The option expiration is on or before the last trading date of the futures, i.e., $\theta \leq t_0$. Premium-type options pay a premium upfront at the transaction date, and are of American type. As such, there is no margining process for them. This type is traded on CBOT for USD bond futures.
2. CBOT Options on USD Bond Futures: The contract months are the first 3 consecutive contract months (2 serial expirations and one quarterly expiration) plus the next 4 months in the March, June, September, and December quarterly cycle. The serials exercise into the first nearby quarterly futures contract. Quarterlies exercise into the futures contracts of the same delivery period. The *Last Trading Day* is the *Last Friday* which precedes by at least 2BD the last business day of the month preceding the option month. The options are quoted in $\frac{1}{64}$th of a point.
3. USD Bond Futures Options Catalog: Volumes quoted here are monthly for October 2013. The codes are for CME Globex (Electronic Platform)/Open Outcry (Trading Floor) Call-Put.

Contract	Maturity	Notional	Codes	Volume
Ultra-Bond	> 25Y	$ 0.1m	OUB/OUL	3,786
Classic Bond	15Y – 25Y	$ 0.1m	OZB/CG-PG	1,247,787
10Y Note	6½Y – 10Y	$ 0.1m	OZN/TC-TP	7,710,256
5Y Note	4Y2M – 5¼Y	$ 0.1m	OZF/FL-FP	1,752,940
2Y Note	1¾Y – 2Y	$ 0.1m	OZT/TUP-TUC	197,574

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

Options On Bond Futures (non AUD/NZD) - Margin

1. Description: An option on futures is described by the underlying future, and expiration date θ, the strike K, and an option type (Call/Put). The option expiration is on or before the last trading date of the futures, i.e., $\theta \leq t_0$. The margin type options are American type and have a future-style method of margining process for the option. This type is traded on EUREX for bond futures.
2. EUR - EUREX Margin Options: The contract months are the first 3 consecutive contract months (2 serial expirations and one quarterly expiration) plus the next month in the March, June, September, and December quarterly cycle. For calendar months, the maturity month of the futures contract is the quarterly month following the expiration month of the option. For quarterly months, the maturity month of the underlying futures contract and the expiration month of the option are identical.
3. Trading/Settlement Days: *Last Trading Day for the Option Series introduced from September 1, 2011 is the last Friday prior to the first calendar day of the option expiration month, followed by at least 2 exchange days prior to the first calendar day of the option expiration month. Exception => If this Friday is not an exchange day, or if this Friday is an exchange day but is followed by only one exchange day prior to the first calendar day of the option expiration month, the exchange day immediately preceding that Friday is the **Last Trading Day**. For the purposes of this exception, an exchange day is an exchange day at both the EUREX exchanges as well as being a Federal work day at the US*. Reference => http://www.eurexchange.com/products/INT/FIX/OGBL_en.html

AUD-NZD Bond Futures

1. Introduction: The Australian and New Zealand futures are settled in cash against a standardized bond. The standardized bond yield is computed as an average of actual bond yields for AUD, and as a linear interpolation of actual bond yields for NZD.
2. Basket Weightings: The average yield cash delivery implies that the futures behave roughly like a weighted average of the underlying. The weights are not exactly equal, but they do not

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Tel. 646-513-3300

change too much with level of the rates. One single bond will never represent the future exactly, but the mixture of bonds that best represent the future does not vary too much with time and rates.

3. Characteristics: Compared to the non AUD/USD bond futures, the AUD bond futures traded in SFE have very different characteristics. The main difference is that they settle in cash vs. the average yield of the underlying bonds. The exact mechanism of the settlement (which is non-trivial) is described below.
4. Maturity Types: There exist 2 maturity types for the SFE Australian Treasury Bond Futures – the 3Y and the 10Y futures. Beyond the maturity the other characteristics of these futures types are similar. Both have a notional of 0.1m AUD per contract. The 3Y futures are more liquid than the 10Y one.
5. Settlement Yield Rules: The general scheme for choosing the yields used in settlement is that a set of randomly chosen dealer quotes is selected (after discarding extreme quotes). The selection of the underlying bonds does not appear to be captured by a very precise rule. A certain number of bonds is chosen by the exchange, often around 3 underlying bonds. The maturities are between 2Y and 4Y for the 3Y futures, and 8Y – 12Y for the 10Y futures. Reference => http://www.asx.com.au
6. Settle Yield Calculation: Suppose there are N bonds underlying the future. Since the contract settles in cash, the settlement is done against the average yield of the underlying bonds. Let $Y_{i,\theta}$ $(1 \leq i \leq n)$ be the yields on the fixing date for the underlying bonds. The reference yield for the settlement is

$$Y_\theta = \frac{1}{N}\sum_{i=1}^{N} Y_{i,\theta}$$

   This yield is used to calculate the final future index yield and the equivalent bond price, as shown below.
7. Reference Price: The time t *Futures Price* (all the caveats regarding the price being a jargon rather than an actual economic quantity applies here) is denoted by Φ_t. All the margining payments related to SFE bond futures are done using a reference bond price R_t is computed

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

from the future index in the following way: Let m be the number of payments ($m = 6$ for semi-annual 3Y futures and $m = 20$ for semi-annual 10Y futures). Then

$$\mathrm{Y}_t = 1 - \Phi_t$$

$$\nu_t = \frac{1}{1 + \frac{Y_t}{2}}$$

$$\mathrm{R}_t = 0.03 \frac{1 - {\nu_t}^m}{Y_t/2} + {\nu_t}^m$$

Finally, this reference price is multiplied by notional value, which is AUD 0.1m per contract.

8. Reference Price Calculation: The expression seen above for R_t is simply a consequence of using semi-annual 3Y/10Y bond with $c = 6.00$ % coupon, and semi-annual yield Y_t. It is just a special case of the expression

$$\mathrm{R}_t = \sum_{i=1}^{m} \frac{c}{2} \frac{1}{\left(1 + \frac{Y}{2}\right)^i} + \frac{1}{\left(1 + \frac{Y}{2}\right)^m} = \frac{c}{2} \frac{\vartheta - \vartheta^{m+1}}{1 - \vartheta} + \vartheta^m = \frac{c}{2} \frac{1 - \vartheta^m}{\frac{Y}{2}} + \vartheta^m$$

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

Section III: Funding and Forward Curve Construction and Customization

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

Curve Builder Features

Overview

1. Smoothness Criterion Evolution: Smoothness formulation is related to the minimization of strain energy (Schwarz (1989)), and the relation to Natural cubic spline (Burden and Faires (1997)), financial cubic spline (Adams (2001)) has been explored.
2. Empirical vs. Theoretical Curve Builder Frameworks: Zangari (1997) and Lin (2002) discuss this in detail.
   - Theoretical Term Structure posit explicit term structure for a variable known as short rate of interest whose values are extracted, possibly, from a statistical analysis of market variables (Vasicek (1977), Cox, Ingersall, and Ross (1985), Rebonato (1998), Barzanti and Corradi (1998), Golub and Tilman (2000)).
   - For bonds/treasuries see Nelson and Siegel (1987), Diament (1993), Svensson (1994), Soderlind and Svensson (1997), Tanggaard (1997). Effectiveness of such treatments is examined in Christensen, Diebold, and Rudebusch (2007), and Coroneo, Nyholm, and Vidova-Koleva (2008).
   - Hybrid methods use empirically determined yield curve inside of a theoretical model (Hull and White (1990), Heath, Jarrow, and Morton (1990), Ron (2000)).
   - A fairly comprehensive (although a bit dated) description of yield curve construction is given in Andersen and Piterbarg (2010).
   - Notes on some of the standard implementations by vendors are available in Jurcaga (2010), Lipman and Mercurio (2010), White (2012a), White (2012b), Gibbs and Goyder (2012), Misys (2012).

Discount Curves

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

1. Exact instrument quote match: Does the builder scheme successfully construct the curve if the quotes do not pose arbitrage? Conversely, for inexact matches, does the builder algorithm converge rapidly, and minimal error (Hagan and West (2006), Hagan and West (2008))?
2. Implied Forward Rates: Taken to be typically 1M or 3M forwards – how much should it matter, and how smooth/positive/continuous are they (McCulloch (1971))?
3. Locality: How local is the interpolating builder? If an input is changed, does the interpolator change only nearby, or is there spillover to non-adjacent far-off segments?
4. Stability of the Forward Rates: How sensitive are the forward rates to change in the inputs? The Jacobian analysis below shows the results for several splining scenarios.
    a. Forward rates are chosen for the curve behavior examination because it is the most elemental entity whose continuous/smooth behavior is meaningful to the practitioner.
5. Hedge Locality: Does most of the delta risk for a given instrument get assigned to the hedging instruments that have maturities close to the given instrument?
6. Sequential vs. Tenor Delta: Does the cumulative tenor delta equal to the aggregate (i.e., parallel shifted) delta? Le Floc'h (2013) examines the importance of this.

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135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

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135 E. 57th Street |25th Floor |New York, NY 10022
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Curve Construction Methodology

Approach

1. Instrument Setup: Identify the calibration instruments, and set up the instrument baseline. This includes initializing the span/segments, as well as the "tuning parameter" to achieve the desired "inner" and the "outer" calibrations.
2. Span/segment stretch set up: Calibrate the segments one by one using the calibration measures/inputs.
3. Tuning Adjustment: Adjust tuners to achieve the desired "boundary" condition.

State Span Design Components

1. Base Quantification Metric Retrieval: This refers to the functionality for retrieval of the State Quantification Metric Response Value at different predictor ordinates, the relative values, and canonical (possibly categorical) representations.
2. Targeted State Metric Computation: This functionality computes state/model specific targeted state metrics (e.g., LIBOR for a discount Curve, I Spread etc.) that may be absolute or relative.
3. Sensitivity Jacobian: This functionality provides for the ability to extract sensitivity Jacobian at the following levels:
    - Cross Quantification Metric (Quantification Metric 1 to Quantification Metric 2) Sensitivity Jacobian
    - External Manifest Metric to Quantification Metric Sensitivity Jacobian
4. Calibration Input Manifest Measure Retrieval: This functionality records and retrieves the calibration input manifest measure set and other relevant calibration details.
    - It needs to be remembered that the calibration input manifest measure set need not just be instrument quotes, but also "event" rates such as user specified turns meant to account for

items such as year-end yield adjustments, periods of high activity etc. (Ametrano and Bianchetti (2009), Kinlay and Bai (2009)). In the case of turns, they may be modeled as discrete latent state jumps across specific pairs of dates, of a user-specified magnitude.

- Exogenously specified State Differentials => As just noted, certain state attributes maybe exogenously specified (e.g., turns, bases, etc.). These state shift differentials may be applied before or after the calibration step.

5. Scenario State Span Re-construction: This functionality re-constructs the state using adjusted, bumped, or otherwise scenario-tweaked quantification metrics and/or manifest measures.
6. Boot State Span: This functionality is used in boot state spans. Here, there needs to be the ability to set the boot values at the node knots, and the build the segment.
7. Non-linear State Span: This functionality sets up the non-linear fixed-point extraction process and the corresponding target match criterion evaluator.

Curve Calibration From Instruments/Quotes

1. Instrument Conventions: Market Conventions for all the typical calibration instruments such as deposits, futures, FRA, IRS, float-float basis swaps, OIS, cross currency swaps etc. are available in a wide variety of publications (e.g., Open Gamma (2012)).
2. Construction from Single Instrument/Quote Set: If there is only one type instrument/quote set to be calibrated from, you can simply “spline” through the constituent segments. In particular, if there are no value limitations/constraints, then spline construction may be achieved directly from the points (e.g., bond yield curve).
   - Questionable if quote interpolation is necessary for even the single instrument set, since this results in double interpolation – the first on the quote space, and the second on the span/segment canonical space.
3. Construction from Diverse/Multiple Instrument/Quote Set: Given a diverse set of instruments and/or quotes, we need canonical quote-independent/quote-transforming measure formulation that is valid across the full instrument stretch.

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Tel. 646-513-3300

4. Curve Span/Segment Latent State Quantification Metric: This is the metric used to quantify the latent state represented by the curve.
    - For discount curves, this can be the discount factor/zero rate/forward rate.
    - For forward curves, this can be the absolute forward rate/forward rate basis.
    - For credit curves, this can be survival factor/cumulative hazard rate/ forward hazard rate.
    - For recovery curves, this can be the expected loss/recovery, of the forward loss/recovery.
5. Cumulative vs. Incremental Quantification Metric: The incremental segment quantification metric Φ may be extracted from an appropriate transformation of the cumulative span quantification metric Z:

$$\Phi \to \frac{\partial(ZS)}{\partial S}$$

    where S is the span variate (specifically, time in this case).
6. Relation between the Quantification Metric: More generally

$$\Phi \to \Im\left(Z, S, \frac{\partial Z}{\partial S}\right)$$

    where $\Im$ comes from the physics of the process. For the discount curve, the credit curve, and the recovery curve

$$\Im\left(Z, S, \frac{\partial Z}{\partial S}\right) \to \frac{\partial(ZS)}{\partial S}$$

7. Cumulative Quantification Metric from Incremental Quantification: Cumulative may be extracted from the incremental forwards using the quadrature formulation, as they are integrands over the segment dimension. For survival/discount/recovery curves

$$\mathrm{Z} = \frac{\int_0^t \Phi(S)dS}{t}$$

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

8. Structure of the Cumulative vs. Incremental Forward: Forward quantification metric is more sharp-edged/swinging than cumulative quantification metric, which, by virtue of the quadrature construct, is smoother.
   - Therefore, single instrument/quote interpolation may be able to use the forward quantification metric, and imply the cumulative quantification metric.
   - Multiple instrument/quote should use the cumulative manifest metric, and perhaps imply the forward quantification metric using the segment <-> span transformation relationship.
9. Constraints on the Forward Quantification Metric: Depends on the physics of the underlying process.
   - For survival curve, $\Phi \geq 0$, and this is a hard constraint.
   - For discount curve, there are no such constraints.
   - For recovery curve, the constraint is that $\Phi \geq 0$.
10. Constraints on the Cumulative Quantification Metric: Again depends on the underlying process behind the corresponding stochastic state variate (i.e., the QM).
    - For survival curve, if Z is the cumulative survival/hazard rate, $Z \geq 0$, and it should be monotonically decreasing - this is a hard constraint.
    - For discount curve, if Z is the discount factor, then $Z \geq 0$. Beyond this there are no constraints.
11. Interpolating in the Forward Quantification Metric Space: For survival/discount, due to the exponential nature of the formulation, splining on Φ can very often cause the prior two constraints to be violated – so relatively speaking, the choice is less stable.
12. Span/Segment Quantification Metric Relationship:
    - Discontinuity in the cumulative quantification metric automatically implies discontinuity in the forward quantification metric.
    - Continuous, but non-differentiable cumulative quantification metric implies discontinuity in the forward quantification metric.
    - Continuity in the first derivative of cumulative quantification metric implies continuous, non-differentiable forward quantification metric.
    - Most generally, C^k continuity of cumulative quantification metric (represented using, e.g., C^k splines) implies continuous, C^{k-1} continuity of the forward quantification metric.

135 E. 57th Street |25th Floor |New York, NY 10022
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- Certain splines become problematic for highly uneven segment lengths, e.g., cubic splines will be unsatisfactory for the situation where you start with close set of nodes and move to a sparser set (Burden and Faires (1997)). This is because the curve is too convex and bulging for points far away from each other.

13. Span Quantification Metric – "Effective" Rate/Hazard Rate: This can simply be defined as

$$\zeta = -\frac{\log Z}{t}$$

where Z is either the discount factor (for the discount curve) or the survival factor (for the survival curve). This needs to be matched for 4 powers (quartic) for polynomial spline, or for three derivatives for non-polynomial (e.g., tension) splines.

Calibration Considerations

1. Exponential/Hyperbolic Tension Splines as a Natural Basis for DF representation: This is popular (Sankar (1997), Securities Industry and Financial Markets Association (2004), Andersen (2005)) because the discount factor often simply monotonically decreases in time (e.g., as in an exponential). Obviously this basis will not be suitable for forward/zero rates.
   - The Trouble with the High-Tension Tension Splines is: This causes the segment responses to be almost linear with the predictor, therefore:
     - For big gaps in the predictor ordinates, "linear" can soon become a huge problem.
     - NASTY, NASTY low-tenor forward's starting near the segment edges.
     - High Tension implies high local forward interest (using above).
     - While Renka (1987) shows an automatic way to extract to specify the tension parameter, the resulting C^1 presents fundamentally no more of an advantage than a C^1 cubic (Le Floc'h (2013)).
     - Other issues with the impact of automatic selection (see Preuss (1978)) and the corresponding implications for sensitivities remain.

2. Sensitivity of the Forward Rate to the Spot Measure: The forward rate/DF sensitivity to the spot quote is not just low, but also ends up producing multiple matching results.
    - In particular, the presence of root multiplicity within a single segment (as is the case for polynomial splines) reduces the calibration to a needle in a haystack search – with huge demands on intelligent heuristics placed on the searcher.
3. Pay Date DF Pre-computation: This method is outlined in Kinlay/Bai, and is NOT a robust method, for the following reasons:
    - It starts by estimating the DF's parametrically (using constant forwards) between dates.
    - Fine pay date grids (owing to, say, diverse/overlapping instrument types, and diverse/overlapping quote types) means that the interpolation grid becomes highly clustered, and this produces challenges for many splining techniques.
4. Non-linear DV01: The DV01-type terms

$$\sum_{j=1}^{n} l_j \Delta_j D_f(t_j)$$

are non-linear on both the discount factor and the forward rate for the generic (i.e., non-telescoping) interest-rate product – this is what makes the curve calibration using the Kinlay/Bai and the Andersen schemes difficult.

    a. While relating the discount factor the LIBOR rate as

$$D_f(t) = \left\{ \prod_{i=1}^{\eta(t)-1} \left[\frac{1}{1 + L_i(t_i - t_{i-1})} \right] \right\} \frac{1}{1 + L_{\eta(t)-1}\left(t - t_{\eta(t)-1}\right)}$$

    may help simplify the formulation, it still does not reduce non-linearity. Here $\eta(t) - 1$ refers to the instrument maturity that precedes the time t.

5. No Arbitrage Conditions:
    - Forward Rates => No Arbitrage for Forwards implies that

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Tel. 646-513-3300

$$\frac{\partial}{\partial t}[t \cdot r(t)] \geq 0$$

although this can easily seen to be violated in several legitimate instances.

- Options => Arbitrage free Implied Volatility Surface for Call Options (Homescu (2011))

$$\frac{\partial}{\partial t}[C(t, K)] \geq 0$$

and

$$\frac{\partial^2}{\partial t^2}[C(t, K)] \geq 0$$

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Curve Construction Formulation

Introduction

1. Cash flow PV Linearity in Discount Factor and Survival: Simply put, the PV of a single cash flow is

$$PV = C_f \times D_f$$

or more generally

$$PV = C_f \times D_f \times S_P \times X$$

where C_f is the cash flow, D_f is the discount factor, S_P is the survival probability, and X is the FX rate. The challenge is to re-cast the measure computation in a manner that retains the formulation linearity in the latent state (it is already linear in D_f, S_P, and X, so that simplifies things a bit).

   - Re-casting all the product/measure calibration as a linear equation depends on the product/manifest measure combination, but many typical formulations satisfy this criterion.

2. Linearized Discount Curve Formulation Schemes:
   - Single Segment Giant Spline => Use all the market observations to construct all the linearization constraints to synthesize one giant multi-basis spline.
   - One Spline Segment per adjacent cash flow pair => This gives maximal control, but ends up being way too computationally involved, as their will be as many spline segments as there are cash flow pairs.
   - One Spline Segment per Instrument Maturity => Here a unique spline segment will be used between adjacent calibration instrument maturities. This ordering is identical to typical instrument level bootstrapping.

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- Transition Spline => This retains the spline cluster per each instrument group. This representation is valuable when you have instruments assembling in cluster (as cash/EDF/swaps etc., which is obviously a typical arrangement). Judicious choice of knots and instruments etc. reduce the chances of jumps/bumps, although can still be a challenge.

3. Nomenclature:
   - Instrument Set => $l = 1, \cdots, a$
   - Segment exclusive to instrument l spans the times $\tau_{l-1} \rightarrow \tau_l$
   - Instrument l has b cash flows indexed by $j = 0, \cdots, b - 1$
   - Segment l's spline coefficients α_{il} are determined by l's cash flows and market quotes.
   - Each Segment has $i = 0, \cdots, n - 1$, i.e., n basis function set representing the discount factor.
   - Instrument l's cash flow j has a pay date of τ_{jl}.
4. Importance of some of the Linear Algebra Operations: While most of what is used in spline systems for linearized curve building can be achieved using a robust linear system solver (e.g., Gauss Elimination, see Press, Teukolsky, Vetterling, and Flannery (1992)), robust matrix inversion algorithms are needed for Jacobian estimation.

Segment Linear Discount Curve Calibration

1. Step #1: Identify and sort instruments by their maturities.
   - In between two maturities lies a segment, and the curve start date demarcates the start of the first (exclusive) segment.
2. Step #2: For each instrument, extract the coefficient of each discount factor (which corresponds to the net cash flow at that node).
3. Step #3: Say that the market PV quote of instrument l is Q_l. This indicates

$$Q_l = \sum_{j=0}^{b-1} c_{jl} D_f(t_{jl}) = \sum_{\substack{j=0 \\ t_{jl} \le \tau_{l-1}}}^{b-1} c_{jl} D_f(t_{jl}) + \sum_{\substack{j=0 \\ t_{jl} > \tau_{l-1}}}^{b-1} c_{jl} D_f(t_{jl})$$

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5. Step #4: Given that all segment l cash flows whose pay date is less than τ_{l-1} belong to the prior periods, their discount factors should be computable. Thus

$$P_l = \sum_{\substack{j=0 \\ t_{jl} \le \tau_{l-1}}}^{b-1} c_{jl} D_f(t_{jl})$$

should be pre-computed.

6. Step #5: The segment specific constraint now becomes

$$Q_l = P_l + \sum_{\substack{j=0 \\ t_{jl} > \tau_{l-1}}}^{b-1} c_{jl} D_f(t_{jl}) \Rightarrow \sum_{\substack{j=0 \\ t_{jl} > \tau_{l-1}}}^{b-1} c_{jl} D_f(t_{jl}) = Q_l - P_l$$

7. Step #6: In terms of the segment spline coefficients α_{il} and the segment basis functions f_{il}, the constraint gets re-specified as follows:

$$D_f(t_{jl}) = \sum_{i=0}^{n-1} \alpha_{il} f_{il}(t_{jl})$$

$$Q_l - P_l = \sum_{\substack{j=0 \\ t_{jl} > \tau_{l-1}}}^{b-1} c_{jl} D_f(t_{jl}) = \sum_{\substack{j=0 \\ t_{jl} > \tau_{l-1}}}^{b-1} c_{jl} \left[\sum_{i=0}^{n-1} \alpha_{il} f_{il}(t_{jl}) \right]$$

Notice that

$$\Omega_l = \sum_{\substack{j=0 \\ t_{jl} > \tau_{l-1}}}^{b-1} \alpha_{jl} f_{jl}(t_{jl})$$

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can be pre-computed. Thus, the above becomes

$$\sum_{\substack{j=0 \\ t_{jl}>\tau_{l-1}}}^{b-1} \alpha_{jl} f_{jl}(t_{jl})\Omega_l = Q_l - P_l$$

8. Step #7: Of course, in general Q_l need not just be the P – it just needs to be any measure linearizable in the discount factor.
9. Cash D_f Loading:
   - Given a rate calibration measure r_l

$$D_f(\tau_l) = \frac{1}{1 + r_l \tau_l}$$

10. Futures D_f Loading:
    - Given a rate calibration measure r_l

$$\frac{-D_f(\tau_{l-1})}{r_l(\tau_l - \tau_{l-1})} + D_f(\tau_l) = 0$$

    - Given a price based calibration measure P_l

$$-P_l D_f(\tau_{l-1}) + D_f(\tau_l)$$

11. Fixed Stream D_f Loading: Given a price measure P_l

$$P_l = \sum_{j=0}^{b-1} c\Delta(t_{j-1}, t_j) D_f(t_j)$$

    where c is the coupon.
12. Floating Stream D_f Loading: Given a price measure P_l

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$$P_l = \sum_{j=0}^{b-1} s\Delta(t_{j-1}, t_j) D_f(t_j) - [D_f(t_0) - D_f(t_m)]$$

where s is the floater spread.

13. <u>IRS D_f Loading</u>:

- For a par swap IRS

$$PV_{Fixed} - PV_{Floating} = 0$$

$$\sum_{j=0}^{b-1} c\Delta(t_{j-1}, t_j) D_f(t_j) - \sum_{j=0}^{b-1} s\Delta(t_{j-1}, t_j) D_f(t_j) + [D_f(t_0) - D_f(t_m)] = 0$$

- Given a price measure P_l

$$P_l = \sum_{j=0}^{b-1} c\Delta(t_{j-1}, t_j) D_f(t_j) - \sum_{j=0}^{b-1} s\Delta(t_{j-1}, t_j) D_f(t_j) + [D_f(t_0) - D_f(t_m)]$$

14. <u>Bond D_f Loading</u>:

- Given a dirty price measure P_l

$$P_l = \sum_{j=0}^{b-1} c\Delta(t_{j-1}, t_j) D_f(t_j) + \sum_{k=0}^{a-1} N_k(t_k) D_f(t_k)$$

- Given a yield measure, the yield can be converted to the dirty price measure P_l.
- Given a spread over TSY measure, it may also be converted to the dirty price measure P_l through the yield.

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Curve Jacobian

1. Representation Jacobian: Every curve implementation needs to generate the Jacobian of the following latent state metric using its corresponding latent state quantification metric:
   - Forward Rate Jacobian to Quote Manifest Measure
   - Discount Factor Jacobian to Quote Manifest Measure
   - Zero Rate Jacobian to Quote Manifest Measure
2. Importance of the Representation Self-Jacobian: Representation Self-Jacobian computation efficiency is critical, since Jacobian of any function $F(Y)$ is going to be dependent on the self-Jacobian $\frac{\partial Y(t)}{\partial Y(t_k)}$ because of the chain rule.
3. Forward Rate - DF Jacobian:

$$F(t_A, t_B) = \frac{1}{t_B - t_A} \log \frac{\partial D_f(t_A)}{\partial D_f(t_B)}$$

$$\frac{\partial F(t_A, t_B)}{\partial D_f(t_k)} = \frac{1}{t_B - t_A} \left\{ \frac{1}{D_f(t_A)} \frac{\partial D_f(t_A)}{\partial D_f(t_k)} - \frac{1}{D_f(t_B)} \frac{\partial D_f(t_B)}{\partial D_f(t_k)} \right\}$$

   where $F(t_A, t_B)$ is the forward rate between t_A and t_B, and $D_f(t_k)$ is the discount factor at time t_k.
4. Zero Rate to Forward Rate Equivalence: This equivalence may be used to construct the Zero Rate Jacobian From the Forward Rate Jacobian. Thus the above equation may be used to extract the Zero Rate micro-Jacobian.
5. Zero Rate - DF Jacobian:

$$\frac{\partial Z(t)}{\partial D_f(t_k)} = \frac{1}{t - t_0} \left\{ \frac{1}{D_f(t)} \frac{\partial D_f(t)}{\partial D_f(t_k)} \right\}$$

   where $Z(t)$ is the zero rate at time t.
6. Analytical Sensitivity vs. Quote Bumped Sensitivity: In general, when dealing with the splined mechanisms for curve cooking, it may not be accurate to depend on the quote

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Tel. 646-513-3300

bumped sensitivity, because it may end up throwing it to a totally different curve builder scheme (Le Floc'h (2013)).

- Also, analytical sensitivities may be estimated right during the calibration itself. However, analytical-to-quote sensitivities implies two-stage Jacobian – the Jacobian of the quote to the state representations, then the Jacobian of the state representation to the sensitivity measure.
- In-situ Calibration Sensitivities => Measure to state sensitivities maybe generated quiet readily, depending on the calibration mode.
    - For linear calibrator, this is simply the state Jacobian inverse.
    - In some non-linear search techniques (esp. open ones like the Newton's method, but with the closed schemes as well), sensitivity Jacobians are automatically (or using light adjustment) generated as part of the calibration itself.
- Spline coefficient sensitivity to segment/node inputs => High sensitivity of the spline coefficients to the node inputs across specific stretches indicates instability in curve (re-) construction and the corresponding deltas (i.e., spurious deltas and leakage). Le Floc'h (2013) examines this for several standard interpolating estimators in use.

7. <u>Quote Jacobian via the Discount Factor Latent State</u>:
    - $c \Longrightarrow 0, \cdots, d-1$ Calibration Components
    - $q_c \Longrightarrow q_0, \cdots, q_{d-1}$ Corresponding Quotes
    - Let's say the Derivative PV is

$$P = \sum_{j=1}^{m} \Upsilon_j D_f(t_j) \Longrightarrow \frac{\partial P}{\partial q_c} = \sum_{j=1}^{m} \Upsilon_j \frac{\partial D_f(t_j)}{\partial q_c}$$

Thus what is typically needed to estimate product-to-quote sensitivities via the Discount Factor latent state is $\frac{\partial D_f(t_j)}{\partial q_c}$.

8. <u>Quote->Zero Rate Jacobian</u>:

$$\frac{\partial Q_j(t)}{\partial Z(t_k)} = (t_k - t_0)\left\{D_f(t_k)\frac{\partial Q_j(t)}{\partial D_f(t_k)}\right\}$$

where $Z(t_k)$ is the zero-rate at time t_k.

9. PV - Quote Jacobian:

$$\frac{\partial PV_j(t)}{\partial Q_k} = \sum_{i=1}^{n}\left\{\frac{\partial PV_j(t)}{\partial D_f(t_i)} \Bigg/ \frac{\partial Q_j(t)}{\partial D_f(t_i)}\right\}$$

10. Cash Rate DF micro-Jacobian:

$$\frac{\partial r_j}{\partial D_f(t_k)} = \frac{1}{D_f(t_j)}\frac{1}{t_j - t_{START}}\frac{\partial D_f(t_j)}{\partial D_f(t_k)}$$

where r_j is the cash rate quote for the j^{th} Cash instrument, and $D_f(t_j)$ is the discount factor at time t_j.

11. Cash Instrument PV-DF micro-Jacobian:

$$\frac{\partial PV_{CASH,j}}{\partial D_f(t_k)} = -\frac{1}{D_f(t_{j,SETTLE})}\frac{\partial D_f(t_j)}{\partial D_f(t_k)}$$

There is practically no performance impact on construction of the PV-DF micro-Jacobian in the adjoint mode as opposed to the forward mode, due to the triviality of the adjoint.

12. Futures Quote-DF micro-Jacobian:

$$\frac{\partial Q_j}{\partial D_f(t_k)} = \frac{1}{D_f(t_{j,START})}\frac{\partial D_f(t_j)}{\partial D_f(t_k)} - \frac{D_f(t_j)}{{D_f}^2(t_{j,START})}\frac{\partial D_f(t_{j,START})}{\partial D_f(t_k)}$$

where Q_j is the Quote for the j^{th} Futures with start date of $t_{j,START}$ and maturity of t_j.

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Tel. 646-513-3300

13. Futures PV-DF micro-Jacobian:

$$\frac{\partial PV_j}{\partial D_f(t_k)} = \frac{1}{D_f(t_{j,START})}\frac{\partial D_f(t_j)}{\partial D_f(t_k)} - \frac{D_f(t_j)}{{D_f}^2(t_{j,START})}\frac{\partial D_f(t_{j,START})}{\partial D_f(t_k)}$$

There is practically no performance impact on construction of the PV-DF micro-Jacobian in then adjoint mode as opposed for forward mode, due to the triviality of the adjoint.

14. Interest Rate Swap DF micro-Jacobian:

$$Q_j DV01_j = PV_{Floating,j}$$

where Q_j is the quote for the j^th IRS maturing at t_j, $DV01_j$ is the DV01 of the IRS, and $PV_{Floating,j}$ is the floating PV of the IRS.

$$\frac{\partial [Q_j DV01_j]}{\partial D_f(t_k)} = \frac{\partial [PV_{Floating,j}]}{\partial D_f(t_k)}$$

$$\frac{\partial [Q_j DV01_j]}{\partial D_f(t_k)} = \frac{\partial Q_j}{\partial D_f(t_k)} DV01_j + Q_j \frac{\partial DV01_j}{\partial D_f(t_k)}$$

$$\frac{\partial DV01_j}{\partial D_f(t_k)} = \sum_{i=1}^{j} N(t_i)\Delta_i \frac{\partial D_f(t_i)}{\partial D_f(t_k)}$$

$$PV_{Floating,j} = \sum_{i=1}^{j} L_i N(t_i)\Delta_i D_f(t_i)$$

$$\frac{\partial PV_{Floating,j}}{\partial D_f(t_k)} = \sum_{i=1}^{j} \frac{\partial L_i}{\partial D_f(t_k)} N(t_i)\Delta_i D_f(t_i) + \sum_{i=1}^{j} L_i N(t_i)\Delta_i \frac{\partial D_f(t_i)}{\partial D_f(t_k)}$$

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15. Interest Rate Swap PV-DF micro-Jacobian: See Hull (2002) for the preliminaries.

$$\frac{\partial PV_{IRS,j}}{\partial D_f(t_k)} = \sum_{i=1}^{j} N(t_i)\Delta(t_{i-1},t_i)\left\{(c_j - L_i)\frac{\partial D_f(t_i)}{\partial D_f(t_k)} - D_f(t_i)\frac{\partial L_i}{\partial D_f(t_k)}\right\}$$

There is no performance impact on construction of the PV-DF micro-Jacobian in then adjoint mode as opposed for forward mode, due to the triviality of the adjoint. Either way the performance is $\mathcal{O}(n \times k)$, where n is the number of cash flows, and k is the number of curve factors.

16. Credit Default Swap DF micro-Jacobian:

$$PV_{CDS,j} = PV_{Coupon,j} - PV_{LOSS,j} + PV_{ACCRUED,j}$$

where j refers to the j^{th} CDS Contract with a maturity t_j, c_j is its Coupon, $PV_{CDS,j}$ is the full contract PV, $PV_{Coupon,j}$ is the PV of the coupon leg of the CDS contract, $PV_{LOSS,j}$ is the PV of the loss leg of the CDS contract, and $PV_{ACCRUED,j}$ is the PV of the accrual paid on default.

$$PV_{Coupon,j} = c_j \sum_{i=1}^{j} N(t_i)\Delta_i S_P(t_i) D_f(t_i)$$

$$\frac{\partial PV_{Coupon,j}}{\partial D_f(t_k)} = c_j \sum_{i=1}^{j} N(t_i)\Delta_i S_P(t_i)\frac{\partial D_f(t_i)}{\partial D_f(t_k)} + \frac{\partial c_j}{\partial D_f(t_k)}\sum_{i=1}^{j} N(t_i)\Delta_i S_P(t_i) D_f(t_i)$$

$$PV_{LOSS,j} = \int_0^{t_j} N(t)[1 - R(t)] D_f(t)\, dS_P(t)$$

$$\frac{\partial PV_{LOSS,j}}{\partial D_f(t_k)} = \int_0^{t_j} N(t)[1 - R(t)]\frac{\partial D_f(t)}{\partial D_f(t_k)}\, dS_P(t)$$

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Tel. 646-513-3300

$$PV_{ACCRUED,j} = c_j \sum_{i=1}^{j} \left[\int_{t_{j-1}}^{t_j} N(t)\Delta(t_{i-1}, t_i) D_f(t) dS_P(t) \right]$$

$$\frac{\partial PV_{ACCRUED,j}}{\partial D_f(t_k)} = \frac{\partial c_j}{\partial D_f(t_k)} \sum_{i=1}^{j} \left[\int_{t_{j-1}}^{t_j} N(t)\Delta(t_{i-1}, t_i) D_f(t) dS_P(t) \right] + c_j \sum_{i=1}^{j} \left[\int_{t_{j-1}}^{t_j} N(t)\Delta(t_{i-1}, t_i) \frac{\partial D_f(t)}{\partial D_f(t_k)} dS_P(t) \right]$$

17. <u>Credit Default Swap DF micro-Jacobian</u>:

$$\frac{\partial PV_{CDS,j}}{\partial D_f(t_k)} = c_j \sum_{i=1}^{j} N(t)\Delta(t_{i-1}, t_i) \frac{\partial D_f(t_i)}{\partial D_f(t_k)} S_P(t_i) + \frac{\partial c_j}{\partial D_f(t_k)} \sum_{i=1}^{j} \left[\int_{t_{j-1}}^{t_j} N(t)\Delta(t_{i-1}, t_i) D_f(t) dS_P(t) \right] + \sum_{i=1}^{j} \left[\int_{t_{j-1}}^{t_j} N(t)\{c_j\Delta(t_{i-1}, t_i) - [1 - R(t)]\} \frac{\partial D_f(t)}{\partial D_f(t_k)} dS_P(t) \right]$$

There is no performance impact on construction of the PV-DF micro-Jacobian in then adjoint mode as opposed for forward mode, due to the triviality of the adjoint. Either way the performance is $\mathcal{O}(n \times k)$, where n is the number of cash flows, and k is the number of curve factors.

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Stream-based Calibration

Latent State Formulation Metric (LSFM)

1. Case for LSFM: In addition to the quantification metric employed as described above for quantifying the latent state, we also need a "Latent State Formulation Metric". The LSFM is the metric that dictates the formulation specification for the predictor/response constraint relation for the latent state at hand. For e.g., commonly price/PV based formulation (i.e., predictor/response relation determination) is used in the discount curve construction using swap calibration instruments, whereas direct manifest measure observations (e.g., the observed FRA rate or the par forward deposit rate etc.) are used for forward curve construction (if zero coupon bond prices are available, they are form direct manifest measure maps of the discount factor quantification metric). While the quantification metric representation is chosen the same across all the constituent segments/stretches to facilitate ancillary objectives (e.g., smoothness/C^k requirements), the LSFM chosen need not be subject to such limitations. The only demand is that, using the manifest measure, the formulation metric result in a linear relation involving the LSQM's corresponding to a given segment/stretch/span.
2. Latent State QM as the Formulation Metric: In this case the relation becomes trivial, as $QM \equiv FM$, and the $QM \leftrightarrow FM$ Jacbian reduces to unity (thus producing a unit QM loading).

Stream Inference Setup

1. The Calibration Entities: The principle quantities involved in the latent state calibration are the latent state response variables, the manifest quote measure, and the formulation metric. Typical latent state calibration relations are set up so that the linearity between the latent state quantification metric and the formulation metric are maintained (there are notable exceptions, however - e.g., $PV_{Deposit}\ vs.\ FwdRate_{Deposit}$). However, the relationship between the

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manifest measure quote and the latent state quantification/formulation metric WILL NOT be linear, generally speaking.

2. Latent State Quantification Metric Ordinate Affixation or Predictor Tagging: Typical quantification metrics for latent states (such as collateral, funding, FX etc.) affix/tag their responses to the pay date predictor ordinate node. The forward latent state (quantified using, say, the forward rate) is an exception, as seen below.
3. Forward Rate Quantification Metric "Affixation Ordinate" Choice: While the "affixation ordinate" for the discount factor is point-wise unique (i.e., it corresponds to the pay date), similar affixation for the forward rate is unique only to within the segment range (i.e., either the start/end of the period). This would allow the choice of any of reset/start/end as a viable nominal affixation ordinate. However, the specific choice of the inference routines (e.g., a boot calibrator) may render some choices of the affixation ordinate more convenient than the others. For instance, the boot calibrator utilizes the notion of sequential segment build-out, thus a particular choice of the affixation ordinate (namely the start/end date against the reset date) may fit in very well with the marking scheme applied to attach the quote to its corresponding exclusive manifest segment.

Coupon Period-Based Calibration Specification

1. Period Latent State Loading: The period formulation metric may require the latent state response values at one/more time predictor ordinates. Recalling that the latent state loading at the appropriate predictor ordinate represents the linear calibration coefficient for the latent states, single point formulation metric requires single point state loading, and multi-point formulation metric requires multi-point state loading.
2. Single Point State Formulation Metric: Whenever the formulation metric is dependent only on a single latent state response value realization, we require just a single corresponding loading. Examples include single period single reset forward rate, period terminal discount factor, period survival probability, period pay FX rate, etc.
3. Single Point State Loading: In the case of PV formulation metric, the PV for period j is

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Tel. 646-513-3300

$$PV_j = \Delta(t_{j-1}, t_j) \times \mathcal{L}_{Compounded,X}(t_{j-1}, t_j) \times S_P(t_j) \times D_f(t_j) \times FX(t_j) \times Convexity(t, t_{j-1}, t_j)$$

a. Credit Loading =>

$$\Delta(t_{j-1}, t_j) \times \mathcal{L}_{Compounded,X}(t_{j-1}, t_j) \times D_f(t_j) \times FX(t_j) \times Convexity(t, t_{j-1}, t_j)$$

b. Funding Loading =>

$$\Delta(t_{j-1}, t_j) \times \mathcal{L}_{Compounded,X}(t_{j-1}, t_j) \times S_P(t_j) \times FX(t_j) \times Convexity(t, t_{j-1}, t_j)$$

c. Forward Loading =>

$$\Delta(t_{j-1}, t_j) \times S_P(t_j) \times D_f(t_j) \times FX(t_j) \times Convexity(t, t_{j-1}, t_j)$$

d. FX Loading =>

$$\Delta(t_{j-1}, t_j) \times \mathcal{L}_{Compounded,X}(t_{j-1}, t_j) \times S_P(t_j) \times D_f(t_j) \times Convexity(t, t_{j-1}, t_j)$$

e. Predictor Ordinate Anchoring => As will be seen later, it is common to anchor the credit loading and the forward loading to period end date predictor ordinate, while funding and FX loading are anchored to period pay date predictor ordinate.

4. <u>Multi-Point Loading - Multi-Reset</u>: This corresponds to the joint cases of a) multi-reset periods per coupon period, and b) the reset periods get compounded arithmetically. In this situation a compounding adjustment identical to the typical non-merged state forward/funding convexity adjustment is applied to each reset period k as

$$ForwardLoading_k = \Delta(t_{j,k-1}, t_{j,k}) \times S_P(t_j) \times D_f(t_j) \times FX(t_j) \times Convexity(t, t_{j,k-1}, t_{j,k}) \qquad (10.6)$$

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Tel. 646-513-3300

5. Multi-Point Loading - Quadrature: Quadrature-based multi-point loading results from the state response realizations being evaluated using a quadrature routine, e.g., loss quadrature grid – and therefore is continuous. While this is the primary distinction between multi-point reset and multi-point quadrature loadings, in practice the quadrature loadings also tend to be discretized – although a finer granularities.

Stream-Based Calibration Specification

1. Calibration State Loadings and Stream Sensitivities Sought: We consider the case of curve construction for the discount rates and the forward rate latent state as a concrete example. In this case we seek:
    a. Distinct Discount State Segment-local Quantification Metric Loading
    b. Distinct Discount State Segment-local Quantification Metric Jacobian Loading
    c. Distinct Forward State Segment-local Quantification Metric Loading
    d. Distinct Forward State Segment-local Quantification Metric Jacobian Loading
    e. Merged Discount/Forward State Segment-local Quantification Metric Loading
    f. Merged Discount/Forward State Segment-local Quantification Metric Jacobian Loading
2. Boot Stretch Calibration: In addition to the above, given that we are going to be focused on a boot stretch with manifest measure exclusivity $[a_+ \rightarrow b]$, we also seek to determine the leading formulation metric contribution from the leading segments/regimes of the stretch $[0 \rightarrow a]$. Given the boot framework, we presume no contributions arising out of the trailing segments.
3. Fixed and Floating Streams: We treat each stream as the calibration unit, since the potential state merging and telescoping occur at this level. The fixed and the floating streams are:

$$\mathfrak{I}_x(a,b) = c\sum_{i=1}^{a} \Delta(t_{i-1},t_i)\, D_f(t_i) + c\sum_{i=b+1}^{b} \Delta(t_{i-1},t_i)\, D_f(t_i)$$

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Tel. 646-513-3300

$$\Im_l(a,b) = \sum_{i=1}^{a} [\mathcal{L}(t_{i-1}, t_i) + \beta] \Delta(t_{i-1}, t_i)\, D_f(t_i) + \sum_{i=a+1}^{b} \mathcal{L}(t_{i-1}, t_i) \Delta(t_{i-1}, t_i)\, D_f(t_i) + \beta \sum_{i=a+1}^{b} \Delta(t_{i-1}, t_i)\, D_f(t_i)$$

β is the floating stream basis, and we've partitioned the manifest measure exclusive segment $[a_+ \to b]$ into the floater and the basis parts.

4. <u>Leading Stream Contribution</u>:
    a. Merged/non-merged Fix =>

$$\Im_x(a,b) = c \sum_{i=1}^{a} \Delta(t_{i-1}, t_i)\, D_f(t_i)$$

    b. Merged/non-merged Floater =>

$$\Im_l(a,b) = \sum_{i=1}^{a} [\mathcal{L}(t_{i-1}, t_i) + \beta] \Delta(t_{i-1}, t_i)\, D_f(t_i)$$

    c. The Merged Floater reduces to

$$\Im_l(a,b) = D_f(t_0) - D_f(t_a) + \beta \sum_{i=1}^{a} \Delta(t_{i-1}, t_i)\, D_f(t_i)$$

5. <u>Fixed Stream D_f Loading</u>:

$$\frac{\partial \Im_x(a,b)}{\partial D_f(t_k)} = \begin{cases} c\Delta(t_{k-1}, t_k) & k \in [a_+ \to b] \\ 0 & k \notin [a_+ \to b] \end{cases}$$

6. <u>Floating Stream D_f Loading</u>:

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Tel. 646-513-3300

$$\frac{\partial \Im_l(a,b)}{\partial D_f(t_k)} = \begin{cases} [\mathcal{L}(t_{k-1},t_k) + \beta]\Delta(t_{k-1},t_k) & k \in [a_+ \to b] \\ 0 & k \notin [a_+ \to b] \end{cases}$$

7. <u>Distinct Forward Rate</u>: Given that the fixed stream relies on no floating stream payments, the forward latent quantification metric loadings will be NULL. However, for the floating stream

$$\frac{\partial \Im_l(a,b)}{\partial F_f(t_k)} = \begin{cases} \Delta(t_{k-1},t_k) D_f(t_k) & k \in [a_+ \to b] \\ 0 & k \notin [a_+ \to b] \end{cases}$$

8. <u>Loadings vs. Constraint for the Stream</u>: While a given stream may not have explicit dependence on the specified latent state, it may still participate in the constraint generation process. For instance, in the case of the fix-float swap seen above, the fixed stream does not have dependence on the floating rate, but will still contribute to the net PV (in this case through the PV formulation metric).

Calibration of Multi-Stream Component

1. <u>Loadings Consolidation as Linear Overlays</u>: In the case of linear formulation metric, the loadings and the constraints of the individual streams are simply overlaid across onto the component level. Thus, for the component index $j = 1, \ldots, m$, the consolidated constraint becomes

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \alpha_{ij} \mathcal{L}_{ij} = \sum_{j=1}^{m} \mathbb{C}_j$$

where α_{ij} corresponds to the loading for node i and component i, and $\mathbb{C}_j$ is the corresponding constraint wo within the containing segment . Clearly this generalizes well to the case of more than 2 components, with the only limitation being that there can only be one

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Tel. 646-513-3300

outstanding quantification metric to be inferred (of course, in the case of merged latent states, as single quantification metric suffices to uniquely quantify multiple latent states).

2. <u>Formulation Metric Consistency across the Streams and the Component</u>: As is obvious, the inference within a single segment (for all streams contained within the component's manifest measure exclusive segment) NEED to share the same formulation metric. While the unitary loadings generation entity is still the stream, the latent state sequence builder interacts (via the specified manifest measure quotes) only with the component. This implies that the component needs to maintain an intimate awareness of the layout/metric of the corresponding constituent streams, and may "create/translate/introduce" stream-specific manifest measures during the calibration run.
3. <u>Fixed Income Product Aggregations</u>: Just as cash flows get aggregated into streams, streams get aggregated onto components, and components onto products. Also cash flows inside a stream get telescoped out to simplify valuation/loadings generation. Likewise, entire streams may get telescoped off inside constituent product components – in particular this feature is utilized in "package calibrations" – as in CCBS discount/forward, and USD OIS using LIBOR-OIS and LIBOR-fixed swaps.

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Spanning Spline

Formulation and Set up

1. Spline vs. Boot Span: For the purposes of this discussion, the main difference between spline span and boot span is that, in boot span, the segment boundaries HAVE to line up with the instrument maturity edges. In spline spans, however, additional criterion-based knots may be used to determine the boundaries (e.g., parametric knot insertion in line with regression spline approaches).
2. Basic Setup: All instruments and quotes fall into one set of constraints as

$$\sum_{j=0}^{b-1} c_{jl} D_f(t_{jl}) = Q_l$$

   where $l = 1, \cdots, a$.

   - In general, $a < b$, so you have $b - a$ degrees of freedom.
3. Local Ordinate Re-formulation: The spline extends from $t_{START} \to t_{b-1}$. Setting

$$x_i = \frac{t_i - t_{START}}{t_{b-1} - t_{START}}$$

$$\sum_{j=0}^{b-1} c_{jl} D_f(t_{jl}) \Longrightarrow \sum_{j=0}^{b-1} c_{jl} D_f(x_{jl})$$

   Further

$$D_f(t_{START}) = D_f(x = 0) = 1$$

4. Basis Formulation: Setting

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$$D_f(x) = \sum_{i=0}^{n-1} \alpha_i f_i(x)$$

$$\sum_{j=0}^{b-1} c_{jl} \sum_{i=0}^{n-1} \alpha_i f_i(t_j) = Q_l \Rightarrow \sum_{i=0}^{n-1} \alpha_i \left\{ \sum_{j=0}^{b-1} c_{jl} f_i(t_j) \right\} = Q_l$$

Thus, if

$$n = a$$

there now are a equations and a unknowns.

5. Monotonicity Preservation in Spanning Splines: The heterogeneity of the calibration instruments demands special techniques for monotonicity maintenance (Hagan West (2006) described in detail earlier was a sample).
    - Stringent monotonic constraints introduced by Hyman (1983) was relaxed by Dougherty, Edelman, and Hyman (1989), and this was works well in practice in its ability to maintain monotonicity (Ametrano and Bianchetti (2009a), Le Floc'h (2013), also implemented in Quantlib (2009)).
    - Intermediate filter constraints introduced by Steffen (1990) and their variants treated in some detail by Huynh (1993) – all suffer from the same unnatural dips or cook bumps.
6. Pros: As always, the degrees of freedom may be expanded beyond a to allow for optimizing spline construction (covered in the spline builder section).
7. Cons: With many basis functions (esp. for polynomials), the inevitable Runge's phenomenon takes over.

Challenges with the Spanning Spline Approach

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Tel. 646-513-3300

1. Problems with Cubic Polynomial Spline: Too well known to documented – spurious inflection, too much concavity/convexity at widely separated predictor nodes (esp. in long end), and no guarantee of positivity where desired.
    - As noted in Le Floc'h (2013), monotone variants (including Hagan and West (2006), Wolberg and Alfy (1999), Hyman (1983)) of the standard cubic spline have differing degrees of problems since they are attempt to model the entire span with a single representation.
2. Problems with Quartic Spline: While this makes the interpolation very smooth (Adams and van Deventer (1994), van Deventer and Inai (1997), Adams (2001), Lim and Xiao (2002), Quant Financial Research (2003)), the stiffness needed for shape-preservation is completely lost. Other troubles as with cubic splines (spurious inflection, too much concavity/convexity at widely separated predictor nodes (esp. in long end), and no guarantee of positivity where desired) as well Runge's swings are also present.

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Monotone Decreasing Splines

Motivation

1. These are spline basis functions that monotonically decrease over the given interval. Valuable for representing discount factors.
2. Why represent discount factors? Because the pay-offs are linearizable in them, so working with them implies working with the linear rates space representation, and all the advantages that come with that.

Exponential Rational Basis Spline

1. Basis Function Set:

$$\left\{1, \frac{1}{1+t}, e^{-t}, \frac{e^{-t}}{1+t}\right\}$$

2. Monotone Decreasing Nature: Each of the above basis functions is decreasing. If

$$\beta_i > 0 \; \forall \; i$$

then, conservatively speaking, the functional form is monotonically decreasing.

- Alternatively, we may also require that no inflection exist within the given segment, but that is hard to enforce with this set.

Exponential Mixture Basis Set

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1. Motivation: Since the discounting function goes as e^{-t}, an exponential mixture basis such as $e^{-\lambda_i t}$ may be a good choice, as they are both intuitively monotone, and linear combinations of them produce convexity/concavity.
2. Basis Function Set:

$$\{e^{-\lambda_i t}\}; i \in [0, \cdots, n-1]$$

- Choosing λ_i: Since for C^2 continuity we require 4 basis functions, we choose $\lambda_{Floor} = 0$, λ_{Low}, λ_{Medium}, and λ_{High}. $\lambda_{Floor} = 0$ accounts for adjusting jumps.
- Typical values can be: $\lambda_{Low} = 0.01\%$, $\lambda_{Medium} = 5\%$, and $\lambda_{High} = 25\%$.
- Parallel with Tension Splines => λ_i are comparable to tension splines.
- With this choice, C^k may be maintained for $k \geq 2$, thereby making the forwards continuous, preserving locality, imparting segment convexity/concavity. Thus all the smoothing schemes may be maintained.

3. Similarity with exponential/hyperbolic tension splines: Very similar in formulation. However, given that with exponential/hyperbolic basis set spline at one of basis functions has a non-negative exponential argument, that basis function becomes monotonically increasing.
   - Further, while estimation of the exponential tension needs to be done extraneously (Renka (1987)), here we appeal to the intuitive physics, as shown.

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Hagan West (2006) Smoothness Preserving Spanning Spline

Monotone/Convexity Preserving Estimator

1. Premise: This is primarily focused on a quadratic interpolant, but it also contains heterogeneously inserted sub-segment knots in effect to achieve the desired monotonicity, convexity, and positivity effect.
2. Philosophy:
   - This is mainly meant for forward rates inside finance, although bit more general outside of it.
   - The observation set $\{z_i\}_{i=1}^{n}$ is simply a quantity conserved on a per-segment basis, e.g., the segment mean of the state variate response, i.e.,

$$z_i = \frac{1}{\tau_i - \tau_{i-1}} \int_{\tau_{i-1}}^{\tau_i} y(t)dt$$

   - $y(t)$ is positive and piece-wise quadratic inside of $\{\tau_{i-1}, \tau_i\}$.
   - The node response value y_i at the predicate ordinate τ_i is linearly interpolated from the observations at z_i and z_{i+1} (obviously edges will be treated slightly differently).
   - Based on the specified monotonicity maintenance and convexity preservation criteria, the algorithm identifies and inserts knots. Zero or more knots may need to be inserted.
   - The quadratic interpolant is essentially a Bessel C^1 Hermite interpolant.
   - Finally, similarity response value may be applied for positivity, and range-bounded-ness.
3. Steps:
   - Infer the response node value y_i at the predicate ordinate τ_i is linearly interpolated from the observations at z_i and z_{i+1} as:

$$y_i = \frac{\tau_i - \tau_{i-1}}{\tau_{i+1} - \tau_{i-1}} z_{i+1} + \frac{\tau_{i+1} - \tau_i}{\tau_{i+1} - \tau_{i-1}} z_i, i \neq 0, n$$

$$y_0 = z_1 - \frac{1}{2}[y_1 - z_1]$$

$$y_n = z_n - \frac{1}{2}[y_{n-1} - z_n]$$

- Work out the "Z-score" metric within $\{\tau_{i-1}, \tau_i\}$:

$$g_{i-1} = y(\tau_{i-1}) - z_i = y_{i-1} - z_i$$

$$g_i = y(\tau_i) - z_i = y_i - z_i$$

Further, we work in the local predictor ordinate space x, where

$$x = \frac{\tau - \tau_{i-1}}{\tau_i - \tau_{i-1}}$$

- Apply the appropriate adjustments for the monotonicity/convexity enforcement at the appropriate zones:
  - Case I => This case arises when either

$$g_{i-1} > 0 \ AND \ \frac{1}{2}g_{i-1} \geq g_i \geq -2g_{i-1}$$

or

$$g_{i-1} < 0 \ AND \ -\frac{1}{2}g_{i-1} \leq g_i \leq -2g_{i-1}$$

is true. In this case the function

$$g(\tau) = g_{i-1}(1 - 4x + 3x^2) + g_i(-2x + 3x^2)$$

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can be used unchanged, as the original construct is already monotone and convex.

- Case II => This case arises when either

$$g_{i-1} > 0 \ AND \ g_i \geq -2g_{i-1}$$

or

$$g_{i-1} < 0 \ AND \ g_i \leq -2g_{i-1}$$

is true. Here, insert a knot at

$$\eta = \frac{g_i + 2g_{i-1}}{g_i - g_{i-1}}$$

The segment univariate becomes

$$g(\tau) = g_{i-1}; 0 \leq x \leq \eta$$

and

$$g(\tau) = g_{i-1} + (g_i - g_{i-1})\left[\frac{x - \eta}{1 - \eta}\right]^2; \eta < x \leq 1$$

- Case III => This case arises when either

$$g_{i-1} > 0 \ AND \ 0 > g_i > -\frac{1}{2}g_{i-1}$$

or

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$$g_{i-1} < 0 \; AND \; 0 < g_i \leq -\frac{1}{2} g_{i-1}$$

is true. Here, insert a knot at

$$\eta = \frac{3 g_{i-1}}{g_i - g_{i-1}}$$

The segment univariate becomes

$$g(\tau) = g_{i-1} + (g_i - g_{i-1}) \left[\frac{\eta - x}{\eta}\right]^2 ; 0 \leq x < \eta$$

and

$$g(\tau) = g_i; \eta \leq x \leq 1$$

- Case IV => This case arises when either

$$g_{i-1} \geq 0 \; AND \; g_i \geq 0$$

or

$$g_{i-1} \leq 0 \; AND \; g_i \leq 0$$

is true. Here, insert a knot at

$$\eta = \frac{g_i}{g_i + g_{i-1}}$$

Setting

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Tel. 646-513-3300

$$A = -\frac{g_i g_{i-1}}{g_i + g_{i-1}}$$

the segment univariate becomes

$$g(\tau) = A + (g_i - A)\left[\frac{\eta - x}{\eta}\right]^2 ; 0 \leq x < \eta$$

and

$$g(\tau) = A + (g_i - A)\left[\frac{x - \eta}{1 - \eta}\right]^2 ; \eta \leq x \leq 1$$

Positivity Preserving Estimator

1. Positivity of the interpolant: Hagan and West (2006) guarantee this by setting the following bounds:

$$y_0 = bound[0, y_0, 2z_1]$$

$$y_n = bound[0, y_n, 2z_n]$$

$$y_i = bound[0, y_i, 2 \times min(z_i, z_{i+1})] ; i \neq 0, n$$

Ameliorating Estimator

1. Amelioration (i.e., Smoothing) of the Interpolant - Steps:
   - #1: Expand the Range at the edges => Add an interval at the beginning and at the end. Define

$$\tau_{-1} = \tau_0 - (\tau_1 - \tau_0)$$

$$z_0 = z_1 - \frac{\tau_1 - \tau_0}{\tau_2 - \tau_0}(z_2 - z_1)$$

$$\tau_{n+1} = \tau_n + (\tau_n - \tau_{n-1})$$

$$z_{n+1} = z_n - \frac{\tau_n - \tau_{n-1}}{\tau_n - \tau_{n-1}}(z_n - z_{n-1})$$

Complete the linear interpolation of the response variate across all the intervals as before.

- #2: Set the Extraneous Bounds Parametrically/Empirically => Assume that the left and the right mini-max bounds are set extraneously for each segment, i.e., $y_{i,LeftMin}$, $y_{i,LeftMax}$, $y_{i,RightMin}$, and $y_{i,RightMax}$ are extraneously set. They may be set either point-by-point, or using another parametrization. This ensures locality, at expense of C^k, however.
    - Check if the given response value is inside of the specified range, i.e.,

      $$\min(y_{i,LeftMax}, y_{i,RightMax}) \geq y_i \geq \max(y_{i,LeftMax}, y_{i,RightMax})$$

      set as follows:
        - If $y_i < \min(y_{i,LeftMax}, y_{i,RightMax})$ $y_i = \min(y_{i,LeftMax}, y_{i,RightMax})$.
        - If $y_i > \max(y_{i,LeftMin}, y_{i,RightMin})$ $y_i = \max(y_{i,LeftMin}, y_{i,RightMin})$.
    - Otherwise:
        - If $y_i < \min(y_{i,LeftMax}, y_{i,RightMax})$ $y_i = \min(y_{i,LeftMax}, y_{i,RightMax})$.
        - If $y_i > \max(y_{i,LeftMin}, y_{i,RightMin})$ $y_i = \max(y_{i,LeftMin}, y_{i,RightMin})$.
- #3: Re-work the edges =>
    - If $|y_0 - z_0| > \frac{1}{2}|y_1 - z_0|$ then $y_0 = z_1 - \frac{1}{2}|y_1 - z_0|$.

- If $|y_n - z_n| > \frac{1}{2}|y_{n-1} - z_n|$ then $y_n = z_n + \frac{1}{2}|y_{n-1} - z_n|$.
- If y_0 is already explicitly specified (as the zero-day rate in some markets) use that instead.
- Finally, if needed re-apply the positivity enforcement across all the segments as before.

Harmonic Spline Extension to the Framework above

1. Harmonic Splines and Continuous Limiters extension: Le Floc'h (2013) applies the harmonic splines originally introduced by Fritsch and Butland (1984), and extends the monotonicity preserving limiters of Van Leer (1974) and Huynh (1993) by using rational functions.
2. Harmonic Forwards in Hagan-West: Couple of interesting items to note: Given

$$m_{i+1} = \frac{y_{i+1} - y_i}{t_{i+1} - t_i}$$

on substituting

$$y_i = -z_i t_i$$

you get

$$z_{i+1} = -m_i$$

and

$$-s_i = +f_i$$

3. Estimation of the node forwards using Harmonic mean: Apply the above now to get

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$$\frac{1}{f_i} = \frac{t_i - t_{i-1} + 2(t_{i+1} - t_i)}{3(t_{i+1} - t_{i-1})}\frac{1}{z_i} + \frac{t_{i+1} - t_i + 2(t_i - t_{i-1})}{3(t_{i+1} - t_{i-1})}\frac{1}{z_{i+1}}$$

if $z_i z_{i+1} > 0$, and $f_i = 0$ otherwise. After this, the regular Hagan-West may be applied without the need to enforce monotonic or convexity constraints, as it now is monotonic/convex by construction.

Minimal Quadratic Estimator

1. Design Philosophy: The algorithm extracts the spline coefficients keeping in mind the following:
   - Formulate using a 2nd degree quadratic polynomial for each segment
   - Maintain the Conserved Quantities
   - Maintain the Segment Edge Continuities
   - Optimize for the linear combination of two penalties:
     - Jump of the inter-segment discontinuities on the first derivatives
     - Curvature of the second derivative
2. Step #1: Preservation of the Conserved Quantity Set: This results in the following equation:

$$z_i = a_i + \frac{1}{2}b_i h_i + \frac{1}{3}c_i {h_i}^2$$

3. Step #2: Edge Continuity Constraint:

$$a_{i+1} = a_i + b_i h_i + c_i {h_i}^2$$

4. Step #3: Minimize the Penalty:
   - Jump of the inter-segment discontinuities on the first derivatives

$$J_{1i} = [b_i + 2c_i h_i - b_{i+1}]^2 = [(b_i - b_{i+1}) + 2c_i h_i]^2$$

$$= (b_i - b_{i+1})^2 + 4c_i h_i (b_i - b_{i+1}) + 4{c_i}^2 {h_i}^2$$

- Curvature of the second derivative

$$J_{2i} = [h_i \cdot 2c_i]^2 = 4{c_i}^2 {h_i}^2$$

- Complete Penalty Formulation =>

$$P(w) = wJ_{2i} + (1-w)J_{1i} = 4wc_i h_i (b_i - b_{i+1}) + 4{c_i}^2 {h_i}^2$$

- Minimizing $P(w)$ =>

$$\frac{\partial P(w)}{\partial c_i} = 0 \Longrightarrow 4wc_i h_i (b_i - b_{i+1}) + 8c_i {h_i}^2 = 0$$

$$\frac{\partial^2 P(w)}{\partial {c_i}^2} = 8{h_i}^2 > 0$$

so minimum exists.

5. <u>Equation Set and Unknowns Analysis</u>:

- $z_i = a_i + \frac{1}{2} b_i h_i + \frac{1}{3} c_i {h_i}^2$ => One per segment => $n-1$ Equations
- $a_{i+1} = a_i + b_i h_i + c_i {h_i}^2$ => One per common edge => $n-2$ Equations
- $4wc_i h_i (b_i - b_{i+1}) + 8c_i {h_i}^2 = 0$ => One each for all c_i up to c_{n-2} => $n-2$ Equations
- Total number of linear equations => $3n-5$
- Total number of unknowns => $3n-3$
- As always, the final 2 conditions from natural, financial, or the not-a-knot clamped boundary conditions.

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Tel. 646-513-3300

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135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

Extrapolation in Curve Construction

1. Latent State Choice for the Extrapolator: The quantification metric used to extrapolate the latent state may be completely different from that used to infer within the span.
    - This clearly indicates that the span spans the extrapolated range as well. Further, the extrapolator should be a property of the Span, not any stretch.
2. Extrapolator Construction: At the span edges, the C^k continuity constraints may be passed onto the extrapolator as well. These may take the form of the stretch boundary conditions (natural/financial etc.).
3. State Space Extrapolation using Synthetic Observations: This is really what it is. In particular, to get the desired left/right boundary behavior, you may insert synthetic observations at either end to produce the desired custom behavior (this may also be used in lieu of the explicit boundary condition specification).

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

Multi-Pass Curve Construction

Motivation

1. Introduction: This is composed of one shape preserving pass on the inferable state quantification metric, followed by one or more "smoothing passes".
2. Shape Preserving Pass: The shape preservation pass occurs on the "native designate" measure, preferably one that is linearly inferred from the manifest measure. The primary objective of the shape preservation pass is to maintain the monotonicity, the convexity, the locality, and possibly the positivity of the quantification metric.
   - The output of the shape-preserving pass is a span on the quantification metric that is "well-behaved", and one that contains a new set of "truthness" nodes on which the eventual smoothing can be done.
3. Shape Preservation Variants:
   - Linear in the Discount Factor Quantification Metric => They are obviously the best shape preserver (owing to the perfection in the match and zero curvature penalty), but they no inherent convexity/concavity in them, so it gets harder for the smoothing stage.
   - Constant forward rate bootstrapping may also be used.
4. Smoothing Pass: Here you smooth on the appropriate quantification metric that is deemed to be a better hidden-state characterizer.
5. Advantages of the Shape-Preserving Pass:
   - Separation between Shape-preservation and smoothing.
   - Choice of convenient, yet potentially different metrics across shape-preserving and smoothing.
   - The final state representation quantification metric need not be linear on the manifest measure.
   - The granularity/precision of fit of the curve automatically adjusts with information (i.e., cash flow event dates such as pay dates), thereby making it inherently more precise.
   - PCHIP techniques may be applied more conveniently on the smoothing pass.

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

- Other closeness of fit techniques (such as least squares methodologies etc.) become much more relevant on the smoothing pass.

6. Disadvantages of the Shape-Preserving Pass:
   - Calculation overhead penalty associated with the dual pass (although, by choosing linearity between manifest measure/quantification metric and the quantification metric/ quantification metric combinations this adverse impact maybe reduced).
   - Artifacts produced during shape-preservation (again, there will be artifacts associated with just about any basis representation).

Bear Sterns Multi-Pass Curve Building Techniques

1. DENSE Methodology: This method is outlined in Nahum (2004).
   - Cash/Forwards => Piece-wise constant forwards. Turn Spreads imposed as needed.
   - Swaps => Shape Preserving uniform tension splines.
   - RAW Swaps Inputs => Quarterly swap rates are now re-implied from the curve constructed in the earlier stage.
   - From these new swap quotes, a new curve is constructed using quarterly constant forward rates (constant forward rates methodology is called RAW).
2. DUAL DENSE Methodology: Again, this method is outlined in Nahum (2004).
   - Short end (Cash/Futures) => Daily forwards (i.e., constant daily forwards or cdf) latent state implied.
   - Long End => Same methodology as DENSE, except for the non-uniform tension that is applied across quarterly swap contracts.

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135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

Transition Spline (Or Stitching Spline)

Motivation

1. Spline per Instrument Grouping: Another possibility is to use transition spline to bridge across different instrument groups – this simply needs to adjust to the smoothness/truthness constraints of each of the instrument groups.
    - Essentially, transition splines connect spline families across instrument group (each instrument essentially belongs to its own spline cluster).
2. Design:
    - May use discontinuous Hermite splines in the transition area, or higher order basis (say, with an appropriate C^k constraint), or even an optimizing transition spline.
    - Instrument choice is critical if we are to avoid steep transition slopes (esp. tight group gaps, and steep measure drops). These are challenges in any mechanism, but possibly a lot more here.
    - Construct single instrument spanning spline curves, then demarcate/spec out the instrument range, finally bridge in the transition splines.
    - Transition splines may also be used to stitch in arbitrary instruments together, each belonging to its own separate group, although it is hard to find a practical need for such a construct.
    - In general, instrument group boundaries need not strictly coincide with the instrument termination nodes (esp. in case of stitch-in splines). Boundaries may be inserted using any of the appropriate knot insertion techniques.
3. Advantages:
    - These preserve the curve character embedded in each instrument grouping, which can be a sub-set of a vaster instrument set.
    - By retaining the localization to the corresponding instrument grouping, the hedges produced by the transition spline may, in principle, be better than those produced by the typical ones.

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Tel. 646-513-3300

4. Disadvantages:
    - Of course, by construction, they do not allow for overlapping instrument groups (which, however, may not be a problem in the practical world). This forces a decision on the instrument set choices and boundaries.
    - Technically, the single "natural spline boundary condition" is not applicable across all the unprocessed instrument groups – this is really what is compromised.
        - How much the effectiveness is compromised due to the above may be estimated using targeted metrics, say the span DPE.
5. Transition Segment in the Transition Spline: This needs at least $2k + 2$ basis functions for representation, as it needs to "mate out" the left stretch and the right stretch ($k + k$ for each of the C^k continuity spec - plus 2 more, one at each end to match up the point node).
6. Using Transition Splines for Calibration Instrument Selection: As shown in Figures 2 and 3 below, the transition stretch represented in figure 2 is narrower, and therefore more abrupt/jumpy (with corresponding implications for the forward rates) than that in Figure 3. A criteria based approach is necessary to develop this.

Stretch Modeling Using Transition Splines

1. Information Propagation across Stretches: All the truthness/smoothness information of the predecessor stretch is captured by the stretch's calibrated span parameters. Any state inference for predictors in a given domain needs to be deferred to the domain's span stretch.
    - The corollary to the above is that trailing stretches will typically need information from the leading stretches for state inference/estimation (leading/trailing here are set in regards to the inference flow (or information flow)). Applied to discount curve cooking, the leading stretch that uses cash instruments is essentially self-calibrating, whereas the trailing stretch of swap instruments is going to rely on information that comes out of the cash calibration. Going into swap segments, the information will propagated in the form of RVC's, so they will need to be handled right from the left-most segment of each stretch.

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

- Regular Stretches vs. Finance Curve Stretches => For typical stretch construction, all you need is the transmission of the segment-to-segment continuity constraints through C^k. For segment curve builders, however,

$$Constraint\ (Segment_N) = f(Segment_i, \cdots, Segment_{N-1})$$

  i.e., more construction information in addition to just the C^k is required (mostly via explicit evaluation of arbitrary points in earlier segments' stretches).

2. Response Stretches: Markov response state variables may follow distinct behavior in different predictor stretches. For example, the discount factor/zero rate/swap rate may be characterized using one set of representations for the cash stretch, whereas the swap stretch may use a different set.
3. Why Response Stretches exist: Is it simply because of the instrument choice (cash for the front end, swap for the back end, etc.), or is there a more fundamental driver? Can't say one way or the other, but the fact is we empirically attempt to match point-by-point in a left to right manner (we do this today) without compromising the empirical characteristics of each instrument group. We call each of these groups manifest groups, since they could be result of specific product manifest measures).
4. Manifest Group Contribution to the Response Signal Strength: Say that a signal strength contribution to a specific response signal is proportional to its liquidity (to improve accuracy, you may make it sided liquidity). As you move from left to right in the predictor space, by working in terms of the liquidity-fade of the left stretch to the liquidity-explode of the right stretch, you may be able to characterize the response space more naturally (with less dependence on explicit stitching splines, or on artificially inserted knots).
5. Liquidity-Fade and Liquidity-Explosion in practice: In practice the actual predictor ordinates across the manifest stretches will be too discrete for tracking the liquidity-fade and liquidity-explosion. Thus, it may be more appropriate to operate on predictor windows. If convenient and admissible, the predictor window boundaries may also coincide with the segment boundaries.

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

Stretch Partition/Isolation in Transition Splines

1. Definition: A given calibratable predictor ordinate/response realization space is called a span. The span is partitioned into stretches. Stretches can be either core stretches or transition stretches. Both the core stretches and the transition stretches are built from segments (within which the response values may be represented using basis splines). Core stretch are inferred to truthness and the smoothness signals, and the transition stretches provide the explicit bridge between the core stretches that may not be possible using the plain core stretch representations.
2. Information Patterns: With a higher unit, information propagation is associated with each sub-unit entities below. Across peer units, information exchange is materially similar in nature. Across higher units, information exchange may be more parsimonious (although it may still happen between lower entities belonging to the higher units).
3. Information Localization and Transmission: Intra-segment information propagation occurs through smoothness constraints such as C^k.
4. Stretch-Level Information Localization: In the spline case, this happens though boundary-condition delimitation/isolation (i.e., natural/financial/clamped boundary conditions based isolation is applicable to within a single stretch).
5. Stretch-Stretch Transmission: These are not bound by the equivalent isolation constraints, therefore the connecting/transition splines need to have a qualitatively different nature.
6. Transition/Connecting Splines: By definition, since they are the bridge between the stretches, they need to have greater degrees of freedom for a complete bridge.

Knot Insertion vs. Transition Splines

1. Equivalence: In some sense, they are equivalent in that inserting knots also attempts to complete the bridge. However, transition splines are more customizable, since the splines that flank the knots are assumed in the literature to be variants of the others.

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

2. Advantages on Knot Insertion: Remember that transition splines need $2k + 2$ basis function. Thus, for high k, you are stuck with higher-order polynomials (for e.g.), along with all the Runge's oscillations/instabilities that it brings. Suitable choice of knots may minimize this.
3. Advantage of Transition Spline: Knots are stretch response altering (via their C^k criteria), whereas transition splines enable each stretch to retain their character.

Overlapping Stretches

1. Premise: By definition, stretch fade-out and stretch explode axiomatizations imply predictor ordinate overlapping stretches.
2. Stretch Boundaries: Each stretch constituting an overlapping stretch needs to have its boundaries identified. What ***do*** not necessarily overlap are the smoothness constraints.
3. Overlapping Stretch – Problem Statement:
   - Predictor Ordinate Stretches overlap.
   - Stretches (and by implication, their predicate ranges) are contained/telescoped.
   - Smoothness constraints may not overlap, in which case they are posited to be distinct in each of the constituent stretches.
   - Truthness should be strictly telescopically contained/localized, i.e., there is a ***manifest measurement exclusivity*** to each stretch.
   - A consequence of this is that the inferred state response variable will be propagated, but not (necessarily) the smoothness criterion.

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

Penalizing Exact/Closeness of Fit and Curvature Penalty

Motivation

1. Least Squares Exact Fit vs. Best Fit: Unlike in functional analysis/financial curve construction, in machine learning "exact fit" is treated as a rarity in machine learning, as there is presumed to be an irreducible manifest measure generation error. Here we assume that there are processes the result in "zero manifest measure uncertainties" – in other words, these are "quotes" that are explicitly honored.
2. Basic Setup: As described in the companion Spline Library Documentation, the regularized regression loss/penalizer may be decomposed and worked out as

$$Gross\ Penalizer = Fitness\ Match\ Penalizer + Curvature\ Penalizer$$

$$\mathfrak{R} = \frac{1}{q}\mathfrak{R}_F + \lambda\mathfrak{R}_C$$

$$\mathfrak{R}_F = \sum_{p=0}^{q-1} W_p\left(y_p - Y_p\right)^2$$

$$\mathfrak{R}_C = \int_{x_l}^{x_{l+1}} \left(\frac{\partial^m y}{\partial x^m}\right)^2 dx$$

3. Estimation of λ: While the segment spline coefficients are computed by minimizing $\mathfrak{R}$, λ is often extraneously supplied as a tuner that trades the prefect high degree of fit to the curvature. Tanggaard (1997) suggests using a few methods to estimate λ:
    - Using the GCV criterion as demonstrated by Craven and Wahba (1979) and Wahba (1990).

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

- From the smoothing spline viewpoint, set the number of basis functions, then search for the corresponding λ using the technique listed in Tanggaard (1997).

4. Measurement Filtering vs. Best Fit Weighted Response: These approaches are very similar, in that the Best Fit Weighted Response "steers" the calibrated spline basis and their coefficients to accommodate the measurements in the uncertain sense (potentially by incorporating measurement uncertainty).
    a. If the measurement uncertainty/variance is explicitly known, the Andersen (2005), the Tanggaard (1997), and/or the GCV techniques may be used to extract better estimate for λ - through Andersen RMS γ^2 estimator, Craven/Wahba's GCV, or Tanggaard's trace-based λ estimator.
    b. Differences => However, it needs to be remembered that, for current curve construction methodologies, a key requirement is the γ^2 matches (i.e., exactly reproducing state estimations) – which is not the typical case for the filtered state estimations.
5. Effectiveness of State Representation Quantification Metric: The combination of curvature penalty, the length penalty, and the closeness of fit penalty must be taken together to gauge the effectiveness of the chosen Quantification Metric/Smoothing spline scheme set. Alternatively, full simulations of the manifest metric (with induced noise terms as explained in for e.g., Fisher, Nychka, and Zervos (1994)) and their corresponding evaluations are also appropriate, although they tend to be time consuming (and possibly overkill).

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135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

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135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

Index/Tenor Basis Swaps

Component Layout and Motivation

1. Basis Swap Market: Although Basis Swaps did exist even earlier (Tuckman and Porfirio (2003), Morini (2008)), post-crisis segmentation (attributable, among other things, to the preference towards receiving higher frequency payments) intensified these differentials (Mercurio (2009)).
2. Origins of Basis Swap Existence: In principle, these are expected to represent embedded duration counter-party credit risk. The "good" model should couple embedded credit risk with the sided flow dynamics (i.e., the credit quality of the counter-party that enters into the long/short side of the greater frequency leg, etc.)
3. Float-Float Swap as a Combination of Two Fix-Float Swaps: The proxy of the float-float as two fix-floats would be perfect if both the fixed legs had the same frequency and day count conventions. In terms of the tenor basis swap conventions, pre-2008 the convention was to quote the float-float basis directly – post-2008, however, it was quoted as a combination listed above.
4. The Discounting Curve: Challenges regarding the uniqueness in relation to the instrument choice for building the discount curve have been identified by Henrard (2007). The issues stem primarily from the uncollateralized nature of deposits and forwards, therefore, these are typically replaced by OIS/EONIA and Futures (Madigan (2008)).
    - Interest Rate Swap continues to be used for the discount curve calibration, as it possesses the following characteristics:
        - Par IRS'es are collateralized at inception.
        - Collateral margining may be applied over time.
        - IRS is the only liquidly available fix-float swap, and as such effectively implies just a single forward curve.
    - Convexity adjustment for extracting the rate from future/forward price => Since futures/forwards act effectively as a zero coupon bond, the transformation of price to the latent zero/forward rate requires a dynamical volatility based curve evolution model.

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

Sophisticated, comprehensive approaches are available in literature (see for e.g., Kirikos and Novak (1997), Jackel and Kawai (2005), Brigo and Mercurio (2006), Piterbarg and Renedo (2006)); common practitioner approaches, however, employ simpler approaches such as the Hull-White one-factor short-rate model (Hull and White (1990)).

5. Multi Curve vs. Forward Smoothness: Given that the discount curve and the forward curve are essentially distinct in the multi-curve latent state, the stringent demands that all forwards stay smooth (as in the single discount curve that covers all the basis curve scenarios) may be relaxed.
   - Forwards Implied in the Discount Curve => Since the forwards are used only for the "core" tenor pillars in the discount curve, only those forwards need to be subject to the extra discounting constraints (e.g., 6M forwards). By discount curve construction this will typically be the case, as the forwards period will always straddle/span fully a single reset pillar.
6. Point- vs. Convolved-Measure State Transform:
   - Point-Measure transform refers to the one-to-one transform between a state measure at a predictor ordinate and its corresponding observation (e.g., discount factor from zero-coupon bond price observations). Since these may be expressed as straightforward transformations, the observation-state non-linearity may be easily accommodated.
   - Convolved measure-state transforms introduce what are effectively observation constraints across predictor ordinate/state response combinations. Non-linearity introduces complications, therefore usage of spline-based linearization constraints are highly effective.
7. Reset-Date Forward-Rate Pair Constraint in Discount Curve Building: The yM tenor (e.g., $yM \Longrightarrow 6M$) may be extracted only at the reset start/end date (depending on the reset rate-time axis label) from the discount curve, i.e., only the pair $\left[yM, Forward_{yM}\right]$ makes sense. In other words, this is the only set of dates for which the information on forward rates is available. Splining is an obvious option at the other dates.
   - yM Tenor/DF Relationship =>

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Tel. 646-513-3300

$$PV_{yM} = \sum_{j=1}^{m} \Delta(j-1, j) F_{yM}(t_j) D_f(t_j)$$

For PV_{yM} to be telescoped away into

$$PV_{yM} = D_f(t_0) - D_f(t_i)$$

the requirements are: Period Accrual End Date == Period Reset End Date == Period Pay Date. This is the main reason why the period dates are adjusted before the cash flows are rolled out.

8. Alternative View: Discount Curve IS the yM Forward Curve: To automatically ensure uniqueness and consistency of the latent state space, it may also be more restrictively imposed that the native yM Forward Curve be implied entirely off of the discount curve. Thus, the native yM Forward Curve may now be implied at all nodes, not just at the reset nodes as postulated earlier. This automatically eliminates the state basis between these measures; further, this is still not too restrictive in terms of the native yM Forward Curve smoothness for same reasons as before.
9. Basis between the yM Forward Curve and the Discount Curve: Given that basis constraints are of paramount consideration in other markets, why not look at the basis between discount curve and its native forward curve? This is because neither the latent state underpinning the forward curve or that underpinning the discount curve is entirely observable (unlike, say basis between a bond and the issuer's underlying CDS). Thus an extraneous observation model is necessary. By convention, the current practice achieves this by construction – the formulation mandates that the discount curve and the "discounting-native" forward curve be alternate quantification metrics of the same latent state.

Formulation

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Tel. 646-513-3300

1. Float-Float Swap Setup: The phenomenology and flow details laid out in Figure 5 are based off of descriptions and details provided by ISDA (2006), Ametrano and Bianchetti (2009a), Bianchetti (2012)). The two swap legs are:
   - The "known" or the "Reference" leg. Forwards of this leg come from the discount curve's IRS contracts, and 6M LIBOR/EURIBOR is the most common such tenor. We generalize this with a basis spread, i.e., the "effective" forward is $F_{6M} + S_{6M}$, where F_{6M} and S_{6M} stand for the corresponding forward and the spread.
   - The "unknown" or the "Derived" leg with a tenor of xM. Forwards of this leg are computed from the corresponding basis market quotes. We generalize this with a basis spread, i.e., the "effective" forward is $F_{xM} + S_{xM}$, where F_{xM} and S_{xM} stand for the corresponding forward and the spread.
2. Basic Formulation Setup:

$$PV_{xM} = \sum_{j=1}^{m} \Delta(j-1, j)\left[F_{xM}(t_j) + S_{xM}\right] D_f(t_j)$$

$$PV_{6M} = \sum_{a=1}^{b} \Delta(a-1, a)[F_{6M}(t_a) + S_{6M}] D_f(t_a)$$

   - Equivalence of S_{xM} and S_{6M} => Since both S_{xM} and S_{6M} are additive, we work in a space that is essentially an adjusted forward rate space, with

$$F_{6M,Adj} \rightarrow F_{6M} + S_{6M}$$

   and

$$F_{xM,Adj} \rightarrow F_{xM} + S_{xM}$$

   While this is straightforward to accommodate in the case of $6M$, from a calibration point-of-view, we can work off of a basis $6M$ space, and re-adjust back after splining.

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

3. Basis Swap Calibration Formulation:

$$PV_{xM} = PV_{6M}$$

implies that

$$\sum_{j>m_l}^{m} \Delta(j-1,j) F_{xM,Adj}(t_j) D_f(t_j)$$
$$= \sum_{a=1}^{b} \Delta(a-1,a) F_{6M,Adj}(t_a) D_f(t_a) - \sum_{j=1}^{m_l} \Delta(j-1,j) F_{xM,Adj}(t_j) D_f(t_j)$$

For all but the left most basis swap, $m_l > 0$.

4. Basis Swap Calibration Constraint Specification: Set

$$\aleph_m = \sum_{a=1}^{b} \Delta(a-1,a) F_{6M,Adj}(t_a) D_f(t_a) - \sum_{j=1}^{m_l} \Delta(j-1,j) F_{xM,Adj}(t_j) D_f(t_j)$$

Notice that $\aleph_m$ maybe fully computed from before. Recognize that

$$F_{xM,Adj}(t) = \sum_{i=1}^{n} \beta_i f_i(t)$$

- Combine above to get the calibration constraint

$$\aleph_m = \sum_{i=1}^{n} \beta_i \left\{ \sum_{j>m_l}^{m} \Delta(j-1,j) f_i(t_j) D_f(t_j) \right\}$$

5. Reference/Derived Par Spread Relations: For parity,

$$PV_{Reference} + DV01_{Reference}S_{Reference} + PV_{Derived} + DV01_{Derived}S_{Derived} = 0$$

Setting

$$S_{Derived} = 0$$

$$S_{Reference} = -\frac{PV_{Reference} + PV_{Derived}}{DV01_{Reference}}$$

Likewise

$$S_{Derived} = -\frac{PV_{Reference} + PV_{Derived}}{DV01_{Derived}}$$

Remember that both $S_{Reference}$ and $S_{Derived}$ can be negative.

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135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

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Multi-Stretch Merged Curve Construction

Motivation

1. Discount Curve composed of Forward Rate Stretches: The discount curve span may be viewed as being composed of overlapping/non-overlapping forward rate stretches, i.e., adjacent or otherwise 3M Tenor forward stretch, 6M Tenor forward stretch, etc. This visualization is a consequence of the representation of the "single discount curve latent state", whose alternate/parallel quantification metrics are composed off of those stretches of forward rates that share the latent state space with the global discount curve.
2. Out-of-Native Stretch Arbitrage: If one seeks a forward rate outside these stretches for the given tenor/index combination, there can be no expectations of no-arbitrage, i.e., there will be a basis between the forward implied by this latent space quantification metric and the forward rate under consideration.
    - Likewise, inside the stretch there should be no implied basis, since the diver latent state is identical/fully correlated.
3. Merging/de-merging of the Latent State along the Predictor Ordinates: If you imagine the rates state space being characterized by a set of latent states (which may be highly correlated), each state may ideally be characterized by a quantification metric that is native to the state physical view. Thus, the unification of the sub-states in a stretch may be viewed as state-merging (i.e., one quantification metric may be inferred from another within a merged space via a trivial transformation).
4. Probit-based Latent State Merger Analysis: Given that the discount/forward latent states merge/de-merge, it might it particularly amenable to a common-factor probit (or even a logistic) analysis of the merger driver dynamics. The challenge would then be to link the driver dynamics to the maturity based predictor ordinate.

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Merge Stretch Calibration

1. Cross-Stretch Calibration: Clearly the latent state span characterized by multiple stretches will in turn be composed of latent state merge sub-stretches. The merged stretch may be followed by de-merged stretch, etc.
2. Calibration Challenges:
    a. What would be most optimal cross-representation inside the merge sub-stretch (i.e., the state representation needs to be smooth for both the discount factor latent state as well as the forward curve latent state)?
    b. On the other hand in the solitary segment sub-stretch, you may have more representation freedom, but may still need to carry over the smoothness constraints from the merged sub-stretch. How can this be done? Can the transition spline treatment above be effectively employed here? In other words, what would be appropriate transition zone applicable to the sub-stretch?

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Latent State Manifest Measure Sensitivity

Introduction

1. $\frac{\partial PV_{Swap}}{\partial q_c}$: Remember that the floater leg PV goes as $D_f(t_0) - D_f(t_n)$. Thus, these terms tend to dominate both the PV and the manifest measure sensitivity calculations. The fixed annuity per-coupon-date cash-flow is smaller comparatively, and that is reflected in the Jacobians.
2. Latent State Sensitivities to the Product Segments:
   - Cash/Deposit => Here the sensitivities are to within a single segment, since it is spot starting.
   - Future/FRA => Here the manifest sensitivities are to the two straddling segments, since it is forward starting.
   - IRS => Sensitivities through multiple segment/preceding segments, but concentrated a more on the edges for the reasons seen above.
3. Latent State Sensitivities Signs: The far end is always negative, since the PV decreases with the increase in the manifest measure sensitivity – this is valid across all products. The near end is positive for Future/FRA as well as IRS, as that corresponds to the shorted side. There is no shorted side for cash.
4. Latent State Segment Manifest Measure Sensitivity: It may be appropriate to imagine that, for a given segment, the latent state sensitivity is contributed to only from the current and the prior segment manifest quotes.
   - Justification for the above => As may be observed from Figure 6, sensitivity has to be zero at the start of the current and at the end of the next. "Current" is completely determined through the constraints and the matches corresponding to intra-segment observations, but the "next" dependence propagation is exclusively via C^k transmission constraints, and devoid of targeted segment-specific contributions.
5. Design of Manifest Measure Sensitivity Segment Tail: Since the tail is, technically, a strict fade, any monotonically decreasing-to-zero function will work – the smoother the better.

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6. Preceding Manifest Measure Sensitivity Basis Function: Given the shape in figure 6, a choice of sinusoidal function would serve as appropriate set of basis. Alternately, the same basis as manifest measure sensitivity (which should, strictly speaking, be the same basis used for quantifying the latent state response) may also be used – the head/tail C^k may provide additional customization.
7. Preceding Manifest Measure Sensitivity Fade off/Retain: The two possibilities of the transmission of the preceding manifest measure stem from the differing nature of the current manifest metric. If the current manifest measure is of the "retain" type, the preceding manifest measure sensitivity is retained as is (i.e., uses a flat through transmission of the right edge value). If the current manifest measure demands that the preceding manifest measure sensitivity fade, then the preceding manifest measure sensitivity is faded off/decayed, as seen above.
8. Fade off/retention sequence: Further, if the current is "retain", all the segment manifest measure sensitivities of the earlier segments since the last fade-off are transmitted, and replicated.
9. Preceding Manifest Measure Sensitivity Customization: The fade-off/retain preceding manifest measure sensitivity customization should be applied on a stretch-by-stretch basis:
    - Cash/Deposit Stretch => Use Fade-off
    - Futures/FRA Stretch => Use Retain
    - IRS Stretch => Use Fade-off

Float-Float Manifest Measure Sensitivities

1. Float-Float Reference Leg Sensitivity to the Derived Leg Basis:

$$V_{Ref} = \sum_{i=1}^{n} D_f(t_i)\left[L_{Ref}(t_{i-1}, t_i) + b_{Ref}\right]\Delta_{Ref}(t_{i-1}, t_i) \Longrightarrow \frac{V_{Ref}}{b_{Der}} = 0$$

2. Float-Float Derived Leg Sensitivity to the Derived Leg Basis:

$$\sum_{j=1}^{n} D_f(t_j)[L_{Der}(t_{j-1}, t_j) + b_{Der}]\Delta_{Der}(t_{j-1}, t_j)$$

$$= \sum_{j=1}^{k} D_f(t_j)[L_{Der}(t_{j-1}, t_j) + b_{Der}]\Delta_{Der}(t_{j-1}, t_j)$$

$$+ \sum_{j=k+1}^{n} D_f(t_j)[L_{Der}(t_{j-1}, t_j) + b_{Der}]\Delta_{Der}(t_{j-1}, t_j)$$

where $(k+1, \cdots, n)$ cash flow instances belong to the manifest measure exclusive segment c.

- Closed Form => The non-exclusive (i.e., the earlier segments and stretches) do NOT contribute to the current manifest measure sensitivity. Thus, the sensitivity becomes

$$\sum_{j=k+1}^{n} D_f(t_j)\frac{\partial L_{Der}(t_{j-1}, t_j)}{\partial b_{Der}}\Delta_{Der}(t_{j-1}, t_j) = -\sum_{j=k+1}^{n} D_f(t_j)\Delta_{Der}(t_{j-1}, t_j)$$

  where the right hand side is the manifest measure exclusive segment incremental derived floating leg annuity.

3. Float-Float Derived Basis Sensitivity Transmission Rule: Given that the floating leg sensitivity could potentially be OVERLAPPING, the preceding manifest measure sensitivity choice will be FADE ON, not RETAIN.
4. Tenor Basis Swap Sensitivity:

$$\frac{\partial L_{Der,j}}{\partial b_{Ref}} = -DV01_{Ref}(0, \cdots, m-1)$$

$$\frac{\partial L_{Der,j}}{\partial b_{Der}} = -DV01_{Der}(0, \cdots, m-1)$$

   Their magnitudes must be similar, save for the annuity flow differences.

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6. Multi Leg Basis Sensitivity Points: For each of the constituent legs and their corresponding manifest measure, the symbolic sensitivities need to be computed/splined in. The sensitivity nodes will be the payment dates, along with an additional cross-leg "final upfront'. The current manifest measures are the derived leg basis, the reference leg basis, and the interest rate sensitivity.
7. Multi-Metric Latent State Calibration: As long as the latent state is linearizable among the multiple metrics, such a calibration is possible. Further, a chain sweep multi-metric sensitivity Jacobian is estimable on the calibration pass. Of course, the preceding quote sensitivity control must be customizable on a per-manifest measure basis.

Multi-Reset Floating Period

1. The Setup: This small sections concerns itself with the case where the reset tenor that is different from the floating period tenor. In this situation, there are 2 specific impacts to be considered:
    a. The compounding rule to accumulate the reset periods onto the floating period, and
    b. The associated convexity correction mismatch since the terminal measure numeraire for the floater period pay is different from that of the reset period terminal measure.
2. Convexity Correction vs. Quanto Adjustment: Remember that the forward and the discount latent states form part of the same shared latent state in the case of convexity correction, and the convexity adjustment stems purely from the terminal measure mismatch, as just observed. Quanto adjustment, however, is applied across multiple distinct latent states that are non-merged, e.g., funding vs. collateral vs. forward vs. FX latent states.
3. Origin of the Convexity Correction: In practical settings, the convexity correction occurs only when a) the floater periods encapsulates the multiple reset periods, AND b) these reset periods DO NOT compound geometrically (e.g., the compounding is arithmetic). As examples, these reset periods include overnight fixings applied via a corresponding index, 3M reset vs. 6M floater etc. The latter is the case for certain standard sovereign IRS'es (CAD).

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4. <u>Merged/non-merged Latent State Convexity/Quanto Estimation</u>: Each constituent segment/stretch/regime is still expressed using distinct stochastic (e.g., Brownian) component partitions, in practice the merged state convexity adjustment ends up looking very similar to that of quanto adjustment. Thus, generalizing from the above, the PV of the forward rate $\mathcal{L}(t \to\ t+\tau)$ between t and $t+\tau$ paid at time T in a different currency X looks identical (save for the specific volatilities involved) in both cases (please note the integration time limit differences):

$$E_0^{Q_D^T}\left[\mathcal{L}(t \to\ t+\tau)D_f(T)X(T)\right] =$$

$$\mathcal{L}(0; t \to\ t+\tau)D_f(0;T)X(0;T) \times$$

$$e^{\left\{\int_0^t \rho_{D\mathcal{L}}(s)\sigma_D(s)\sigma_{\mathcal{L}}(s)ds + \int_0^T \rho_{DX}(s)\sigma_D(s)\sigma_X(s)ds + \int_0^t \rho_{\mathcal{L}X}(s)\sigma_{\mathcal{L}}(s)\sigma_X(s)ds\right\}}$$

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OIS Valuation and Curve Construction

Base Framework and Environment Setup

1. The OIS Model: Given the compounded overnight rate $R_{ON}(T_{i-1}, T_i)$, the par OIS Rate $R_{ON}^{OIS}(t, T, S)$ is given as

$$R_{ON}^{OIS}(t, T, S) = \frac{\sum_{i=1}^{n} D_f(t, T_i) R_{ON}(T_{i-1}, T_i) \tau_{ON}(T_{i-1}, T_i)}{Fixed\ Leg\ DV01}$$

As expected, given that this corresponds to the par OIS, this telescopes to

$$R_{ON}^{OIS}(t, T, S) = \frac{D_f(t, T_0) - D_f(t, T_n)}{Fixed\ Leg\ DV01}$$

The compounded rate $R_{ON}(T_{i-1}, T_i)$ is computed from the daily overnight fixes as (Mercurio (2011))

$$R_{ON}(T_{i-1}, T_i) = \frac{1}{\tau_{ON}(T_{i-1}, T_i)} \prod_{k=1}^{n_i} [\{1 + R_{ON}(T_{i,k-1}, T_{i,k}) \tau_{ON}(T_{i,k-1}, T_{i,k})\} - 1]$$

2. Stringency on the OIS Spline Construction: Since the OIS has shown itself to dip into the negative territory (Whitall (2010), Cameron (2011), Atkins and Jones (2012), Carver (2012), Lipman (2012)), the corresponding demands on the shape preserving splines need to be accommodative.

OIS Valuation Extensions and Approximations

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1. OIS Extensions Using Fed Fund Basis Quotes: For some jurisdictions (say, USD), the OIS quotes are not widely available beyond the 10Y tenor. Therefore the OIS discount curve is constructed using the USD LIBOR – Fed Fund Basis Swap Quotes that trade till the 30Y tenor. Since both the OIS and the Fed Funds Basis Swap Quotes are projected from the forwards of the Fed Funds Effective Rate FEDL01, no arbitrage arguments may be used to extract the OIS Curve (Bloomberg (2011a), Bloomberg (2012), Bloomberg (2013)).
2. USD OIS Curve Construction: As discussed partly in White (2012a), the USD OIS curve construction can occur in the same manner as that done for building discount curves from CCBS and IRS quotes – i.e., the OIS long end OIS curve can be constructed from fed funds-LIBOR basis swap (a float-float swap) and IRS.
3. Fed Fund OIS Basis Swap:

$$PV_{Fed\ Fund\ Leg} = \sum_{i=0}^{n_{FF}-1} \Delta_{FF}(t_i, t_{i+1}) \left[\mathcal{L}_{FF}(t_i, t_{i+1}) + b_{FF}\right] D_f(t_{i+1})$$

$$PV_{LIBOR\ Leg} = \sum_{j=0}^{n_L-1} \Delta_L(t_j, t_{j+1}) \left[\mathcal{L}_L(t_j, t_{j+1}) + b_L\right] D_f(t_{j+1})$$

$$\begin{aligned} PV_{Fed\ Fund\ LIBOR} &= PV_{Fed\ Fund\ Leg} + S_L PV_{LIBOR\ Leg} \\ &= \sum_{i=0}^{n_{FF}-1} \Delta_{FF}(t_i, t_{i+1}) \left[\mathcal{L}_{FF}(t_i, t_{i+1}) + b_{FF}\right] D_f(t_{i+1}) \\ &+ S_L \sum_{j=0}^{n_L-1} \Delta_L(t_j, t_{j+1}) \left[\mathcal{L}_L(t_j, t_{j+1}) + b_L\right] D_f(t_{j+1}) \end{aligned}$$

   a. Basis Free LIBOR Leg PV: Setting

$$PV_{Fed\ Fund\ LIBOR} = 0$$

   we get

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$$\sum_{j=0}^{n_L-1} \Delta_L(t_j, t_{j+1}) \left[\mathcal{L}_L(t_j, t_{j+1})\right] D_f(t_{j+1})$$

$$= -\frac{1}{S_L}\left\{\sum_{i=0}^{n_{FF}-1} \Delta_{FF}(t_i, t_{i+1}) \left[\mathcal{L}_{FF}(t_i, t_{i+1}) + b_{FF}\right] D_f(t_{i+1})\right\}$$

$$- b_L \sum_{j=0}^{n_L-1} \Delta_L(t_j, t_{j+1}) D_f(t_{j+1})$$

Working from an IRS point-of-view, by matching the fixed and the floating legs we get

$$\sum_{j=0}^{n_L-1} \Delta_L(t_j, t_{j+1}) \left[\mathcal{L}_L(t_j, t_{j+1})\right] D_f(t_{j+1}) = -\frac{s}{S_L} \sum_{k=0}^{m-1} \Delta(t_k, t_{k+1}) D_f(t_{k+1})$$

Here S_L stands for the notional sign (typical convention is to assume -1 for the floater side), and s stands for the swap rate.

4. Consolidated Discount Curve Builder Relation:

$$\sum_{i=0}^{n_{FF}-1} \Delta_{FF}(t_i, t_{i+1}) \left[\mathcal{L}_{FF}(t_i, t_{i+1}) + b_{FF}\right] D_f(t_{i+1}) + S_L b_L \sum_{j=0}^{n_L-1} \Delta_L(t_j, t_{j+1}) D_f(t_{j+1})$$

$$= s \sum_{k=0}^{m-1} \Delta(t_k, t_{k+1}) D_f(t_{k+1})$$

5. Curve/Quote Dependence for the Cross DC Builder: Remember

$$\sum_{i=0}^{n_{FF}-1} \Delta_{FF}(t_i, t_{i+1}) \mathcal{L}_{FF}(t_i, t_{i+1}) D_f(t_{i+1}) = D_f(t_0) - D_f(t_{i+1})$$

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Tel. 646-513-3300

This clearly shows that there is no explicit market curve dependence for building the OIS curve – the only quote dependences are on b_{FF}/b_L (one of them, typically b_L, is zero) and s. What the equation provides is to create a sequence of linear constraints over $\{D_f(t_i)\}$, $\{D_f(t_j)\}$, and $\{D_f(t_k)\}$.

6. Bloomberg (2013) Approximation for the OIS Curve Rate, given LIBOR Level: If, say, the 10Y LIBOR is known, and so is the FF-LIBOR basis, Bloomberg (2013) approximates the OIS rate as

$$\hat{s}_N{}^{OIS} = \left[1 + \frac{r_Q - b_N{}^{FF}}{4}\right]^4 - 1$$

where $b_N{}^{FF}$ is the Fed Funds OIS Basis, $\hat{s}_N{}^{OIS}$ is the corresponding 10Y LIBOR, and

$$r_Q = \left\{\left(1 + \frac{s_N{}^{LIBOR} \times \frac{360}{365}}{2}\right)^{\frac{2}{4}} - 1\right\} \times 4$$

7. Bloomberg (2013) Enhanced Approximation: The approximation above is in place because the daily discrete compounding applied over specific holidays, weekends etc. becomes very expensive to compute. Therefore Bloomberg (2013) introduces an additional accuracy enhancement to compensate for the daily compounding of the FF using a flat curve to get $\hat{s}_N{}^{OIS,ENH}$ as:

$$\hat{s}_N{}^{OIS,ENH} = \hat{s}_N{}^{OIS} + \left[\left(1 + \frac{\hat{s}_N{}^{OIS}}{360}\right)^{90} - 1 - \frac{\hat{s}_N{}^{OIS}}{4}\right] \times 4 = \left[\left(1 + \frac{\hat{s}_N{}^{OIS}}{360}\right)^{90} - 1\right] \times 4$$

OIS FX-Basis Swap Valuation and Approximations

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1. OIS-FX-Basis Swap Definition: Consider 2 OIS floating streams of the corresponding currencies CCY1 and CCY2 respectively. This package of both the legs together is called the OIS FX-Basis Cross-Currency Swap. Further, assume that this is collateralized in currency #1.
2. OIS-FX-Basis Swap Valuation: This valuation is straightforward.

$$PV_{1,N} = \sum_{i=1}^{N} {f_{1,i}}^{OIS} \Delta(t_{i-1}, t_i) {D_{1,f}}^{OIS}(t_i) + {D_{1,f}}^{OIS}(t_N)$$

   and

$$PV_{1,N} = \gamma(0)\left[\sum_{j=1}^{M} \left\{{f_{2,j}}^{OIS} \Delta\left(t_{j-1}, t_j\right) + {b_{2,N}}^{OIS,FX}\right\} {D_{2,f}}^{OIS}\left(t_j\right) + {D_{2,f}}^{OIS}(t_M)\right]$$

   where $\gamma(0)$ is the appropriate starting FX Rate.
3. LIBOR FX-Basis Swap: This is identical to the OIS-FX-Basis Swap, except that the floating leg now pays LIBOR floating plus a basis. The valuation is done precisely as in the case of the OIS-FX-Basis Swap, with the OIS part replaced by the LIBOR part.
4. Approximating the OIS FX Basis using the LIBOR FX Basis: Since LIBOR FX-Basis is more widely traded than OIS-FX Basis, Bloomberg (2011b) claim to have developed an approximation for the OIS-FX-Basis Spread in terms of the LIBOR-FX-Basis Spread that is intuitive, simple, easy to use, and very accurate (they demonstrate this using comparative reconciliations):

$${b_{2,N}}^{OIS,FX} = {b_{2,N}}^{LIBOR,FX} - \left[{s_{1,N}}^{LIBOR} - {s_{1,N}}^{OIS}\right] + \left[{s_{2,N}}^{LIBOR} - {s_{2,N}}^{OIS}\right]$$

Arithmetic Accrual Convexity Correction

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1. <u>One Floater Unit Paid out at the non-terminal Time</u>: As shown in Figure 7, this corresponds to the classic change of measure paradigm. The payout time is T, and the forward period is $(t, t+\tau)$. The PV of the accrual unit becomes

$$PV = \langle L_\tau(t) D_f(T) \rangle = E_0^{Q_D^T}\left[L_\tau(t) D_f(T)\right]$$

2. <u>Equivalent Martingale Forward Valuation</u>:

$$\langle L_\tau(t) D_f(T) \rangle = D_f(0,T) E_0^{Q_D^T}\left[{L_\tau}'(t) {D_f}'(T)\right]$$

where ${D_f}'(T)$ is simply the de-drifted martingale devoid of the drift $D_f(0,T)$, and where we assume (by the fundamental theorem) that $L_\tau(t)$ is a martingale itself. Setting up the dynamics for $L_\tau(t)$ and ${D_f}'(T)$ as

$$\Delta L_\tau = \sigma_L L_\tau \Delta W_L$$

and

$$\Delta D_f' = \sigma_D D_f' \Delta W_D$$

along with their joint moves

$$\Delta W_D \Delta W_L = \rho_{DL} \sigma_D \sigma_L \Delta t$$

we get

$$\langle L_\tau(t) D_f(T) \rangle = L_\tau(0,t) D_f(0,T) e^{\int_0^t \rho_{DL}(s) \sigma_D(s) \sigma_L(s) ds}$$

where $L_\tau(0,t)$ and $D_f(0,T)$ stand for today's expectations of $L_\tau(t)$ and $D_f(T)$ respectively.

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3. Convexity Adjusted Accrual PV: The model above for $L_\tau(t)$ and $D_f(T)$ is pretty generic – but deterministic (i.e., non-local). This may now be applied to assess the convexity adjustment to be used for each of the daily payments on the overnight fund index for the period j as

$$PV_j = D_f(T_j)\sum_i L_{ON}(t_{j,i})\Delta(t_{j,i}, t_{j,i+1})\, e^{\int_0^{t_{j,i}} \rho_{ON,D}(t)\sigma_D(t)\sigma_{ON}(s)dt}$$

Composed Period Latent State Loadings

1. Composed Period Compounding – Arithmetic: In literature, the notional of arithmetic/geometric accruals is often spelt out in terms of rates averaging, i.e., arithmetic vs. geometric averaging over the composable rates periods. The arithmetic accrual over the composable periods (with the index i running over the composable periods) is given as

$$\left[\sum_{i=1}^{n} L_i(t_i - t_{i-1})\,\vartheta_i(\tau_n)\right] D_f(\tau_n)S_P(\tau_n)X(\tau_n)$$

thereby clearly inducing the convexity adjustment $\vartheta_i(\tau_n)$ across each composable period.

2. Composed Period Compounding - Geometric: Here the period PV becomes

$$\left\{\prod_{i=1}^{n}[1 + \ L_i(t_i - t_{i-1})] - 1\right\} D_f(\tau_n)S_P(\tau_n)X(\tau_n)$$

Using the fact that

$$L_i(t_i - t_{i-1}) = \frac{\widetilde{D}_f(t_{i-1})}{\widetilde{D}_f(t_i)} - 1$$

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where $\widetilde{D}_f$ refers to the discount factor quantification metric of the corresponding forward rate L, we reduce the above to

$$\prod_{i=1}^{n}[1 + L_i(t_i - t_{i-1})] = \frac{\widetilde{D}_f(t_{i-1})}{\widetilde{D}_f(t_i)}$$

Thus, this induces a single convexity correction, at the end of the composite period.

3. Composite Period - Accruals: The difference between the above analysis and the one for accruals is that, in the case of accruals, all the unit periods' rates preceding the accrual valuation date have been realized. This makes the analysis more straightforward. Given that the fixings have been realized, we have ONLY ONE convexity adjustment across all the realized periods for both arithmetic and geometric compounding – the one at the period terminal anchor.
4. Merged Forward/Funding State Loading Under Arithmetic Compounding: Using the discount factor quantification metric above for the case of arithmetic compounding above, we get

$$\left[\sum_{i=1}^{n} L_i(t_i - t_{i-1})\,\vartheta_i(\tau_n)\right] D_f(\tau_n)S_P(\tau_n)X(\tau_n)$$
$$= \left[\sum_{i=1}^{n} \frac{\widetilde{D}_f(t_{i-1}) - \widetilde{D}_f(t_i)}{\widetilde{D}_f(t_i)}\vartheta_i(\tau_n)\right] D_f(\tau_n)S_P(\tau_n)X(\tau_n)$$

This demonstrates that, since there will be a mismatch between the terminal measures at t_i and τ_n, the telescoping will not occur in this case. Thus the merged forward/funding calibration trial becomes a non-linear exercise, thereby not making itself amenable to much of the linear scenarios seen above.
5. Separated Forward and Funding State Loadings under Arithmetic Compounding: Since the expression

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$$\left[\sum_{i=1}^{n} L_i(t_i - t_{i-1})\,\vartheta_i(\tau_n)\right] D_f(\tau_n) S_P(\tau_n) X(\tau_n)$$

is linear separately in both L_i and D_f (as well as S_P and X), linear state loadings should absolutely be generatable for those states, the basis spline representations described above are applicable.

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Spline-Based Credit Curve Calibration

1. <u>Overview</u>: Andersen (2003) has made an initial effort in this regard.

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Section IV: Multi-Curve Construction and Product Valuation

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Correlated Multi-Curve Build-out

Introduction

1. Regime Segmentation: Indicators of regime changes in the interest rate markets, e.g., divergence between XIBOR-based deposits vs. OIS/EONIA, FRA's vs. forwards implied by consecutive deposits etc. are discussed in Ametrano and Bianchetti (2009b), Goldman Sachs (2009), and Bianchetti (2012) among others.
2. Pre-Crisis Segmentation: Segmentation was already present and well-understood pre-2008, but ignore since the effects were small (Fruchard, Zammouri, and Willems (1995), Tuckman and Porfirio (2003)).
3. Prior Multi-Curve Frameworks: The cross-currency swap multi-curve framework was proposed by Boenkost and Schmidt (2005), and was extended to the 3-curve case (i.e., the discount curve, the LIBOR curve, and the bond rates instance) by Kijima, Tanaka, and Wong (2008).
    - Other Two-Curve Extensions =>
        - Morini (2008) and Morini (2009) approach this from the point-of-view of counter-party risk
        - Mercurio (2009) approaches this in terms of an extended LMM
        - Henrard (2009) approaches this using a more foundational axiomatization framework setup

Standard FRA Setup

1. Standard FRA Setup Basis:

$$F_x(t;\ T_1, T_2) = \frac{1}{\tau_x(T_1, T_2)}\left[\frac{1}{P_x(t;\ T_1, T_2)} - 1\right]$$

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Tel. 646-513-3300

for tenor x.

$$P_x(t;\ T_1, T_2) = \frac{P_x(t;\ T_2)}{P_x(t;\ T_1)}$$

is the discount factor for tenor x. From Brigo and Mercurio (2006),

$$FRA_x(T_2;\ T_1, T_2, K, N) = N\tau_x(T_1, T_2)[L_x(T_1, T_2) - K]$$

where

$$L_x(T_1, T_2) = \frac{1 - P_x(T_1, T_2)}{\tau_x(T_1, T_2)P_x(T_1, T_2)}$$

i.e., $L_x(T_1, T_2)$ is the T_1-spot LIBOR maturing at

$$T_2 = T_1 + xM$$

2. <u>LIBOR-Standard FRA Specification</u>:

$$\begin{aligned} &FRA_x(t;\ T_1, T_2, K, N) \\ &= P_x(t, T_2)\mathbb{E}_t^{Q_x^{T_2}}[FRA_x(T_2;\ T_1, T_2, K, N)] \\ &= N\tau_x(T_1, T_2)P_x(t, T_2)\mathbb{E}_t^{Q_x^{T_2}}[L_x(T_1, T_2) - K] \\ &= N\tau_x(T_1, T_2)P_x(t, T_2)\mathbb{E}_t^{Q_x^{T_2}}[F_x(t;\ T_1, T_2) - K] \end{aligned}$$

where

$$F_x(t;\ T_1, T_2) = \mathbb{E}_t^{Q_x^{T_2}}[L_x(T_1, T_2)]$$

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Tel. 646-513-3300

is the corresponding Standard FRA rate, $\mathbb{E}_t^{Q_x^{T_2}}$ denotes the expectation taken at time t with respect to the measure Q_t (within the filtration $\mathcal{F}_t$ that encodes the market information available at t) over the $\mathbb{E}_t^{Q_x^{T_2}}$-forward measure corresponding to the numeraire $P_x(t, T_2)$.

3. Multiplicative Standard FRA Basis:

$$\mathcal{N}(t;\ T_1, T_2) = \frac{F_f(t;\ T_1, T_2)}{F_d(t;\ T_1, T_2)} \frac{\tau_f(T_1, T_2)}{\tau_d(T_1, T_2)} = \frac{P_f(T_1, T_2)}{P_d(T_1, T_2)} \left[\frac{P_f(t, T_2) - P_f(t, T_1)}{P_d(t, T_2) - P_d(t, T_1)}\right]$$

which simply results in

$$P_f(t;\ T_1, T_2) = \frac{1}{1 + F_d(t;\ T_1, T_2)\mathcal{N}(t;\ T_1, T_2)\tau_d(T_1, T_2)}$$

4. Additive Standard FRA Basis:

$$P_f(t;\ T_1, T_2) = \frac{1}{1 + [F_d(t;\ T_1, T_2) + \mathcal{N}'(t;\ T_1, T_2)]\tau_d(T_1, T_2)}$$

implies

$$\mathcal{N}'(t;\ T_1, T_2) = F_d(t;\ T_1, T_2)[\mathcal{N}(t;\ T_1, T_2) - 1]$$

5. Forward Basis Bootstrapping Relations: These come from Bianchetti (2012)

$$P_{d,i} = \frac{P_{f,i}\mathcal{N}_i}{P_{f,i-1} - P_{f,i} + P_{f,i}\mathcal{N}_i} P_{d,i-1} = \frac{P_{f,i}\mathcal{N}_i}{P_{f,i-1} - P_{f,i}\mathcal{N}_i{}'\tau_{d,i}} P_{d,i-1}$$

$$P_{f,i} = \frac{P_{d,i}}{P_{d,i} + \left(P_{d,i-1} + P_{d,i}\right)\mathcal{N}_i} P_{f,i-1} = \frac{P_{d,i}\mathcal{N}_i}{P_{d,i-1} + P_{d,i}\mathcal{N}_i{}'\tau_{d,i}} P_{f,i-1}$$

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Tel. 646-513-3300

6. Quanto-Adjusted Standard FRA Evolution Dynamics: The forward rates are martingales in their own measure ${Q_t}^f$:

$$\frac{\Delta F(t;\ T_1, T_2)}{F(t;\ T_1, T_2)} = \sigma_F(t)\Delta {W_F}^{T_2} : t \leq T_1$$

Likewise, we employ an analogy with the quanto-based approaches used in the FX world to derive the dynamics of a multiplicative quanto-adjustment X_{fd} as

$$\frac{\Delta X_{fd}(t;\ T_1, T_2)}{X_{fd}(t;\ T_1, T_2)} = \sigma_F(t)\Delta {W_X}^{T_2} : t \leq T_2$$

with

$$\Delta {W_F}^{T_2}\Delta {W_X}^{T_2} = \rho_{FX}\Delta t$$

- FX Quanto Analogy Application => The Standard FRA Payoff at T_2 is $F(t;\ T_1, T_2)$. The payoff over the "local/domestic" currency using the domestic numeraire is

$$\frac{F(t;\ T_1, T_2)}{X_{fd}(t;\ T_1, T_2)}$$

  This quantity, by using the FX quanto analogy, is a martingale in the domestic/discounting measure.
- Quanto Drift Adjustment => This sets us up for the application of the change of numeraire that produces an additional drift, i.e.,

$$\mathbb{E}_t^{{Q_d}^{T_2}}\left[\frac{F(t;\ T_1, T_2)}{X_{fd}(t;\ T_1, T_2)}\right] \approx -\rho_{FX}\sigma_F\sigma_X$$

  i.e.,

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Tel. 646-513-3300

$$\mathbb{E}_t^{Q_d^{T_2}}\left[\frac{F_f(t;\ T_1,T_2)}{X_{fd}(t;\ T_1,T_2)}\right] = F_f(t;\ T_1,T_2)QA_{fd}(t;\ T_1,\sigma_F,\sigma_X,\rho_{FX}) = e^{-\int_t^{T_1}\rho_{FX}(u)\sigma_F(u)\sigma_X(u)du}$$

where $QA_{fd}(t;\ T_1,\sigma_F,\sigma_X,\rho_{FX})$ is the multiplicative quanto adjustment (Jamshidian (1989), Geman, El Karoui, and Rochet (1995), Brigo and Mercurio (2006)).

- Additive Quanto Adjustment => Define this as Bianchetti (2012) does:

$$\begin{aligned} QA'_{fd}(t;\ T_1,\sigma_F,\sigma_X,\rho_{FX}) &= \mathbb{E}_t^{Q_d^{T_2}}\left[F_f(T_1;\ T_1,T_2)\right] - F_f(t;\ T_1,T_2) \\ &= F_f(t;\ T_1,T_2)\left[QA_{fd}(t;\ T_1,\sigma_F,\sigma_X,\rho_{FX}) - 1\right] \end{aligned}$$

- Additive/Multiplicative Basis Adjustment =>
  - Multiplicative Basis Adjustment $BA_{fd}(t;\ T_1,T_2)$ is given as:

$$BA_{fd}(t;\ T_1,T_2)QA_{fd}(t;\ T_1,\sigma_F,\sigma_X,\rho_{FX}) = \frac{\tau_f(T_1,T_2)}{\tau_d(T_1,T_2)}\frac{\mathbb{E}_t^{Q_d^{T_2}}\left[L_f(T_1,T_2)\right]}{\mathbb{E}_t^{Q_d^{T_2}}\left[L_d(T_1,T_2)\right]}$$

  - Additive Basis Adjustment $BA'_{fd}(t;\ T_1,T_2)$ is given as:

$$\begin{aligned} &BA'_{fd}(t;\ T_1,T_2) + QA'_{fd}(t;\ T_1,\sigma_F,\sigma_X,\rho_{FX}) \\ &\qquad = \tau_f(T_1,T_2)\mathbb{E}_t^{Q_d^{T_2}}\left[L_f(T_1,T_2)\right] - \tau_d(T_1,T_2)\mathbb{E}_t^{Q_d^{T_2}}\left[L_d(T_1,T_2)\right] \end{aligned}$$

- Mean-reverting Deterministic Volatility Form => This is outlined in Andersen and Piterbarg (2010), and it possesses distinctive properties that enable it to capture certain kinds of physics:

$$\sigma(t,T) = \frac{\alpha(t)}{a}\left[1 - e^{-a(T-t)}\right]$$

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In his collateral choice option calculation, Piterbarg (2012) uses $\alpha(t) = 0.50\%$ and $a = 40\%$.

7. Standard FRA Price:

$$FRA_d(t;\ T_1, T_2, K, N) = NP_d(t, T_2)\tau_f(T_1, T_2)\left\{\mathbb{E}_t^{Q_d^{T_2}}\left[F_f(T_1;\ T_1, T_2)\right] - K\right\}$$

$$FRA_d(t;\ T_1, T_2, K, N) = NP_d(t, T_2)\tau_f(T_1, T_2)\{QA_{fd}(t;\ T_1, \sigma_F, \sigma_X, \rho_{FX}) - K\}$$

8. Standard FRA Quanto-Adjusted Par Swap Rate:

$$S_f(T, S) = \frac{\sum_{i=1}^{n} P_d(t, T_i)\tau_f(T_{i-1}, T_i)F_f(t;\ T_{i-1}, T_i)QA_{fd}\left(t;\ T_{i-1}, \sigma_{F,i}, \sigma_{X,i}, \rho_{FX,i}\right)}{\sum_{j=1}^{m} P_d\left(t, S_j\right)\tau_d\left(S_{j-1}, S_j\right)}$$

9. Applicability of the Quanto Adjustment Formulas: It is important to remember that these quanto adjustments above are primarily for textbook/standard FRAs. Further, for these FRA's, the observed prices/rates can be worked out explicitly in the discounting measure in itself anyway, thus rendering the quanto correlation effects irrelevant. Obviously, both the discounting and the native FRA measure converge at the FRA exercise date due to the fixing.

Standard FRA Options

1. Caplet/Floorlet Options: Caplet/Floorlet Options on a T_1-spot rate exercised date with the payoff maturity at T_2 is given by (Bianchetti (2012), Bianchetti and Carlicchi (2012))

$$cf(T_2;\ T_1, T_2, K, w, N) = N\tau_f(T_1, T_2)\max\left\{w\left(L_f(T_1, T_2) - K\right)\right\}$$

Thus,

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$$cf(T_2;\ T_1, T_2, K, w, N) = N\tau_f(T_1, T_2)\mathbb{E}_t^{Q_d^{T_2}}\left[\max\left\{w\left(L_f(T_1, T_2) - K\right)\right\}\right]$$

- Closed Form Expression =>

$$cf(T_2;\ T_1, T_2, K, w, N)$$
$$= N\tau_f(T_1, T_2)Black\left(F_f(t;\ T_1, T_2)QA_{fd}(t;\ T_1, \sigma_F, \sigma_X, \rho_{FX}), K, \mu_f, \sigma_f, w\right)$$

where

$$Black\left(F, K, \mu_f, \sigma_f, w\right) = w[F\Phi(wd_+) - K\Phi(wd_-)]$$

with

$$d_+ = \frac{\log\frac{F}{K} + \mu(t,T) \pm \frac{1}{2}\sigma^2(t,T)}{\sigma(t,T)};\ w = \begin{cases} +1 & Caplet \\ -1 & Floorlet \end{cases}$$

$$\mu(t,T) = \int_t^T \mu(u)du = -\int_t^T \sigma_f(u)\sigma_X(u)\rho_{fX}(u)du$$

$$\sigma^2(t,T) = \int_t^T \sigma^2(u)du = \int_t^T {\sigma_f}^2(u)du$$

- Cap/Floor Option Prices =>

$$CF(t;\ T, K, w, N) = \sum_{i=1}^{n} cf(t;\ T_{i-1}, T_i, K_i, w_i, N_i)$$

Plug in the earlier developed relation for the cap/floor prices.

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Tel. 646-513-3300

- SABR/LIBOR Cap Volatility Functional Form => The claim is that the industry uses the following humped function for capturing the cap volatility:

$$\sigma(t) = [a + b(T - t)]e^{-c(T-t)} + d$$

Rebonato, McKay, and White (2009) assign physical meanings to a, b, c, and d, as well as how to calibrate this model using caplet volatilities.

No Arbitrage and Counter-party Risk Based Standard FRA Formulation

1. Setup: Following Mercurio (2009), we set

$$P_f(t, T) = P_d(t, T)R\left(t;\ t, T, R_f\right)$$

where

$$R\left(t;\ t, T, R_f\right) = R_f + \left(1 - R_f\right)\mathbb{E}_t^{Q_d}[q_d(T_1, T_2)]$$

given R_f is the recovery rate, and

$$q_d(T_1, T_2) = \mathbb{E}_t^{Q_d}\left[\mathrm{I}_{\tau(t)>T}\right]$$

is the counter-party default probability associated with the default time $\tau(t)$.

2. Counter-party risk based Risky XIBOR and Standard Forward:

$$L_f(T_1, T_2) = \frac{1}{\tau_f(T_1, T_2)}\left[\frac{1}{P_f(T_1, T_2)} - 1\right] = \frac{1}{\tau_f(T_1, T_2)}\left[\frac{1}{P_d(T_1, T_2)}\frac{1}{R\left(T_1;\ T_1, T_2, R_f\right)} - 1\right]$$

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

$$F_f(t;\ T_1,T_2) = \frac{1}{\tau_f(T_1,T_2)}\left[\frac{P_f(t,T_1)}{P_f(t,T_2)} - 1\right] = \frac{1}{\tau_f(T_1,T_2)}\left[\frac{P_d(t,T_1)}{P_d(t,T_2)}\frac{R\left(t;\ t,T_1,R_f\right)}{R\left(t;\ t,T_2,R_f\right)} - 1\right]$$

The corresponding Standard FRA price is

$$F_f(t;\ T_1,T_2) = \frac{P_d(t,T_1)}{R\left(t;\ T_1,T_2,R_f\right)} - P_d(t,T_2)\left[1 + K\tau_f(T_1,T_2)\right]$$

3. Counter-party risk Quanto Adjustment:

$$QA_{fd}(t;\ T_1,T_2) = \frac{P_d(t,T_1)\dfrac{1}{R\left(t;\ T_1,T_2,R_f\right)} - P_d(t,T_2)}{P_d(t,T_1)\dfrac{R\left(t;\ t,T_1,R_f\right)}{R\left(t;\ t,T_2,R_f\right)} - P_d(t,T_2)}$$

$$QA'_{fd}(t;\ T_1,T_2) = \frac{1}{\tau_f(T_1,T_2)}\frac{P_d(t,T_1)}{P_d(t,T_2)}\left[\frac{1}{R\left(t;\ T_1,T_2,R_f\right)} - \frac{R\left(t;\ t,T_1,R_f\right)}{R\left(t;\ t,T_2,R_f\right)}\right]$$

Morini (2009) expresses the counter party risk spot exchange rate in terms of the credit variables.

4. Counter-party risk Basis Adjustment:

$$BA_{fd}(t;\ T_1,T_2) = \frac{P_d(t,T_1)R\left(t;\ t,T_1,R_f\right) - P_d(t,T_2)R\left(t;\ t,T_2,R_f\right)}{\left[P_d(t,T_1) - P_d(t,T_2)\right]R\left(t;\ t,T_2,R_f\right)}$$

$$BA'_{fd}(t;\ T_1,T_2) = \frac{1}{\tau_d(T_1,T_2)}\frac{P_d(t,T_1)}{P_d(t,T_2)}\left[\frac{R\left(t;\ t,T_1,R_f\right)}{R\left(t;\ t,T_2,R_f\right)} - 1\right]$$

Market FRA Setup

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Tel. 646-513-3300

1. Standard FRA vs. Market FRA: Using the time T payoff, the standard FRA value is

$$FRA_{STD}\left(t,T,L_{x,i},K\right)=\tau_L(T_{i-1},T_i)P_C(t,T_i)\left[F_{x,i}(t)-K\right]$$

The payoff at time T for the market FRA is

$$FRA_{MKT}(T_{i-1},T_i,K)=\frac{[L_x(T_{i-1},T_i)-K]\tau_L(T_{i-1},T_i)}{1+L_x(T_{i-1},T_i)\tau_L(T_{i-1},T_i)}$$

2. Pricing of the Market FRA: We employ the formulation presented in Mercurio (2010), simplifying the notation a little. Setting

$$\tau_L(T_{i-1},T_i)\to\tau_{1,2}$$

and

$$L(T_{i-1},T_i)\to L(T_1,T_2)$$

we get

$$FRA_{MKT}(T_1;\ T_1,T_2,K)=\frac{[L(T_1,T_2)-K]\tau_{1,2}}{1+L(T_1,T_2)\tau_{1,2}}$$

   - Reduced discounting measure representation for Market FRA => Unfortunately, given the above payoff definition for the market FRA's, further discounting measure representations are not possible.

3. The Par FRA: Since FRA's are fully collateralized, we work assuming that overnight rate is the collateral rate. The par FRA rate at time t is the K above that results in the net value of zero, i.e.,

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Tel. 646-513-3300

$$\mathbb{E}_t^{Q_C^{T_1}}\left[\frac{[L(T_1,T_2)-FRA_{PAR}(t;\ T_1,T_2)]\tau_{1,2}}{1+L(T_1,T_2)\tau_{1,2}}|\mathcal{F}_t\right]=0$$

or

$$\mathbb{E}_t^{Q_C^{T_1}}\left[1-\frac{FRA_{PAR}(t;\ T_1,T_2)\tau_{1,2}}{1+L(T_1,T_2)\tau_{1,2}}|\mathcal{F}_t\right]=0$$

Thus

$$FRA_{PAR}(t;\ T_1,T_2)=\frac{1}{\tau_{1,2}\mathbb{E}_t^{Q_C^{T_1}}\left[\frac{1}{1+L(T_1,T_2)\tau_{1,2}}|\mathcal{F}_t\right]}-\frac{1}{\tau_{1,2}}$$

4. <u>Measure Change from $\mathbb{E}_t^{Q_C^{T_1}}$ to $\mathbb{E}_t^{Q_C^{T_2}}$</u>: If the terminal payoff only depended on T_1, the above evaluation can be made in the terminal measure $\mathbb{E}_t^{Q_C^{T_1}}$. However, since $L(T_1,T_2)$ also depends on the collateral account's numeraire evolution from T_1 to T_2, we need to apply the appropriate risk neutral measure $P_C(T_1,T_2)$ and change the measure to $\mathbb{E}_t^{Q_C^{T_2}}$.
    - T_2-Forward Measure Numeraire Changes => Every cash flow needs to be discounted at its terminal/payout date, a consequence of the basic Arrow replication principle. Thus, the "inner" contingent claims need to be evaluated using their own terminal measure. This necessitates a measure change, which, using the corresponding forward numeraire change, results in the deterministic discount factor being pulled out (this numeraire corresponds to the discount factor spanning the inner start and the end dates).
5. <u>Expectation Under $\mathbb{E}_t^{Q_C^{T_2}}$</u>:

$$\mathbb{E}_t^{Q_C^{T_1}}\left[\frac{1}{1+L(T_1,T_2)\tau_{1,2}}|\mathcal{F}_t\right]=P_C(T_1,T_2)\mathbb{E}_t^{Q_C^{T_2}}\left[\frac{1}{P_C(T_1,T_2)}\frac{1}{1+L(T_1,T_2)\tau_{1,2}}|\mathcal{F}_t\right]$$
$$=\frac{P_C(t,T_2)}{P_C(t,T_1)}\mathbb{E}_t^{Q_C^{T_2}}\left[\frac{1+L_C(T_1,T_2){\tau^C}_{1,2}}{1+L(T_1,T_2)\tau_{1,2}}|\mathcal{F}_t\right]$$

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Tel. 646-513-3300

where

$$L_C(T_1, T_2) = \frac{1}{\tau^C{}_{1,2}}\left[\frac{P_C(t, T_2)}{P_C(t, T_1)} - 1\right]$$

6. FRA Par Rate Expression: Thus, the par FRA Rate becomes

$$FRA_{PAR}(t;\ T_1, T_2) = \frac{1}{\tau_{1,2}\frac{P_C(t, T_2)}{P_C(t, T_1)}\mathbb{E}_t^{Q_C^{T_2}}\left[\frac{1 + L_C(T_1, T_2)\tau^C{}_{1,2}}{1 + L(T_1, T_2)\tau_{1,2}}|\mathcal{F}_t\right]} - \frac{1}{\tau_{1,2}}$$

7. FRA Par Rate in Terms of the Collateral Forward Rate: Remembering that

$$F_C(t;\ T_1, T_2) = \frac{1}{\tau^C{}_{1,2}}\left[\frac{P_C(t, T_2)}{P_C(t, T_1)} - 1\right]$$

we get

$$FRA_{PAR}(t;\ T_1, T_2) = \frac{1 + F_C(t;\ T_1, T_2)\tau^C{}_{1,2}}{\tau_{1,2}\mathbb{E}_t^{Q_C^{T_2}}\left[\frac{1 + L_C(T_1, T_2)\tau^C{}_{1,2}}{1 + L(T_1, T_2)\tau_{1,2}}|\mathcal{F}_t\right]} - \frac{1}{\tau_{1,2}}$$

It is easy to see that under the single curve case

$$FRA_{PAR}(t;\ T_1, T_2) = F_C(t;\ T_1, T_2)$$

8. Modeling the Dynamics of $F_C(t;\ T_1, T_2)$ and $FRA(t;\ T_1, T_2)$: We choose the convenient shifted log normal evolution form for $F_C(t;\ T_1, T_2)$ and $FRA(t;\ T_1, T_2)$:

$$\Delta F_C(t;\ T_1, T_2) = \sigma_{1,2}^C\left[\frac{1}{\tau_{1,2}^C} + F_C(t;\ T_1, T_2)\right] = \Delta Z_2^C$$

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Tel. 646-513-3300

and

$$\Delta F_{FRA}(t;\ T_1,T_2) = \sigma_{1,2}\left[\frac{1}{\tau_{1,2}} + F_{FRA}(t;\ T_1,T_2)\right] = \Delta Z_2$$

where $\sigma_{1,2}$ and $\sigma_{1,2}^C$ are the constant instantaneous volatilities, and ΔZ_2 and ΔZ_2^C are the $\mathcal{F}_t$-adapted $Q_C^{T_2}$ Brownians with instantaneous correlations $\rho_{1,2}$.

9. Evaluating the Closed-Form for the Dynamics:

$$F_C(T_1;\ T_1,T_2) = F_C(t;\ T_1,T_2)e^{-\frac{1}{2}{\sigma_{1,2}^C}^2(T_1-t)+\sigma_{1,2}^C[Z_2^C(T_1)-Z_2^C(t)]}$$

and

$$FRA(T_1;\ T_1,T_2) = FRA(t;\ T_1,T_2)e^{-\frac{1}{2}{\sigma_{1,2}}^2(T_1-t)+\sigma_{1,2}[Z_2(T_1)-Z_2(t)]}$$

The evolution stops at T_1, since both F_C and FRA cease their evolutions beyond their fixing time, i.e., T_1.

10. Connection to Fixings: Remember that $L_C(T_1,T_2)$ and $L(T_1,T_2)$ are linked to their corresponding market FRA via

$$L_C(T_1,T_2) = F_C(T_1;\ T_1,T_2)$$

and

$$L(T_1,T_2) = FRA(T_1;\ T_1,T_2)$$

Also, since this caters only to the evolution of the forward rates, they are still in their stochastic forms, with the expectations taken to within the T_1 measure only at the final stage.

11. Evaluation of the Expectation:

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Tel. 646-513-3300

$$\mathbb{E}_t^{Q_C^{T_2}}\left[\frac{1+L_C(T_1,T_2){\tau^C}_{1,2}}{1+L(T_1,T_2)\tau_{1,2}}|\mathcal{F}_t\right]$$

$$= \mathbb{E}_t^{Q_C^{T_2}}\left[\frac{1+F_C(t;\ T_1,T_2){\tau^C}_{1,2}}{1+FRA(t;\ T_1,T_2)\tau_{1,2}}|\mathcal{F}_t\right]e^{-\frac{1}{2}\left({\sigma^C_{1,2}}^2-{\sigma_{1,2}}^2\right)(T_1-t)+\sigma^C_{1,2}[Z^C_2(T_1)-Z^C_2(t)]-\sigma_{1,2}[Z_2(T_1)-Z_2(t)]}$$

Evaluation of this stochastic integral leads to

$$\mathbb{E}_t^{Q_C^{T_2}}\left[\frac{1+L_C(T_1,T_2){\tau^C}_{1,2}}{1+L(T_1,T_2)\tau_{1,2}}|\mathcal{F}_t\right]=\frac{1+L_C(T_1,T_2){\tau^C}_{1,2}}{1+L(T_1,T_2)\tau_{1,2}}e^{-\left({\sigma^C_{1,2}}^2-\rho_{1,2}\sigma_{1,2}\sigma^C_{1,2}\right)(T_1-t)}$$

12. <u>Closed Form Par FRA and Convexity Correction</u>:

$$FRA_{PAR}(t;\ T_1,T_2)=\frac{1}{\tau_{1,2}}\left[1+F_C(t;\ T_1,T_2){\tau^C}_{1,2}\right]\left(e^{-({\sigma_{1,2}}^2-\rho_{1,2}\sigma_{1,2}\sigma^C_{1,2})(T_1-t)}-1\right)$$

Thus the convexity correction is computed as

$$FRA_{PAR}(t;\ T_1,T_2)-FRA_{STD}(t;\ T_1,T_2)$$
$$=\frac{1}{\tau_{1,2}}\left[1+F_C(t;\ T_1,T_2){\tau^C}_{1,2}\right]\left(e^{-({\sigma_{1,2}}^2-\rho_{1,2}\sigma_{1,2}\sigma^C_{1,2})(T_1-t)}-1\right)$$

13. <u>Constant FRA-Collateral Forwards Basis</u>: Mercurio (2010) shows that the dual log-normal formulation above results in corrections of the order of one bp in most cases. Thus, a case is made for analyzing the impact of using a small/constant FRA-forward collateral basis.
14. <u>Constant FRA-Collateral Forwards Basis Formulation</u>:

$$FRA_{PAR}(t;\ T_1,T_2)-FRA_{STD}(t;\ T_1,T_2)=S_{1,2}$$

a small positive constant. The $FRA_{PAR}(t;\ T_1,T_2)$ then becomes

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$$FRA_{PAR}(t;\ T_1,T_2) = \frac{1+\left[FRA(t;\ T_1,T_2)-S_{1,2}\right]\tau^C{}_{1,2}}{\tau_{1,2}\mathbb{E}_t^{Q_C^{T_2}}\left[\frac{1+\{L_C(T_1,T_2)-S_{1,2}\}\tau^C{}_{1,2}}{1+L(T_1,T_2)\tau_{1,2}}|\mathcal{F}_t\right]}-\frac{1}{\tau_{1,2}}$$

$$= \frac{1+\left[FRA(t;\ T_1,T_2)-S_{1,2}\right]\tau^C{}_{1,2}}{\tau^C{}_{1,2}-\left(\tau^C{}_{1,2}-\tau_{1,2}+\tau_{1,2}\tau^C{}_{1,2}S_{1,2}\right)\mathbb{E}_t^{Q_C^{T_2}}\left[\frac{1}{1+L(T_1,T_2)\tau_{1,2}}|\mathcal{F}_t\right]}-\frac{1}{\tau_{1,2}}$$

15. Shifted Log Normal Dynamics for $L(T_1,T_2)$: Assuming shifted log normal dynamics for $L(T_1,T_2)$, and assuming

$$\tau^C{}_{1,2}=\tau_{1,2}$$

- an equivalent simplifying basis assumption would correspond to

$$\tau_{1,2}FRA_{PAR}(t;\ T_1,T_2)-\tau^C{}_{1,2}F_C(t;\ T_1,T_2)=S_{1,2}$$

- we get

$$FRA_{PAR}(t;\ T_1,T_2) = \frac{1+\left[FRA(t;\ T_1,T_2)-S_{1,2}\right]\tau_{1,2}}{\tau_{1,2}-\left(\tau_{1,2}{}^2S_{1,2}\right)\frac{1}{1+FRA(t;\ T_1,T_2)\tau_{1,2}}e^{\sigma_{1,2}{}^2T_1}}-\frac{1}{\tau_{1,2}}$$

16. Convexity Correction: The corresponding convexity correction is

$$FRA_{PAR}(t;\ T_1,T_2)-FRA(t;\ T_1,T_2)$$
$$= S_{1,2}\left[e^{\sigma_{1,2}{}^2T_1}-1\right]\frac{\tau_{1,2}S_{1,2}{}^2e^{\sigma_{1,2}{}^2T_1}\left[e^{\sigma_{1,2}{}^2T_1}-1\right]}{1+FRA(t;\ T_1,T_2)\tau_{1,2}-\tau_{1,2}S_{1,2}{}^2e^{\sigma_{1,2}{}^2T_1}}$$
$$\approx S_{1,2}\left[e^{\sigma_{1,2}{}^2T_1}-1\right]$$

to the leading order in $S_{1,2}$. Since the convexity adjustment goes as $e^{\sigma_{1,2}{}^2T_1}$, it may appear that the straightforward dependence on linear, unadjusted T_1 may cause the correction to

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blow up at sufficient maturities. However, the shifted log-normal volatility is at least one order of magnitude smaller than the corresponding log normal volatility, i.e.,

$$\sigma_{1,2} \approx \frac{\tau_{1,2}{\sigma_{1,2}}^{LN} FRA(t;\ T_1, T_2)}{1 + FRA(t;\ T_1, T_2)}$$

where ${\sigma_{1,2}}^{LN}$ is the log normal volatility.

Futures

1. Futures Terminal Price/Payoff: Futures Payoff is

$$1 - L(T_{k-1} - T_k)$$

Thus

$$\mathbb{E}_t^{Q_C^{T_{k-1}}}[1 - L(T_{k-1} - T_k)] = 1 - \mathbb{E}_t^{Q_C^{T_{k-1}}}[L(T_{k-1} - T_k)]$$

Here the measure $Q_C^{T_{k-1}}$ will be treated as the collateralized discounting measure, as the futures are collateralized transactions (the collateralized discounting measure can be a discretely re-balanced bank account numeraire – called the spot-T measure).

2. Treatments of Convexity Adjustment: Several treatments of the convexity adjustment exist in the literature:
   - Both Kirikos and Novak (1997) and Henrard (2005) use the 1-factor Hull and White (1990) model
   - Piterbarg and Renedo (2006) use the stochastic volatility model
   - Mercurio (2009) and Mercurio (2010) use the multi-curve extended Market Model – that is what we consider here.

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3. <u>Terminology for the Extended Multi-Curve Market Model</u>: Borrowing from Mercurio (2009) and Mercurio (2010) we get:
   - $Z^d = \{Z_1{}^d, \cdots, Z_M{}^d\} \Longrightarrow$ The M-dimensional discretely balanced bank-account numeraire measure Q_C^T, and its Brownian components
   - $$F_h^C(t) = F^C(t;\ T_{k-1}, T_k) = \frac{1}{\tau_k^c}\left[\frac{P_C(t, T_{k-1})}{P_C(t, T_k)} - 1\right]$$

     is the discounting curve forward rate, and τ_k^c is the time interval.
   - σ_k and σ_h^c are respective deterministic volatilities of L_k and F_h^c (instantaneous)
   - $\rho_{k,h}^{L,F}$ is the instantaneous correlation between L_k and F_h^c
4. <u>Change of Numeraire to get to $(T_{k-1}, T_k]$</u>: Applying the change of measure on successive segments to get to $(T_{k-1}, T_k]$, the extended market model predicts that

$$\Delta L_k(t) = \sigma_k L_k(t) \sum_{h=\beta(t)}^{k} \frac{\rho_{k,h}^{L,F} \tau_k^c \sigma_h^c F_h^C(t)}{1 + \tau_k^c F_h^C(t)} \Delta t + \sigma_k L_k(t) \Delta Z_k^d(t)$$

where

$$\beta(t) = m \quad T_{m-2} < t \le T_{m-1}; m \ge 1$$
$$\beta(0) = 0 \quad t \in \left(T_{\beta(t)-2}, T_{\beta(t)-1}\right]$$

5. <u>Full Drift Freeze</u>: For computational convenience, we freeze the drift evolution at its time 0 value as

$$\mu_k = \sigma_k \sum_{h=0}^{k} \frac{\rho_{k,h}^{L,F} \tau_k^c \sigma_h^c F_h^C(0)}{1 + \tau_k^c F_h^C(0)}$$

Now we can evolve $L_k(t)$ as

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$$\Delta L_k(t) = \mu_k L_k(t)\Delta t + \sigma_k L_k(t)\Delta Z_k^d(t)$$

- Future Price => The Futures Price Valuation now becomes straightforward, as

$$V_t = 1 - \mathbb{E}_t^{Q_C^{T_{k-1}}}[L(T_{k-1}, T_k)|\mathcal{F}_t] \approx 1 - L_k(t)e^{\mu_k(T_{k-1}-t)}$$

$e^{\mu_k(T_{k-1}-t)}$ then is the convexity adjustment, and given that $L_k(0)$ is a market observable, V_0 may be computed from

$$L_k(0) = (1 - V_0)e^{\mu_k(T_{k-1}-t)}$$

6. Drift Freeze Adjustment #2: Here, only the values of the forward rates at frozen at time 0, not $\beta(t)$, and

$$\mu_k = \sigma_k \sum_{h=\beta(t)}^{k} \frac{\rho_{k,h}^{L,F}\tau_k^c\sigma_h^c F_h^C(0)}{1 + \tau_k^c F_h^C(0)}$$

Thus

$$V_t = 1 - \mathbb{E}_t^{Q_C^{T_{k-1}}}[L(T_{k-1}, T_k)|\mathcal{F}_t] \approx 1 - L_k(t)e^{\int_t^{T_{k-1}}\mu_k(u)du}$$

7. Price/Convexity Adjustment:

$$\mathbb{E}_t^{Q_C^{T_{k-1}}}[L(T_{k-1}, T_k)|\mathcal{F}_t] = L_k(0)e^{\sigma_k\sum_{h=0}^{k}\left[\int_{T_{h-1}}^{T_h}\sum_{j=h+1}^{k}\frac{\rho_{k,h}^{L,F}\tau_h^c\sigma_h^c F_h^C(0)}{1+\tau_k^c F_h^C(0)}du\right]}$$

$$= L_k(0)e^{\sigma_k\sum_{h=0}^{k}\sum_{j=h+1}^{k}\frac{\rho_{k,h}^{L,F}\tau_h^c\sigma_h^c F_h^C(0)}{1+\tau_k^c F_h^C(0)}(T_h-T_{h-1})} = L_k(0)e^{\sigma_k\sum_{j=1}^{k}\frac{\rho_{k,j}^{L,F}\tau_j^c\sigma_j^c F_j^C(0)}{1+\tau_j^c F_j^C(0)}T_{j-1}}$$

Thus

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$$L_k(0) = (1 - V_0)e^{\sigma_k \sum_{j=1}^{k} \frac{\rho_{k,j}^{L,F} \tau_j^c \sigma_j^c F_j^C(0)}{1+\tau_j^c F_j^C(0)} T_{j-1}}$$

Multi-Curve Swap Valuation

1. Interest Rate Futures: Just like the standard FRA, interest rate futures are key for the discovery of the forward swap rates and volatilities, and therefore for their calibration. These do trade, and appear to referred to literature under different names – STIR (short-term interest rate) future (Henrard (2013)), or FSIRS (forward starting IRS) (Bianchetti (2012)).
2. Swap Annuities in the Discount/Forward Measure:

$$A_d(t, S) = \sum_{i=1}^{m} \tau_d(S_{i-1}, S_i) P_d(t, S_i)$$

and

$$A_f(t, S) = \sum_{j=1}^{m} \tau_d(T_{j-1}, T_j) P_f(t, T_j)$$

The swap rate in the forward measure would be

$$S_f(t;\ T, S) = \frac{\sum_{i=1}^{m} \tau_f(S_{i-1}, S_i) F_f(S_{i-1}, S_i) P_f(t, S_i)}{A_d(t, S)}$$

and similar expressions may be computed for $S_d(t;\ T, S)$.

3. Dynamics of $S_f(t;\ T, S)$: In its own measure, $S_f(t;\ T, S)$ is a martingale, i.e.,

$$\frac{\Delta S_f(t;\ T, S)}{S_f(t;\ T, S)} = \nu_f(t; T, S) \Delta W_F^{T,S}$$

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with $t \geq T_0$, T_0 being the starting/effective date of the swap contract.

4. Swap Annuity Exchange Rate: Given that the swap annuity is the numeraire, we may introduce the swap annuity exchange rate quanto adjustment by resorting to FX-type quanto adjustment we have seen before, i.e.,

$$Y_{fd}(t,S) = \frac{A_f(t,S)}{A_d(t,S)}$$

Further, we consider the dynamics of $Y_{fd}(t,S)$

$$\frac{\Delta Y_{fd}(t,S)}{Y_{fd}(t,S)} = \nu_Y(t,S)\Delta W_Y^S$$

along with

$$\Delta W_F^{T,S}\Delta W_Y^S = \rho_{FY}(t;T,S)\Delta t$$

for

$$t \geq T_0$$

5. Swap Annuity Quanto Adjustment:

$$\mathbb{E}_t^{Q_d^{T,S}}[SwapPV] = A_d(t,S)\mathbb{E}_t^{Q_d^{T,S}}\left[\frac{S_f(t;T,\ S)}{Y_{fd}(t,S)}\right]$$

Given that, by construction, $\mathbb{E}_t^{Q_d^{T,S}}\left[\frac{S_f(t;T,S)}{Y_{fd}(t,S)}\right]$ is a martingale in the $Q_d^{T,S}$ measure, this produces the drift adjustment

$$QA_{fd}(t;T,S,\nu_F,\nu_Y,\rho_{FY}) = e^{-\lambda_f} = e^{-\int_t^{T_0}\rho_{FX}(u,T,S)\nu_Y(u,S)\nu_F(u,T,S)du}$$

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6. Quanto Adjusted Par Swap Rate: Setting

$$\mathbb{E}_t^{Q_d^{T,S}}[SwapPV] = A_d(t,S)\mathbb{E}_t^{Q_d^{T,S}}\left[S_f(T_0;T,S)\right]$$

we get

$$\mathbb{E}_t^{Q_d^{T,S}}\left[S_f(t;T,S)\right] = S_f(T_0;T,S)QA_{fd}(t;T,S,\nu_F,\nu_Y,\rho_{FY})$$

and

$$QA'_{fd}(t;T,S,\nu_F,\nu_Y,\rho_{FY}) = S_f(t;T,S)\left[QA_{fd}(t;T,S,\nu_F,\nu_Y,\rho_{FY}) - 1\right]$$

where $QA_{fd}(t;T,S,\nu_F,\nu_Y,\rho_{FY})$ is the multiplicative swap quanto adjustment, and $QA'_{fd}(t;T,S,\nu_F,\nu_Y,\rho_{FY})$ is the additive swap quanto adjustment.

7. Swaption Pricing: The T_0-Spot swap rate with an exercise date at T_0 is given by

$$SwaptionPrice(T_0;T,S,K,w,N) = NA_d(T_0,S)\max\left(w\left[S_f(t;T,S) - K\right]\right)$$

The corresponding price at $t < T_0$ is

$$SwaptionPrice(t;T,S,K,w,N) \\ = NA_d(t,S)Black\left(wS_f(t;T,S)QA_{fd}(t;T,S,\nu_F,\nu_Y,\rho_{FY}),w,\lambda_f,\nu_f,\rho_{fY}\right)$$

where λ_f is the drift seen before.

8. Multiple Underlying Interest Rates: When two or more underlying interest rate are present, the pricing expressions can become considerably more complicated (e.g., spread options – see Brigo and Mercurio (2006)).

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9. <u>Generalization for Joint Multi Factor Numeraire</u>: Say that the Brownian dynamics of evolution of n latent states are

$$\frac{\Delta A_i}{A_i} = \mu_i \Delta t + \sigma_i \Delta W_i;\ \Delta W_i \Delta W_i = \rho_{ij} \Delta t$$

for $i, j = 1, \dots, n$. The evolution for $\prod_{i=1}^{n} A_i$ is guided by

$$\frac{\Delta\left[\prod_{i=1}^{n} A_i\right]}{\left[\prod_{i=1}^{n} A_i\right]} = \left[\sum_{i=1}^{n} \mu_i(t) + \sum_{i=1}^{n} \sum_{j>i}^{n} \rho_{ij}(t) \sigma_i(t) \sigma_j(t)\right] \Delta t$$

From this it is easy to derive the joint numeraire

$$\prod_{i=1}^{n} A_i = e^{\int_{t_0}^{t} \left\{\sum_{i=1}^{n} \mu_i(s) + \sum_{i=1}^{n} \sum_{j>i}^{n} \rho_{ij}(s) \sigma_i(s) \sigma_j(s)\right\} ds}$$

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Cross Currency Basis Swap

Product Details and Valuation

1. Background: From Fujii, Shimada, and Takahashi (2010a, 2010b, 2010c, 2010d), for a USDJPY CCS that uses a USD discounting, the set of constitutive equations are

$$CCS_{USD} = \sum_{n=1}^{N} \delta_{n,USD} \mathbb{E}_{t,USD}^{T_n}\left[L_{a,USD}(T_{n-1}, T_n)\right] P_{USD}(t, T_n) + \sum_{m=1}^{M} \delta_{m,USD} \mathbb{E}_{t,USD}^{T_m}\left[L_{b,USD}(T_{m-1}, T_m)\right] P_{USD}(t, T_m)$$

i.e., the USD Leg in itself is a full basis swap. Likewise, the JPY leg is also a full basis swap. Thus,

$$PV_{JPY,USD}(t) = CCS_{JPY} + X_{JPY,USD} CCS_{USD}$$

- The CCS basis swap is quoted typically on the non-funding leg. In the above case that would be the JPY leg.

2. Valuation: Generalizing from the approach of Fujii, Shimada, and Takahashi (2010), we use the following nomenclature:
   - a, b are the corresponding LIBOR legs.
   - $i = 1, \cdots, N$ for leg a, and $j = 1, \cdots, M$ for leg b, where i and j are the cash flow indices.
   - V_{ab} is the PV of the USD segment of the CCS, i.e.,

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$$V_{ab} = \sum_{i=1}^{N} \Delta_{i,USD} \mathbb{E}_{t,USD}^{T_i} [L_{a,USD}(T_{i-1}, T_i)] P_{USD}(t, T_i) + \sum_{j=1}^{M} \Delta_{j,USD} \mathbb{E}_{t,USD}^{T_j} [L_{b,USD}(T_{j-1}, T_j)] P_{USD}(t, T_j)$$

This need not be zero.

3. CCS Valuation Base Setup: From the USDJPY CCS market information we get for the JPY basis swap

$$\sum_{i=1}^{N} \Delta_{i,JPY} [L_{a,JPY}(T_{i-1}, T_i) + l_{aC}] P_{JPY}(t, T_i) + \sum_{j=1}^{M} \Delta_{j,JPY} [L_{b,JPY}(T_{j-1}, T_j) + l_{bC}] P_{JPY}(t, T_j) = V_{ab}$$

From the JPY tenor basis swap market information we get

$$\sum_{i=1}^{N} \Delta_{i,JPY} [L_{a,JPY}(T_{i-1}, T_i) + l_{aT}] P_{JPY}(t, T_i) + \sum_{j=1}^{M} \Delta_{j,JPY} [L_{b,JPY}(T_{j-1}, T_j) + l_{bT}] P_{JPY}(t, T_j) = V_{ab}$$

Here we've removed all expectation operators $\mathbb{E}_{t,JPY}^{T_i}$ etc. l_{aC}, l_{bC} are the basis quotes in the cross currency markets, and l_{aT}, l_{bT} are the tenor basis market quotes.

4. Basis Quotes Inputs: Notice that both swaps need to refer to the same tenor/maturity set a, b and T_N, T_M - that is the only restriction. If the quote for the tenor swap is not available, they will need to be implied off of others – this provides a very strong motivation to mark the forward curve off of the forwards basis space. This may also require extrapolation – we'll get to that messy challenge later.

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Building the CCS Discount Curve

1. Setup: Subtracting the basis quote legs from the CCS legs, we get

$$[l_{aC} - l_{aT}] \sum_{i=1}^{N} \Delta_{i,JPY} P_{JPY}(t, T_i) + [l_{bC} - l_{bT}] \sum_{j=1}^{M} \Delta_{j,JPY} P_{JPY}\left(t, T_j\right) = V_{ab}$$

Actually,

$$l_{aC} \equiv l_{aC}(T_N)$$

$$l_{bC} \equiv l_{bC}(T_M)$$

$$l_{aT} \equiv l_{aT}(T_N)$$

and

$$l_{bT} \equiv l_{bT}(T_M)$$

to explicitly spell out the bootstrapping dependence. In particular, if

$$l_{bC} = 0$$

and

$$l_{bT} = 0$$

then we have the really simple following bootstrapping relationship:

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Tel. 646-513-3300

$$\sum_{i=1}^{N} \Delta_{i,JPY} P_{JPY}(t, T_i) = \frac{V_{ab}}{l_{aC}(T_N) - l_{aT}(T_N)}$$

2. Symmetry in the Discount Curve Calibration from CCS: Notice that V_{ab}, when taken to the other side, becomes $-\frac{V_{ab}}{X_{JPY,USD}}$. Thus from the same market quotes we may calibrate the USD discount curve using the JPY discount curve as well.
3. Parallel between Forward and Discount Curve Construction:
    - Relevant leg for the Forward Curve Extraction => Derived Component Derived Leg (which corresponds to a Floating stream, as the derived component is a float-float swap).
    - Relevant leg for the Discount Curve Construction => Again derived component Derived Leg, this now corresponds to a Fixed Stream, as the derived component is a fix-float (IRS) component.
    - Remember that the basis is always placed on the reference component – either the reference leg or the derived leg.

4. CCBS Cross Bases Computation: Say $\kappa_{\alpha,\beta}$ is the cross-currency basis.

$$\kappa_{\alpha,\beta} = \frac{PV_{DC} + XPV_{RC}}{DV01_{\alpha,\beta}}$$

where α indicates the component side (i.e., reference or derived), and β indicates the component stream under consideration (i.e., the reference stream or the derived stream). $DV01_{\alpha,\beta}$ corresponds to the DV01 of stream β inside of component α. X is the FX of the derived/reference cross, i.e., units of the derived currency in terms of the reference currency.

Custom CCBS Based Curve Construction SKU

1. Curve Construction from CCBS: For each currency leg, there are 3 potential latent states – the forward latent states 1 & 2, and the discounting latent state. Thus there are 6 of them in all. This contributes to the cognitive confusion – the number of latent states.

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2. Latent State Determination using CCBS: Typically we are given a one set of currency parameters, and made to determine the other. The practical use case would be for:
    a. Computing the funding curve parts – this includes the merge-stretched discount curve and its corresponding forward curve
    b. Computing the non-funding forward curve – in this case we need to have either the merge-stretch discount and forward, or the distinct discount/forward curves.
3. Separated "Derived Product" Latent States: Since we infer the latent states of the derived sides from the inputs, in we general we are required to extract:
    a. The derived forward state
    b. The reference forward latent state
    c. The discounting latent state

    Thus the state composites are built strictly on the availability of the additional contingent inputs (beyond the CCBS quotes).
4. CCBS Derived Forward vs. Discount Curve Construction: If the derived product discount curve and the derived product reference leg forward curves are available, the derived product derived leg products then become spline constructible using the CCBS quotes alone. Otherwise, additional external inputs and/or simplifying assumptions are needed (these assumptions help set the merge states).
5. Joint Latent State Estimations: If there are non-linear couplings between 2/more latent states above, the linear splined state extractions become infeasible without some kind of kernel transformation. This is among the 2 troubles with Fujii, Takahashi, and Shimada (2010a, 2010b, 2010c, 2010d):
    a. The instrument set pair of CCBS/IRS used for calibration purposes needs to be completely paired up
    b. Linear reductions (either directly or by using kernel transformations) need to be possible by the elimination of the correspondingly paired CCBS/IRS streams
6. CCBS Deep-Drill Cognitive Challenge: CCBS is possibly the biggest challenge so far in terms of back-and-forth cognitive switch-in/switch-out, owing to the following:
    a. The variety of the latent states
    b. The manner in which these latent states interleave

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c. Estimation of each of them at each segment level using the corresponding "deep-drill" instrument quotes
d. Cross influence via the C^k transmission criterion across the latent states
e. Interleaving of the sensitivities as well

Mark-To-Market Cross-Currency Swap Valuation

1. The "MTM" in MTM Cross Currency Swap: In this context the term "MTM" is a misnomer in that it is simply an MTM in the FX dimension, i.e., it is an FX-MTM cross-currency swap. The floating nature ensures "MTM" in the floater dimension, however, it does not address MTM in the FX/floating and the forward cross-currency dimensions.
2. MTM is not Collateralization: For the reasons above, the MTM cross-currency may not be viewed as a collateralized transaction – not even in the non-continuous collateralization limit. Collateralization impact is essentially undetermined in account of the reasons above – even in the absence of a stochastic cross-currency basis, under-collateralization/over-collateralization is determined entirely by the sign of the FX/floater correlation.
3. MTM for Dual Stream Instruments: Instruments in this category include the fix-float and the float-float swaps. As noted above, while explicit MTM'ing using the forward construction is not necessary in this case, the instruments are still exposed to collateral/forward, collateral/funding, and forward/funding volatilities/correlations, which are not easy to hedge for – thereby making a case for "MTM" type agreements (esp. for the basis).

Mark-To-Market Cross-Currency Swap – Valuation Formulation

1. Setup:

$$PV_{RC} = PV_{RL} + PV_{DL} \rightarrow X_0 PV_{RC} = X_0 PV_{RL} + X_0 PV_{DL}$$

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$$X_0 PV_{RC} = X_0 \left[\sum_i \mathcal{L}_{RL}(t_{i-1}, t_i) DCF_{RL}(t_{i-1}, t_i) D_f(t_i) \right] + X_0 \left[\sum_j \mathcal{L}_{DL}(t_{j-1}, t_j) DCF_{DL}(t_{j-1}, t_j) D_f(t_j) \right] + X_0 \kappa_{\alpha,\beta} DV01_{\alpha,\beta}$$

- The last term simply indicates that the basis can be either on the reference leg or on the derived leg.
- We will assume that $\kappa_{\alpha,\beta}$ is indepdendent of the FX process or the discount factor process.

2. Reference Component PV in Derived Currency: Consider a forward starting CCBS, maturing at time t_0. The PV of the reference component in the derived currency at time t can be expressed as:

$$\begin{aligned} X_0 PV_{RC}(t, t_0) = & \left[\sum_i \mathcal{L}_{RL}(t_{i-1}, t_i) DCF_{RL}(t_{i-1}, t_i) D_f(t, t_i) \right] X(t, t_0) \\ & + \left[\sum_j \mathcal{L}_{DL}(t_{j-1}, t_j) DCF_{DL}(t_{j-1}, t_j) D_f(t, t_j) \right] X(t, t_0) \\ & + X(t, t_0) \kappa_{\alpha,\beta} DV01_{\alpha,\beta}(t) \\ & = X(t, t_0) D_f(t, t_0) \left[\sum_i \mathcal{L}_{RL}(t_{i-1}, t_i) DCF_{RL}(t_{i-1}, t_i) D_f(t_0, t_i) \right. \\ & \left. + \sum_j \mathcal{L}_{DL}(t_{j-1}, t_j) DCF_{DL}(t_{j-1}, t_j) D_f(t_0, t_j) + \kappa_{\alpha,\beta} DV01_{\alpha,\beta}(t_0) \right] \\ & = X(t, t_0) D_f(t, t_0) PV_{RC}(t_0, t_0) \end{aligned}$$

3. t_m-Forward MTM CCBS PV: From

$$PV_{CCBS,m} = PV_{DC,m} + XPV_{RC,m}$$

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we get

$$E_t^{Q_D^{tm}}\left[PV_{CCBS,m}\right] = E_t^{Q_D^{tm}}\left[PV_{DC,m}\right] + E_t^{Q_D^{tm}}\left[XPV_{RC,m}\right]$$

which implies

$$E_t^{Q_D^{tm}}\left[PV_{CCBS,m}\right] = E_t^{Q_D^{tm}}\left[PV_{DC,m}\right] + E_t^{Q_D^{tm}}\left[PV_{RC,m}\right]E_t^{Q_D^{tm}}\left[X(t,t_m)D_f(t,t_m)\right]$$

4. <u>Joint X, D_f Evolution</u>:

$$\Delta X = \mu_X X\Delta t + \sigma_X X\Delta W_X$$

$$\Delta D_f = \mu_D D_f\Delta t + \sigma_D D_f\Delta W_D$$

$$\Delta X\Delta D_f = \rho_{DX}\sigma_X\sigma_D X D_f\Delta t$$

$$\Delta\left(XD_f\right) = XD_f\{(\mu_X + \mu_D + \rho_{XD}\sigma_X\sigma_D)\Delta t + \sigma_X\Delta W_X + \sigma_D\Delta W_D\}$$

$$E_t^{Q_D^{tm}}\left[\Delta\left(XD_f\right)\right] = E_t^{Q_D^{tm}}\left[XD_f\right]\{(\mu_X + \mu_D + \rho_{XD}\sigma_X\sigma_D)\Delta t\}$$

$$E_t^{Q_D^{tm}}\left[XD_f\right] = e^{\int_t^{tm}[\mu_X(s)+\mu_D(s)+\rho_{XD}(s)\sigma_X(s)\sigma_D(s)]ds}$$

5. <u>MTM Reference Component Adjustment</u>:

$$\begin{aligned} E_t^{Q_D^{tm}}\left[X(t,t_m)D_f(t,t_m)\right] &= E_t^{Q_D^{tm}}\left[X(t,t_m)\right]E_t^{Q_D^{tm}}\left[D_f(t,t_m)\right]e^{\int_t^{tm}[\rho_{XD}(s)\sigma_X(s)\sigma_D(s)]ds} \\ &= X(t,t_m)D_f(t,t_m)M_A(t,t_m) \end{aligned}$$

Here

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$$X(t, t_m) = E_t^{Q_D^{t_m}}[X(t, t_m)] = e^{\int_t^{t_m} \mu_X(s) ds}$$

$$D_f(t, t_m) = E_t^{Q_D^{t_m}}\left[D_f(t, t_m)\right] = e^{\int_t^{t_m} \mu_D(s) ds}$$

The MTM Adjuster $M_A(t, t_m)$ is given from

$$M_A(t, t_m) = e^{\int_t^{t_m} [\rho_{XD}(s) \sigma_X(s) \sigma_D(s)] ds}$$

6. <u>Aggregated MTM CCBS PV</u>:

$$PV_{CCBS,MTM}(t) = \sum_m PV_{CCBS,m}(t) = \sum_m PV_{DC,m}(t) + \sum_m X(t, t_m) PV_{RC,m}(t)$$

$$PV_{CCBS,MTM}(t) = \sum_m D_f(t, t_m) PV_{DC,m}(t_m) + \sum_m D_f(t, t_m) X(t, t_m) M_A(t, t_m) PV_{RC,m}(t_m)$$

$$PV_{CCBS,MTM}(t) = \sum_m D_f(t, t_m) PV_{DC,m}(t_m) + \sum_m D_f(t, t_m) X(t, t_m) PV_{RC,m}(t_m) + \sum_m D_f(t, t_m) X(t, t_m) [M_A(t, t_m) - 1] PV_{RC,m}(t_m)$$

$$PV_{CCBS,MTM}(t) = PV_{CCBS,non-MTM}(t) + \sum_m D_f(t, t_m) X(t, t_m) [M_A(t, t_m) - 1] PV_{RC,m}(t_m)$$

7. <u>MTM Adjustment to MTM CCBS PV</u>: From above

$$PV_{CCBS,MTM}(t) = PV_{CCBS,non-MTM}(t) + MTM_{Adjustment}(t)$$

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where

$$MTM_{Adjustment}(t) = \sum_m D_f(t, t_m) X(t, t_m) [M_A(t, t_m) - 1] PV_{RC,m}(t_m)$$

The $MTM_{Adjustment}(t)$ correction term vanishes as $\rho_{DX} \to 0$ as one would expect. Further owing to the presence of the exponentials, when $\rho_{DX} > 0$ the correction term dominates as $\rho_{XD}, \sigma_X, and\ \sigma_D$ tend higher. If negatively correlated (i.e. $\rho_{DX} < 0$) the correction essentially reduces towards $-\sum_m D_f(t, t_m) X(t, t_m) PV_{RC,m}(t_m)$, and the contribution to $PV_{CCBS,MTM}(t)$ from $PV_{RC,m}(t)$ diminishes.

8. Absence of the explicit Cross Currency Basis in the MTM Correction: Since the basis is confined exclusively to the reference component, it is automatically incorporated into $PV_{RC,m}$ and its derivative terms above. Throughout the only assumption made about the cross currency basis is that it evolve independently of the discount factor and the FX rate.

Absolute/Relative MTM Application

1. Component Pair Relative MTM Generalization:
    a. The "MTM"able component pair consists of 2 components – the reference and the derived.
    b. "MTM"ing occurs at discrete MTM dates – which is most typically set to the coupon dates of either of the streams.
    c. The MTM process can be absolute or relative, i.e., in the relative MTM'ing the reference is MTM'ed w.r.t. the derived.
    d. The Reference Component is decomposed into forward components – each forward being built out from the forward MTM dates.
    e. Valuation of each of the stripped forward component may be customized to correspond to employ its own model/assumptions/market data/evolution dynamics.
2. Component Pair Absolute MTM Generalization: Situation here is identical to the "Relative" Case, except for items c) (the reference and the derived are MTM'ed independent of each

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other, not relative to one another) and e) (of course, the forward evolution/valuation is still going to be based off of one set market parameters and their realization).

3. Generalized Absolute/Relative Valuation Adjustment Market Data: Models that drive the valuation in this case would rely on the following external market data:
    a. Funding/FX Volatility/Correlation
    b. Funding/Forward Volatility/Correlation
    c. Collateral/Funding Volatility/Correlation
    d. Collateral/FX Volatility/Correlation
    e. Collateral/Forward Volatility/Correlation
    f. Joint modes implied from the combinations above
4. No Convexity Adjustment for non-MTM Contracts: Given that the convexity adjustment is applicable only to dynamic (i.e., MTM) jointly-evolved state-specification in the contract, no joint state convexity corrections with respect to the given state will be applied to non-MTM counterparts of a specific numeraire.

Per-trade Risk Isolation Components

1. Underlier Security Price Market Risk
2. Discount Factor Risk
3. Forward Rate Risk
4. Currency/FX Risk
5. Basis Risk (on any Risk Factor)
6. Funding Risk
7. Collateral Risk
8. Counter-party Risk

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Section V: Collateralized Valuation

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Collateralized Valuation

Background

1. Background: While economies without risk-free rates have been considered in the past (Black (1972)), typical derivatives pricing treatments have assumed the existence of such rates as a matter of course (e.g., Duffie (2001)).
2. Holy Grail of Curve Construction: Combining multiple curves, partial collateralization involving multiple currencies, with liquidity, counter party risk, funding, and credit risk factored in into a dynamic approach is treated in a variety of papers (Pallavicini and Tarenghi (2010), Fujii, Shimada, and Takahashi (2010a), Fujii, Shimada, and Takahashi (2010b), Fujii, Shimada, and Takahashi (2010c), Fujii and Takahashi (2011a, 2011b), Castagna (2012), Henrard (2013)).
3. Treatments of CVA/DVA: Partial collateralization results in non-zero counter-party risk, and these cases are covered in Burgard and Kjaer (2011a, 2011b), Brigo, Pallavicini, Buescu, and Liu (2012), Crpey (2012a), Crpey (2012b). Considerations regarding the risk of an “average” counter-party are treated in Morini (2009).

Introduction and Motivation

1. Counter party Credit Risk Free Asset: Closest to a credit-risk free asset is an asset that is fully collateralized on a continuous basis (ISDA (2009), ISDA (2011), Sawyer (2011), Piterbarg (2012)), i.e., the collateralized asset produces cash flows that are continuous with changes in both the derivative MTM and the collateral coupon. Macey (2011) and Piterbarg (2012) illustrate how to retain the traditional risk-neutral valuation in a collateralized context.
2. Collateralized Asset Process: At the inception of a fully collateralized trade, there is no cash exchange, i.e., the upfront payment amount is returned back as collateral. Further, in

exchange for the continuous pay streams above, the trade can be cancelled at any time with zero net value for either side.

3. Price of a Collateralized Asset: The price of a collateralized asset is effectively the outstanding level of the collateral account, i.e., a collateralized transaction is an asset with a zero-drift price process and with the given cumulative dividend flows (Duffie (2001)).
4. Collateral Cash Flows: $V(t)$ is the asset price paid by A to B, and B posts this amount back as collateral. A now pays the contractual collateral coupon flow $c(t)$ back to B. In time unit Δt, the cash flow that is exchanged (i.e., paid to A) is $V(t+\Delta t) - V(t) - c(t)V(t)\Delta t$, i.e.,

$$\Delta\chi(t) = \Delta V(t) - c(t)V(t)\Delta t$$

Once this is exchanged, the transaction can terminate, and A can keep the collateral.

Two Collateralized Assets

1. Setup: Assume that each of the assets follows its corresponding real-world measures, but are exposed to the same risk factor ΔW, i.e.,

$$\Delta V_i(t) = \mu_i(t)V_i(t)\Delta t + \sigma_i(t)V_i(t)\Delta W$$

for

$$i = 1, 2$$

2. Hedge Portfolio: Say that the corresponding collateralized account for each of these assets has the dynamics

$$\Delta\chi_i(t) = \Delta V_i(t) - c(t)V_i(t)\Delta t$$

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Construct a hedge portfolio using $-\sigma_1(t)V_1(t)$ of asset 2 and $+\sigma_2(t)V_2(t)$ of asset 1. The net change in the real world collateralized portfolio of these two assets is:

$$\Delta\chi_{12}(t) = \sigma_2(t)V_2(t)[\Delta V_1(t) - c(t)V_1(t)\Delta t] - \sigma_1(t)V_1(t)[\Delta V_2(t) - c(t)V_2(t)\Delta t]$$

$$\Delta\chi_{12}(t) = V_1(t)V_2(t)[\sigma_2(t)\{\mu_1(t) - c(t)\} - \sigma_1(t)\{\mu_2(t) - c(t)\}]\Delta t$$

3. Application of the Collateral Rules: The above amount is known at time t, and maybe exchanged at $t + \Delta t$, at zero additional cost to either party. Thus, the only way both can enter into this transaction is if the net cash flow is zero (this is the collateralized version of no arbitrage). This produces

$$\frac{\mu_1(t) - c(t)}{\sigma_1(t)} = \frac{\mu_2(t) - c(t)}{\sigma_2(t)}$$

4. Differences with Traditional Risk Neutral Pricing: The main difference is: in the traditional risk-neutral pricing, the hedged portfolio grows at the "risk-free" rate. In collateralized pricing, the COLLATERALIZED + HEDGED portfolio grows at ZERO rate (i.e., does not grow at all) after incremental netting! Therefore the "risk-free" rate does not enter into this setting at all.
5. Measure Change: Create a new measure t where

$$\Delta W_Q = \Delta W + \frac{\mu_i(t) - c(t)}{\sigma_i(t)}$$

In this new measure, the individual assets grow as

$$\Delta V_i(t) = c(t)V_i(t)\Delta t + \sigma_i(t)V_i(t)\Delta W_Q$$

using which we estimate $V_i(t)$ as

$$V_i(t) = \mathbb{E}_t^Q\left[e^{-\int_t^T c(s)ds}V_i(T)\right]$$

As may be observed, measure Q looks like the traditional risk neutral measure.

6. Different Collateral Rates: The collateral rates $c_i(t)$ can be asset-specific within changing any of our principal conclusions, and $V_i(t)$ now becomes

$$V_i(t) = \mathbb{E}_t^Q\left[e^{-\int_t^T c_i(s)ds}V_i(T)\right]$$

Examples would be, say, a stock collateralized at its repo rate (or other funding rate), while the derivative would be collateralized at it collateral rate (e.g., Piterbarg (2010)).

7. Other Variants: Other collateralization variants include varying collateral processes, different counter-parties etc. Typically all these only end up varying the drift, thus you get

$$\frac{P_1(t,T)}{P_2(t,T)} = \frac{\mathbb{E}_t^{Q_1}\left[e^{-\int_t^T c_1(s)ds}V(T)\right]}{\mathbb{E}_t^{Q_2}\left[e^{-\int_t^T c_2(s)ds}V(T)\right]}$$

Of course, the collateralization drift can also be stochastic. This measure change from collateralization scheme #1 to collateralization scheme #2 induces a drift to the scheme #2 as

$$\mathbb{E}_t^Q\left[e^{-\int_t^T [c_2(s)-c_1(s)]ds}V(T)\right]$$

8. Many Collateralized Assets: Will quickly flip through this, as Piterbarg (2012) spells out the details. N-dimensional asset $\vec{V}$ possesses the real-world dynamics

$$\Delta\vec{V}(t) = \vec{\mu}^T(t)\vec{V}(t)\Delta t + \vec{\sigma}^T(t)\vec{V}(t)\Delta\vec{W}$$

A linearly combined weight set $\vec{w}$ of the hedge portfolio satisfies the constraint

$$\vec{w}^T\vec{\sigma} = 0$$

Using the collateral cash flow matching arguments presented above, we get

$$\vec{w}^T\left[\vec{\mu}^T\vec{V} - \vec{c}^T\vec{V}\right] = 0$$

Measure Change => As before, there exists a measure Q with the drift vector $\vec{c}$, one for each asset, such that an adjustment $\vec{\lambda}$ can be made to the real world measure making it

$$\Delta\vec{V}(t) = \vec{c}^T(t)\vec{V}(t)\Delta t + \vec{\sigma}^T(t)\vec{V}(t)\left[\Delta\vec{W} + \vec{\lambda}\Delta t\right]$$

such that $\Delta\vec{W} + \vec{\lambda}\Delta t$ can become drift-less. Once again, in this new measure, the individual assets follow

$$V_i(t) = \mathbb{E}_t^Q\left[e^{-\int_t^T c_i(s)ds}V_i(T)\right]$$

Collateral PDE Formulation

1. PDE Collateralization Treatments: Bjork (2009), Piterbarg (2010), Castagna (2011), Fujii and Takahashi (2011a, 2011b), Henrard (2012), Piterbarg (2012), Ametrano and Bianchetti (2013), and Han, He, and Zhang (2013) extend the no-arbitrage to the collateralization case.
2. Review of Derivative PDE Using Replication: The derivative that is replicated using n assets and a bond via

$$V = nS + B$$

undergoes the evolution through the self-financing formulation

$$\Delta V = n\Delta S + \Delta B$$

This is matched to the derivative change

$$\Delta V = \left[\frac{\partial V}{\partial t} + \frac{1}{2}{\sigma_S}^2 S^2 \frac{\partial^2 V}{\partial S^2}\right]\Delta t + \frac{\partial V}{\partial S}\Delta S$$

Equating the two, setting

$$\frac{\partial V}{\partial S} = n$$

to eliminate stochasticity, and noticing that

$$\Delta B = rB\Delta t$$

we get

$$\left[\frac{\partial V}{\partial t} + \frac{1}{2}{\sigma_S}^2 S^2 \frac{\partial^2 V}{\partial S^2}\right]\frac{1}{r} = B$$

Using the expression for V, this may be re-composed as the Black-Scholes PDE from

$$\frac{\partial V}{\partial t} + \frac{1}{2}{\sigma_S}^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV$$

(Harrison and Kreps (1979), Harrison and Pliska (1981), Harrison and Pliska (1983)).

3. Derivative Replication with Collateral Account: The replication strategy now involves the assets, the bank funding account, and the collateral account.

$$V = aS + B + C$$

where a and B are the number of assets and the bank funding notional account, respectively, and C is the collateral account. Under perfect collateralization

$$C \equiv V$$

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Further

$$\Delta C = r_C V \Delta t$$

$$\Delta B = r_f B \Delta t$$

and

$$\Delta S = \mu_S S \Delta t + \sigma_S S \Delta W$$

4. Derivative Value Change: Applying the self-financing condition

$$\Delta V = a\Delta S + \Delta B + \Delta C$$

Using the perfect collateral condition we get

$$V = aS + B + V$$

which implies

$$B = -aS$$

Thus

$$\Delta V = a\Delta S + r_f B \Delta t + r_C V \Delta t$$

results in

$$\Delta V = a\Delta S - a r_f S \Delta t + r_C V \Delta t$$

We refer to the quantity

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$$\Gamma(r_C, r_f) = -ar_f S\Delta t + r_C V \Delta t$$

as the cash account.

5. The Collateralization PDE:

$$\Delta V = \left[\frac{\partial V}{\partial t} + \frac{1}{2}{\sigma_S}^2 S^2 \frac{\partial^2 V}{\partial S^2}\right]\Delta t + \frac{\partial V}{\partial S}\Delta S = a\Delta S - ar_f S\Delta t + r_C V\Delta t$$

Setting

$$a = \frac{\partial V}{\partial S}$$

we get

$$\frac{\partial V}{\partial t} + \frac{1}{2}{\sigma_S}^2 S^2 \frac{\partial^2 V}{\partial S^2} + r_f S\frac{\partial V}{\partial S} = r_C V$$

Re-casting using the appropriate measure terminology, we get

$$V(S,t) = \mathbb{E}_t^{Q^f}[D_C(S,T)V(S,T)]$$

where

$$D_C(t,T) = e^{-\int_t^T r_C(u)du}$$

Cross Currency Model

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1. LCH.ClearNet Collateral Rules: Single currency trades (currently mostly swaps) are collateralized in their own currencies, but multi-currency trades (e.g., cross currency swaps) are typically collateralized in USD.
2. Building Blocks: The building blocks typically are a) Domestic-Currency Collateralized Domestic Zero Coupon Bonds, b) Foreign-Currency Collateralized Foreign Zero Coupon Bonds, c) Collateralized FX Contracts. In practice, the former (the collateralized zeros) may not trade, whereas collateralized FX contracts typically do.
3. Foreign Bonds Collateralized in Domestic Currency: Consider a foreign zero-coupon bond collateralized with domestic collateral. The price of this zero coupon bond in foreign currency is $P_{f,d}(t)$. If $X(t)$ is the forex rate (i.e., the number of domestic units per foreign unit), the collateral account cash flow growth is

$$\Delta\chi_i(t) = \Delta\left[P_{f,d}(t)X(t)\right] - c_d(t)\left[P_{f,d}(t)X(t)\right]\Delta t$$

   where $c_d(t)$ is the domestic collateral rate.
4. Collateralization of the FX: If $r_{d,f}(t)$ is the rate agreed on a domestic loan collateralized by foreign collateral, then the FX collateral account cash flow growth is

$$\Delta\chi_X(t) = \Delta X(t) - X(t)\Delta t$$

   The contention by Piterbarg (2012) is that there is no relation between the collateralization rates $r_{d,f}(t)$, $c_d(t)$, and $c_f(t)$.
   5. Collateralization Using Domestic Collateral:

$$\begin{bmatrix} \dfrac{\Delta X(t)}{X(t)} \\ \dfrac{\Delta P_{d,d}(t)}{P_{d,d}(t)} \\ \dfrac{\Delta P_{f,d}(t)}{P_{f,d}(t)} \end{bmatrix} = \begin{bmatrix} r_{f,d}(t) \\ c_d(t) \\ c_f(t) \end{bmatrix} \Delta t + \sigma_d \Delta W_d$$

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Thus, under the domestic collateralization risk-neutral measure Q_d we have the following:

$$X(t) = \mathbb{E}_t^{Q_d}\left[e^{-\int_t^T r_{d,f}(s)ds} X(T)\right]$$

$$P_{d,d}(t,T) = \mathbb{E}_t^{Q_d}\left[e^{-\int_t^T c_d(s)ds}\right]$$

$$P_{f,d}(t,T) = \frac{1}{X(t)}\mathbb{E}_t^{Q_d}\left[e^{-\int_t^T c_d(s)ds} X(T)\right]$$

6. Collateralization Using Foreign Collateral:

$$\begin{bmatrix} \dfrac{\Delta\left[\frac{1}{X(t)}\right]}{\left[\frac{1}{X(t)}\right]} \\ \dfrac{\Delta P_{f,f}(t)}{P_{f,f}(t)} \\ \dfrac{\Delta P_{d,f}(t)}{P_{d,f}(t)} \end{bmatrix} = \begin{bmatrix} -r_{d,f}(t) \\ c_f(t) \\ c_d(t) \end{bmatrix} \Delta t + \sigma_f \Delta W_f$$

Thus, under the foreign collateralization risk-neutral measure Q_f we have the following:

$$\frac{1}{X(t)} = \mathbb{E}_t^{Q_f}\left[e^{\int_t^T r_{d,f}(s)ds}\frac{1}{X(T)}\right]$$

$$P_{f,f}(t,T) = \mathbb{E}_t^{Q_f}\left[e^{-\int_t^T c_f(s)ds}\right]$$

$$P_{d,f}(t,T) = X(t)\mathbb{E}_t^{Q_d}\left[e^{-\int_t^T c_d(s)ds}\frac{1}{X(T)}\right]$$

7. $\underline{P_{d,f}(t,T)}$ and $\underline{P_{f,d}(t,T)}$ Numeraires: These are effectively the cross-currency, oppositely collateralized numeraires, i.e., one unit of domestic/foreign currency collateralized using the corresponding foreign/domestic collateral. Thus these numeraires, as such, can form the basis for cross-currency discount curves employed in cross-currency swaps. Further, while these building blocks are primarily only discounting oriented – securities with forward/floater leg may also require a quanto adjustment to be applied.
8. Cross Currency Model Parameters: All the model parameters and the process dynamical parameters in the set of equations above can be independently observed.
9. "Implied" Cross Currency Risk Free Rate: The measure change from Q_d to Q_f under the Q_d measure is captured by the Q_d martingale

$$M(t) = \frac{\partial Q_f}{\partial Q_d} = e^{-\int_t^T r_{d,f}(s)ds} \frac{X(t)}{X(0)}$$

   Thus, the corresponding growth rate $r_{d,f}(t)$ also helps clarify the references to the "cross-currency risk-free Rates" (e.g., Fujii and Takahashi (2011a, 2011b)) – viz., they are instantaneous FX collateralization rate using the foreign collateral.
10. Forward Forex Contract Collateralized with Domestic Collateral: This contract pays

$$X(T) - K$$

   in the domestic currency, and is collateralized using the domestic collateral. Thus

$$\mathbb{E}_t^{Q_{f,d}}[X(T) - K] = X(t)P_{f,d}(t,T) - KP_{d,d}(t,T)$$

   Therefore, the par strike K for this contract is

$$K = \frac{X(t)P_{f,d}(t,T)}{P_{d,d}(t,T)}$$

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Tel. 646-513-3300

11. Forward Forex Contract Collateralized with Foreign Collateral: This contract pays $1 - \frac{K}{X(T)}$ in the foreign currency, and is collateralized using the foreign collateral. Thus

$$\mathbb{E}_t^{Q_f}\left[1 - \frac{K}{X(T)}\right] = P_{f,f}(t,T) - K\frac{P_{d,f}(t,T)}{X(T)}$$

Therefore, the par strike K for this contract is

$$K = \frac{X(T)P_{f,f}(t,T)}{P_{d,f}(t,T)}$$

12. Same Currency Collateralization:

$$V_{d,d}(t,T) = \mathbb{E}_t^{{Q_{d,d}}^T}[X(T) - K] = P_{d,d}(t,T)[X(T) - K]$$

and

$$V_{f,f}(t,T) = \mathbb{E}_t^{{Q_{f,f}}^T}\left[1 - \frac{K}{X(T)}\right] = \left[1 - \frac{K}{X(T)}\right]P_{f,f}(t,T)$$

No rocket science, really, with simple forwards. Question, however, is that whether $X(T)$ would ever be domestically collateralized, and that whether $\frac{1}{X(T)}$ would ever be collateralized in foreign currency. Same currency collateralization is uncommon presumably for these reasons.

13. Market Quotes for Collateralized Forex Forwards: Strictly speaking, all Forex Forwards should always be collateralized using either foreign or domestic collateral. Thus, the Forward Prices should be different depending on the collateralization currency. However, this DOES NOT appear to be the market practice, as the quotes are independent of collateral.

135 E. 57th Street |25th Floor |New York, NY 10022
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Collateral Choice Model

1. Setup: Here, an American style path-dependent collateral is chosen at every incremental step by opting for the collateral among the choices available that maximizes the incremental collateral cash flow.
2. Motivation: Collateralization at the domestic collateral accrual rate is c_d. On switching over to the foreign collateral, the rate becomes $c_f + r_{d,f}$. Thus at each time step we want to maximize the incremental collateral cash flow

$$\max\left(c_d, c_f + r_{d,f}\right) = c_d + \max\left(c_f + r_{d,f} - c_d, 0\right)$$

We begin by setting

$$q_{d,f} = c_f + r_{d,f} - c_d$$

3. Dynamics of $Q_{d,f}$: Consider the dynamics of

$$Q_{d,f} = \frac{P_{d,f}}{P_{f,f}}$$

This entity has a drift

$$q_{d,f} = c_f + r_{d,f} - c_d$$

First of all, the dynamics of $c_d(t)$, $r_{d,f}(t)$, and $c_f(t)$ may be worked out using one of several typically accepted practices – e.g., the HJM-type dynamics, or an even more simplified Hull-White type dynamics.

- Using

$$Q_{d,f} = \frac{P_{d,f}}{P_{f,f}}$$

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Tel. 646-513-3300

it is fairly straightforward to show that

$$\frac{\Delta Q_{d,f}}{Q_{d,f}} = q_{d,f}(t)\Delta t + \sigma_q(t)\Delta W_q$$

where

$$\sigma_q(t)\Delta W_q = \sigma_f(t)\Delta W_f + \sigma_X(t)\Delta W_X - \sigma_d(t)\Delta W_d$$

4. Piterbarg (2012) Expression for $q_{d,f}(t)$: Piterbarg (2012) employs a combination of HJM machinery as listed above and additional techniques outlined in Andersen and Piterbarg (2010) to obtain $q_{d,f}(t)$.
5. Collateral Choice - Deterministic $q_{d,f}(t)$: If $q_{d,f}(t)$ is deterministic, there will be no optionality involved; however, depending upon the sign of $q_{d,f}(t)$, there will be a collateral switch at each time increment. Piterbarg (2012) demonstrates this in his framework by turning the volatility explicitly down to zero.
6. Deterministic and Incremental Curve Decay Collateral: If the collateral discounting path choice can be proxied using a "curve roll up" phenomenon, the collateral choice discount factor becomes

$$P_{d,CC}(t_0, t_n) = \prod_{i=1}^{n} \min\left(\{P_{d,j}(t_{i-1}, t_i)\}_{j=1}^{r}\right)$$

where

$$j = 1, \cdots, r$$

are the r possible collateral choices, $j = 0$ is the domestic collateral curve, $P_{d,j}(t_{i-1}, t_i)$ is the discount factor between t_{i-1} and t_i for one unit of domestic currency collateralized using

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Tel. 646-513-3300

the foreign collateral j, and $P_{d,CC}(t_{i-1}, t_i)$ is the collateral choice discount factor between t_{i-1} and t_i for one unit of domestic currency collateralized using the most appropriate incremental collateral. Note that this discount curve is artificial and deterministic.

- Advantages of using deterministic collateral choices => All the advantages stem from the computational simplicity. They are:
    - More than one collateral currency may be used, thus optimizing over the multiple collateral choices (USD, GBP, EUR, JPY, etc.)
    - Empirical Curve Representations using splining techniques may be usable

7. Valuing the Collateral Choice Option: The value we seek is of the form

$$P_{d,d}(0,T)\mathbb{E}_t^{Q_d^T}\left[e^{-\int_t^T \max(q_{d,f}(s),0)ds}V(T)\right]$$

where $V(T)$ is the terminal payoff at the time instant T. It may be a fixed amount (i.e., the fixed swap rate) or a variable amount (the floating swap coupon).

- Closed Form => Typically

$$\mathbb{E}_t^{Q_d^T}\left[e^{-\int_t^T \max(q_{d,f}(s),0)ds}\right]$$

has to be computed using Monte-Carlo or a PDE, therefore we seek an alternative fast analytic approximation. By Jensen's inequality, Piterbarg (2012) noticed that

$$\mathbb{E}_t^{Q_d^T}\left[e^{-\int_t^T \max(q_{d,f}(s),0)ds}\right] \geq e^{-\int_t^T \max(q_{d,f}(s),0)ds}$$

This approximation may be used to compute the fixed leg value for the swap above. For the floater leg, the term $V(T)$ may be pushed outside to a separated expectation to get

$$P_{d,d}(0,T)\mathbb{E}_t^{Q_d^T}[V(T)]\mathbb{E}_t^{Q_d^T}\left[e^{-\int_t^T \max(q_{d,f}(s),0)ds}\right]$$

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Piterbarg (2012) performs a full set of comparison to demonstrate that these approximations behave favorably with the Monte-Carlo under several situations.

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135 E. 57th Street |25th Floor |New York, NY 10022
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Section VI: Assorted Calibration, Hedging, and Valuation Considerations

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Convexity Corrections Associated with Margining

1. Origin of Convexity Corrections in Margining: Certain exchanges (esp. CME, on the futures) expect posting of the collateral (full or the maintenance amount) on moving out of the money. However, when in the money, you get nothing. This results in a returns mismatch asymmetry between in-the-money/out-of-the-money time snaps of the trade, i.e., this happens because you need to fund your margin gaps. This is also sometimes referred to as a one-way CSA.
2. Literature Confusion on Margin Convexity: The above asymmetry, of course, is a drag on the position value, and needs to be accounted for – and it is also referred to as convexity correction for some reason, and requires a dynamic rates model to value.
3. Modern CSA's and CCP's: CSA's essentially provide for symmetric collateral cash flow payments, therefore the situation listed above does not arise explicitly. Further some of the newer margining rules in CCP's treat in/out symmetrically (through a concept referred to as PAI – price alignment interest rate – which essentially appears like a collateral rate), so again in these term contracts these issues vanish.

135 E. 57th Street |25th Floor |New York, NY 10022
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Hedging Considerations

1. Curve Construction vs. Product Hedging Instrument Manifest Measure Choices: It may be preferable to incorporate a vaster universe of input instruments and their manifest measures (the manifest measures maybe both exact matches as well as the imperfect best-fit matches) in the latent state calibration. Hedging, however, may use only the most liquid set of products and their manifest measures.
2. Hedge Manifest Measure Moves: As a better approximation, of you can work out the β's of the manifest measure moves for the "non-hedging" product observation set to the manifest measure moves of the "hedging" manifest measure set, you may achieve a better and more complete PnL explain.

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Product Curve Effect Attribution

Market Value Change Explain Components

1. Linear Daily Market Value Change Components: The 3 main linear principal components to the market value change are:
    - Coupon Accruals
    - Time Value Market Parameters Intrinsic
    - Market Parameters Extrinsic.

    Time is an implicit factor entity across all the three components, simply because PnL explains are conducted across distinct time entity snapshots.

Coupon Accrual Intrinsic

1. Motivation: This is applicable only to coupon/dividend bearing securities – here the coupon payout is part of the security value, and therefore causes a security value jump at the payout dates.
2. Modern Accrual Intrinsic: This should include funding flows, re-investment flows, as well as collateral flows. Collateral flows should include initial, maintenance, and valuation margins. Switches on the collateral choice numeraire generate their own flows, depending on the corresponding rolling numeraires (analogous to the traditional CTD's). Accrual flows only relate to realized cash flows, and are therefore deterministic.

Market Parameters Intrinsic

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1. Motivation: This refers to the “riding the market” effects. The baseline level corresponds to the world where all the market levels stay frozen at the current instant levels. This is closes to what is referred to as θ (the intrinsic time value change).
2. Computation: Effectively this calculation addresses the question “How does the value of the derivative change as the market parameters stay frozen over the incremental period under consideration at the initial levels”. STAY FROZEN is NOT the same as riding the curve. Simply put, this principal component quantifies the incremental period market curve set effects, owing to the component’s maturity shrinkage by the corresponding time horizon.
3. Related Market Parameter Intrinsic Computation: The principal component measured precisely as above is referred to as the “maturity roll down” principal component. Related to this are the other ones:
    a. Maturity Roll Up => Here the derivative is valued by rolling/riding up the market curve
    b. Time Roll => Here the derivative values differences are estimated as the difference between the values at 2 distinct time snaps of the same latent state projected at the respective instants.
4. Modern Time Value Intrinsic: Roll up, roll down, and time roll are all computed on the instantaneous valuation market parameter set that determine the security value. Thus, there should be one time value intrinsic corresponding to each latent state.
5. Shape Sensitive Explain Component: The market parameter intrinsic component happens to be the most shape sensitive explain component (this includes roll down, roll up, and time roll). As a consequence, this component ends up being the most sensitive to the splined latent state representation scheme.

Market Parameters Extrinsic

1. Motivation: This principal component aims to capture the first order market move impact on the security value. This leading linear order is referred to as the “curve shift” effect, i.e., impact of the change in quote that intuitively corresponds closest to the product’s extrinsic market move impact (if a single such manifest measure quote uniquely exists). Subsequent

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Tel. 646-513-3300

orders (such as twist/tilt, farther quote manifest measure re-calibration impact etc.) cause higher order change impacts (e.g., convexity, butterflies, etc.)

2. Modern Market Parameters Extrinsic: Since the extrinsic market parameters simply correspond in reality to the full variety of the calibrated latent states, each of these latent state metric change triggers the corresponding linear principal component shift, and thereby a non-zero corresponding explain component. The additional "modern" latent states contributing to these factors are the collateral curves, collateral switch curves, funding curves, and re-investment curves.

Market Value Change Effects Formulation

1. The Linear Explain Components:

$$\Delta V(t_2, t_1) = V(t_2) - V(t_1)$$

$$= \sum_{t_i = t_1}^{t_i < t_2} C_f(t_i) + (t_2 - t_1) \sum_j \mathcal{R}_j(t_1)$$

$$+ \sum_j \left[\mathbb{C}_j(t_2, t_1) \left(\mathcal{L}_j(t_2) - \mathcal{L}_j(t_1) \right) \right]$$

$$\sum_{t_i = t_1}^{t_i < t_2} C_f(t_i)$$

is the cumulative carry.

$$\mathcal{R}_j(t_1) = \left\{ \frac{\partial V}{\partial \mathcal{L}_j} \frac{\partial \mathcal{L}_j}{\partial t} | \mathfrak{I}_{t_1} \right\}$$

is the per-market parameter specific roll-down.

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$$\mathbb{C}_j(t_2, t_1) = \left\{\frac{\partial V}{\partial \mathcal{L}_j} | \mathfrak{I}_{t_2}\right\}$$

is the per-market parameter specific "curve shift". $\mathcal{L}_j$ refers to the latent state designated by the market value, and may need to be computed as

$$\frac{\partial V}{\partial \mathcal{L}_j} = \sum_k \frac{\partial V}{\partial q_k} \frac{\partial q_k}{\partial t}$$

where $\{q_k\}$ corresponds to the quote set required for the calibration, and $\frac{\partial V}{\partial q_k}$ is the corresponding Jacobian.

2. <u>The Linear Explain Using Value Quote Jacobian</u>: In practice what we want is

$$\Delta V(t_2, t_1) = V(t_2) - V(t_1)$$
$$= \sum_{t_i = t_1}^{t_i < t_2} C_f(t_i) + (t_2 - t_1) \sum_j \mathcal{R}_j(t_1) + \sum_j \left[\widetilde{\mathbb{Q}}_j(t_2, t_1) \left(\mathbb{Q}_j(t_2) - \mathcal{L}_j(t_1)\right)\right]$$

where

$$\widetilde{\mathbb{Q}}_j(t_2, t_1) = \left\{\frac{\partial V}{\partial \mathbb{Q}_j} | \mathfrak{I}_{t_2}\right\}$$

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Section VII: Statistical Learning in Curve Construction

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Inference-Based Curve Construction

Curve Smoothing in Finance

1. Unconstrained Curve Smoothing:
    - Applicable primarily for rates/semi-liquid FX curves. Smoothing can be done here without constraints.
    - Smoothing may also be applicable to the quotes for a given instrument across several days.
    - Smoothing may also be applied over a single day curve – particularly to model the switch over from instrument to instrument (e.g., between EDF and Swaps).
2. Constrained Curve Smoothing: Applicable, for e.g., to the case of a hazard curve. The smoothing basis functions/weights combination must guarantee, from a formulation PoV, that the implied hazard rate is always greater than zero.
3. Latent State Inference as a Deep Learning Exercise: Multi-stretch, multi-pass latent state inference/representation (esp. in the financial curve construction context) can be essentially construed as a shallow version of the deep neural network.
4. Liquidity Based Weighted Signal Smoothing:
    - Fidelity at the "liquid bonds" / benchmark bond nodes
    - Lower fidelity penalty, but higher smoothness penalty for the less liquid bonds
    - Penalty measure is calculated off of the relative liquidity ranking measure (for e.g., TRACE)
5. Non Bayesian Liquidity Based Smoothing:
    - Liquidity indicator serves as a roughness/fidelity magnifier/dampener
    - Also need to penalize for over-parameterized fits using AIC/BIC (also CV/GCV – given that this is essentially a frequentist case).
    - These can be applied not just for bonds, but also CDS, rates, FX – even less liquid ones.
6. Bayesian Extension to the above: Any parametrically specified distribution needs to evolve using a hyper-prior, and the Wahba parametric Bayesian priors need evolving too.

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7. Nodal Jacobian/Sensitivity Impact: As always study the impact on the locality of the perturbation, as well as the ease of Jacobian estimation – esp. if the calibration needs to occur through MCMC, non-linear optimization etc.
8. Mixtures of splines and smoothness penalties: As always estimate the impact on monotonicity, convexity, shape preservation etc. - the category item checks in Goodman's paper.
9. Knot Selection Tips: Need some tips in both situations – frequentist and Bayesian.
10. Suggestion on the locally adaptive Parametric Form: Examine the knot-to-knot smoothness and penalty by using additional locally adaptive microstructure parameters and their implications.
11. Goodman and Eilers/Marx Talking Point Issues: Criterion check for these specific "goodness" checks.

Bayesian Curve Calibration

1. Bayesian based past knowledge incorporation of survival probabilities: Given that the prior's, the posterior's and the likelihood's are all probabilities, perhaps the best starting point is for applying it to the problem of updating the survival probabilities and recovery rates based on price observations.
2. Curve Updating techniques: Need grand new formulation techniques that are based on AD and Bayesian methodologies as part of the curve updating strategies based upon individual incoming observations and their strength signals.
3. Curve Construction off of hard/soft signals: Hard Signals are typically the truthness signals. Typically reduce to one calibration parameter per hard observation, and they include the following:
    - Actual observations => Weight independent true truthness signals
    - Weights => Potentially indicative of the truthness hard signal strength

    Soft signals are essentially signals extracted from inference schemes. Again, typically reduce to one calibration parameter per soft inference unit, and they include the following:

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- Smoothness signals => Continuity, first, second, and higher-order derivatives match – one parameter per match.
- Bayesian update metrics => Inferred using Bayesian methodologies such as maximum likelihood estimates, variance minimization, and error minimization techniques.

4. No-arbitrage hard signals: Simply indicates that ***the*** given hard observation is out of bounds and irreconcilable (i.e., no solution can be found) within the axiomatic inference space dictated by:
    - The parameter sequence implied by the other set of hard signals.
    - The model axiom schemes.
    - The inference rules.

- Directionality "bias" is inherent in calibration (e.g., left to right, ordered sequence set, etc.) – this simplifies the problem space significantly. Therefore, the same directional bias also exists in the calibration nodal sequence.

5. Parameter Space Explosion: Generally not a problem as long as it is segment-localized (in matrix parlance, as long the transition matrix is tri-diagonal, or close to it), i.e., local information discovery does not affect far away nodes/segments.

- Also maybe able to use optimization techniques to trim them.

6. Live Calibrated Parameter Updating: Use automatic differentiation to:

- Estimate parametric Jacobians (or sub-coefficient micro-Jacobians) to the observed product measures.
- Re-adjust the shifts using the hard-signal strength.
- Update the parameters from the calculated shifts.
- Re-construct the curve ever so periodically (for a full re-build, as opposed to the incremental).
- Remember that AD based parametric updates break smoothness (including continuity as Bayesian MLE's) – so use a tolerance in the shift if this is acceptable.

7. Causality Bayesian Network DAG For Credit Curve Building: See Figure 1.

- DAG searches are not really needed, since here they maybe formulated conceptually/axiomatically, as opposed to being established through a search mechanism.

8. Financial First Principles SKU: Following concepts are the core components that can be used to create the curve construction SKU:

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- Time Value of Money.
- Latent Default Indicator.
- Recovery on Default.
- Imbalance premium/discount (for FX, Basis Swaps, etc.)

9. Financial Signal Analysis: Need special analysis techniques to pick out "event trends" from "concept jumps", even for highly liquid instruments.
    - Liquidity-based Signal Extraction =>
        - Identify a liquidity metric
        - Imply the "perfect liquidity" – the point at which there is no premium
        - Compute the liquidity metric for each security
        - Regress (or conceptually determine, or fit) the bid-ask spread to inverse liquidity (remember that even benchmarks only have finite liquidity, not infinite) for each security
        - Try to slap in a secular "event premium" across all the instruments, over and above liquidity
10. Systemic Finance Variables Evolution: Given that every measurement is uncertain to within bounds, it stands to reason that every distribution is also a true distribution (to within the tolerance provided by the corresponding sufficient statistics, and over a finite observation window) of the technical state of the world (i.e., technical = fundamental + a bias).
11. Technical to Fundamental Bias Estimation: This should result from the flow of the information. Non-technical/Fundamental may possibly be estimated using a bias correction applied to the technical signal – Bayesian/frequentist techniques may be of value here.
    - Proxy for non-technical behavior => Identify the non-market proxies for the fundamental drivers, and estimate market drivers as possibly lagging indices.
12. Bayesian Decomposition of Technical Signals: In general, the signal core drivers are limited (like systemic/idiosyncratic factors – alternatively, the latent state quantification metric), but the product specific manifest measures are more varied. Bayesian frameworks well suited for these.
13. Financial Stretch Identification: Bayesian classification techniques can be readily adapted for these purposes – in fact, with abundance of data, these techniques are very appropriate now.

135 E. 57th Street |25th Floor |New York, NY 10022
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Sequential Curve Estimation

1. <u>Calibration Framework Drivers</u>: Calibration is considered to occur FOR a hidden state $\vec{S}$, which is quantified using the quantification metric $\vec{X}$. $\vec{X}$ is estimated from the manifest measure $\vec{X}$.
2. <u>Product-Measure Point-of-View</u>: From the Dempster-Shaefer/Kalman Filter/Linear Quadratic Estimator point-of-view, the Kalman $\vec{H}$ matrix probabilistically transforms the hidden state quantification metric to an observation measure, e.g., the latent forward rate manifests itself through the swap rate.
3. <u>Segment/Span Nomenclature vs. Curve Calibrator Nomenclature</u>: Call the Curve Calibrator the Dempster-Shaefer Calibrator. Under this:
   - LSQM (Latent State Quantification Metric) => Elastic Variate
   - State Dimensions (Tenor Axis, X/Y Axis of predictors) => Inelastic Variates
   - Thus, the predictors are inelastic, and the responses are elastic.
4. <u>Linearization of LSQM over the predictor axes</u>: The Kalman $\vec{H}$ observation transformer should just linearize $\vec{M}$ onto the space of $\vec{X}$ over the predictor dimensions. Non-linearity of $\vec{X}$ over the predictors is handled through basis splines.
5. <u>Hidden State Evolution vs. Hidden State Representation</u>: The Kalman $\vec{H}$ matrix is more of a state modeling and state representation matrix (i.e., the update part that is fully local to the current time slice) that already brings in the manifest measure ⇔ LSQM transformation model.
6. <u>The Curve Builder $\vec{H}$ matrix</u>: Due to the above, the curve builder $\vec{H}$ matrix needs to accommodate the 2 possible uncertainties:
   - Uncertainty in the manifest measure
   - Uncertainty in the manifest measure ⇔ LSQM transformation model. If this transformation is non-parametric, then treat it as certain/deterministic. If it is parametric, then use MLE/MAP to the handle the parameter estimation.

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7. UKF Techniques applied to evolve the Curve Builder $\vec{H}$ matrix: Potential non-linearity in the curve builder $\vec{H}$ may be handled using the Jacobian EKF and/or the sigma-point UKF schemes.
8. The Curve Builder $\vec{F}$ matrix: The Curve builder $\vec{F}$ Matrix dictates the evolution from t_i to t_{i+1} as $LSQM_{i+1} = \vec{F} \times LSQM_i$. This should be explicitly posited/formulated. Again, use splining to linearize.
9. Financial Noise Covariance Estimation: May be able to extraneously determine these covariance independent of the state evolution model (if not, we may have to rely on techniques such as ALS (Rajamani (2007), Rajamani and Rawlings (2009)).

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Section VIII: Bond Relative Value Metrics Generation

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Credit Analytics Bond RV Calculation Methodology

Introduction

This document outlines the methodology used in Credit Analytics (release 1.4 and above) for the calculation of the bond curve-based relative value measures.

The Bond RV Measure Set

Classification of a given bond measure as an "RV" measure is somewhat arbitrary. In general, it is used (here) to refer to any of the measure that is in use for spotting relative value across bonds for a given issuer (or any similar category), and which is usually determined straight from a bond market measure (price/yield/spread to treasury). Specifically, it excludes such bond measures as DV01, loss PV, principal PV etc.

Following is the list of the RV measures - refer to the section below for a precise definition of these terms.

- Bond Basis
- Convexity
- Credit Basis
- Discount Margin
- Duration
- DV01
- G Spread (Spread to the Government/Treasury Discount Curve)
- I Spread (Interpolated Spread to the Discount Curve)
- Option Adjusted Spread

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- Macaulay Duration
- Modified Duration
- Par Asset Swap Spread
- Par Equivalent CDS Spread (PECS)
- Price
- Spread over Treasury (TSY) benchmark
- Yield
- Yield Basis
- Yield Spread
- Yield01
- Zero Discount Margin (ZDM)
- Zero (Z) Spread

Asset Swap Spread

Asset swap is an estimate of the spread over a matching swap maturing at the bond's maturity. For a non-par swap, an additional spread is added by dividing the departure from par by the swap annuity.

Bond Basis

Bond Basis to Exercise (B_E) is a bond RV metric capturing the basis in the yield space. It is defined as the difference between the yield to exercise computed from the market price and the yield to exercise computed from the theoretical price off of the risk-free discount curve.

Convexity

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Convexity to Exercise (C_E) measures the rate of change of duration with yield. It is defined as the change in market duration on 1 basis point increase in yield.

Credit Basis

Credit Basis to Exercise (Φ_E) captures the adjustment needed to the input credit curve to account for the bond market price. It is defined as the parallel shift needed to be applied across the input credit curves quotes to make create the credit curve that produces the market price.

Credit Basis can be negative; given that the credit curve does not typically calibrate for negative hazard rates, the credit basis may not be calculable for market prices above a certain range.

Discount Margin

Discount Margin to Exercise (Δ_E) measures that spread earned above the reference rate. For fixed coupon bonds, it is computed as the difference between market yield and the initial implied discount rate to the bond's frequency. For floaters, it is computed as the difference between market yield and the initial reference index rate.

Duration

Duration to Exercise (D_E) captures the relative rate of change of bond price with yield. It is defined as the fractional change of price as the market yield increases by 1 basis point.

DV01

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DV01 to exercise captures the cash flow present-value weighted pay date durations.

G Spread

G Spread to Exercise (G_E) accounts for the Spread over the Government/Treasury Discount Curve. It is defined as the difference between the market yield to exercise of the bond and the rate calculated to the exercise date, implied from the specified discount curve constructed from the government debt instruments.

I Spread

I-Spread to Exercise (I_E) measures the spread over the specified Discount Curve interpolated to the exercise date. It is defined as the difference between the market yield to exercise of the bond and the rate interpolated to the exercise date, implied from the specified discount curve.

Macaulay Duration

Macaulay duration to exercise captures the cash flow present-value weighted pay date durations.

Modified Duration

Modified duration to Exercise (D_E) captures the relative rate of change of bond price with yield. It is defined as the fractional change of price as the market yield increases by 1 basis point.

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Option Adjusted Spread

Option adjusted to Exercise (O_E) spread captures the value of the option embedded in the bond. It is calculated identical to the Z-Spread (see Z-Spread for details), although it may be based off of a different discount curve.

Par Asset Swap Spread

Par asset swap spread to Exercise (P_E) estimates the spread implied by the price that a par floater would be expected to pay. It is defined as the difference between the market price and the theoretical price computed using the discount curve, computed in units of the bond PV01 (duration times price).

Par Spread

Par spread to Exercise (Ω_E) estimates the fair fixed coupon implied by the market price that an equivalent fixed coupon bond trading at par would pay. It is defined as the difference between the market price and par, computed in units of the bond PV01 (duration times price).

Par Equivalent CDS Spread (PECS)

The PECS to Exercise (Θ_E) measures the flat credit spread premium implied by the bond price. It is computed as the implied flat spread of the fictitious CDS needed to recover the market price of the bond.

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Price

The theoretical exercise price of the bond can be computed from the bond cash flows, the discount curve and/or the credit curve and recovery using the methodology described below.

Spread Over Treasury Benchmark

Treasury Spread to Exercise (S_{TSY}) accounts for the returns over the given benchmark bond. It is defined as the difference between the market yield to exercise of the bond and the yield to maturity of the specified benchmark treasury bond.

Yield

The yield to exercise (y_E) implied from the bond market price is calculated according to the equations shown below.

Yield Basis

Yield basis to Exercise is defined identically as the bond basis. See Bond Basis for details.

Yield Spread

Yield spread is defined identically as the bond basis. See Bond Basis for details.

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Yield01

Yield01 (also called YV01) to Exercise (D_E) captures the relative rate of change of bond price with yield. It is defined as the fractional change of price as the market yield increases by 1 basis point.

Zero Discount Margin (ZDM)

Zero Discount Margin to Exercise (Ψ_E) estimates the excess spread over the reference index curve. It is a measure valid only for floaters; it is defined as the extra coupon spread to be applied to the reference index rate curve so as to be able to recover the market price.

Zero (Z) Spread

Z Spread to Exercise (Z_S) captures the excess spread over the discount curve. The details of implying the zero-curve and the corresponding calculation of the Z Spread are described below.

Symbol	Description
B_E	Bond Basis to Exercise
C_E	Convexity to Exercise
Φ_E	Credit Basis to Exercise
Δ_E	Discount Margin to Exercise
D_E	Duration to Exercise
G_E	G-Spread to Exercise
I_E	I Spread to Exercise
O_E	Option Adjusted Spread to Exercise
P_E	Par Asset Swap Spread to Exercise

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Ω_E	Par Spread to Exercise
Θ_E	Par Equivalent CDS Spread to Exercise
Ψ_E	Zero Discount Margin to Exercise
ϵ_i	The Full Period Coupon Rate between t_{i-1} and t_i
φ_E	Government Curve implied Rate to Exercise
$\Gamma_C(i-1,i)$	Coupon Day Count Fraction between t_{i-1} and t_i
$\Gamma_Y(i-1,i)$	Yield Quote Day Count Fraction between t_{i-1} and t_i
δ_{IR}	Spread applied to the Interest Rate Curve
d_c	Coupon Day Count Convention
d_{yc}	Yield Quote Day Count Convention
f_c	Coupon Frequency
f_y	Yield Quote Frequency
t_i	Time at coupon flow #i
t_E	Exercise Date Time
y_E	Yield To Exercise
$C_f(t_i)$	Coupon Flow at Date Time t_i
$D_f(t_i)$	Discount Curve based Discount Factor at Date Time t_i
$D_f(\delta,t_i)$	δ Bumped Discount Curve based Discount Factor at Date Time t_i
$D_f(y_E,f_y,d_{yc},t_i)$	Discount Factor at Date Time t_i given Yield To Exercise y_E, Quote Frequency f_y, Quote Day Count Convention d_{yc}
$D_f(z_S,f_y,d_{yc},t_i)$	Discount Factor at date time t_i given the Z Spread z_S, the quote frequency f_y, Quote Day Count Convention d_{yc}
N_E	Notional at Exercise
N_j	Outstanding Notional at Date Time N_j
ΔN_j	Principal Notional Payout at Date Time N_j
$P_{Dirty}(IR_{Theo})$	Theoretical Dirty Price calculated from the input IR Curve
$P_{CR,Dirty}(IR_{Theo},CR_{Theo})$	Theoretical Dirty Price calculated from the input IR and Credit Curves
$P_{Dirty}(\delta,IR_{Theo})$	Theoretical Dirty Price calculated from the input IR Curve with a spread adjustment

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$P_{CR,Dirty}(\lambda_{CR}, IR_{Theo}, CR_{Theo})$	Theoretical Dirty Price calculated from the input IR Curve and Credit Curve, where the Credit Curve is created off of a flat spread λ_{CR}
$P_{CR,Dirty}(\delta_{CR}, IR_{Theo}, CR_{Theo})$	Theoretical Dirty Price calculated from the input IR Curve and Credit Curve, with a spread adjustment δ_{CR} applied to the Credit Curve
R_E	Discount Curve implied Rate to Exercise
$S_P(t)$	Survival Probability to time t
S_{TSY}	Treasury Benchmark Spread to Exercise (done)
y_{BMK}	Yield of the Specified Treasury Benchmark
y_E	Yield to Exercise
$y_E(IR_{Theo}, CR_{Theo})$	Theoretical Yield to exercise
$\{z_i\}$	Collection of the ordered nodes $\{z_i\}$ that constitute the Zero Curve
z_i	Zero Rate to the Date Time t_i
z_S	Z Spread

Basic Measures

The Coupon Cash Flow of the bond at coupon date time t_i is given as

$$C_f(t_i) = \epsilon_i \Gamma_c(i-1, i) d_c$$

The Discount Factor at date time t given the yield to exercise y_E, the quote frequency f_y, and the annualized quote day count based time fraction $\Gamma_y(i-1, i)$ is given as

$$D_f(y_E, f_y, d_{yc}, t_i) = \left[\frac{1}{1 + \frac{y_E}{f_y}}\right]^{f_y \Gamma(0,t)}$$

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The Zero Rate z_i to a date time t_i is determined by the solution to z_i that computes the discount factor $D_f(t_i)$ given the quote frequency f_y, and the annualized quote day count based time fraction $\Gamma_y(i-1,i)$ is given as

$$D_f(t_i) = \left[\frac{1}{1+\frac{z_i}{f_y}}\right]^{f_y\Gamma(0,t)}$$

The Discount Factor at date time t_i given the zero rate z_i, the Z Spread z_S, the quote frequency f_y, and the annualized quote day count based time fraction $\Gamma_y(i-1,i)$ is given as

$$D_f\left(z_S, f_y, d_{yc}, t_i\right) = \left[\frac{1}{1+\frac{z_i+z_S}{f_y}}\right]^{f_y\Gamma(0,t_i)}$$

The Principal redeemed, amortized, or capitalized at time t_j is given as

$$\Delta N_j = N_j - N_{j-1}$$

The Dirty Price of the bond at exercise given an exercise yield y_E is given as

$$P_{Dirty}(y_E) = \sum_i C_f(t_i)D_f\left(y_E, f_y, d_{yc}, t_i\right) + \Delta\sum_j N_j D_f\left(y_E, f_y, d_{yc}, t_j\right) + N_E D_f\left(y_E, f_y, d_{yc}, t_E\right)$$

The Dirty Price of the bond at exercise given a Z spread (z_S) is given as

$$P_{Dirty}(z_S) = \sum_i C_f(t_i)D_f\left(z_S, f_y, d_{yc}, t_i\right) + \sum_j \Delta N_j D_f\left(z_S, f_y, d_{yc}, t_j\right) + N_E D_f\left(z_S, f_y, d_{yc}, t_E\right)$$

The Theoretical IR implied Dirty Price $P_{Dirty}(IR_{Theo})$ of the bond at exercise calculated using the discount factors from the input discount curve is given as

$$P_{Dirty}(IR_{Theo}) = \sum_{i} C_f(t_i)D_f(t_i) + \sum_{j} \Delta N_j D_f(t_j) + N_E D_f(t_E)$$

The IR implied Dirty Price $P_{Dirty}(\delta_{IR}, IR_{Theo})$ of the bond at exercise calculated using the discount factors from the input discount curve bumped by a rate δ_{IR} is given as

$$P_{Dirty}(\delta_{IR}, IR_{Theo}) = \sum_{i} C_f(t_i)D_f(\delta_{IR}, t_i) + \sum_{j} \Delta N_j D_f(\delta_{IR}, t_j) + N_E D_f(\delta_{IR}, t_E)$$

The Theoretical Credit implied Dirty Price $P_{CR,Dirty}(IR_{Theo}, CR_{Theo})$ of the bond at exercise calculated using the discount factors and the survival probabilities from the input discount curve and the credit curve respectively is given as

$$P_{CR,Dirty}(IR_{Theo}, CR_{Theo}) = \sum_{i} C_f(t_i)D_f(t_i)S_P(t_i) + \sum_{j} \Delta N_j D_f(t_j)S_P(t_j) + N_E D_f(t_E)S_P(t_E)$$

The Theoretical Credit implied Dirty Price $P_{CR,Dirty}(\delta_{CR}, IR_{Theo}, CR_{Theo})$ of the bond at exercise calculated using the discount factors and the survival probabilities from the input discount curve and the credit curve respectively, where the credit curve is bumped by a rate δ_{CR} is given as

$$\begin{aligned} P_{CR,Dirty}(\delta_{CR}, IR_{Theo}, CR_{Theo}) \\ &= \sum_{i} C_f(t_i)D_f(t_i)S_P(\delta_{CR}, t_i) + \sum_{j} \Delta N_j D_f(t_j)S_P(\delta_{CR}, t_j) \\ &+ N_E D_f(t_E)S_P(\delta_{CR}, t_E) \end{aligned}$$

The Credit Basis to Exercise Φ_E of the bond given the market price (P_{MKT}) is given as the solution of δ_{CR} in (30.11):

$$P_{CR,Dirty}(\Phi_E, IR_{Theo}, CR_{Theo}) = \sum_i C_f(t_i) D_f(t_i) S_P(\Phi_E, t_i) + \sum_j \Delta N_j D_f\left(t_j\right) S_P\left(\Phi_E, t_j\right) + N_E D_f(t_E) S_P(\Phi_E, t_E)$$

The Theoretical Credit implied Dirty Price of the bond at exercise $P_{CR,Dirty}(\lambda_{CR}, IR_{Theo}, CR_{Theo})$ is calculated using the discount factors and the survival probabilities from the input discount curve and the credit curve respectively, where the credit curve is created off of a flat spread λ_{CR}, is given as

$$P_{CR,Dirty}(\lambda_{CR}, IR_{Theo}, CR_{Theo}) = \sum_i C_f(t_i) D_f(t_i) S_P(\lambda_{CR}, t_i) + \sum_j \Delta N_j D_f\left(t_j\right) S_P\left(\lambda_{CR}, t_j\right) + N_E D_f(t_E) S_P(\lambda_{CR}, t_E)$$

The Par Equivalent CDS Spread to Exercise of the bond given the market price (P_{MKT}) is given as the solution of δ_{CR} in (30.13):

$$\begin{aligned} P_{CR,Dirty}(\delta_{CR}, IR_{Theo}, CR_{Theo}) &= \sum_i C_f(t_i) D_f(t_i) S_P(\delta_{CR}, t_i) + \sum_j \Delta N_j D_f\left(t_j\right) S_P\left(\delta_{CR}, t_j\right) \\ &+ N_E D_f(t_E) S_P(\delta_{CR}, t_E) \end{aligned}$$

The Bond Spread to Treasury Benchmark at exercise S_{TSY} is computed from the Bond Yield to Exercise y_E and the given Treasury Benchmark Yield y_{BMK} as

$$S_{TSY} = y_E - y_{BMK}$$

The Bond I Spread to exercise I_E is computed from the Bond Yield to Exercise y_E and the Discount rate to Exercise implied from the input Interest Rate Curve R_E as

$$I_E = y_E - R_E$$

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The Bond G Spread to exercise G_E is computed from the Bond Yield to Exercise y_E and the Discount rate to Exercise implied from the input Government Rate Curve φ_E as

$$G_E = y_E - \varphi_E$$

The Theoretical Yield to exercise $y_E(IR_{Theo})$ of the bond at exercise calculated using the discount factors from the input discount curve is given as the solution of y_E in (30.6), where the dirty price P_{Dirty} is substituted by $P_{Dirty}(IR_{Theo})$ of (30.8).

The Bond Basis at exercise B_E (also referred to as yield basis or as yield spread) is computed from the Bond Yield to Exercise y_E and the Bond Yield to Exercise y_E as

$$B_E = y_E - y_E(IR_{Theo})$$

The Bond Duration to exercise D_E is computed as the fractional change in bond market price (P_{MKT}) to the change in the market yield (Y_{MKT}) as

$$D_E = \frac{1}{P_{MKT}} \frac{\Delta P_{MKT}}{\Delta Y_{MKT}}$$

The Bond Convexity to exercise C_E is computed as the change in bond market duration to exercise (D_E) to the change in the market yield (Y_{MKT}) as

$$C_E = \frac{\Delta D_E}{\Delta Y_{MKT}}$$

The Discount Margin to Exercise Δ_E of the bond given the market yield to exercise (y_E) is given as:

$$\Delta_E = y_E - R_E$$

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The Par Asset Swap Spread to Exercise (P_E) of the bond given the market price (P_{MKT}) is given as:

$$P_E = \frac{1}{P_{MKT}} \frac{P_{Dirty}(IR_{Theo}) - P_{MKT}}{D_E}$$

The Option Adjusted Spread to Exercise O_E is calculated identical to Z Spread, as a solution to z_S in (30.7).

Some Trivial Closed-Form Analytical Bond Math Results

1. Price when Yield Equals Coupon: Given the annualized coupon r, payment frequency f, period yield y, period coupon payment $c = \frac{r}{f}$, number of coupon periods to maturity n, the PV is computed from

$$PV = \sum_{m=1}^{n} \frac{\frac{r}{f}}{(1+y)^m} + \frac{1}{(1+y)^n} = \sum_{m=1}^{n} \frac{c}{(1+y)^m} + \frac{1}{(1+y)^n} = cd\frac{1-d^n}{1-d} + d^n$$

where

$$d = \frac{1}{1+y}$$

If you are just past a coupon pay so that

$$PV_{Clean} = PV_{Dirty}$$

and if

$$PV = 1$$

then we get

$$1 = cd\frac{1-d^n}{1-d} + d^n$$

which implies that

$$d = \frac{1}{1+c}$$

and thus

$$y = c$$

2. Par Yield Dirty Price at a non-coupon Date: If ξ is the accrual fraction corresponding to the accruing period, then

$$PV_{Dirty} = \frac{\xi c}{(1+y)^{\xi}} + \frac{c}{(1+y)^{\xi+1}} + \cdots + \frac{c}{(1+y)^{\xi+n}} + \frac{1}{(1+y)^{\xi+n}}$$

which reduces to

$$PV_{Dirty} = \frac{\xi c}{(1+y)^{\xi}} + \frac{1}{(1+y)^{\xi}}\left[\sum_{m=1}^{n}\frac{c}{(1+y)^m} + \frac{1}{(1+y)^n}\right] = \frac{1+\xi c}{(1+y)^{\xi}}$$

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Section IX: Stochastic Evolution and Option Pricing

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Stochastic Calculus

Single Factor Stochastic Calculus

1. The Principal Brownian Stochastic Differential Equation: Given

$$\Delta S = \mu_S(S, t)\Delta t + \sigma_S(S, t)\Delta W_S$$

   the Brownian SDE that accounts for the evolution of

$$F \equiv F(S, t)$$

   is

$$\Delta F(S, t) = \left[\frac{\partial F(S, t)}{\partial t} + \mu_S(S, t)S\frac{\partial F(S, t)}{\partial S} + \frac{1}{2}{\sigma_S}^2(S, t)S^2\frac{\partial^2 F(S, t)}{\partial S^2}\right]\Delta t + \sigma_S(S, t)S\Delta W_S$$

   ONLY in situations where $F(S, t)$ is second order or higher in S does the Brownian SDE have a non-trivial contribution (arising from the $\frac{\partial^2 F(S,t)}{\partial S^2}$ term).

2. Incorporation of the Weiner Process in the Brownian SDE: This is incorporated exclusively at the point

$$\underset{\Delta t \to 0}{Limit} \langle {\Delta W_S}^2 \rangle \cong \Delta t$$

   Notice that in the SDE above $\langle {\Delta W_S}^2 \rangle$ shows up only in conjunction with $\frac{\partial^2 F(S,t)}{\partial S^2}$! Consistent with the limit above, one may often find similar meanings behind statements such as

$$\underset{\Delta t \to 0}{Limit} \langle \Delta W_i \Delta W_j \rangle \cong \rho_{ij}\Delta t$$

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3. Non-Brownian Evolution SDE's: In general,

$$\Delta F(S,t) = \frac{\partial F(S,t)}{\partial t}\Delta t + \frac{\partial F(S,t)}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 F(S,t)}{\partial S^2}(\Delta S)^2 + \frac{1}{2}\frac{\partial^3 F(S,t)}{\partial S^3}(\Delta S)^3 + \cdots$$

As we saw earlier, for Brownian we observed that $\underset{\Delta t \to 0}{Limit} \langle \Delta W_S{}^2 \rangle \cong \Delta t$. Of course, there could be other SDE's where the higher order terms may have specific stochastic expectations applied in the limit.

4. General Purpose Validity of the Stochastic Differential: The power behind the validity of $\Delta S = \mu_S(S,t)\Delta t + \sigma_S(S,t)\Delta W_S$ is as follows: Since both μ_S and σ_S can, in general, be functions of S and t, the incremental ΔS still **ABSOLUTELY** follows the stochastic dynamics dictated by the driver process ΔW_S.
   - Further, given the stochastic shock for ΔS may go in the opposite direction from ΔW_S, σ_S can be absolutely negative as well.
5. Convenience of the log form: By setting

$$\Delta F(S,t) = \log S$$

the driver equation now becomes

$$\Delta(\log S) = \left[\mu_S(S,t) - \frac{1}{2}\sigma_S{}^2(S,t)\right]\Delta t + \sigma_S(S,t)\Delta W_S$$

The advantages to this are:

- $S > 0$ always (by demands of the log normal form), and
- The RHS may be independent of S, and dependent only on t, which enables explicit evaluation of the dynamics.

The reason for the above advantages is that, in

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$$\Delta F(S,t) = \left[\frac{\partial F(S,t)}{\partial t} + \mu_S(S,t)S\frac{\partial F(S,t)}{\partial S} + \frac{1}{2}\sigma_S{}^2(S,t)S^2\frac{\partial^2 F(S,t)}{\partial S^2}\right]\Delta t + \sigma_S(S,t)S\Delta W_S$$

the only explicitly S-dependent terms are $S\frac{\partial F(S,t)}{\partial S}$ and $S^2\frac{\partial^2 F(S,t)}{\partial S^2}$. The $\log S$ form above reduces these terms to

$$S\frac{\partial F(S,t)}{\partial S} = 1$$

and

$$S^2\frac{\partial^2 F(S,t)}{\partial S^2} = -1$$

In the case where

$$\mu_S(S,t) \equiv \mu_S(t)$$

and

$$\sigma_S(S,t) \equiv \sigma_S(t)$$

(i.e., μ_S and σ_S depend only on t), an explicit expression for S may be worked out as

$$S = S_0 e^{\int_{t_0}^{t}\left[\mu_S(t)-\frac{1}{2}\sigma_S{}^2(t)\right]dt} e^{\int_{t_0}^{t}\sigma_S(t)dW_S}$$

6. Universality of the results above: Despite a "particular" choice for the transformation of S for the solution seen above, the evolution dynamics specified above are ABSOLUTELY VALID UNIVERSALLY for the latent state dynamics of S. The only axiomatic stipulation is that W be Brownian with variance $\sqrt{\Delta t}$.

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Multi-Factor Stochastic Calculus

1. Multi-variate non-Stochastic Evolution: Consider 2 driver deterministic processes

$$\frac{\Delta B}{B} = \mu_B \Delta t$$

and

$$\frac{\Delta F}{F} = \mu_F \Delta t$$

From above

$$\Delta(BF) = \Delta B . F + \Delta F . B = \mu_B BF \Delta t + \mu_F BF \Delta t$$

results in

$$\frac{\Delta(BF)}{BF} = \mu_B \Delta t + \mu_F \Delta t = \frac{\Delta B}{B} + \frac{\Delta F}{F}$$

2. Multi-variate Stochastic Evolution: The 2 driver processes now become

$$\frac{\Delta B}{B} = \mu_B \Delta t + \sigma_B \Delta W_B$$

and

$$\frac{\Delta F}{F} = \mu_F \Delta t + \sigma_F \Delta W_F$$

From above

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$$\Delta(BF) = \Delta B.F + \Delta F.B + \Delta B.\Delta F = BF\frac{\Delta B}{B} + BF\frac{\Delta F}{F} + \Delta B.\Delta F$$

results in

$$\frac{\Delta(BF)}{BF} = \frac{\Delta B}{B} + \frac{\Delta F}{F} + \frac{\Delta B \Delta F}{BF}$$

The final component $\frac{\Delta B \Delta F}{BF}$ is attributable to the stochastic correlative cross product.

3. <u>$\frac{\Delta B \Delta F}{BF}$ Simplification</u>: Consider the situation where ΔW_B and ΔW_F are both Brownian such that

$$\Delta W_B \Delta W_F = \rho_{BF} \Delta t$$

Then

$$\Delta B \Delta F = \rho_{BF} \sigma_B \sigma_F BF \Delta t$$

Thus

$$\frac{\Delta B \Delta F}{BF} = \rho_{BF} \sigma_B \sigma_F \Delta t$$

which is the pure stochastic drift term. Of course, if ΔW_B and ΔW_F are related in other covariance forms (i.e., non-Brownian forms), other terms will enter the formulation.

4. <u>Numeraire</u>: “Numeraire” refers to the multiplicative unit that has the impact of “localizing” the corresponding stochastic factor. Implicit in our usage of the numeraire here is its multiplicative nature, thereby requiring the log-normal dynamics in the formulation as in above.

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- "Change of numeraire" => This has a more formal and comprehensive treatment later, but the term here refers to the impact of the cross-numeraire correlative component which simply contributes to the drift, i.e.,

$$\frac{\Delta(BF)}{BF} = \frac{\Delta B}{B} + \frac{\Delta F}{F} + \frac{\Delta B \Delta F}{BF} = \frac{\Delta B}{B} + \frac{\Delta F}{F} + \rho_{BF}\sigma_B\sigma_F\Delta t$$

$\rho_{BF}\sigma_B\sigma_F$ is the incremental cross numeraire drift. The corresponding expression for "divided" numeraire formulation that is common in stochastic finance is

$$\frac{\Delta\left(B/_F\right)}{\left(B/_F\right)} = \frac{\Delta B}{B} - \frac{\Delta F}{F} - \frac{\Delta B \Delta F}{BF} = \frac{\Delta B}{B} - \frac{\Delta F}{F} - \rho_{BF}\sigma_B\sigma_F\Delta t$$

5. Challenges with the log numeraire formulation: It needs to be remembered that working in the log numeraire format space does not result in the cross-correlation term coming out explicitly. We'll see that below.
6. Evolution Formulation: The driver processes are

$$\frac{\Delta B}{B} = \mu_B\Delta t + \sigma_B\Delta W_B$$

and

$$\frac{\Delta F}{F} = \mu_F\Delta t + \sigma_F\Delta W_F$$

We will examine the behavior of

$$\phi \equiv \phi(B, F, t)$$

given

$$\Delta W_B \Delta W_F = \rho_{BF} \Delta t$$

- SDE => Remembering that

$$\mu_B = \mu_B(F, t)$$

$$\mu_F = \mu_F(F, t)$$

$$\sigma_B = \sigma_B(F, t)$$

and

$$\sigma_F = \sigma_F(F, t)$$

$$\Delta\phi = \left[\frac{\partial\phi}{\partial t} + \mu_B B \frac{\partial\phi}{\partial B} + \frac{1}{2}{\sigma_B}^2 B^2 \frac{\partial^2\phi}{\partial B^2} + \mu_F F \frac{\partial\phi}{\partial F} + \frac{1}{2}{\sigma_F}^2 F^2 \frac{\partial^2\phi}{\partial F^2} + \rho_{BF}\sigma_B\sigma_F BF \frac{\partial^2\phi}{\partial B \partial F}\right]\Delta t + \sigma_B B \Delta W_B + \sigma_F F \Delta W_F$$

7. $\underline{\phi = BF}$: Setting

$$\phi = BF$$

we get the following:

$$\frac{\partial\phi}{\partial B} = F$$

$$\frac{\partial^2\phi}{\partial B^2} = 0$$

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$$\frac{\partial \phi}{\partial F} = B$$

$$\frac{\partial^2 \phi}{\partial F^2} = 0$$

$$\frac{\partial^2 \phi}{\partial B \partial F} = 1$$

The SDE now becomes

$$\Delta\phi = [\mu_B + \mu_F + \rho_{BF}\sigma_B\sigma_F]\Delta t + \sigma_B B\Delta W_B + \sigma_F F\Delta W_F$$

Applying the original Ito's Lemma on $\log \phi$, we now get

$$\Delta(\log \phi) = [\mu_B + \mu_F]\Delta t + \sigma_B B\Delta W_B + \sigma_F F\Delta W_F$$

Notice that the correlation cross product vanishes.

Likewise, if

$$\phi = \frac{B}{F}$$

(the typical divided numeraire), and working through the partials, the SDE becomes

$$\Delta\phi = [\mu_B - \mu_F - \rho_{BF}\sigma_B\sigma_F]\Delta t + \sigma_B B\Delta W_B - \sigma_F F\Delta W_F$$

Applying the original Ito's Lemma on this $\log \phi$, we get

$$\Delta(\log \phi) = [\mu_B - \mu_F]\Delta t + \sigma_B B\Delta W_B - \sigma_F F\Delta W_F$$

As before, the correlation cross product vanishes.

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Risk-Neutral Pricing Framework

1. Probability Measure: This is typically specified in stochastic Brownian terms, using the drift μ and volatility σ. As an example, for a given stock asset, the dynamics are specified using the real-world μ and σ (i.e., potentially the realized μ and σ).
2. Risk-Neutral Probability Measure: This has its own μ and σ, the only difference being that

$$\mu \equiv r$$

   the risk free rate of return, and σ is the implied future volatility, i.e., the volatility extracted by calibrating to the market prices.

   - Real-world <-> Risk-Neutral Transforms => The risk-neutral μ/σ maybe mapped over to the real-world μ/σ using the standard measure transforms such as the continuous Girsanov and Radon-Nikodym transformations (Bjork (2004)).
   - The Drift in Risk-Neutral => In reality, the risk-neutral drift is not entirely unique. If the derivatives are collateralized, the drift should correspond to the collateral numeraire drift. Under dynamic replication/hedging, however, the drift should correspond to the appropriate numeraire that accounts for the funding cost (e.g., treasury account numeraire, etc.)

3. Replication Principle: This simply states that the derivative price is computed using the risk-free discounting of the terminal payoff sequence. The terminal payoff may be a function of the asset price, but it **MAY NOT** imply a specific asset path, or a particular payoff asset distribution.
4. Replication of the Terminal Payoff: Given that we typically attempt to replicate a known pay-off, what we are attempting to estimate (and, therefore, taken to be unknown) is today's asset price. In particular, this is true for zero-coupon bond prices ($P(t,T)$ is the unknown, and $P(T,T)$ is taken to be 1); for FX ($X(t)$ is the unknown, and $X(T)$ is known), etc. NB: P/X are derivatives, and not primary securities. Primary securities evolve on their own (real-

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world or risk-neutral, as the usage may dictate), and their time value is not estimated using replication arguments.

5. Asset Price as a Martingale in its own Risk-Neutral Measure: This is because the risk-neutral asset growth rate occurs at the risk-free rate due to the multi-period self-financing requirement of the Arrow prices owing to dynamic replication (Taleb (1997), Gisiger (2010).
    - State Price Density => This refers to the distribution of the security's prices on its own measure. Unique risk-neutral state measure (i.e., the measure under which the state prices are martingales) implies that the state price density (i.e., distribution) is unique under the same measure.
    - Uniqueness of the Risk-Neutral State Measure => Why will the risk-neutral state measure be unique? This is because, as seen above, the measure's drift and volatility will be unique – the drift being unique because the asset prices are martingales, as this corresponds to **ZERO** price-of-market-risk premium (thereby making the portfolio return the risk-free rate). This is also applicable to non-Brownian scenarios, but with the stipulation that the drift still correspond to the risk-free rate. As in the case of Brownian motion, the measure parameters may be inferred/calibrated using the terminal payoff prices calibrated to the market (using the given model).
6. Unique Risk Neutral Asset Price: As can be seen above, this is a martingale. The no-arbitrage state resulting from dynamic replication causes a) the risk-free drift, and b) the unique state price.
7. Zero Coupon Bond as the Perfect Replicator: The scenario specific cash flow may be generated only by the corresponding zero-coupon bond – further, it generates ONLY the specific cash flow, and nothing else. Thus, this is the ideal Arrow replicator.
8. Measure Change: If the underlying asset's stochastic drivers and the risk-free numeraire's stochastic drivers are identical, the net volatility is simply the difference, and the asset continues to be a martingale in the numeraire's measure. Likewise, if the drivers between the asset and the risk-free numeraire's processes are orthogonal, they should be able to grow consistently independent of each other (with the asset continuing being a martingale in its own measure). If these drivers are correlated, however, the measure change amount needs to be applied as an adjustment to the asset's risk-neutral drift (this may be easily verified using straightforward application of Ito's lemma).

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9. Equivalent Martingale Formulation: Let's say

$$y_1 = \alpha_1 \Delta t + \beta \Delta W$$

and

$$y_2 = \alpha_2 \Delta t + \beta \Delta W$$

Then, on setting

$$\widetilde{W} = W + \frac{\alpha_2 - \alpha_1}{\beta} t$$

you get

$$y_1 = \alpha_1 \Delta t + \beta \Delta W = \alpha_2 \Delta t + \beta \Delta \widetilde{W}$$

Thus the drift is now the same for y_1 and y_2, but under different measures (W and $\widetilde{W}$).

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- Gisiger, N. (2010): Risk-Neutral Probabilities Explained, eSSRN.
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Black-Scholes Methodology

Overview and Base Derivation

1. Components of the Black Scholes Pricing Framework:
   - Terminal Payoff Replication (this may show itself up as a boundary condition, really)
   - Instantaneously non-stochastic replication
   - Self-financing, indicating that the portfolio grows at the risk-free (i.e., financing/unique "risk free") rate
2. Black-Scholes Portfolio: Let V be the value of the option, S the value of the underlying, N the number of short units of the underlying, and Π the Options Portfolio. Then

$$\Pi = V - nS$$

   results in

$$\Delta\Pi = \Delta V - n\Delta S$$

$$\Delta V = \frac{\partial V}{\partial S}\Delta S + \left[\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2}\right]\Delta t$$

3. Delta-Hedged Portfolio:

$$n = \frac{\partial V}{\partial S}$$

   The time change for Π is the same as that for V, i.e.,

$$\frac{\mathrm{dV}}{dt} = \frac{\mathrm{d}\Pi}{dt} = \frac{\partial \mathrm{V}}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2} = r\Pi = r[V - nS]$$

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Finally, since the dependence $\Delta\Pi$ on ΔS is zero, the dependence ΔV on ΔS is non-zero! Note that

$$\frac{\partial V}{\partial \mu_S} = 0$$

as there is no explicit dependence.

4. Relation between Delta, Gamma, Theta, and Option Value: A simple linear relation exists between these:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2} - rV = 0$$

where

$$\frac{\partial V}{\partial t} = \theta$$

$$\frac{\partial V}{\partial S} = \Delta$$

and

$$\frac{\partial^2 V}{\partial S^2} = \Gamma$$

5. Base Equation Interpretation: The quantity $\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2}$ represents the ***Option Total Time Value Change***, and $r\left[V - S\frac{\partial V}{\partial S}\right]$ represents the ***Hedged Portfolio Value Return***. Thus

$$\textbf{Option Total Time Value Change} = \textbf{Hedged Portfolio Value Return}$$

The Replication Technology

1. Replication Technique:
    - Blind expectation performed over the distribution of the stochastic underlier (in this case S) will not work, since the investors' expectation of the premium (i.e., drift) over the base drift of the underlier can vary dependent on many factor (e.g., the investors' risk tolerance profile).
    - Pick the Arrow securities – the set of securities that replicate the pay-off of the asset you are pricing – at least synthesize these securities if they are not real.
    - Replication done using one per each risk factor (Stock Option needs to be replicated using one stock and one bond).
        - More strongly, it is the hedged portfolio that guarantees replication.
        - For a delta-hedged replicator, need $\frac{\partial V}{\partial S}$ short units of the underlier for one unit of the derivative.
        - Remember that, by definition, the S in Black-Scholes refers to $S(0)$ - today's value for S, so all prices, deltas, and hedge-ratios are for today.
        - Finally, remember that the actual terminal pay-off in itself could simply become a boundary condition. At exercise this pay-off can be non-differentiable (even discontinuous), but differentiable earlier.
2. Cause for the drift Elimination in the lack Scholes Portfolio: Anything that eliminates the explicit ΔW coefficient (i.e., makes the portfolio non-stochastic) also eliminates the explicit Δt coefficient. Of course in the case of Brownian motion, the $(\Delta W)^2$ term contributes to the time dependence.
3. Pricing/Payoff Replication Portfolio:

$$\Pi = V - nS - mB$$

implies that

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$$\Delta\Pi = \Delta V - n\Delta S - m\Delta B$$

with n and m fixed. Note that the instantaneous non-stochastic dynamics constraint results in a value for n and m. This is called the self-financing portfolio, in that

$$\Delta\Pi \rightarrow 0$$

as it is hedged across both S and B.

4. Pricing vs. Hedging Portfolio: Notice that the pricing/replication portfolio is instantaneous, in that it is valid on for the specific n

$$n = -\frac{\partial V}{\partial t}$$

Hedging is done using individual securities, therefore does not constitute a portfolio in that sense; further it is CERTAINLY not self-financing it the portfolio is not re-balanced.

5. Pricing as a Hedging Portfolio: Remember that the stochastic factor hedging here is only strictly instantaneous. Therefore, an incremental time instant later this same portfolio will not be hedged, that is, it will not be stochastically invariant, and will therefore need to be re-composed.
6. PRICE is only based off of terminal payoff replication and instantaneous stochasticity elimination: Other approaches (e.g., pre-Black Scholes strategies) may say that the price of a derivative is sum total of all the cash-flows that form a part of the derivative product life-cycle valued individually, and in a non-risk neutral manner, i.e., derivative product cash flows, hedging cash-flows, collateralization cash-flows, funding cash-flows, etc., i.e., cash-flows associated with a given strategy/set of strategies in the future through to maturity. Not so in the case of Black-Scholes, where only instantaneous risk-neutrality and terminal payoff replication for a given cash-flow (stochastic/deterministic) are considered.
7. "Rates" in the Black Scholes Portfolio: This corresponds to:
    - The re-investment/investment returns rate if the Portfolio Value is positive.
    - The funding rate if the Portfolio Value is negative.

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- The funding Rate if the activity is for hedging and/or futures replication.
- The collateral rate, if the rate refers to the cash flow associated with collateralization.

8. Interpreting the Replicating Portfolio from the BS Call Formula:

$$V(S, K, T) = S\Phi(d_1) - KB\Phi(d_2)$$

where

$$B = e^{-r\tau}$$

$$d_1 = \frac{\log\frac{S}{K} + \tau\left(r + \frac{1}{2}\sigma^2\right)}{\sigma\sqrt{\tau}}$$

$$d_2 = d_1 - \sqrt{\sigma}$$

and

$$\tau = T - t$$

From this we can say that:

- $\Phi(d_1)$ is the number of replicating stocks (therefore the hedge ratio)
- $\Phi(d_2)$ is the probability of the call expiring in the money (also the bond numeraire units scaled by the strike).

9. Put-Call Parity:

$$C(K) = S\Phi(d_1) - Ke^{-\int_t^T r(t)dt}\Phi(d_2)$$

Buying a call while simultaneously selling a put with the same strike is equivalent to buying a stock and simultaneously borrowing $Ke^{-\int_t^T r(t)dt}$. Thus

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$$C - P = S - Ke^{-\int_t^T r(t)dt}$$

thereby resulting in

$$P(K) = Ke^{-\int_t^T r(t)dt}\Phi(-d_2) - S\Phi(-d_1)$$

Capital Asset Pricing Model

1. Definition: The Capital Asset Pricing Model (CAPM) stipulates that the expected return of the security i in excess of the risk-free rate is

$$\mathbb{E}[r_i] - r = \beta_i\{\mathbb{E}[r_M] - r\}$$

where r_i is the return on the asset, r_M is the return on the market, and

$$\beta_i = \frac{Covariance\langle r_i, r_M\rangle}{Variance\langle r_M\rangle}$$

is the security's β to the market.

2. CAPM for the Assets: In a time increment Δt, the expected asset return is

$$\mathbb{E}[r_S\Delta t] = \mathbb{E}[\Delta S]$$

where S follows

$$\Delta S = rS\Delta t + \sigma S\Delta W$$

Using the CAPM setup, we see that

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$$\mathbb{E}\left[\frac{\Delta S}{S}\right] = r\Delta t + \beta_S\{\mathbb{E}[r_M] - r\}\Delta t$$

for the asset. Likewise, for the derivative

$$\mathbb{E}\left[\frac{\Delta V}{V}\right] = r\Delta t + \beta_V\{\mathbb{E}[r_M] - r\}\Delta t$$

- Starting from

$$\Delta V = \frac{\partial V}{\partial t}\Delta t + \frac{\partial V}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(\Delta S)^2$$

substituting for $(\Delta S)^2$ and dividing throughout by V, we get

$$\frac{\Delta V}{V} = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right]\Delta t + \frac{\partial V}{\partial S}\frac{\Delta S}{S}\frac{S}{V}$$

Applying the CAPM rule, we get

$$r_V\Delta t = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right]\Delta t + \frac{\partial V}{\partial S}\frac{S}{V}r_S\Delta t$$

Taking covariance on both sides, and dropping the $\left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right]\Delta t$ term (since it is non-stochastic), we get

$$Covariance\langle r_V, r_M\rangle = \left(\frac{\partial V}{\partial S}\frac{S}{V}\right)Covariance\langle r_S, r_M\rangle$$

which implies

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$$\beta_V = \left(\frac{\partial V}{\partial S}\frac{S}{V}\right)\beta_S$$

(Rouah (2010a)).

- Using the β_V in terms of β_S: From the expression for $\mathbb{E}\left[\frac{\Delta V}{V}\right]$, multiplying by V we get

$$\mathbb{E}[\Delta V] = rV\Delta t + \beta_V V\{\mathbb{E}[r_M] - r\}\Delta t = rV\Delta t + \frac{\partial V}{\partial S}\beta_S S\{\mathbb{E}[r_M] - r\}\Delta t$$

- Comparing this with

$$\mathbb{E}[\Delta V] = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right]\Delta t + \mathbb{E}\left[\frac{\partial V}{\partial S}\Delta S\right]$$

  using the CAPM expression for ΔS in terms of β_S, we get the Black Scholes equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

3. Black Scholes PDE from the Binomial Model: Cox, Ross, and Rubinstein (1979) derive the Black-Scholes PDE as a limit of the binomial tree model in the limit of the discretized time interval evolution.

Multi-numeraire Formulation

1. Setup: Say that the Brownian dynamics of evolution of n latent states are

$$\frac{\Delta A_i}{A_i} = \mu_i \Delta t + \sigma_i \Delta W_i$$

$$\Delta W_i \Delta W_i = \rho_{ij}\Delta t$$

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for $i, j = 1, \dots, n$. The evolution for $\prod_{i=1}^{n} A_i$ is guided by

$$\frac{\Delta[\prod_{i=1}^{n} A_i]}{[\prod_{i=1}^{n} A_i]} = \left[\sum_{i=1}^{n} \mu_i(t) + \sum_{i=1}^{n} \sum_{j>i}^{n} \rho_{ij}(t)\sigma_i(t)\sigma_j(t)\right] \Delta t$$

From this it is easy to derive the joint numeraire

$$\prod_{i=1}^{n} A_i = e^{\int_{t_0}^{t}\left\{\sum_{i=1}^{n} \mu_i(s) + \sum_{i=1}^{n} \sum_{j>i}^{n} \rho_{ij}(s)\sigma_i(s)\sigma_j(s)\right\} ds}$$

References

- Cox, J. C., S. A. Ross, and M. Rubinstein (1979): Option Pricing: A Simplified Approach *Journal of Financial Economics* **7** 229-263.

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Log-Normal Black Scholes Greeks

First Order Greeks

1. Notation: In all the treatments below we use ϕ and Φ to represent the point-wise Gaussian distribution density and cumulative Gaussian respectively. Further we assume that there exists no dividends discrete/continuous.
2. Vega:

$$\nu = \frac{\partial V}{\partial \sigma}$$

   Sometimes the symbol κ is used instead. Products such as straddles are extremely sensitive to changes in volatility.

$$\nu_{Call} = \nu_{Put} = S\phi(d_1)\sqrt{\tau} = Ke^{-r\tau}\phi(d_2)\sqrt{\tau}$$

3. Theta:

$$\theta = -\frac{\partial V}{\partial \tau}$$

   is the sensitivity with respect to time (time decay). Except for deep out-of-the money puts, most options have negative θ. θ is composed of the intrinsic (which is always positive) and the time value (which is negative).

$$\theta_{Call} = -\frac{S\sigma\phi(d_1)}{2\sqrt{\tau}} - rKe^{-r\tau}\Phi(d_2)$$

$$\theta_{Put} = -\frac{S\sigma\phi(d_1)}{2\sqrt{\tau}} - rKe^{-r\tau}\Phi(-d_2)$$

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4. Rho:

$$\rho = \frac{\partial V}{\partial r}$$

is the sensitivity with respect to the interest rate.

$$\rho_{Call} = Ke^{-r\tau}\Phi(d_2)$$

$$\rho_{Put} = -Ke^{-r\tau}\Phi(-d_2)$$

Second Order Greeks

1. Gamma:

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 \mathrm{V}}{\partial S^2}$$

is always positive for long options.

$$\Gamma_{Call} = \Gamma_{Put} = \frac{\phi(d_1)}{S\sigma\sqrt{\tau}}$$

2. Vanna:

$$\mathrm{Vanna} = \frac{\partial \Delta}{\partial \sigma} = \frac{\partial \mathrm{V}}{\partial S} = \frac{\partial^2 \mathrm{V}}{\partial \sigma \partial S}$$

where the equality strictly holds if the partial second derivative exists.

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$$\text{Vanna}_{Call} = \text{Vanna}_{Put} = -\frac{\phi(d_1)d_2}{\sigma} = \frac{\nu}{S}\left[1 - \frac{d_1}{\sigma\sqrt{\tau}}\right]$$

3. Vomma and Charm:
   a. Vomma/Volga/Vega Convexity is the 2nd order sensitivity to the volatility.

$$\text{Vomma} = \frac{\partial \nu}{\partial \sigma} = \frac{\partial^2 \nu}{\partial S^2} = S\phi(d_1)\sqrt{\tau}\frac{d_1 d_2}{\sigma} = \nu\frac{d_1 d_2}{\sigma}$$

   b. Charm is the rate of decay of delta, i.e., delta decay.

$$\text{Charm} = -\frac{\partial \Delta}{\partial \tau} = \frac{\partial \theta}{\partial S} = -\frac{\partial^2 V}{\partial S \partial \tau}$$

$$Charm_{Call} = Charm_{Put} = -\phi(d_1)\frac{2r\tau - \sigma d_2\sqrt{\tau}}{2\tau\sigma\sqrt{\tau}}$$

4. Veta and Vera:
   a. Veta => Vega Decay, given from

$$Vega_{Decay} = \frac{\partial \nu}{\partial \tau} = \frac{\partial^2 V}{\partial \sigma \partial \tau} = S\phi(d_1)\sqrt{\tau}\left[\frac{rd_1}{\sigma\sqrt{\tau}} - \frac{1 + d_1 d_2}{2\tau}\right]$$

   b. Vera is also called rhova, and is the rate of change of ρ with respect to volatility.

$$\text{Vera} = \frac{\partial \rho}{\partial \sigma} = \frac{\partial^2 V}{\partial \sigma \partial r}$$

Third Order Greeks

1. Color: Measures the time decay of gamma.

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$$Color = \frac{\partial \Gamma}{\partial \tau} = \frac{\partial^3 \mathrm{V}}{\partial^2 S \partial \tau} = -\frac{\phi(d_1)}{2S\tau\sigma\sqrt{\tau}}\left[1 + d_1 \frac{2r\tau - \sigma d_2\sqrt{\tau}}{\sigma\sqrt{\tau}}\right]$$

2. <u>Speed</u>: Speed is the delta of the gamma.

$$Speed = \frac{\partial^3 \mathrm{V}}{\partial S^3} = \frac{\partial^2 \Delta}{\partial S^2} = \frac{\partial \Gamma}{\partial S} = -\frac{\phi(d_1)}{S^2\sigma\sqrt{\tau}}\left[\frac{d_1}{\sigma\sqrt{\tau}} + 1\right] = -\frac{\Gamma}{S}\left[\frac{d_1}{\sigma\sqrt{\tau}} + 1\right]$$

3. <u>Ultima</u>:

$$Ultima = \frac{\partial^3 \mathrm{V}}{\partial \sigma^3} = -\frac{\nu}{\sigma^2}\left[d_1 d_2(1 - d_1 d_2) + {d_1}^2 + {d_2}^2\right]$$

4. <u>Zomma</u>:

$$Zomma = \frac{\partial \Gamma}{\partial \sigma} = \frac{\partial \mathrm{Vanna}}{\partial S} = \frac{\partial^3 \mathrm{V}}{\partial^2 S \partial \sigma} = \phi(d_1)\frac{d_1 d_2 - 1}{\sigma^2 S\sqrt{\tau}} = \Gamma\left(\frac{d_1 d_2 - 1}{\sigma}\right)$$

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Black-Scholes Extensions

Time-Dependent Black Scholes

1. Pricing Expression: If r and σ in the Black-Scholes expression are strictly time-dependent (i.e., not asset value dependent), the adjustment to the Black-Scholes expression is trivial:

$$d_1 = \frac{\log\frac{S}{K} + \int_0^T r(t)dt + \frac{1}{2}\int_0^T \sigma^2(t)dt}{\sqrt{\int_0^T \sigma^2(t)dt}}$$

$$d_2 = \frac{\log\frac{S}{K} + \int_0^T r(t)dt - \frac{1}{2}\int_0^T \sigma^2(t)dt}{\sqrt{\int_0^T \sigma^2(t)dt}}$$

Effectively the original Black-Scholes equation may be used by applying

$$\bar{r} = \int_0^T r(t)dt$$

and

$$\bar{\sigma} = \sqrt{\int_0^T \sigma^2(t)dt}$$

$\bar{\sigma}$ is referred to as the root-mean squared volatility (Rebonato (2004)).

2. Time-Dependent Volatility from Implied Volatility: From

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$$\sigma_{IMP}(T) = \sqrt{\int_0^T \sigma^2(t)dt}$$

you get

$$\int_0^T \sigma^2(t)dt = T{\sigma_{IMP}}^2(T)$$

which implies

$$\sigma^2(T) = \frac{\partial[T{\sigma_{IMP}}^2(T)]}{\partial T} = {\sigma_{IMP}}^2(T) + 2T\sigma_{IMP}(T)\frac{\partial\sigma_{IMP}(T)}{\partial T}$$

Thus if $\sigma_{IMP}(T)$ is known at selected time grid nodes, you may use a spline form to fit $\sigma_{IMP}(T)$ vs. T, and extract the time-dependent volatility $\sigma(T)$ from there.

3. Shortcoming of time-dependent volatility: Time-dependent Black Scholes formulation can be calibrated to the ATM implied volatility term structure $\sigma_{IMP}(T)$, but cannot reproduce as-is the complete implied volatility surface $\sigma_{IMP}(T, K)$.

Local Volatility Models

1. Local Volatility Model Definition: Here, under the risk-neutral measure

$$\Delta S = r(t)S\Delta t + \sigma(S, t)\Delta W_S$$

Thus, while $r(t)$ can be deterministic, $\sigma(S, t)$ is stochastic. Implied volatilities may be fitted using local Volatility models.

2. Risk neutral Distribution of the Asset Price at T: As shown in popular, the call-price may be computed from the risk-neutral probability density function $\sigma(S_T, T)$ as

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$$C(K,T) = e^{-rT}\int_K^{\infty}(S_T - K)\varphi(S_T,T)dS_T$$

results in

$$\frac{\partial C(K,T)}{\partial K} = -e^{-rT}\int_K^{\infty}\varphi(S_T,T)dS_T$$

and

$$\frac{\partial^2 C(K,T)}{\partial K^2} = e^{-rT}\varphi(S_T,T)$$

1. Implication of the T derivation => This shows that the risk-neutral distribution of the asset price at T can be entirely determined from the market quotes of European options.
3. Extracting Local Volatility from Market Prices: If you have an option price surface, you typically have $\frac{\partial C(K,T)}{\partial T}$, $\frac{\partial C(K,T)}{\partial K}$, and $\frac{\partial^2 C(K,T)}{\partial K^2}$. The analysis above demonstrated how to extract $\varphi(S_T,T)$ from the option price surface, now we demonstrate how to extract $\sigma(K,T)$ from the same.

$$\frac{\partial C(K,T)}{\partial T} = -rC(K,T) + e^{-rT}\int_K^{\infty}(S_T - K)\frac{\partial\varphi(S_T,T)}{\partial T}dS_T$$

$$\frac{\partial C(K,T)}{\partial T} = -rC(K,T) + e^{-rT}\int_K^{\infty}(S_T - K)\left\{\frac{1}{2}\frac{\partial^2}{\partial {S_T}^2}\left[{S_T}^2\sigma^2\varphi\right] - \frac{1}{2}\frac{\partial}{\partial S_T}\left[rS_T\varphi\right]\right\}dS_T$$

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where we have made use of the Fokker-Planck relation in the second step. Notice the dependence on $\frac{\partial}{\partial S_T}$ and $\frac{\partial^2}{\partial {S_T}^2}$in the integral above, so integrating the above by parts twice yields

$$\frac{\partial C}{\partial T} = -rC + \frac{1}{2} e^{-rT} \sigma^2 K^2 \varphi + r e^{-rT} \int_K^{\infty} S_T \varphi d S_T$$

2. Using the expressions for $\frac{\partial C(K,T)}{\partial K}$ and $\frac{\partial^2 C(K,T)}{\partial K^2}$ from the previous point, you can eliminate all dependence on φ to get

$$\frac{\partial C}{\partial T} = -rK \frac{\partial C}{\partial K} + \frac{1}{2} \sigma^2 K^2 \frac{\partial^2 C}{\partial K^2}$$

which results in

$$\sigma_{LOCAL} = \sqrt{\frac{\frac{\partial C}{\partial T} + rK \frac{\partial C}{\partial K}}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}}}$$

Thus, once a 2D spline surface representing

$$C \equiv C(K,T)$$

is constructed, the corresponding σ_{LOCAL} is readily determined.

Black Normal Model Specification and Dynamics

1. Setup: Here

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$$\Delta F(t;\ T,S) = \sigma_N \Delta W$$

This leads to

$$F(T;\ T,S) = L(T,S) \approx \mathcal{N}(F(t;\ T,S), {\sigma_N}^2 T)$$

and the corresponding probability density function is

$$\varphi(F) = \frac{1}{\sqrt{2\pi{\sigma_N}^2 T}} e^{-\frac{(F-F_0)^2}{2\pi{\sigma_N}^2 T}}$$

2. <u>Call/Put Prices</u>:

$$C(S,K) = P(0,S)\mathbb{E}_0[(F-K)^+] = P(0,S)\sigma_N\sqrt{T}\left[\frac{1}{\sqrt{2\pi}}e^{-\frac{d^2}{2}} + d\Phi(d)\right]$$

where

$$d = \frac{F_0 - K}{\sigma_N\sqrt{T}}$$

and

$$\Phi(d) = \int_K^{+\infty} \varphi(F)dF$$

Setting

$$\phi(x) = \mathcal{N}'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

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we get

$$C(S,K) = P(0,S)\sigma_N\sqrt{T}[\phi(d) + d\mathcal{N}(d)]$$

and

$$P(S,K) = P(0,S)\sigma_N\sqrt{T}[\phi(d) - d\mathcal{N}(-d)]$$

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Options on Forward

Theoretical Framework and Background

1. Forwards as a Martingale: Forward is defined as that entity whose price/value at a forward time T is under consideration at a time $t < T$. The forward is treated as an entity whose expectation is time invariant under its own measure (as opposed to "spot" whose expectation grows typically with time even under its own native measure). This makes the forward a martingale. Projection of the forward from the spot occurs via the risk-neutral discounting.

- The forward is a martingale simply because the payoff has to be a martingale in its own measure. Of course the payoff may be correlated with the discounting measure, in which case the correlation adjustment needs to be applied.

2. Replication of the forward from the spot: This needs a SPOT and a funding account, thus bringing the funding measure in. Of source the funding measure may be correlated with the spot measure, in which case an adjustment needs to be applied again along the path (not on the terminal payoff distribution).

Valuation

1. Black76 Model: This is the most straightforward extension to what we saw before. Given

$$\Delta S = r(t)S\Delta t + \sigma_S(S,t)\Delta W_S$$

define

$$F(t,T) = S(t)e^{-\int_t^T r(t)dt}$$

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It is straightforward to see that $F(t,T)$ is a martingale with the same volatility as that of $S(t)$. Thus, a call option on $F(t,T)$ with the strike K becomes

$$C(K) = e^{\int_t^T r(t)dt}\{F(t,T)\Phi(d_1) - K\Phi(d_2)\}$$

2. <u>Forward Evolution</u>: The forward evolves according to

$$\frac{\Delta F}{F} = \mu_F \Delta t + \sigma_F \Delta W_F$$

Call it the Q^f measure – all you know are μ_F and σ_F. As noticed earlier we assume

$$\mu_F = 0$$

so that martingale property of the forward is maintained.

3. <u>Numeraire Evolution</u>: The numeraire evolves according to

$$\frac{\Delta B}{B} = \mu_B \Delta t + \sigma_B \Delta W_B$$

- call it the Q^b measure. Q^b and Q^f are correlated via

$$\langle \Delta W_B \Delta W_F \rangle = \rho_{BF} \Delta t$$

Remember that if B refers to the discrete bond numeraire, there will be several ΔW_i's.

4. <u>What do we seek</u>: We seek $B_0 \int d\left(\frac{F}{B}\right)$, i.e. the PV of the Arrow security that pays F units replicated using B, where B and F follows the evolution dynamics above. In essence, we seek the evolution of $\Delta\left(\frac{F}{B}\right)$.

5. <u>$\Delta\left(\frac{F}{B}\right)$ Evolution</u>:

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$$\Delta\left(\frac{F}{B}\right) = \left(\frac{\mathrm{B}\Delta F - \mathrm{F}\Delta B - \Delta\mathrm{B}\Delta F}{B^2}\right)$$

implies that

$$\frac{\Delta\left(\frac{F}{B}\right)}{\left(\frac{F}{B}\right)} = \frac{\Delta F}{F} - \frac{\Delta B}{B} - \frac{\Delta B}{B}\frac{\Delta F}{F}$$

which results in

$$\frac{\Delta\left(\frac{F}{B}\right)}{\left(\frac{F}{B}\right)} = (\mu_F - \mu_B - \rho_{BF}\sigma_B\sigma_F)\Delta t + \sigma_F\Delta W_F - \sigma_B\Delta W_B$$

On integrating the above, you get

$$\Delta\left(\frac{F}{B}\right) = \frac{F_0}{B_0}\Phi\left[(\mu_F - \mu_B)\Delta t - Variance\left(\frac{\Delta B}{B},\frac{\Delta F}{F}\right)\cdot Covariance\left(\frac{\Delta B}{B},\frac{\Delta F}{F}\right)\right]$$

6. <u>Change of Measure</u>: Denoted as $\frac{\partial Q^f}{\partial Q^b}$, in cases above ends up picking on the drift term $-Covariance\left(\frac{\Delta B}{B},\frac{\Delta F}{F}\right)$. Thus,

$$\mathbb{E}^{Q^b}\left[\frac{\partial Q^f}{\partial Q^b}\right] = e^{-\int(\mu_F-\mu_B)\mathrm{d}t - Covariance\left(\frac{\mathrm{d}B}{B},\frac{\mathrm{d}F}{F}\right)}$$

Notice the terminology/language/notation.

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Stochastic Volatility Models: The Heston Model

Model Specification and Dynamics

1. Stochastic Volatility Model: Examples of stochastic volatility models are the Heston (1993) and SABR models. These are NOT local volatility models in the Dupire sense, as the volatilities here do not have an explicit dependence on the asset levels, strike, or the money-ness. Therefore they are widely applicable in markets where the common quotes are at-the money, e.g., in caps/floors/IR Options markets.
2. Stochastic Volatility Dynamics: The Heston model (Heston 1993)) assumes that the underlying S follows a Brownian stochastic process with variance $v(t)$ that follows a Cox-Ingersoll-Ross stochastic volatility dynamics:

$$\Delta S = rS\Delta t + \sqrt{v}S\Delta W_S$$

$$\Delta v = \kappa(\theta - v) + \sqrt{v}\sigma\Delta W_v$$

$$\Delta W_v \Delta W_S = \rho_{vS}\Delta t$$

3. The Heston Portfolio: The Heston portfolio consists of the first option V, the second option U, and the underlier S (Gatheral (2006)):

$$\Pi = V + \delta S + \phi U$$

From the self-financing criterion, we have

$$\Delta\Pi = \Delta V + \delta\Delta S + \phi\Delta U$$

4. Portfolio Dynamics: Applying Ito's lemma to ΔV, ΔU, and ΔS, we get

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$$\Delta V = \frac{\partial V}{\partial t}\Delta t + \frac{\partial V}{\partial S}\Delta S + \frac{\partial V}{\partial \upsilon}\Delta \upsilon + \frac{1}{2}\upsilon S^2 \frac{\partial^2 V}{\partial S^2}\Delta t + \frac{1}{2}\upsilon \sigma^2 \frac{\partial^2 V}{\partial \upsilon^2}\Delta t + \rho\sigma\upsilon S \frac{\partial^2 V}{\partial \upsilon \partial S}\Delta t$$

$$\Delta U = \frac{\partial U}{\partial t}\Delta t + \frac{\partial U}{\partial S}\Delta S + \frac{\partial U}{\partial \upsilon}\Delta \upsilon + \frac{1}{2}\upsilon S^2 \frac{\partial^2 U}{\partial S^2}\Delta t + \frac{1}{2}\upsilon \sigma^2 \frac{\partial^2 U}{\partial \upsilon^2}\Delta t + \rho\sigma\upsilon S \frac{\partial^2 U}{\partial \upsilon \partial S}\Delta t$$

and for the portfolio increment

$$\begin{aligned} &\left\{\frac{\partial V}{\partial t} + \frac{1}{2}\upsilon S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma\upsilon S \frac{\partial^2 V}{\partial \upsilon \partial S} + \frac{1}{2}\upsilon\sigma^2 \frac{\partial^2 V}{\partial \upsilon^2}\right\}\Delta t \\ \Delta\Pi = \phi &\left\{\frac{\partial U}{\partial t} + \frac{1}{2}\upsilon S^2 \frac{\partial^2 U}{\partial S^2} + \rho\sigma\upsilon S \frac{\partial^2 U}{\partial \upsilon \partial S} + \frac{1}{2}\upsilon\sigma^2 \frac{\partial^2 U}{\partial \upsilon^2}\right\}\Delta t \\ &\left\{\frac{\partial V}{\partial S} + \phi \frac{\partial U}{\partial S} + \delta\right\}\Delta S + \left\{\frac{\partial V}{\partial \upsilon} + \phi \frac{\partial U}{\partial \upsilon} + \delta\right\}\Delta \upsilon \end{aligned}$$

5. Risk-less Portfolio: To eliminate the dependence on ΔS and $\Delta \upsilon$, we require

$$\phi = -\frac{\frac{\partial V}{\partial \upsilon}}{\frac{\partial U}{\partial \upsilon}}$$

and

$$\delta = -\frac{\partial V}{\partial S} - \phi \frac{\partial U}{\partial S}$$

As is typical in these treatments, the risk-less portfolio should then evolve according to

$$\Delta\Pi = r\Pi\Delta\mathrm{t}$$

6. Risk-less Portfolio Dynamics: Set

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$$\Delta\Pi = (A + \phi B)\Delta\mathrm{t}$$

where

$$A = \frac{\partial V}{\partial t} + \frac{1}{2}\upsilon S^2\frac{\partial^2 V}{\partial S^2} + \rho\sigma\upsilon S\frac{\partial^2 V}{\partial\upsilon\partial S} + \frac{1}{2}\upsilon\sigma^2\frac{\partial^2 V}{\partial\upsilon^2}$$

and

$$B = \frac{\partial U}{\partial t} + \frac{1}{2}\upsilon S^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma\upsilon S\frac{\partial^2 U}{\partial\upsilon\partial S} + \frac{1}{2}\upsilon\sigma^2\frac{\partial^2 U}{\partial\upsilon^2}$$

Putting this back into $\Delta\Pi = (A + \phi B)\Delta\mathrm{t}$ we get

$$\mathrm{r}\Pi\Delta\mathrm{t} = r(V + \delta S + \phi U)\Delta\mathrm{t}$$

7. Dynamics of the Market Price of Volatility Risk: Plug in for ϕ and δ from the portfolio hedging relations above, and re-arrange them to produce

$$\frac{A - rV + rS\frac{\partial V}{\partial S}}{\frac{\partial V}{\partial \upsilon}} = \frac{B - rU + rS\frac{\partial U}{\partial S}}{\frac{\partial U}{\partial \upsilon}} = -\kappa(\theta - \upsilon) + \lambda(S, \upsilon, t)$$

where $\lambda(S, \upsilon, t)$ is the market price of the volatility risk. Heston (1993) sets

$$\lambda(S, \upsilon, t) = \lambda\upsilon$$

i.e., the market price of volatility risk to be linear in the volatility.

8. Sources of Market Price of Risk in the Heston Model: There are two sources of market risk – one for ΔS and one for $\Delta\upsilon$. Instantaneous hedge across ΔS produces a drift for ΔS of r - the risk-free rate – essentially

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$$\mathrm{r} = \lambda_S - \mu_S$$

Likewise, for $\Delta \upsilon$ it is

$$\frac{\lambda(S, \upsilon, t) - \kappa(\theta - \upsilon)}{\upsilon}$$

which is what is seen above.

- In the final option price PDE, the risk-adjusted drifts always appear as the coefficients of the corresponding option price <-> stochastic entity derivatives. Thus, $\frac{\partial U}{\partial S}$'s coefficient is rS; and $\frac{\partial U}{\partial \upsilon}$'s coefficient is $\lambda(S, \upsilon, t) - \kappa(\theta - \upsilon)$.
- One consequence of the above formulation is the need to calibrate multiple groups of parameters – the real-world ones (the mean reversion intensity κ and the steady state mean reverted level θ), and the market price of risk $\lambda(S, \upsilon, t)$.

9. Heston Option Price PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \upsilon S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma \upsilon S \frac{\partial^2 V}{\partial \upsilon \partial S} + \frac{1}{2} \upsilon \sigma^2 \frac{\partial^2 V}{\partial \upsilon^2} - rV + rS \frac{\partial V}{\partial S} + [\kappa(\theta - \upsilon) - \lambda(S, \upsilon, t)] \frac{\partial V}{\partial \upsilon} = 0$$

Expressing the above in terms of

$$x = \log S$$

we get

$$\frac{\partial V}{\partial t} + \frac{1}{2} \upsilon \frac{\partial^2 V}{\partial x^2} + \left[r - \frac{1}{2} \upsilon\right] \frac{\partial V}{\partial x} + \rho \sigma \upsilon \frac{\partial^2 V}{\partial \upsilon \partial x} + \frac{1}{2} \upsilon \sigma^2 \frac{\partial^2 V}{\partial \upsilon^2} - rV + [\kappa(\theta - \upsilon) - \lambda(S, \upsilon, t)] \frac{\partial V}{\partial \upsilon} = 0$$

As noted above Heston (1993) set $\lambda(S, \upsilon, t) = \lambda \upsilon$, simplifying the above slightly.

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- Drift Terms for ΔS and $\Delta \upsilon$ => As expected the drift terms for ΔS and $\Delta \upsilon$ do not show up explicitly, since we are dealing with instantaneously hedged portfolios. However, for the reasons seen above, Heston appears to have explicitly accommodated the elements of υ dynamics via the market price of risk approach, essentially making the formulation "non-local" in υ.

Price Estimation Through Characteristic Functions

1. <u>Call Price re-cast of the Derivative Price above</u>: The Call Price

$$C_T(K) = e^{-r\tau}\mathbb{E}[(S_T - K)^+] = e^x K P_1(x, \upsilon, \tau) - e^{-r\tau} K P_2(x, \upsilon, \tau)$$

where

$$x = \log S$$

and

$$\tau = T - t$$

Using

$$\frac{\partial V}{\partial t} = -\frac{\partial V}{\partial \tau}$$

the pricing equation above becomes

$$-\frac{\partial V}{\partial \tau} + \frac{1}{2}\upsilon\frac{\partial^2 V}{\partial x^2} + \left[r - \frac{1}{2}\upsilon\right]\frac{\partial V}{\partial x} + \rho\sigma\upsilon\frac{\partial^2 V}{\partial \upsilon \partial x} + \frac{1}{2}\upsilon\sigma^2\frac{\partial^2 V}{\partial \upsilon^2} - rV + [\kappa(\theta - \upsilon) - \lambda(S, \upsilon, \tau)]\frac{\partial V}{\partial \upsilon} = 0$$

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2. Derivative Value in terms of $P_j(x, \upsilon, \tau)$: Substituting for $C_T(K)$ using $P_1(x, \upsilon, \tau)$ and $P_2(x, \upsilon, \tau)$, Heston (1993) derived

$$e^x f\big(P_1(x, \upsilon, \tau)\big) - e^{-r\tau} f\big(P_2(x, \upsilon, \tau)\big) = 0$$

where

$$f\big(P_j\big) = -\frac{\partial P_j}{\partial \tau} + \rho\sigma\upsilon \frac{\partial^2 P_j}{\partial \upsilon \partial x} + \frac{1}{2}\upsilon \frac{\partial^2 P_j}{\partial x^2} + \frac{1}{2}\upsilon\sigma^2 \frac{\partial^2 P_j}{\partial \upsilon^2} + \big[r + u_j \upsilon\big] \frac{\partial P_j}{\partial x} - rV + \big[a + b_j \upsilon\big] \frac{\partial P_j}{\partial \upsilon}$$

Here

$$u_1 = \frac{1}{2}$$

$$u_2 = -\frac{1}{2}$$

$$a = \kappa\theta$$

$$b_1 = \kappa + \lambda - \rho\sigma$$

and

$$b_2 = \kappa + \lambda$$

Given that P_1 and P_2 are state evolution probabilities, their evolution is governed by the Fokker-Planck equation

$$-\frac{\partial P_j}{\partial \tau} + \rho\sigma\upsilon \frac{\partial^2 P_j}{\partial \upsilon \partial x} + \frac{1}{2}\upsilon \frac{\partial^2 P_j}{\partial x^2} + \frac{1}{2}\upsilon\sigma^2 \frac{\partial^2 P_j}{\partial \upsilon^2} + \big[r + u_j \upsilon\big] \frac{\partial P_j}{\partial x} - rV + \big[a + b_j \upsilon\big] \frac{\partial P_j}{\partial \upsilon} = 0$$

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This results in

$$f(P_1) = 0$$

and

$$f(P_2) = 0$$

so we seek a solution for them.

3. <u>Functional Basis Form for $P_j(x, \upsilon, \tau, \alpha)$</u>: Use the form

$$P_j(x, \upsilon, \tau, \alpha) = e^{C_j(\tau,\alpha)+\upsilon D_j(\tau,\alpha)+i\alpha x}$$

for

$$j = 1, 2$$

This choice is attractive for several reasons:

- The dependence of $P_j(x, \upsilon, \tau, \alpha)$ on x and α is exponentially partitioned.
- Given that $P_j(x, \upsilon, \tau, \alpha)$ shows up linearly on the pricing PDE, it gets factored out under such a representation, leaving behind only the coefficients.
- As will be seen below, this results in expressing the equation set as

$$p\upsilon + q = 0$$

across all υ, so that

$$p = 0$$

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and

$$q = 0$$

can be solved separately.

- Finally

$$p = 0$$

may be solved entirely for D_j alone, and

$$q = 0$$

may be solved for C_j using a trivial dependence on D_j.

4. Insight into the $e^{i\alpha x}$ choice: Remember that

$$x = \log S$$

and this translates into

$$e^{i\alpha x} = i\alpha S$$

Further by imposing

$$D(0, \alpha) = 0$$

and

$$C(0, \alpha) = 0$$

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as boundary conditions, Heston (1993) ensures that the time T price is either directly proportional to S or 0, as you would want.

- In effect, this choice of basis function that involves α is simply a Fourier transform (plus function/field partition) over the asset "frequency" range α, so we need a final Inverse Fourier Transform step that reverts this.

5. <u>Exponential Basis Expansion for $P_j(x, v, \tau)$</u>: You get

$$\left[-\frac{\partial D_j}{\partial \tau} + i\rho\sigma\alpha D_j - \frac{1}{2}\alpha^2 + \frac{1}{2}\sigma^2 {D_j}^2 + iu_j\alpha + b_j D_j\right] v + \left[-\frac{\partial C_j}{\partial \tau} + ir\alpha + aD_j\right] = 0$$

This is, of course, of the form

$$pv + q = 0$$

above, thus

$$p = 0$$

and

$$q = 0$$

may be solved separately. This results in

$$-\frac{\partial D_j}{\partial \tau} + i\rho\sigma\alpha D_j - \frac{1}{2}\alpha^2 + \frac{1}{2}\sigma^2 {D_j}^2 + iu_j\alpha + b_j D_j = 0$$

and

$$-\frac{\partial C_j}{\partial \tau} + ir\alpha + aD_j = 0$$

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The first is a second degree ODE depending exclusively on D_j (this is referred to as a Riccatti's equation); the second is a first order ODE on C_j that may be solved using a straightforward integration of D_j.

6. Solution for D_j: The Riccatti equation is of the form

$$\frac{\partial D_j}{\partial \tau} = M_j + N_j D_j + R_j {D_j}^2$$

Heston (1993) and Rouah (2010b) demonstrate that the solution that incorporates the boundary condition

$$D_j(0, \alpha) = 0$$

is

$$D_j = \frac{b_j - i\rho\sigma\alpha + d_j}{\sigma^2}\left[\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right]$$

where

$$d_j = \sqrt{\left(i\rho\sigma\alpha - b_j\right)^2 - \sigma^2\left(2i\alpha u_j - \alpha^2\right)^2}$$

and

$$g_j = \frac{b_j - i\rho\sigma\alpha + d_j}{b_j - i\rho\sigma\alpha - d_j}$$

7. Solution for C_j:

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$$C_j = \int_0^\tau [ir\alpha + aD_j]d\tau$$

Applying the boundary condition

$$C_j(0, \alpha) = 0$$

we get

$$C_j = ir\alpha\tau + \frac{a}{\sigma^2}\left[(b_j - i\rho\sigma\alpha + d_j)\tau - 2\log\left|\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right|\right]$$

8. <u>Enhancement by Albrecher, Mayer, Schoutens, and Tistaert (2007)</u>: They propose the following tweaks to improve performance of the Heston model (a similar proposal was also made by Gatheral (2006)). Reset

$$g_j' = \frac{1}{g_j} = \frac{b_j - i\rho\sigma\alpha - d_j}{b_j - i\rho\sigma\alpha + d_j}$$

Apparently this modest formulation adjustment makes a significant impact on the numerical stability. This results in

$$C_j' = ir\alpha\tau + \frac{a}{\sigma^2}\left[(b_j - i\rho\sigma\alpha - d_j)\tau - 2\log\left|\frac{1 - g_j' e^{-d_j\tau}}{1 - g_j'}\right|\right]$$

and

$$D_j' = \frac{b_j - i\rho\sigma\alpha - d_j}{\sigma^2}\left[\frac{1 - e^{-d_j\tau}}{1 - g_j' e^{-d_j\tau}}\right]$$

All other formulation components remain the same.

9. Solution for $P_j(x, \upsilon, \tau)$:

$$P_j(x, \upsilon, \tau) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} Real\left[\frac{e^{-i\alpha x} P_j(x, \upsilon, \tau, \alpha)}{i\alpha}\right] d\alpha$$

Fourier Inversion in Characteristic Function

1. Motivation for the Fourier Inversion Form in Heston (1993): The formulation uses the result that the Fourier transform of the Heaviside function is composed simply of a Dirac component and a hyperbolic component:

$$\int e^{-2\pi i\alpha x} h(x) dx = \frac{1}{2}\delta(x) + \frac{1}{2\pi i\alpha}$$

2. Problem #1 Multi Valued Complex Log: The Fourier transform of the multi-valued complex logarithm may end up getting switched away from the principal branch. One suggestion to remedy this is to carefully keep track of the branch (Schobel and Zhu (1999), Mikhailov and Nogel (2003), Sepp (2004), Lee (2005)) along the discretized path integral of

$$f_j(x, \upsilon, \tau, \alpha) = \frac{e^{-i\pi\alpha} P_j(x, \upsilon, \tau, \alpha)}{i\alpha}$$

as α goes from 0 to ∞.

3. Problem #2 Branch Switching of the Complex Power Function: Related to the above, but the distinction is highlighted in Kahl and Jackel (2009). By setting the complex variable in the Fourier transform

$$z = re^{i\phi}$$

we get

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$$z\varpi = re^{i\varpi\phi}$$

As the phase of z changes from $-\pi$ to $+\pi$, the phase of z^{ϖ} changes from $-\varpi\pi$ to $+\varpi\pi$. If $\varpi \in Z$, this is a clearly not a problem, but if $\varpi \notin Z$, the branch switching can occur.

4. Phase Correction: Kahl and Jackel (2009) narrow down on the $\frac{1-g_j e^{d_j\tau}}{1-g_j}$ term, since that is the logarithm operand, and apply the phase rotation corrections separately to the numerator and the denominator for each α evaluation – this eliminates the need to track the phase (and its jumps/discontinuities) across all subsequent α evaluations.

- The algorithm => Phase adjustment for $g_j - 1$ is $2\pi n$, where

$$n = int\left[\frac{Phase(g_j) + \pi}{2\pi}\right]$$

This states that if

$$Phase(g_j) > 0$$

then

$$n = 1$$

otherwise

$$n = 0$$

Thus, the adjustment amount is entirely determined by $Phase(g_j)$.

- The full correction => Phase adjustment for $g_j e^{d_j\tau} - 1$ is $2\pi m$. The full correction, therefore, becomes $\frac{1-g_j e^{d_j\tau}}{1-g_j}$, which gives

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$$\log\left|\frac{1-g_j e^{d_j\tau}}{1-g_j}\right| = \log\left|\frac{\|1-g_j e^{d_j\tau}\|}{\|1-g_j\|}\right| + Phase\left(1-g_j e^{d_j\tau}\right) - Phase\left(1-g_j\right) + 2\pi(n-m)$$

5. Applicability of the Phase Corrections: Although discussed in a specific context here, these techniques may be used for all inverse Fourier transforms across all fields – in particular among the option pricing models that use the Fourier inversion integral approach (Carr and Madan (1999), Carr (2003)) based on the log-characteristic function.
6. Fourier Integration Quadrature Schemes: Adaptive quadrature schemes are required for Fourier Inversion Integrands. Since the integration limit for α goes from 0 to ∞, we seek to transform the limits to 0 to 1, which works well for adaptive schemes such as the Gauss-Labatto algorithm (Gander and Gautschi (2000)). Thus

$$\int_0^\infty f_j(x,v,\tau,\alpha)d\alpha = \int_0^1 \frac{f_j\left(x,v,\tau,\alpha(\xi)\right)}{\xi C_\infty} d\xi$$

where

$$\alpha(\xi) = -\frac{\log\xi}{C_\infty}$$

resulting in

$$\xi = e^{-\alpha(\xi)C_\infty}$$

- Determination of C_∞ => C_∞ above is estimated in the limit of $\alpha\to\infty$ for all the coefficients in the Heston formulation, giving

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$$C_{\infty} = \frac{\sqrt{1-\rho^2}}{\sigma}(v_0 + \kappa\theta\lambda)$$

- Kahl and Jackel (2009) work out the limiting expressions for $f_1(x, v, \tau, \alpha)$ and $f_2(x, v, \tau, \alpha)$ as $\alpha \to 0$ and $\alpha \to \infty$, but unfortunately do not explicitly spell out the appropriate lower/upper bounds for α (and therefore ξ).

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Dynamical Latent State Calibration

Fokker-Planck Equations

1. Introduction: Consider a random variable x that follows

$$\Delta x = \mu(x,t)\Delta t + \sigma(x,t)\Delta W$$

with

$$x(t_0) = x_0$$

The transition probability of reaching x at $t > t_0$ is $f(x, t : x_0, t_0)$, and is given using the Fokker-Planck version of the Kolmogorov equation (see, e.g., Wang (2010a)) as

$$\frac{\partial}{\partial t} f(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2(x,t)f(x,t)] - \frac{1}{2}\frac{\partial}{\partial x}[\sigma(x,t)f(x,t)]$$

2. Extension to Options: There is explicit requirement that x follow Brownian motion. For options, if $\varphi(S_T, T)$ is the probability of reaching S_T at T given

$$S(t_0) = S_0$$

we get

$$\frac{\partial}{\partial t}\varphi(S_T,T) = \frac{1}{2}\frac{\partial^2}{\partial {S_T}^2}\left[{S_T}^2\sigma^2(x,t)\varphi(S_T,T)\right] - \frac{1}{2}\frac{\partial}{\partial S_T}[S_T\sigma(x,t)\varphi(S_T,T)]$$

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3. Feynman-Kac Relation: This is a generalization of the Fokker-Planck equation (see, for e.g., Karatzas and Shreve (1997)): Give the boundary value problem

$$\left[\frac{\partial}{\partial t} + \mu(t,X)\frac{\partial}{\partial X} + \frac{1}{2}\sigma^2(t,X)\frac{\partial^2}{\partial X^2}\right] f(t,X) = A(t,X)f(t,X) + B(t,X)$$

and the boundary condition

$$f(t,X) = \psi(X)$$

we can solve for $f(t,X)$ as

$$f(t,X) = \mathbb{E}_t{}^Q\left[D(t,T,X)\psi(X) - \int_t^T D(u,T,X)B(u,X)du\right]$$

with

$$D(t,T,X) := e^{-\int_t^T A(u,X)du}$$

where, under the measure Q, X now follows

$$\Delta X = \mu(t,X)X(t)\Delta t + \sigma(t,X)X(t)\Delta W^Q(t)$$

with

$$S(0) = S_0$$

Volatility Observations vs. Calibrations

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1. Latent State Quantification Metric (LSQM) Calibration vs. Manifest Measure (MM) Observation Quotes in the Volatility Space: Both prices and certain implied volatilities are manifest measure quotes, so there is less empirical significance in interpolating their "intermediate" nodes. Calibrated deterministic/local volatility surfaces are the corresponding latent state quantification metrics, so their splined latent state representations are of greater significance.
2. LSQM Extraction from MM: Both the deterministic volatility term structures as well as Dupire's calibrated local volatility surface depend on the derivatives of the manifest measure quotes. Thus, this is significantly distinct from the equivalent treatments of the splined discount/forward/credit/basis curve latent states. Correspondingly, the C^k continuity criterion for the volatility term structure/surface requires $k + r + 1$ basis functions to denote the manifest measure representation.
    - For local volatility surfaces, in addition, multi-dimensional splines are required. Wire mesh 2D spline proxies may work in some cases (esp., for extracting term structures at the strike/term nodal anchors), but are poor alternatives for full surface splines (e.g., bi-cubic splines).
3. Deterministic Volatility – No Arbitrage Criterion: Given that

$$\sigma_{Det}(T) = \sqrt{\sigma_{IMP}{}^2(T) + 2T\sigma_{IMP}(T)\frac{\partial\sigma_{IMP}(T)}{\partial T}}$$

$\sigma_{Det}(T)$ stays valid as long as

$$\sigma_{IMP}(T) > 2T\frac{\partial\sigma_{IMP}(T)}{\partial T}$$

This can get violated for either a steeply upward sloping curve where

$$\frac{\partial\sigma_{IMP}(T)}{\partial T} > 0$$

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or a humped curve. Thus this forms the basis behind no arbitrage detection in construction of deterministic volatility term structures, and of the corresponding market quotes (prices OR implied volatilities).

4. Volatility Surface Bootstrapping: Over any incremental time, the probability of moving into the money (and therefore, the corresponding option call/put payoff) increases monotonically. This monotonic in-the-money probability can be mathematically related to the incremental σ, and hence can form the basis for volatility bootstrapping.
5. State Dynamical Parameters Bootstrapping: The local volatility surface enhanced by the generalized Fokker-Planck formulation (through the Feynman-Kac formalism) is the basis behind the volatility dynamics forward diffusion. This formulation is ABSOLUTELY general and POWERFUL in its validity, with the stationary latent-state inferences falling out as a consequence of applying steady-state treatment to this behavior.
6. Risk-Neutral Forward Measures as a Volatility Surface Boot-strapper Unit: Since this measure captures the dynamics of the forward risk-neutral numeraire, the boot-strapper can serve as a suitable incremental dynamics parameters inferrer. The corresponding no-arbitrage drifts may be splined at every time snapshot, or an even better approach may be incorporated.
7. Market Volatility Quote Transform: Remember that cap/floor volatilities etc. are log-normal level quotes (often ATM). Convexity correction using shifted log-normal volatilities (i.e., volatility of $1 + \tau F$) needs to use the corresponding shifted log-normal volatility, which scaled by $\sim\tau F$, the log-normal one. Since $\tau F \sim \frac{1}{100}$ or less is common, this leads to extremely small convexity corrections.
8. Option Price Manifest Measure Quotes: In general, the option manifest measure quotes are with respect to the contract description. Thus, price would be the contract PV. In this case there is no ambiguity – thus, the price manifest measure quote maybe used to "uniquely" calibrate the latent state.
9. Option Implied Volatility Manifest Measure Quote: This quote can be a severe problem. Risk neutrality apart, implied volatility of what – the forward? The terminal payoff metric? The latter is the most sensible interpretation (i.e., implied volatility of the terminal forward payoff).
10. Latent State Dynamics Estimation: Using the combination of the "Current" Latent State and the specific suite of option manifest measures, you may infer the latent state dynamical

parameters. All that is available from the “current” latent state metric would be the central forward metrics (such as par ATM levels, etc.).

11. Discount Curve Latent State Quantification Metric as Forward Rate: This provides an additional motivation towards representing the discounting latent state using the forward rate quantification metric, as the ATM levels automatically fall out of the latent state representation.
12. Options Manifest Measure Quote: Ultimately, like the “steady state” case, the option valuation has to get formulated and eventually represented along a spline formalism (a la local volatility model) no matter what the dynamical state latent state quantification metric parameter set is. Thus, a generalized forward linear formulation would be of great value.

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Section X: Interest Rate Dynamics and Option Pricing

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HJM Model

Introduction

1. Background: Heath, Jarrow, and Morton (1992) discovered that the no-arbitrage condition for zero-coupon bond prices under the risk-neutral measure Q, whose numeraire is the bank account numeraire, implies the existence of a simple constraint between the instantaneous volatility and the instantaneous drift of the instantaneous forward rate.
2. Instantaneous Forward Rate: It is assumed that, for a *fixed Maturity* T, the instantaneous forward rate $f(t,T)$ under Q follows

$$\Delta f(t,T) = \alpha(t,T)\Delta t + \vec{\sigma}(t,T) \cdot \Delta\vec{W}(t)$$

   with the initial condition

$$f(0,T) = f_M(0,T)$$

   where $\vec{\sigma}(t,T) \cdot \Delta\vec{W}(t)$ is the inner product of the two vectors

$$\vec{\sigma}(t,T) = [\sigma_1(t,T), \dots, \sigma_N(t,T)]^T$$

   and

$$\Delta\vec{W}(t) = [W_1(t), \dots, W_N(t)]^T$$

   (Wang (2010a)).
3. Gaussian HJM: If, in addition, $\vec{\sigma}(t,T)$ does not depend on $f(t,T)$, then it is known as the Gaussian HJM, where the instantaneous forward rates are normally distributed.

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Formulation

1. Instantaneous Forward Rate and Bond Price: By definition

$$f(t,T) = -\frac{\partial \log P(t,T)}{\partial T}$$

which implies that

$$P(t,T) = e^{-\int_t^T f(t,u)du} = e^{Q(t,T)}$$

where

$$Q(t,T) = -\int_t^T f(t,u)du$$

2. Derivative on $Q(t,T)$:

$$\Delta Q(t,T) = f(t,t)\Delta t - \int_t^T [\Delta f(t,u)]du = r(t)\Delta t - \int_t^T \left[\alpha(t,u)\Delta t + \vec{\sigma}(t,u) \cdot \Delta\vec{W}(t)\right]du$$

$$= r(t)\Delta t - \left[\int_t^T \alpha(t,u)du\right]\Delta t - \left[\int_t^T \vec{\sigma}(t,u)du\right] \cdot \Delta\vec{W}(t)$$

3. Price Relationship to $Q(t,T)$: Setting

$$\alpha^*(t,T) = \int_t^T \alpha(t,u)du$$

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and

$$\vec{\sigma}^*(t,T) = \int_t^T \vec{\sigma}(t,u)du$$

$$\Delta Q(t,T) = r(t)\Delta t - \alpha^*(t,T)\Delta t - \vec{\sigma}^*(t,T) \cdot \Delta\vec{W}(t)$$

From this we get

$$\Delta P(t,T) = P(t,T)\left\{\left[r(t) - \alpha^*(t,T) + \frac{1}{2}\vec{\sigma}^*(t,T) \cdot \vec{\sigma}^*(t,T)\right]\Delta t - \vec{\sigma}^*(t,T) \cdot \Delta\vec{W}(t)\right\}$$

4. <u>Discounted Zero-Coupon Bond Price Process</u>: Since the zero-coupon bond, as a tradeable asset, must have its discounted price $\frac{P(t,T)}{B(t)}$ to be a martingale under the risk-neutral measure, the drift term of $P(t,T)$ should be $r(t)P(t,T)$, therefore

$$\alpha^*(t,T) = \frac{1}{2}\vec{\sigma}^*(t,T) \cdot \vec{\sigma}^*(t,T)$$

Taking derivative with respect to T produces

$$\alpha(t,T) = \vec{\sigma}(t,T) \cdot \int_t^T \vec{\sigma}(t,u)du$$

In other words, the drift term of the instantaneous forward rate is completely determined by the volatility term. This is the main result of Heath, Jarrow, and Morton (1992).

5. <u>Differential Form of the Instantaneous Forward Rate</u>: Therefore, under the risk-neutral world, HJM model says that

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$$\Delta f(t,T) = \left\{ \vec{\sigma}(t,T) \cdot \int_t^T \vec{\sigma}(t,u)du \right\} \Delta t + \vec{\sigma}(t,T) \cdot \Delta \vec{W}(t)$$

with

$$f(0,T) = f_M(0,T)$$

where $f_M(0,T)$ are the exogenous inputs that ensure that these models are automatically consistent with discount bond prices at

$$t = 0$$

6. HJM Integral Forms:

$$f(0,T) = f_M(0,T) + \int_0^t \left\{ \vec{\sigma}(u,T) \cdot \int_u^T \vec{\sigma}(u,s)ds \right\} du + \int_0^t \vec{\sigma}(s,T) \cdot d\vec{W}(s)$$

$$\frac{\Delta P(t,T)}{P(t,T)} = r(t)\Delta t - \left\{ \int_t^T \vec{\sigma}(t,s)ds \right\} \cdot \Delta \vec{W}(t) = r(t)\Delta t - \vec{\sigma}^*(t,T) \cdot \Delta \vec{W}(t)$$

$$P(t,T) = P(0,T) + \int_0^t P(s,T)r(s)ds - \int_0^t P(s,T)\vec{\sigma}(s,T) \cdot d\vec{W}(s)$$

$$r(t) = f(t,t) = f(0,t) + \int_0^t \left\{ \vec{\sigma}(u,t) \cdot \int_u^t \vec{\sigma}(u,s)ds \right\} dt + \int_0^t \vec{\sigma}(s,t) \cdot d\vec{W}(s)$$

From the last equation on $r(t)$, it follows that the short rate process $r(t)$ is not Markovian in general.

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Hull-White From HJM

1. HW Case: Hull-White is a special case of the one-factor HJM:

$$f(0,T) = f_M(0,T) + \int_0^t \left\{ \vec{\sigma}(u,T) \cdot \int_u^T \vec{\sigma}(u,s)ds \right\} dt + \int_0^t \vec{\sigma}(s,T) \cdot d\vec{W}(s)$$

with the additional assumption for the volatility as a time-homogenous of the form

$$\vec{\sigma}(t,T) = \vec{\sigma}_{HW} e^{-a(T-t)}$$

2. HJM HW Short Rate Formulation: Then

$$r(t) = f(0,t) + \int_0^t \left\{ \vec{\sigma}(u,t) \cdot \int_u^t \vec{\sigma}(u,s)ds \right\} du + \int_0^t \vec{\sigma}(s,t) \cdot d\vec{W}(s)$$

$$= f(0,t) + \int_0^t \left\{ \vec{\sigma}_{HW} e^{-a(t-u)} \cdot \int_u^t \vec{\sigma}_{HW} e^{-a(s-u)} ds \right\} du$$

$$+ \int_0^t \vec{\sigma}_{HW} e^{-a(t-s)} \cdot d\vec{W}(s)$$

$$= f(0,t) + \frac{\left\|\vec{\sigma}_{HW}{}^2\right\|}{2a^2}[1 - e^{-at}]^2 + \vec{\sigma}_{HW} \cdot \int_0^t e^{-a(t-s)} d\vec{W}(s)$$

$$= r(0)e^{-at} + f(0,t) + \frac{\left\|\vec{\sigma}_{HW}{}^2\right\|}{2a^2}[1 - e^{-at}]^2 - f(0,0)e^{-at} + \vec{\sigma}_{HW}$$

$$\cdot \int_0^t e^{-a(t-s)} d\vec{W}(s)$$

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3. Comparison with the Standard HW Model: Comparison with the regular HW model, with the observation that

$$f(0,0) = r(0)$$

shows that

$$\Delta r(t) = [\theta(t) - ar(t)]\Delta t + \vec{\sigma}(t,T) \cdot \Delta\vec{W}(t)$$

where

$$\theta(t) = \frac{\partial f(0,t)}{\partial T} + af(0,t) + \frac{\left\|\vec{\sigma}_{HW}{}^{2}\right\|}{2a}[1 - e^{-2at}]$$

G2++ - A 2-Factor HJM Model

1. Setup: G2++ is a special case of HJM where there are 2 factors. It becomes

$$\Delta f(t,T) = \alpha(t,T)\Delta t + \vec{\sigma}_1(t,T) \cdot \Delta\vec{W}_1(t) + \vec{\sigma}_2(t,T) \cdot \Delta\vec{W}_2(t)$$

and

$$\Delta\vec{W}_1(t) \cdot \Delta\vec{W}_2(t) = \rho\Delta t$$

The two factors may be determined in practice using PCA – e.g., $\Delta\vec{W}_1(t)$ stands for the change of slope, while $\Delta\vec{W}_2(t)$ can stand for the change of curvature.

2. Formulation: Assume time-homogenous exponential form

$$\vec{\sigma}(t,T) \cdot \Delta\vec{W}(t) = \vec{\sigma}e^{-a(T-t)} \cdot \Delta\vec{W}_1(t) + \vec{\eta}e^{-b(T-t)} \cdot \Delta\vec{W}_2(t)$$

Using the same approach as for the 1-factor HJM we get

$$r(t) = f(0,t) + r(0)e^{-at} + \left\{\frac{\|\vec{\sigma}^2\|}{2a^2}[1-e^{-at}]^2 - f(0,0)e^{-at}\right\} + \vec{\sigma}\cdot\int_0^t e^{-a(t-s)}d\overrightarrow{W_1}(s)$$
$$+ r(0)e^{-bt} + \left\{\frac{\|\vec{\eta}^2\|}{2b^2}[1-e^{-bt}]^2 - f(0,0)e^{-bt}\right\} + \vec{\eta}\cdot\int_0^t e^{-a(t-s)}d\overrightarrow{W_2}(s)$$

3. <u>G2++ Model Short Rate</u>: Let

$$\varphi(t) = f(0,t) + r(0)e^{-at} + \left\{\frac{\|\vec{\sigma}^2\|}{2a^2}[1-e^{-at}]^2 - f(0,0)e^{-at}\right\} + r(0)e^{-bt}$$
$$+ \left\{\frac{\|\vec{\eta}^2\|}{2b^2}[1-e^{-bt}]^2 - f(0,0)e^{-bt}\right\}$$

Setting

$$x(t) = \vec{\sigma}\cdot\int_0^t e^{-a(t-s)}d\overrightarrow{W_1}(s)$$

and

$$y(t) = \vec{\eta}\cdot\int_0^t e^{-a(t-s)}d\overrightarrow{W_2}(s)$$

we get

$$\Delta x(t) = -ax(t)\Delta t + \vec{\sigma}\cdot\Delta\overrightarrow{W_1}(t); x(0) = 0$$

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$$\Delta y(t) = -by(t)\Delta t + \vec{\eta} \cdot \Delta\overrightarrow{W_2}(t); y(0) = 0$$

and

$$\Delta\overrightarrow{W_1}(t) \cdot \Delta\overrightarrow{W_2}(t) = \rho\Delta t$$

and we arrive at the G2++ model under the risk-neutral measure as

$$r(t) = x(t) + y(t) + \varphi(t)$$

HJM to LMM

1. The Forward Rate: Consider the forward rate

$$F(t; S, T) = \frac{1}{\tau(S, T)}\left[\frac{P(t, S)}{P(t, T)} - 1\right]$$

Applying the Ito lemma, we get

$$\Delta F(t; S, T) = \frac{1}{\tau(S, T)}\left[\frac{1}{P(t, T)}\Delta P(t, S) + P(t, S)\Delta\left\{\frac{1}{P(t, T)}\right\} + \Delta P(t, S)\Delta\left\{\frac{1}{P(t, T)}\right\}\right]$$

2. Application of the HJM to the Forward Rate: From the above, we get

$$\Delta P(t, T) = P(t, T)\left\{r(t)\Delta t - \vec{\sigma}^*(t, T) \cdot \Delta\overrightarrow{W}(t)\right\}$$

which implies that

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$$\Delta\left\{\frac{1}{P(t,T)}\right\} = -\frac{1}{P(t,T)^2}\Delta P(t,T) + \frac{1}{P(t,T)^3}\Delta\langle P(t,T), \Delta P(t,T)\rangle$$
$$= \frac{1}{P(t,T)}\left\{[-r(t) + \|\vec{\sigma}(t,T)^*\|^2]\Delta t + \vec{\sigma}^*(t,T)\cdot\Delta\vec{W}(t)\right\}$$

so that $\Delta F(t;S,T)$ can be re-written as

$$\Delta F(t;S,T) = \frac{1}{\tau(S,T)}\left[\frac{1}{P(t,T)}\Delta P(t,S) + P(t,S)\Delta\left\{\frac{1}{P(t,T)}\right\} + \Delta P(t,S)\Delta\left\{\frac{1}{P(t,T)}\right\}\right]$$
$$= \frac{1}{\tau(S,T)}\frac{P(t,S)}{P(t,T)}\left\{[\|\vec{\sigma}(t,T)^*\|^2 - \vec{\sigma}^*(t,S)\cdot\vec{\sigma}^*(t,T)]\Delta t\right.$$
$$\left. + [\vec{\sigma}^*(t,T) - \vec{\sigma}^*(t,S)]\cdot\Delta\vec{W}(t)\right\}$$

3. <u>Shifted LIBOR Forward Rate</u>: From the above, we see that

$$\Delta F(t;S,T) = \left[F(t;S,T) + \frac{1}{\tau(S,T)}\right][\vec{\sigma}^*(t,T) - \vec{\sigma}^*(t,S)]\cdot\left[\vec{\sigma}^*(t,T)\Delta t + \Delta\vec{W}(t)\right]$$

Now define the shifted LIBOR forward rate as

$$F^*(t;S,T) = F(t;S,T) + \frac{1}{\tau(S,T)}$$

This leads to the following log-normal process

$$\Delta F^*(t;S,T) = F^*(t;S,T)\left[\int_S^T \vec{\sigma}(t,u)du\right]\cdot\left[\vec{\sigma}^*(t,T)\Delta t + \Delta\vec{W}(t)\right]$$

4. <u>T-Forward Measure</u>: We know that under the T-forward measure the shifted LIBOR forward rate is a martingale. To change the measure from risk-neutral to the T-forward measure, we use Girsanov theorem (Girsanov Theorem (Wiki)) as follows:

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$$X(t) = -\int_0^t \left[\int_s^T \vec{\sigma}(s,u)du\right] \cdot d\vec{W}(s) = -\int_s^T \vec{\sigma}^*(s,u) \cdot d\vec{W}(s)$$

implies that

$$\Delta X(t) = -\left[\int_s^T \vec{\sigma}(s,u)du\right] \cdot \Delta\vec{W}(s) = -\vec{\sigma}^*(s,u) \cdot \Delta\vec{W}(s)$$

resulting in

$$\vec{W}_T(t) = \vec{W}(t) - X(t)$$

Thus

$$\Delta\vec{W}_T(t) = \Delta\vec{W}(t) - \left[\Delta\vec{W}(t), \Delta X(t)\right] = \Delta\vec{W}(t) + \vec{\sigma}^*(t,T)\Delta t$$

So, under the T-forward measure

$$\Delta F^*(t;S,T) = F^*(t;S,T)\left[\int_S^T \vec{\sigma}(t,u)du\right] \cdot \Delta\vec{W}_T(t)$$

5. <u>Instantaneous Forward Rate under T-forward Measure</u>: Using the fact that

$$\Delta f(t,T) = \vec{\sigma}(t,T) \cdot \vec{\sigma}^*(t,T)\Delta t + \vec{\sigma}(t,T) \cdot \Delta\vec{W}(t)$$
$$= \vec{\sigma}(t,T) \cdot \vec{\sigma}^*(t,T)\Delta t + \vec{\sigma}(t,T) \cdot \left[\Delta\vec{W}_T(t) - \vec{\sigma}^*(t,T)\Delta t\right] = \vec{\sigma}(t,T) \cdot \Delta\vec{W}_T(t)$$

we can see that, under the T-forward measure the instantaneous forward rate is a martingale, or

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$$\Delta f(t,T) = \vec{\sigma}(t,T) \cdot \Delta \overrightarrow{W_T}(t)$$

Thus, from the T-forward measure for $F^*(t;S,T)$ and $f(t,T)$ we can see that the expectation hypothesis holds under the T-forward measure, i.e.,

$$\mathbb{E}_t^{Q^T}[r(t)|\mathcal{F}_t] = f(t,T)$$

6. Caplet/Floorlet on $F(t;S,T)$: The caplet/Floorlet on rate $F(S;S,T)$ with strike K can be re-defined on $F^*(S;S,T)$ with a strike $K + \frac{1}{\tau(S,T)}$. The Black volatility seen above becomes $\int_0^S \left\| \int_S^T \vec{\sigma}(t,u)du \right\|^2 dt$.
7. Blowup of the T-forward Instantaneous Forward Rate: Anderson and Piterbarg (2010a, 2010b) pointed out that the log-normal proportional volatility HJM process

$$\sigma\big(t,T,f(t,T)\big) = \sigma(t,T)f(t,T)$$

leads to log-normally distributed instantaneous forward rates under the T-forward measure, i.e.,

$$\Delta f(t,T) = \vec{\sigma}(t,T) \cdot \Delta \overrightarrow{W_T}(t)$$

but the forward rates can explode in finite time to ∞ with non-zero probability. The LIBOR market model addresses this drawback.
8. Generalization of LMM: The typical LIBOR market model can be generalized as follows. Suppose there exists a deterministic function $\hat{\sigma}(t,T)$ such that

$$\int_S^T \sigma(s,u)du = \frac{F(t;S,T)}{F(t;S,T) + \frac{1}{\tau(S,T)}} \hat{\sigma}(t,T)$$

Then

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$$\Delta F(t;S,T) = \hat{\sigma}(t,T)F(t;S,T)\Delta W_T(t)$$

HJM PCA

1. Introduction: In this section we introduce the historical estimation/calibration for HJM. We then follow it up by applying PCA for the calibration of HJM.
2. Time Homogeneity: Considering the Gaussian HJM model, time homogeneity means that volatility $\vec{\sigma}(t,T)$ depends on the remaining time to maturity, or

$$\vec{\sigma}(t,T) = \vec{\sigma}(T-t)$$

   so that the Gaussian HJM may be written as

$$\Delta f(t,T) = \mu(t,T)\Delta t + \vec{\sigma}(T-t)\cdot\Delta\vec{W}(t)$$

   This enables us to estimate $\vec{\sigma}$ (and hence the drift) from the historical data, e.g., to estimate the 3M volatility, we retain the time-to-maturity at 3M across observations.
3. HJM Estimation on the Spot Rate: The historical estimation maybe performed on the instantaneous forward rate, or on the more observable continuous zero rate. Recall that the continuously compounded spot rate is defined by

$$R(t,T) = -\frac{\log P(t,T)}{\tau(t,T)}$$

   Applying the Ito lemma, we get

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Tel. 646-513-3300

$$\Delta R(t,T) = -\Delta\left[\frac{1}{\tau(t,T)}\right]\log P(t,T)$$
$$-\frac{1}{\tau(t,T)}\left\{\frac{1}{P(t,T)}\Delta P(t,T) - \frac{1}{2P(t,T)^2}\Delta\langle P(t,T), P(t,T)\rangle\right\}$$
$$= -\frac{\log P(t,T)}{(T-t)^2}\Delta t - \frac{1}{\tau(t,T)}\left\{\left[r(t) - \frac{1}{2}\|\vec{\sigma}^*(t,T)\|^2\right]\Delta t - \vec{\sigma}^*(t,T)\cdot\Delta\vec{W}(t)\right\}$$
$$= \frac{1}{\tau(t,T)}\left\{\left[R(t,T) - r(t) + \frac{1}{2}\|\vec{\sigma}^*(t,T)\|^2\right]\Delta t + \vec{\sigma}^*(t,T)\cdot\Delta\vec{W}(t)\right\}$$

which yields

$$\Delta R(t,T) = \frac{1}{\tau(t,T)}\left\{\left[R(t,T) - r(t) + \frac{1}{2}\|\vec{\sigma}^*(t,T)\|^2\right]\Delta t + \vec{\sigma}^*(t,T)\cdot\Delta\vec{W}(t)\right\}$$

4. Variance/Covariance of the Instantaneous Forward Rate: Here we consider general processes of the form

$$\Delta X_i(t) = \mu_i \Delta t + \sum_{j=1}^{N} \sigma_{ij}\Delta W_j(t)$$

where X can be the instantaneous forward rate or the continuously compounded spot rate. Further the time $T-t$ is replaced by the subscript i to indicate different tenors (3M/6M etc.) Work in the correlation space eventually to determine the constituent components.

5. Dimensionality Reduction via PCA: Usually the first 3 PCA's a repacked, and they represent the 3 main yield curve movements – parallel shifts, twists (steepener/flattener), and curvature (butterfly).

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Tel. 646-513-3300

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135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

Hull-White Model

Short Rate Formulation

1. Basic Relation: This section adapted from Brigo and Mercurio (2006) and Wang (2010). As an extension of the Vasicek model, the Hull-White model assumes that the short rate follows the mean-reverting SDE

$$\Delta r(t) = [\theta(t) - ar(t)]\Delta t + \sigma W(t)$$

where σ and a are positive constants, and $\theta(t)$ is a time-dependent function that will be used to fit the current zero curve.

2. Solution to the Hull-White SDE: To solve this SDE, first apply the Ito lemma to re^{at};

$$\Delta(re^{at}) = e^{at}\Delta r + are^{at}\Delta t = (\theta\Delta t + \sigma\Delta W)e^{at}$$

Then integrate both sides over (s, t);

$$r(t)e^{at} - r(s)e^{as} = \int_s^t \theta(u)e^{au}du + \sigma\int_s^t e^{au}dW(u)$$

resulting in

$$r(t) = r(s)e^{-a(t-s)} + \int_s^t \theta(u)e^{-a(t-u)}du + \sigma\int_s^t e^{-a(t-u)}dW(u)$$

3. Fitting to the Initial Term Structure: In order to fit to the initial term structure of interest rates the time-dependent $\theta(t)$ must satisfy

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Tel. 646-513-3300

$$\theta(t) = \frac{\partial f_M(0,t)}{\partial t} + a f_M(0,t) + \frac{\sigma^2}{2a}[1 - e^{-2at}]$$

where $f_M(0,t)$ is the market observed instantaneous forward rate at time 0 for maturity t.

4. <u>Re-cast of $r(t)$</u>: From the above, $r(t)$ can be written as

$$r(t) = r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW(u)$$

where

$$\alpha(t) = f_M(0,t) + \frac{\sigma^2}{2a^2}[1 - e^{-at}]^2$$

5. <u>Mean and Variance of $r(t)$</u>: Therefore $r(t)$ conditional on $\mathcal{F}_s$ is normally distributed with mean and variance given respectively by

$$\mathbb{E}[r(t)|\mathcal{F}_s] = r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)}$$

and

$$Var[r(t)|\mathcal{F}_s] = \sigma \int_s^t e^{-2a(t-u)} du = \frac{\sigma^2}{2a}[1 - e^{-2at}]$$

6. <u>Hull-White as an Affine Term Structure Model</u>: Hull-White model is an *affine term structure model* where the continuously compounded spot rate is an affine function on the short rate, i.e.,

$$R(t,T) = \alpha'(t,T) + \beta(t,T)r(t)$$

7. <u>Hull-White Based Product Valuation</u>: The zero-coupon price is given by

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$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

where

$$A(t,T) = \frac{P_M(0,T)}{P_M(0,t)} e^{B(t,T)f_M(0,t) - \frac{\sigma^2}{4a}[1-e^{-2at}]B(t,T)^2}$$

and

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a}$$

We can also find closed-form formulas for zero-coupon bond options, caps/floors, and swaptions (Brigo and Mercurio (2006)).

Hull-White Trinomial Tree

1. Decomposition: To construct the Hull-White tree, it is useful to decompose the short rate into the following format:

$$r(t) = x(t) + \alpha(t)$$

where

$$\alpha(t) = f_M(0,t) + \frac{\sigma^2}{2a^2}[1 - e^{-at}]^2$$

$$\Delta x(t) = -ax(t)\Delta t + \sigma \Delta W(t); x(0) = 0$$

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$$x(t) = x(s) + e^{-a(t-s)} + \sigma \int_{s}^{t} e^{-a(t-u)} dW(u)$$

2. Trinomial Tree Construction Steps: With this decomposition in hand, the tree construction can be achieved in 2 steps. In the first, one constructs a trinomial tree for $x(t)$. Then in the next, one shifts the tree by $\alpha(t)$ to being it in line with the initial term structure.

Construction of the Symmetric Trinomial Tree

1. Setting up the Tree Nodes: Denote the tree nodes by (i, j) where the time index i ranges from 0 to N and the space index j ranges from some $\underline{j}_i$ to some $\overline{j}_i$. Recall that the expectation and the variance of $x(t)$ are

$$\mathbb{E}[x(t)|\mathcal{F}_s] = x(s)e^{-a(t-s)}$$

and

$$Var[r(t)|\mathcal{F}_s] = \frac{\sigma^2}{2a}\left[1 - e^{-2a(t-s)}\right]$$

respectively. Using these in our discretized nodes (Brigo and Mercurio (2006)), we get

$$\mathbb{E}\left[x(t_{i+1})|x(t_i) = x_{i,j}\right] = x_{i,j}e^{-a\Delta t_i} \doteqdot M_{i,j}$$

and

$$Var\left[x(t_{i+1})|x(t_i) = x_{i,j}\right] = \frac{\sigma^2}{2a}\left[1 - e^{-2a\Delta t_i}\right] \doteqdot {V_i}^2$$

where

$$\Delta t_i = t_{i+1} - t_i$$

2. Incorporating the Transition Probabilities: Now, given the node $x_{i,j}$, we need to locate its subsequent nodes $x_{i+1,k+1}$, $x_{i+1,k}$, and $x_{i+1,k-1}$, associated with their transition probabilities p_u, p_m, and p_d. First we find the spatial displacement as

$$\Delta x_{i+1} = V_i\sqrt{3} = \sigma\sqrt{\frac{3}{2a}[1 - e^{-2a\Delta t_i}]}$$

Then locate the node k using

$$k = round\left(\frac{M_{i,j}}{\Delta x_{i+1}}\right)$$

where $round(x)$ indicates the integer closest to the real number x. We then set the following:

$$x_{i+1,k+1} = (k+1)\Delta x_{i+1}$$

$$x_{i+1,k} = k\Delta x_{i+1}$$

$$x_{i+1,k-1} = (k-1)\Delta x_{i+1}$$

3. Choice of Transition Probabilities: Finally the transition probabilities are chosen in such a way as to match the conditional mean and the variance:

$$p_u = \frac{1}{6} + \frac{{\eta_{j,k}}^2}{6{V_i}^2} + \frac{\eta_{j,k}}{2\sqrt{3}V_i}$$

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Tel. 646-513-3300

$$p_m = \frac{2}{3} - \frac{{\eta_{j,k}}^2}{3{V_i}^2}$$

$$p_d = \frac{1}{6} + \frac{{\eta_{j,k}}^2}{6{V_i}^2} - \frac{\eta_{j,k}}{2\sqrt{3}V_i}$$

Here

$$\eta_{j,k} = M_{i,j} - x_{i+1,k} = M_{i,j} - k\Delta x_{i+1}$$

Displacing the Nodes of the Trinomial Tree

1. Calculation of the displacement α: An easy way to do this is by noticing that

$$\alpha(t) = f_M(0,t) + \frac{\sigma^2}{2a^2}[1 - e^{-at}]^2$$

   and approximating the instantaneous forward rate by

$$f_M(0,t) \approx F_M(0;\ t, t + 0.5\ bp)$$

   (Wang (2010)). This approach has to approximate the continuously compounded rate $R(0,t)$ by the short rate $r(0)$, therefore doesn't fit the zero curve exactly.

2. Using Arrow-Debreu Prices: The alternate approach is to use the Arrow-Debreu prices. Denote α_i as the displacement at t_i, and let Q_{ij} be the Arrow-Debreu price at node (i,j). A *State-Price Security* or the *Arrow-Debreu Security* is defined as the contract that pays $1 at a particular date and a particular time, and pays $0 in all other states. The corresponding price (i.e., the NPV) is referred to as the Arrow-Debreu Price.
3. α_i/Q_{ij} Calculation Step #1: Initialize

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Tel. 646-513-3300

$$Q_{ij} = 0$$

4. $\underline{\alpha_i / Q_{ij}}$ Calculation Step #2: Find

$$\alpha_0 = -\frac{\log P_M(0, t_1)}{t_1}$$

5. $\underline{\alpha_i / Q_{ij}}$ Calculation Step #3: With the given α_i (where $i = 0$ at the start), calculate

$$Q_{i+1,j} = \sum_h Q_{i,h} q(h, j) e^{-(\alpha_i + j\Delta x_i)\Delta t_i}$$

where $q(h, j)$ is the probability of migrating from node (i, h) to node $(i + 1, j)$.

6. $\underline{\alpha_i / Q_{ij}}$ Calculation Step #4: With $Q_{i,j}$ in hand, find α_i by solving

$$P(0, t_{i+1}) = \sum_{j=\underline{j}_i}^{\bar{j}_i} Q_{i,j} e^{-(\alpha_i + j\Delta x_i)\Delta t_i}$$

which leads to

$$\alpha_i = \frac{1}{\Delta t_i} \log \frac{\sum_{j=\underline{j}_i}^{\bar{j}_i} Q_{i,j} e^{-j\Delta x_i \Delta t_i}}{P(0, t_{i+1})}$$

7. $\underline{\alpha_i / Q_{ij}}$ Calculation Step #5: Loop through steps 3 and 4 to discover α_i and $Q_{i,j}$ in the eventual steps. The short rate at each node is

$$r_{i,j} = x_{i,j} + \alpha_i$$

In general, remember that if σ and/or a is a function of f, r, the tree will be non-recombining.

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Market Model of Interest Rate Dynamics

Problems with Conventional Market Practice

1. Typical Swap Derivatives Book: In most markets, caps and floors form the largest component of the average swap derivatives book, caps/floors being considered here are strips of caplets/floorlets, each of which is a call/put on the forward rate.
2. Market Pricing Practice: Conventional market practice has been to price the option assuming that the underlying forward rate is distributed log-normally with zero drift. Thus the option price is given by the Black's formula, discounted from the settlement date.
3. Trouble with the Market Practice: In an arbitrage-free setting, forward rates over consecutive time intervals are related to one another and cannot all be lognormal under one arbitrage-free measure.
4. The BGM Approach: Brace, Gatarek, and Musiela (1997) show that the market practice above can be made consistent with an arbitrage-free term structure model. Consecutive quarterly or semi-annual rates can all be lognormal while the model remains arbitrage-free. This is possible because each forward rate is lognormal under the corresponding forward (to the settlement date) arbitrage-free measure rather than under a single spot arbitrage-free measure. Lognormality under the appropriate forward and not the spot arbitrage-free measure is required to justify the use of the Black futures formula with discount for the caplet pricing.

Nomenclature and Notation

1. Origins of the Term Structure Parametrization: The term structure parametrizations considered here were originally proposed by Musiela (1993), and developed further later by Musiela and Sondermann (1993), Brace and Musiela (1994), Goldys, Musiela, and Sondermann (1994), and Musiela (1994).

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2. The Continuously Compounded Forward Rate: We denote by $r(t, x)$ the continuously compounded forward rate prevailing at time t over the interval $[t + x, t + x + \Delta t]$. There is an obvious relationship between the Heath, Jarrow, and Morton (1992) forward rates $f(t, T)$ and $r(t, x)$, namely

$$r(t, x) = f(t, t + x)$$

For all $T > 0$ the process

$$P(t, T) = e^{-\int_0^{T-t} r(t,u)du} = e^{-\int_t^T f(t,u)du}$$

for

$$0 \leq t \leq T$$

describes the price evolution of a zero-coupon bond maturing at T.

3. Evolution of $r(t, x)$: We make the usual mathematical assumptions, i.e., all processes are defined on the probability space $[\Omega, \{\mathcal{F}_t \geq 0\}, \mathbb{P}]$, where the filtration $\{\mathcal{F}_t \geq 0\}$ is the $\mathbb{P}$-augmentation of the natural filtration generated by a d-dimensional Brownian motion

$$\vec{W} = \{\vec{W}(t);\ t \geq 0\}$$

We then assume that the process

$$\{r(t, x);\ x, t \geq 0\}$$

satisfies

$$\Delta r(t, x) = \frac{\partial}{\partial x}\left[\left\{r(t, x) + \frac{1}{2}|\vec{\sigma}(t, x)|^2\right\}\Delta t + \vec{\sigma}(t, x) \cdot \Delta\vec{W}(t)\right]$$

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where for all

$$x \geq 0$$

the volatility process

$$\{\vec{\sigma}(t,x);\ t \geq 0\}$$

is $\mathcal{F}_t$-adapted with values in $\mathbb{R}^d$, while $|\cdot|$ and $\cdot$ stand for the usual norm and the inner product in $\mathbb{R}^d$, respectively. We also assume that the function

$$x \longmapsto \vec{\sigma}(t,x)$$

is absolutely continuous, and the derivative

$$\vec{\tau}(t,x) = \frac{\partial}{\partial x}\vec{\sigma}(t,x)$$

is bounded on $\mathbb{R}_+^2 \times \Omega$.

4. The Discount Function: The time evolution of the discount function

$$D(t,x) = P(t,t+x) = e^{-\int_0^x r(t,u)du}$$

is described by the discount process

$$\{D(t,x);\ x,t \geq 0\}$$

From the dynamics of $r(t,x)$, it easily follows that

$$\Delta D(t,x) = D(t,x)\left\{[r(t,0) - r(t,x)]\Delta t - \vec{\sigma}(t,x) \cdot \Delta\overrightarrow{W}(t)\right\}$$

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Tel. 646-513-3300

and hence $\vec{\sigma}(t,x)$ can be interpreted as price volatility. Obviously we have

$$\vec{\sigma}(t,0) = 0$$

5. <u>Non-Markovian Nature of the Spot Rate Process</u>: The spot rate process

$$\{r(t,0);\ t \geq 0\}$$

satisfies

$$\Delta r(t,0) = \frac{\partial}{\partial x} r(t,x)\bigg|_{x=0} \Delta t + \frac{\partial}{\partial x}\vec{\sigma}(t,x)\bigg|_{x=0} \cdot \Delta\vec{W}(t)$$

and hence is, in general, not Markovian.

6. <u>Spot Rate Savings Account</u>: The process

$$\beta(t) = e^{\int_0^t r(u,0)du}\ \forall\ t \geq 0$$

represents the amount generated at time

$$t \geq 0$$

by continuously re-investing $1 at the spot rate

$$r(s,0)\ \forall\ \ 0 \leq s \leq t$$

7. <u>Martingale Nature of $P(t,T)$ under $\beta(t)$</u>: It is well-known that if for all

$$T > 0$$

the process

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$$\left\{\frac{P(t,T)}{\beta(t)}; 0 \le t \le T\right\}$$

is a martingale under $\mathbb{P}$, then there is no arbitrage possible between the zero coupon bonds $P(t,T)$ of all maturities $T > 0$ and the savings account $\beta(t)$. Thus, from the $r(t,x)$ evolution equation, one can write that

$$\frac{P(t,T)}{\beta(t)} = P(0,T)\, e^{-\int_0^t \vec{\sigma}(s,T-s)\cdot d\vec{W}(s) - \frac{1}{2}\int_0^t |\vec{\sigma}(s,T-s)|^2 ds}$$

where the RHS is a martingale under $\mathbb{P}$. It also follows that

$$\Delta P(t,T) = P(t,T)\left\{r(t,0)\Delta t - \vec{\sigma}(t,T-t)\cdot \Delta\vec{W}(t)\right\}$$

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135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

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135 E. 57th Street |25th Floor |New York, NY 10022
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The BGM Model

LIBOR Rate Dynamics

1. Lognormal LIBOR Rate: To specify the model, or equivalently, to define the volatility process $\vec{\sigma}(t,x)$ in

$$\Delta r(t,x) = \frac{\partial}{\partial x}\left[\left\{r(t,x) + \frac{1}{2}|\vec{\sigma}(t,x)|^2\right\}\Delta t + \vec{\sigma}(t,x) \cdot \Delta\vec{W}(t)\right]$$

we fix

$$\delta > 0$$

and assume that for each

$$x \geq 0$$

the LIBOR rate process

$$\{L(t,x);\ t,x \geq 0\}$$

defined by

$$1 + \delta L(t,x) = e^{\int_x^{x+\delta} r(t,u)du}$$

has a lognormal volatility structure, i.e.,

$$\Delta L(t,x) = \cdots \Delta t + L(t,x)\vec{\gamma}(t,x) \cdot \Delta\vec{W}(t)$$

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where the deterministic function

$$\vec{\gamma}: \mathbb{R}_+^2 \longrightarrow \mathbb{R}^d$$

is bounded and piecewise continuous.

2. Formulation of the LIBOR Rate Dynamics: From Ito's formula and the equation for $\Delta r(t, x)$ dynamics above, we get

$$\Delta L(t, x) = \frac{1}{\delta}\Delta\left[e^{\int_x^{x+\delta} r(t,u)du}\right]$$

$$= \frac{1}{\delta}e^{\int_x^{x+\delta} r(t,u)du}\Delta\left[e^{\int_x^{x+\delta} r(t,u)du}\right] + \frac{1}{\delta}\frac{1}{2}\left[e^{\int_x^{x+\delta} r(t,u)du}\right]|\vec{\sigma}(t, x+\delta) - \vec{\sigma}(t, x)|^2\Delta t$$

$$= \frac{1}{\delta}\left[e^{\int_x^{x+\delta} r(t,u)du}\right]\left\{\left[r(t, x+\delta) - r(t, x) + \frac{1}{2}|\vec{\sigma}(t, x+\delta)|^2 - \frac{1}{2}|\vec{\sigma}(t, x)|^2\right]\Delta t\right.$$

$$\left. + [\vec{\sigma}(t, x+\delta) - \vec{\sigma}(t, x)] \cdot \Delta\overrightarrow{W}(t)\right\}$$

$$+ \frac{1}{\delta}\frac{1}{2}\left[e^{\int_x^{x+\delta} r(t,u)du}\right]|\vec{\sigma}(t, x+\delta) - \vec{\sigma}(t, x)|^2\Delta t$$

$$= \left\{\frac{\partial}{\partial x}L(t, x) + \frac{1}{\delta}[1 + \delta L(t, x)]\vec{\sigma}(t, x+\delta) \cdot [\vec{\sigma}(t, x+\delta) - \vec{\sigma}(t, x)]\right\}\Delta t$$

$$+ \frac{1}{\delta}[1 + \delta L(t, x)][\vec{\sigma}(t, x+\delta) - \vec{\sigma}(t, x)] \cdot \Delta\overrightarrow{W}(t)$$

3. The LIBOR Recurrence: For the above LIBOR dynamics to be lognormal for all

$$x \geq 0$$

we need

$$\vec{\sigma}(t, x+\delta) - \vec{\sigma}(t, x) = \frac{\delta L(t, x)}{1 + \delta L(t, x)}\vec{\gamma}(t, x)$$

This recurrence defines the HJM volatility process $\vec{\sigma}(t, x)$ for all

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$$x \geq \delta$$

provided $\vec{\sigma}(t,x)$ is defined on the interval

$$0 \leq x < \delta$$

We set

$$\vec{\sigma}(t,x) = 0$$

for all

$$0 \leq x < \delta$$

and the recursive solution for $\vec{\sigma}(t,x)$ for $x \geq \delta$ is

$$\vec{\sigma}(t,x) = \sum_{k=1}^{\left\lfloor \frac{x}{\delta} \right\rfloor} \frac{\delta L(t,x-k\delta)}{1+\delta L(t,x-k\delta)} \vec{\gamma}(t,x-k\delta)$$

4. Evolution of $L(t,x)$: From the recursive lognormal constraint, the equation for $L(t,x)$ becomes

$$\Delta L(t,x) = \left[\frac{\partial}{\partial x} L(t,x) + L(t,x)\vec{\gamma}(t,x) \cdot \vec{\sigma}(t,x+\delta)\right] \Delta t + L(t,x)\vec{\gamma}(t,x) \cdot \Delta \overrightarrow{W}(t)$$

5. Recursion Dynamics of $L(t,x)$: Using the recursion solution for $\vec{\sigma}(t,x)$, the process $\{L(t,x);\ t,x \geq 0\}$ satisfies

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Tel. 646-513-3300

$$\Delta L(t,x) = \left[\frac{\partial}{\partial x}L(t,x) + L(t,x)\vec{\gamma}(t,x)\cdot\vec{\sigma}(t,x) + \frac{\delta L^2(t,x)}{1+\delta L^2(t,x)}|\vec{\gamma}(t,x)|^2\right]\Delta t + L(t,x)\vec{\gamma}(t,x)\cdot\Delta\overrightarrow{W}(t)$$

Relation to the HJM Dynamics

1. Introduction: The BGM approach to the term structure modeling is quite different from the ones based on instantaneous continuously compounded spot or forward rates, and therefore its motivations/origins are worth examining.
2. Instantaneous Effective Annual Rates: The change of focus from the instantaneously compounded forward rates to the instantaneous compounded effective annual rates was first proposed by Sandmann and Sondermann (1993) in response to the impossibility of being able to price a Eurodollar futures contract with a lognormal mode of the continuously compounded spot rate.
3. Lognormal Effective Annual Rate: An HJM-type model based on instantaneous effective annual rates was introduced by Goldys, Musiela, and Sondermann (1994). A lognormal volatility structure was assumed on the effective annual rate $j(t,x)$ which is related to the instantaneous continuously compounded forward rate $r(t,x)$ via the formula

$$1 + j(t,x) = e^{r(t,x)}$$

4. Nominal Annual Rates: The case of the nominal annual rates $q(t,x)$ corresponding to $r(t,x)$, i.e.,

$$[1+\delta j(t,x)]^{\frac{1}{\delta}} = e^{r(t,x)}$$

was studied by Musiela (1994). It turns out that the HJM volatility process takes the form

$$\vec{\sigma}(t,x) = \int_0^x \frac{1}{\delta}\left[1 - e^{-\delta r(t,u)}\right]\vec{\gamma}(t,u)du$$

Obviously for

$$\delta = 1$$

we get the Goldys, Musiela, and Sondermann (1994) model, and for

$$\delta = 0$$

we get

$$\vec{\sigma}(t,x) = \int_0^x r(t,u)\vec{\gamma}(t,u)du$$

and hence the HJM lognormal model, which is known to explode – for

$$\delta > 0$$

no explosion occurs.

5. Option Pricing Challenges with the above Representation: Unfortunately, these formulas do not give the closed form pricing formulas for the options. In order to price a caplet, for example, one would have to use some numerically intensive algorithms. This would not be practical for model calibration, where an iterative procedure would be needed to identify the volatility $\vec{\gamma}(t,x)$ which returns the market prices for a large number of caps and swaptions.
6. Discrete Annual Rates: A key piece of the term structure puzzle was found by Miltersen, Sandmann, and Sondermann (1994). First, attention was shifted from the instantaneous forward rates $q(t,x)$ to the nominal annual rates $f(t,x,\delta)$ defined by

$$[1 + f(t, x, \delta)]^{\delta} = e^{\int_x^{x+\delta} r(t,u)du}$$

More importantly, however, it was shown for $\delta = 1$ the model prices a yearly caplet according to the market standard.

7. Effective Discrete Rates: Unfortunately, the volatility $\vec{\sigma}(t, x)$ was not completely identified above, leaving open the question of model selection for maturities different from

$$x = i\delta$$

as well as the solution to $\Delta r(t, x)$. These problems were partially addressed by Miltersen, Sandmann, and Sondermann (1995), where a model based on the effective rates $f(t, T, \delta)$ defined by

$$1 + \delta f(t, T, \delta) = e^{\int_{T-t}^{T+\delta-t} r(t,u)du}$$

was analyzed.

8. The BGM Approach: As indicated earlier, the BGM approach assumes a log-normal volatility structure on the LIBOR rate $L(t, x)$ defined by

$$1 + \delta L(t, x) = e^{\int_x^{x+\delta} r(t,u)du}$$

for all

$$x \geq 0$$

and a fixed

$$\delta > 0$$

This leads to the $\vec{\sigma}(t, x)$ for $L(t, x)$ given by

$$\vec{\sigma}(t,x) = \sum_{k=1}^{\left\lfloor \frac{x}{\delta} \right\rfloor} \frac{\delta L(t, x - k\delta)}{1 + \delta L(t, x - k\delta)} \vec{\gamma}(t, x - k\delta)$$

and

$$\Delta L(t,x) = \left[\frac{\partial}{\partial x} L(t,x) + L(t,x)\vec{\gamma}(t,x) \cdot \vec{\sigma}(t,x) + \frac{\delta L^2(t,x)}{1 + \delta L^2(t,x)} |\vec{\gamma}(t,x)|^2\right] \Delta t$$
$$+ L(t,x)\vec{\gamma}(t,x) \cdot \Delta\vec{W}(t)$$

Existence, Uniqueness, and Regularity of the LIBOR Dynamics Solution

1. Uniqueness Statement: For all

$$x \geq 0$$

let

$$\left\{\vec{\xi}(t,x);\ t \geq 0\right\}$$

be an adapted, bounded stochastic process with values in $\mathbb{R}^d$,

$$\vec{a}(\cdot,x): \mathbb{R}_+ \to \mathbb{R}^d$$

be a deterministic and piece-wise continuous function, and let

$$M(t,x) = \int_0^t \vec{a}(s,x) \cdot d\vec{W}(s)$$

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For all

$$x \geq 0$$

the equation

$$\Delta y(t,x) = y(t,x)\vec{a}(t,x) \cdot \left\{\left[\frac{\delta y(t,x)}{1+\delta y(t,x)}\vec{a}(t,x) + \vec{\xi}(t,x)\right]\Delta t + \Delta\vec{W}(t)\right\} \; \forall \; y(0,x) > 0$$

where

$$\delta > 0$$

is a constant, has a unique and strictly positive solution on $\mathbb{R}_+$. Moreover, if for some

$$k \in \{0,1,2,\cdots\}$$

$$y(0,x) \in \mathbb{C}^k(\mathbb{R}_+)$$

for all

$$t \geq 0$$

$$\vec{a}(t,x), M(t,x), \vec{\xi}(t,x) \in \mathbb{C}^k(\mathbb{R}_+)$$

then

$$y(t,x) \in \mathbb{C}^k(\mathbb{R}_+) \; \forall \; t \geq 0$$

2. <u>Proof</u>:

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a. Uniqueness => Since the RHS of the equation for $\Delta y(t,x)$ is locally Lipschitz-continuous with respect to y on $\mathbb{R} - \frac{1}{\delta}$ and Lipschitz-continuous on $\mathbb{R}_+$, there exists a unique (possibly exploding) strictly positive solution to $\Delta y(t,x)$.

b. Applying Ito Integral => By the Ito formula

$$y(t,x) = y(0,x)e^{\int_0^t \vec{a}(s,x)\cdot d\vec{W}(s)+\int_0^t \vec{a}(s,x)\cdot\left[\frac{\delta y(s,x)}{1+\delta y(s,x)}\vec{a}(s,x)+\vec{\xi}(s,x)-\frac{1}{2}\vec{a}(s,x)\right]ds}$$

for all

$$t < \tau = \inf\{t: y(t,x) = \infty, or\ y(t,x) = 0\}$$

But if

$$y(t,x) = 0$$

for some

$$t < \infty$$

then

$$y(s,x) = 0$$

for all

$$s \geq t$$

and hence

$$\tau = \inf\{t: y(t,x) = \infty\}$$

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Moreover, because

$$\int_{0}^{t} |\vec{a}(s,x)|^2 ds < \infty$$

for all

$$t < \infty$$

we deduce that

$$\tau = \infty$$

c. Ito Integral to the Volterra Integral Form => Thus the Ito integral form for $y(t,x)$ above is equivalent to the following Volterra-type integral equation for

$$l(t,x) = \log y(t,x)$$

$$l(t,x) = l(0,x) + \int_{0}^{t} \vec{a}(s,x) \cdot d\vec{W}(s) + \int_{0}^{t} \vec{a}(s,s) \cdot \left[\frac{\delta e^{l(s,x)}}{1 + \delta e^{l(s,x)}} \vec{a}(s,x) + \vec{\xi}(s,x) - \frac{1}{2}\vec{a}(s,x)\right] ds$$

d. Lipschitz-continuity of the RHS => Because the RHS in the Volterra-type integral equation is globally Lipschitz continuous with respect to l, we deduce using the standard fixed-point arguments that exists a unique path-wise solution to the Volterra-type integral equation above. Moreover, for any

$$t \geq 0$$

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Tel. 646-513-3300

$$l(t,x) \in \mathbb{C}^k(\mathbb{R}_+)$$

provided

$$l(0,x), \vec{a}(t,x), \vec{\xi}(t,x) \in \mathbb{C}^k(\mathbb{R}_+) \ \forall \ t \geq 0$$

3. <u>Smoothness of the Solution - Statement</u>: Let

$$\vec{\gamma}: {\mathbb{R}_+}^2 \to \mathbb{R}^d$$

be a deterministic, bounded, and piecewise continuous function

$$\delta > 0$$

be a constant, and let

$$M(t,x) = \int_0^t \vec{\gamma}(s, x+t-s) \cdot d\vec{W}(s)$$

Then the equation

$$\Delta L(t,x) = \left[\frac{\partial}{\partial x} L(t,x) + L(t,x)\vec{\gamma}(t,x) \cdot \vec{\sigma}(t,x) + \frac{\delta L^2(t,x)}{1+\delta L^2(t,x)} |\vec{\gamma}(t,x)|^2\right] \Delta t$$
$$+ L(t,x)\vec{\gamma}(t,x) \cdot \Delta\vec{W}(t)$$

admits a unique non-negative solution $L(t,x)$ for any

$$t \geq 0$$

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and any non-negative initial condition

$$L(0, x) = L_0$$

If

$$L_0 > 0$$

then

$$L(t, x) > 0 \; \forall \; t > 0$$

If

$$L_0 \in \mathbb{C}^k(\mathbb{R}_+) \; \forall \; t \geq 0$$

$$\vec{\gamma}(t, x) \in \mathbb{C}^k(\mathbb{R}_+)$$

$$M(t, x) \in \mathbb{C}^k(\mathbb{R}_+)$$

$$\left. \frac{\partial^j}{\partial x^j} \vec{\gamma}(t, x) \right|_{x=0} = 0 \; \forall \; j = 0, 1, \dots, k$$

then

$$L(t, x) \in \mathbb{C}^k(\mathbb{R}_+) \; \forall \; t \geq 0$$

4. Proof:
   a. Mild Solution to $\Delta L(t, x)$ => By the solution to $\Delta L(t, x)$ we mean the so-called mild solution (Da Prato and Zabczyk (1992)), i.e., $L(t, x)$ is a solution if

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Tel. 646-513-3300

$$L(t,x) = L(0,x+t) + \int_0^t L(s,x+t-s)\vec{\gamma}(s,x+t-s)\cdot\vec{\sigma}(s,x+t-s)ds$$

$$+ \int_0^t \frac{\delta L^2(s,x+t-s)}{1+\delta L(s,x+t-s)}|\vec{\gamma}(s,x+t-s)|^2 ds$$

$$+ \int_0^t L(s,x+t-s)\vec{\gamma}(s,x+t-s)\cdot d\vec{W}(s)\ \ \forall\, x,t \ge 0$$

b. Validity of the Solution to $0 \le t \le x$ => The above integral form holds true for

$$0 \le x < \delta$$

because the process

$$L(t,x-t), 0 \le t \le x, x > 0$$

is a solution to

$$\Delta y(t,x) = y(t,x)\vec{a}(t,x)\cdot\left\{\left[\frac{\delta y(t,x)}{1+\delta y(t,x)}\vec{a}(t,x) + \vec{\xi}(t,x)\right]\Delta t + \Delta\vec{W}(t)\right\}$$

$$y(0,x) > 0$$

with

$$\vec{a}(t,x) = \vec{\gamma}(t,(x-t)\vee 0)$$

and

$$\vec{\xi}(t,x) = 0$$

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Tel. 646-513-3300

c. Validity for $\delta \leq x \leq 2\delta$ => For

$$\delta \leq x \leq 2\delta$$

the process

$$L(t, x-t), 0 \leq t \leq x$$

satisfies the solution to $\Delta y(t,x)$ with

$$\vec{a}(t,x) = \vec{\gamma}(t, (x-t) \vee 0)$$

and

$$\vec{\xi}(t,x) = \frac{\delta L(t, (x-\delta-t) \vee 0)}{1 + \delta L(t, (x-\delta-t) \vee 0)}$$

Thus by induction, we prove that $\Delta y(t,x)$ admits a unique solution for any

$$x > 0$$

and

$$0 \leq t \leq x$$

d. Using Induction to Complete the Smoothness Proof => Further by induction, from the recursion relation for $\vec{\sigma}(t,x)$, we deduce that the corresponding $\vec{a}(t,x)$ and $\vec{\xi}(t,x)$ satisfy the assumptions of regularity in the statement for uniqueness and existence, and hence $L(t,x)$ is smooth as well.

5. Application of the above Result to $r(t,x)$: If for some

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$$k \in \mathbb{N}$$

and all

$$t \geq 0$$

$$\vec{\gamma}(t,x) \in \mathbb{C}^{k}(\mathbb{R}_{+})$$

and

$$\left.\frac{\partial^{j}}{\partial x^{j}}\vec{\gamma}(t,x)\right|_{x=0} = 0 \;\; j = 0,1,\dots,k$$

then

$$\Delta r(t,x) = \frac{\partial}{\partial x}\left[\left\{r(t,x) + \frac{1}{2}|\vec{\sigma}(t,x)|^{2}\right\}\Delta t + \vec{\sigma}(t,x)\cdot\Delta\vec{W}(t)\right]$$

has a unique solution

$$r(t,x) \in \mathbb{C}^{k-1}(\mathbb{R}_{+})$$

for any positive initial condition

$$r(0,x) \in \mathbb{C}^{k-1}(\mathbb{R}_{+})$$

6. Proof of Smooth, Unique Solution to $r(t,x)$: Consider the $\Delta r(t,x)$ evolution equation above with the fixed volatility process $\vec{\sigma}(t,x)$ given by

$$\vec{\sigma}(t,x) = \sum_{k=1}^{\left\lfloor\frac{x}{\delta}\right\rfloor}\frac{\delta L(t,x-k\delta)}{1+\delta L(t,x-k\delta)}\vec{\gamma}(t,x-k\delta)$$

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Tel. 646-513-3300

$$\Delta L(t,x) = \left[\frac{\partial}{\partial x} L(t,x) + L(t,x)\vec{\gamma}(t,x) \cdot \vec{\sigma}(t,x) + \frac{\delta L^2(t,x)}{1 + \delta L^2(t,x)} |\vec{\gamma}(t,x)|^2\right] \Delta t + L(t,x)\vec{\gamma}(t,x) \cdot \Delta \vec{W}(t)$$

Applying the above result, the proof follows.

7. Non-smooth $\vec{\gamma}(t,x)$: The volatility $\vec{\sigma}(t,x)$ given by the recurrence relation may not be differentiable with respect to x for some functions $\vec{\gamma}$ (for example, piece-wise constant with respect to x). In such cases, the term structure dynamics cannot be analyzed in the HJM framework of

$$\Delta r(t,x) = \frac{\partial}{\partial x}\left[\left\{r(t,x) + \frac{1}{2}|\vec{\sigma}(t,x)|^2\right\}\Delta t + \vec{\sigma}(t,x) \cdot \Delta \vec{W}(t)\right]$$

8. Validity of the Savings Account Numeraire No-Arbitrage: The difficulty presented above, however, is just technical, since the relation

$$\frac{P(t,T)}{\beta(t)} = P(0,T) e^{-\int_0^t \vec{\sigma}(s,T-s) \cdot d\vec{W}(s) - \frac{1}{2}\int_0^t |\vec{\sigma}(s,T-s)|^2 ds}$$

is still sufficient to eliminate arbitrage. By setting

$$T = t$$

we may also define the savings account numeraire purely in terms of the price volatility $\vec{\sigma}(t,x)$.

9. Recurrence for the Spot Account Numeraire: It is also easy to see that

$$P(t,t+\delta) = \beta(t) P(0,t+\delta) e^{-\int_0^t \vec{\sigma}(s,t+\delta-s) \cdot d\vec{W}(s) - \frac{1}{2}\int_0^t |\vec{\sigma}(s,t+\delta-s)|^2 ds}$$

$$= \beta(t) P(0,t+\delta) e^{-\int_0^{t+\delta} \vec{\sigma}(s,t+\delta-s) \cdot d\vec{W}(s) - \frac{1}{2}\int_0^{t+\delta} |\vec{\sigma}(s,t+\delta-s)|^2 ds} = \frac{\beta(t)}{\beta(t+\delta)}$$

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since

$$\vec{\sigma}(t, x) = 0$$

for

$$0 \leq x < \delta$$

10. Establishment of the No-Arbitrage Criterion: Solving the recurrence relationship

$$\frac{\beta(t)}{P(t, t + \delta)} = \beta(t + \delta)$$

we get

$$\beta(t) = \prod_{k=0}^{\left\lfloor \frac{t}{\delta} \right\rfloor} \frac{1}{P(\{t - [k + 1]\delta\}^{+}, t - k\delta)}$$

Thus, the zero coupon prices

$$\{P(t, T); 0 \leq t \leq T\}$$

discounted by

$$\{\beta(t); t \geq 0\}$$

satisfy the evolution dynamics for $\Delta r(t, x)$, and hence there is no arbitrage.

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Upper/Lower Bounds for the LIBOR Rate

1. <u>Non-Markovian Nature of all Rates</u>: The regularity of $\vec{\gamma}$ has an important influence on the short rate $r(t, 0)$ dynamics. If the process

$$\{r(t, 0); t \geq 0\}$$

is a semi-martingale, then it satisfies

$$\Delta r(t, 0) = \frac{\partial}{\partial x} r(t, x)\bigg|_{x=0} \Delta t$$

Thus, the short rate is a process of finite variation, and therefore it cannot be strong Markov, except in the deterministic case (Cinlar and Jacod (1981)). The LIBOR process

$$\{L(t, 0); t \geq 0\}$$

satisfies an equivalent $\Delta L(t, 0)$ relation as well.

2. <u>Bounding $|\vec{\gamma}(s, x + t - s) \cdot \vec{\sigma}(s, x + t - s)|$</u>: It follows from the existence/uniqueness criterion and the relation

$$\Delta y(t, x) = y(t, x)\vec{a}(t, x) \cdot \left\{\left[\frac{\delta y(t, x)}{1 + \delta y(t, x)} \vec{a}(t, x) + \vec{\xi}(t, x)\right] \Delta t + \Delta \overrightarrow{W}(t)\right\}; \quad y(0, x) > 0; \quad \delta > 0$$

that the process

$$\{L(t, x); t, x \geq 0\}$$

satisfies

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$$L(t,x)$$

$$= L(0,x+t)e^{\int_0^t \vec{\gamma}(s,x+t-s)\cdot d\vec{W}(s)+\int_0^t \vec{\gamma}(s,x+t-s)\cdot\left[\frac{\delta L(s,x+t-s)}{1+\delta L(s,x+t-s)}\vec{\gamma}(s,x+t-s)+\vec{\sigma}(s,x+t-s)-\frac{1}{2}\vec{\gamma}(s,x+t-s)\right]ds}$$

and

$$|\vec{\gamma}(s,x+t-s)\cdot\vec{\sigma}(s,x+t-s)| \le \sum_{k=1}^{\left\lfloor\frac{x+t-s}{\delta}\right\rfloor} |\vec{\gamma}(s,x+t-s)||\vec{\gamma}(s,x+t-s-k\delta)|$$

3. <u>Explicit Lower and Upper Bounds for $L(t,x)$</u>: Therefore

$$L_1(t,x) \le L(t,x) \le L_2(t,x)$$

where

$$L_1(t,x) = L(0,x+t)e^{\int_0^t \vec{\gamma}(s,x+t-s)\cdot d\vec{W}(s)-\int_0^t\left[\alpha(s,x+t-s)+\frac{1}{2}|\vec{\gamma}(s,x+t-s)|^2\right]ds}$$

and

$$L_2(t,x) = L(0,x+t)e^{\int_0^t \vec{\gamma}(s,x+t-s)\cdot d\vec{W}(s)+\int_0^t\left[\alpha(s,x+t-s)+\frac{1}{2}|\vec{\gamma}(s,x+t-s)|^2\right]ds}$$

where

$$\alpha(s,x+t-s) = \sum_{k=1}^{\left\lfloor\frac{x+t-s}{\delta}\right\rfloor} |\vec{\gamma}(s,x+t-s)||\vec{\gamma}(s,x+t-s-k\delta)|$$

4. <u>Application of the Bound above</u>: Consequently, the LIBOR rate is bounded from below and above by log-normal processes. The estimate from the above can be used to show that the Euro-dollar futures price is well-defined. The most common Euro-dollar contract relates to

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the LIBOR rate. The futures payoff at time T is equal to $\delta L(T,0)$, and hence the Euro-dollar futures price at time

$$t \le T$$

is

$$\mathbb{E}[\delta L(T,0)|\mathcal{F}_t]$$

Because

$$L(T,0) \le L_2(T,0)$$

and

$$\mathbb{E}[\delta L_2(T,0)] = L(T,0)e^{\int_0^T[\alpha(s,T-s)+|\vec{\gamma}(s,T-s)|^2]ds} < \infty$$

we conclude that the expectation is finite.

5. <u>Stochastic System of Equations for Rates</u>: For

$$n = 1, 2, \cdots$$

and

$$t \ge 0$$

define

$$y_n(t) = L(t,[n\delta - t\ \vee 0]), \vec{\gamma}_n(t) = \vec{\gamma}(t,[n\delta = t\ \vee 0])$$

and assume that

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$$\vec{\gamma}(t,0) = 0$$

It easily follows that the processes satisfy the following closed system of stochastic equations

$$\Delta y_n(t) = y_n(t)\vec{\gamma}_n(t) \cdot \left[\sum_{j=\left\lfloor \frac{t}{\delta} \right\rfloor + 1}^{n} \frac{\delta y_j(t)}{1 + \delta y_j(t)} \vec{\gamma}_j(t)\Delta t + \Delta \overrightarrow{W}(t) \right]$$

6. Mean Reversion of Rates: We now examine whether the model for $y_n(t)$ above implies mean-reverting behavior. For this, we assume the following;

$$|\vec{\gamma}(t,x)| \leq \beta(x)$$

where

$$\underset{0 \leq x \leq \delta}{sup} \sum_{k=0}^{\infty} \beta(x + k\delta) < \infty$$

wnd

$$\int_{0}^{\infty} (x+1)\beta^2(x)dx < \infty$$

7. Bounding the p^{th} Raw Moment of LIBOR: Under the $|\vec{\gamma}(t,x)|$ bounding conditions above, for any

$$p \geq 1$$

and any deterministic initial condition

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$$L(0,x) \in \mathbb{C}_b(\mathbb{R}_+)$$

we have

$$\sup_{t \geq 0} \sup_{x \geq 0} \mathbb{E}[L^p(t,x)] < \infty$$

a. Proof => We use α and L_2 as defined above. By the $|\vec{\gamma}(t,x)|$ bounding conditions,

$$\sup_{t \geq 0} \sup_{x \geq 0} \int_0^t \left[\alpha(s, x+t-s) + \frac{1}{2}|\vec{\gamma}(s, x+t-s)|^2\right] ds < \infty$$

and

$$\mathbb{E}\left[\left\{\int_0^t \vec{\gamma}(s, x+t-s) \cdot d\overrightarrow{W}(s)\right\}^2\right] \leq \int_0^\infty \beta^2(x+s)dx \leq \int_0^\infty \beta^2(x)dx < \infty$$

Since $\log L_2$ is Gaussian,

$$\sup_{t \geq 0} \sup_{x \geq 0} \mathbb{E}[{L_2}^p(t,x)] < \infty$$

for any

$$p \geq 1$$

Since

$$L \leq L_2$$

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Tel. 646-513-3300

$$\sup_{t \geq 0} \sup_{x \geq 0} \mathbb{E}[L^p(t,x)] < \infty$$

Invariant Measure for the LIBOR Rate

1. Additional Bounding Assumptions: Additionally, we assume

$$\vec{\gamma}(t,x) = \vec{\gamma}(x)$$

$$\int_0^\infty x \left|\frac{\partial \vec{\gamma}(x)}{\partial x}\right|^2 dx < \infty$$

$$\sup_{0 \leq x \leq \delta} \sum_{k=0}^{\infty} \left|\frac{\partial \vec{\gamma}(x + k\delta)}{\partial x}\right| < \infty$$

$$\int_0^\infty |\vec{\gamma}(x)| dx = C < \frac{1}{K}$$

2. Existence of an Invariant Measure - Approach: The assumption

$$\vec{\gamma}(t,x) = \vec{\gamma}(x)$$

implies that L is a time-homogenous Markov process. Hence we can examine the notion of invariant measures. The proof of the existence of an invariant measure will follow the standard Krylov-Bogoliubov scheme – the Feller property and the tightness of the family of distributions $\mathcal{L}[L(t)]_{t \geq 0}$ implies the existence of an invariant measure (Da Prato and Zabczyk (1992)).

3. Setup and Definitions: Let

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$$C_0(\mathbb{R}) = \{u \in C(\mathbb{R}) : u(x) \to 0 \; as \; x \to \infty\}$$

and let

$$C^{\alpha}(\mathbb{R}) = \{u \in C(\mathbb{R}) : |u(x) - u(z)| \leq C|x - z|^{\alpha}\}$$

for any

$$0 < \alpha \leq 1$$

We represent the Holder norm in $C^{\alpha}(\mathbb{R})$ by $\|\cdot\|_{\alpha}$.

4. Relative Compactness Criteria: A family of functions

$$\Gamma \subset C_0(\mathbb{R}_+)$$

is relatively compact in $C_0(\mathbb{R}_+)$ if and only if the following conditions are satisfied:

   a. The family Γ is equi-continuous on any bounded set;
   b. There exists a function

$$R : \mathbb{R}_+ \to \mathbb{R}_+$$

such that

$$R(u) \to 0$$

as

$$u \to \infty$$

and

$$|f(u)| \leq R(u)$$

for any

$$f \in \Gamma$$

and

$$u \geq 0$$

5. <u>Existence of a Concentrated/Tight Invariant Measure - Statement</u>: We use the additional bounds above, and let $L(t, x)$ be the solution to

$$\Delta L(t,x) = \left[\frac{\partial}{\partial x} L(t,x) + L(t,x)\vec{\gamma}(t,x) \cdot \vec{\sigma}(t,x) + \frac{\delta L^2(t,x)}{1 + \delta L^2(t,x)} |\vec{\gamma}(t,x)|^2\right] \Delta t$$
$$+ L(t,x)\vec{\gamma}(t,x) \cdot \Delta\overrightarrow{W}(t)$$

and

$$\begin{matrix} sup \\ 0 \leq x < \infty \end{matrix} |\log L(0,x)| < \infty$$

Then

$$\begin{matrix} sup \\ t \geq 0 \end{matrix} \mathbb{E}[\|\log L(t)\|] < \infty$$

If, moreover,

$$\int_0^\infty |\vec{\gamma}(x)| dx = C < \frac{1}{K}$$

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is satisfied, then there exists an invariant measure for the process $L(t,x)$ concentrated on the closed set

$$U = \{u \in C(\mathbb{R}) : u > 0, and\ u(x) \to 0\ as\ x \to \infty\}$$

6. Proof:
   a. Step #1 - Log Representation of $L(t,x)$ => Consider the process

$$l(t,x) = \log L(t,x)$$

which can be represented as

$$l(t,x) = l_0(x+t) + \int_0^t \vec{F}\big(l(t-s)\big)(x+t-s) \cdot \vec{\gamma}(x+t-s)ds$$

$$-\frac{1}{2}\int_0^t |\vec{\gamma}(x+t-s)|^2 ds + M(t,x)$$

for any

$$t \geq 0$$

where M is defined by

$$M(t,x) = \int_0^t \vec{\gamma}(x+t-s) \cdot d\overrightarrow{W}(s)$$

and

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$$\vec{F}(l)(x) = \sum_{k=0}^{\left\lfloor\frac{x}{\delta}\right\rfloor} \frac{\delta e^{l(x-k\delta)}}{1+\delta e^{l(x-k\delta)}} \vec{\gamma}(x-k\delta)$$

for any

$$l \in C(\mathbb{R})$$

b. Step #2 - l as a Feller Process => Using

$$\sup_{0 \le x \le \delta} \sum_{k=0}^{\infty} \beta(x+k\delta) < \infty$$

we can see that

$$\vec{\gamma} \cdot \vec{F} : \; C_0(\mathbb{R}_+) \to C_0(\mathbb{R}_+)$$

is a Lipschitz-continuous function. By the standard fixed-point method, $l(t)$ depends continuously on the initial condition in the space $C_0(\mathbb{R}_+)$. Therefore the process l is a Feller process.

c. Step #3 - Bounding $\{M(t,x)\}^2$ and $\left[\frac{\partial M(t,x)}{\partial x}\right]^2$ => Notice that

$$\frac{\partial M(t,x)}{\partial x} = \int_0^t \frac{\partial \vec{\gamma}(x+t-s)}{\partial x} \cdot d\vec{W}(s)$$

By the Ito formula, we have

$$\mathbb{E}\left[\int_0^{\infty} \{M(t,x)\}^2 dx\right] \le \int_0^{\infty} \int_0^{\infty} |\vec{\gamma}(x+s)|^2 dx\, ds = \sqrt{2} \int_0^{\infty} x |\vec{\gamma}(x)|^2 dx$$

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and

$$\mathbb{E}\left[\int_0^\infty \left\{\frac{\partial M(t,x)}{\partial x}\right\}^2 dx\right] \le \int_0^\infty \int_0^\infty \left|\frac{\partial \vec{\gamma}(x+s)}{\partial x}\right|^2 dx\, ds = \sqrt{2}\int_0^\infty x \left|\frac{\partial \vec{\gamma}(x)}{\partial x}\right|^2 dx$$

d. Step #4 - Bounding $\underset{x \ge u}{sup}\{M(t,x)\}^2$ => Using the expressions for $\mathbb{E}\left[\int_0^\infty \{M(t,x)\}^2 dx\right]$ and $\mathbb{E}\left[\int_0^\infty \left\{\frac{\partial M(t,x)}{\partial x}\right\}^2 dx\right]$, using Sobolev embedding along with the additional bounds above, we get

$$\mathbb{E}\left[\underset{x \ge u}{sup}\{M(t,x)\}^2\right] \le C_1\left\{\int_0^\infty \int_0^\infty |\vec{\gamma}(x+s)|^2 dx\, ds + \int_0^\infty \int_0^\infty \left|\frac{\partial \vec{\gamma}(x+s)}{\partial x}\right|^2 dx\, ds\right\}$$
$$\le R(u) < \infty$$

and for

$$\alpha < \frac{1}{2}$$

$$\mathbb{E}\left[\|M(t,x)\|_\alpha{}^2\right] \le C_1\left\{\int_0^\infty \int_0^\infty |\vec{\gamma}(x+s)|^2 dx\, ds + \int_0^\infty \int_0^\infty \left|\frac{\partial \vec{\gamma}(x+s)}{\partial x}\right|^2 dx\, ds\right\} \le R(u)$$
$$< \infty$$

where R and $R(u)$ are independent of t, and

$$R(u) \to 0$$

as

$$u \to \infty$$

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e. Step #5 - Bounding $\mathbb{E}[\|l(t)\|]$ => From

$$l(t,x) = l_0(x+t) + \int_0^t \vec{F}\big(l(t-s)\big)(x+t-s) \cdot \vec{\gamma}(x+t-s)ds$$

$$-\frac{1}{2}\int_0^t |\vec{\gamma}(x+t-s)|^2 ds + M(t,x)$$

for any

$$t \geq 0$$

we get

$$\mathbb{E}[\|l(t)\|] \leq \|l(0)\| + K\int_0^\infty |\vec{\gamma}(x)|dx + \frac{1}{2}\int_0^\infty |\vec{\gamma}(x)|^2 dx + \mathbb{E}\left[\begin{matrix} sup \\ x \geq 0 \end{matrix} |M(t,x)|\right]$$

Using the bounds for

$$\begin{matrix} sup \\ t \geq 0 \end{matrix} \mathbb{E}\left[\begin{matrix} sup \\ x \geq 0 \end{matrix} |M(t,x)|\right]$$

above, we see that

$$\begin{matrix} sup \\ t \geq 0 \end{matrix} \mathbb{E}[\|l(t)\|] < \infty$$

f. Step #6 - Approach for Proving Tightness of $\mathbb{E}[l(t)]_{t\geq 0}$ => From

$$\int_0^\infty |\vec{\gamma}(x)|dx = C < \frac{1}{K}$$

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and assuming that

$$l_0 = 0$$

in order to prove the existence of an invariant measure for the process l, we will need to prove that the family of laws $\mathbb{E}[l(t)]_{t \geq 0}$ is tight.

g. Step #7 - Bounding $|l(t,x)|_{l_1}$ Unconditionally => Again from the expression for $l(t,x)$, we get

$$\sup_{t \geq 0} \mathbb{E}\left[\sup_{x \geq u} |l(t,x)|\right] \leq K \int_u^\infty |\vec{\gamma}(x)| dx + \frac{1}{2} \int_u^\infty |\vec{\gamma}(x)|^2 dx + \mathbb{E}\left[\sup_{x \geq u} |M(t,x)|\right] \rightarrow 0$$

as

$$u \rightarrow \infty$$

h. Step #8 - Lipschitz Criterion on $\vec{F}(\psi)$ => From the Lipschitz criterion listed earlier, we can see that for any

$$\psi \in C(\mathbb{R})$$

$$\frac{\left|\vec{F}(\psi)(x) - \vec{F}(\psi)(u)\right|}{|x-u|^\alpha} \leq C_1 + CK \sup_{x, u \geq 0} \frac{|\psi(x) - \psi(u)|}{|x-u|^\alpha}$$

for a certain constant C_1.

i. Step #9 - Explicit Bounds for $\mathbb{E}[\|l(t)\|_\alpha]$ => From

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$$\int_0^\infty (x+1)\beta^2(x)dx < \infty$$

$$\int_0^\infty x\left|\frac{\partial\vec{\gamma}(x)}{\partial x}\right|^2 dx < \infty$$

$$\int_0^\infty |\vec{\gamma}(x)|dx = C < \frac{1}{K}$$

we get

$$\sup_{t \geq 0} \mathbb{E}[\|l(t)\|_\alpha] \leq C_1 + CK \sup_{t \geq 0} \mathbb{E}[\|l(t)\|_\alpha]$$

Since

$$CK < 1$$

we see that

$$\sup_{t \geq 0} \mathbb{E}[\|l(t)\|_\alpha] \leq \frac{C_2}{1 - CK}$$

j. Step #10 - Proof that the Family is Tight => Using

$$\sup_{t \geq 0} \mathbb{E}\left[\sup_{x \geq u} |l(t,x)|\right] \leq K\int_u^\infty |\vec{\gamma}(x)|dx + \frac{1}{2}\int_u^\infty |\vec{\gamma}(x)|^2 dx + \mathbb{E}\left[\sup_{x \geq u} |M(t,x)|\right]$$
$$\to 0$$

as

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$$u \to \infty$$

and

$$\underset{t \geq 0}{sup}\, \mathbb{E}[\|l(t)\|_{\alpha}] \leq \frac{C_2}{1 - CK}$$

since

$$CK < 1$$

as well as the relative compactness criterion, we can see that the family $\mathcal{L}[L(t)]_{t \geq 0}$ is tight on $C_0(\mathbb{R})$.

k. Step #11 - Existence of Invariant Measures on $L(t, x)$ => Since $l(t, x)$ is a Feller process, by the standard Krylov-Bogoliubov technique there exists an invariant measure for the process l, concentrated on $C_0(\mathbb{R})$. Existence of invariant measures for l on $C_0(\mathbb{R})$ is equivalent to existence of invariant measures for L on U.

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Application of BGM to Derivatives Pricing

Cap/Floor Pricing

1. Payer Forward Swap Fixing: Consider a payer forward swap on principal \$1 settled quarterly in arrears at times

$$T_j = T_0 + j\delta \; \forall \; j = 1, \cdots, n$$

The LIBOR rate received at T_j is set at T_{j-1} at the level

$$L(T_{j-1}, 0) = \frac{1}{\delta}\left[\frac{1}{P(T_{j-1} - T_j)} - 1\right]$$

2. Payer Forward Swap Cash-flows and Pricing: The swap cash flows at $T_j, j = 1, \cdots, n$ are $\delta L(T_{j-1}, 0)$ and $-\kappa\delta$ and hence the time $t(t \leq T_0)$ value of the swap is (Brace and Musiela (1994b))

$$\mathbb{E}\left[\sum_{j=1}^{n} \frac{\beta(t)}{\beta(T_j)}\{L(T_{j-1}, 0) - \kappa\}\delta|\mathcal{F}_t\right] = P(t - T_0) - \sum_{j=1}^{n} C_j P(t - T_j)$$

where

$$C_j = \kappa\delta$$

for $j = 1, \cdots, n - 1$ and

$$C_n = 1 + \kappa\delta$$

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3. The Par Forward Swap Rate: The forward swap rate $\omega_{T_0}(t,n)$ at time t for the forward/futures maturity/expiry T_0 is that value of the fixed rate κ which makes the value of the forward swap zero, i.e.,

$$\omega_{T_0}(t,n) = \frac{P(t,T_0) - P(t,T_n)}{\delta \sum_{j=1}^{n} P(t,T_j)}$$

4. Cap/Floor Pricing Formulation: In a forward cap (resp. floor) on principal \$1 settled in arrears at times $T_j, j = 1, \cdots, n$, the cash flows at times T_j are $\delta\{L(T_{j-1},0) - \kappa\}^+$ (resp. $\delta\{\kappa - L(T_{j-1},0)\}^+$). The cap price at time $t \le T_0$ is

$$Cap(t) = \mathbb{E}\left[\sum_{j=1}^{n} \frac{\beta(t)}{\beta(T_j)}\{L(T_{j-1},0) - \kappa\}^+ \delta|\mathcal{F}_t\right] = \sum_{j=1}^{n} P(t,T_j)\,\mathbb{E}_{T_j}\left[\{L(T_{j-1},0) - \kappa\}^+ \delta|\mathcal{F}_t\right]$$

where $\mathbb{E}_T$ stands for the expectation under the forward measure $\mathbb{P}_T$ defined by (Musiela (1995))

$$\mathbb{P}_T = \mathbb{P}_0 e^{-\int_0^t \vec{\sigma}(s,T-s)\cdot d\vec{W}(s) - \frac{1}{2}\int_0^t |\vec{\sigma}(s,T-s)|^2 ds} = \frac{\mathbb{P}_0}{P(0,T)\beta(t)}$$

5. The LIBOR Rate Process Equation: The LIBOR rate process equation

$$K(t,T) = L(t,T-t), 0 \le t \le T$$

satisfies

$$\Delta K(t,T) = K(t,T)\vec{\gamma}(t,T-t) \cdot \left\{\left[\vec{\sigma}(t,T-t) + \frac{\delta K(t,T)}{1+\delta K(t,T)}\vec{\gamma}(t,T-t)\right]\Delta t + \Delta\vec{W}(t)\right\}$$
$$= K(t,T)\vec{\gamma}(t,T-t) \cdot \left[\vec{\sigma}(t,T+\delta-t)\Delta t + \Delta\vec{W}(t)\right]$$

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6. $K(t,T)$ Under the Forward Measure: The process

$$\overrightarrow{W_T}(t) = \vec{W}(t) + \int_0^t \vec{\sigma}(s, T-s) \cdot d\vec{W}(s)$$

is a Brownian motion under $\mathbb{P}_T$. Consequently

$$\Delta K(t,T) = K(t,T)\vec{\gamma}(t, T-t) \cdot \overrightarrow{W_{T+\delta}}(t)$$

and hence $K(t,T)$ is log-normally distributed under $\mathbb{P}_{T+\delta}$.

7. The Caplet Pricing Relation: From the above, it follows that

$$\mathbb{E}_{T+\delta}[\{L(T,0) - K\}^+ \delta | \mathcal{F}_t] = \mathbb{E}_{T+\delta}[\{K(T,T) - K\}^+ \delta | \mathcal{F}_t]$$
$$= K(t,T)N[h(t,T)] - KN[h(t,T) - \varsigma(t,T)]$$

where

$$h(t,T) = \frac{\log \frac{K(t,T)}{K} + \frac{1}{2}\varsigma^2(t,T)}{\varsigma(t,T)}$$

and

$$\varsigma^2(t,T) = \int_t^T |\vec{\gamma}(s, T-s)|^2 ds$$

8. The Cap Pricing Relation: The cap price at time $t \leq T_0$ is

$$Cap(t) = \sum_{j=1}^{n} \delta P(t, T_j)\{K(t, T_{j-1})N[h(t, T_{j-1})] - KN[h(t, T_{j-1}) - \varsigma(t, T_{j-1})]\}$$

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9. Comparison of the Caplet Price with Black's Formula: The preceding $Cap(t)$ formula corresponds to the market Black futures formula with discount from settlement date. It was originally derived using a different approach and model setup by Miltersen, Sandmann, and Sondermann (1994).

Payer Swap Option Pricing

1. Payer Swap Option Details Recap: The payer swap option at strike κ maturing at T_0 gives the right to receive at T_0 the cash flows corresponding to the payer swap settled in arrears, or alternately, discounted from the settlement dates

$$T_j = T_0 + j\delta \; \forall \; j = 1, \cdots, n$$

to T_0 the value of the cash flows defined by $\left\{\omega_{T_0}(t, n) - K\right\}^+$, where

$$\omega_{T_0}(T_0, n) = \frac{P(t, T_0) - P(t, T_n)}{\delta \sum_{j=1}^{n} P\left(t, T_j\right)}$$

2. Payer Swap Option Pricing: Hence at time $t \leq T_0$ the price of the option is

$$\mathbb{E}\left[\frac{\beta(t)}{\beta(T_0)}\mathbb{E}\left[\sum_{j=1}^{n}\frac{\beta(T_0)}{\beta(T_j)}\left\{\omega_{T_0}(T_0, n) - \kappa\right\}^+ \delta | \mathcal{F}_{T_0}\right] | \mathcal{F}_t\right]$$

$$= \mathbb{E}\left[\frac{\beta(t)}{\beta(T_0)}\left\{1 - \sum_{j=1}^{n} C_j P\left(T_0, T_j\right)\right\}^+ | \mathcal{F}_t\right]$$

$$= \mathbb{E}\left[\frac{\beta(t)}{\beta(T_0)}\mathbb{E}\left[\sum_{j=1}^{n}\frac{\beta(T_0)}{\beta(T_j)}\left\{L\left(T_{j-1}, 0\right) - \kappa\right\}^+ \delta | \mathcal{F}_{T_0}\right] | \mathcal{F}_t\right]$$

where

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$$C_j = \kappa\delta$$

for $j = 1, \cdots, n-1$ and

$$C_n = 1 + \kappa\delta$$

(Brace and Musiela (1994b)).

3. Spot Measure Option in-the-money Probability: Let

$$A = \{\omega_{T_0}(T_0, n) \geq \kappa\}$$

be the event that the swaption ends in the money. Then the payer swap option price is

$$\begin{aligned}
P_{Swaption}(t) &= \mathbb{E}\left[\frac{\beta(t)}{\beta(T_0)}\left\{1 - \sum_{j=1}^{n} C_j P(T_0, T_j)\right\}^{+} | \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\frac{\beta(t)}{\beta(T_0)}\left\{1 - \sum_{j=1}^{n} C_j P(T_0, T_j)\right\} \mathcal{I}_A | \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\frac{\beta(t)}{\beta(T_0)} \mathcal{I}_A | \mathcal{F}_t\right] - \mathbb{E}\left[\frac{\beta(t)}{\beta(T_0)}\left\{\sum_{j=1}^{n} C_j P(T_0, T_j)\right\} \mathcal{I}_A | \mathcal{F}_t\right] \\
&= P(t, T_0)\mathbb{P}_{T_0}(A|\mathcal{F}_t) - \sum_{j=1}^{n} C_j \mathbb{E}\left[\frac{\beta(t)}{\beta(T_0)} \mathbb{E}\left[\frac{\beta(T_0)}{\beta(T_j)} | \mathcal{F}_{T_0}\right] \mathcal{I}_A | \mathcal{F}_t\right] \\
&= P(t, T_0)\mathbb{P}_{T_0}(A|\mathcal{F}_t) - \sum_{j=1}^{n} C_j P(t, T_j)\mathbb{P}_{T_j}(A|\mathcal{F}_t)
\end{aligned}$$

4. Forward Measure Option in-the-money Probability: Also for all $j = 1, \cdots, n$,

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$$P(t,T_{j-1})\mathbb{P}_{T_{j-1}}(A|\mathcal{F}_t) = \mathbb{E}\left[\frac{\beta(T_0)}{\beta(T_{j-1})}\mathcal{I}_A|\mathcal{F}_t\right] = \mathbb{E}\left[\frac{\beta(T_0)}{\beta(T_j)}\frac{1}{P(T_{j-1},T_j)}\mathcal{I}_A|\mathcal{F}_t\right]$$

$$= \mathbb{E}\left[\frac{\beta(T_0)}{\beta(T_j)}\{1+\delta K(T_{j-1},T_{j-1})\}\mathcal{I}_A|\mathcal{F}_t\right]$$

$$= P(t,T_j)\mathbb{P}_{T_j}(A|\mathcal{F}_t) + \delta P(t,T_j)\mathbb{E}[K(T_{j-1},T_{j-1})\mathcal{I}_A|\mathcal{F}_t]$$

$$= P(t,T_j)\mathbb{P}_{T_j}(A|\mathcal{F}_t) + \delta P(t,T_j)\mathbb{E}[K(T_0,T_{j-1})\mathcal{I}_A|\mathcal{F}_t]$$

where the last equality holds because the process

$$\{K(t,T_{j-1}); 0 \le t \le T_{j-1}\}$$

is a martingale under the $\mathbb{P}_{T_j}$ measure, and the event A is $\mathcal{F}_{T_0}$-measurable.

5. Payer Swap Option Pricing Formula: The payer swap option price at time $t \le T_0$ is

$$P_{Swaption}(t) = \delta\sum_{j=1}^{n} P(t,T_j)\mathbb{E}_{T_j}[\{K(T_0,T_{j-1})-\kappa\}\mathcal{I}_A|\mathcal{F}_t]$$

Payer Swap Option Pricing Simplification

1. The Approach: To simplify the payer swap option pricing formula, we need to first analyze the relationships between the forward measures $\mathbb{P}_{T_j}$ given from

$$\mathbb{P}_{T_j} = \mathbb{P}_0 e^{-\int_0^t \vec{\sigma}(s,T_j-s)\cdot d\vec{W}(s) - \frac{1}{2}\int_0^t |\vec{\sigma}(s,T_j-s)|^2 ds} = \frac{\mathbb{P}_0}{P(0,T_j)\beta(t)}$$

as well as the corresponding forward Brownian motions $\overrightarrow{W_T}(t)$ given by

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$$\overrightarrow{W_T}(t) = \vec{W}(t) + \int_0^t \vec{\sigma}(s, T - s) \cdot d\vec{W}(s)$$

for $j = 1, 2, \cdots$.

2. Consequent Period Measure Change: We have

$$\Delta\overrightarrow{W_{T_j}}(t) = \Delta\vec{W}(t) + \vec{\sigma}(s, T_j - s)\Delta t = \Delta\overrightarrow{W_{T_{j-1}}}(t) + \left[\vec{\sigma}(t, T_j - t) - \vec{\sigma}(t, T_{j-1} - t)\right]\Delta t$$

$$= \Delta\overrightarrow{W_{T_{j-1}}}(t) + \frac{\delta K(t, T_{j-1})}{1 + \delta K(t, T_{j-1})}\vec{\gamma}(t, T_{j-1} - t)\Delta t$$

3. Change in the LIBOR PV $\Delta\left[\frac{\delta K(t, T_{j-1})}{1 + \delta K(t, T_{j-1})}\right]$: Also because the process

$$\{K(t, T_{j-1}); 0 \le t \le T_{j-1}\}$$

satisfies

$$\Delta K(t, T_{j-1}) = K(t, T_{j-1})\vec{\gamma}(t, T_{j-1} - t) \cdot \Delta\overrightarrow{W_{T_j}}(t)$$

we have

$$\begin{aligned}\Delta&\left[\frac{\delta K(t, T_{j-1})}{1 + \delta K(t, T_{j-1})}\right] \\ &= \frac{\delta K(t, T_{j-1})}{\left[1 + \delta K(t, T_{j-1})\right]^2}\vec{\gamma}(t, T_{j-1} - t) \cdot \Delta\overrightarrow{W_{T_j}}(t) \\ &- \frac{\delta^2 K^2(t, T_{j-1})}{\left[1 + \delta K(t, T_{j-1})\right]^3}\left|\vec{\gamma}(t, T_{j-1} - t)\right|^2\Delta t \\ &= \frac{\delta K(t, T_{j-1})}{\left[1 + \delta K(t, T_{j-1})\right]^2}\vec{\gamma}(t, T_{j-1} - t) \cdot \Delta\overrightarrow{W_{T_{j-1}}}(t)\end{aligned}$$

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and hence

$$\left\{\frac{\delta K\left(t, T_{j-1}\right)}{1+\delta K\left(t, T_{j-1}\right)} ; 0 \leq t \leq T_{j-1}\right\}$$

is a super-martingale under the measure $\mathbb{P}_{T_j}$ and a martingale under the measure $\mathbb{P}_{T_{j-1}}$.

4. Forward Bond Pricing: Let for $t \leq T_0$

$$F_{T_0}(t, T_k) = \frac{P(t, T_k)}{P(t, T_0)}$$

denote the forward price at time t for settlement at time T_0 on a T_k maturity zero coupon bond.

5. Option in-the-money Probability in Terms of LIBOR: Because we have

$$F_{T_0}(t, T_k) = \prod_{i=1}^{k} F_{T_{i-1}}(t, T_i) = \frac{1}{\prod_{i=1}^{k}[1+\delta K(t, T_{i-1})]}$$

$$A = \left\{\sum_{k=1}^{n} C_k P(T_0, T_k) \leq 1\right\}$$

becomes

$$A = \left\{\sum_{k=1}^{n} \frac{C_k}{\prod_{i=1}^{k}[1+\delta K(t, T_{i-1})]} \leq 1\right\}$$
$$= \left\{\sum_{k=1}^{n} \frac{C_k}{\prod_{i=1}^{k}[1+\delta K(t, T_{i-1})] e^{\int_t^T \vec{\sigma}(s, T_{i-1}-s)\cdot d\vec{W}_{T_i}(s) + \frac{1}{2}\int_0^t |\vec{\sigma}(s, T_{i-1}-s)|^2 ds}} \leq 1\right\}$$

6. Coupon Period Measure Change: Using the period T_0 left pivot for both $i, j = 1, \cdots, n$, for $t \leq T_0$, we get

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Tel. 646-513-3300

$$\Delta\overrightarrow{W_{T_i}}(t) = \Delta\overrightarrow{W_{T_j}}(t) + \sum_{l=0}^{i-1}\frac{\delta K(t,T_l)}{1+\delta K(t,T_l)}\vec{\gamma}(t,T_l - t)\,\Delta t - \sum_{l=0}^{j-1}\frac{\delta K(t,T_l)}{1+\delta K(t,T_l)}\vec{\gamma}(t,T_l - t)\,\Delta t$$

7. <u>Simplification of the $\Delta\overrightarrow{W_{T_i}}(t)$ Component of A under $\Delta\overrightarrow{W_{T_j}}(t)$</u>: Using the above expression for $\Delta\overrightarrow{W_{T_i}}(t)$, we can write

$$\begin{aligned} X_i &= \int_t^T \vec{\gamma}(s, T_{i-1} - s) \cdot d\overrightarrow{W}_{T_i}(s) \\ &= \int_t^T \vec{\gamma}(s, T_{i-1} - s) \cdot d\overrightarrow{W}_{T_j}(s) \\ &+ \sum_{l=1}^{i}\int_t^T \frac{\delta K(t,T_{l-1})}{1+\delta K(t,T_{l-1})}\vec{\gamma}(s,T_{i-1}-s)\cdot\vec{\gamma}(t,T_{l-1}-t)ds \\ &- \sum_{l=1}^{j}\int_t^T \frac{\delta K(t,T_{l-1})}{1+\delta K(t,T_{l-1})}\vec{\gamma}(s,T_{i-1}-s)\cdot\vec{\gamma}(t,T_{l-1}-t)ds \end{aligned}$$

8. <u>Approximation of X_i by ${X_i}^j$</u>: We will approximate the conditional on $\mathcal{F}_t$ distribution of $X_1, \cdots, X_n$ under the measure $\mathbb{P}_{T_j}$ (for each $j = 1, \cdots, n$) by the distribution of the random vector ${X_1}^j, \cdots, {X_n}^j$ where

$${X_i}^j = \int_t^T \vec{\gamma}(s, T_{i-1} - s) \cdot d\overrightarrow{W}_{T_j}(s) + \sum_{l=1}^{i}\frac{\delta K(t,T_{l-1})}{1+\delta K(t,T_{l-1})}\Delta_{li} - \sum_{l=1}^{j}\frac{\delta K(t,T_{l-1})}{1+\delta K(t,T_{l-1})}\Delta_{li}$$

and

$$\Delta_{li} = \int_t^T \vec{\gamma}(s, T_{i-1} - s) \cdot \vec{\gamma}(t, T_{l-1} - t)ds$$

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9. Approximation of the Change in LIBOR PV: In view of the expression for the change in LIBOR PV, i.e.,

$$\Delta\left[\frac{\delta K(s,T_l)}{1+\delta K(s,T_l)}\right]=\frac{\delta K(s,T_l)}{[1+\delta K(s,T_l)]^2}\vec{\gamma}(s,T_l-s)\cdot\Delta\overrightarrow{W}_{T_l}(s)$$

the approximations above correspond to Wiener chaos 0 approximation of the process

$$\frac{\delta K(s,T_l)}{1+\delta K(s,T_l)}, s\le T_l$$

under the measure $\mathbb{P}_{T_l}$.

10. Higher Order Approximation of X_i: A more accurate approximation involving Wiener chaos of order 0 and 1 may be used as well. However, since the order 1 Wiener chaos contribution is not significant, so $K(s,T_l)$ can be replaced by its t value $K(t,T_l)$, or because of

$$\Delta\left[\frac{\delta K(t,T_l)}{1+\delta K(t,T_l)}\right]=\frac{\delta K(t,T_l)}{[1+\delta K(t,T_l)]^2}\vec{\gamma}(t,T_l-t)\cdot\Delta\overrightarrow{W}_{T_l}(t)$$

by the conditional expectation under $\mathbb{P}_{T_l}$ given $\mathcal{F}_t$.

11. Distribution of ${X_1}^j,\cdots,{X_n}^j$ under $\mathbb{P}_{T_t}$: The conditional-on-$\mathcal{F}_t$ distribution of ${X_1}^j,\cdots,{X_n}^j$ under the measure $\mathbb{P}_{T_j}$ is $N\left(\mu^j,\Delta\right)$, where

$${\mu_i}^j=\sum_{l=1}^{i}\frac{\delta K(t,T_{l-1})}{1+\delta K(t,T_{l-1})}\Delta_{li}-\sum_{l=1}^{j}\frac{\delta K(t,T_{l-1})}{1+\delta K(t,T_{l-1})}\Delta_{li}$$

12. Estimation of μ and Δ for the Distribution: In practice, the first eigenvalue for the matrix Δ is approximately 50 times larger than the second, and therefore we can assume that Δ is of rank 1, or equivalently

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$$\Delta_{li} = \Gamma_l \Gamma_i$$

for some positive constants $\Gamma_1, \cdots, \Gamma_n$. Setting

$$\mathrm{d}_0 = 0$$

and for $i \geq 1$

$$\mathrm{d}_i = \sum_{l=1}^{i} \frac{\delta K(t, T_{l-1})}{1 + \delta K(t, T_{l-1})} \Gamma_l$$

it follows that

$$\mu_i{}^j = \Gamma_i \left(\mathrm{d}_i - \mathrm{d}_j \right)$$

13. <u>Solution to the Lower Cutoff for A</u>: For all $j = 1, \cdots, n$, the function

$$\mathrm{f}_j(x) = 1 - \sum_{k=1}^{n} \frac{C_k}{\prod_{i=1}^{k} [1 + \delta K(t, T_{i-1})] e^{\Gamma_i (x + \mathrm{d}_i - \mathrm{d}_j) - \frac{1}{2} \Gamma_i{}^2}}$$

satisfies

$$\mathrm{f}_j{}'(x) > 0$$

$$\mathrm{f}_j(-\infty) = -n\delta\kappa$$

and

$$\mathrm{f}_j(\infty) = 1$$

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Hence there is a unique point s_j such that

$$\mathrm{f}_j\left(\mathrm{s}_j\right) = 0$$

Moreover, if s_0 is the solution with $j = 0$, clearly

$$\mathrm{s}_j = \mathrm{s}_0 + \mathrm{d}_j$$

14. <u>In-the-money Probability in the T_j-Measure</u>: Also

$$\mathrm{f}_j(x) \geq 0$$

for

$$x \geq \mathrm{s}_j$$

and therefore, using

$$A = \left\{ \sum_{k=1}^{n} \frac{C_k}{\prod_{i=1}^{k} [1 + \delta K(t, T_{i-1})] e^{\int_t^{T_0} \vec{\gamma}(s, T_{i-1} - s) \cdot d\vec{W}_{T_i}(s) - \frac{1}{2} \int_t^{T_0} |\vec{\gamma}(s, T_{i-1} - s)|^2 ds}} \leq 1 \right\}$$

$$X_i{}^j = \int_t^T \vec{\gamma}(s, T_{i-1} - s) \cdot d\vec{W}_{T_j}(s) + \sum_{l=1}^{i} \frac{\delta K(t, T_{l-1})}{1 + \delta K(t, T_{l-1})} \Delta_{li} - \sum_{l=1}^{j} \frac{\delta K(t, T_{l-1})}{1 + \delta K(t, T_{l-1})} \Delta_{li}$$

and

$$\mu_i{}^j = \Gamma_i\left(\mathrm{d}_i - \mathrm{d}_j\right)$$

we deduce that

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$$\mathbb{P}_{T_j}(A|\mathcal{F}_t) = \mathbb{P}_{T_j}\left(X_j{}^j \geq \Gamma_j \mathrm{s}_j\right) = N\left(-\mathrm{s}_0 - \mathrm{d}_j\right)$$

15. <u>T_j-Measure in-the-money LIBOR Expectation</u>: From standard arguments

$$\mathbb{E}_{T_j}\left[K\left(T_0, T_{j-1}\right)\mathcal{I}_A|\mathcal{F}_t\right] = \mathbb{E}_{T_j}\left[K\left(t, T_{j-1}\right)e^{\int_t^{T_0}\vec{\gamma}(s,T_{j-1}-s)\cdot d\vec{W}_{T_i}(s)-\frac{1}{2}\int_t^{T_0}\left|\vec{\gamma}(s,T_{j-1}-s)\right|^2 ds}\mathcal{I}_A|\mathcal{F}_t\right]$$
$$= K\left(t, T_{j-1}\right)N\left(-\mathrm{s}_0 - \mathrm{d}_j + \Gamma_j\right)$$

which leads to the approximated formula for the payer swaption.

16. <u>Payer Swap Option Formula Approximation</u>: The price at time $t \geq T_0$ of the payer swaption can be approximated by

$$P_{Swaption,Approximation}(t) = \delta\sum_{j=1}^{n} P\left(t, T_j\right)\left[K\left(t, T_{j-1}\right)N\left(-\mathrm{s}_0 - \mathrm{d}_j + \Gamma_j\right) - \kappa N\left(-\mathrm{s}_0 - \mathrm{d}_j\right)\right]$$

where s_0 is given by

$$\sum_{k=1}^{n}\frac{C_k}{\prod_{i=1}^{k}[1 + \delta K(t, T_{i-1})]e^{\Gamma_i(x+\mathrm{d}_i-\mathrm{d}_j)-\frac{1}{2}\Gamma_i{}^2}} = 1$$

$$C_k = \kappa\delta \ \forall\ k = 1, \cdots, n-1$$

$$C_n = 1 + \kappa\delta$$

while Γ_i is given from

$$\int_t^T \vec{\gamma}(s, T_{i-1} - s)\cdot\vec{\gamma}(t, T_{l-1} - t)ds = \Gamma_i\Gamma_l$$

and

$$\mathrm{d}_i = \sum_{l=1}^{i} \frac{\delta K(t, T_{l-1})}{1 + \delta K(t, T_{l-1})} \Gamma_l$$

Mismatched Periods Cap/Swaption Pricing

1. Introduction: In the US, the UK, and the Japanese markets, caps correspond to rates compounded quarterly, while swaptions are semi-annual. In the German market caps are quarterly and swaptions annual. We deal with these mismatched periods by assuming lognormal volatility structure on the quarterly rates.
2. Lognormal Volatility Swaption Formulation: The forward swap rate at time $t \le T_0$ is

$$\omega_{T_0}{}^{(k)}(t, n) = \frac{P(t, T_0) - P(t, T_{kn})}{k\delta \sum_{j=1}^{n} P\left(t, T_{kj}\right)}$$

and hence the time $t \le T_0$ price of a payer swaption at strike κ maturing at T_0 is

$$P_{Swaption,k}(t) = \mathbb{E}\left[\sum_{j=1}^{n} \frac{\beta(t)}{\beta\left(T_{jk}\right)} \left\{\omega_{T_0}{}^{(k)}(T_0, n) - \kappa\right\}^{+} k\delta | \mathcal{F}_t\right]$$

$$= \mathbb{E}\left[\frac{\beta(t)}{\beta(T_0)} \left\{1 - \sum_{j=1}^{n} C_j{}^{(k)} P\left(T_0, T_{jk}\right)\right\}^{+} | \mathcal{F}_t\right]$$

$$= P(t, T_0) \mathbb{P}_{T_0}(A | \mathcal{F}_t) - \sum_{j=1}^{n} C_j{}^{(k)} P\left(T_0, T_{jk}\right) \mathbb{P}_{T_{jk}}(A | \mathcal{F}_t)$$

where

$$C_j{}^{(k)} = \kappa k \delta \; \forall \; j = 1, \cdots, n-1$$

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and

$$C_n{}^{(k)} = 1 + \kappa k \delta$$

while

$$A = \left\{\omega_{T_0}{}^{(k)}(T_0, n) \geq \kappa\right\} = \left\{\sum_{j=1}^{n} C_j{}^{(k)} P\left(T_0, T_{jk}\right) \leq 1\right\}$$

3. Mismatched Periods in-the-money Probability in the Forward Measure: Extending the formulation

$$P\left(t, T_{j-1}\right)\mathbb{P}_{T_{j-1}}(A|\mathcal{F}_t) = P\left(t, T_j\right)\mathbb{P}_{T_j}(A|\mathcal{F}_t) + \delta P\left(t, T_j\right)\mathbb{E}_{T_j}\left[K\left(T_0, T_{j-1}\right)\mathcal{J}_A|\mathcal{F}_t\right]$$

it follows that for all j

$$\begin{aligned} P\left(t, T_{k(j-1)}\right)&\mathbb{P}_{T_{k(j-1)}}(A|\mathcal{F}_t) \\ &= P\left(t, T_{kj}\right)\mathbb{P}_{T_{kj}}(A|\mathcal{F}_t) \\ &+ \delta \sum_{i=1}^{k} P\left(t, T_{k(j-1)+i}\right)\mathbb{E}_{T_{k(j-1)+i}}\left[K\left(T_0, T_{k(j-1)+i-1}\right)\mathcal{J}_A|\mathcal{F}_t\right] \end{aligned}$$

and so

$$P_{Swaption,k}(t) = \delta \sum_{j=1}^{n} \left\{\sum_{i=k(j-1)+1}^{kj} P(t, T_i)\mathbb{E}_{T_i}[K(T_0, T_{i-1})\mathcal{J}_A|\mathcal{F}_t] - k\kappa P\left(t, T_{jk}\right)\mathbb{P}_{T_{jk}}(A|\mathcal{F}_t)\right\}$$

4. Mismatched Payer Swap Option Approximation: The full set of arguments used for the matched period option pricing above may be employed here.

5. <u>Mismatched Period Swaption Approximation Formula</u>: Let k and δ be such that $\frac{1}{k\delta}$ is the compounding frequency per year of the swap rate ${\omega_{T_0}}^{(k)}(t,n)$ given by

$$ {\omega_{T_0}}^{(k)}(t,n) = \frac{P(t,T_0) - P(t,T_{kn})}{k\delta \sum_{j=1}^{n} P\left(t,T_{kj}\right)} $$

The time $t \le T_0$ price of a payer swaption can be approximated by the formula

$$ \begin{aligned} &P_{Swaption\ Approximation,k}(t) \\ &\quad = \delta \sum_{j=1}^{n} \left\{ \sum_{i=k(j-1)+1}^{kj} P(t,T_i)K(t,T_{i-1})N\left(-{s_0}^{(k)} - d_i - \Gamma_i\right) \right. \\ &\quad \left. - k\kappa P\left(t,T_{jk}\right)N\left(-{s_0}^{(k)} - d_{kj}\right) \right\} \end{aligned} $$

where ${s_0}^{(k)}$ is given by

$$ \sum_{j=1}^{n} \frac{{c_j}^{(k)}}{\prod_{i=1}^{kj}[1+\delta K(t,T_{i-1})]e^{\Gamma_i\left({s_0}^{(k)}+d_i\right)-\frac{1}{2}{\Gamma_i}^2}} = 1 $$

where

$$ {C_j}^{(k)} = \kappa k\delta \ \forall\, j = 1,\cdots,n-1 $$

and

$$ {C_n}^{(k)} = 1 + \kappa k\delta $$

and Γ_i is defined from

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$$\int_t^T \vec{\gamma}(s, T_{i-1} - s) \cdot \vec{\gamma}(t, T_{l-1} - t) ds = \Gamma_i \Gamma_l$$

and

$$\mathrm{d}_i = \sum_{l=1}^{i} \frac{\delta K(t, T_{l-1})}{1 + \delta K(t, T_{l-1})} \Gamma_l$$

6. Mismatched Period Joint Volatility Calibration: If one chooses $\delta = 0.25$, for example, in a market with quarterly and semi-annual caps and swaptions, then the mismatched period payer swaption formula above can be used to price the semi-annual caps and the swaptions, and hence it can also be used to jointly calibrate both quarterly and semi-annual volatility inputs.

Approximate vs. Full Simulation Comparisons

1. Cross-Verification of Simulation against Approximation: To analyze the differences between the exact swaption value computed by simulation, and the approximation for the mismatched payer swaption formula with $k = 1$ and $t = 0$, Brace, Gatarek, and Musiela (1997) fit a one-factor model to the US cap and swaption date for 12 July 1994, generating a typical volatility structure.
2. Generation of the Simulation Prices: Simulation prices were generated under the $\mathbb{P}_{T_n}$ measure using the exact formula

$$P(0, T_n) \mathbb{E}_{T_n} \left[\left\{ \sum_{j=0}^{n-1} C_j \prod_{i=j+1}^{n} \{1 + \delta K(T_0, T_{i-1})\} + C_n \right\}^+ \right]$$

with

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$$C_0 = 1$$

$$C_j = -\kappa\delta \ \forall\ j = 1, \cdots, n-1$$

$$C_n = -(1 + \kappa\delta)$$

and

$$K(t, T_{i-1}) = K(0, T_{i-1}) e^{\int_0^t \vec{\gamma}(s, T_{i-1} - s) \cdot d\vec{W}_{T_i}(s) - \frac{1}{2}\int_0^t |\vec{\gamma}(s, T_{i-1} - s)|^2 ds}$$

$$\overrightarrow{W_{T_{i-1}}}(t) = \overrightarrow{W_{T_i}}(t) - \int_0^t \frac{\delta K(s, T_{i-1})}{1 + \delta K(s, T_{i-1})} \vec{\gamma}(s, T_{i-1} - s) ds$$

3. Simulation of the T_n-Brownians: The simulation equations above permit the recursive calculation of the Brownian motions

$$\overrightarrow{W_{T_0}}(t), \cdots, \overrightarrow{W_{T_{n-1}}}(t)$$

for

$$0 \le t \le T_0$$

For each simulation of $\overrightarrow{W_{T_i}}(t)$ on $[0, T_0]$ that gives values of

$$K(T_0, T_{i-1})\ \forall\ i = 1, \cdots, n$$

substitution into the exact formula above provides the value of the swaption.

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4. Validation of the Simulation Sequence: The simulation procedure, which involves Reimann and stochastic integration steps, was checked by back-calculating the cap prices used in the parametrization. The simulation prices coincided with the closed form prices calculated using

$$Cap(0) = \sum_{j=1}^{n} \delta P(0, T_j) \{K(t, T_{j-1}) N[h(t, T_{j-1})] - \kappa N[h(t, T_{j-1}) - \varsigma(t, T_{j-1})]\}$$

5. Comparison with Lognormal Black Closed Form: Comparison can also be done with the approximate/simulated swaption formula, along with the market formula based on assuming that the underlying swap-rate is lognormal, and given as

$$P_{Swaption,Market}(t) = \delta \sum_{j=1}^{n} P(t, T_j) \{\omega_{T_0}(t, n) N(h) - \kappa N(h - \gamma T_0)\}$$

where

$$h = \frac{\log \frac{\omega_{T_0}(t, n)}{\kappa} + \frac{1}{2}\gamma^2 T_0}{\gamma \sqrt{T_0}}$$

6. Reduction to the Black Lognormal Closed Form: Note that because

$$\mathbb{E}\left[\sum_{j=1}^{n} \frac{1}{\beta(T_j)} \delta \{\omega_{T_0}(T_0, n) - \kappa\}^{+}\right] = \delta \sum_{j=1}^{n} P(0, T_j) \mathbb{E}_{T_j}\left[\{\omega_{T_0}(T_0, n) - \kappa\}^{+}\right]$$

the market seems to identify the forward measures

$$\mathbb{P}_{T_j} \ \forall\, j = 1, \cdots, n$$

with the forward measure $\mathbb{P}_{T_0}$ and assumes log-normality of the swap rate processes

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$$\omega_{T_0}(t,n), 0 \le t \le T_0$$

under the measure $\mathbb{P}_{T_0}$. In fact

$$P_{Swaption\ Approximation,k}(0) = \delta \sum_{j=1}^{n} \left\{ \sum_{i=k(j-1)+1}^{kj} P(0,T_i)K(0,T_{i-1})N\left(-s_0{}^{(k)} - d_i - \Gamma_i\right) - k\kappa P\left(0,T_{jk}\right)N\left(-s_0{}^{(k)} - d_{kj}\right) \right\}$$

where $s_0{}^{(k)}$ is given by

$$\sum_{j=1}^{n} \frac{C_j{}^{(k)}}{\prod_{i=1}^{kj}[1+\delta K(0,T_{i-1})]e^{\Gamma_i\left(s_0{}^{(k)}+d_i\right)-\frac{1}{2}\Gamma_i{}^2}} = 1$$

reduces to

$$P_{Swaption,Market}(0) = \delta \sum_{j=1}^{n} P\left(0,T_j\right)\left\{\omega_{T_0}(0,n)N(h) - \kappa N(h-\gamma T_0)\right\}$$

where

$$h = \frac{\log \frac{\omega_{T_0}(0,n)}{\kappa} + \frac{1}{2}\gamma^2 T_0}{\gamma\sqrt{T_0}}$$

if

$$d_i = 0$$

$$\Gamma_i = \sqrt{\Delta_{ii}} = \gamma\sqrt{T_0}$$

and

$$K(0, T_i) = \kappa$$

Typical Model Calibration Results

1. Two-Factor Calibration: Brace, Gatarek, and Musiela (1997) calibrate the model above to the data from the UK market using a two-factor model with piecewise constant volatility structure

$$\gamma(t, x) = f(t)\gamma(x)$$

where

$$\gamma(x) = [\gamma_1(x), \gamma_2(x)]$$

and

$$f: \mathbb{R}_+ \to \mathbb{R}$$

Thus if

$$f \equiv 1$$

the volatility is time-homogenous, so f represents the term structure of volatility (Brace and Musiela (1994a)).

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2. Normal HJM Fit: The normal HJM model can almost always be fit to the UK and the US caps and the swaptions data with a one-factor homogenous volatility; fitting the correlation with the second factor improves the overall fit.
3. Log-normal HJM Fit: The log-normal HJM model frequently cannot fit a term structure of volatility in the log-normal case, and may this indicate that the price volatility of the normal HJM is more stable than the yield volatility of the log-normal HJM.
4. Comparison of the Implied Black Volatilities: The implied Black volatilities of the caps and the swaptions for both models are quiet similar, with log-normal volatilities being 1.0% to 1.5% greater at longer swaption maturities – possibly reflecting the different impact of correlation on the two models.

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The SABR Model

Introduction

1. Definition: SABR model, or “Stochastic Alpha, Beta, Rho” model, is a stochastic volatility model for forward LIBOR rates. Consider the forward rate

$$f_t = F(t; S, T)$$

Under the T-forward measure Q^T with numeraire $P(t, T)$ this forward rate is a martingale. In addition, we also assume its volatility is also a martingale under Q^T.

2. The SDE: The SDE is specified as (Wang (2010)):

$$\Delta f_t = \sigma_t (f_t)^\beta \Delta z_t$$

$$\Delta \sigma_t = v \sigma_t \Delta w_t$$

$$\mathbb{E}^{\mathbb{Q}^T}[\Delta z_t \Delta w_t] = \rho \Delta t$$

$$f_{t=0} = f_0$$

and

$$\sigma_{t=0} = \sigma_0$$

where the current forward price $f_{t=0} = f_0$ is observed in the market.

3. Model Parameters: The model has four parameters;

$$\sigma_0 > 0$$

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$$0 \leq \beta \leq 1$$

$$-1 < \rho < 1$$

$$v \geq 0$$

In terms of the model name, stochastic alpha stands for σ_t, beta and rho for their respective parameters. If

$$\beta = 0$$

the forward rate is normal; if

$$\beta = 1$$

the forward rate is log-normal. If the *Volatility of Volatility Parameter*

$$v = 0$$

the model is reduced to the constant elasticity of variance (CEV) model.

4. SABR Model Closed Form: SABR models the implied volatility curve directly, which is then used to obtain the European option prices using the Black-76 model. The Black implied volatility is modeled as

$$\sigma_{Model}(K, f_0) = A.\left(\frac{z}{\chi(z)}\right).B$$

where

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$$A = \frac{\sigma_0}{(f_0 K)^{\frac{1-\beta}{2}} \left[1 + \left\{\frac{(1-\beta)^2}{24} log^2 \frac{f_0}{K} + \frac{(1-\beta)^4}{1920} log^4 \frac{f_0}{K} + \cdots \right\}\right]}$$

$$B = \left[1 + \left\{\frac{(1-\beta)^2}{24} - \frac{{\sigma_0}^2}{(f_0 K)^{1-\beta}} + \frac{\rho\beta v \sigma_0}{4(f_0 K)^{\frac{1-\beta}{2}}} + \frac{2-3\rho^2}{24} v^2 \right\} + \cdots \right]$$

$$z = \frac{v}{\sigma_0}(f_0 K)^{\frac{1-\beta}{2}} \log \frac{f_0}{K}$$

$$\chi(z) = \log \frac{\sqrt{z^2 - 2\rho z + 1} + z - \rho}{1 - \rho}$$

5. SABR Approximation: Except for the special cases of $\beta = 0$ and $\beta = 1$, the no closed form approximation is known. But the approximation is very accurate as long as the option is not too out-of-the-money, or T is not too large.

Parameter Estimation

1. Implied Volatility Curve Shape: The four parameters influence the shape of the implied volatility curve differently.

Parameter	Curve Property	Definition
σ_0	Level	The curve shifts upwards as it increases
β	Slope	The curve steepens as it decreases
ρ	Slope	The curve steepens as it decreases
v	Curvature	The curvature increases as it increases

2. Calibration Steps: To estimate the parameters, or in other words, to calibrate the model, it usually takes 3 steps; a) Estimate β; b) Imply σ_0 from ρ and v; c) Calibrate ρ and v.

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According to the parameter table, β and ρ both control the shape of the volatility curve. Thus to some degree, the model is over-determined. A common industry practice is to skip step 1 by choosing directly

$$\beta = 0.5$$

Another way of calibration combines steps 2 and 3 together, and calibrates these 3 parameters directly.

3. Step 1 - Estimate β: For the ATM options, the expression for $\sigma_{ATM,Market}$ can be re-written as

$$\sigma_{ATM,Market} \approx \sigma_{Model}(f_0, f_0) = \frac{\sigma_0}{f_0^{1-\beta}} \left\{ 1 + \left[\frac{(1-\beta)^2}{24} \frac{\sigma_0{}^2}{f_0{}^{2-2\beta}} + \frac{\rho\beta v\sigma_0}{4f_0{}^{1-\beta}} + \frac{2-3\rho^2}{24} v^2 \right] T \right\}$$

Taking logs (Hagan, Kumar, and Lesniewski (2002)), we get

$$\log \sigma_{ATM,Market} = \log \sigma_0 - (1-\beta) \log f_0$$

Therefore β can be estimated from a linear regression between the log ATM volatilities and the log forward rate time series.

4. Step 2 - Imply σ_0 from ρ and v: Given the current ATM market volatility $\sigma_{ATM,Market}$, we can invert the above equation to obtain the following cubic equation in σ_0:

$$\left[\frac{(1-\beta)^2 T}{24 f_0{}^{2-2\beta}} \right] \sigma_0{}^3 + \left[\frac{\rho\beta v T}{4 f_0{}^{1-\beta}} \right] \sigma_0{}^2 + \left[1 + \frac{2-3\rho^2}{24} v^2 T \right] \sigma_0 - \sigma_{ATM,Market} f_0{}^{1-\beta} = 0$$

5. Smallest Positive Solution σ_0: So σ_0, as the smallest positive real root to this equation, is then explicitly calibrated to the ATM volatility $\sigma_{ATM,Market}$. It is expressed as a function of the parameters ρ and v, which will be calibrated in the next step. The Tartaglia approach to the cubic equation solution can be found in Flannery, Press, Teukolsky, and Vetterling (1992).

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6. <u>Step 3 - Calibrate ρ and v</u>: After step 2, there remain only 2 parameters to be calibrated - ρ and v. The calibration process is a fairly standard one. We choose the parameters that bring the model volatilities down to the market quote implied volatilities. That is

$$(\rho, v) = \underset{\rho, v}{\arg min} \sum_i \left[\hat{\sigma}_i(\rho, v) - \sigma_{i,Market}\right]^2$$

7. <u>Step 3 Full Surface Calibration</u>: Following steps 1, 2, and 3, SABR model in its primitive form can be relatively straightforward to calibrate. In general, if one tries to calibrate a model to a volatility surface $\sigma(K, T)$ (or a volatility cube $\sigma\left(K, T_\alpha, T_\beta\right)$ in the case of swap options), the process is usually more complicated. So SABR fixes the forward rate (and time T) and calibrates itself to a smile/skew curve with respect to the strike K.
8. <u>Forward Rate Dynamics under the T-Forward Measure</u>: The forward rate

$$f_t = F(t; S, T)$$

is treated in its own K-forward measure and does not interact with other forward rates. Compare this with the cap volatility calibration where the ATM volatility curve with respect to time T is considered. In both cases, the curve is one dimensional. In sum, as long as we don't consider the forward rate and its volatility dynamics under other T-forward measures, the calibration process demands much less effort.

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Section XI: Algorithmic Differentiation

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Algorithmic Differentiation

Glossary

1. Wengert List: List of all the non over-writable program variables (Wengert (1964)) – can also be seen as a linearization of the computational graph. By construction, it is an intermediate variable.
2. Intermediate Wengert Canonical Variable: These are intermediate financial variables those are fixed from the point-of-view of the output Jacobians and the input parameters that serve as computation graph parsimonious optimizers (Figures 8 and 9).
3. Wengert fan-in and fan-out: Reduction of a set of initial/intermediate Wengert variates onto the subsequent set is called fan-in; the opposite is fan-out.
4. Wengert funneling: Same as Wengert fan-in.
5. Micro-Jacobian: Change in the calibrated instrument measure coefficients to unit change in the quoted instrument measures.
6. Self-Jacobian: Self-Jacobian refers to the Jacobian of the Objective Function at any point in the variate to the Objective Function at the segment nodes, i.e., $\frac{\partial Y(t)}{\partial Y(t_K)}$. Self-Jacobian is a type of micro-Jacobian.
7. Derivative Entity: The entity whose dynamics are determined by the evolution of a stochastic variate, and whose specific facets/measures are observable.
8. Path-wise Derivative Estimator: $\frac{\partial V}{\partial X_i(0)}$, where V is the value of the derivative, and $X_i(0)$ is the starting value for a specific stochastic variate.
9. Non-Parsimonized Parameters: Parameters that map one-to-one with the input instrument set, e.g., typical curve bootstrapping.
10. Parsimonization: Reduction of the parameter space from the input measure space.

Overview

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1. AD History: Iri (1991)
2. Mathematical Foundations: Griewank (2000)
3. Survey: Berz (1996)
4. Implementation Tools, Methodologies, Processes, and Techniques (Bischof, Hovland, and Norris (2005))
5. AD Resource: [*http://www.autodiff.org/*](http://www.autodiff.org/)

Algorithmic Differentiation in Finance

1. Focus has been primarily on Monte-Carlo methodologies.
2. Although path-wise optimized sensitivity generation had been employed earlier (Glasserman (2004)), Giles and Glasserman (2006) first discussed adjoint methods in path-wise sensitivity generation.
3. Full extension to LMM based stochastic variate evolution and a corresponding exotic (in this case Bermudan) swap option evaluation (Leclerc, Liang, and Schneider (2009)), as well as to correlated defaults and their sensitivities (Capriotti and Giles (2011)).
4. Capriotti (2011) covers automated Greek generation, but with a focus on automatic differentiation, and in the context of Monte-Carlo methods.
5. Finally, algorithmic differentiation has also been applied to addressing the issue of calibration along with sensitivity generation (Schlenkirch (2011)).

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Algorithmic Differentiation - Basics

Motivation and Advantages

1. Definition: Automatic differentiation is a set of techniques for transforming a program that calculates the numerical values of a function into a program that calculates numerical values for derivatives of that function with about the same accuracy and efficiency as the function values themselves (Bartholomew-Biggs, Brown, Christianson, and Dixon (2000)).
2. Symbolic Derivatives: Calculate the local symbolic derivatives rather than the a) divided differences, or b) numerical differentials (*Automatic Differentiation - Wikipedia Entry*).
3. Calculation Speed: Same number of Objective Function Calculation as the original; however, potential "chain rule" multiplication factor effects.
4. Accuracy vs. Performance: Due to the usage of symbolics, accuracy of Automatic Differentiation always better than numerical differentials; however, due to the chain-rule issue, may not be always faster.
5. Scalability at infinitesimal variates: Since Automatic Differentiation is always symbolic and therefore infinitesimal, it will automatically scale to arbitrarily small variate infinitesimals – reduced errors due to bit cancellation etc.
6. Higher-order derivatives: Automatic Differentiation does not need additional objective function evaluations for higher order derivative calculations (beyond the chain-rule issues); therefore, those are infinitesimally correct too.

Program Sequence Construction Modes

1. Forward Automatic Differentiation: Express the final and the intermediate variables as a consequence of a computed forward graph, and derive the symbolic forward derivative graph.

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- Effectively computes the gradient of the intermediate variables to the variates or the "independent variables" and transmits them up the graph.

2. Reverse Automatic Differentiation: Express the intermediate variables and the input variates as nodes in the computed reverse graph, and derive the symbolic reverse derivative graph.
    - Often may still need the forward path to store the calculated intermediates needed on the way back.
    - Effectively computes the gradient of the intermediate variables to the "dependent variables" and transmits them down the graph.
3. Speed:
    - Forward Mode => Speed proportional to n, the number of "independent" variables
    - Reverse Mode => Speed proportional to m, the number of "dependent" variables
4. Memory Usage (Ghaffari, Li, Li, and Nie (2007)):
    - Forward Mode => a) Each Wengert variable, b) Forward Jacobian for each Wengert, c) Forward Dependency Graph
    - Reverse Mode => a) Each Wengert Adjoint, b) Reverse Jacobian for each Wengert, c) Forward/Backward Dependency graph
5. When the difference is minimal: When the dependence of the final Jacobian sensitivity step is the dominating factor, and the adjointing step is not the rate-determining part, then the performance will always be $\Theta(n)$, where n is the number of sensitivities – for e.g., if

$$y = \sum_{i=1}^{n} x_i$$

given that $\frac{\partial y}{\partial x_i}$ is trivial to calculate, the performance will always be $\Theta(n)$.

- For instance, given a univariate objective function (as in constrained/unconstrained optimization (e.g., maximization/minimization) problems), either forward or reverse Automatic Differentiation is an equally good choice for sensitivity generation, owing to its performance.

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Canonicalization - Program Statements Simplification by Decomposition

1. Program Line-level decomposition: Canonicalization decomposes the program/statement units into specific analysis bits.
    - Canonicalization is commonly used in many areas of computer science, e.g., in compiler design/code generation, SKU formulation/synthesis/customization etc.
2. Canonicalization Implementation: In general, canonicalization and other related Automatic Differentiation Source Code generation/transformation techniques should go hand in hand with optimizing compiled code emission techniques, program active variable activity analysis.
    - Canonicalization sequence should include steps (Bischof, Hovland, and Norris (2005)) where you would be able to mark the mathematical "Automatically Differentiable" code segments to separate from the others during, for instance, pre-processing etc.
    - For true program transformation effectiveness, Hot-Spot type dynamic run-time analysis is needed in addition to static compile time data flow analysis etc.
    - In VM oriented languages like Java, the run-time GC already works, so would it might make a candidate for embedding AD execution/selective sensitivity generation in.
3. Equivalence with Wengert Structuring: Given that canonicalization consists of hoisting all the l-value updates separately without side effects, it is effectively the same as Wengert un-rolling and DAG linearization.
4. Limitations with the implementation: For many of the reasons above, automated implementations of canonicalization (like other automated code generation/re-structuring) might result in "invisible inefficiencies", and the had-drafted techniques those are based upon essentially the same principles may be more optimal.
5. Post canonicalized Enhancement Cost: Given that the worst case operation is division, going from

$$c = \frac{a}{b}$$

to

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$$dc = \frac{da}{b} - \frac{a}{b^2} db$$

results in going from 1 function unit execution cost to 4 automatic differentiation execution unit costs. Typically due to “weird” functions, the worst-case addition to a single post-canonicalized statement is a factor of 5, not 4.

6. Divided Differences based Differentiation Fall back:

$$\frac{\partial^n y}{\partial x^n} \cong \sum_{i=1}^{n} \frac{(-1)^i C_i{}^n y\big(x + \delta(n - 2i)\big)}{(2\delta)^n}$$

Challenges of Automating the Differentiation

1. Deep-dig perspective: Re-purposed Automatic Differentiation perspective forces the visualization of the computation at the granularity of the symbolic functional forms of the objective function.
    a. Objective Function evaluator over-loading => This requires propagation of the inner most symbolic graph nodes through the graph chain => causes additional cognitive SKU export!!
    b. Objective Function Neighborhood Behavior => With every Wengert variable, calculation of the set of forward sensitivities and the reverse Jacobians builds a local picture of the Objective Function without having to evaluate it.
2. Block-level View Fixation: Source code transformation techniques are very invasive, and require highly locally frozen view fixation, and are therefore less cognitive. Operator overloading techniques enable retention of the domain focus, and are therefore more cognitive.
    a. Naïve operator overloading would simply generate a block-level (or function call level) adjoint. This can explode the required storage, in addition to generating sub-optimal reverse-mode code. Needless to mention, source code transformation

techniques can be built to overcome this – in practice, however, many may not quite do it.

3. Complied language Automatic Differentiation implementation: Without the usage of obfuscating "versatile" templates, auto-generation of very generic forward/reverse accumulation code is impossible. Therefore source level function overloading and automated program instrumentation techniques are very hard.
    a. Further, compiled language source code transformation appears to be a vestige of "smart compiler" efforts of the '90s – classic instance of where a simple idea is "intellectually transmitted" than "built out-of-the-box".
4. Symbolic Differentiation Challenges with certain Unit Functional Forms: When you consider functions such as

$$y = \frac{1}{f(x)}$$

and you seek $\frac{dy}{dx}$ symbolically, the higher order symbolic differentiations become much more challenging:

$$\frac{dy}{dx} = -\frac{1}{f^2(x)}\frac{df(x)}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{2}{f^3(x)}\frac{df(x)}{dx} - \frac{1}{f^2(x)}\frac{d^2f(x)}{dx^2}$$

and so on for higher orders. Thus symbolically handling these series this way gets out of control fast!

Wengert Representation and Optimal Program Structure Synthesis

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1. Combination of Forward/Reverse Modes: Forward (n inputs) and reverse (m outputs) mode represent just two possible (extreme) ways of recursing through the chain rule. For $n > 1$ and $m > 1$ there is a golden mean, but finding the optimal way is probably an NP-hard problem (Berland (2006)) – optimal Jacobian accumulation is NP-complete (Naumann (2008)).
2. Wengert Intermediate Fan-in and possibly fan-out: See Figures 8 to 10 for illustrate this.
    - Wengert Intermediate Performance Enhancement => If there exists an intermediate quantity that is fixed from the point-of-view of the output Jacobians and the input parameters, the performance may be improved (see Figure 1).
    - Reusable Intermediate Performance Improvement => If the input/output computation leads to sufficient commonality among the Wengert intermediate calculation, that may also reduce computation by promoting reuse, thereby improving efficiency.
    - Wengert Funneling Criterion => For non-optimizing, non-parsimonized Wengert funnels,

$$\frac{\partial P_i}{\partial MI_j} \rightarrow \delta_{ij}$$

    for the Wengert fan to be a funneling fan-in – otherwise rippling out causes huge non-diagonal state evolution matrices. This is true for

$$I \rightarrow P$$

$$P \rightarrow W$$

    and

$$W \rightarrow O$$

3. Standardized Computational Finance Structures: In computational finance (esp. computational fixed income finance), the payout/product/pricer object serves the function of the intermediate Wengert variate indicated above. From below this variate you have the

inputs/parameters rippling up, and from above you have the Jacobians/output measure adjoints feeding down (Figure 9).

- Reactive Tree Upticks => Every intermediate element in Figure 9 is a reactive tree dependent node from the entity below, so forwarding/adjointing should happen with every real-time uptick.
- Automatic Differentiation for the Wengert Canonicals => This involves the following:
    a. Identifying the abstractable financial canonical/reusable common object structures (market parameters, product parameters, pricer parameters, etc.)
    b. Working out their forward differentials and the reverse adjoints.
- One Financial Automatic Differentiation view => The Intermediate Wengert Canonical View is the conceptual parsimonisation of the variate parameters space and the Jacobian measure space.

Optimization using Pre-accumulation and Check Pointing

1. Pre-accumulation: Aggregation (and possibly caching) of the sensitivity Jacobian over all the intermediate Wengert's inside a routine/block/module – thereby only exposing $\frac{\partial Output_i}{\partial Input_j}$ for the group unit (not each Wengert inside).
    a. Pre-accumulation also provides a suitable boundary for parallelization.
    b. It may also be looked at as the appropriate edge at which the source code transformation technique and operator overloading technique may "merge".
2. Cross-country Accumulation: Same as pre-accumulation, but pre-accumulation occurs in a specified (forward/reverse), Cross-country accumulation need not – in fact it may be guided by program analysis using Optimal Wengert intermediate composition techniques.
    a. This is also referred to as check pointing.
    b. This typically also requires snapshotting the program global and other execution context parameters at the checkpoint boundaries.
    c. Works best when the program state is easily and minimally savable, and quickly recoverable.

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d. Will also work well in conjunction with traditional kernel level check pointing schemes for fail-over etc.

Algorithmic Differentiation Financial Application Space Customization

1. Math Modules:
   - Forward differentials and auto-adjointing of math modules => May be needed for most of them.
   - Every block, compute the base "value", forward differential, and reverse adjoint.
   - In fact, for every active double-precision variable v, source code transformation automatic differentiation techniques recursively automatically generate the doublet $(v, \dot{v})$. Further, this calculation may also be parallelized.
     - This particular calculation may also be propagated at the function call level, so that the assignment outputs are automatically generated for the doublet/multiple.
     - Computational structures => Design restrictions may also be imposed by the computability of the AD of a math module, i.e., would the financial **MarketParamsContainer** be broken down into further parameter units?
2. Stochastic Variate Automatic Differentiation: Evolution of stochastic variates and their derivative entities may be further optimized by exploiting sparse-ness of the multi-factor co-variance matrix, thereby evolving the variate/derivative matrix that is sparse optimally (as opposed to blind delta bumps that may happen when computing differentials).
   - Variance Reduction along the forward path => If a specific forward path a) does not need to be traveled, or b) certain forward Wengert intermediates automatically compute to zero, then these produce zero path derivatives. Further, external pre-computations can be done during the adjoint generation.
   - Delta effects on the Optimal Exercise Dates => This imposes restrictions on how the path derivatives maybe computed using automatic differentiation. This may also be used in conjunction with regression analysis for estimating optimal exercise times. That certainly enables adjoint automatic differentiation techniques to be used.
   - Tangent multi-mode arc derivatives =>

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a. Identifying the circumstances under which they are re-usable
b. Arc derivatives extraction intermediates may also be re-used
c. Depends (as always) on the speed up and memory used.

3. Quasi-analytic Computation Models: No Monte-Carlo evolution needed at all, but still Wengert intermediate level reformulation necessary to enhance the quasi-analytics analysis (e.g., Copula methods).
    - Adjoint-Natural Formulation Mode => Typical formulation works out the Wengerts backwards from the final measure (e.g., say from PV), so they are automatically amenable to the adjoint mode of automatic differentiation.
4. Latent State Calibration from Observed Manifest Measures:
    - Formulation of the de-convolution the latent state from the observed manifest measure is necessary for the extraction of the latent state parameter set (this is accomplished by the calibration process).
    - Of course, latent state calibration occurs among the elastic and the inelastic dimensions, and the inelastics are parameter set!
    - Latent state calibration/parameterization etc. inherently involve parsimonization – this is where the models come in.

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Sensitivity Generation During Curve Construction

Introduction

1. Advantages: In addition to the usual advantage that Automatic Differentiation provides on doing accurate Greeks on the same run as pricing, there is no need for multiple bumped curves anymore – but the proper Jacobians need to be calculated.
   - Further speed up => The segment micro-Jacobian needs to be pre-calculated right during the calibration - we need to calculate the Jacobian $\frac{\partial C_i}{\partial q_j}$ where C_i is the ith coefficient, and q_j is the jth input.
2. Curve Calibration Deltas: Typical deltas are with respect to the
   - dynamical latent state stochastic variates (e.g., the forward rates)
   - calibrated parameters (e.g., the segment spline coefficients)
   - unit change in the quoted instrument measures (e.g., 1 bp change) - here the Jacobians need to ripple upwards from the quoted instrument manifest measures.
3. Span/Segment Elastic Variates: Consider the situation where the latent state in itself (not its transformation) is explicitly measured. There are 5 different kinds of latent state proxies to consider:
   - Φ => Span stochastic latent state evolution variate.
   - Φ_k => Stochastic latent state evolution variate for segment k.
   - ϕ => Implied Span Quoted Instrument Manifest Measure.
   - ϕ_k => Implied Quoted Instrument Manifest Measure for Segment k.
   - φ_k => Observed Quoted Instrument Manifest Measure for Segment k at precisely a single variate point – typically, the observations are done at the anterior/posterior terminal ends of the segment.
4. Span/Segment variate relations: For a given calculated/formulated output manifest measure Ξ, the following are true by definition:

$$\Phi_k(t = t_k) = \Phi(t = t_k)$$

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implies that

$$\left|\frac{\partial \Xi}{\partial \Phi}\right|_{t=t_k} = \left|\frac{\partial \Xi}{\partial \Phi_k}\right|_{t=t_k}$$

$$\varphi_k = \phi_k(t = t_k) = \phi(t = t_k)$$

implies that

$$\frac{\partial \Xi}{\partial \varphi_k} = \left|\frac{\partial \Xi}{\partial \phi}\right|_{t=t_k} = \left|\frac{\partial \Xi}{\partial \phi_k}\right|_{t=t_k}$$

5. Sensitivities to the elastic variates:
    - Sensitivity to Stochastic Evolution Variate => $\frac{\partial \Xi}{\partial \Phi}$
    - Sensitivity to Implied Span Quoted Instrument Measure => $\frac{\partial \Xi}{\partial \phi}$
    - Sensitivity to Observed Span Quoted Instrument Measure => $\frac{\partial \Xi}{\partial \varphi_k}$
    - $\frac{\partial \Xi}{\partial \varphi_k}$ (Case c) above) is what you need to calculate the hedge ratio
6. Piece-wise constant segment variate: In this case,

$$\frac{\partial \Xi}{\partial \Phi_k} = \frac{\partial \Xi}{\partial \phi_k} = \frac{\partial \Xi}{\partial \varphi_k}$$

7. Splined segment variate: Recall that segment spline coefficient calibration is simply a problem of matching to a terminal node (which is the quoted instrument measure at the terminal node). Thus, for a formulated output Ξ, at node k, it is obvious that

$$\frac{\partial \Xi}{\partial \Phi_k} \neq \frac{\partial \Xi}{\partial \phi_k}$$

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- Stochastic Evolution Variate Derivative => For the case where Ξ refers to the discount factor, it can be shown that

$$D_F(t) = e^{-\int \Phi(t)dt} = e^{-\sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} \Phi_i(t)dt - \int_{t_j}^{t} \Phi_j(t)dt}$$

where

$$t_j < t < t_{j+1}$$

Thus

$$\frac{\partial D_F(t)}{\partial \Phi_k} = -D_F(t) \times \begin{cases} t_{k+1} - t_k & k < j \\ t - t_k & k = j \\ 0 & k > j \end{cases}$$

- Quoted Instrument Manifest Measure Derivative => This depends on the actual details of the quadrature. Thus

$$\frac{\partial D_F(t)}{\partial \phi_k} = -D_F(t) \times \begin{cases} \int_{t_k}^{t_{k+1}} \frac{\partial \Phi}{\partial \varphi_k} dt & k < j \\ \int_{t_k}^{t} \frac{\partial \Phi}{\partial \varphi_k} dt & k = j \\ 0 & k > j \end{cases}$$

8. Linear Dependence of Integrand Quadrature: For many functional formulations in finance, the calculated product measure (Ξ) has a linear dependence on the stochastic evolution variate, i.e.,

$$\Xi \Longrightarrow \Psi\left(\int_{t_a}^{t_b} \Phi(t)\, dt\right)$$

This implies that

$$\frac{\partial \Xi}{\partial \Phi_k} = \delta_{ik} \frac{\partial \Xi}{\partial \Psi_i} (t_{i+1} - t_i)$$

i.e.,

$$\frac{\partial \Xi}{\partial \Phi_k} = \alpha \delta_{ik}$$

only, and not on the quadrature details.

Curve Jacobian

1. Representation Jacobian: Every Curve implementation needs to generate the Jacobian of the following latent state metric using its corresponding latent state quantification metric:
   - Forward Rate Jacobian to Quote Manifest Measure
   - Discount Factor Jacobian to Quote Manifest Measure
   - Zero Rate Jacobian to Quote Manifest Measure
2. Calibration Jacobian vs. Monte-Carlo Automatic Differentiation Delta: Both of these are actually path-wise, the difference being that:
   - Jacobian generated during calibration is part of inference, therefore iterative.
   - Jacobian of Monte-Carlo Automatic Differentiation is typically path-wise and non-iterative, therefore it is technically part of prediction.
3. Importance of the representation Self-Jacobian: Representation Self-Jacobian computation efficiency is critical, since Jacobian of any function $F(Y)$ is going to be dependent on the self-Jacobian $\frac{\partial Y(t)}{\partial Y(t_k)}$ because of the chain rule.
4. Forward Rate->DF Jacobian: Using $D_f(t_k)$ to represent the discount factor at t_k and to represent the forward rate $F(t_A, t_B)$ between times t_A and t_B, we get

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$$F(t_A, t_B) = \frac{1}{t_B - t_A} \log \frac{\partial D_f(t_A)}{\partial D_f(t_B)}$$

$$\frac{\partial F(t_A, t_B)}{\partial D_f(t_k)} = \frac{1}{t_B - t_A} \left\{ \frac{1}{D_f(t_A)} \frac{\partial D_f(t_A)}{\partial D_f(t_k)} - \frac{1}{D_f(t_B)} \frac{\partial D_f(t_B)}{\partial D_f(t_k)} \right\}$$

5. Zero Rate to Forward Rate Equivalence: This equivalence may be used to construct the Zero Rate Jacobian From the Forward Rate Jacobian. Thus the above equation may be used to extract the Zero Rate micro-Jacobian.
6. Zero Rate->DF Jacobian: Using $Z(t)$ to represent the discount factor at t, we get

$$\frac{\partial Z(t)}{\partial D_f(t_k)} = \frac{1}{t - t_0} \left\{ \frac{1}{D_f(t)} \frac{\partial D_f(t)}{\partial D_f(t_k)} \right\}$$

7. Quote->Zero Rate Jacobian:

$$\frac{\partial Q_j(t)}{\partial D_f(t_k)} = (t_k - t_0) \left\{ D_f(t_k) \frac{\partial Q_j(t)}{\partial D_f(t_k)} \right\}$$

8. PV->Quote Jacobian:

$$\frac{\partial PV_j(t)}{\partial Q_k} = \sum_{i=1}^{n} \left\{ \frac{\partial PV_j(t)}{\partial D_f(t_i)} \div \frac{\partial Q_j(t)}{\partial D_f(t_i)} \right\}$$

9. Cash Rate DF micro-Jacobian: Using r_j to represent the Cash Rate Quote for the jth Cash instrument, we get

$$\frac{\partial r_j}{\partial D_f(t_k)} = \frac{1}{t_j - t_{START}} \left\{ \frac{1}{D_f(t_j)} \frac{\partial D_f(t_j)}{\partial D_f(t_k)} \right\}$$

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10. Cash Instrument PV-DF micro-Jacobian:

$$\frac{\partial PV_{CASH,j}}{\partial D_f(t_k)} = \frac{1}{D_f\left(t_{j,SETTLE}\right)} \frac{\partial D_f\left(t_j\right)}{\partial D_f(t_k)}$$

There is practically no performance impact on construction of the PV-DF micro-Jacobian in then adjoint mode as opposed for forward mode, due to the triviality of the adjoint.

11. Euro-dollar Future DF micro-Jacobian: Setting Q_j to be the quote for the j[th] EDF with start date of $t_{j,START}$ and maturity of t_j, we get

$$\frac{\partial Q_j}{\partial D_f(t_k)} = \frac{\partial D_f\left(t_j\right)}{\partial D_f(t_k)} \frac{1}{D_f\left(t_{j,START}\right)} - \frac{D_f\left(t_j\right)}{{D_f}^2\left(t_{j,START}\right)} \frac{\partial D_f\left(t_{j,START}\right)}{\partial D_f(t_k)}$$

12. Euro-dollar Future PV-DF micro-Jacobian:

$$\frac{\partial PV_{EDF,j}}{\partial D_f(t_k)} = \frac{\partial D_f\left(t_j\right)}{\partial D_f(t_k)} \frac{1}{D_f\left(t_{j,START}\right)} - \frac{D_f\left(t_j\right)}{{D_f}^2\left(t_{j,START}\right)} \frac{\partial D_f\left(t_{j,START}\right)}{\partial D_f(t_k)}$$

There is practically no performance impact on construction of the PV-DF micro-Jacobian in then adjoint mode as opposed for forward mode, due to the triviality of the adjoint.

13. Interest Rate Swap DF micro-Jacobian: Setting Q_j to be the quote for the j[th] IRS maturing at t_j, $DV01_j$ to be the $DV01$ of the swap, and $PV_{Floating,j}$ as the floating PV of the swap, we get

$$Q_j \cdot DV01_j = PV_{Floating,j}$$

$$\frac{\partial\left(Q_j \cdot DV01_j\right)}{\partial D_f(t_k)} = \frac{\partial\left(PV_{Floating,j}\right)}{\partial D_f(t_k)}$$

$$\frac{\partial\left(Q_j \cdot DV01_j\right)}{\partial D_f(t_k)} = \frac{\partial\left(Q_j\right)}{\partial D_f(t_k)} \cdot DV01_j + \frac{\partial\left(DV01_j\right)}{\partial D_f(t_k)} \cdot Q_j$$

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

$$\frac{\partial(DV01_j)}{\partial D_f(t_k)} = \sum_{i=1}^{j} N(t_i)\Delta_i \frac{\partial D_f(t_i)}{\partial D_f(t_k)}$$

$$PV_{Floating,j} = \sum_{i=1}^{j} \mathcal{L}_i N(t_i)\Delta_i D_f(t_i)$$

$$\frac{\partial PV_{Floating,j}}{\partial D_f(t_k)} = \sum_{i=1}^{j} \frac{\partial \mathcal{L}_i}{\partial D_f(t_k)} N(t_i)\Delta_i D_f(t_i) + \sum_{i=1}^{j} \mathcal{L}_i N(t_i)\Delta_i \frac{\partial D_f(t_i)}{\partial D_f(t_k)}$$

14. <u>Interest Rate Swap PV-DF micro-Jacobian</u>:

$$\frac{\partial PV_{IRS,j}}{\partial D_f(t_k)} = -\sum_{i=1}^{j} \frac{\partial \mathcal{L}_i}{\partial D_f(t_k)} N(t_i)\Delta_i D_f(t_i) + \sum_{i=1}^{j} [c_j - \mathcal{L}_i] N(t_i)\Delta_i \frac{\partial D_f(t_i)}{\partial D_f(t_k)}$$

There is no performance impact on construction of the PV-DF micro-Jacobian in then adjoint mode as opposed for forward mode, due to the triviality of the adjoint. Either way the performance is $\Theta(n \times k)$, where n is the number of cash flows, and k is the number of curve factors.

15. <u>Credit Default Swap DF micro-Jacobian</u>: Setting c_j to be the coupon for the j^{th} CDS maturing at t_j, $PV_{CDS,j}$ to be the PV of the CDS contract, $PV_{Coupon,j}$ as the Coupon Leg of the CDS, $PV_{LOSS,j}$ as the PV of the Loss Leg of the CDS, and $PV_{ACCRUED,j}$ as the PV of the Accrual Paid on Default, we have

$$PV_{CDS,j} = PV_{Coupon,j} - PV_{LOSS,j} + PV_{ACCRUED,j}$$

$$PV_{Coupon,j} = c_j \sum_{i=1}^{j} N(t_i)\Delta_i S_P(t_i) D_f(t_i)$$

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Tel. 646-513-3300

$$\frac{\partial PV_{Coupon,j}}{\partial D_f(t_k)} = c_j \sum_{i=1}^{j} N(t_i)\Delta_i S_P(t_i) \frac{\partial D_f(t_i)}{\partial D_f(t_k)} + \frac{\partial c_j}{\partial D_f(t_k)} \sum_{i=1}^{j} N(t_i)\Delta_i S_P(t_i) D_f(t_i)$$

$$PV_{LOSS,j} = \int_0^{t_j} N(t)[1 - R(t)] D_F(t) dS_P(t)$$

$$\frac{\partial PV_{LOSS,j}}{\partial D_f(t_k)} = \int_0^{t_j} N(t)[1 - R(t)] \frac{\partial D_F(t)}{\partial D_f(t_k)} dS_P(t)$$

$$PV_{ACCRUED,j} = c_j \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} N(t)\Delta(t, t_{i-1}) D_F(t) dS_P(t)$$

$$\frac{\partial PV_{ACCRUED,j}}{\partial D_f(t_k)} = \frac{\partial c_j}{\partial D_f(t_k)} \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} N(t)\Delta(t, t_{i-1}) D_F(t) dS_P(t) + c_j \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} N(t)\Delta(t, t_{i-1}) \frac{\partial D_F(t)}{\partial D_f(t_k)} dS_P(t)$$

16. Credit Default Swap DF micro-Jacobian:

$$\frac{\partial PV_{CDS,j}}{\partial D_f(t_k)} = \frac{\partial c_j}{\partial D_f(t_k)} \sum_{i=1}^{j} \left\{ N(t_i)\Delta_i S_P(t_i) D_f(t_i) + \int_{t_{i-1}}^{t_i} N(t)\Delta(t, t_{i-1}) D_F(t) dS_P(t) \right\} + c_j \sum_{i=1}^{j} \left[N(t_i)\Delta_i S_P(t_i) \frac{\partial D_f(t_i)}{\partial D_f(t_k)} + \int_{t_{i-1}}^{t_i} N(t)\Delta(t, t_{i-1}) \frac{\partial D_F(t)}{\partial D_f(t_k)} dS_P(t) \right] - \int_0^{t_j} N(t)[1 - R(t)] \frac{\partial D_F(t)}{\partial D_f(t_k)} dS_P(t)$$

135 E. 57th Street |25th Floor |New York, NY 10022
Tel. 646-513-3300

There is no performance impact on construction of the PV-DF micro-Jacobian in then adjoint mode as opposed for forward mode, due to the triviality of the adjoint. Either way the performance is $\Theta(n \times k)$, where n is the number of cash flows, and k is the number of curve factors.

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Stochastic Entity Evolution

Stochastic Entity Evolution – Sensitivity Formulation

1. Evolution Dynamics: Simplest evolution of stochastic variables $\mathcal{L}_i(t)$ will be ones with constant forward volatilities. Once the dynamics is formulated according to

$$\Delta\mathcal{L}_i(t) = \mu_i(\mathcal{L}_i, t)\Delta t + \sum_j \sigma_{ij}(\mathcal{L}_i, t)\Delta W_j$$

where $\mu_i(\mathcal{L}_i, t)$ is the component drift, and σ_{ij} is the component co-variance to the factor W_j, subsequent evolution can be determined.

- The Eulerized version of the above is

$$\Delta x_j(t) = h\mu_j(\vec{x}, t)\Delta t + \sqrt{h}\sum_l \sigma_{jl}(\vec{x}, t)\Delta Z_l$$

  where h is the time-step, and $\vec{Z}$ is the Weiner random variable.

- In the case of forward rates, e.g., the drifts can be established by a no-arbitrage condition binding the forward rate drifts to their variances.

2. Evolution of the derivative entity: Once the stochastic variate dynamics is established, the dynamics of the observed derivative entity can be progressively determined.
3. Derivative Entity Measure path-wise evolution: Evolution sequence can be determined for the individual pay-off measures as well. These measures may further be dependent on the differentials of the derivative entity, so those may also need to be evolved using automatic differentiation.
4. Derivative Entity Computational efficiency enhancement:
   - Using the adjoint automatic differentiation methods
   - Using optimal combination of forward and adjoint automatic differentiation methods

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- Further optimizations using sparse-ness of the multi-factor co-variance matrix, thereby evolving the variate/derivative matrix that is sparse optimally (as opposed to blind delta bumps that may happen when computing differentials).
- Quasi-analytic computation models and automatic differentiation techniques => No Monte-Carlo evolution needed at all, but still Wengert intermediate level reformulation necessary to enhance the quasi-analytics analysis (e.g., Copula methods).

5. Derivative Entity Measure Calculation: $[Instrument, Manifest\ Measure]$ input to $[Instrument, Manifest\ Measure]$ output is equivalently maintained in the Jacobian. Alternately, the computation may also hold $[Latent\ State\ Calibrated\ Parameter]$ to $[Instrument, Manifest\ Measure]$ Output map.

Sensitivities to Stochastic State Variates and Dynamical Parameters

1. State Variates: These are base stochastic entities that characterize the actual system statics/dynamics.
   - Sensitivities to the state variates are typically sensitivities to the "current" (or starting) realization of these variates – e.g., delta, gamma.
2. Dynamic Parameters**:** Model parameters that govern the evolution/equilibrium behavior of the state variates, and thereby the system dynamics.
   - Examples would be sensitivities to volatility, correlation, etc.
3. Segment/Span Coefficients: These are the additional coefficients serve act as the interpolated "PROXY" for the segment latent state at the unobserved points in the segment.
   - Sensitivities may also be sought to the coefficients.

Stochastic Variate Evolution Constrained by Splines

1. The forward rates (or indeed any term instrument measures) need to evolve such that
   - They are continuous at the boundaries
   - The first (and possibly the second) derivatives are continuous at the boundaries

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- The boundary conditions (either financial or tensional) are retained intact

2. For e.g., the evolution dynamics of the forward rates (or indeed any term instrument measures) can still be via LMM, but splines may still be applicable to the intermediate nodes, as the segment spline coefficients adjust to the forward rate nodes.
3. Splines may also be used for any term instrument measure determinant (e.g., the volatility surface maybe also be interpolatively constructed using splines), so as to preserve the continuity/smoothness, as opposed to piece-wise discreteness.

Formulation of the Evolution of Stochastic Variate Self-Jacobian

1. Evolution Formulation:

$$\Delta x_j(t) = \mu_j(x_1, \cdots, x_n, t)\Delta t + \sum_{l=1}^{m} \sigma_{jl}(x_1, \cdots, x_n, t)\Delta W_l(t)$$

2. Definition of Self-Jacobian Delta:

$$J_{ij} = \frac{\partial x_i(t)}{\partial x_j(0)}$$

3. Evolution Sensitivity Formulation:
   a. i => Index over the number of underliers $(1, \cdots, n)$
   b. l => Index over the number of independent stochastic factors $(1, \cdots, m)$
   c. Then

$$\frac{\partial \Delta x_j(t)}{\partial x_k(0)} = \sum_{i=1}^{n} \left[\frac{\partial \mu_j(x_1, \cdots, x_n, t)}{\partial x_i(t)} \frac{\partial x_i(t)}{\partial x_k(0)}\right] \Delta t + \sum_{l=1}^{m} \left[\frac{\partial \sigma_{jl}(x_1, \cdots, x_n, t)}{\partial x_i(t)} \frac{\partial x_i(t)}{\partial x_k(0)}\right] \Delta W_l(t)$$

   d. Eulerized version of the above is:

$$\frac{\partial \Delta x_j(t)}{\partial x_k(0)} = \sum_{i=1}^{n} \left[\frac{\partial \mu_j(x_1, \cdots, x_n, t)}{\partial x_i(t)} \frac{\partial x_i(t)}{\partial x_k(0)}\right] h + \sqrt{h} \sum_{l=1}^{m} \left[\frac{\partial \sigma_{jl}(x_1, \cdots, x_n, t)}{\partial x_i(t)} \frac{\partial x_i(t)}{\partial x_k(0)}\right] Z_l(t)$$

e. First re-jig:

$$\frac{\partial x_j(t+h)}{\partial x_k(0)} = \sum_{i=1}^{n} \left[\delta_{ij} + h \frac{\partial \mu_j(t)}{\partial x_i(t)} + \sqrt{h} \sum_{l=1}^{n} \left\{Z_l(t+h) \frac{\partial \sigma_{jl}(t)}{\partial x_i(t)}\right\}\right] \frac{\partial x_i(t)}{\partial x_k(0)}$$
$$= \sum_{i=1}^{n} D_{ij}(k,t) \frac{\partial x_i(t)}{\partial x_k(0)}$$

where

$$D_{ij}(k,t) = \delta_{ij} + h \frac{\partial \mu_j(t)}{\partial x_i(t)} + \sqrt{h} \sum_{l=1}^{n} \left\{Z_l(t+h) \frac{\partial \sigma_{jl}(t)}{\partial x_i(t)}\right\}$$

f. Second re-jig:

$$\left[\frac{\partial x(t+h)}{\partial x_k(0)}\right] = [D(k,t)] \left[\frac{\partial x(t)}{\partial x_k(0)}\right]$$

where $\left[\frac{\partial x(t+h)}{\partial x_k(0)}\right]$ and $\left[\frac{\partial x(t)}{\partial x_k(0)}\right]$ are column matrices, and $[D(k,t)]$ is an $n \times n$ square matrix.

g. Third re-jig:

$$\left[\frac{\partial x(t+h)}{\partial x_k(0)}\right] = [D(k,t)][D(k,t-h)] \cdots [D(k,0)] \left[\frac{\partial x(0)}{\partial x_k(0)}\right]$$

- This is still forward automatic differentiation mode and is $\Theta(n)$, but you can optimize this using specific techniques shown in Glasserman and Zhao (1999).
- Another significant optimization can be achieved by adjointing techniques [Griewank (2000), Giles and Pierce (2000)].

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- To achieve further significant optimization, transpose this, to get the following adjoint form:

$$\left[\frac{\partial x(t+h)}{\partial x_k(0)}\right]^{\mathrm{T}} = \left[\frac{\partial x(0)}{\partial x_k(0)}\right]^{\mathrm{T}} [D(k,0)]^{\mathrm{T}}[D(k,h)]^{\mathrm{T}} \cdots [D(k,t-h)]^{\mathrm{T}}[D(k,t)]^{\mathrm{T}}$$

which actually reduces to vector/matrix as opposed to matrix/matrix in the non-transposed version – this would be $\Theta(n^2)$, as opposed to $\Theta(n^3)$.

h. The matrix nature of $[D(k,t)]$ simply arises from the chain rule summation over i. Similar chain rules may be set for the different cash flow Jacobians, etc.

i. Re-casting $D_{ij}(k,t)$ from above as

$$D_{ij}(k,t) = D_{ij,PRIOR}(k,t) + D_{ij,DRIFT}(k,t) + D_{ij,VOLATILITY}(k,t)$$

we can separate out the different contributions to $D_{ij}(k,t)$. a) The term

$$D_{ij,PRIOR}(k,t) = \delta_{ij}(k,t)$$

is the contribution due to the previous D, i.e., $D_{ij}(k,t-h)$. b) The term

$$D_{ij,DRIFT}(k,t) = h\frac{\partial \mu_j(t)}{\partial x_i(t)}$$

is the contribution from the derivative of the drift term. c) The term

$$D_{ij,VOLATILITY}(k,t) = \sqrt{h}\sum_{l=1}^{n}\left\{Z_l(t+h)\frac{\partial \sigma_{jl}(t)}{\partial x_i(t)}\right\}$$

is the contribution from the volatility derivative.

4. <u>Definition of Self-Jacobian Gamma</u>:

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Tel. 646-513-3300

$$\Gamma_{ij} = \frac{\partial^2 x_C(t)}{\partial x_A(0)\partial x_B(0)}$$

$$\frac{\partial^2 x_C(t+h)}{\partial x_A(0)\partial x_B(0)} = \sum_{i=1}^{n}\sum_{j=1}^{n}\left[S_{ij}(C,\vec{x}(t),t)\frac{\partial x_i(t)}{\partial x_A(0)}\frac{\partial x_j(t)}{\partial x_B(0)}\right] + \sum_{i=1}^{n}\left[M_i(C,\vec{x}(t),t)\frac{\partial^2 x_i(t)}{\partial x_A(0)\partial x_B(0)}\right]$$

$$S_{ij}(C,\vec{x}(t),t) = h\frac{\partial^2 \mu_C(\vec{x}(t),t)}{\partial x_i(t)\partial x_j(t)} + \sqrt{h}Z_C(t+h)\frac{\partial^2 \sigma_C(\vec{x}(t),t)}{\partial x_i(t)\partial x_j(t)}$$

$$M_i(C,\vec{x}(t),t) = h\frac{\partial \mu_C(\vec{x}(t),t)}{\partial x_i(t)} + \sqrt{h}Z_C(t+h)\frac{\partial \sigma_C(\vec{x}(t),t)}{\partial x_i(t)}$$

Correlated Stochastic Variables Evolution

1. Continuous Evolution of LMM-type Quantities: Let $\vec{X}$ be the vector of financial variables that need to be mapped to the corresponding Weiner variates $\vec{Z}$. In LMM, for e.g., start with

$$\vec{X}(0) = \{X_1(0), X_2(0), \cdots, X_n(0)\}$$

then the LMM evolutionary techniques generate $\vec{Z}$ and update $\vec{X}(t)$.

- Any continuous entity can be chosen to model correlations, not just LMM-type asset movements. For instance, if a default process can be correspondingly transformed to an asset indicator variable, that may be correlated with the other asset variables too.
- For a set of correlated variates, the stochastic evolution equation is

$$X_j(t+h) = X_j(t) + h\mu_j(\vec{X},t) + \sqrt{h}\sigma_j(\vec{X},t)\sum_{l=1}^{m}\rho_{jl}(\vec{X},t)Z_l(t+h)$$

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- Here $\sigma_j(\vec{X},t)$ is the variance, and $\rho_{jl}(\vec{X},t)$ is the correlation matrix – the variance is factored out of the covariance matrix to produce the correlation grid. $Z_l(t+h)$ is in the usual i.i.d. $\mathcal{N}(0,1)$.
- The corresponding delta is

$$\left[\frac{\partial \vec{X}(t+h)}{\partial X_k(0)}\right] = [D]\left[\frac{\partial \vec{X}(t)}{\partial X_k(0)}\right]$$

- The entry in matrix D is given as

$$D_{ij} = \delta_{ij} + h\frac{\partial \mu_j(\vec{X},t)}{\partial X_i(t)} + \sqrt{h}\sum_{l=1}^{m} Z_l(t+h)\left\{\rho_{jl}(\vec{X},t)\frac{\partial \sigma_j(\vec{X},t)}{\partial X_i(t)} + \sigma_j(\vec{X},t)\frac{\partial \rho_{jl}(\vec{X},t)}{\partial X_i(t)}\right\}$$

- The corresponding parameter sensitivity is:

$$\left[\frac{\partial \vec{X}(t+h)}{\partial \alpha}\right] = [D]\left[\frac{\partial \vec{X}(t)}{\partial \alpha}\right]$$

  This may be simplified in cases where α is an explicit function ONLY of the state evolution variables as

$$\frac{\partial X_j(t+h)}{\partial \alpha} = \frac{\partial X_j(t)}{\partial \alpha} + h\frac{\partial \mu_j(\vec{X},t)}{\partial \alpha}$$
$$+ \sqrt{h}\sum_{l=1}^{m} Z_l(t+h)\left\{\rho_{jl}(\vec{X},t)\frac{\partial \sigma_j(\vec{X},t)}{\partial \alpha} + \sigma_j(\vec{X},t)\frac{\partial \rho_{jl}(\vec{X},t)}{\partial \alpha}\right\}$$

2. <u>Correlated Default Times</u>: Unlike the continuous variables above, if we are to consider the correlations between default times ONLY, it is much more efficient to draw correlated

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default times – again this correlation is different from that of continuous asset value times that results in default.

3. Generation of Correlated Default Times:
   - Generate the vector $\vec{Z}_{INDEPENDENT}$.
   - Factorize the correlation matrix ρ_{jk} to create the Cholesky diagonal matrices C and C^{T}.
   - Use the Cholesky transformation to create $\vec{Z}_{CORRELATED}$ from $\vec{Z}_{INDEPENDENT}$ using

$$\vec{Z}_{CORRELATED} = C\vec{Z}_{INDEPENDENT}$$

   - For each entity $\tilde{z}_i$ in $\vec{Z}_{CORRELATED}$:
      i. Evaluate the cumulative normal

$$y_i = \int_{x=-\infty}^{\tilde{z}_i} \mathcal{N}(0,1)dx$$

where $\mathcal{N}(0,1)$ is a Normal distribution with unit mean and zero variance.

      ii.

$$\tau_{i,DEFAULT} = S_i^{-1}(y_i)$$

where S_i is the survival probability for the entity i.

      iii. More generally remember that

$$X_i = M_i^{-1}(y_i)$$

LMM Forward Rate Evolution

1. Importance of the LMM Formulation: 2 reasons why it is important:

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- LMM is one of the most popularly used formulation, and it is essential to evaluate the impact the no-arbitrage constrained drift has on the evolution and the impact on the greeks.
- The lognormal nature of the forward rate $\vec{L}(t)$ is important in its own right.

2. No-arbitrage constraint specification:

$$\mathcal{L}_j(t+h) = \mathcal{L}_j(t) + h\mu_j\big(\vec{\mathcal{L}}(t), t\big) + \sqrt{h} Z_l(t+h)\sigma_j\big(\vec{\mathcal{L}}(t), t\big)$$

where

$$\mu_j\big(\vec{\mathcal{L}}(t), t\big) = b_j \mathcal{L}_j(t) \sum_{p=\eta(t)}^{j} \frac{b_p \Delta\big(t_{p-1}, t_p\big)\mathcal{L}_p(t)}{1 + \Delta\big(t_{p-1}, t_p\big)\mathcal{L}_p(t)}$$

$$\sigma_j\big(\vec{\mathcal{L}}(t), t\big) = b_j \mathcal{L}_j(t)$$

and $\eta(t)$ is the maturity of the first instrument that matures after t [Brace, Gatarek, and Musiela (1997), Jamshidian (1997)].

3. Forward Rate Volatility vs. At-the-Money Swap Option Volatility: LMM uses forward rate volatilities, so there needs to be a conversion step that involves converting the market observed at-the-money swap option volatility onto LMM forward rate volatility [Brigo and Mercurio (2001)].

- Self-Jacobian of the extended LMM Formulation: As shown in Denson and Joshi (2009a) and Denson and Joshi (2009b)

$$\frac{\partial \mathcal{L}_j(t+h)}{\partial \mathcal{L}_k(0)} = \frac{\partial \mathcal{L}_j(t)}{\partial \mathcal{L}_k(0)} + \sum_{i=1}^{n} \left\{ h \frac{\partial \mu_j\big(\vec{\mathcal{L}}(t), t\big)}{\partial \mathcal{L}_i(t)} + \sqrt{h} Z_l(t+h) \frac{\partial \sigma_j\big(\vec{\mathcal{L}}(t), t\big)}{\partial \mathcal{L}_i(t)} \right\} \frac{\partial \mathcal{L}_i(t)}{\partial \mathcal{L}_k(0)}$$

where

$$\frac{\partial\sigma_j\left(\vec{\mathcal{L}}(t),t\right)}{\partial\mathcal{L}_i(t)}=\delta_{ij}b_i$$

$$\frac{\partial\mu_j\left(\vec{\mathcal{L}}(t),t\right)}{\partial\mathcal{L}_i(t)}(\eta(t)\le i)=\delta_{ij}b_j\sum_{p=\eta(t)}^{j}\frac{b_p\Delta\left(t_{p-1},t_p\right)\mathcal{L}_p(t)}{1+\Delta\left(t_{p-1},t_p\right)\mathcal{L}_p(t)}+\frac{b_j\mathcal{L}_j(t)}{[1+\Delta(t_{i-1},t_i)\mathcal{L}_i(t)]^2}$$

$$\frac{\partial\mu_j\left(\vec{\mathcal{L}}(t),t\right)}{\partial\mathcal{L}_i(t)}(\eta(t)>i)=\delta_{ij}b_j\sum_{p=\eta(t)}^{j}\frac{b_p\Delta\left(t_{p-1},t_p\right)\mathcal{L}_p(t)}{1+\Delta\left(t_{p-1},t_p\right)\mathcal{L}_p(t)}$$

4. <u>Forward-Rate Evolution Matrix</u>: As expected

$$\left[\frac{\partial x(t+h)}{\partial x_k(0)}\right]^{\mathrm{T}}=\left[\frac{\partial x(0)}{\partial x_k(0)}\right]^{\mathrm{T}}[D(k,0)]^{\mathrm{T}}[D(k,h)]^{\mathrm{T}}\cdots[D(k,t-h)]^{\mathrm{T}}[D(k,t)]^{\mathrm{T}}$$

where

$$\begin{aligned}&D_{ij}\left(\vec{\mathcal{L}}(t),t\right)(\eta(t)\le i)\\&\qquad=\delta_{ij}\left\{1+hb_j\sum_{p=\eta(t)}^{j}\frac{b_p\Delta\left(t_{p-1},t_p\right)\mathcal{L}_p(t)}{1+\Delta\left(t_{p-1},t_p\right)\mathcal{L}_p(t)}+\sqrt{h}b_jZ_j(t+h)\right\}\\&\qquad+\frac{hb_j\mathcal{L}_j(t)}{[1+\Delta(t_{i-1},t_i)\mathcal{L}_i(t)]^2}\end{aligned}$$

and

$$D_{ij}\left(\vec{\mathcal{L}}(t),t\right)(\eta(t)>i)=\delta_{ij}\left\{1+hb_j\sum_{p=\eta(t)}^{j}\frac{b_p\Delta\left(t_{p-1},t_p\right)\mathcal{L}_p(t)}{1+\Delta\left(t_{p-1},t_p\right)\mathcal{L}_p(t)}+\sqrt{h}b_jZ_j(t+h)\right\}$$

5. <u>Variate Jacobian Parameter Sensitivity</u>:

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$$\frac{\partial \mathcal{L}_j(t+h)}{\partial \alpha} = \frac{\partial \mathcal{L}_j(t)}{\partial \alpha} + h\frac{\partial \mu_j\big(\vec{\mathcal{L}}(t),t\big)}{\partial \alpha} + \sqrt{h}Z_j(t+h)\frac{\partial \sigma_j\big(\vec{\mathcal{L}}(t),t\big)}{\partial \alpha} + \sum_{i=1}^{n} D_{ij}\big(\vec{\mathcal{L}}(t),t\big)\frac{\partial \mathcal{L}_i(t)}{\partial \alpha}$$

where $D_{ij}\big(\vec{\mathcal{L}}(t),t\big)$ is available from above for the two scenarios. Re-casting the above, we get

$$\frac{\partial \mathcal{L}_j(t+h)}{\partial \alpha} = B_j\big(\vec{\mathcal{L}}(t),t\big) + \sum_{i=1}^{n} D_{ij}\big(\vec{\mathcal{L}}(t),t\big)\frac{\partial \mathcal{L}_i(t)}{\partial \alpha}$$

where

$$B_j\big(\vec{\mathcal{L}}(t),t\big) = \mathcal{L}_j(t)\left\{\sqrt{h}Z_j(t+h)\frac{\partial b_j}{\partial \alpha} + h\sum_{p=\eta(t)}^{j}\frac{b_p\Delta\big(t_{p-1},t_p\big)\mathcal{L}_p(t)}{1+\Delta\big(t_{p-1},t_p\big)\mathcal{L}_p(t)}\left[b_p\frac{\partial b_j}{\partial \alpha} + b_j\frac{\partial b_p}{\partial \alpha}\right]\right\}$$

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Formulation of Sensitivities for Pay-off Functions

Formulation of Pay-off Function Stochastic Evolution

1. Monte-Carlo Path-wise Derivatives: Path-wise derivatives are typically forward derivatives, not adjoint [Giles and Glasserman (2006)]. Therefore computation time is proportional to the number of inputs. Further, not easy to accommodate these in complex payouts [Capriotti (2011)].
2. Payoff Expectation Formulation: $V = E_Q\left[P\left(\vec{X}\right)\right]$ [Harrison and Kreps (1979)], where $\vec{X}$ is the vector of financial variables.
   - Path Payoff Expectation [Kallenberg (1997)] => $V = \frac{1}{N_{MC}} \sum_{i_{MC}=1}^{N_{MC}} P\left(\vec{X}\left[i_{MC}\right]\right)$ and

$$Variance = \frac{N_{MC}{}^{2} \sum_{i_{MC}=1}^{N_{MC}} \left[\left\{P\left(\vec{X}\left[i_{MC}\right]\right)\right\}^{2}\right] - \left\{\sum_{i_{MC}=1}^{N_{MC}} P\left(\vec{X}\left[i_{MC}\right]\right)\right\}^{2}}{N_{MC}{}^{2}}$$

Path Greek

1. Unbiased Estimate of Path Sensitivity: Estimate is unbiased [Kunita (1990), Broadie and Glasserman (1996), Glasserman (2004)] if $\left\langle \frac{\partial Y(x)}{\partial x(0)} \right\rangle = \frac{\partial}{\partial x(0)} \left\langle \partial Y(x) \right\rangle$ where $x(0)$ is the starting point for the variate.

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2. Monte-Carlo Greek Definition: Greek is defined at the change in Y with respect to the starting value of x, i.e., $x(0)$. $\frac{\partial Y(x(t))}{\partial x(0)} = \frac{\partial Y(x(t))}{\partial x(t)} \frac{\partial x(t)}{\partial x(0)}$. If x is a multi-component vector $\vec{X}$, then $\frac{\partial Y\left[\vec{X}(t)\right]}{\partial x_j(0)} = \sum_{i=1}^{n} \frac{\partial Y\left[\vec{X}(t)\right]}{\partial x_j(t)} \frac{\partial x_j(t)}{\partial x_j(0)}$.

3. Pay-off Function Delta: $\frac{\partial V(t)}{\partial x_k(0)} = \sum_{j=1}^{n} \frac{\partial V(t)}{\partial x_j(t)} \frac{\partial x_j(t)}{\partial x_k(0)}$. Now use the earlier formulation for $\left[\frac{\partial x(t)}{\partial x_k(0)}\right]$ to establish the path delta. In particular, using above, $\left[\frac{\partial V(t)}{\partial x_k(0)}\right]^T = \left[\frac{\partial V(0)}{\partial x_k(0)}\right]^T [D(k,0)]^T [D(k,h)]^T \ldots [D(k,t-h)]^T [D(k,t)]^T$, so all the speed up advantages associated with the adjoint formulation above follows.

4. Variance in the Greeks in addition to the base Greeks:
   - Cluster all the Path-wise Greeks calculated for a given input (either $x_k(0)$ or a parameter θ).
   - Within that cluster estimate the corresponding Greek.
   - Usual population sampling variance techniques applied to compute the variance in the Greek.

5. Path Parameter (α) Sensitivity: $\frac{dV(t)}{d\alpha} = \frac{\partial V(t)}{\partial \alpha} + \sum_{j=1}^{n} \frac{\partial V(t)}{\partial x_j(t)} \frac{\partial x_j(t)}{\partial \alpha}$. Now use the earlier formulation for $\left[\frac{\partial x(t)}{\partial \alpha}\right]$ to establish the path parameter sensitivity.

6. Explicit Pay-off Greek Formulation:

$$\frac{\partial x_j(t+h)}{\partial \alpha} = h\frac{\partial \mu_j(t)}{\partial \alpha} + \sqrt{h}\sum_{l=1}^{m} Z_l(t+h)\frac{\partial \sigma_{jl}(t)}{\partial \alpha} + \sum_{i=1}^{n}\left\{\delta_{ji} + h\frac{\partial \mu_j(t)}{\partial x_i(t)} + \sqrt{h}\sum_{l=1}^{m} Z_l(t+h)\frac{\partial \sigma_{jl}(t)}{\partial x_i(t)}\right\}\frac{\partial x_i(t)}{\partial \alpha}$$

   - Notice that it has additional terms since the explicit dependence of μ, σ on α is, in general, non-zero: otherwise, $B_j(t,\alpha) = 0$, and the pay-off parameter sensitivity formulation proceeds precisely along the same lines as delta formulation.

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a. <u>Rewrite #1</u>: $\frac{\partial x_j(t+h)}{\partial \alpha} = B_j(t,\alpha) + \sum_{i=1}^{n} D_{ji}(t,\alpha)\frac{\partial x_i(t)}{\partial \alpha}$ where $D_{ji}(t,\alpha)$ is exactly the same as earlier, and $B_j(t,\alpha) = h\frac{\partial \mu_j(t)}{\partial \alpha} + \sqrt{h}\sum_{l=1}^{m} Z_l(t+h)\frac{\partial \sigma_{jl}(t)}{\partial \alpha}$.

b. <u>Rewrite #2</u>: $\left[\frac{\partial \vec{x}(t+h)}{\partial \alpha}\right] = [B(\alpha,t)] + [D(\alpha,t)]\left[\frac{\partial \vec{x}(t)}{\partial \alpha}\right]$ where $\left[\frac{\partial \vec{x}(t+h)}{\partial \alpha}\right]$, $\left[\frac{\partial \vec{x}(t)}{\partial \alpha}\right]$, and $[B(\alpha,t)]$ are n x 1 column matrices, and $[D(k,t)]$ is an n x n square matrix.

c. <u>Rewrite #3</u>: Generalizing over all the j's, we get

$$\left[\frac{\partial \vec{x}(t+h)}{\partial \alpha}\right] = \sum_{e=0}^{s}\left[\left\{\prod_{f=1}^{e}[D(t-fh)]\right\}[B(t-eh)]\right] + \left\{\prod_{e=1}^{s}[D(t-eh)]\right\}\left[\frac{\partial \vec{x}(t)}{\partial \alpha}\right].$$

d. <u>Rewrite #4</u>: Transposing the above we get

$$\left[\frac{\partial \vec{x}(t+h)}{\partial \alpha}\right]^{\mathrm{T}} = \sum_{e=s}^{1}[B(\alpha,t-eh)]^{\mathrm{T}}\left[\left\{\prod_{f=e}^{1}[D(\alpha,t-fh)]^{\mathrm{T}}\right\}\right] + \left[\frac{\partial \vec{x}(t)}{\partial \alpha}\right]^{\mathrm{T}}\left\{\prod_{e=s}^{1}[D(\alpha,t-eh)]^{\mathrm{T}}\right\}.$$

e. <u>Implications of rewrite #4</u>: Given that $[B(\alpha,t)]^{\mathrm{T}}$ and $\left[\frac{\partial \vec{x}(t)}{\partial \alpha}\right]^{\mathrm{T}}$ are now row matrices, and that they are the preceding terms in the series, all the adjoint advantages indicated earlier continue to be valid. Further the previous formulations for $[D(\alpha,t)]$ can be re-used at the same Eulerian time step.

f. <u>Adjoint Storage Demands</u>: Remember that $[B(\alpha,t)]$ and $[D(\alpha,t)]$ still need to be retained in memory during the forward evolutionary sweep for $\left[\frac{\partial \vec{x}(t)}{\partial \alpha}\right]$, so this represents a corresponding increase on the storage requirements.

Payoff Sensitivity to the Correlation Matrix

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1. Payoff Sensitivity Formulation: Irrespective of where the stochastic process is diffusive or not, $\frac{\partial V}{\partial \rho_{jk}} = \sum_{i=1}^{n} \frac{\partial V}{\partial \tilde{Z}_i} \frac{\partial \tilde{Z}_i}{\partial \rho_{jk}}$, where ρ_{jk} is the correlation matrix.

2. Financial Variable to Correlated Random Partial:

   - Remember the general theorem that if $y(z) = \int_{x=-\infty}^{x=z} \Phi(x) dx$, then $\frac{\partial y}{\partial z} = \Phi(z)$.
   - From this, and using $X_i = M_i^{-1}(y_i)$, you can derive $\frac{\partial X_i}{\partial \tilde{Z}_i} = \Phi\left(\tilde{Z}_i\right) \frac{\partial X_i}{\partial \Phi(X_i)}$.

3. Differential of the Cholesky Factorization Matrix:

   - $\frac{\partial \tilde{Z}_i}{\partial \rho_{jk}} = \sum_{l=1}^{n} \sum_{m=1}^{n} \frac{\partial \tilde{Z}_i}{\partial C_{lm}} \frac{\partial C_{lm}}{\partial \rho_{jk}}$ where $\frac{\partial C_{lm}}{\partial \rho_{jk}}$ is readily computed.
   - Therefore $\frac{\partial V}{\partial \rho_{jk}} = \sum_{i=1}^{n} \frac{\partial V}{\partial \tilde{Z}_i} \frac{\partial \tilde{Z}_i}{\partial \rho_{jk}} = \sum_{i=1}^{n} \frac{\partial V}{\partial X_i} \frac{\partial X_i}{\partial \tilde{Z}_i} \frac{\partial \tilde{Z}_i}{\partial \rho_{jk}} = \sum_{i=1}^{n} \frac{\partial V}{\partial X_i} \Phi\left(\tilde{Z}_i\right) \frac{\partial X_i}{\partial \Phi(X_i)} \frac{\partial \tilde{Z}_i}{\partial \rho_{jk}}$ where $\frac{\partial \tilde{Z}_i}{\partial \rho_{jk}}$ is given from above.

Algorithmic Differentiation in Payoff Sensitivities Calculation

1. Monte-Carlo Path-wise Derivatives: Path-wise derivatives are typically forward derivatives, not adjoint (Giles and Glasserman (2006)). Therefore computation time is proportional to the number of inputs.
2. Forward Monte-Carlo evolution variates: The full set forward evolution variates is still needed for extracting the fields/parameters required for the delta estimation of the adjoint path.
3. Corresponding storage requirements: All the variates set (the transition matrices etc.) still need to be maintained, so this represents an increase in the storage needed.

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4. Adjointing vs. Reverse Mode: Typically adjoint refers **ONLY** to the intermediate/dynamical matrices [Giles (2007), Giles (2009)], whereas **REVERSE** refers to calculation of only the relevant outputs and their sensitivities [Griewank (2000)].
    - Adjointing deals with the evolved latent state space parameters left to right, therefore technically it is still forward in the time sense – and achieves optimization by minimizing the matrix<->matrix computations.
    - In the non-matrix sense (as in adjoint automatic differentiation), the term reverse and adjoint are analogous, i.e., adjoint/reverse refer to a scan backwards from right to left inside the SAME step, for e.g., a time step.
    - Finally, formalized pure "forward" and pure "reverse" is often theoretical constructs. Just like hand-rolled code can beat generic optimizers, hand-rolled algorithmic differentiation code will be better – even for Monte-Carlo sensitivity runs. However, development productivity gains to be attained by using automated AD tools are well documented.
5. Systematic Design Paradigm for using Automatic Differentiation for Path-wise Monte-Carlo Derivatives: Capriotti and Giles (2011) detail several techniques for this.
6. Cost:
    - Forward Automatic Differentiation Cost => $\frac{Cost[B+F]}{Cost[B]} = [2,2.5]$
    - Reverse Automatic Differentiation Cost => $\frac{Cost[B+F+R]}{Cost[B]} = [4,5]$
    - B => Base; F => Forward; R => Reverse.
7. Calibration along with Automatic Sensitivities Generation: Automatic Differentiation is natural performance fit in these situations (Kaebe, Maruhn, and Sachs (2009), Schlenkirch (2011)). Many approaches in this regard end up utilizing intermediate value theorem to facilitate the formulation (Christianson (1998), Giles and Pierce (2000)).

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Bermudan Swap Option Sensitivities

Base Formulation

1. Option Valuation under Monte-Carlo: Unlike typical closed forms (such Black-Scholes, Black etc.), volatility does not explicitly show up in the PV generation part for options. Instead, it features intrinsically, through the evolution dynamics, and from the valuation of the underlying that needs to be valued under a specific exercise scenario.
2. H x M Bermudan Swap Option Details:
    - Define the M swap exercise/pay date tenor grids $T_0 < T_1 < \ldots < T_M$.
    - Option exercise dates T_r start from date T_H onwards, i.e., $T_r \in \{T_H, T_{H+1}, \ldots, T_{M-1}\}$.
    - The cash flow stream after the exercise is the payment stream $\vec{X} = \{X_r, X_{r+1}, \ldots, X_M\}$.
3. H x M Exercised Bermudan Swap Valuation: $X_i = N(T_i)\Delta(t_{i-1}, t_i)[L_i - R]$, where R is the fixed rate. The Bermudan Swap PV is $PV_{Berm}(T_r) = \mathrm{E}\left[\sum_{i=r}^{M} D_f(t_i) X_i\right]$, where $\mathrm{E}[\ldots]$ is the expectation operator.
4. H x M Bermudan Swap Valuation SKU:
    - Simulate a single path sequence of $\vec{L}$.
    - For this path, evaluate $PV(X_i)$ for each $\vec{X} = \{X_r, X_{r+1}, \ldots, X_M\}$.
    - For this path, generate a vector of $PV_{Berm}(T_p)$ corresponding to each possible exercise date $T_p \in \{T_H, T_{H+1}, \ldots, T_{M-1}\}$.
    - Find T_r that maximizes $PV_{Berm}(T_p)$.
    - Record $\{T_r, PV_{Berm}(T_r)\}$.

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Greek Estimation

1. H x M Exercised Bermudan Swap Option Delta/Parameter Sensitivity [Piterbarg (2004), Capriotti and Giles (2011)]: $\frac{\partial PV_{Berm}(T_r)}{\partial L_K(0)} = \frac{\partial\left\{E\left[\sum_{i=r}^{M} D_f(t_i)X_i\right]\right\}}{\partial L_K(0)} = E\left[\sum_{i=r}^{M}\frac{\partial\{D_f(t_i)X_i\}}{\partial L_K(0)}\right]$

$$\frac{\partial PV_{Berm}(T_r)}{\partial \alpha} = \frac{\partial\left\{E\left[\sum_{i=r}^{M} D_f(t_i)X_i\right]\right\}}{\partial \alpha} = E\left[\sum_{i=r}^{M}\frac{\partial\{D_f(t_i)X_i\}}{\partial \alpha}\right]$$

2. Individual Cash-flow PV and Greeks [Leclerc, Liang, and Schneider (2009)]:
   - $PV_j = D_f(t_j)\Delta(t_{j-1},t_j)[L_j - R]$
   - $D_f(t_j) = \prod_{p=1}^{j}\frac{1}{1+\Delta(t_{p-1},t_p)L_p} \Rightarrow PV_j = \left\{\prod_{p=1}^{j}\frac{1}{1+\Delta(t_{p-1},t_p)L_p}\right\}\Delta(t_{j-1},t_j)[L_j - R]$
   - Remember that $\frac{\partial PV_j(t)}{\partial L_K(0)} = \sum_{i=1}^{n}\frac{\partial PV_j(t)}{\partial L_i(t)}\frac{\partial L_i(t)}{\partial L_K(0)}$ where $\frac{\partial L_i(t)}{\partial L_K(0)}$ is given by the LMM formulation presented earlier.
3. Cash-flow PV Delta:
   - $\frac{\partial PV_j(t)}{\partial L_i(t)} = \frac{\partial}{\partial L_i(t)}\left[\left\{\prod_{p=1}^{j}\frac{1}{1+\Delta(t_{p-1},t_p)L_p}\right\}\Delta(t_{j-1},t_j)[L_j - R]\right]$
   - $\frac{\partial PV_j(t)}{\partial L_i(t)}[j \ge i] = \left[\delta_{ij} - \frac{\Delta(t_{i-1},t_i)[L_j - R]}{1+\Delta(t_{i-1},t_i)L_i(t)}\right]\Delta(t_{j-1},t_j)D_f(t_j)$
   - $\frac{\partial PV_j(t)}{\partial L_i(t)}[j < i] = 0$

LSM Methodology

1. Curve-Fitting to Extract Optimal Exercise: Since the simple model of maximizing $PV_{Berm}(T_r)$ across T_r gets too cumbersome if the exercise dates are numerous – LSM based optimal

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exercise determination laid out in [Longstaff and Schwartz (2001)] can be used – regress T_r against $PV_{Berm}(T_r)$.

2. Continuous or Fine-grained Call Schedules: LSM is highly effective in these situations. Sampling is reduced to a few evenly spaced-out grid points – such that the full sample scoping is eliminated.
3. Interpolation between Sampled Nodes: Any appropriate inter-nodal interpolating/splining technique to determine $PV_{Berm}(T_r)$ as a function of T_r is valid – e.g., constant $PV_{Berm}(T_r)$ over T_r, linear/quadratic/polynomial $PV_{Berm}(T_r)$ over T_r, or even exponential/hyperbolic tension spline-based $PV_{Berm}(T_r)$ over T_r.

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Basket Sensitivities

NTD Product Formulation

1. Running Index Details:
   - $p = 1 \rightarrow n$ => Number of Components
   - $j, k = 1 \rightarrow n$ => Row, column index of the correlation matrix for each of the n components
   - $l, m = 1 \rightarrow n$ => Factorized Cholesky diagonal matrix for the n components
   - r => rth component in the current draw of ordered default times; it corresponds to the current nth-to-default.
   - N => The "N" in NTD ($\tau_N \equiv \tau_r$).
2. Base NTD Pricing:
   - $V_{NTD} = V_{Loss} + V_{\mathrm{Premium}} + V_{Accrued}$
   - $V_{Loss} = [1 - R_r(\tau_r)] D_f(\tau_r) N(\tau_r)$
   - $V_{\mathrm{Premium}} = c \sum_{i=1}^{n} N(t_i) D_f(t_i) \Delta(t_{i-1}, t_i) \perp (t_i \le \tau_r)$
   - $V_{Accrual} = c \sum_{i=1}^{n} N(\tau_r) D_f(\tau_r) \Delta(t_{i-1}, \tau_r) \perp (t_{i-1} \le \tau_r) \perp (t_i \ge \tau_r)$
   - $\perp (t \le \tau)$ => Default Indicator that is 1 if $t \le \tau$, and 0 otherwise.
     i. To make the computation convenient [Capriotti and Giles (2010), Capriotti and Giles (2011), Giles (2009), Chen and Glasserman (2008)] $\perp (t \le \tau)$ is regularized and smeared out using an appropriate proxy, i.e., $\perp (t \le \tau) \cong \mathrm{H}(t \le \tau)$.
     ii. $\mathrm{H}(t \le \tau)$ can be the Heaviside function.
     iii. The proxy $\mathrm{H}(t \le \tau)$ has a bias, but it can be designed to be much tighter than the Monte-Carlo accuracy.

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3. NTD Sensitivity:

- $\dfrac{\partial V_{NTD}}{\partial \rho_{jk}} = \sum_{p=1}^{n} \dfrac{\partial V_{NTD}}{\partial \tau_p} \dfrac{\partial \tau_p}{\partial \rho_{jk}}$
- $\dfrac{\partial V_{NTD}}{\partial \tau_p} = \dfrac{\partial V_{Loss}}{\partial \tau_p} + \dfrac{\partial V_{\Pr emium}}{\partial \tau_p} + \dfrac{\partial V_{Accrued}}{\partial \tau_p}$
- $\dfrac{\partial V_{Loss}}{\partial \tau_p} = \delta_{rp} \dfrac{\partial \{[1 - R_r(\tau_r)] D_f(\tau_r) N(\tau_r)\}}{\partial \tau_p}$
- $\dfrac{\partial V_{\Pr emium}}{\partial \tau_p} = c\delta_{rp} \sum_{i=1}^{n} N(t_i) D_f(t_i) \Delta(t_{i-1}, t_i) \dfrac{\partial \mathrm{H}(t_i \le \tau_r)}{\partial \tau_p}$
- $V_{Accrual} = c\delta_{rp} \sum_{i=1}^{n} \dfrac{\partial \{N(\tau_r) D_f(\tau_r) \Delta(t_{i-1}, \tau_r) \mathrm{H}(t_{i-1} \le \tau_r) \mathrm{H}(t_i \ge \tau_r)\}}{\partial \tau_p}$

Basket Options

1. Base Pricing Formulation: $V = D_f(T) \sum_{i=1}^{n} [W_i X_i(T) - S]^+$

2. Basket Options Delta:

- Remember from earlier that $\dfrac{\partial V(t)}{\partial x_k(0)} = \sum_{j=1}^{n} \dfrac{\partial V(t)}{\partial x_j(t)} \dfrac{\partial x_j(t)}{\partial x_k(0)}$. Here:

    i. $t = T$

    ii. $V(t) = V_{BO}(T)$

- $\dfrac{\partial V_{BO}\left(\vec{X}(T), T\right)}{\partial X_i(T)} = \dfrac{\partial D_f(T)}{\partial X_i(T)} \left[\sum_{p=1}^{n} W_p X_p(T) - S \right]^+ + W_i D_f(T) * Black\_Scholes\_Delta(Strike_p, T)$

  where $Strike_p = \dfrac{S - \sum_{p \ne i, p=1}^{n} W_p X_p(T)}{W_i}$

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Leibnitz Integral Rule

1. Differentiation of the Limits of Integrals:

$$\frac{\partial}{\partial y}\int_{a(y)}^{b(y)} f(x,y)dx = \frac{\partial b(y)}{\partial y} f(x,y) - \frac{\partial a(y)}{\partial y} f(x,y) + \int_{a(y)}^{b(y)} \frac{\partial}{\partial y} f(x,y)dx \cdot$$

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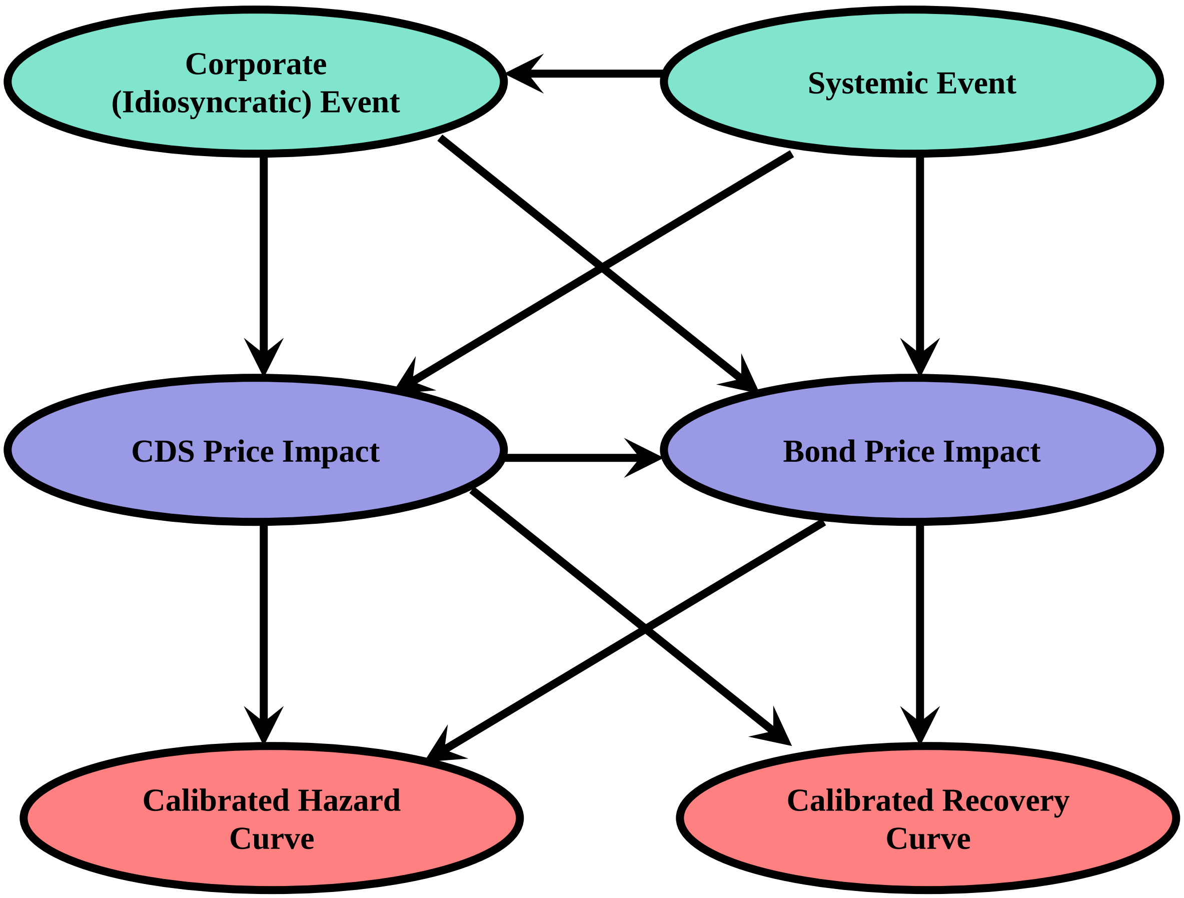

Figure 1: Causality Bayesian Network DAG For Credit Curve Building

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Figure 2: Transition Splines – Low Width Transition Stretch

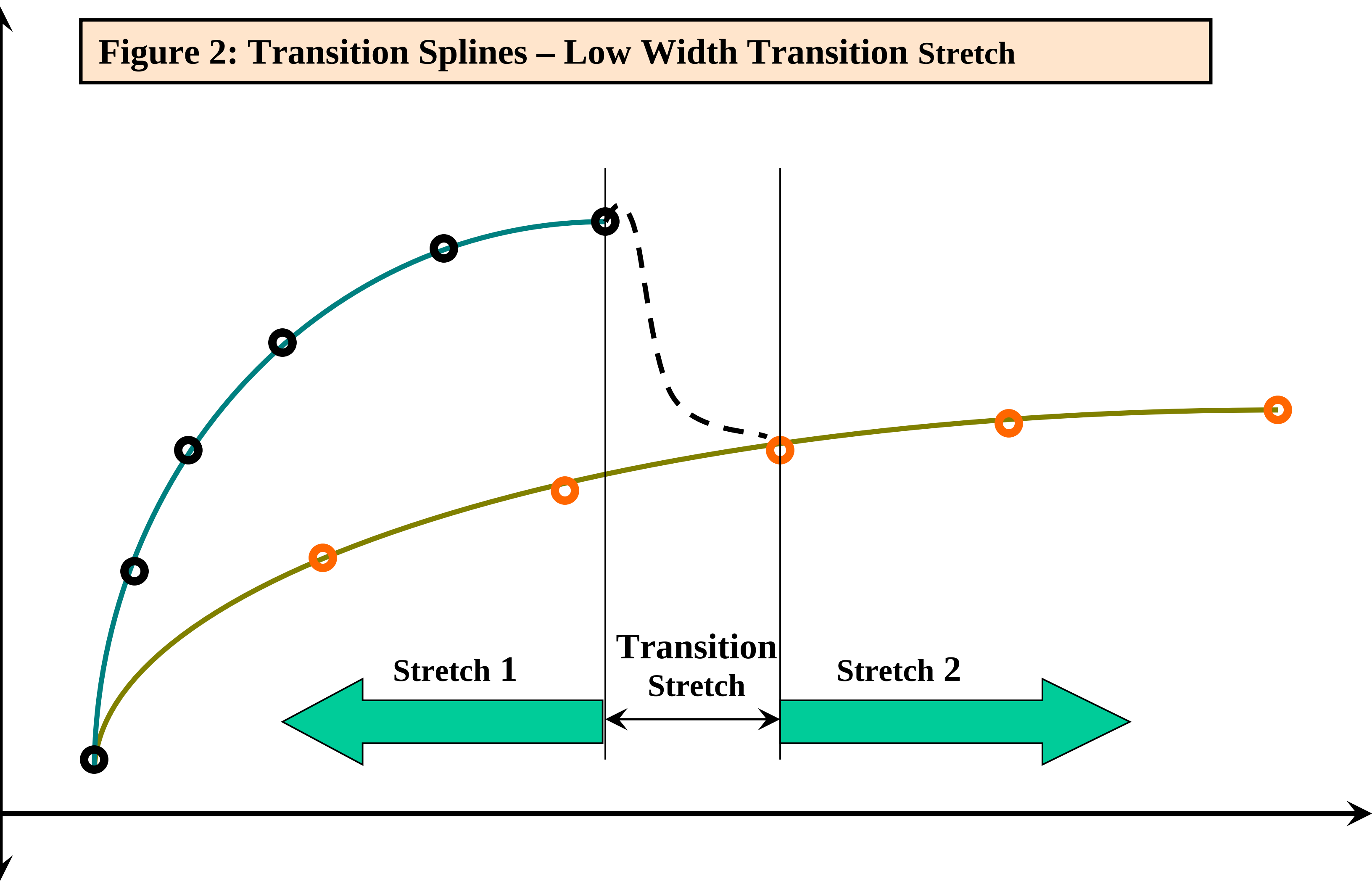

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Figure 3: Transition Splines – High Width Transition Stretch

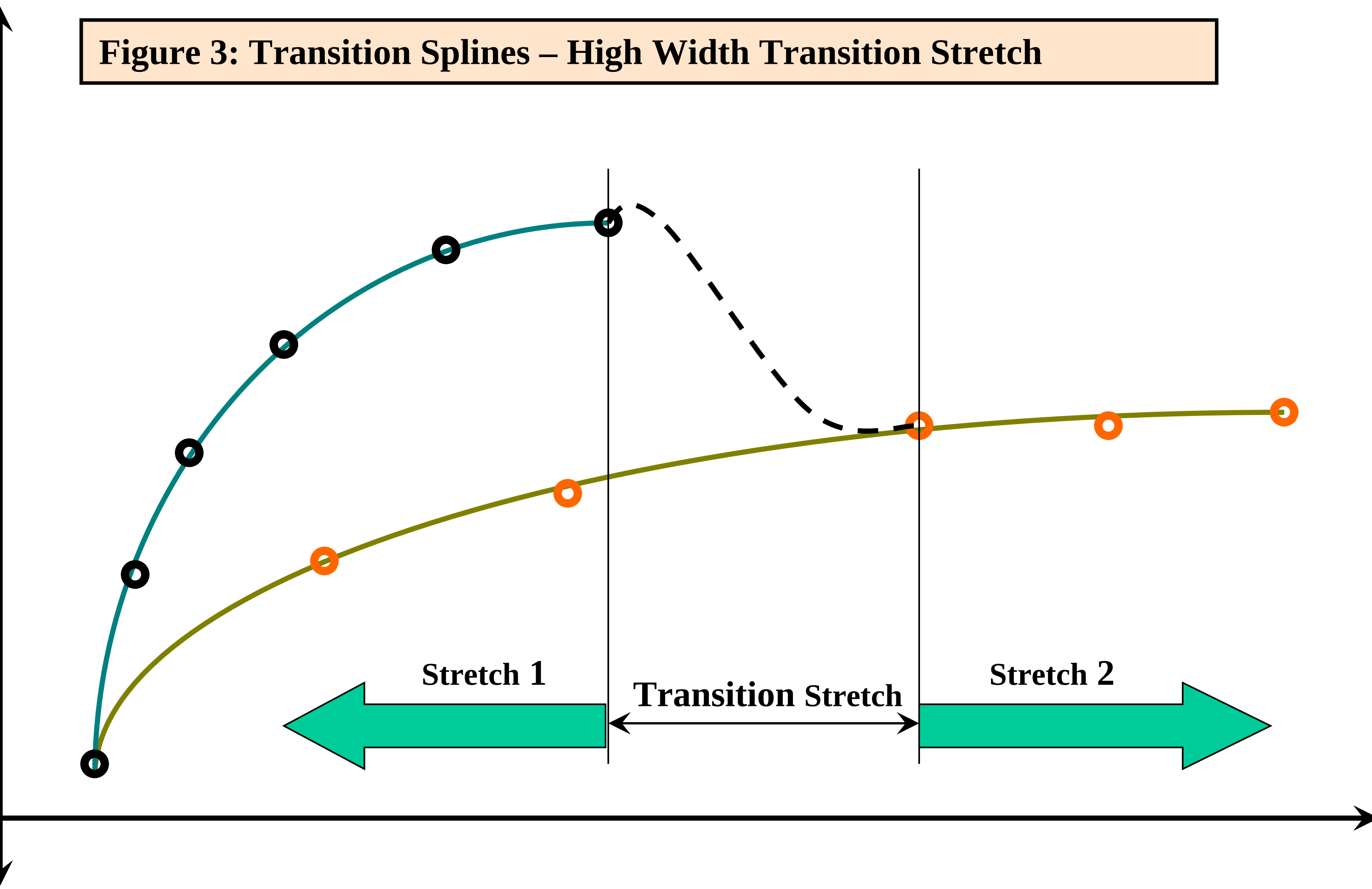

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Figure 4: Transition Splines – Segment <-> Stretch Layout

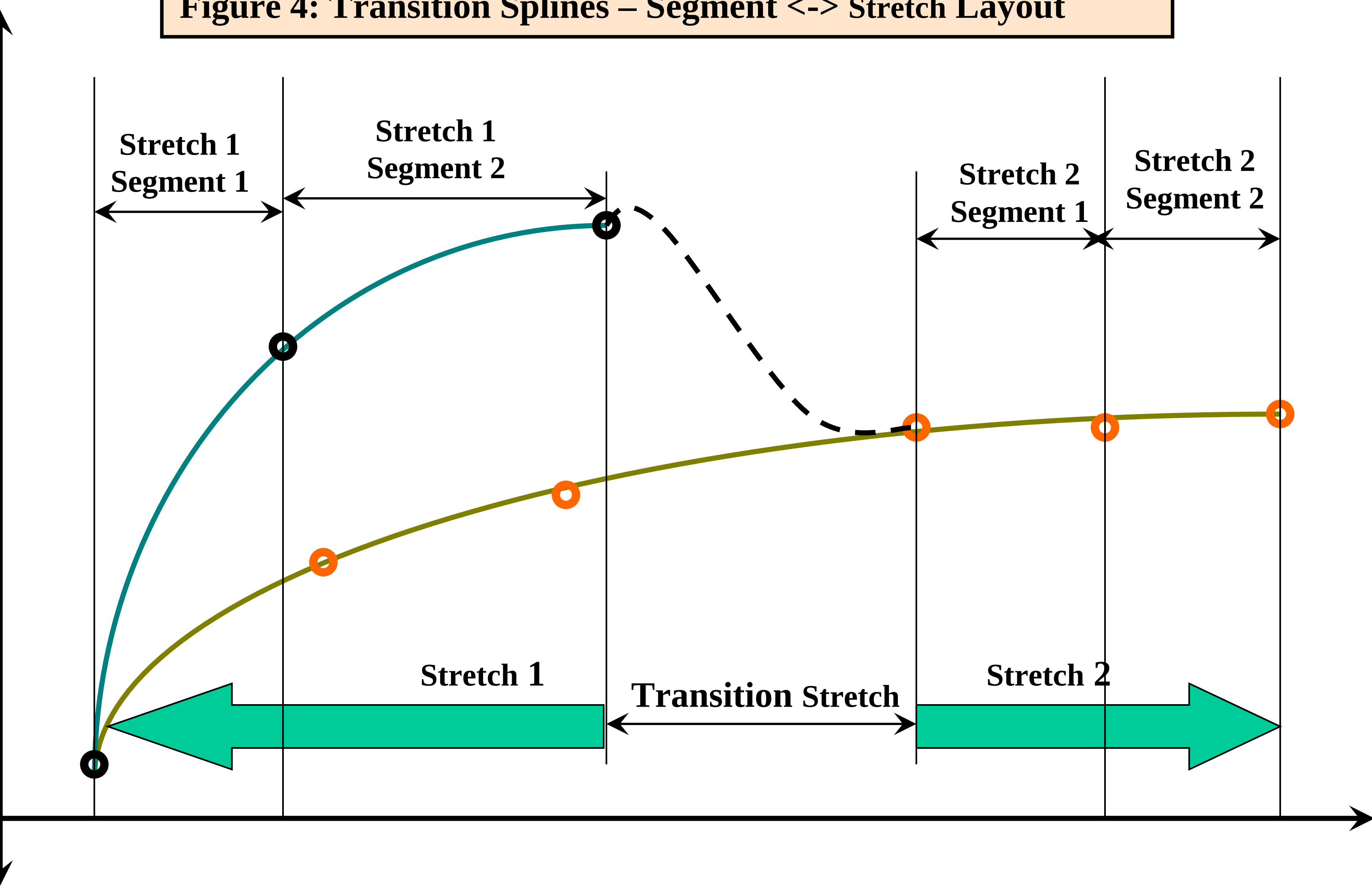

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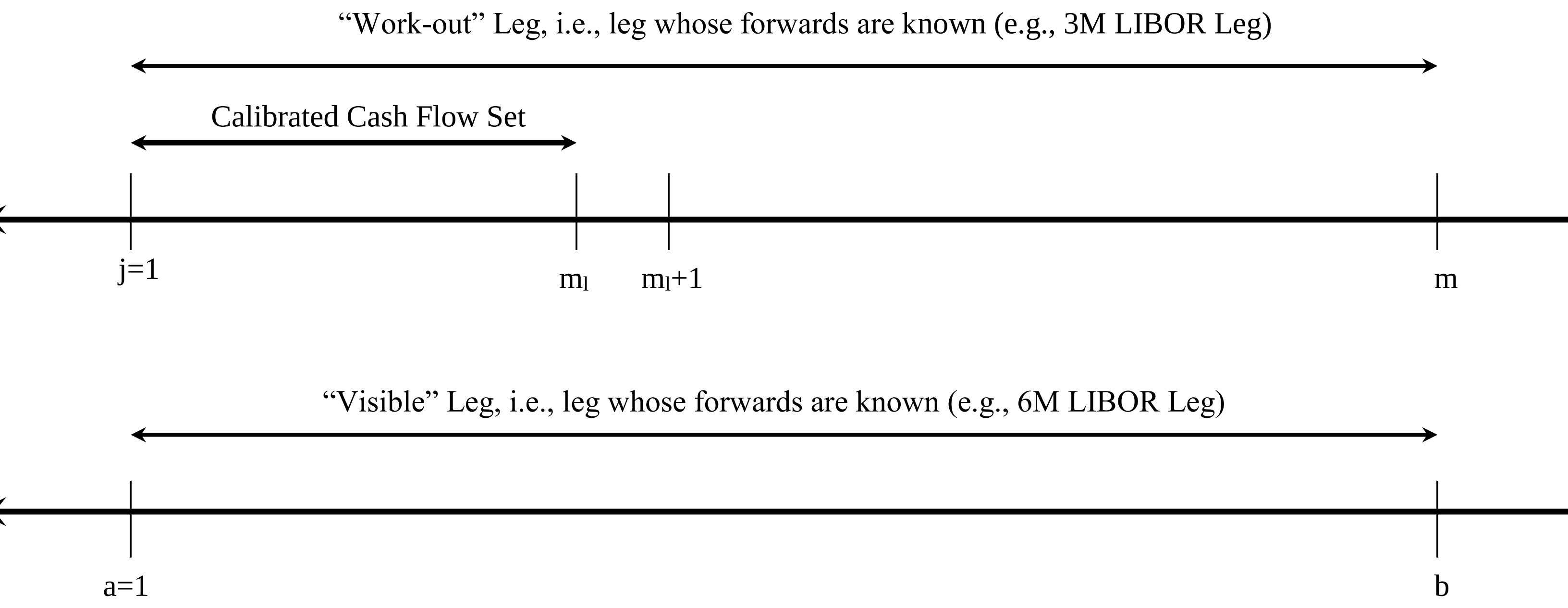
Figure 5: Float-Float Swap Set-up



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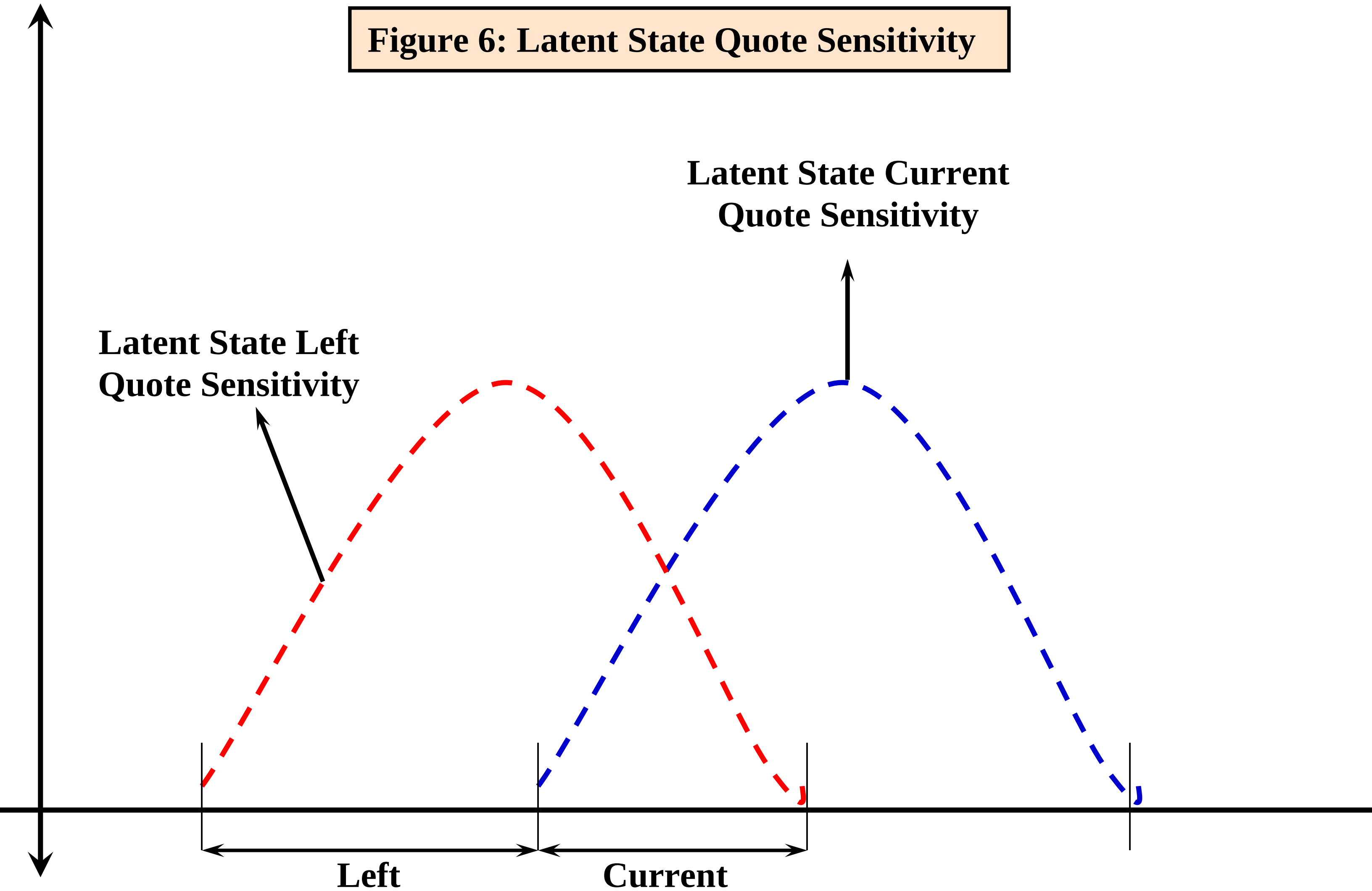

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Figure 7: Measure Change Setup

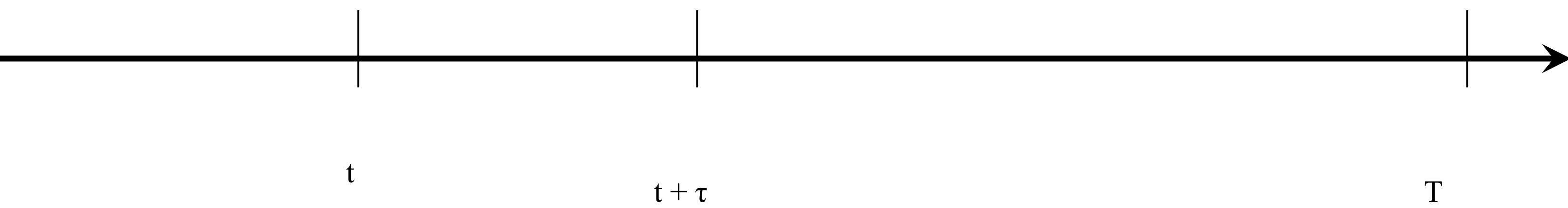

Figure 8: Optimal Intermediate Wengert Variable

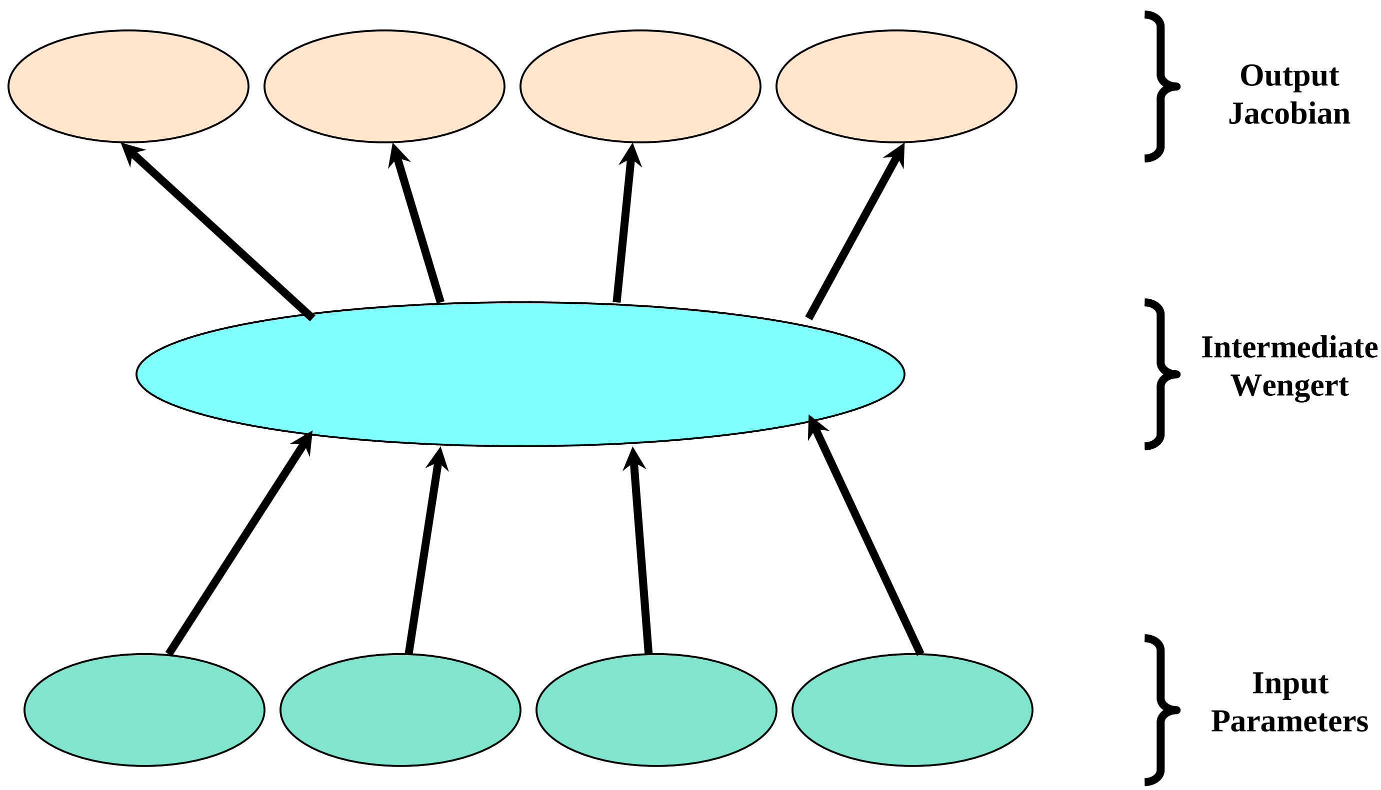

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Figure 9: Computation Financial Object Scheme

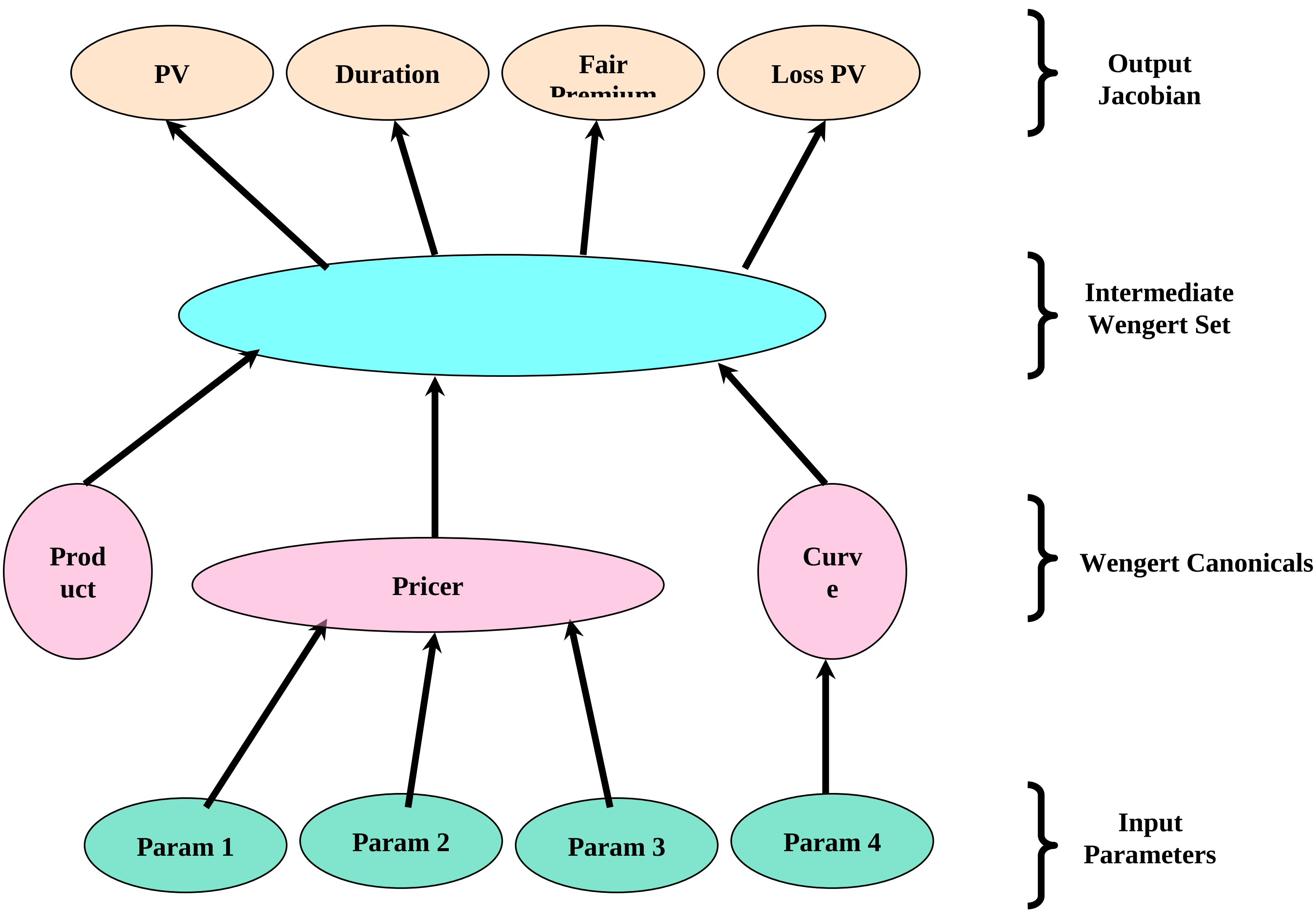

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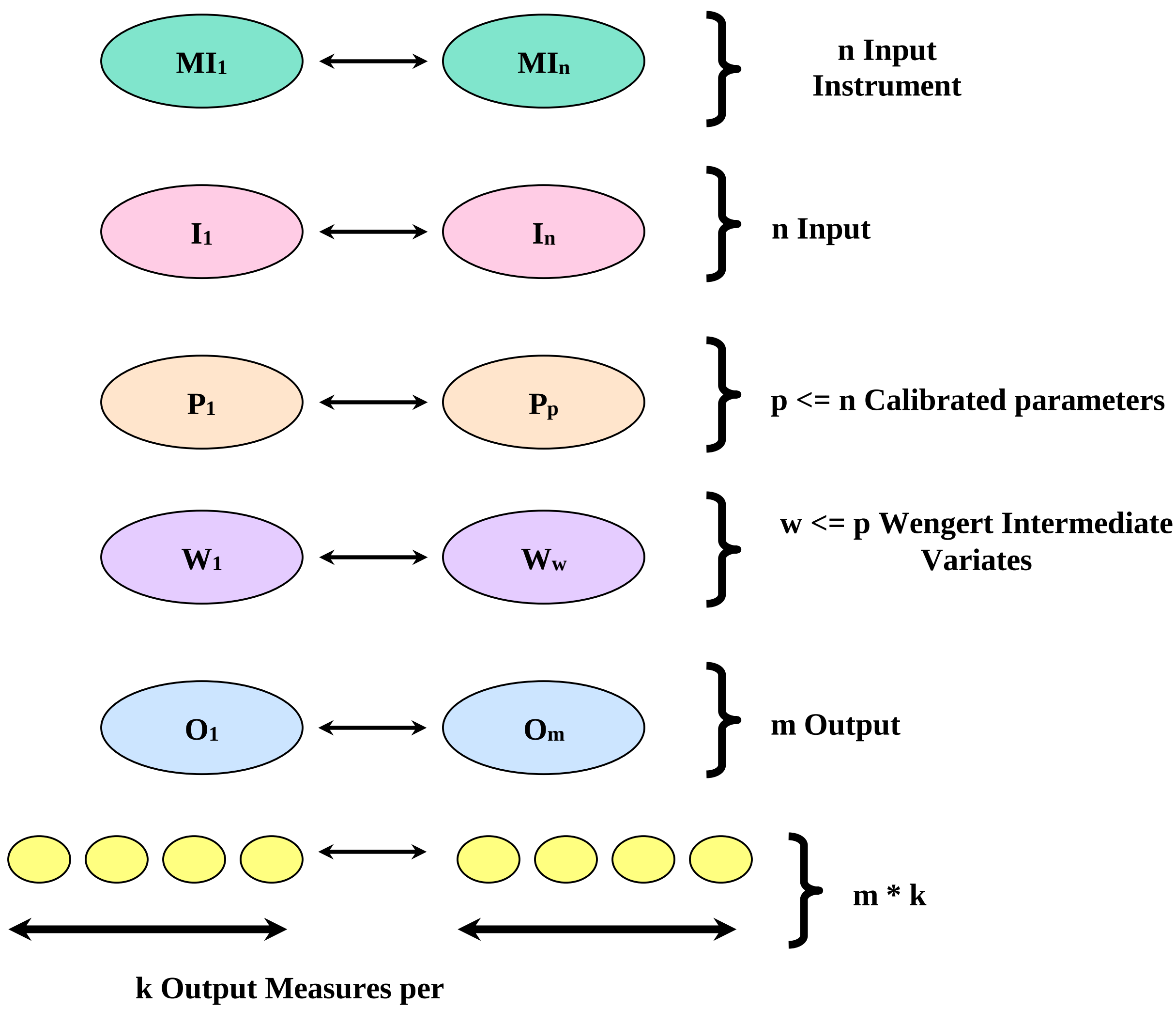
Figure 10: Wengert Fan-in and fan-out



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