MASTER of SCIENCE PROGRAM
Mathematics in Finance

NEW YORK UNIVERSITY
Courant Institute of Mathematical Sciences

Mathematics in Finance Working Paper Series

Optimal Trading in a Dynamic Market

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Working Paper #2009-2

May 26, 2009

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251 Mercer Street, New York, NY 10012
http://www.cims.nyu.edu/working_paper_series/

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Abstract

We consider the problem of mean-variance optimal agency execution strategies, when the market liquidity and volatility vary randomly in time. Under specific assumptions for the stochastic processes satisfied by these parameters, we construct a Hamilton-Jacobi-Bellman equation for the optimal cost and strategy. We solve this equation numerically and illustrate optimal strategies for varying risk aversion. These strategies adapt optimally to the instantaneous variations of market quality.

A fundamental part of agency algorithmic trading in equities and other asset classes is trade scheduling. Given a trade target, that is, a number of shares that must be bought or sold before a fixed time horizon, trade scheduling means planning how many shares will be bought or or sold by each time instant between the beginning of trading and the horizon.

Grinold and Kahn (1995) and Almgren and Chriss (2000) suggested that the optimal trajectory could be determined by balancing market impact cost, which leads toward slow trading, against volatility risk which pushes toward rapid completion of the order. This framework leads to an "efficient frontier," in which the trade schedule is selected from a one-parameter family based on a risk aversion parameter that must be specified by the trading client. While other factors such as anticipated price drift, serial correlation or other short-term signals, and daily patterns, are certainly important, this fundamental "arrival price" framework has proven remarkably robust and useful in designing practical trading systems.

A fundamental assumption of most of this work has been that the market parameters, such as liquidity and volatility, are constant or at least have known predictable profiles. This assumption is reasonably accurate for large-cap US stocks. Under that assumption, optimal strategies are *static;* that is, the trade schedule can be determined before trading starts and is not modified by the new information revealed by price moves during trading. (Almgren and Chriss (2000)

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did consider a model in which the market parameters updated at a single time to one of a known set of possible new values.)

Over the last few years, a major push of providers of algorithmic trading services has been to extend their functionality to smaller-cap stocks and other less liquid assets. (This author was closely involved with the design of the Instinct™ algorithm at Bank of America.) A distinguishing feature of these assets is that their liquidity and volatility vary randomly in time. That is, there will be times during the trading day when trading is very expensive, and times when trading is cheap; similarly there will be times when delaying trading introduces large amounts of volatility risk and other times when delay is relatively costless. The modeling challenge is to determine optimal strategies that adapt to the instantaneous market state, while retaining the mean-variance tradeoff inherent in the arrival price framework. Walia (2006) solved this problem in a discrete-time discrete-state model. This paper provides a systematic mathematical solution to this problem in continuous time and continuous state.

In Section 1 we present the basic price and impact models that we use, and we present the optimal trading problem. We also present the "coordinated variation" approximation in which liquidity and volatility vary together, which is very realistic and which greatly simplifies the mathematical problem. In Section 2 we use dynamic programming to construct a Hamilton-Jacobi-Bellman partial differential equation (PDE) describing the optimal cost function and trade rate. In Section 3 we describe some aspects of the numerical solution of this PDE and present example solutions.

1 The liquidation problem

The trader begins at time $t = 0$ with a purchase target of X shares, which must be completed by time $t = T$. The number of shares remaining to purchase at time t is the trajectory $x(t)$, with $x(0) = X$ and $x(T) = 0$. The rate of buying is $v(t) = -dx/dt$. Thus for a buy program, $X > 0$, and we expect $x(t) \geq 0$ decreasing and $v(t) \geq 0$ (a sell program may be modeled similarly). In general, the trajectory $x(t)$ may be determined depending on price motions and on market conditions that are discovered during trading, so it is a random variable.

The price $S(t)$ follows arithmetic Brownian motion

$$dS(t) = \sigma(t)\, dB(t), \qquad S(0) = S_0,$$

where $B(t)$ is a standard Brownian motion, and the instantaneous volatility $\sigma(t)$ depends on time either deterministically or randomly. Note that σ is an absolute volatility rather than fractional; it contains an implicit factor of the reference price S_0. It is possible to include permanent market impact into the price equation but it is not central to our problem.

The price actually received on each trade is

$$\tilde{S}(t) \;=\; S(t) \;+\; \eta(t)\, v(t)$$

where $\eta(t)$ is the coefficient of temporary market impact, also time-varying. Again, η is a absolute coefficient rather than fractional. We assume that both $\sigma(t)$ and $\eta(t)$ are observable.

We may address two broad classes of problems:

1. First is the case in which $\sigma(t)$ and $\eta(t)$ are known non-random functions of time. This would accomodate the well-known intraday profiles of liquidity and volatility: generally markets are more active in the morning and near the close than in the middle of the day. This case is not our primary focus.

2. The second case is when volatility and liquidity vary randomly through the day, so that $\sigma(t)$ and $\eta(t)$ follow some stochastic processes. This effect is very important in algorithmic trading of small- and medium-capitalization stocks and other assets that are less heavily traded than large-cap US stocks.

1.1 Cost of trading

The *cost of trading* is the total dollars paid to purchase X shares, relative to the initial market value $X S_0$:

$$\begin{aligned} C \;&=\; \int_0^T \tilde{S}(t)\, v(t)\, dt \;-\; X S_0 \\ &=\; \int_0^T \sigma(t)\, x(t)\, dB(t) \;+\; \int_0^T \eta(t)\, v(t)^2\, dt \end{aligned}$$

using $x(t) = \int_t^T v(s)\, ds$. The cost C is a random variable, both because of the price uncertainty $B(t)$ in the first term and because of the liquidity uncertainty. We determine the strategy $x(t)$ to tailor the properties of this random variable to meet some optimality criterion.

More generally, starting at time $t \ge 0$ with $x(t)$ shares remaining to purchase, the cost of a strategy $x(s)$ on $t \le s \le T$ is

$$C \;=\; \int_t^T \sigma(s)\, x(s)\, dB(s) \;+\; \int_t^T \eta(s)\, v(s)^2\, ds. \tag{1}$$

We determine the optimal trajectory by the mean-variance criterion

$$\min_{x(s):t\le s\le T} \mathbb{E}(C) \;+\; \lambda\, \mathrm{Var}(C)$$

where $\lambda \geq 0$ is a risk-aversion coefficient. Note that

$$\mathbb{E}(C) \;=\; \int_t^T \eta(s)\, v(s)^2\, ds$$

since the first term is an Itô integral, and

$$\mathrm{Var}(C) \;=\; \int_t^T \sigma(s)^2\, x(s)^2\, ds \;+\; \{\text{terms from uncertainty in } \eta(s) \text{ and } v(s)\} \qquad (2)$$

We shall argue below that the first term in (2) dominates the other terms.

1.2 Constant coefficients

The "classic" problem (Almgren and Chriss 2000) takes σ and η constant. Then for a strategy $x(t)$ that is fixed in advance and does not adapt to price motions,

$$\mathbb{E}(C) + \lambda\,\mathrm{Var}(C) \;=\; \int_t^T \Big(\eta\, v(s)^2 \;+\; \lambda\sigma^2 x(s)^2 \Big)\, ds. \qquad (3)$$

Using the calculus of variations to minimize this over trajectories $x(s)$ gives the second-order ODE

$$\frac{d^2x}{ds^2} \;=\; \kappa^2\, x(s), \qquad \text{with} \quad \kappa^2 = \frac{\lambda\sigma^2}{\eta}.$$

The solution is a combination of exponentials $\exp(\pm\kappa s)$

$$x(s) \;=\; x(t)\,\frac{\sinh\big(\kappa(T-s)\big)}{\sinh\big(\kappa(T-t)\big)}, \qquad v(s) \;=\; \kappa\, x(t)\,\frac{\cosh\big(\kappa(T-s)\big)}{\sinh\big(\kappa(T-t)\big)}.$$

Thus $1/\kappa$ is the characteristic time scale of liquidation.

This strategy may also be expressed as the rule for $v(t)$

$$v(t) \;=\; \kappa\, x(t)\, \coth\big(\kappa(T-t)\big). \qquad (4)$$

The cost function is

$$C(x,t,\eta,\sigma) \;=\; \eta\,\kappa\, x^2\, \coth\big(\kappa(T-t)\big) \;=\; \eta v\, x. \qquad (5)$$

The total cost is equal to the impact cost component (neglecting the volatility term) incurred by trading x shares at a price concession given by the instantaneous velocity v. The actual trajectory slows down as the position size decreases, thus reducing market impact costs, but the total cost includes volatility risk as well as impact cost, giving the above value.

The shape of the solution is governed by the nondimensional quantity $\kappa(T-t)$, the ratio of time remaining to the intrinsic time scale determined by the market parameters and the trader's risk aversion.

In the infinite-horizon limit $\kappa(T-t) \gg 1$, the strategy has the limit $x(s) = x(t)\exp\bigl(-\kappa(s-t)\bigr)$, with $v(t) = \kappa x(t)$, and the cost function $C \to \eta\kappa x^2$. Trading is substantially completed well before expiration, and the precise value of T is not controlling.

In the short-horizon limit $\kappa(T-t) \ll 1$, the strategy has the linear form

$$x(s) \;=\; x(t)\,\frac{T-s}{T-t}, \qquad v(s) \;=\; \frac{x(t)}{T-t}$$

and the cost function is essentially nonrandom with the value

$$C \;\sim\; \frac{\eta x^2}{T-t}, \qquad \kappa(T-t) \to 0. \tag{6}$$

If a time change is applied to match the market average profile, then this strategy is equivalent to a volume-weighted average price (VWAP) execution.

Whether adaptive strategies are better than fixed ones is a subtle question. Almgren and Chriss (2000) showed that if the strategy is re-evaluated at an intermediate time, using mean and variance measured at that time, then the optimal strategy is the remaining part of the initial strategy and hence the optimal strategy is fixed. That is the context in which we work here, since it is appropriate for dynamic programming. Almgren and Lorenz (2007) and Lorenz (2008) showed that adaptive strategies are optimal if mean and variance are measured at the initial time, for portfolios that are large enough so that their impact is a substantial fraction of volatility. This framework is appropriate for *ex post* measurement of historical mean and variance. Schied and Schöneborn (2009) showed that improvement from adaptivity depends on the risk aversion profile; for example, it vanishes for a CARA utility function.

1.3 Coordinated variation

Suppose $\sigma(t)$ and $\eta(t)$ vary perfectly inversely, so that

$$\sigma(t)^2\,\eta(t) \;=\; \text{constant} \;=\; \bar\sigma^2\,\bar\eta$$

where $\bar\sigma$ and $\bar\eta$ are constant reference values. For example, this relationship would be a natural consequence of a "trading time" model (Geman, Madan, and Yor 2001; Jones, Kaul, and Lipson 1994) in which the single source of uncertainty is the arrival rate of trade events. If each trade event brings both a fixed amount of price variance, and the opportunity to trade a fixed number of shares for a particular cost, then we obtain the above relation.

We may then change time to an artificial variable $\hat{t}(t)$ defined by

$$d\hat{t} \;=\; \sigma(t)^2\, dt.$$

In this time frame, we have a modified Brownian motion $\hat{B}(\hat{t})$ with

$$d\hat{B}(\hat{t}) \;=\; \sigma(t)\, dB(t).$$

The share holdings are the same trajectory at different times, so $\hat{x}(\hat{t}) = x(t)$. The trade rate is modified to

$$\hat{v}(\hat{t}) \;=\; -\frac{d\hat{x}}{d\hat{t}} \;=\; \frac{v(t)}{\sigma(t)^2}.$$

In terms of these new variables, the trading cost is

$$C \;=\; \int_0^{\hat{T}} \hat{x}(\hat{t})\, d\hat{B}(\hat{t}) \;+\; \bar{\sigma}^2\bar{\eta} \int_0^{\hat{T}} \hat{v}(\hat{t})^2\, d\hat{t}$$

where $\hat{T} = \hat{t}(T)$ is the time horizon in the changed variable. This is easy to solve in two cases:

1. If the time-varying liquidity and volatility have known non-random profiles, then we can compute the upper bound $\hat{T}$ explicitly. The problem reduces exactly to the constant-coefficient problem (3) and the solution is the exponential functions computed there. The rule (4) becomes

$$v(t) \;=\; \kappa(t)\, x(t)\, \coth\!\left(\frac{\kappa(t)}{\sigma(t)^2}\int_t^T \sigma(s)^2\, ds\right)$$

   where $\kappa(t) = \sqrt{\lambda\sigma(t)^2/\eta(t)}$ is the time scale formed with the instantaneous values of the parameters.

2. If the coefficients vary randomly, then the problem is not the same as the constant-coefficient problem, because of the uncertainty in the end time. But in the infinite-horizon case $T = \infty$, we have also $\hat{T} = \infty$ (under very mild assumptions on $\sigma(t)$). The trade rate is $v(t) = \kappa(t)x(t)$ and the cost $C = \eta(t)\kappa(t)x^2 = x^2\sqrt{\lambda\bar{\sigma}^2\bar{\eta}}$.

Somewhat surprisingly, the optimal cost in the coordinated-variation random-market infinite-horizon case does not depend on the instantaneous market state $\eta(t)$ and $\sigma(t)$, though the instantaneous rate of trading certainly does depend on the market state. In effect, since volatility is low whenever impact is high, the strategy is always able to wait for favorable market conditions without incurring very much risk from the delay.

Thus the interesting problems come from two sources:

- With non-random coefficients, the variation of the profiles of $\sigma(t)$ and $\eta(t)$ away from the "base case" $\sigma(t)^2\eta(t)$ = constant. See Kim and Boyd (2008) for a fuller discussion of optimal trading with general market profiles.
- With random coefficients, the proper handling of uncertainty as the end time is approached. For example, if liquidity is temporarily low, is it worthwhile to wait for a better opportunity, or is there a large risk that this opportunity will not come before expiration?

1.4 Rolling horizon approximate strategy

One way to determine a plausible strategy is to use the formula (4) to compute $v(t)$, using the instantaneous values of $\eta(t)$, $\sigma(t)$, and hence $\kappa(t)$. That is, we assume that the values observed at each instant will remain constant through the end of the liquidation period, and determine the statically optimal strategy using those values. When the values change, we recompute a stationary solution. This strategy is strictly optimal only in the infinite-horizon case, and only when the market parameters co-vary in the appropriate way. In general it is not optimal, but provides a reasonable solution that is easy to implement.

1.5 Small-impact approximation

In order to do dynamic programming when $\eta(t)$ and $\sigma(t)$ vary randomly, we need to approximate the variance term in (2). We approximate it as

$$\mathrm{Var}(C) \;\approx\; \mathbb{E}\int_t^T \sigma(s)^2\, x(s)^2\, ds.$$

That is, the variance comes primarily from the price volatility represented by σ, with lesser contributions from the uncertainty in $\sigma(t)$ and $\eta(t)$. In the language of Almgren and Lorenz (2007) and Lorenz (2008), this is a small value of the "market power:" the portfolio is small enough that price changes due to impact are small compared to volatility. Then we take as the value function

$$c(t,x,\eta,\sigma) \;=\; \min_{v(s),\, t\le s\le T} \mathbb{E}\int_t^T \Big(\lambda\sigma(s)^2 x(s)^2 \;+\; \eta(s)\, v(s)^2 \Big)\, ds.$$

From now on we shall make this approximation.

2 Dynamic Programming

Since $\eta(t)$ and $\sigma(t)$ are positive, it is convenient to use

$$\xi(t) \;=\; \log\frac{\eta(t)}{\bar\eta}, \qquad\qquad \zeta(t) \;=\; \log\frac{\sigma(t)}{\bar\sigma}$$

as state variables. Here $\bar\eta, \bar\sigma$ are typical values of σ and η, and $\xi(t), \zeta(t)$ are nondimensional values that fluctuate around zero. We shall also write $\bar\kappa = \sqrt{\lambda\bar\sigma^2/\bar\eta}$ for the intrinsic time scale in the mean market state.

We take $\xi(t)$ and $\zeta(t)$ to evolve according to some stochastic differential equations (SDE) of the forms

$$d\xi = a_\xi\, dt + b_\xi\, dB_{\mathrm{L}} \qquad \text{and} \qquad d\zeta = a_\zeta\, dt + b_\zeta\, dB_{\mathrm{V}}, \tag{7}$$

where a_ξ, b_ξ, a_ζ, and b_ζ are coefficients whose values may depend on ξ and ζ. $B_{\mathrm{L}}(t)$ and $B_{\mathrm{V}}(t)$ are Brownian motions, independent from the process $B(t)$ driving the price process but possibly correlated with each other, with $\mathbb{E}(dB_{\mathrm{L}}\, dB_{\mathrm{V}}) = \rho\, dt$.

Then by dynamic programming we write

$$c(t,x,\xi,\zeta) = \min_v \Big[\; \lambda\sigma^2 x^2 dt \;+\; \eta v^2 dt \;+\; \mathbb{E} c\big(t+dt, x+dx, \xi+d\xi, \zeta+d\zeta\big) \;\Big]$$

giving the PDE (subscripts on c denote partial derivatives)

$$\begin{aligned} 0 \;=\; c_t \;+\; \lambda\sigma^2 x^2 \;+\; \min_v \Big[\; \eta v^2 \;-\; v c_x \;\Big] & \\ +\; a_\xi c_\xi + a_\zeta c_\zeta \;+\; \frac{1}{2} b_\xi^2 c_{\xi\xi} + \rho b_\xi b_\zeta c_{\xi\zeta} + \frac{1}{2} b_\zeta^2 c_{\zeta\zeta}. & \end{aligned}$$

The minimum is clearly

$$v \;=\; \frac{c_x}{2\eta} \tag{8}$$

and then the PDE for c is

$$-c_t \;=\; \lambda\sigma^2 x^2 \;-\; \frac{c_x^2}{4\eta} \;+\; a_\xi c_\xi + a_\zeta c_\zeta \;+\; \frac{1}{2} b_\xi^2 c_{\xi\xi} + \rho b_\xi b_\zeta c_{\xi\zeta} + \frac{1}{2} b_\zeta^2 c_{\zeta\zeta}$$

Near expiration, we must liquidate on a linear trajectory $v = x/(T-t)$. The cost comes entirely from market impact in the market conditions at that time, since volatility risk is negligible across a short time. Thus (6) applies and

$$c(t,\xi,\zeta) \;\sim\; \frac{\eta x^2}{T-t} \;=\; \frac{\bar\eta e^\xi x^2}{T-t}, \qquad T-t \to 0.$$

To nondimensionalize the cost function and the differential equation, we need to define a time scale, which will also help us to define a cost scale; the market parameters are already nondimensionalized by their mean values. So far, the only two time scales are the intrinsic liquidation time in the meam market state $1/\bar{\kappa}$, and the imposed horizon T. Since both of these depend on the trader's preferences for a particular trade order, it will be more natural to use a time scale based on market dynamics.

2.1 Lognormal model

We now assume that $\xi(t)$ and $\zeta(t)$ evolve according to Ornstein-Uhlenbeck mean-reverting processes of mean zero. Thus we set

$$a_\xi(\xi) = -\frac{\xi}{\delta_\mathrm{L}} \qquad b_\xi = \frac{\beta_\mathrm{L}}{\sqrt{\delta_\mathrm{L}}} \tag{9}$$

$$a_\zeta(\zeta) = -\frac{\zeta}{\delta_\mathrm{V}} \qquad b_\zeta = \frac{\beta_\mathrm{V}}{\sqrt{\delta_\mathrm{V}}}. \tag{10}$$

Here δ_L and δ_V are relaxation times for liquidity and volatility, and β_L and β_V are nondimensional "burstiness" parameters. In steady state $\xi(t)$ and $\zeta(t)$ are normal, with

$$\begin{aligned} \mathbb{E}\big(\xi(t)\big) &\to 0, & \mathbb{E}\big(\zeta(t)\big) &\to 0, \\ \mathrm{Var}\big(\xi(t)\big) &\to \frac{1}{2}\beta_\mathrm{L}^2, & \mathrm{Var}\big(\zeta(t)\big) &\to \frac{1}{2}\beta_\mathrm{V}^2 \end{aligned}$$

as $t \to \infty$. Thus β_L and β_V describe the dispersion of liquidity and volatility around their average levels.

We may then nondimensionalize using δ_L as the time scale. That is, we define

$$\tau = \frac{T-t}{\delta_\mathrm{L}} \qquad \text{and set} \qquad c(t,x,\xi,\zeta) = \frac{\bar{\eta}\, x^2}{\delta_\mathrm{L}}\, u\left(\frac{T-t}{\delta_\mathrm{L}}, \xi, \zeta\right),$$

where $u(\tau,\xi,\zeta)$ is a nondimensional function of nondimensional variables, with

$$\begin{aligned} u_\tau + \xi u_\xi + \mu\,\zeta u_\zeta = {} & K^2 e^{2\zeta} - e^{-\xi} u^2 \\ & + \frac{1}{2}\beta_\mathrm{L}^2 u_{\xi\xi} + \rho\sqrt{\mu}\,\beta_\mathrm{L}\beta_\mathrm{V} u_{\xi\zeta} + \frac{1}{2}\mu\beta_\mathrm{V}^2 u_{\zeta\zeta}. \end{aligned} \tag{11}$$

We write $\mu = \delta_\mathrm{L}/\delta_\mathrm{V}$ and the nondimensional risk aversion parameter is

$$K = \bar{\kappa}\,\delta_\mathrm{L} = \frac{\text{relaxation time of market liquidity}}{\text{trade time scale in mean market state}}.$$

Of course, τ is the time remaining to expiration, measured as a multiple of the market relaxation time.

The initial condition is

$$u(\tau,\xi,\zeta) \sim \frac{e^{\xi}}{\tau}, \qquad \tau \to 0.$$

From (8), the dimensional trade velocity is

$$v = \frac{x}{\delta_{\mathrm{L}}} e^{-\xi} u(\tau,\xi,\zeta). \tag{12}$$

Constant market The steady-state market takes $\beta_{\mathrm{L}} = \beta_{\mathrm{V}} = 0$. Along the line $\xi = \zeta = 0$, the PDE (11) reduces to the ODE

$$u_\tau = K^2 - u^2, \qquad \text{with} \qquad u(\tau) \sim \frac{1}{\tau} \quad \text{as } \tau \to 0,$$

whose solution is

$$u(\tau) = K \coth K\tau.$$

On undoing the changes of variables, this reduces exactly to (5).

To generalize the above solution, we consider the limit $\xi \to -\infty$ and $\zeta \to \infty$. That is, market impact is temporarily very small, and volatility is very large: the optimal strategy is to trade very quickly. Since the market relaxation time scales are fixed, fast trading means that the program is completed before the market parameters have time to change. Thus the cost is the static cost (5) using the instantaneous market parameters, which in the transformed functions becomes

$$u(\tau,\xi,\zeta) \sim K \exp\Big(\zeta + \frac{1}{2}\xi\Big) \coth\Big(K \exp\Big(\zeta - \frac{1}{2}\xi\Big)\,\tau\Big), \qquad \xi \to -\infty,\ \zeta \to +\infty. \tag{13}$$

The corresponding trade rate is the "rolling horizon" strategy of Section 1.4, which is always an admissible though suboptimal strategy. The expression (13) accurately describes the optimal cost only in the indicated limit, when indeed the market coefficients do not change substantially before trading is completed.

2.2 Coordinated variation

Rather than solve the full PDE (11) in two "space" dimensions plus time, we can get more insight by using the coordinated variation model described above. Thus we make the following assumptions on the stochastic processes:

- The time scales of liquidity and volatility are equal: $\delta_{\mathrm{L}} = \delta_{\mathrm{V}} = \delta$ so $\mu = 1$.

- The Brownian motions driving liquidity and volatility have perfect positive correlation: $\rho = 1$.
- For now, the fluctuation magnitudes β_{L} and β_{V} are arbitrary. We set

$$\gamma \;=\; -\frac{2\beta_{\mathrm{V}}}{\beta_{\mathrm{L}}} \;=\; \frac{\text{signed fractional variation of } \sigma^2}{\text{signed fractional variation of } \eta}.$$

so that the coordinated variation case takes $\gamma = 1$. We shall assume $\gamma > 0$.

Then from (7,9,10),

$$d\Big(\beta_{\mathrm{V}}\xi - \beta_{\mathrm{L}}\zeta\Big) \;=\; -\Big(\beta_{\mathrm{V}}\xi - \beta_{\mathrm{L}}\zeta\Big)\frac{dt}{\delta}$$

and hence after at most an initial transient, the solutions satisfy

$$\beta_{\mathrm{V}}\,\xi \;-\; \beta_{\mathrm{L}}\,\zeta \;=\; 0 \qquad \text{or} \qquad \zeta = -\frac{\gamma}{2}\,\xi.$$

In the PDE (11)

$$u_\tau \;+\; \Big(\xi\,\partial_\xi + \zeta\,\partial_\zeta\Big)\,u \;=\; K^2\,e^{2\zeta} \;-\; e^{-\xi}u^2 \;+\; \frac{1}{2}\Big(\beta_{\mathrm{L}}\partial_\xi + \beta_{\mathrm{V}}\partial_\zeta\Big)^2 u, \tag{14}$$

we may neglect the cross-variation[1] and the PDE for $u(\tau,\xi)$ is (with $\beta = \beta_{\mathrm{L}}$)

$$u_\tau \;+\; \xi u_\xi \;=\; K^2\,e^{-\gamma\xi} \;-\; e^{-\xi}u^2 \;+\; \frac{1}{2}\beta^2 u_{\xi\xi}. \tag{15}$$

The initial condition is still $u(\tau,\xi) \sim e^\xi/\tau$ as $\tau \to 0$.

2.3 Asymptotic behavior

Short time To study the behavior as $\tau \to 0$ for fixed ξ, we write

$$u(\tau,\xi) \;\sim\; \frac{e^\xi}{\tau} \;+\; u_0(\xi) \;+\; \tau\,u_1(\xi) \;+\; \cdots, \qquad \tau \to 0. \tag{16}$$

[1] To trace the change of variables in detail, introduce $w(\tau,\xi,\chi)$ with

$$u(\tau,\xi,\zeta) \;=\; w(\,\tau,\;\xi,\;\beta_{\mathrm{L}}\zeta - \beta_{\mathrm{V}}\xi\,)$$

so that

$$\xi u_\xi + \zeta u_\zeta = \xi w_\xi + \chi w_\chi \qquad \text{and} \qquad (\beta_{\mathrm{L}}\partial_\xi + \beta_{\mathrm{V}}\partial_\zeta)^2 u = \beta_{\mathrm{L}}^2 w_{\xi\xi}.$$

Thus

$$w_\tau \;+\; \Big(\xi\,\partial_\xi + \chi\,\partial_\chi\Big)\,w \;=\; K^2 e^{2\zeta} - e^{-\xi}\,w^2 + \tfrac{1}{2}\,\beta_{\mathrm{L}}^2\,w,$$

and on the plane $\chi = 0$ this reduces to (15).

We find at $\mathcal{O}(1/\tau)$

$$u_0(\xi) \;=\; -\frac{1}{2}\left(\xi - \frac{1}{2}\beta^2\right)e^{\xi}, \tag{17}$$

and at $\mathcal{O}(1)$

$$3u_1 = K^2\,e^{-\gamma\xi} - e^{-\xi}u_0^2 - \xi u_0' + \frac{1}{2}\beta^2 u_0''$$

or

$$u_1(\xi) \;=\; \frac{1}{3}K^2 e^{-\gamma\xi} + \frac{1}{12}\,e^{\xi}\left(\left(\xi + 1 - \frac{1}{2}\beta^2\right)^2 - (1+\beta^2)\right). \tag{18}$$

Thus $u(\xi,\tau)$ has a regular expansion in powers of τ. This reassures us that the singular initial data is indeed enough to define the solution.

Long time As $\tau \to \infty$, presumably there is a steady state cost and strategy in which the time horizon is not controlling. The steady solution $u(\xi)$ will be determined by the nonlinear second order ODE

$$\xi\,u' \;=\; K^2\,e^{-\gamma\xi} \;-\; e^{-\xi}\,u^2 \;+\; \frac{1}{2}\beta^2 u''. \tag{19}$$

In the coordinated variation case $\gamma = 1$, this has the constant solution $u(\xi) = K$. For $\gamma \neq 1$ we cannot give an explicit solution, but for $\gamma > -1$ we can identify the asymptotic behavior $u(\xi) \sim K\exp(-\frac{1}{2}(\gamma-1)\xi)$ as $\xi \to -\infty$, based on the balance $0 = K^2 e^{-\gamma\xi} - e^{-\xi}u^2$.

Liquid market As $\xi \to -\infty$, with $\gamma > 0$ we also have $\zeta \to \infty$ and thus the asymptotic solution (13) is valid and simplifies to

$$u(\tau,\xi) \;\sim\; K\,e^{-\frac{1}{2}(\gamma-1)\xi}\coth\!\left(K\,e^{-\frac{1}{2}(\gamma+1)\xi}\tau\right), \qquad \xi \to -\infty. \tag{20}$$

At leading order, this is consistent with both the short-time and the long-time behavior above.

3 Numerical solution

Since we are unable to give explicit analytic solutions, we resort to numerical calculations to give solutions of the PDE (15) for a range of relevant parameters:

- The coordination parameter γ must be chosen as part of the model structure. Since we have no particular reason to choose other values, we shall always consider $\gamma = 1$. (Similarly, we will not illustrate numerical solutions of the two-variable problem (11), since a truly complete study would also consider a broader range of market dynamic models.)

- The burstiness parameter β is stock-specific. A large-cap stock will have β near zero, for a near-uniform profile. A small-cap stock will have $\beta = 1$ or larger. For our example calculations, we will fix $\beta = 2$, a relatively large value to better illustrate the effects of market variation.
- The risk-aversion parameter K must range across nonnegative values, since the actual choice of trajectory will be determined by the trader's risk preference. Values of K smaller than unity are the most realistic, so that the algorithm has time to adapt to at least one market reversion time.

We briefly discuss a few technical issues with time and space discretization, and then present example solutions.

3.1 Time discretization

The first obstacle is that the initial data is given as singular behavior. A simple modification to the forward Euler scheme handles this problem. We illustrate this modification using an ordinary differential equation (ODE).

Consider

$$\frac{du}{dt} = -(u-a)(u-b), \qquad u(t) \sim \frac{1}{t} \text{ as } t \to 0, \tag{21}$$

whose exact solution is

$$u(t) = \frac{ae^{-bt} - be^{-at}}{e^{-bt} - e^{-at}}. \tag{22}$$

Either by expanding this solution, or directly from the ODE, we determine the local expansion

$$u(t) \sim \frac{1}{t} + \frac{1}{2}(a+b) + \frac{t}{12}(a-b)^2 + \cdots, \qquad t \to 0 \tag{23}$$

For the numerics, we apply a forward Euler scheme to $w(t) = tu(t)$, which is regular near $u = 0$. With $w'(t) = tu'(t) + u(t)$ and denoting $u_n \approx u(t_n)$, this gives

$$u_{n+1} = u_n + \frac{t_n}{t_{n+1}}(t_{n+1} - t_n)\, u_n'. \tag{24}$$

Thus we apply a correction to the Euler update formula, which becomes small as we move away from the initial singular time and $t_n/t_{n+1} \to 1$.

We test on a regular grid with $t_n = n\,\Delta t$, starting at $n = k \geq 1$. We choose k to satisfy the stability condition for the forward Euler scheme. For an ODE $u_t = f(u)$, we require $\Delta t < 1/|f'(u)|$. In this case, $f'(u) \sim 2u \sim 2/t$, so we need $\Delta t < t/2$ or $t > 2\,\Delta t$. We thus expect the scheme should be stable for $k \geq 2$.

We explore four cases, given by all combinations of two parameters.

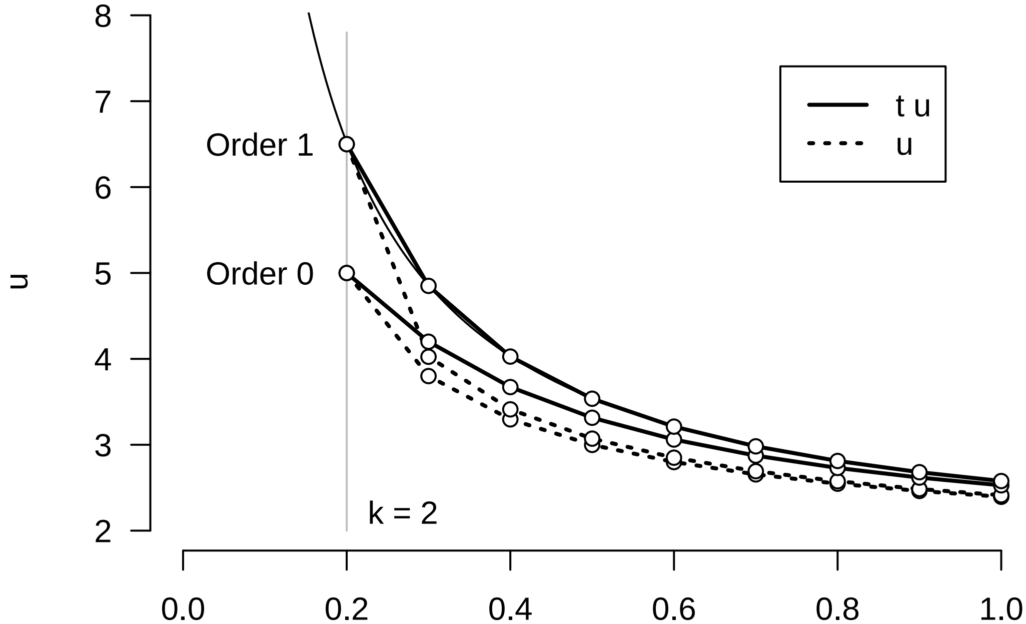

Figure 1: Numerical solution of the ODE (21) with $a = 2$ and $b = 1$, discretized with $\Delta t = 0.1$ beginning at the second step $k = 2$. The light curve is the exact solution (22). "Order 0" initial data uses only the given initial condion $u(t) \sim 1/t$; "Order 1" initial data adds the first correction term from (23). The dashed curves ("`u`") apply a forward Euler scheme directly to $u(t)$, the solid curves ("`t u`") apply a forward Euler scheme to $tu(t)$ as in (24).

1. Discretization scheme:
   (a) Forward Euler applied directly to u (dashed lines "`u`" in Figure 1), or
   (b) Forward Euler applied to tu as above (solid lines "`t u`").
2. Data at the first time step:
   (a) $u_k = (1/t_k)$ using the given initial condition ("`Order 0`"), or
   (b) $u_k = (1/t_k) + \frac{1}{2}(a + b)$ using the local expansion (23) ("`Order 1`").

Figure 1 shows example solutions. The combination of improved initial data, with a time discretization that takes account of the initial singularity, yields far more accurate results than naïve discretization.

3.2 Space discretization

We use the standard 3-point discretization for the diffusion term, and upwind differencing for the convection term. We use forward Euler time discretization with the correction above; thus we use a small time step for stability. For initial data we use the asymptotic expression (16,17,18) at an initial time $t = k\,\Delta t$.

It is more convenient to discretize $v(\tau,\xi) = e^{-\xi}u(\tau,\xi)$ rather than u directly; by (12) this is the instantaneous trade rate except for the dimensional factor. The PDE for v is easily derived from (15).

We use a finite spatial domain $-\Xi \le \xi \le \Xi$. At the left boundary $\xi = -\Xi$, we use the far-field solution (20). At the right boundary $\xi = \Xi$ we use "natural" boundar conditions $v_{\xi\xi} = 0$. Since the convective term is flowing outwards, the effect of the boundary conditions is confined to a narrow boundary layer.

3.3 Example solutions

Figure 2 shows the computed solution of the PDE for $\gamma = 1$, $\beta = 1$, and $K = 0.1$ (the computational grid is much finer than what is plotted). As noted above, the quantity that is computed and plotted is not the value function u, but $e^{-\xi}u$, the dimensionless trade rate as a fraction of the shares remaining (we plot its natural log). As expected, when τ is small, the trade rate becomes large like $1/\tau$. When ξ is large positive, market impact is high and volatility is low, so the optimal strategy trades very slowly except near expiration. When ξ is large negative, market impact is low and volatility is high, so the optimal strategy trades rapidly. As $\tau \to \infty$ the solution approaches the steady state $\log(e^{-\xi}u) = \log K - \xi$.

Figure 3 shows a realization of the market state process $\xi(t)$ used for the trajectory simulations. With the "coordinated variation" approximation the market moves back and forth between a high-activity regime with low impact and high volatility (small ξ) and a low-activity regime with high impact and low volatility (large ξ). We have taken $\beta = 1$ so that the root-mean square fluctation of $\xi(t)$ is $\frac{1}{2}$. The mean-reversion time $\delta = 1$, so that with a time $T = 10$ we experience several market cycles.

Figure 4 show optimal trading trajectories for risk-aversion parameters $K =$ 0.1, 0.3, and 1, compared to the nonadaptive trajectories computed in the mean market state. The dynamic response is very clear. For example, around $t = 1$ the market state is poor, so all the trajectories trade slowly and fall behind the static curves. Around $t = 1.5$ there is a brief burst of liquidity, and all the strategies accelerate in response. The trajectories with lower urgency (higher on the plots) have more shares remaining to trade so they are able to react more than the high-urgency trajectory (lower on the plots) which has completed a substantial fraction of its goal by that time. The lowest urgency strategy continues to adapt and is able to benefit from the large and prolonged liquidity burst around $t = 7$.

The dotted lines in Figure 4 show the "rolling horizon" strategy of Section 1.4. For large risk-aversion (fast trading), this approximate strategy is almost identical to the optimum. For smaller risk-aversion (slow trading), the rolling horizon strategy almost rigidly follows a straight-line trajectory, while the true optimum is able to adapt to varying market state even when its profile is nearly linear. In general, the rolling horizon strategy seems to be an adequate approximation when risk aversion is relatively high.

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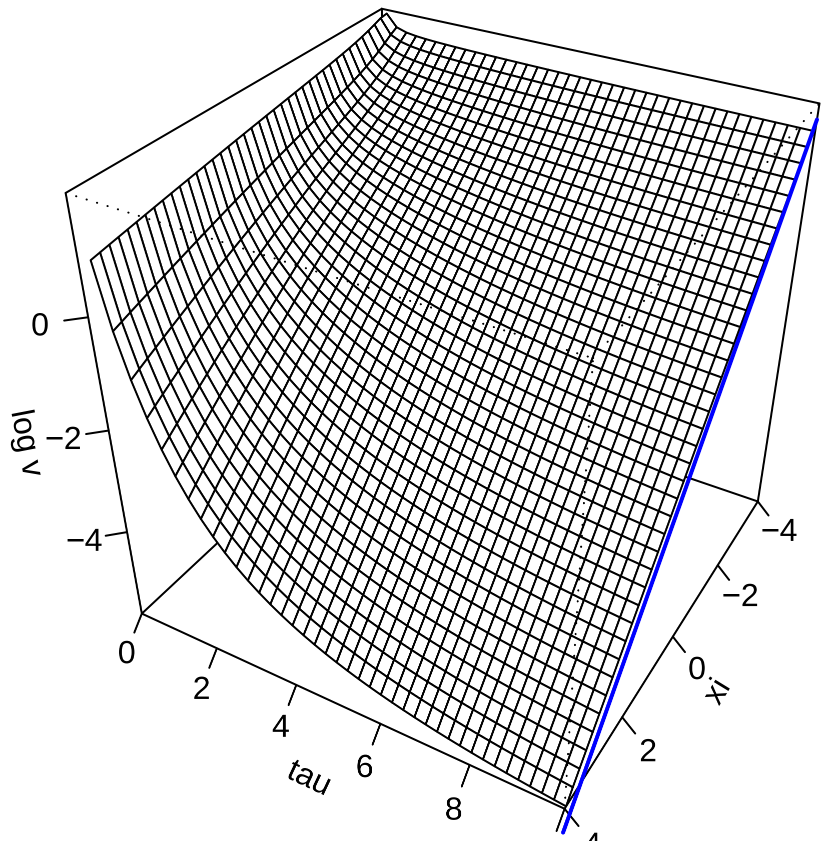

Figure 2: Solution of the PDE for $\gamma = 1$, $K = 0.1$, and $\beta = 1$. The horizontal axes are the nondimensional time remaining to expiration τ and the market state ξ; the quantity plotted is the natural log of the instantaneous trade rate as a fraction of shares remaining. The thick line is the long-time asymptotic solution. When the market is liquid and volatile (ξ large negative), trading is fast; when the market is illiquid and nonvolatile (ξ large positive), trading is slow. As expiration approaches ($\tau \to 0$), trading is forced to be fast.

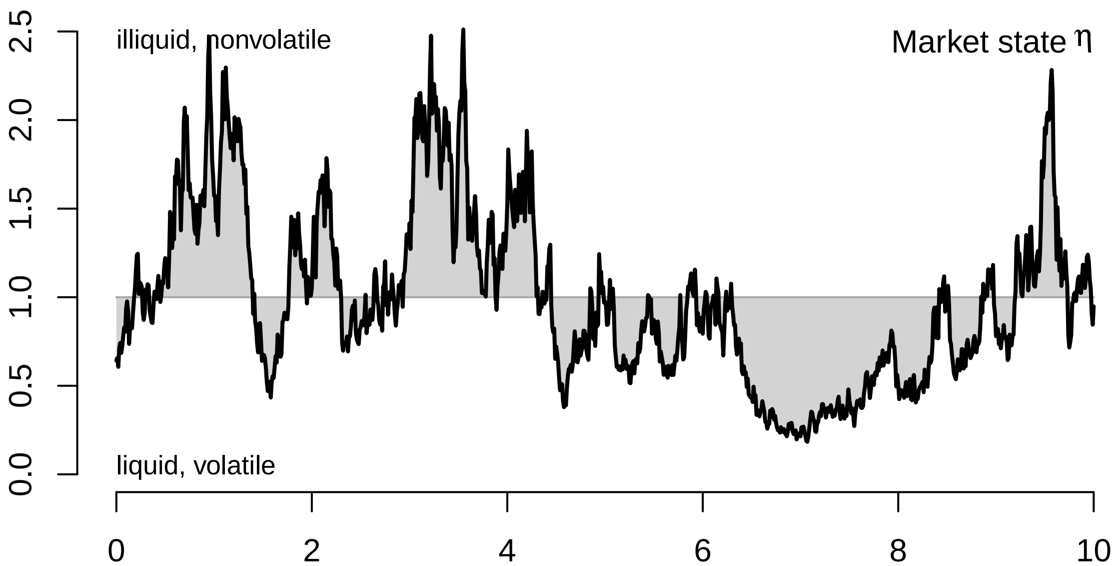

Figure 3: Realization of the market state trajectory used for the example numerical simulations, with $\beta = 1$. The horizontal axis is time measured in units of the mean reversion time. The value plotted is the market impact coefficient $\eta(t)$, so large values mean an illiquid market. In our simplified "coordinated variation" market assumption, the volatility $\sigma(t)$ varies inversely with $\eta(t)$: the market is either liquid and volatile, or illiquid and nonvolatile.

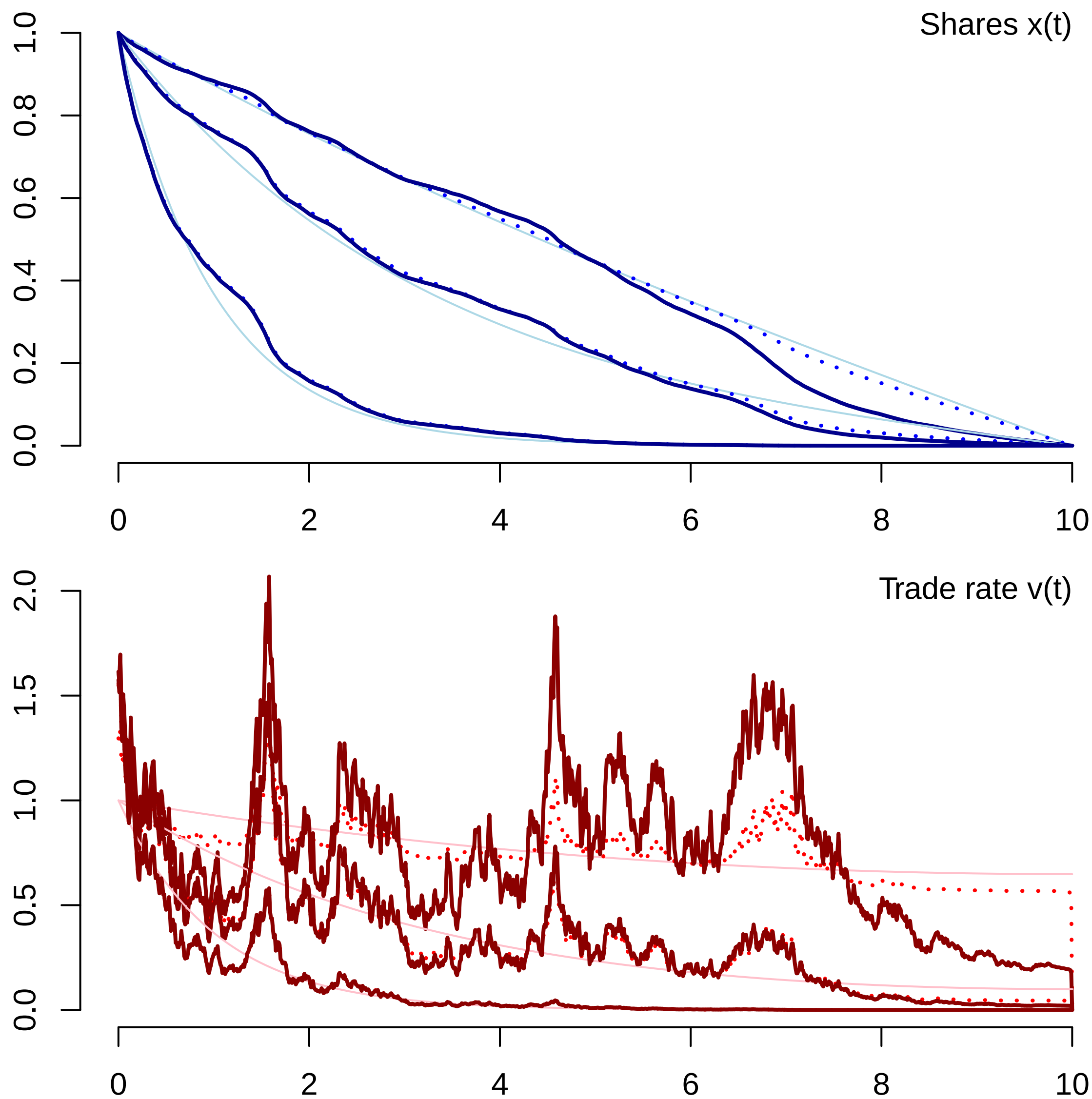

Figure 4: Optimal trajectories for $\lambda = 0.1$ (highest curves), 0.3, and 1 (lowest curves). The horizontal axis is time t forward from the initial time, measured in units of the market mean reversion time: the program must be completed by $t = T = 10$. Upper panel is shares remaming; lower panel is trade rate. Light lines are the non-adaptive solution in the mean market state; dotted lines are the "rolling horizon" approximation. In the lower panel, the trade rate is normalized so that each non-adaptive solution has the same initial trade rate.