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Option Basics

Contents

Time-Dependent Black-Scholes Formula

Let's begin with re-discovering the BSM formula. Only this time, we assume time-dependent parameters.

Assume under EMM measure, stock price follows SDE

$$dS_t = r_t S_t dt + \sigma_t S_t dW_t$$

Here risk-free rate and volatility are deterministic (but not constant) functions of time t. The bank account follows an ODE

$$dB_t = r_t B_t dt \Rightarrow B_t = \exp\left(\int_0^t r_t dt\right)$$

Now suppose we are at time 0. Use Ito lemma on lnS_t,

$$dlnS_t = \frac{1}{S_t}dS_t - \frac{1}{2S_t^2}\langle dS_t, dS_t\rangle = \left(r_t - \frac{1}{2}\sigma_t^2\right)dt + \sigma_t dW_t$$

$$\Rightarrow lnS_t - lnS_0 = \int_0^t \left(r_t - \frac{1}{2}\sigma_t^2\right)dt + \int_0^t \sigma_t dW_t$$

In particular, at time T, lnS_t is normally distributed, or S_t is log-normally distributed,

$$lnS_t \sim N\left(lnS_0 + \int_0^T \left(r_t - \frac{1}{2}\sigma_t^2\right)dt, \int_0^T \sigma_t^2 dt\right)$$

This is reached by using the fact

$$Var\left(\int_0^T \sigma_t dW_t\right) = \int_0^T \sigma_t^2 dt$$

Once we know the probability density function (pdf) $\varphi(S_T)$ of S_T, a European call with strike K is priced at

$$C(K) = E\left[e^{-\int_0^T r_t dt}(S_T - K)^+\right] = e^{-\int_0^T r_t dt}\left\{\int_K^{+\infty}(S_T - K)\varphi(S_T)\,\mathrm{d}S_T\right\}$$

$$= e^{-\int_0^T r_t dt}\left\{e^{lnS_0 + \int_0^T r_t dt} N\left(\frac{lnS_0 + \int_0^T r_t dt + \frac{1}{2}\int_0^T \sigma_t^2 dt - lnK}{\sqrt{\int_0^T \sigma_t^2 dt}}\right) - KN\left(\frac{lnS_0 + \int_0^T r_t dt - \frac{1}{2}\int_0^T \sigma_t^2 dt - lnK}{\sqrt{\int_0^T \sigma_t^2 dt}}\right)\right\}$$

where to reach the second line, partial expectation formula of lognormal distribution (Wikipedia:Lognormal distribution) is used.

By defining

$$d_1 = \frac{ln\left(\frac{S_0}{K}\right) + \int_0^T r_t dt + \frac{1}{2}\int_0^T \sigma_t^2 dt}{\sqrt{\int_0^T \sigma_t^2 dt}}$$

$$d_2 = \frac{ln\left(\frac{S_0}{K}\right) + \int_0^T r_t dt - \frac{1}{2}\int_0^T \sigma_t^2 dt}{\sqrt{\int_0^T \sigma_t^2 dt}}$$

The call option price can be re-written as

$$C = S_0 N(d_1) - Ke^{-\int_0^t r_t dt}(d_2) \qquad (1)$$

It is the Black-Scholes formula where r and σ are replaced respectively by

$$\bar{r} = \frac{1}{T}\int_0^T r_t dt \text{ and } \bar{\sigma} = \sqrt{\frac{1}{T}\int_0^T \sigma_t^2 dt}$$

$\bar{\sigma}$ is referred to as the **root-mean-squared volatility** (Rebonato 2004).

Buying a call while simultaneously selling a put with the same strike is equivalent to buying a stock while borrowing $Ke^{-\int_0^T r_t dt}$. This observation leads to **put-call parity**

$$C - P = S - Ke^{-\int_0^T r_t dt} \qquad (2)$$

From equations (1) and (2), the price of a put is given by

$$P = N(-d_2)Ke^{-\int_0^T r_t dt} - N(-d_1)S \qquad (3)$$

Back out time-dependent volatility from implied volatility

That is to say, to calibrate time-dependent Black-Scholes model to the market observed implied volatility term structure $\sigma_{imp}(T)$ observed at time 0. Usually in this case $\sigma_{imp}(T)$ stands for at-the-money (ATM) volatility term structure. In other words,

$$\bar{\sigma}(T) = \sqrt{\frac{1}{T}\int_0^T \sigma_t^2 dt} = \sigma_{imp}(T) \Rightarrow \int_0^T \sigma_t^2 dt = T\sigma_{imp}(T)^2$$

$$\overset{\frac{\partial}{\partial T}}{\Rightarrow} \sigma_T^2 = \sigma_{imp}(T)^2 + 2T\sigma_{imp}(T)\frac{\partial \sigma_{imp}(T)}{\partial T}$$

$$\Rightarrow \sigma_T = \sqrt{\sigma_{imp}(T)^2 + 2T\sigma_{imp}(T)\frac{\partial \sigma_{imp}(T)}{\partial T}}$$

In theory, this will give volatility at any time T. In practice, because we don't have continuous-time (and differentiable) implied volatility quotes, some additional assumptions and interpolation methods (such as piecewise constant) are needed in order to back out the time-dependent volatility model parameter.

Black-76 Model

Let's continue to assume that under risk-neutral measure, the stochastic process of stock price is

$$dS_t = r_t S_t dt + \sigma_t S_t dW_t$$

From no-arbitrage argument, the forwards price must satisfy

$$F(t,T) = S_t e^{\int_t^T r_t dt}$$

Then, from Ito lemma

$$dF(t,T) = e^{\int_t^T r_t dt} dS_t + S_t e^{\int_t^T r_t dt}(-r_t dt)$$
$$= r_t S_t e^{\int_t^T r_t dt} dt + \sigma_t S_t e^{\int_t^T r_t dt} dW_t - r_t S_t e^{\int_t^T r_t dt} dt$$
$$= \sigma_t F(t,T) dW_t$$

That is, **forward price is a martingale** under risk-neutral measure. In addition, **its volatility is exactly the stock volatility**.

Similarly, the forward price follows lognormal distribution as

$$lnF(t,T) \sim N\left(lnF(0,T) - \frac{1}{2}\int_0^T \sigma_t^2 dt, \int_0^T \sigma_t^2 dt\right)$$

Follow the same logic in the BS model and use the fact $F(T,T) = S(T)$, we have the Black-76 model

$$C(K) = e^{-\int_0^T r_t dt}\{F(0,T)N(d_1) - KN(d_2)\} \qquad (4)$$

where

$$d_1 = \frac{ln\left(\frac{F(0,T)}{K}\right) + \frac{\bar{\sigma}^2}{2}T}{\bar{\sigma}\sqrt{T}}, \qquad d_2 = \frac{ln\left(\frac{F(0,T)}{K}\right) - \frac{\bar{\sigma}^2}{2}T}{\bar{\sigma}\sqrt{T}}$$

Buying a call while simultaneously selling a put with the same strike is equivalent to entering a stock forward contract fixed at K. This observation leads to **put-call parity**

$$C - P = e^{-\int_0^T r_t dt} E\{F(T,T) - K\} = e^{-\int_0^T r_t dt}(F(0,T) - K) \qquad (5)$$

In the sense of stock forward contract, the at-the-money ($F = K$) call option and put option have the same price ($C = P$).

From put-call parity, the put option price is

$$P(K) = e^{-\int_0^T r_t dt}\{KN(-d_2) - F(0,T)N(-d_1)\} \qquad (6)$$

Local Volatility Model

Time dependent Black-Scholes model can be calibrated to the ATM implied volatility term structure $\sigma_{imp}(T)$, but it can't produce the implied volatility surface $\sigma_{imp}(K,T)$.

Local volatility model is described under risk-neutral measure as

$$dS = rSdt + \sigma(S,t)SdW$$

The volatility term is not deterministic any more. It depends on state variable S and therefore is stochastic. This leads to a stochastic quadratic variation which leads to implied volatility surface (Rebonato 2008).

Local volatility model is complete.

Backing out the local volatility from call option price

This section follows closely Wilmott (2006).

Given risk-neutral pdf function $\varphi(S_T,T)$, the call option price is

$$C(K,T) = e^{-rT}\int_K^{+\infty}(S_T - K)\,\varphi(S_T,T)dS_T$$

Apply Leibniz rule,

$$\frac{\partial C}{\partial K} = -e^{-rT}\int_K^{+\infty}\varphi(S_T,T)\,dS_T$$

Apply Leibniz rule again,

$$\frac{\partial^2 C}{\partial K^2} = e^{-rT}\varphi(S_T,T)$$

That is, the risk-neutral distribution of stock price at time T can be completely backed out from the market quotes of European options (long butterfly with close strikes).

Now differentiate C with respect to T and use the Fokker-Planck equation,

$$\frac{\partial C}{\partial T} = -rC + e^{-rT}\int_K^{+\infty}(S_T - K)\frac{\partial \varphi}{\partial T}dS_T$$
$$= -rC + e^{-rT}\int_K^{+\infty}(S_T - K)\left[\frac{1}{2}\frac{\partial^2}{\partial {S_T}^2}(\sigma^2 S_T^2\varphi) - \frac{\partial}{\partial S_T}(rS_T\varphi)\right]dS_T$$

Integrating this by parts twice, we get

$$\frac{\partial C}{\partial T} = -rC + \frac{1}{2}e^{-rT}\sigma^2K^2\varphi + re^{-rT}\int_K^{+\infty}S_T\varphi dS_T$$

Notice that

$$e^{-rT}\sigma^2K^2\varphi = \sigma^2K^2\frac{\partial^2 C}{\partial K^2}$$

$$e^{-rT}\int_K^{+\infty} S_T\varphi dS_T = C + e^{-rT}\int_K^{+\infty} K\varphi dS_T$$

The equation can be re-written as

$$\frac{\partial C}{\partial T} = \frac{1}{2}\sigma^2K^2\frac{\partial^2 C}{\partial K^2} - rK\frac{\partial C}{\partial K}$$

Inverting this gives

$$\sigma(K,T) = \sqrt{\frac{\frac{\partial C}{\partial T} + rK\frac{\partial C}{\partial K}}{K^2\frac{\partial^2 C}{\partial K^2}}}$$

Stochastic Volatility Model

This section follows Lewis (2000) and Gatheral (2006).

Consider a general SV model under measure P

$$\begin{gathered} dS_t = \mu_t S_t dt + \sqrt{V_t}S_t dW_t^S \\ dV_t = \alpha(S_t,V_t,t)dt + \beta(S_t,V_t,t)dW_t^V \\ \langle dW_t^S, dW_t^V\rangle = \rho dt \end{gathered}$$

The process can be written as

$$\begin{gathered} dS_t = \mu_t S_t dt + \sqrt{V_t}S_t dB_t^1 \\ dV_t = \alpha(S_t,V_t,t)dt + \beta(S_t,V_t,t)\left[\rho dB_t^1 + \sqrt{1-\rho^2}dB_t^2\right] \\ \langle dB_t^1, dB_t^2\rangle = 0 \quad dW_t^S = dB_t^1 \quad dW_t^V = \rho dB_t^1 + \sqrt{1-\rho^2}dB_t^2 \end{gathered}$$

Let λ_t^S and $\lambda_t^\perp$ be the market risk premium and hedging risk premium, respectively. They correspond respectively to dB_t^1 and dB_t^2. In addition, λ_t^S is the original market price of risk in BSM model. That is,

$$\lambda_t^S = \frac{\mu_t - r_t}{\sqrt{V_t}}$$

Because the **market is incomplete**, $\lambda_t^\perp$ is a free variable and depends on the market utility function.

Apply **Girsanov Theorem** (Wikipedia:Girsanov Theorem) and let (using the notation on Wikipedia for a moment)

$$dX_t = -\lambda_t^S dB_t^1 - \lambda_t^\perp dB_t^2$$

$$d\tilde{B}_t^1 = dB_t^1 - \langle dB_t^1, dX_t\rangle = dB_t^1 + \lambda_t^S dt$$
$$d\tilde{B}_t^2 = dB_t^2 - \langle dB_t^2, dX_t\rangle = dB_t^2 + \lambda_t^\perp dt$$
$$\left.\frac{dQ}{dP}\right|_{Ft} = \mathcal{E}(X_t) = exp\left(X_t - \frac{1}{2}[X_t]\right)$$
$$= exp\left\{-\int_0^t \lambda_t^S dt - \int_0^t \lambda_t^\perp dt - \frac{1}{2}\int_0^t [(\lambda_t^S)^2 + (\lambda_t^\perp)^2]dt\right\}$$

Then W_t^V under risk-neutral measure Q is

$$d\tilde{W}_t^V = \rho d\tilde{B}_t^1 + \sqrt{1-\rho^2}d\tilde{B}_t^2$$
$$= \rho(dB_t^1 + \lambda_t^S dt) + \sqrt{1-\rho^2}(dB_t^2 + \lambda_t^\perp dt)$$
$$= dW_t^V + \left(\rho\lambda_t^S + \sqrt{1-\rho^2}\lambda_t^\perp\right)dt$$
$$= dW_t^V + \lambda_t^V dt$$

where λ_t^V, defined by

$$\lambda_t^V = \rho\lambda_t^S + \sqrt{1-\rho^2}\lambda_t^\perp$$

is known as the **market price of volatility risk**.

The SV model under risk neutral measure Q is usually written in terms of λ_t^V as

$$dS_t = r_t S_t dt + \sqrt{V_t}S_t d\tilde{W}_t^S$$
$$dV_t = [\alpha - \lambda_t^V\beta]dt + \beta d\tilde{W}_t^V$$
$$\langle d\tilde{W}_t^S, d\tilde{W}_t^V\rangle = \rho dt$$

Let’s drop the tilts and re-write the SV model **in risk-neutral measure Q** as

$$dS_t = r_t S_t dt + \sqrt{V_t}S_t dW_t^S$$
$$dV_t = \alpha(S_t, V_t, t)dt + \beta(S_t, V_t, t)dW_t^V$$
$$\langle dW_t^S, dW_t^V\rangle = \rho dt$$

For any payoff $F = F(S_t, V_t, t)$, it satisfies the following PDE

$$\frac{\partial F}{\partial t} + \frac{1}{2}VS^2\frac{\partial^2 F}{\partial S^2} + \rho\beta S\sqrt{V}\frac{\partial^2 F}{\partial V\partial S} + \frac{1}{2}\beta^2\frac{\partial^2 F}{\partial V^2} + rS\frac{\partial F}{\partial S} + \alpha\frac{\partial F}{\partial V} = rF$$

Heston Model

The **Heston model** (Heston 1993) is a SV model that has closed-form solutions for European options.

Under risk-neutral measure Q, Heston model assumes,

$$dS_t = r_t S_t dt + \sqrt{V_t}S_t dW_t^S$$
$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^V$$
$$\langle dW_t^S, dW_t^V\rangle = \rho dt$$

The square root process of variance stays positive if $2\kappa\theta > \sigma^2$.

Apply the general SV model to this special (square root) form, the Heston PDE is

$$\frac{\partial F}{\partial t} + \frac{1}{2} V S^2 \frac{\partial^2 F}{\partial S^2} + \rho\sigma S V \frac{\partial^2 F}{\partial V \partial S} + \frac{1}{2}\sigma^2 V \frac{\partial^2 F}{\partial V^2} + rS\frac{\partial F}{\partial S} + \kappa(\theta - V)\frac{\partial F}{\partial V} = rF$$

Heston (1993) found the solution by using Fourier transformation. Given current states (S_0, V_0) and $\tau = T - t$, the European call and put option is priced as

$$C(K,\tau; S_t, V_t) = S_t\left[P_1(K,\tau) + \frac{1}{2}\right] - Ke^{-r\tau}\left[P_2(K,\tau) + \frac{1}{2}\right] \qquad (7)$$

$$P(K,\tau; S_t, V_t) = S_t\left[P_1(K,\tau) - \frac{1}{2}\right] - Ke^{-r\tau}\left[P_2(K,\tau) - \frac{1}{2}\right] \qquad (8)$$

where

$$P_j(K,\tau) = \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi \ln K}\varphi_j(\tau,\phi)}{i\phi}\right]d\phi \qquad j = 1,2$$

and the characteristic functions φ_j are given by

$$\varphi_j(\tau,\phi) = exp\{C_j(\tau,\phi) + D_j(\tau,\phi)V_t + i\phi \ln S_t\} \qquad j = 1,2$$

and

$$C_j(\tau,\phi) = r\phi i\tau + \frac{\kappa\theta}{\sigma^2}\left\{(b_j - \rho\xi\phi i + d_j)\tau - 2\ln\left[\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right]\right\}$$

$$D_j(\tau,\phi) = \frac{b_j - \rho\sigma\phi i + d_j}{\sigma^2}\left[\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right]$$

where

$$b_1 = \kappa - \rho\sigma \qquad b_2 = \kappa$$

$$u_1 = 0.5, \qquad u_2 = -0.5$$

$$d_j = \sqrt{(\rho\sigma\phi i - b_j)^2 - \sigma^2(2u_j\phi i - \phi^2)}$$

$$g_j = \frac{b_j - \rho\sigma\phi i + d_j}{b_j - \rho\sigma\phi i - d_j}$$

As pointed out by Mikhailov and Nogel (2003), C++ standard library provides complex<> template to handle complex numbers; and Quantlib has Gaussian Quadrature integration in its math library.

Albrecher et al. (2007) claimed that the following formulation is identical to Heston's original formulation, but offers better numerical performance. This is also documented in Gatheral (2006).

$$g'_j = \frac{1}{g_j} = \frac{b_j - \rho\sigma\phi i - d_j}{b_j - \rho\sigma\phi i + d_j}$$

$$C'_j(\tau,\phi) = r\phi i\tau + \frac{\kappa\theta}{\sigma^2}\left\{(b_j - \rho\sigma\phi i - d_j)\tau - 2ln\left[\frac{1 - g'_j e^{-d_j\tau}}{1 - g'_j}\right]\right\}$$

$$D'_j = \frac{b_j - \rho\sigma\phi i - d_j}{\sigma^2}\left[\frac{1 - e^{-d_j\tau}}{1 - g'_j e^{-d_j\tau}}\right]$$

The **put-call parity** still holds:

$$C - P = S - Ke^{-r\tau}$$

Example

The accompanying C++ example illustrates Heston model calibration on SPX index options.

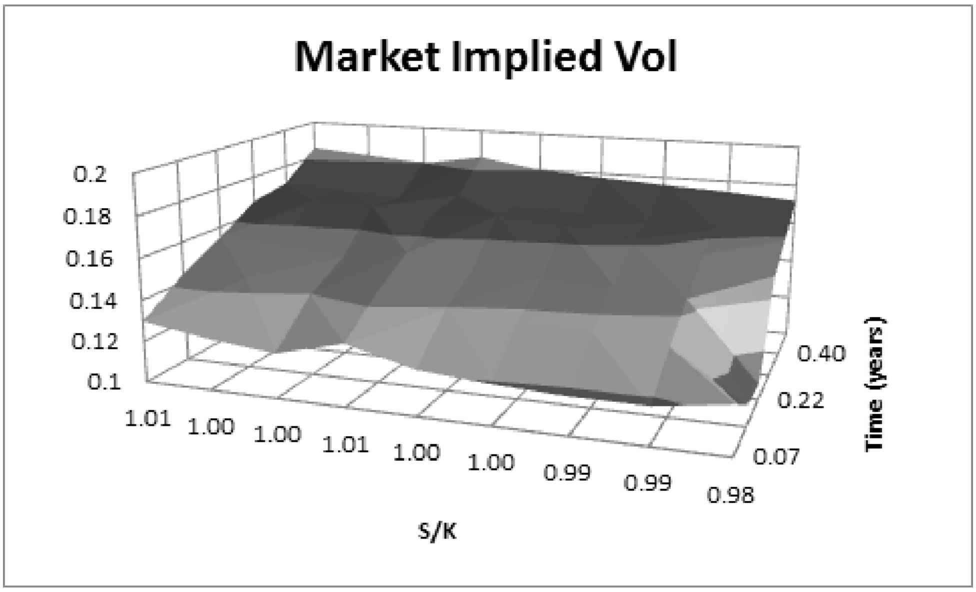
Unlike SPY options, SPX options are trade solely in CBOE. They are European style, may be exercised only on the last business day before expiration. The Expiration date is set on the Saturday following the third Friday of the expiration month. They are quoted in decimals, settled in cash. One contract contains 100 shares of index (multiplier is 100).

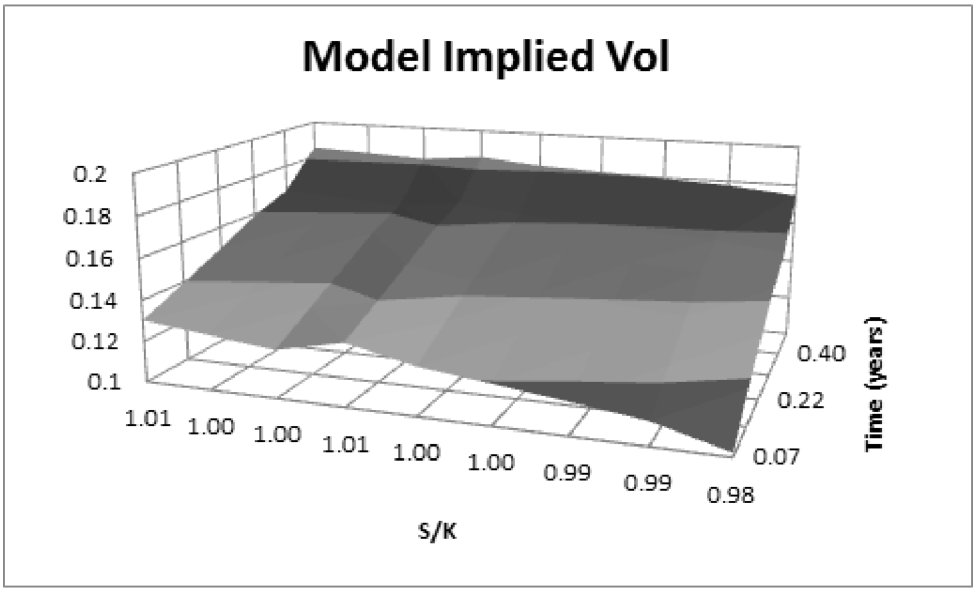
On Monday, 2011-04-25, the call option that matures in May is quoted as

ID	Maturity	Strike	Bid	Ask	Last
SPX 5/21/11 C1325	2011-5-21	1325	22.00	23.00	22.65

On that day, S&P 500 index closed at 1335.25. Therefore the call option has intrinsic value of $10.25 and time value of $12.4.

The results show that calibrated model volatility surface fits the market implied volatility surface and is smoother.





Related Bloomberg Commands: SPX US <Eqty> then OMON

Appendix

Fokker-Planck equation

Consider a random variable that follows SDE

$$dx = \mu(x,t)dt + \sigma(x,t)dW$$

and its value at time t_0 is x_0. Denote probability of reaching x at time $t > t_0$ by its transition probability function $f(x,t;x_0,t_0)$. Then the evolution of this transition probability function follows the 1-D **Fokker-Planck (Kolmogorov forward) equation** (Wikipedia: Fokker-Planck equation)

$$\frac{\partial}{\partial t}f(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma(x,t)^2 f(x,t)] - \frac{\partial}{\partial x}[\mu(x,t)f(x,t)]$$

Particularly in the case of BSM,

$$dS = \mu S dt + \sigma S dW$$

Denote $\varphi(S_T, T)$ the probability of reaching S_T at time T from S_0 at time 0. Then its evolution follows

$$\frac{\partial \varphi}{\partial T} = \frac{1}{2}\frac{\partial^2}{\partial {S_T}^2}(\sigma^2 S_T^2 \varphi) - \frac{\partial}{\partial S_T}(\mu S_T \varphi)$$

Leibniz integral rule (wikipedia:Leibniz integral rule)

$$\frac{d}{dy}\int_{a(y)}^{b(y)} f(x,y)\,dx = \frac{db(y)}{dy} f(b(y),y) - \frac{da(y)}{dy} f(a(y),y) + \int_{a(y)}^{b(y)} \frac{\partial f(x,y)}{\partial y}\,dx$$

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