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LIBOR Volatility

Contents

Black Lognormal Vol and Black Normal Vol

Consider a call option on the forward rate (this is called caplet that is to be introduced in the next section). The option matures at time T but the payoff occurs at time S with amount

$$max(L(T,S)-K,0)=(L(T,S)-K)^+$$

for the call and

$$max(K-L(T,S),0)=\left(K-L(T,S)\right)^+$$

for the put. Unit notional is assumed here.

Then the call price is given by

$$P(0,S)E_0\{(L(T,S)-K)^+\}$$

Also it is known that the forward rate $F(t,T,S)$ is a martingale under T-forward measure with numeraire $P(t,S)$.

First let's consider the **Black-76/Black lognormal model**. Assume the forward rate follows

$$dF(t;T,S) = \sigma F(t;T,S)dW$$

From the discussion in Chapter 2, the call/put option prices at time 0, and put-call parity are as follows

$$C = P(0,S)\{F(0;T,S)N(d_1) - KN(d_2)\} \quad (1)$$

$$P = P(0,S)\{KN(-d_2) - F(0;T,S)N(-d_1)\} \quad (2)$$

$$C - P = P(0,S)\{F(0;T,S) - K\} \quad (3)$$

where

$$d_1 = \frac{ln\left(\frac{F(0;T,S)}{K}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}, \qquad d_2 = \frac{ln\left(\frac{F(0;T,S)}{K}\right) - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$$

Now consider the **Black-Normal model**, who gains its popularity in recent years. In the model the forward rate is assumed to follow

$$dF(t;T,S) = \sigma_N dW$$

which leads to

$$F(T;T,S) = L(T,S) \sim N(F(0;T,S), \sigma_N^2 T)$$

Therefore, the forward rate at time T is no longer lognormally distributed. Instead, it is normally distributed.

To simply notation, let $F = L(T,S) = F(T;T,S)$ and $F_0 = F(0;T,S)$. Then F is normally distributed and its pdf function $\varphi(F)$ is given by

$$\varphi(F) = \frac{1}{\sqrt{2\pi\sigma_N^2 T}} e^{-\frac{(F-F_0)^2}{2\sigma_N^2 T}}$$

So that the call option price is calculated as

$$
\begin{aligned}
C &= P(0,S)E_0\{(F-K)^+\} \\
&= P(0,S)\int_K^{+\infty}(F-K)\varphi(F)dF \\
&= P(0,S)\left\{\int_K^{+\infty}F\varphi(F)dF - K\int_K^{+\infty}\varphi(F)dF\right\} \\
&= P(0,S)\left\{\int_K^{+\infty}(F-F_0)\varphi(F)dF - (K-F_0)\int_K^{+\infty}\varphi(F)dF\right\}
\end{aligned}
$$

where the first integral is

$$
\begin{aligned}
\int_K^{+\infty}(F-F_0)\varphi(F)dF &= \frac{1}{\sqrt{2\pi\sigma_N^2T}}\int_K^{+\infty}(F-F_0)e^{-\frac{(F-F_0)^2}{2\sigma_N^2T}}dF \\
&= \frac{-\sigma_N^2T}{\sqrt{2\pi\sigma_N^2T}}\int_K^{+\infty}e^{-\frac{(F-F_0)^2}{2\sigma_N^2T}}d\left(-\frac{(F-F_0)^2}{2\sigma_N^2T}\right) \\
&= \frac{-\sigma_N^2T}{\sqrt{2\pi\sigma_N^2T}}\left(0-e^{-\frac{(K-F_0)^2}{2\sigma_N^2T}}\right) \\
&= \frac{1}{\sqrt{2\pi}}\sigma_N\sqrt{T}e^{-\frac{(F_0-K)^2}{2\sigma_N^2T}}
\end{aligned}
$$

and the second integral becomes

$$
\begin{aligned}
&\int_K^{+\infty}\varphi(F)dF = Pr(F \geq K) \\
&= Pr\left(\frac{F-F_0}{\sigma_N\sqrt{T}} \geq \frac{K-F_0}{\sigma_N\sqrt{T}}\right) = N\left(\frac{F_0-K}{\sigma_N\sqrt{T}}\right)
\end{aligned}
$$

Therefore the call option price becomes

$$
C = P(0,S)\sigma_N\sqrt{T}\left\{\frac{1}{\sqrt{2\pi}}e^{-\frac{d^2}{2}} + dN(d)\right\} \tag{4.1}
$$

where

$$
d = \frac{F_0-K}{\sigma_N\sqrt{T}}
$$

Denote $n(\cdot)$ be the pdf function of standard normal distribution, or

$$
n(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} = N'(x)
$$

Then equation (4.1) can be re-written as

$$
C = P(0,S)\sigma_N\sqrt{T}\{n(d) + dN(d)\} \tag{4.2}
$$

From put-call parity of equation (3), we have

$$P = P(0,S)\sigma_N\sqrt{T}\{n(d) - dN(-d)\} \qquad (5)$$

Market uses Black model to back out the implied volatility from option price quotes. If Black lognormal model is used, one gets **implied lognormal volatility (σ^{imp})**, or Black log vol; If Black normal model is used, one gets **implied normal volatility (σ_N^{imp})**, or black normal vol.

Cap/Floor

First let's introduce caplet/floorlet. They are European options on a forward rate, typically a 3m LIBOR. Consider the forward rate $F(t;T_1,T_2)$, where typically $T_2 = T_1 + 3m$. The mechanism of caplet/floorlet is similar to that of FRA introduced in Chapter LIBOR rates. The fixing date is two business days before day T_1, when the forward rate is fixed at

$$F(T_1;T_1,T_2) = L(T_1,T_2)$$

Now the caplet (floorlet) is a call (put) option on the forward rate, having payoff only when it is greater (smaller) than a **strike rate K** which is preset at time 0, or

$$\begin{aligned} caplet\ payoff &= [L(T_1,T_2) - K]^+\tau(T_1,T_2) \\ floorlet\ payoff &= [K - L(T_1,T_2)]^+\tau(T_1,T_2) \end{aligned}$$

per unit notional, where $\tau(T_1,T_2)$ is the year fraction between T_1 and T_2. **The maturity date of this option is T_1, and the payment is made at time T_2**.

Market uses Black-76 model to price/quote the caplet/floorlet. By change of numeraire, the price of a caplet is given by

$$\begin{aligned} C &= E^B\left[\frac{1}{B(T_2)}\tau(F(T_1;T_1,T_2) - K)^+\right] \\ &= E^{T_2}\left[\frac{P(0,T_2)}{P(T_2,T_2)}\tau(F(T_1;T_1,T_2) - K)^+\right] \\ &= P(0,T_2)\tau E^{T_2}[(F(T_1;T_1,T_2) - K)^+] \end{aligned}$$

In particular, if we assume

$$dF(t;T_1,T_2) = \sigma F(t;T_1,T_2)dW_t$$

under Q^2 T-forward martingale, then we can immediately obtain the pricing formula from equations (1)-(3) (**by adding the year fraction term**).

A caplet/floorlet is said to be at-the-money (ATMF) if $K = F(0;T_1,T_2)$.

A cap/floor is simply a portfolio of caplets/floorlets, all having the same strike. Using the notation in Chapter LIBOR rate, a cap with first reset date T_α and payment dates $T_{\alpha+1},\dots,T_\beta$ is given by

$$\begin{aligned} Cap &= E\left\{\sum_{i=\alpha+1}^{\beta}\tau(T_{i-1},T_i)D(0,T_i)(F(T_{i-1};T_{i-1},T_i)-K)^+\right\} \\ &= \sum_{i=\alpha+1}^{\beta}\tau(T_{i-1},T_i)P(0,T_i)E^i[(F(T_{i-1};T_{i-1},T_i)-K)^+] \\ &= \sum_{i=\alpha+1}^{\beta} Caplet_i \end{aligned}$$

The floor is defined in a similar way. From the put-call parity, we have the following relationship

$$\text{Long cap} + \text{short floor} = \text{payer swap}$$

A cap or floor is said to be at-the-money (ATMF) if the strike rate equals the swap rate, or

$$K = S_{\alpha,\beta}(0) = \frac{P(0,T_\alpha) - P(0,T_\beta)}{\sum_{i=\alpha+1}^{\beta}\tau(T_{i-1},T_i)P(0,T_i)}$$

A spot-start cap/floor ($T_\alpha = 0$) has its first caplet/floorlet ignored. For example, a 1y quarterly cap is really a 3mx1y cap, or a forward-start cap starting in three months. It includes three caplets: [3m, 6m], [6m, 9m], and [9m, 1y]. In contrast, a forward cap/floor ($T_\alpha > 0$) has full caplets. For example, a 1x2 cap is a 1y cap, starting in 1y, and has 4 caplets.

Bloomberg Commands: **SWPM, VCUB, WCV**

Eurodollar Option

CME lists eight quarterly options and two front month serial options on Eurodollar futures. These options are American style, quoted in the same way as Eurodollar futures (1MM notional, quote in percentage, 1 tick = 1bps = $25).

Options on Euro-dollar futures are treated as caplets/floorlets, and priced by equation (1) and (2). Note that **a call on the price is a floor on the implied rate**, so that a Eurodollar call option corresponds to Black floorlet formula (2).

Example: On 2011-05-16, EDZ1 is quoted at 99.6. EDZ1P with strike 99.625 is quoted at 0.1100. All these numbers are in percentage. So that the forward rate and strike are

$$F = \frac{100 - 99.6}{100} = 0.004, K = \frac{100 - 99.625}{100} = 0.00375$$

The option quoted price is 0.1100%=11bps, or $11 \times 25 = \$275$ dollars (1bps = $25).

The option expires in 217 days, and $P(0{,}127d) = 0.997186$. Therefore

$$275 = 0.997186 \times \$1M \times \{FN(d_1) - KN(d_2)\} \times \frac{90}{360}$$

The implied Black log vol is 82.9(%). Note that EDZ1P is a put option priced via Black caplet formula.

Bloomberg Commands: **EDA <CMDTY> OMON**

Swaption

Swaption, or swap option, refers to option on interest rate swap (IRS). A European payer (receiver) swaption gives the right to enter a payer (receiver) IRS at the option maturity date.

Suppose we are at time 0 and assume unit notional amount. Consider a payer swaption with strike K and maturity T_α giving the holder the right to enter at time T_α a payer IRS with payment dates $T_{\alpha+1}, \dots, T_\beta$ and associated year fractions $\tau_{\alpha+1} = \tau(T_\alpha, T_{\alpha+1}), \dots, \tau_\beta = \tau(T_{\beta-1}, T_\beta)$.

The **forward-swap rate** has been defined in the IRS chapter as

$$S_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)}$$

The payer IRS discounted payoff at time 0 can be expressed in terms of forward swap rate as

$$
\begin{aligned}
IRS\,payoff &= \sum_{i=\alpha+1}^{\beta} D(0,T_i)\tau_i(F(T_\alpha;T_{i-1},T_i)-K) \\
&= D(0,T_\alpha)\left(S_{\alpha,\beta}(T_\alpha)-K\right)\sum_{i=\alpha+1}^{\beta}\tau_i P(T_\alpha,T_i)
\end{aligned}
$$

Consequently the payer swaption payoff is

$$
PS\,payoff = D(0,T_\alpha)\left(S_{\alpha,\beta}(T_\alpha)-K\right)^+\sum_{i=\alpha+1}^{\beta}\tau_i P(T_\alpha,T_i)
$$

Now, define annuity $C_{\alpha,\beta}(t)$ as

$$
C_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta}\tau_i P(T_\alpha,T_i)
$$

and use it as the numeraire. By change of numeraire,

$$
\begin{aligned}
PS &= E\left\{D(0,T_\alpha)\left(S_{\alpha,\beta}(T_\alpha)-K\right)^+ C_{\alpha,\beta}(T_\alpha)\right\} \\
&= E^B\left\{\frac{B(0)}{B(T_\alpha)}\left(S_{\alpha,\beta}(T_\alpha)-K\right)^+ C_{\alpha,\beta}(T_\alpha)\right\} \\
&= E^{\alpha,\beta}\left\{\frac{C_{\alpha,\beta}(0)}{C_{\alpha,\beta}(T_\alpha)}\left(S_{\alpha,\beta}(T_\alpha)-K\right)^+ C_{\alpha,\beta}(T_\alpha)\right\} \\
&= C_{\alpha,\beta}(0)E^{\alpha,\beta}\left\{\left(S_{\alpha,\beta}(T_\alpha)-K\right)^+\right\}
\end{aligned}
$$

Now assume that forward-swap rate follows a lognormal dynamics under **annuity numeraire**, or

$$
dS_{\alpha,\beta}(t) = \sigma S_{\alpha,\beta}(t)dW(t)
$$

then the price of above payer (receiver) swaption can be priced by the Black equations (1) and (2) as well.

The swaption is said to be at-the-money (ATMF) if

$$
K = S_{\alpha,\beta}(0) = \frac{P(0,T_\alpha)-P(0,T_\beta)}{\sum_{i=\alpha+1}^{\beta}\tau_i P(0,T_i)}
$$

Bloomberg Commands: **SWPM, VCUB**

Example

This example illustrates how to back out Caplet volatility curve from ED futures Options and Cap/Floor quotes. **Caplet volatility** is also referred to as spot volatility; and **cap volatility** is also known as flat volatility.

Caps/floors have been introduced in a way that is in conformity with the **LIBOR market model** notation. Because correlations among LIBOR forward rates have no impact on caps, the discussion on correlation structure is deferred to later chapters. Therefore, if we leave aside the swaption volatility, we can neglect the correlation structure and calibrate cap/floors in a straight-forward way.

This section is the interest rate counterpart of the deterministic/**time dependent volatility model** discussed in the stock option chapter. If we assume that time-dependent instantaneous volatility of forward rates is piecewise constant, we can start backing out the volatility curve by hand.

For example, on May 16, 2011, the ED futures options are quoted as follows.

CUSIP	Days to Maturity	Implied Black Log Vol
EDM1P	28	51.17
EDU1P	126	83.92
EDZ1P	217	78.28

Given constant volatility, the Black variance can be written as

$$\int_a^b \sigma(t)^2 dt = \sigma^2 \int_a^b dt = (b-a)\sigma^2$$

Therefore, the volatility for the first 28 days is 51.17%. For the next 98 = 126 – 28 days, the instantaneous constant volatility satisfies

$$(126-28)\sigma^2 + (28)(51.17^2) = (126)(83.92^2)$$

which yields $\sigma = 91.14\%$. Similarly, for the next 91 = 217-126 days, the instantaneous constant volatility is 69.72%. This is illustrated in the following table.

EDM1P	28D 51.17%	
EDU1P	98D 91.14%	28D 51.17%

EDZ1P	91D 69.72%	98D 91.14%	28D 51.17%

Time

One point needs to point out is that the volatility should be backed out **according to not the time, but the time to maturity**. For example, for the third option EDZ1P, its instantaneous volatility is 69.72% in the beginning; then it becomes 91.14%, and 51.17% for the last 28 days of its life.

Then it ends here. It is hard to continue this process with caps. Instead, industry uses the following hump function [4]

$$\sigma(t) = [a + b(T-t)]e^{-c(T-t)} + d$$

to model the volatility curve. The calibrated parameters have meaningful interpretation (see [4] for details). This model is implemented in *class AbcdFunction*. The accompanying code illustrates how to calibrate caplet volatility to this model.

Reference

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