OpenGamma Quantitative Research

Multi-Curve Framework with Collateral

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Abstract

This note is dedicated to the impact of collateral on the multi-curve framework. The pricing formula in presence of collateral are described in a generic way encompassing several financial realities. The multiple currency collateral case is also described, including the convexity adjustment required. The pricing of STIR futures in this framework is analysed in detail.

Contents

1 Introduction

The post-crisis multi-curve framework is nowadays the standard pricing framework for interest rate derivatives. The framework relies on credit risk-free discounting curves and does not incorporate collateral and funding. In this article we use the framework as described in Henrard (2010) which is based on pre-crisis developments proposed in Henrard (2007). Other multi-curve framework descriptions can be found in Ametrano and Bianchetti (2009), Kijima et al. (2009) and Bianchetti (2010).

The goal of this article is to extend the multi-curve framework to incorporate collateral. We prove generic collateral results that extend those presented in Kijima et al. (2009), Macey (2011), Pallavicini et al. (2012) and Piterbarg (2012) and specialise them to the multi-curve framework. Using techniques similar to the ones developed in Henrard (2013b) for the multi-curve framework with stochastic spread we propose explicit formulas to evaluate the impact of the collateral corrections or convexity adjustments. Our analysis of collateral framework has also benefited from the insights of McCloud (2013), Ametrano and Bianchetti (2013), Pallavicini and Brigo (2013) and Fujii and Takahashi (2013).

The question of the collateral on pricing of derivatives products has certainly been analysed in the literature, especially since the beginning of the 2007 crisis. Some research predates the crisis and we cite in that category Johannes and Sundaresan (2007). Obviously with the increasing importance of collateral and the widening of spreads between different rates, the literature on the subject has boomed since the start of the crisis and we cite as a short list Piterbarg (2010), Fujii et al. (2010a), Fujii et al. (2010b), Fujii and Takahashi (2010), Brigo et al. (2011), Pallavicini et al. (2011), Brigo et al. (2012b), Brigo et al. (2012a), Castagna (2012), and Han et al. (2013).

Our first goal is to compute the theoretically consistent amount to be transferred for a given collateral agreement. By theoretically consistent amount we mean that the amount transfered is the exact compensation for the rest of the contract, including the impact of the collateral itself. If at any date, a party break the contract, the contract stops to exists and the collateral is kept by the other party, the two parties do not occur any financial loss. There is a feedback loop between the collateral contract details and the value computed. The collateral contract creates extra cash flows from the stand alone derivative contract. The daily payments of collateral and potentially of interest have to be taken into account when computing the "value". The term of "value" has to be reviewed in line with the above description. As explained at the end of the introduction, we will use the term "quote" to qualify the theoretical number computed.

Some of the above-mentioned articles and books propose frameworks significantly more general that the one we describe below, in particular regarding funding and credit risk. We concentrate mainly on the collateral impact. On the other hand our definition of collateral situation is *more general* than the one used in the above literature as it incorporates not only collateral, repo, no-collateral and futures but also proportional under- and over-collateral, collateral by other assets, with and without interest, and collateral in foreign currency. Our definition of collateral is given in the first section. The specificity of our approach is also to describe the impacts of those generic results for the multi-curve framework and to provide explicit results on the convexity adjustments.

The starting point of the multi-curve framework is the existence of (credit) risk-free fixed cash flows (and the associated rates). Part of the collateral approach can be described in theory without supposing the existence of such rates (see Macey (2011) and Piterbarg (2012)). Pallavicini and Brigo (2013) suppose the existence of a risk-free rate but obtain pricing formulas based on the market quoted instruments only. It is probably debatable from a philosophical point of view if

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risk-free rates exist and if we need them at all. Nevertheless from a practical point of view, in some cases it is easier to suppose they exists, even if we don't suppose they are used in actual market contracts. We will do so when it simplifies the approach.

The multi-curve framework is the foundation for pricing Ibor related instruments. Most of the liquid instruments are collateralised in one way or another. It is thus paramount to have a detailed framework that combines collateral and multi-curve frameworks. We provide such a framework here. Similar definition related to a multi-curve framework with collateral can be found in Fries (2013).

The combination of collateral and multi-curve has often been reduced to an "OIS discounting" formula. I will not use that term here as it is, in my opinion, a misleading term. A collateral at overnight is not enough to justify the use of the technique. It requires several other conditions, including OIS themselves to be collateralised with the same rule, a condition not always satisfied in practice. The exact requirements for "OIS discounting" are detailed in this note.

It is important to insist that our approach to collateralised instrument valuation suppose a feedback loop. The quote computed from the perfect collateral framework should be used to transfer the collateral amount. If the collateral transferred is computed using a different rule, like an old one-curve framework valuation or an accounting rule, then it does not fit our definition of collateral. We are describing a world where the computations of quantitative finance are imposed to the rest of the process, and in particular to back-offices and accounting departments. In other words, our collateral world is "perfect" if quants are ruling the world!

An approach with only one collateral rule for each currency is not general enough to cover all practical cases. Collateral with foreign currencies, often described in the literature, is only one special case. Collateral with bonds, leading to "collateral square"-type results, is often used. Modified collateral rates, like overnight plus a fixed spread, are also found in practice.

The collateral practice leads to a somewhat confusing terminology. In this note we use the term which has to be has explained in the collateral framework. Due to collateral rules, when an asset is purchased with perfect collateral, the amount obtained from the market quote is paid but returned immediately through the collateral, resulting in a net zero payment at the start of the trade. Only the change of quote and interests on the collateral account are exchanged (daily). We will use the more neutral term *quote*. Personally I prefer the term *reference number* or *reference index* but it is a little bit too long to use.

We use the standard term *quote*, even if it would be more correct to speak of a *reference number* or *reference index*. In the cash collateral case, the quote is never actually paid, only the quote change and the reference amount on which the interests are paid are important. The quote could be shifted by an arbitrary figure without impact on the economy if the reference amount for the collateral was similarly adapted. This consideration on prices was originally proposed in Henrard (2006) for futures and later adapted for collateral in Piterbarg (2012).

STIR futures are an important instrument to include in any multi-curve and collateral framework discussion. Not only is it a very liquid instrument and an important source of market information, but from a technical point of view its pricing can be viewed as the pricing with a very specific collateral rule paying a rate of 0. Understanding the pricing of those instruments in a multi-curve and collateral framework is the first step to a more general understanding of change of collateral. We dedicate one section of the "Modelling with collateral" chapter to those instruments.

2 Collateral: rate, asset or both

In general, we are interested in analysing the collateral impact not only in one currency but in a multi-currency framework. The currencies will be denoted X and Y. In the first part of this section we restrict ourself to a unique currency and will extend the results to collateral with assets and more than one currency later.

2.1 Definition of collateral rate

We need to define *collateral rate c for a quote V*. All the cash flows are expressed in the same currency that we call the domestic currency. If a cash flow is paid in another currency, it is supposed to be converted into the domestic currency at the exchange rate prevailing at the cash flow time.

Definition 1 (Collateral rate) *The quote V_t has a collateral rate c_t if a continuous dividend*

$$dD_t = dV_t - c_t V_t dt \tag{1}$$

without any other payments (not upfront or final payment) can be created.

In the naming convention of Duffie (2001), which is also used in Brigo et al. (2012a), this means that it is possible to create an asset package with price process constant at 0 and a dividend process D_t given above.

The dividend payment is not necessarily coming from one asset on its own; it can be provided by a package of instruments including collateral contracts which, combined together, create the above dividend cash flow. This is the case for example for uncollateralized transactions where the dividend is created by a combination of an asset and its funding.

This may look as a convoluted way to describe collateral but it covers several financial realities and allows us to have only one treatment for all of them. Several financial realities are covered, as described below, nevertheless by abuse of language, we refer to it everywhere with the name *collateral rate*. To our knowledge, this generalised definition of collateral has not been used before. We will create even more general definition of collateral in the next sections to include collateral by assets, including foreign currencies, with or without interest payments. In some sense, through that definition we bundle together the instrument and its financing (collateral in this case).

> La mathématique est l'art de donner le même nom à des choses différentes.
>
> *Henri Poincaré.*

In case of perfect cash collateral in domestic currency, if one purchases an asset, he pays V_t but the amount is immediately paid back to him as collateral. So there is no cash flow at the start. Then he over the interval of time dt pays the interest $c_t V_t dt$ on collateral and receives at the same time as new collateral the change of value of the asset dV_t. At any stage the contract can be cancelled without further cash flow; the collateral compensate for the instrument value due to the perfect collateral. The collateral amount compensates for the asset. The resulting cash flows satisfy our definition above.

The situation of an asset for which a special repo rate is available is also covered by the definition. One can purchase the asset paying the amount V_t, immediately repoing it out, receiving the same cash flow V_t (if we ignore haircut). No net cash flow occurs for the buying party. At the

end, the buyer receives the asset back and pays the repo (principal plus interest) $-V_t(1+c_t dt)$. He can sell the asset and receive the proceeds $V_t + dV_t$. In total the continuous cash flow is $dV_t - c_t V_t dt$. This situation also falls within our general collateral definition.

The situation where there is no special repo for that asset and the asset can be financed only by a naked borrowing at rate c_t is also covered. The rate is the risk-free rate if we suppose that it is possible to borrow at that rate or the funding rate if not. Here we ignore the possibility of our own default. The funding rate is becoming the equivalent of the risk-free rate, the one at which one can borrow and lend. This equivalence was one of the starting arguments in Henrard (2007). One purchases the asset at V_t, borrowing the cash. No net cash flow is paid at the start by the purchasing party. The next payment at the end is the borrowing reimbursement $-V_t(1+c_t dt)$ and the simultaneous sell of the asset for the price $V_t + dV_t$. Again the total net payment is $dV_t - c_t V_t dt$. This situation also falls in our general definition. Note that the funding cash account itself is an asset with collateral rate equal to the funding rate. The account satisfies $dV_t = c_t V_t dt$, which means that the continuous payment $dV_t - c_t V_t dt$ is a continuous payment of 0. Pallavicini (2013) described initial amount corresponding to initial margins in a very general setting including credit risk.

The definition also covers the case of proportional cash collateral (under-collateral or over-collateral). Suppose that the collateral value in cash is αV_t. If $\alpha = 1$ we have the perfect collateral case, if $\alpha = 0$ we have the no-collateral case, if $0 < \alpha < 1$ it is a proportional under-collateral, and if $\alpha > 1$ it is a proportional over-collateral. The situation is covered by the above definition with as equivalent collateral rate $\alpha c + (1 - \alpha)f$ where c is the collateral rate paid on the cash and f is the risk-free/funding rate for the extra amount borrowed or invested to fill the collateral account. The proportional over/under-collateral is discussed in Pallavicini and Brigo (2013) in the CCPs framework. Our approach does not cover the case where the over-/under-collateral is not proportional. This would be the case for a CCP where an initial margin is required which is risk proportional, not value proportional.

Finally we mention the margined futures and futures options. In a future, one can enter into a contract without any upfront payment. There is an initial margin, but we ignore it here; it corresponds to an imperfect collateral which is beyond the scope of our analysis. The next futures payment is done at each margining time where the amount dV_t is paid. The futures situation enters into our collateral definition with $c_t = 0$. This equivalence between futures and zero rate collateral is mentioned in Johannes and Sundaresan (2007), Henrard (2012) and Piterbarg (2012).

Obviously in the five cases, the market rate, denoted c_t in all cases, will not be the same. In the first case, it is the collateral rate as agreed in the contract terms and conditions (often through a CSA); in the second case, it is the market repo rate for the particular asset; in the third case, it is the risk-free rate or funding rate specific to the institution; and in the futures case it is 0. The unique notation covers different situations, allowing us to deal with all of them in the same unified framework. This unified approach between repo and funding rate is implicit in (Han et al., 2013, Section 3); we prefer to have all our hypotheses and choices explicit.

One asset can have a collateral rate that changes from one category to another through time. A fully collateralised swap at overnight rate will have a collateral rate equal to the overnight rate up to its payment. After that, the payment exits the legal framework of the collateral agreement and becomes a non-collateral cash, with funding rate. With our extended definition of collateral, an instrument can still be modelled after a coupon payment. With a restricted version of collateral, at the first coupon payment, the instrument exits the framework and the theoretical world ends; the modelling is not possible anymore.

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Without our generalised definition of collateral, it is difficult to deal with instruments paying an actual cash flow. For example the generic dynamic given by Equation (2) is written as a diffusion and would not be valid at a coupon payment time. At the payment time, there would be a jump in the quote. For the framework to be coherent, te amount paid needs to stay in the framework and an uncollateralized amount needs to satisfy the collateral definition. Otherwise, after the end of the contract there is no collateral in place and no equation left. The time horizon for each instrument would be its maturity date. This means also that the pricing would not be truly additive; the equations would be meaningful only to the first payment date in the package. But if the equations are meaningful only to the first payment date, what is the value of the instrument at that date? The value cannot be computed by discounting from the end date of the other payments, as our economy does not exist any more after the first payment date. The pricing of a swap as the sum of its coupons would then be debatable. The additive property can be brought in at the cost of extending the valuation beyond the maturity and this can be done by reintroducing the funding rate and using a generalised definition of collateral.

Note also that we use the term cash flow to denote any change of value. It can be achieved by an actual cash flow, a cash flow in another currency converted in the domestic currency or a change of value which could be realised by selling part of the asset. All the values are no-arbitrage values, not accounting figures. We do not allow hiding profit, losses or future cash flows in banking book like places or historical cost accounting.

The first assumption of the multi-curve framework concerns risk-free discounting. Here we do not work with a risk-free rate but with collateral. Our hypothesis is the existence of collateralised assets:

A There exists packages in our economy satisfying Definition 1.

At this stage this hypothesis is very vague, but we will give more precision later.

The general filtration of the economy is denoted $(\mathcal{F}_t)_{t \in [0,T]}$. Later we will introduce other filtrations to describe the randomness of some specific components. When we do so we will do it through sub-filtrations $\mathcal{G}_t^i$ $(i = 1, \ldots)$.

Suppose that the continuously compounded rate paid on the collateral is the rate c_t and let

$$N_t^c = \exp\left(\int_0^t c_\tau d\tau\right)$$

be the collateral cash-account. The rate c can be stochastic (and in most cases will be). Even if N_t^c is called the collateral cash-account, we do not suppose that we can invest or borrow through that account freely. Only the cash used for collateral will be in that account, according to the quotes associated with the collateralised trades.

Results similar to the one described below can be found, when applied to a restricted definition of collateral, in Macey (2011), Piterbarg (2012), Pallavicini et al. (2012) and Fujii and Takahashi (2013). The proof approaches in those papers are quite different between them but the results specialised to a common framework are similar. Note that Macey (2011) restricts the collateral to be the overnight index, a restriction we do not impose. In our proof, we follow the general techniques used in Macey (2011) and later in Piterbarg (2012) even if we add some ingredients missing in the original proofs' sketches and apply them to the more general definitions of collateral introduced here.

Let $V_u^{X,c,Y}$ be the *quote* in currency X at time u of an asset with collateral in currency Y at rate c. At some stage, the value $V_u^{X,c,Y}$ will be independent of the collateral. For example a fixed cash flow has a value on its payment date independent of the collateral currency and rate and equal

to the amount paid. In that case we don't indicate the currency and the collateral rate and denote the price $V_u^{X,c,Y} = V_u$. When the asset and collateral currencies are the same and the context is clear, we write $V_u^{X,c,X} = V_u^c$.

We now develop the generic pricing formula under collateral. The proof is based on portfolio hedging, not dissimilar to the one used in standard Black-Scholes pricing formula. In this case we hedge one package generating the dividend (1) with n other packages as source of randomness. The dimension n is the dimension of the Brownian motion underlying the economy. This extra asset is important to create the hedging procedure. In the Black-Scholes proof the underlying asset of the option and a cash-account are used; here in general we don't have the same flexibility to access the collateralised account as the instrument and the collateral account are tied together.

2.2 Single currency collateral

Suppose that you have $n + 1$ collateralised assets with collateral rates c. By rate c, we mean one rate for each asset and potentially different between the assets. The collateral cash-account for multiple assets and rates has to be understood as a multidimensional value with each element corresponding to one asset and given by the above defined cash-account. When a multiplication of vector is done element by element, like a multidimensional rate multiplied by a multidimensional quote, we use the notation $*$, like $c_t * V_t$.

The quotes of the assets depend of an n-dimensional Brownian motion W_t through the equations

$$dV_t^c = \mu_t dt + \Sigma_t dW_t. \tag{2}$$

The quotes V and the drift μ have dimension $n + 1$, the volatility matrix has dimension $n + 1 \times n$. Suppose moreover that the matrix composed of the n first lines of Σ_t is invertible for all t. This means that we have really n assets and not a lower number with some of them recombined in a different way. We denote the square matrix with the first n lines by $\bar{\Sigma}_t$. Similarly, for any vector x of dimension $n+1$, we denote $\bar{x}$ its first n elements. The i-th line of Σ_t is denoted Σ_t^i. The drift and the volatility are potentially stochastic (and potentially dependent of V itself). We suppose also that μ and Σ are "sufficiently regular" and in particular that they are $\mathcal{F}_t$-adapted. The measure in which those equations are valid can be the physical measure or any measure; we don't restrict the result to the physical measure like Macey (2011) and Piterbarg (2012). The only important point at this stage is that the measure is the same for all instruments.

Like in the Black-Scholes approach, the instrument valuation is based on a replicating portfolio approach. The portfolios used for hedging will be self-financing. The intuition of self-financing is that there is no cash appearing or disappearing from the portfolio. The change of value of the portfolio is triggered only by the change of value of the assets of the portfolios.

Definition 2 *The portfolio with $\mathcal{F}_t$-adapted quantities w_t in each package, where the packages have only a dividend process D_t and a null price process, is said to be* self-financing *when*

$$w_t^T dD_t = 0.$$

This condition or its equivalent adapted to their framework is missing in the sketches of Macey (2011) and Piterbarg (2012), as it was missing in the original Black and Scholes (1973) sketch. The self-financing property is not strictly speaking required to do the computations in the proof but if the property is not satisfied, the hedging portfolio can not be put in place by itself. The portfolio would require external financing that would need to be taken into account in the hedging.

The strategy described is a valid replication strategy only if it is self-financing. It is paramount to verify this requirement to obtain a real replicable price.

The self-financing definition above is the one used in Duffie (2001) and Brigo et al. (2012a) adapted to our situation. Our description starts from (asset) packages with zero price process and a dividend process. The equality between the gain process and the price process reduces in our case to null dividend process as defined above.

Theorem 1 (Cash collateral formula) *In presence of collateral with rate c in currency X for a quote in currency X, the quote in time t of an asset with quote V_u^c in time u is*

$$V_t^c = N_t^c \, \mathrm{E}^{\mathbb{X}} \left[(N_u^c)^{-1} V_u^c \, \middle| \, \mathcal{F}_t \right] \tag{3}$$

for some measure $\mathbb{X}$ (identical for all assets, but potentially currency dependent).

Proof: Let $\bar{w}_t^T = -\Sigma_t^{n+1}(\bar{\Sigma}_t)^{-1}$ and $w_t^T = (\bar{w}_t^T, 1)$. By definition, the vector is such that $w_t^T \Sigma_t = 0$.

We create a portfolio with quantities w_t for the packages associated to V_t and producing a dividend $dV_t - c_t * V_t dt$. Using the equations for V_t, the portfolio dividend satisfies

$$w_t^T dD_t = w_t^T (dV_t - c_t * V_t dt) = w^T(\mu_t - c_t * V_t)dt + w_t^T \Sigma_t dW_t = w_t^T(\mu_t - c_t * V_t)dt.$$

In absence of arbitrage, the dividend of a portfolio with zero price process cannot have only drift, no stochastic part and be non zero. Consequently we have $w_t^T(\mu_t - c_t * V_t) = 0$ for all t. With that property, the portfolio with quantities w_t is self-financing.

Let $x = \mu_t - c_t * V_t$. As $\bar{\Sigma}_t$ is invertible, there exists λ_t with values in $\mathbb{R}^n$ such that $\bar{x} = \bar{\Sigma}_t \lambda_t$. Using the definition of w and λ, we have $0 = w_t^T x = -\Sigma_t^{n+1}\lambda_t + x^{n+1}$. This shows that $\mu_t - c_t V_t = \Sigma_t \lambda_t$. The equations for V_t become

$$dV_t = (\Sigma_t \lambda_t + c_t * V_t)dt + \Sigma_t dW_t = c_t * V_t dt + \Sigma_t(dW_t + \lambda_t dt) = c_t * V_t dt + \Sigma_t dW_t^X \tag{4}$$

where dW_t^X is a new Brownian motion defined by $dW_t + \lambda_t dt$. In the numeraire associated to that Brownian motion, all the assets have a drift c_t. We denote the numeraire by N^X. In that numeraire, the prices rebased by the quantity N_t^c are martingales and the result follows. $\qquad\square$

At this stage the question is: What is this still mysterious measure X? To analyse it, remember that we used $n+1$ assets to create a hedged portfolio but we have only a dimension n source of randomness. Remember that λ_t is defined by

$$\lambda_t = (\bar{\Sigma}_t)^{-1}(\bar{\mu}_t - \bar{c}_t * \bar{V}_t).$$

The quantity λ_t that defines the new Brownian motion and the measure is uniquely defined by any n assets with volatility matrix of full rank. It is possible to remove the last asset and replace it by any other asset (with its own collateral rule). If the asset is such that another sub-matrix of full rank n including that asset exists, we can change the order of the lines and have the same resulting λ_t from another set of assets. We can do this for any number of asset and replace all the original assets by new assets, one by one without changing λ_t. The λ_t is intrinsic to the economy and the starting measure. If the starting measure is the physical one, the quantity λ_t is usually called the price of risk.

There is one type of collateral situation for which we know the formula: when there exists a risk-free rate and uncollateralized asset. In that case, the formula above is valid for X the measure associated to the risk-free cash account numeraire and c the risk-free rate. The measure is the one associated to the risk-free cash account if we suppose that such a concept exists.

Here we don't explicitly need this existence, so we keep the vague name, using only the currency as indication. Later in particular cases, for the valuation of futures in particular, we may reintroduce the risk-free rate existence.

It is also important to notice that the quantity N_t^c is not the numeraire associated to the standard measure. First there are infinitely many N_t^c, one for each rate c and only one N^X. Also the quantities V_t^c analysed and that we call *quotes* are not "present values" in the traditional sense of quantitative finance. They are quantities used in the collateral framework but cannot be used outside their context. In particular a quote rescaled by a numeraire will never be a martingale (except in the particular case where the collateral is the risk free rate). What we have is that the quote rebased by the collateral cash account associated to that particular quote is a martingale. The pricing situation is more complex and always involves three elements: the quote, the collateral discounting and a numeraire, by opposition to only two, the price and the numeraire, in standard quantitative finance.

From the collateral account, we define the collateral related discount factors

Definition 3 (Collateral pseudo-discount factors) *The collateral (pseudo-)discount factors for collateral rate c paid in currency X are defined by*

$$P_X^c(t,u) = N_t^c \, \mathrm{E}^X \left[(N_u^c)^{-1} \middle| \mathcal{F}_t \right]. \tag{5}$$

The quantity $P_X^c(t,u)$ is a positive random variable. The quantity is the quote in t for paying a cash flow 1 (in currency X) in u collateralised in the currency of the cash flow at rate c_t. Note that N_t^c is not currency dependent but P_X^c is currency dependent. The currency dependency appears through the expected value $\mathrm{E}^X[.]$.

Let $\mathrm{E}^{c,v}[]$ be the expectation associated to the numeraire $P_X^c(.,v)N_.^X(N_.^c)^{-1}$. This is the numeraire such that

$$N_t^c \, \mathrm{E}^X \left[(N_u^c)^{-1} Y_u \middle| \mathcal{F}_t \right] = P_X^c(t,v) \, \mathrm{E}^{c,v} \left[(P_X^c(u,v))^{-1} Y_u \middle| \mathcal{F}_t \right] \tag{6}$$

for any random variable Y. The associated measure is called c-collateralized v-forward measure in Pallavicini and Brigo (2013). In particular we have that any quote with collateral c scaled by $P_X^c(t,v)$ is a martingale in that measure. Moreover a random variable is a martingale in the N^X numeraire measure when rescaled by the collateral account if and only if it is a martingale in the collateralised forward measure when rescaled by the collateral pseudo-discount factor. This equivalence will be used when we model the collateral rates later.

When the collateral rate is the overnight index in the currency of the payment, this valuation formula is often called "OIS discounting". This terminology is correct in the sense that an overnight related discounting is used to discount the pay-off inside the expected value. Nevertheless the terminology is hiding several facts. The first one is that the expectation is done under a measure for which the discounting may *not be the numeraire*. The result is not the same as replacing the risk-free rate of the no-collateral approach by the overnight rate. Moreover for the term to be right, one needs not only the instrument priced to be collateral at overnight, but also the OIS with compounded rate themselves to be collateral with the same rule. Finally you need a formula that links collateral pseudo-discount factors to OIS instrument. We provide that formula in Theorem 7. The "OIS discounting" does not depend only on the collateral rule of the instrument priced, but also on the OIS collateral rule and the OIS pricing formula. For those reasons, the term "OIS discounting" is misleading. We will refrain from using the term further.

To have symmetrical results later, we define forward rates for the collateral and risk free curve.

Definition 4 (Forward collateral rate) *The collateral forward rate over the period $[u, v]$ is given at time t by*

$$F_t^c(u, v) = F_t^{X,c}(u, v) = \frac{1}{\delta} \left(\frac{P_X^c(t, u)}{P_X^c(t, v)} - 1 \right).$$ (7)

Clearly this means that

$$1 + \delta F_t^c(u, v) = \frac{P_X^c(t, u)}{P_X^c(t, v)}.$$

We call the above quantity the "investment factor", by opposition to the discount factor, it represent the amount obtained at the end of the period if you invest 1 at the rate.

Using the definition of collateral v-forward measure and the remark above, the investment factor $1 + \delta F_t^c(u, v)$ is a martingale in the c-collateral v-forward measure. Even if this result seems trivial at this stage, because it is important for rate modelling, we write it as a theorem.

Theorem 2 (Martingale property of forward collateral rate) *In the collateral framework, the collateral forward rate $F^c(t, u, v)$ is a martingale in the c-collateral and v-forward measure.*

2.3 Change of collateral: independent spread

We have the general formula for computing the price in a framework with collateral. A natural question is to know what is the relation between the prices of assets with same final price V_u but different collateral rules.

Suppose that we have two collateral rates with the relation

$$c_t^2 = c_t^1 + s_t$$

with s_t a $\mathcal{G}_t^2$-adapted random variable and V and c_1 both $\mathcal{G}_t^1$-adapted with $\mathcal{G}_t^2$ independent of $\mathcal{G}_t^1$. Often the spread will be constant or deterministic. The situation can be for example that the rate paid on the collateral is the overnight index plus a fixed spread.

By definition of the spread, the collateral accounts are such that $N_t^{c_2} = N_t^{c_1} N_t^s$. Suppose that the price in u is collateral independent (for example it is the cash flow date). With this simple relation between the rates, the relation between the quotes is

$$\begin{aligned}
V_0^{c_2} &= N_0^{c_1} N_0^s \, \mathrm{E}^X \left[(N_u^{c_1} N_u^s)^{-1} V_u \right] \\
&= N_0^s \, \mathrm{E}^X \left[(N_u^s)^{-1} \middle| \mathcal{F}_t \right] N_0^{c_1} \, \mathrm{E}^X \left[(N_u^{c_1})^{-1} V_u \right] \\
&= P_X^s(0, u) V_0^{c_1}
\end{aligned}$$ (8)

where for the second equality we have used properties independence of the variables.

This can be interpreted as the quote when there is a spread paid on the collateral and the spread is independent is the original quote discounted by the spread.

Note also that we have

$$P_X^{c_2}(0, u) = P_X^s(0, u) P_X^{c_1}(0, u)$$ (9)

as a special case of the above result.

2.4 Collateral with asset

In some cases the collateral is not done with cash but with an asset. The asset itself is subject to collateral (remember that no collateral is considered as a form of collateral in our generalised definition). This can be described as *collateral square*[1].

This is the case with interest rate swaps. The standard is cash collateral at overnight rate but in some cases the collateral consists of treasuries. This is often the case with asset managers who prefer to deposit assets instead of cash. This section describes how to price assets with other assets as collateral.

We need to define asset collateral. As before, we do it through a definition of continuous dividends. We denote by V_t^C the quote for an asset with collateral with quote C.

To our knowledge, this generalised definition of collateral has not been used before.

Definition 5 (Collateral with asset) *The quotee V_t has a collateral quote C_t if a continuous dividend*

$$dD_t = dV_t - \frac{V_t}{C_t} dC_t$$

without any other payments (not upfront or final payment) can be created.

The explanation of the definition is the following. If the collateral is an asset with associated price C_t, the collateral cash flow is the change of asset dV_t. There is no payment of interest per se but the change of the value of the collateral asset needs to be compensated. The quantity of asset with value C_t to cover the value V_t is V_t/C_t. The change of value of the asset is dC_t. When the collateral asset increases in value we have to release the extra value to the counterpart, hence a cash flow of minus the change of value. We suppose that the collateral can be rehypothecated. In particular, if you buy the asset and receive the other asset as collateral in guarantee, you can sell the collateral to cancel the initial cash flows.

With that definition, a quote has a collateral rate c_t if the quote has the collateral account as a collateral asset. The new definition is a natural extension of our general definition of collateral with rate.

Like before, we suppose we have $n+1$ assets depending on an n-dimensional Brownian motion W_t. The equations for the assets are the same as before and given by Equation (2). We suppose that the first n assets have collateral rates c_t. We also keep the hypothesis that the volatility matrix $\bar{\Sigma}_t$ for the first n assets is invertible. The difference is for the last asset, we suppose that the last asset is collateralised by one of the previous assets. Without loss of generality, we suppose the collateral is the first asset. The continuous cash flow that can be created by the last asset with collateral the first asset is

$$dV_t^{n+1} - \frac{V_t^{n+1}}{V_t^1} dV_t^1.$$

Theorem 3 (Asset collateral formula) *In presence of collateral with asset of quote C_t which has itself a collateral rate c_t, the quote in time t for an asset with quote V_u^C in time u is*

$$V_t^C = N_t^c \, \mathrm{E}^X \left[(N_u^c)^{-1} V_u^C \,\middle|\, \mathcal{F}_t \right]. \tag{10}$$

Proof: Let

$$\bar{w}_t^T = \left(\frac{V_t^{n+1}}{V_t^1} \Sigma^1 - \Sigma_t^{n+1} \right) (\bar{\Sigma}_t)^{-1}$$

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[1]The name was suggested by Brigo (2013).

and $w_t^T = (\bar{w}_t^T, 1)$.

We create a portfolio $w_t^T V_t$. From the equations of V,

$$\bar{w}_t^T (d\bar{V}_t - \bar{c}_t * \bar{V}_t dt) + dV_t^{n+1} - \frac{V_t^{n+1}}{V_t^1} dV_t^1$$

$$= \left(\bar{w}_t^T (\bar{\mu} - \bar{c}_t * \bar{V}_t) + \left(\mu_t^{n+1} - \frac{V_t^{n+1}}{V_t^1} \mu_t^1 \right) \right) dt + \left(\bar{w}\bar{\Sigma} + \Sigma^{n+1} - \frac{V_t^{n+1}}{V_t^1} \Sigma^1 \right) dW_t$$

By definition of w, the coefficient in front of the Brownian motion is equal to 0. In absence of arbitrage, the dividend can not have only drift and be non zero. Consequently we get the drift $w_t^T x = 0$ with

$$x = \begin{pmatrix} \bar{\mu}_t - \bar{c} * \bar{V} \\ \mu_t^{n+1} - \frac{V_t^{n+1}}{V_t^1} \mu_1 \end{pmatrix}.$$

With that choice, the portfolio is self-financing.

As $\bar{\Sigma}_t$ is invertible, there exists λ_t with values in $\mathbb{R}^n$ such that $\bar{x} = \bar{\Sigma}_t \lambda_t$. Using the definition of w and λ, we have

$$0 = w_t^T x = \bar{w}_t^T \bar{\Sigma}_t \lambda_t + x^{n+1} = \left(\frac{V_t^{n+1}}{V_t^1} \Sigma^1 - \Sigma^{n+1} \right) \lambda_t + x^{n+1}$$

and so $x^{n+1} = \left(\Sigma^{n+1} - V_t^{n+1}/V_t^1 \Sigma^1 \right) \lambda_t$. From this we have $\bar{\mu}_t = \bar{\Sigma}_t \lambda_t + \bar{c} * \bar{V}$. Using the result for x^{n+1} and the above result for μ_1, we obtain $\mu^{n+1} = \Sigma_t^{n+1} \lambda_t + c_t^1 * V_t^{n+1}$.

Define $c_t^{n+1} = c_t^1$. The equations for V_t become

$$dV_t = c_t * V_t dt + \Sigma_t (dW_t + \lambda_t dt) = c_t * V_t dt + \Sigma_t dW_t^X. \tag{11}$$

In the measure associated to the W_t^X Brownian motion, all the assets have a drift c_t. The drift for the $n+1$-th asset is c_t^1, i.e. the collateral rate of the collateral asset. The measure is the same as in the previous section. $\qquad\square$

Note that the definition of λ_t depends only on $\bar{\Sigma}_t$, $\bar{\mu}$ and $\bar{c}$, i.e. only on the first n instruments equation and collateral are used to define the change of measure. The new Brownian motion W^X is the same as the one obtained in the cash only collateral case. Both formulas use the same expectation.

The result could be described as the quote of an asset collateralised by another asset is obtained by discounting the quote at the rate of the collateral collateral, that is discounting by the collateral rate of the collateral.

2.5 Collateral with asset and cash payment

In this section we extend the analysis to the combination of the two previous cases and gearing factor. The collateral is composed of another asset with gearing and at the same time there is the payment of some rate on the collateral value. The gearing factor is used to model haircut as described later. For a collateral asset αC and a collateral rate c, we denote the price by $V_t^{\alpha,C,c}$.

The payment of interest on the collateral is typically the case with the foreign currency collateral. The currency itself changes value and collateral has to be added or removed with that change of value. At the same time a conventional interest has to be paid on the amount. Note that even if the rate is paid in a foreign currency it is still paid on the full collateral value which is V_t. There is no need to know the exchange rate to know the domestic currency value of interest paid, it will be $c_t V_t dt$.

Definition 6 (Collateral with asset and rate) *The asset with quote V_t has a collateral asset C_t and collateral rate c_t if a continuous dividend*

$$dD_t = dV_t - \alpha \frac{V_t}{C_t} dC_t - c_t V_t dt$$

without any other payments (not upfront or final payment) can be created.

As hinted above, the definition covers collateral with haircut. Suppose that collateral C is accepted with a haircut h, i.e. for a value V_t, one has to post assets for a total value of $V/(1-h)$. To use the notation of the definition, a gearing factor of $\alpha = 1/(1-h)$ is required. What is the dividend of such a collateral process? At t, one party pays the instrument V_t, received the collateral for a total value αV_t, sells it and invest the difference $(\alpha - 1)V_t$ in cash of funding rate f. After the time dt, he sells the instrument for $V_t + dV_t$, repays to collateral for $\alpha V_t/C_t(C_t + dC_t)$ and received from the cash investment $(\alpha - 1)V_t(1 + fdt)$. The total dividend created is $dV_t - \alpha V_t/C_t dC_t + (\alpha - 1)fV_t dt$. The definition is satisfied with $\alpha = 1/(1-h)$ and $c = (\alpha - 1)f$.

The same result can be obtained for foreign currency for which haircut is required and a conventional rate $\bar{c}$ is paid. The definition is satisfied with the same gearing factor $\alpha = 1/(1-h)$ and $c = \alpha\bar{c} + (1 - \alpha)f$.

Like before, we suppose we have $n + 1$ assets depending on an n-dimensional Brownian motion W_t. The equations for the assets are the same as before and given by Equation (2). We suppose that the first n asset have collateral rates c_t. We also keep the hypothesis that the volatility matrix $\bar{\Sigma}_t$ for the first n assets is invertible. The difference is for the last asset. We suppose that it is collateralised by an asset and the rate c_t^{n+1}. Without loss of generality, we suppose that the collateral asset is the first one. So the continuous payment that can be created by the last asset is

$$dV_t^{n+1} - \alpha \frac{V_t^{n+1}}{V_t^1} dV_t^1 - c_t^{n+1} V_t^{n+1} dt.$$

Theorem 4 (Asset and rate collateral formula) *Suppose the asset has collateral with asset C_t, gearing α and rate c_t^{n+1} and the collateral asset itself has a collateral rate c_t^1. Then the quote in time t of an asset with quote $V_u^{\alpha, C, c^{n+1}}$ in time u is*

$$V_t^{\alpha, C, c^{n+1}} = N_t^{\alpha(c^1 + c^{n+1})} \mathrm{E}^{\mathbb{X}} \left[(N_u^{\alpha(c^1 + c^{n+1})})^{-1} V_u^{\alpha, C, c^{n+1}} \,\Big|\, \mathcal{F}_t \right]. \tag{12}$$

Proof: Let

$$\bar{w}_t^T = \left(\alpha \frac{V_t^{n+1}}{V_t^1} \Sigma^1 - \Sigma_t^{n+1} \right) (\bar{\Sigma}_t)^{-1}$$

and $w_t^T = (\bar{w}_t^T, 1)$.

We create a portfolio $w_t^T V_t$. From the equation of the assets,

$$\bar{w}_t^T (d\bar{V}_t - \bar{c}_t * \bar{V}_t dt) + dV_t^{n+1} - \alpha \frac{V_t^{n+1}}{V_t^1} dV_t^1 - c_t^{n+1} V_t^{n+1} dt$$

$$= \left(\bar{w}_t^T (\bar{\mu} - \bar{c}_t * \bar{V}_t) + \left(\mu_t^{n+1} - \alpha \frac{V_t^{n+1}}{V_t^1} \mu_t^1 - c_t^{n+1} V_t^{n+1} \right) \right) dt + \left(\bar{w}\bar{\Sigma} + \Sigma^{n+1} - \alpha \frac{V_t^{n+1}}{V_t^1} \Sigma^1 \right) dW_t$$

The coefficient in front of the Brownian motion is equal to 0. In absence of arbitrage, the dividend can not have only drift and be non zero. Consequently we get the drift $w_t^T x = 0$ with

$$x = \begin{pmatrix} \bar{\mu}_t - \bar{c} * \bar{V} \\ \mu_t^{n+1} - \alpha \frac{V_t^{n+1}}{V_t^1} \mu_1 - c_t^{n+1} V_t^{n+1} \end{pmatrix}.$$

With that choice, the portfolio is self-financing.

As $\bar{\Sigma}_t$ is invertible, there exists λ_t with values in $\mathbb{R}^n$ such that $\bar{x} = \bar{\Sigma}_t \lambda_t$. Using the definition of w and λ, we have

$$0 = w_t^T x = \bar{w}_t^T \bar{\Sigma}_t \lambda_t + x_{n+1} = \left(\alpha \frac{V_t^{n+1}}{V_t^1} \Sigma^1 - \Sigma^{n+1} \right) \lambda_t + x_{n+1}$$

and so $x^{n+1} = \left(\Sigma^{n+1} - \alpha V_t^{n+1}/V_t^1 \Sigma^1 \right) \lambda_t$. By construction we have $\bar{\mu}_t = \bar{\Sigma}_t \lambda_t + \bar{c} * \bar{V}$. Using the result for x^{n+1} and the above result for μ_t^1, we obtain $\mu_t^{n+1} = \Sigma_t^{n+1} \lambda_t + (c_t^1 + c_t^{n+1}) V_t^{n+1}$.

Define $\tilde{c} = (\bar{c}_t, \alpha(c_t^1 + c_t^{n+1}))$. The equations for V_t become

$$dV_t = \tilde{c}_t * V_t dt + \Sigma_t(dW_t + \lambda_t dt) = \tilde{c}_t * V_t dt + \Sigma_t dW_t^X. \tag{13}$$

In the numeraire associated to that Brownian motion, all the assets have a drift $\tilde{c}_t$. The drift for the $n + 1$-th asset is $\alpha(c_t^1 + c_t^{n+1})$, i.e. the collateral rate of the collateral asset plus the collateral rate itself multiplied by the gearing factor. □

The above result can be applied to collateral in foreign currencies. For a foreign currency, the asset collateral rate in the sense of Definition 1 is the forex swap annualised points. For a domestic currency X and a foreign currency Y, we denote the points by $p_t^{X,Y}$. Note that the quoted forex forward points in the market are not annualised and not relative (they are absolute numbers to be added to the current exchange rate).

The collateral rate for foreign currency is obtained in the following way. Buying one unit of foreign currency can be covered by an instantaneous forex forward, selling the foreign currency in the near leg and buying the same quantity in the far leg. If the forward points are annualised and relative, the swap far leg will pay the exchange rate f times $1 + p_t^{X,Y} dt$. At the end of the period, one can sell the foreign currency unit of the far leg at $f + df$. The total cash flow is then $df - f p_t^{X,Y} dt$. By Definition 1, $p^{X,Y}$ is the collateral rate of the foreign currency asset Y.

When there exist risk-free rates in both currencies, one has $p_t^{X,Y} = r_t^X - r_t^Y$.

If we define the mixed currency account for the collateral of an asset in currency X by a rate c in currency Y by

$$N_t^{X,c,Y} = \exp \left(\int_0^t c_\tau + p_\tau^{X,Y} d\tau \right).$$

Our previous result specialised to the currency case now reads like (Pallavicini et al., 2012, Proposition 3.7) and (Fujii and Takahashi, 2013, Theorem 6.1).

Theorem 5 (Foreign currency collateral formula) *In presence of collateral in currency Y with rate c_t, the quote in time t of an asset with quote $V_u^{X,c,Y}$ in time u is*

$$V_t^{X,c,Y} = N_t^{X,c,Y} \, \mathrm{E}^{\mathbb{X}} \left[(N_u^{X,c,Y})^{-1} V_u^{X,c,Y} \,\middle|\, \mathcal{F}_t \right] \tag{14}$$

Note that this is simply the result for single currency collateral with the collateral being another currency. The cross-currency value $V_t^{X,c,Y}$ is simply the single currency value $V_t^{X,c+r^X-r^Y,X}$.

3 Multi-curve framework with collateral

3.1 Forward index rate and spread

We apply the general collateral results to the multi-curve framework. Like in Henrard (2010) and Henrard (2013a), the multi-curve framework with collateral supposes the existence of a specific set of assets, the Ibor coupons.

I A j floating coupon in currency X with collateral in currency Y at rate c is an asset for each tenor j, each fixing date θ, each collateral rate c and each currency X and Y.

The curve description approach is based on the following definitions.

Definition 7 (Forward index rate with collateral) *The* forward curve $F_t^{X,c,Y,j}(\theta, u, v)$ *is the continuous function such that,*

$$P_X^{c+p^{X,Y}}(t, v)\delta F^{X,c,Y,j}(t, u, v) \tag{15}$$

is the quote in t of the j-Ibor coupon in currency X with fixing date θ, start date u, maturity date v ($t \le t_0 \le u = \mathrm{Spot}(t_0) < v$) and accrual factor δ collateralised at rate c in currency Y.

Let $I_X^j(t_0)$ denote the fixing rate published on t_0 for the index in currency X for the period $[u, v]$ of length j. The fixing rate is independent of the collateral. On the fixing date we have

$$F^{X,c,Y,j}(\theta, u, v) = I_X^j(\theta)$$

for any collateral rate c and currency Y.

An IRS is described by a set of fixed coupons or cash flows c_i at dates $\tilde{t}_i$ ($1 \le i \le \tilde{n}$). For those flows, the collateral discounting curve is used. It also contains a set of floating coupons over the periods $[t_{i-1}, t_i]$ with $t_i = t_{i-1} + j$ ($1 \le i \le n$). The accrual factors for the periods $[t_{i-1}, t_i]$ are denoted δ_i. The value of a (fixed rate) receiver IRS in $t < t_0$ is

$$\sum_{i=1}^{\tilde{n}} c_i P_X^{c+p^{X,Y}}(t, \tilde{t}_i) - \sum_{i=1}^{n} P_X^{c+p^{X,Y}}(t, t_i)\delta_i F^{X,c,Y,j}(t, t_{i-1}, t_i). \tag{16}$$

In the textbook one curve pricing approach, the IRS are usually priced through either the *discounting forward rate* approach or the *cash flow equivalent* approach. The *discounting forward rate* approach is similar to the above formula.

As with the one curve framework, we can define a forward swap rate. This is the fixed rate for which the vanilla IRS price is 0:

$$S_t^{X,c,Y,j} = \frac{\sum_{i=1}^{n} P_X^{c+p^{X,Y}}(t, t_i)\delta_i F^{X,c,Y,j}(t, t_{i-1}, t_i)}{\sum_{i=1}^{\tilde{n}} P_X^{c+p^{X,Y}}(t, \tilde{t}_i)\tilde{\delta}_i}. \tag{17}$$

3.2 Same currency collateral

Suppose that the coupon currency X is the same as the collateral currency Y.

In the c-collateral v-forward numeraire described after Definition 3, the collateralised forward satisfies

$$F_t^{c,j}(\theta, u, v) = \mathrm{E}^{c,v}\left[F_s^{c,j}(\theta, u, v)\middle| \mathcal{F}_t\right], \tag{18}$$

i.e. the forward rate is a martingale.

Like in Henrard (2010) and Henrard (2013a), we can define the multiplicative spreads between the different curves.

Definition 8 (Multiplicative spread) *The* multiplicative spread $\beta^{A|B}$ *between the forward rate $A \in \{D; c; c, j\}$ and the forward rate B is defined by*

$$(1 + \delta F_t^A(u, v))\beta_t^{A|B}(u, v) = 1 + \delta F_t^B(u, v). \tag{19}$$

Like in the standard multi-curve framework, we write the spread in a multiplicative way. The multiplicative spread is natural in interest rate modelling as it corresponds to the composition of rates.

Using that notation the quote of a collateralised coupon becomes

$$P_X^c(t,v)(1 + \delta F_t^c(u,v))\beta_t^{c|c,j}(u,v) - P_X^c(t,v) = P_X^c(t,u)\beta_t^{c|c,j}(u,v) - P_X^c(t,v).$$

We have a formula similar to the one curve framework where the value of a coupon is obtained by discounting the notional at start date and at end date. Here the start date amount is adjusted (by the ratio β) to take into account the spread between the discounting curve and the index projection curve.

3.3 Overnight-indexed swaps

Up to now we have defined the collateral discount factors, but have not describe how to obtain them from the market. The most commonly used collateral rates are overnight rates. There exist liquid market instruments with pay-off linked to those rates; they are called *overnight indexed swaps* (OIS). We show how to price OISs in an overnight collateral framework.

The proof below is adapted from Quantitative Research (2012) where a similar proof is provided for compounded coupons with deterministic spread between the discounting and forward curves. In Fujii et al. (2010a) a similar result is provided for continuously compounded rates (not periodically compounded) and no spread. A similar-looking result is also proposed in (Macey, 2011, Section 3.3.1) but from undefined notations.

The overnight rate for the period $[u,v]$ is fixed in u and is denoted $I^{X,O}(u,v)$. The date v is u plus one business day. The associated accrual factor is δ.

Suppose the rate paid on the collateral account is the overnight rate plus a deterministic spread, which means that N_v is $\mathcal{F}_u$-adapted and

$$N_v^c(N_u^c)^{-1}\beta_u^{c|O}(u,v) = 1 + \delta I^{X,O}(u,v)$$

with $\beta_u^{c|O}$ deterministic. Usually β will come from a constant spread like $\beta_u = 1 + \delta s$ or $\beta_u(u,v) = \exp(s(v-u))$ with s a constant.

Using the Equation (5) defining the collateral discount factor and the above equality, we have

$$P_X^c(u,v)(1 + \delta F_u^{c,O}(u,v)) = P_X^c(u,v)(1 + \delta I_u^{X,O}(u,v)) = \beta_u^{c|O}(u,v).$$

By Definition 7 with the overnight index O, the above quantity in u is the price of an asset (fixed amount plus overnight coupon). Its price satisfies the generic collateral pricing and

$$P_X^c(s,v)(1 + \delta F_s^{c,O}(u,v)) = N_s^c \, \mathrm{E}^X\left[(N_u^c)^{-1}\big|\mathcal{F}_s\right]\beta_u^{c|O}(u,v) = P_X^c(s,u)\beta_u^{c|O}(u,v).$$

The forward index rate collateralised by the same rate satisfies the familiar looking equation:

Theorem 6 (One period overnight forward rate value) *The rate of a one period (that is with $v = u + 1$ day) overnight coupon, when the collateral rate is the overnight rate multiplied by a deterministic spread, is*

$$1 + \delta F_s^{X,O}(u,v) = \frac{P_X^c(s,u)}{P_X^c(s,v)}\beta_u^{c|O}(u,v).$$

The description of an overnight indexed (compounded) coupon is as follows. The times associated are denoted $(t_i)_{i=0,\dots,n}$. The rate for the period $[t_{i-1}, t_i]$ is fixed in t_{i-1} $(i = 1, \dots, n)$. The accrual factor in the index convention for the period $[t_{i-1}, t_i]$ is δ_i.

The overnight coupon pays the amount

$$\left(\prod_{i=1}^{n}(1 + \delta_i I^{X,O}(t_{i-1}, t_i)) \right) - 1$$

in t_p. In this section we suppose that $t_p = t_n$.

Using the result above, we have, for the expected value of the product in the numeraire defined by Equation 6,

$$
\begin{aligned}
& \mathrm{E}^{c,t_n}\left[\left(\prod_{i=1}^{n}(1 + \delta_i F_{t_{i-1}}^{c,O}(t_{i-1}, t_i)) \right) \right] \\
&= \mathrm{E}^{c,t_n}\left[\left(\prod_{i=1}^{n-1} \frac{P_X^c(t_{i-1}, t_{i-1})}{P_X^c(t_{i-1}, t_i)} \mathrm{E}^{c,t_n}\left[\frac{P_X^c(t_{n-1}, t_{n-1})}{P_X^c(t_{n-1}, t_n)} \middle| \mathcal{F}_{t_{n-2}} \right] \right) \right] \prod_{i=1}^{n-1} \beta_{t_{i-1}}^{c|O}((t_{i-1}, t_i)) \\
&= \mathrm{E}^{c,t_n}\left[\left(\prod_{i=1}^{n-1} \frac{P_X^c(t_{i-1}, t_{i-1})}{P_X^c(t_{i-1}, t_i)} \frac{P_X^c(t_{n-2}, t_{n-1})}{P_X^c(t_{n-2}, t_n)} \right) \right] \prod_{i=1}^{n-1} \beta_{t_{i-1}}^{c|O}((t_{i-1}, t_i)) \\
&= \mathrm{E}^{c,t_n}\left[\left(\prod_{i=1}^{n-2} \frac{P_X^c(t_{i-1}, t_{i-1})}{P_X^c(t_{i-1}, t_i)} \frac{P_X^c(t_{n-2}, t_{n-2})}{P_X^c(t_{n-2}, t_n)} \right) \right] \prod_{i=1}^{n-1} \beta_{t_{i-1}}^{c|O}((t_{i-1}, t_i)) \\
&= \mathrm{E}^{c,t_n}\left[\left(\prod_{i=1}^{n-2} \frac{P_X^c(t_{i-1}, t_{i-1})}{P_X^c(t_{i-1}, t_i)} \frac{P_X^c(t_{n-3}, t_{n-2})}{P_X^c(t_{n-3}, t_n)} \right) \right] \prod_{i=1}^{n-1} \beta_{t_{i-1}}^{c|O}((t_{i-1}, t_i)).
\end{aligned}
$$

In the first equality we use Theorem 6, the fact that F_s^O is $\mathcal{F}_{t_{n-2}}$-measurable for $s \leq t_{n-2}$ and the tower property. In the second equality, we use the martingale property of the c-quotes rescaled by the c-pseudo-discount factors with maturity t_n in the $\mathrm{E}^{c,t_n}[]$ measure. The third equality is simply the simplification of fractions. In the fourth equality we use the same technique as in the first and second but with $\mathcal{F}_{t_{n-3}}$. The same procedure can be repeated down for all factors in the product and we obtain

$$\mathrm{E}^{c,t_n}\left[\frac{P_X^c(t_0, t_0)}{P_X^c(t_0, t_n)} \right] \prod_{i=1}^{n-1} \beta_{t_{i-1}}^{c|O}((t_{i-1}, t_i)).$$

After applying once more the martingale property and the definition of the numeraire, we obtain the result:

Theorem 7 *Under the hypothesis **A**, **I**, the price of the overnight indexed (compounded) coupon described above in the overnight collateral framework is given by*

$$(P_X^c(0, t_0) - P_X^c(0, t_n)) \prod_{i=1}^{n-1} \beta_{t_{i-1}}^{c|O}((t_{i-1}, t_i)).$$

The result means that for collateral equal to the overnight index plus a deterministic spread, the OIS curve can be built like in the one-curve framework (taking the spread into account). The formula is very simple and resembles the one for a deposit. Once we have this first building block,

we can, like in the multi-curve framework without collateral, start to build index curves with collateral. The index (forward) curves are the curves as described in Definition 7 for which the model quote is equal to its market quote (or for which the model par rate is equal to the market par rate).

In this situation the tools developed for curve calibration in the multi-curve framework without collateral can still be used. We refer in particular to Ametrano and Bianchetti (2013) and Henrard (2013a) for more details on challenges of the curve calibration.

As mentioned earlier the "OIS discounting" framework requires that not only the instrument analysed is collateralised at overnight, but that the OIS themselves are collateralised at overnight and that the above formula is correct. Not all those conditions are satisfied in all markets. For example in SGD, there is a (quite illiquid) market on OIS with fixing on SONAR. But most of the cash collateral pays an interest indexed on overnight SIBOR, a different index (Tee (2013)). The "OIS discounting" rule cannot be applied sensu stricto in the SGD market.

3.4 Change of collateral: independent spread

Suppose that the spread between two collateral rates is independent of the initial rate like in Section 2.3, this is

$$c_t^2 = c_t^1 + s_t$$

with s_t a $\mathcal{G}_t^2$-adapted random variable and I_X^j and c_1 are $\mathcal{G}_t^1$ adapted with $\mathcal{G}_t^2$ independent of $\mathcal{G}_t^1$. The forward rates are defined through the value of coupons and are, by Equation (8) and (9), linked by

$$P_X^{c_2}(0,v)\delta F^{c_2,j}(0,u,v) = P_X^s(0,v)P_X^{c_1}(0,v)\delta F^{c_1,j}(0,u,v) = P_X^{c_2}(0,v)\delta F^{c_1,j}(0,u,v)$$

This means that the forward rates are the same:

$$F^{c_2,j}(0,u,v) = F^{c_1,j}(0,u,v)$$

If a forward curve has been calibrated for a given collateral rate it can be used for other collateral rates if the spread is independent. Obviously the equality does not extend to quotes, which are discounting dependent. Also it does not extend to swap rates. The swap rates do not depend only on forward rates but also on discounting. We will come back to not independent spreads later and introduce the convexity adjustment required in those cases.

3.5 Collateral in foreign currency and independent spread

In this section we want to price interest rate products linear on Ibor rates but with collateral in a foreign currency. At a later stage we want to price cross-currency swaps between the domestic currency and a foreign currency with the collateral in the domestic currency (i.e. a foreign currency for the foreign currency leg).

Using the previous notations we want to price $V^{Y,c,X}$. Let the standard collateral used in Y be denoted c^Y. In particular the currency Y IRS quoted in the market uses that collateral rate. From the previous development one can obtain the curve $F^{Y,c^Y,Y,j}$ and $P_Y^{c^Y}$ from the market. Suppose that the standard collateral rule in currency X refers to the rate c^X. In particular we can obtain the discounting curve $P_X^{c^X}$ from the market. Remember that the discounting in currency Y for

collateral in a currency X at rate c^X is done with the rate $c_t^X + p_t^{Y,X}$. Suppose that the collateral and risk free rates are, like in Section 2.3, such that

$$c_t^X + p_t^{Y,X} = c_t^Y + s_t$$

with s_t a $\mathcal{G}_t^2$-adapted random variable and $I^{Y,j}$ and $c^X + p^{Y,X}$ $\mathcal{G}_t^1$-adapted with $\mathcal{G}_t^2$ independent of $\mathcal{G}_t^1$. We don't know s yet (and neither $p^{X,Y}$) but we suppose that the rates have the above relationship. For a price with common value V_u we have, using the remark after Theorem 5 and Equation (8),

$$
\begin{aligned}
V_0^{Y,c^X,X} &= P_Y^s(0,v)N_0^{c^Y} \, \mathrm{E}^Y \left[(N_v^{c^Y})^{-1} I_Y^j(t_0) \right] \\
&= P_Y^s(0,v) P_Y^{c^Y}(0,v) F^{Y,c^Y,Y}(0,u,v) \\
&= P_Y^{s+c^Y}(0,v) F^{Y,c^Y,Y}(0,u,v).
\end{aligned}
$$

The collapse of $P_Y^s P_Y^{c^Y}$ into $P_Y^{s+c^Y}$ is due to the independence hypothesis as noted in Equation (9).

The value of a fixed collateralised cash flow N in v can be written, by definition, as

$$V_0^{Y,c^X,X} = P_Y^{s+c^Y}(0,v)N.$$

In cross-currency swaps, the pricing of fixed coupons is required for the potential spreads and for the notional payments.

Suppose that cross-currency swaps are collateralised in domestic currency. By providing the spread in the cross-currency swap described above, the market is providing the price of the foreign leg with collateral in the domestic currency. On the theoretical side, the price can be obtained from two curves: the new discounting curve including the standard collateral and the unknown spread $P^{Y,s+c^Y}$ and the forward rate curve $F^{Y,c^Y,Y}$. The second curve is known from the foreign currency swap market using the independence hypothesis. We are thus faced with solving a "one curve and one set of constraints" problem. This new curve calibration can be done using the same techniques used for standard multi-curve framework calibration. The inputs are different but the numerical techniques are the same.

The main challenge is probably to keep track of the numerous inputs and relations between them to create a large Jacobian/transition matrix which covers all the inputs and outputs. Such a mechanism was described in Quantitative Research (2013) for the multi-curve framework. We extend it below to the collateral multi-currency framework.

3.6   Keeping track of transition matrices

To obtain the transition matrices between the market quotes and the curve parameters, one can compute directly a huge matrix with all the market quotes as input and all the curve parameters as the output of a very large root-solving problem.

This is possible, but with many currencies, many collateral and many ways to combine them, one obtains easily 10 curves with 20 parameters each. Solving a 200x200 non-linear system of equation and computing the associated Jacobian matrix is probably not the most efficient way to achieve the result. As described above the calibration can be done in an inductive way for most of the curves. The last curves calibrated depend on all the previous curves, but at each step only a reduced system has to be solved. Usually the sub-system, that we call unit in the sequel, contains only one to three curves and not ten or more like the full system.

Suppose that the curves are obtained as a multidimensional root-finding process in an inductive way. The market rates/prices for instrument n are denoted m_n. The parameters to describe the calibrated curves are denoted p_n. There is the same number of market instruments as number of parameters for the curves. The parameters are grouped in units with indices $[n_{i-1}, n_i)$ where $n_{-1} = n_0 = 0$ and there are $n_i - n_{i-1}$ instruments and parameters in each unit.

The goal is to obtain the calibrated curves $(p_n)_{n \in [0,N)}$ from the market rates $(m_n)_{n \in [0,N)}$ but also the generalised transition matrix

$$(D_{m_l} p_n)_{n \in [0,N), l \in [0,N)} .$$

Due to the way the curves are built, we know that a good part of the matrix is full of zeros; the non-zero part, which we are interested in, is made of the sub-matrices

$$(D_{m_l} p_n)_{n \in [n_{i-1}, n_i), l \in [0, n_i)} .$$

The equations solved to obtain the parameters are at the i-th step:

$$s[n_{i-1}, n_i](p[0, n_{i-1}), p[n_{i-1}, n_i)) = 0$$

where the notation $x[i, j]$ represents the vector with values $(x_k)_{k \in [i,j)}$. The functions s_i are the spreads between the market rates and the rates computed from the same instrument implied by the curves with parameters $p[0, n_i)$.

Suppose that at each step, the previous step transition matrices $D_{m_l} p_n$ are available for $n \in [n_{i-2}, n_{i-1})$ and $l \in [0, n_{i-1})$. We can compute $D_{p_n} m_l = D_{p_n} s_l$ for $n, l \in [n_{i-1}, n_i)$ directly. Moreover

$$(D_{m_l} p_n)_{n, l \in [n_{i-1}, n_i)} = \left((D_{p_n} m_l)_{n, l \in [n_{i-1}, n_i)} \right)^{-1} .$$

So we have the derivative with respect to the new market rates. For the previous market rates, we use composition. Suppose that we have $D_{p_l} p_n$ for $n \in [n_{i-1}, n_i)$ and $l \in [0, n_{i-1})$. Then

$$(D_{m_l} p_n)_{n \in [n_{i-1}, n_i), l \in [0, n_{i-1})} = (D_{p_l} p_n)_{n \in [n_{i-1}, n_i), l \in [0, n_{i-1})} (D_{m_k} p_l)_{l \in [0, n_{i-1}), k \in [0, n_{i-1})}$$

The second factor is provided by the previous steps, we still have to obtain the first factor. Using the implicit function theorem, we have

$$(D_{p_l} p_n)_{n \in [n_{i-1}, n_i), l \in [0, n_{i-1})} = - \left((D_{p_n} s_k)_{k \in [n_{i-1}, n_i), n \in [n_{i-1}, n_i)} \right)^{-1} (D_{p_l} s_k)_{k \in [n_{i-1}, n_i), l \in [0, n_{i-1})} .$$

We now have all the required results to keep track of the full transition matrix.

In the following example we use an implementation[2] of the above construction. We build the USD Fed Fund, USD Libor 3M, EUR Eonia and EUR Euribor 3M as in the standard multi-curve framework. The USD Libor 3M is the one for collateral at Fed Fund and the EUR Euribor 3M is the one for the collateral at Eonia. We now want to construct the EUR discounting curve for collateral in USD at Fed Fund. We use the above developments with the hypothesis of independence to calibrate the curve itself and to build the associated extended transition matrix.

The transition matrix will depend of all the previous curves. The EUR Dsc USD FedFund parameters depend on five curves market rates. Using the notation of our implementation, we have the summary dependency in Table 1. It has to be interpreted as follows: the generalised

---

transition matrix depends on five curves (with the listed name), for each curve the numbers are the index of the first parameter of the curve and the number of parameters in the curve. Note that even if we have reused the Euribor 3M curves build with Eonia collateral for the forwards, we give to the copy a different name (EUR Fwd Euribor3M USD FedFund). Even if we do not compute convexity adjustment for that curve, In this simplified example we have 54 parameters.

```
USD Dsc FedFund=[0, 13],
EUR Dsc Eonia=[13, 12],
USD Fwd Libor3M FedFund=[25, 8],
EUR Fwd Euribor3M USD FedFund=[33, 8],
EUR Dsc USD FedFund=[41, 13]
```

Table 1: Summarised representation of the dependency of the collateral curve to the other curves of the example.

The total transition matrix for the last curve has 13 rows and 54 columns and is too large to be represented directly in this document. Instead we give a graphical representation of it in Figure 1. The dark squares indicate a strong dependency (absolute value above 0.10) and the grey squares a low dependency down to the white squares where the dependency is 0.
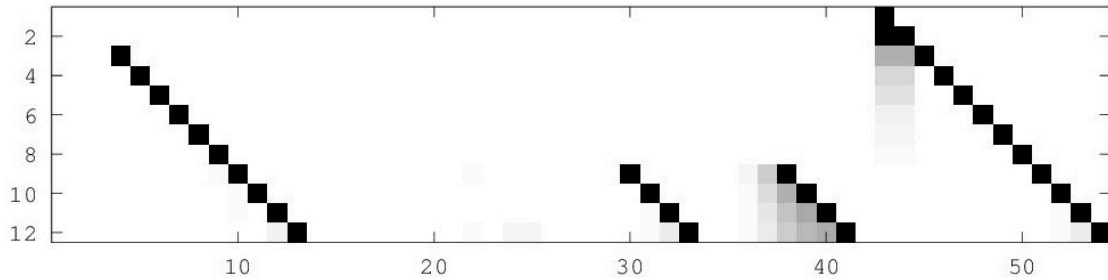

Figure 1: Visual representation of the transition matrix of the EUR discounting for collateral in USD with Fed Fund rates.

The discounting curve in EUR for collateral in USD obviously depends on the cross-currency instruments (FX swaps and cross-currency swaps), but also on the USD discounting (OIS), USD single currency swaps and EUR single currency swaps.

Using this huge quantity of information and managing the dependencies is probably is the main challenge in collateralised cross-currency trading and risk management.

3.7   Change of collateral: general case

In this section we give a general description of the pricing of swap-related products for general change of collateral (including foreign currency collateral). The result is written as a general *convexity adjustment* result. To be really useful in practice, the result should be made more explicit with a modelling of the different components and an estimate of the adjustment value.

Theorem 8 *The forward rate with collateral $c+s$ can be written as a function of the forward rate with collateral c through*

$$F_t^{c+s,j}(u,v) = F_t^{c,j}(u,v)\left(1 + \gamma^{c,v,s}(t,u,v)\right)$$

with

$$\gamma^{c,v,s}(t,u,v) = \frac{P_X^c(t,v)}{P_X^{c+s}(t,v)} \frac{\operatorname{cov}^{c,v}\left[N_t^s(N_v^s)^{-1}, I^{X,j}(u,v)\middle|\mathcal{F}_t\right]}{F_t^{c,j}(u,v)}.$$

Proof: The value of a coupon with collateral $c_t + s_t$ is given, by the definition of $F^{c+s,j}$, by

$$P_X^{c+s}(t,v)F_t^{c+s,j}(u,v).$$

Using Theorem 1, the price is also

$$
\begin{aligned}
V_t^{c+s} &= N_t^{c+s}\operatorname{E}^X\left[(N_v^{c+s})^{-1}I^{X,j}(u,v)\middle|\mathcal{F}_t\right]\\
&= N_t^s P_X^c(t,v)\operatorname{E}^{c,v}\left[(N_v^s)^{-1}I^{X,j}(u,v)\middle|\mathcal{F}_t\right]\\
&= N_t^s\operatorname{E}^{c,v}\left[(N_v^s)^{-1}\middle|\mathcal{F}_t\right]P_X^c(t,v)F_t^{c,j}(u,v)\left(1 + \frac{\operatorname{cov}^{c,v}\left[(N_v^s)^{-1},I^{X,j}(u,v)\middle|\mathcal{F}_t\right]}{\operatorname{E}^{c,v}\left[(N_v^s)^{-1}\middle|\mathcal{F}_t\right]F_t^{c,j}(u,v)}\right).
\end{aligned}
$$

From the definition of $\operatorname{E}^{c,v}[]$, we have

$$
\begin{aligned}
N_t^s\operatorname{E}^{c,v}\left[(N_v^s)^{-1}\middle|\mathcal{F}_t\right] &= N_t^s(P_X^c(t,v))^{-1}P_X^c(t,v)\operatorname{E}^{c,v}\left[(P_X^c(v,v))^{-1}(N_v^s)^{-1}\middle|\mathcal{F}_t\right]\\
&= N_t^s(P_X^c(t,v))^{-1}N_t^c\operatorname{E}^X\left[(N_v^c N_v^s)^{-1}\middle|\mathcal{F}_t\right]\\
&= P_X^{c+s}(t,v)(P_X^c(t,v))^{-1}.
\end{aligned}
$$

Combining the two results, we have

$$F_t^{c+s,j}(u,v) = F_t^c(u,v)\left(1 + \gamma^{c,v,s}(t,u,v)\right)$$

as announced. $\qquad\square$

4 Modelling with collateral: collateral HJM model

4.1 Collateral curve

The results described in this section are inspired by Fujii et al. (2011), Fujii and Takahashi (2013) and Pallavicini and Brigo (2013). From the definition of $P_X^c(t,v)$, the ratio $P_X^c(t,v)/N_t^c$ is a martingale in the N^X numeraire. If we model the ratio by a diffusion equation, we need an equation without drift. We write it, in a way reminding Heath et al. (1992) modelling of the risk free curve, like

$$d\left(\frac{P_X^c(t,v)}{N_t^c}\right) = -\frac{P_X^c(t,v)}{N_t^c}\nu^c(t,v)\cdot dW_t^X \tag{20}$$

where W_t^X is a n-dimensional Brownian motion in the standard measure associated to $\operatorname{E}^X[]$. The volatility ν^c can be stochastic. If we write the equation for $P_X^c(t,v)$, taking into account the definition of N^c, we have

$$dP_X^c(t,v) = P_X^c(t,v)c_t dt - P_X^c(t,v)\nu^c(t,v)\cdot dW_t^X. \tag{21}$$

21

Using the above equation, the definition of the collateral forward rate Definition 4 and Ito's lemma, we obtain the following equation for the collateral investment factor $1 + \delta F_t^c(u,v)$:

$$
\begin{aligned}
&d(1 + \delta F_t^c(u,v)) \\
&= \frac{dP^c(t,u)}{P^c(t,v)} - \frac{P^c(t,u)}{(P^c(t,v))^2} dP^c(t,v) + \left(-\frac{P^c(t,u)}{P^c(t,v)} \nu^c(t,u).\nu^c(t,v) + \frac{P^c(t,u)}{P^c(t,v)} |\nu^c(t,v)|^2 \right) dt \\
&= (1 + \delta F_t^c(u,v)) \Big((\nu^c(t,v) - \nu^c(t,u)) \cdot dW_t^X + \nu^c(t,v) \cdot (\nu^c(t,v) - \nu^c(t,u)) dt \Big).
\end{aligned}
$$

This leads to the following solution for the collateral forward rates.

Lemma 1 (HJM dynamic of collateral forward rates) *In the collateral HJM model, the collateral forward rates satisfy, for $t \geq s$,*

$$
1 + \delta F_t^c(u,v) = (1 + \delta F_s^c(u,v)) \exp\left(-\alpha_c(s,t,u,v) X_{s,t}^c - \frac{1}{2}\alpha_c^2(s,t,u,v) \right) \gamma_c(s,t,u,v) \tag{22}
$$

with

$$
\alpha_c(s,t,u,v) X_{s,t}^c = \int_s^t (\nu^c(\tau,u) - \nu^c(\tau,v)) \cdot dW_\tau^X,
$$

$$
\alpha_c^2(s,t,u,v) = \int_s^t |\nu^c(\tau,u) - \nu^c(\tau,v)|^2 d\tau
$$

and

$$
\gamma_c(s,t,u,v) = \exp\left(\int_s^t \nu^c(t,v) \cdot (\nu^c(\tau,v) - \nu^c(\tau,u)) d\tau \right).
$$

The same lemma can be written in term of the discount factors, instead of the investment factor $1 + \delta F$. By inverting the above result, we have:

Lemma 2 (HJM dynamic of collateral discount factors) *In the collateral HJM model, the collateral discount factors satisfy, for $t \geq s$,*

$$
\frac{P_X^c(t,v)}{P_X^c(t,u)} = \frac{P_X^c(s,v)}{P_X^c(s,u)} \exp\left(\alpha_c(s,t,u,v) X_{s,t}^c - \frac{1}{2}\alpha_c^2(s,t,u,v) \right) \gamma_c(s,t,v,u). \tag{23}
$$

We now detail a specific model where the above equation for dP_X^c is satisfied.

When the discount curve $P_X^c(t,.)$ is absolutely continuous, which is something that is always the case in practice as the curve is constructed by some kind of interpolation from market quotes, the equivalent of the instantaneous forward rate can be defined for the collateral pseudo-discount factors.

Definition 9 *Suppose that the pseudo-discount factor curve $P_X^c(t,.)$ is absolutely continuous. The instantaneous forward collateral rate $f^c(t,u)$ is defined by*

$$
P_X^c(t,u) = \exp\left(-\int_t^u f_X^c(\tau,u) d\tau \right).
$$

This is equivalent to, for a weak differentiation,

$$f_X^c(t, u) = -D_u \ln(P_X^c(t, u)).$$

Let $\sigma^c : \mathbb{R}^2 \to \mathbb{R}^n$ and

$$\nu^c(t, u) = \int_t^u \sigma^c(t, \tau) d\tau.$$

From Equation (21) and Ito's lemma, we have

$$\ln(P_X^c(t, u)) - \ln(P_X^c(0, u)) = \int_0^t \left(c_\tau - \frac{1}{2} \nu^2(\tau, u) \right) d\tau + \nu(\tau, u) \cdot dW_\tau^X.$$

Taking the opposite of the derivative with respect to u on both sides, and exchanging integral and derivative in the right-hand side term, we obtain

$$df^c(t, u) = \sigma^c(t, u) \cdot \nu^c(t, u) dt + \sigma^c(t, u) \cdot dW_t.$$

in the standard measure.

We defined the short rate associated with the instantaneous forward rate by $r_t^c = f_X^c(t, t)$. We will show below that the short rate r^c is equal to the collateral rate c.

By definition of r^c,

$$
\begin{aligned}
r_\tau^c &= f_X^c(\tau, \tau) = f_X^c(u, \tau) + \int_u^\tau df_X^c(s, \tau) ds \\
&= f_X^c(u, \tau) + \int_u^\tau \nu^c(s, \tau) \cdot D_2 \nu^c(s, \tau) ds + \int_u^\tau D_2 \nu^c(s, \tau) \cdot dW_s^X.
\end{aligned}
$$

Then using Fubini, we have

$$\int_u^v r_\tau^c d\tau = \int_u^v f_X^c(u, \tau) d\tau + \frac{1}{2} \int_u^v |\nu^c(s, v)|^2 + \int_u^v \nu^c(s, v) \dot{d}W_s^X$$

where we have used the fact that $\nu^c(s, s) = 0$.

The result is obtained by taking the exponential of the last equality and using the definition of f_X^c:

$$N_u^{r^c} (N_v^{r^c})^{-1} = \exp \left(-\int_u^v r_\tau^c d\tau \right) = P_X^c(u, v) \exp \left(-\int_u^v \nu^c(s, v) \cdot dW_s^X - \frac{1}{2} \int_u^v |\nu^c(s, v)|^2 ds \right).$$

Using the above equality and the above lemma for pseudo discount factors, we have that

$$\frac{P_X^c(t, v)}{N_t^{r^c}} = \frac{P_X^c(0, v)}{N_0^{r^c}} \exp \left(-\int_0^t \nu^c(s, v) \cdot dW_s^X - \frac{1}{2} \int_0^t |\nu^c(s, v)|^2 ds \right).$$

This implies that

$$dP_X^c(t, v) = P_X^c(t, v) r_t^c - P_X^c(t, v) \nu^c(t, v) \cdot dW_t^X.$$

Combined with Equation (21), it proves that $r_t^c = c_t$.

Lemma 3 (HJM dynamic of collateral cash account) *In the collateral HJM model, the collateral cash account N^c satisfies the equation*

$$N_u^c (N_v^c)^{-1} = P_X^c(u, v) \exp \left(-\int_u^v \nu^c(s, v) \cdot dW_s^X - \frac{1}{2} \int_u^v |\nu^c(s, v)|^2 ds \right). \tag{24}$$

The equivalence between the collateral rate c and the short term rate r^c can be obtained more directly (Fujii (2013)). Using the definition of P_X^c and of f_X^c, we have

$$
\begin{aligned}
c(t,u) &= -D_u \ln P_X^c(t,u) \\
&= \frac{1}{P_X^c(t,u)} \mathrm{E}^X \left[\exp\left(-\int_t^u c_\tau d\tau \right) c_u \,\middle|\, \mathcal{F}_t \right]
\end{aligned}
$$

If one takes $u = t$ in the above equation, one obtains

$$
c(t,t) = \mathrm{E}^X \left[c(t) \,\middle|\, \mathcal{F}_t \right] = c(t).
$$

Using the above expression of $c(t,u)$ as an expectation and changing the measure, one has

$$
c(t,u) = \mathrm{E}^{c,u} \left[c(u) \,\middle|\, \mathcal{F}_t \right].
$$

The instantaneous forward collateral rate is the expected value in the c-collateral u-forward measure of the collateral rate.

4.2 STIR futures

STIR futures are an important instrument to include in any multi-curve and collateral framework discussion. Not only is it a very liquid instrument and an important source of market information, but from a technical point of view it involves a mechanism of margin payment equivalent to a 0 rate collateral. An understanding of the collateral impact would not be complete without a detailed analysis of those instruments. Technically, this is a simplified version of the change of collateral presented later.

In this section we provide an explicit pricing formula for STIR futures in a Gaussian HJM model using as data the collateral curve and the collateralised forward curve. By Gaussian HJM, we mean we use the curve modelling described in the previous section with ν_c deterministic. The results described are very close to the one described in Henrard (2013b) and in large part based on Henrard (2005).

For this section, we denote $\mathcal{G}_t^1$ the filtration generated by the collateral rate and the N^X numeraire. This is N_t^c, $P_x^c(t,c)$ and N_t^X are $\mathcal{G}_t^1$-adapted. As F^c is a martingale in the the $\mathrm{E}^{c,v}$ numeraire,

$$
1 + \delta F_t^c(u,v) = (1 + \delta F_s^c(u,v)) \exp\left(-\alpha_c(s,t,u,v) X_{s,t}^{c,v} - \frac{1}{2}\alpha_c^2(s,t,u,v) \right) \tag{25}
$$

with

$$
\alpha_c(s,t,u,v) X_{s,t}^{c,v} = \int_s^t (\nu^c(\tau,u) - \nu^c(\tau,v)) dW_\tau^{c,v}.
$$

and $W^{c,v}$ the Brownian motion associated to the $\mathrm{E}^{c,v}$ [] measure.

The main hypothesis on the index basis is

SjS The multiplicative spread $\beta_t^{c|c,j}$ is written as

$$
\beta_t^{c|c,j}(u,v) = \beta_0^{c|c,j} \mathcal{X}_t^{c|c,j} \left(\frac{1 + \delta F_t^c}{1 + \delta F_0^c} \right)^a x^{c|c,j}(t,u,v)
$$

with $\mathcal{X}_t$ $\mathcal{G}_t^2$-adapted with $\mathcal{G}_t^2$ independent of $\mathcal{G}_t^1$ and $\mathcal{X}_t$ is a martingale in E^X [] with independent relative increments.

By independent relative increments, we mean that $\mathcal{X}_t/\mathcal{X}_s$ is independent of $\mathcal{F}_s$. In particular, this will be the case if $\mathcal{X}_t$ is a geometric Brownian motion.

The deterministic function $x^{c|c,j}(t)$ is selected such that the rate $F^{c,j}$ is a martingale in $\mathrm{E}^{c,v}[]$ as required.

In our modelling assumptions, we do not allow the spread β to depend on the risk-free rate, if such a rate exists. The spread depends only on the collateral rate c and a random variable $\mathcal{X}$ independent of c and of the standard numeraire N^X. We restrict the dependency on quantities provided by the market.

Note that as $\mathcal{X}_t$ is a martingale in $\mathrm{E}^X[]$ and $\mathcal{X}_t$ is $\mathcal{G}_t^2$-adapted and the numeraire of $\mathrm{E}^X[]$ and $\mathrm{E}^{c,v}[]$ are $\mathcal{G}_t^1$ adapted, then $\mathcal{X}_t$ is also a martingale in $\mathrm{E}^{c,v}[]$.

The choice of $x^{c|c,j}$ is similar to the one in Henrard (2013b) and is obtained, for $t < s$, through

$$
\begin{aligned}
1 + \delta F_t^{c,j} &= \mathrm{E}^{c,v}\left[1 + \delta F_s^{c,j}\middle|\mathcal{F}_t\right] \\
&= \mathrm{E}^{c,v}\left[\frac{(1+\delta F_s^c)^{1+a}}{(1+\delta F_t^c)^a}\beta_t\frac{\mathcal{X}_s}{\mathcal{X}_t}\frac{x(s)}{x(t)}\middle|\mathcal{F}_t\right] \\
&= \mathrm{E}^{c,v}\left[\exp\left(-\frac{1}{2}(1+a)\alpha_c(t,s)X_{t,s}^{c,v} - \frac{1}{2}(1+a)^2\alpha_c^2(t,s)\right)\frac{\mathcal{X}_s}{\mathcal{X}_t}\middle|\mathcal{F}_t\right] \\
&\quad (1+\delta F_t^{c,j})\frac{x^{c|c,j}(s)}{x^{c|c,j}(t)}\exp\left(\frac{1}{2}(1+a)a\alpha_c^2(t,s)\right)
\end{aligned}
$$

Given that the exponential is a martingale in the $\mathrm{E}^{c,v}[]$ numeraire and $X_{s,t}$ and $\mathcal{X}_t/\mathcal{X}_s$ are independent of $\mathcal{F}_f$, the expected value is 1. The equality is true only if

$$
x^{c|c,j}(t,u,v) = \exp\left(-\frac{1}{2}(1+a)a\alpha_c^2(0,t,u,v)\right).
$$

Theorem 9 (Futures price with stochastic spread) *Let $0 \le t \le t_0 \le u \le v$. In the multicurve and collateral framework with the collateral HJM model and **SjS** hypothesis on the spread, the price of the futures fixing in t_0 for the period $[u,v]$ with accrual factor δ is given in t by*

$$
\Phi_t^j = 1 + \frac{1}{\delta} - \frac{1}{\delta}(1+\delta F^{c,j}(t,u,v))\gamma_c^{1+a}(t,t_0,v). \tag{26}
$$

Proof: Using the generic pricing futures price process theorem which is equivalent to the collateral pricing formula with 0 collateral rate,

$$
\Phi_t^j = \mathrm{E}^X\left[1 - I_X^j(t_0)\middle|\mathcal{F}_t\right].
$$

The Ibor rate can be written as

$$
1 - I_X^j(t_0) = 1 + \frac{1}{\delta} - \frac{1}{\delta}\beta_t^{c|c,j}(u,v)\frac{\mathcal{X}_{t_0}}{\mathcal{X}_t}\frac{(1+\delta F^c(t_0,u,v))^{1+a}}{(1+\delta F^c(t,u,v)^a}\frac{x^{c|c,j}(t_0)}{x^{c|c,j}(t)}.
$$

Using Lemma 1 and the definition of $x^{c|c,j}$ we have

$$
\begin{aligned}
1 - I_X^j(t_0) &= 1 + \frac{1}{\delta} - \frac{1}{\delta}\beta_t^{c|c,j}(u,v)\frac{\mathcal{X}_{t_0}}{\mathcal{X}_t}(1+\delta F^c(t,u,v)) \\
&\quad \exp\left(-(1+a)\alpha_c(t,t_0)X_{t_0}^c - \frac{1}{2}(1+a)^2\alpha_c^2(t,t_0)\right)\gamma_c^{1+a}(t,t_0).
\end{aligned}
$$

25

The variables X^c_{t,t_0} and $\mathcal{X}_{t_0}/\mathcal{X}_t$ are independent of $\mathcal{F}_t$ and independent of each other, so

$$
\begin{aligned}
\mathrm{E}^N &\left[\frac{\mathcal{X}_{t_0}}{\mathcal{X}_t} \exp\left(-(1+a)\alpha_c(t,t_0)X^c_{t_0} - \frac{1}{2}(1+a)^2\alpha_c^2(t,t_0) \right) \middle| \mathcal{F}_t \right] \\
&= \mathrm{E}^N\left[\frac{\mathcal{X}_{t_0}}{\mathcal{X}_t} \exp\left(-(1+a)\alpha_c(t,t_0)X^c_{t_0} - \frac{1}{2}(1+a)^2\alpha_c^2(t,t_0) \right) \right] \\
&= \mathrm{E}^N\left[\frac{\mathcal{X}_{t_0}}{\mathcal{X}_t} \right] \mathrm{E}^N\left[\exp\left(-(1+a)\alpha_c(t,t_0)X^c_{t_0} - \frac{1}{2}(1+a)^2\alpha_c^2(t,t_0) \right) \right] = 1
\end{aligned}
$$

From there the result is direct. $\qquad\square$

As both collateralised Ibor coupon based instruments, like swaps, and futures are liquid in most markets, the above result can be used to calibrate part of the model. With the market providing information on both the collateralised forward rates $F^{c,j}$ and the futures prices Φ^j_t, one can obtain the market implied quantity γ_c^{1+a}. The quantity can be used as an input to the pricing of other instruments. In particular Henrard (2013b) shows how to use that information (together with cap/floor prices) to obtain the price of futures options coherent with both futures and cap/floor market and very light calibration.

4.3 Change of collateral: explicit formulas in collateral HJM

Suppose that we have two collateral rates c_1 and c_2. We suppose that both those rates satisfy the HJM model described in Section 4.1 with volatilities ν^{c_1} and ν^{c_2}. Let $\mathcal{G}^1_t$ be a filtration such that N^X and the rates c_1, c_2, F^{c^1} and F^{c^2} are $\mathcal{G}^1_t$-adapted.

Our goal is to write the forward rate with respect to one collateral $F^{c_2,j}$ as a function of the rate with the other collateral $F^{c_1,j}$ and some explicit adjustments. To achieve that result, we use a modelling framework of stochastic spreads very similar to the one of the previous section for STIR futures. The multiplicative spread between an Ibor index forward and a collateral forward is written through its dependent parts and independent part.

We use the following hypothesis with $\mathcal{G}^2_t$ a filtration independent of $\mathcal{G}^1_t$

SjS2 The multiplicative spread $\beta^{c_2|c_2,j}_t$ is written as

$$
\beta^{c_2|c_2,j}_t(u,v) = \beta^{c_2|c_2,j}_0 \mathcal{X}^{c_2|c_2,j}_t (1+\delta F^{c_1}_t)^{a_1}(1+\delta F^{c_2}_t)^{a_2} x^{c_2|c_2,j}(t,u,v)
$$

with $\mathcal{X}^{c_2|c_2,j}_t$ $\mathcal{G}^2_t$-adapted and a martingale in the measure associated to N^X. The deterministic function $x^{c_2|c_2,j}$ will be selected to ensure that the forward rate $F^{c_2,j}$ is a martingale. Note that, like in the previous section, as $\mathcal{X}^{c_2|c_2,j}$ is a $\mathrm{E}^N[]$ martingale and using the independence, $\mathcal{X}^{c_2|c_2,j}$ is a $\mathrm{E}^{c,v}[]$ martingale.

The developments are based on the two following results. The first one is equivalent to Equation (22) and the second one describes the dynamic of the collateral account given by Lemma 3 for each rate:

$$
\begin{aligned}
1 + \delta F^{c_i}(t,u,v) &= (1+\delta F^{c_i}(s,u,v)) \exp\left(-\int_s^t (\nu^{c_i}(\tau,u) - \nu^{c_i}(\tau,v))dW^X_\tau \right. \\
&\quad \left. -\frac{1}{2}\int_s^t (|\nu^{c_i}(\tau,u) - \nu^{c_i}(\tau,v)|^2)d\tau \right) \gamma_{c_i}(s,t)
\end{aligned}
$$

and

$$(N_v^{c_i})^{-1} = P_X^{c_i}(0, v) \exp\left(-\int_0^v \nu^{c_i}(\tau, v) \cdot dW_\tau^X - \frac{1}{2}\int_0^v |\nu^{c_i}(\tau, v)|^2 d\tau\right).$$

We first obtain the value of $x^{c_2|c_2,j}$ like in previous section. We know that $1 + \delta F^{c_2,j}$ is a martingale in $E^{c,v}$ []. Using exactly the same technique as in the previous section, we obtain, after a lengthy but relatively straightforward computation

$$\begin{aligned}
x^{c_2|c_2,j}(t) &= \exp\left(-\frac{1}{2}\int_0^t a_1(a_1 - 1)|\nu^{c_1}(\tau, u) - \nu^{c_1}(\tau, v)|^2 \right.\\
&\quad + (1 + a_2)a_2|\nu^{c_2}(\tau, u) - \nu^{c_2}(\tau, v)|^2\\
&\quad \left. + 2a_1(1 + a_2)(\nu^{c_1}(\tau, u) - \nu^{c_1}(\tau, v)) \cdot (\nu^{c_2}(\tau, u) - \nu^{c_2}(\tau, v))d\tau\right)
\end{aligned}$$

If we take $a_1 = 0$ and $a_2 = a$, we obtain $x^{c_2|c_2,j}(t) = x^{c|c,j}(t)$ for the function defined in the previous section.

To obtain the relation between $F^{c_1,j}$ and $F^{c_2,j}$ we use the definition of forward rate and the fact that $F^{c_2,j}(t_0) = F^{c_1,j}(t_0)$ as both are fixing on the same index. We have

$$P^{c_1}(0, v)(1 + \delta F^{c_1,j}(0)) = N_0^{c_1} E^X\left[(N_v^{c_1})^{-1}(1 + \delta F^{c_2,j}(t_0))\right].$$

We can then replace $N_v^{c_1}$ using the result above and develop $F^{c_2,j}$ using the hypothesis **SjS2** and writing the value $F^{c_i}(t_0)$ as function of $F^{c_i}(0)$ using the rate dynamic from above.

We obtain

$$P^{c_1}(0, v)(1 + \delta F^{c_1,j}(0)) = P^{c_1}(0, v)(1 + \delta F^{c_2,j}(0))\gamma_{c_1}^{a_1}(0, t_0)\gamma_{c_2}^{1+a_2}(0, t_0)\gamma_{c_1}^{-a_1}\zeta^{1+a_2}(0, t_0)$$

with

$$\xi(0, t_0, u, v) = \exp\left(\int_0^{t_0} \nu^{c_1}(\tau, v) \cdot (\nu^{c_2}(\tau, u) - \nu^{c_2}(\tau, v))d\tau\right).$$

We summarise all the adjustment in one variable

$$\zeta(a_1, 1 + a_2, u, v) = \gamma_{c_2}^{1+a_2}(0, t_0, u, v)\xi^{1+a_2}(0, t_0, u, v).$$

Theorem 10 (Change of collateral in HJM and stochastic spread) *In the HJM model for both collateral, if the forward rate spread satisfies the hypothesis* **SjS2**, *the forward rates in the different collateral are linked by*

$$(1 + \delta F_0^{c_1,j}(t_0, u, v)) = (1 + \delta F_0^{c_2,j}(t_0, u, v))\zeta(a_1, 1 + a_2, u, v)$$

From a practical point of view, the above result is as explicit as the computation of ζ.

Special cases

If $\nu^{c_1} = 0$, i.e. c_1 is deterministic, then $\zeta = \gamma_{c_2}^{1+a}$ as described in the STIR futures section. When one collateral rate is deterministic or 0, the problem is reduced to the same as the futures margining, which is equivalent to a collateral of 0.

If $\nu^{c_2} = \nu^{c_1}$, $\xi = \gamma_{c_2}^{-1}$ and there is no convexity adjustment. If the rates are moving together, then we are in a case similar to a deterministic spread.

Suppose that ν^{c_1} has two components associated to independent motions: $\nu^{c_1} = (\tilde{\nu}^{c_2}, \nu^s)$ with $\nu^{c_2} = (\tilde{\nu}^{c_2}, 0)$, with s interpreted as a spread. This means that the quantities associated to F^{c_1} evolve with two independent parts. The evolution of $1 + \delta F^{c_1}(\tau)$ is the product of two independent evolutions. In that case $\xi(0, t) = \gamma_{c_2}^{-1}$ and there is no convexity adjustment. We have a result similar to the independent spread result.

4.4 Collateral in foreign currency: correct discount factors with wrong instruments

We work in a set-up similar to the conclusion of last section and foreign currency collateral. Suppose that cross-currency swaps are collateralised in domestic currency and the foreign currency swaps are collateralised in local currency. Can we calibrate the discount factors and the forward rates directly and not by reusing the forward rates of a first calibration like in the previous section?

A priori we can not combine the domestic currency collateralised leg of the cross-currency swaps with foreign currency collateralised instruments to coherently collaborate the curves.

The start of the answer, on the short part of the curve, is yeas. As the forward rates are by hypothesis the same, the quotes of the FRAs collateralised in foreign currency is the same as the quote of the FRAs collateralised in domestic currency. The present value is collateral dependent, but the quote, which is the K such that the present value is zero:

$$P^c(O, v) \frac{\delta}{\beta} (F_0^{c,j}(0, u, v) - K) = 0$$

is the same for any P^c. The part of the curve where FRAs are used to build the foreign currency forward curve can be combine with FX swaps and cross-currency swaps in a coherent way. The quotes are the same for any collateralisation (under the "same forward" hypothesis).

When we move to the part of the curve calibrated to vanilla swaps, the same is not true. The swap rate is a weighted average of the forward rates (this is good) but the weights are discount factor dependent (this is bad). Using the notation of Equation 17,

$$S_t^{X,c,Y,j} = \frac{\sum_{i=1}^n P_X^{c+p^{X,Y}}(t, t_i) \delta_i F^{X,c,Y,j}(t, t_{i-1}, t_i)}{\sum_{i=1}^{\tilde{n}} P_X^{c+p^{X,Y}}(t, \tilde{t}_i) \tilde{\delta}_i}.$$

This weight is independent of the discounting in the particular case where the forward curve is constant and floating and fixed payments have the same frequency and convention (like GBP and AUD). When the forward rates are not constant is the change acceptable for typical change of collateral and a typical curve shape? To answer that question, we have to define "acceptable" and 'typical'. Here we loosely define "totally acceptable" as within the tick value of the instruments "acceptable" within the bid-offer of the instrument. For the graphs, we set the tick value at i.e. 0.1 basis point and the bid-offer at one basis point.

For the typical value of the spread, we have to remember what the spread is. It is not the difference between the interest rates in the two currencies but the basis spread for a cross-currency collateral index swaps which is in most application the cross-currency overnight indexed swaps. The market where the spread is the most liquid is EUR/USD and the spread is currently between -7 bps (short term) and -27.5 bps (long term). The spread were generally lower, i.e. more negative, by as much as 20 bp over last year. For the GBP/USD the spread is currently between -2 and -15 basis points. We can think of "typical" as anything between -30 and +30 bps.

In Figure 2, the difference for swap rates for difference collateral discounting curves are displayed. The figures are computed for GBP swaps (with standard GBP conventions). The initial forward pseudo-discount factors curve is very steep with short rates at 1.00% and long term rates at 4.00%. The difference is acceptable for spreads up to 10 bps. It is also acceptable for swaps below 7 years and spreads below 25 bps.

The lower difference for shorter swaps is understandable as for shorter swaps the forward rate have less variability and the approximation is better. This leads to a possible strategy to to reduce
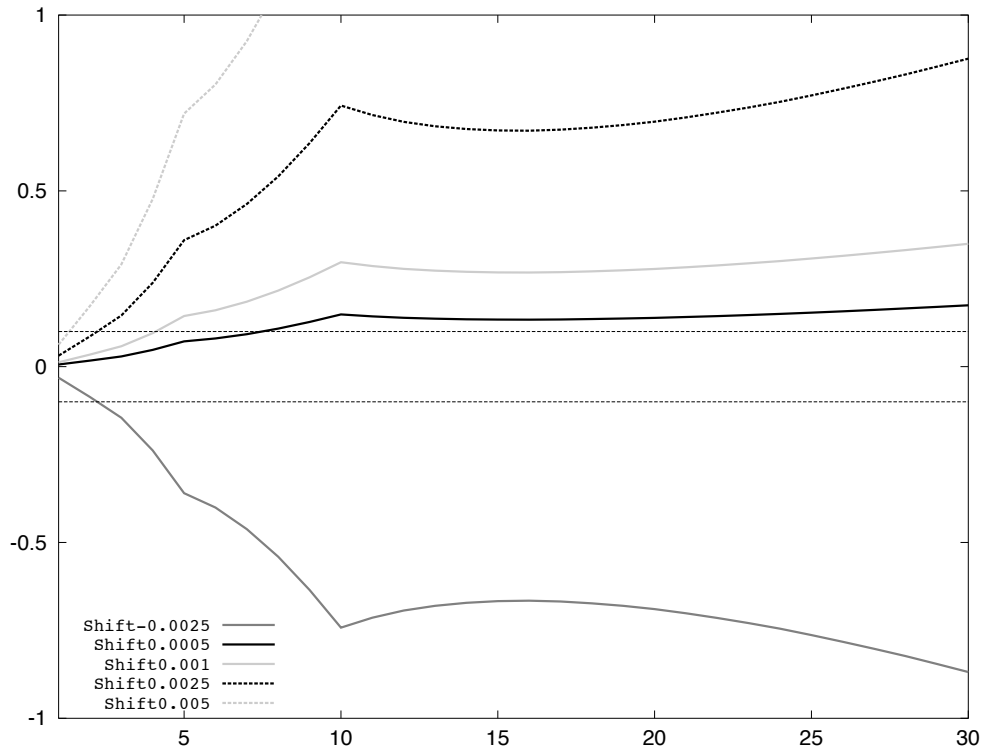
Figure 2: Differences in swap rates for different collateral discounting curves. Swap with tenors between 1 and 30 years. Figures for GBP swaps.

the difference: using shorter swaps for all the curve. This can be achieved by using forward swaps. In Figure 3(a) the difference is displayed for forward swaps of tenor one year (the *X*-axis represents the start tenor of the swap). For those short swaps, the difference is totally acceptable. Even if the instruments are not correct (the are collateralised in the domestic currency) they produce the correct information (the correct swap rates for swap collateralised in foreign currency) for any collateralisation spread.

Using one year swaps is maybe to much to ask. In Figure 3(b) we have display the difference for swaps with tenor 2 years for the starting years 0 to 7 and forward swaps with tenor 5 years for longer starting periods. The differences are totally acceptable for most of the starting points and tenors. The domestic currency collateral pseudo-discount factors in foreign currency can be computed from the foreign currency (forward) swap rates and cross-currency swaps collateralised in domestic currency with small differences.

Aknowledgements

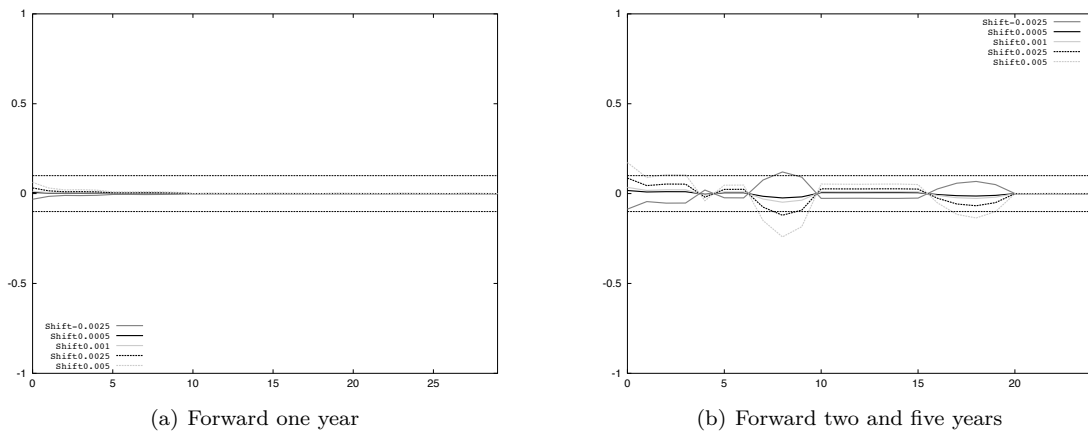(a) Forward one year

(b) Forward two and five years

Figure 3: Differences in swap rates for different collateral discounting curves. Forward swaps with start year given by the X-axis and tenor given in the subfigure. Figures for GBP swaps.

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