Transaction Cost Analytics in DRIP

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Optimal Execution of Portfolio Transactions

Overview, Scope, and Key Results

1. Portfolio Transactions under Market Impact: Almgren and Chriss (2000) consider the execution of portfolio transactions with the aim of minimizing a combination of volatility risk and transaction costs arising from temporary and permanent market impact.
2. Efficient Frontier under Linear Cost: For a simple linear cost model, they explicitly construct an *efficient frontier* in the space of time-dependent liquidation strategies, which have the minimum expected cost for a given level of uncertainty.
3. Choice of the Utility Function: This enables one to select optimal strategies either by minimizing a quadratic utility function, or by minimizing the Value-at-Risk.
4. Liquidity Adjusted Value at Risk: The latter choice leads to the concept of liquidity-adjusted VaR, or L-VaR, that explicitly considers the best trade-off between the volatility risk and the liquidity costs.

Motivation Background, and Synopsis

1. Transactions Changing the Portfolio Composition: Almgren and Chriss (2000) consider the optimal execution of portfolio transactions that move a portfolio from a given staring composition to a specified final composition within a specified period of time.
2. The Bertsimas and Lo Approach: Bertsimas and Lo (1998) define the best execution as the dynamic trading strategy that provides the minimum cost of trading over a fixed period of time, and they also show that under a variety of circumstances one can

find such a strategy by employing a dynamic optimization procedure; but they ignore the volatility of revenues of different trading strategies.

3. Maximization of Expected Trading Revenue: Almgren and Chriss (2000) work in the more general framework of maximizing the *expected revenue* – or equivalently minimizing the costs – with a suitable penalty for the *uncertainty* of revenue (or cost).
4. Market Microstructure Framework: This general framework arises in the market microstructure theory, but with a different purpose in mind. The *uninformed discretionary trader* trades an exogenous endowment over an exogenously specified amount of time to maximize the profits (Admati and Pfleiderer (1988)); the informed strategic trader trades over multiple periods on information not widely available, again to maximize profits (Kyle (1985)). In both cases the literature focuses on the link between the trader and the market maker, and a theory is produced to predict the market clearing price of the security at each period. Thus a trader's optimal strategy is used as a means to study the price formation in the markets, not as an object of interest in itself.
5. Variance of the Trading Cost: Almgren and Chriss (2000) study the variance of the trading cost in optimal execution because it fits in with the intuition that the trader's utility should figure in the definition of *optimal* in "optimal execution".
6. Example: Trading Illiquid Volatile Securities: For example, in trading a highly illiquid, volatile security, there are two extreme outcomes; trade everything now at a known, but high, cost, or trade in equal sized packets over a fixed time at a relatively lower cost. The latter strategy has a lower expected, but this comes at the expense of greater uncertainty in the final revenue.
7. Estimation of the Trading Uncertainty: How to evaluate the above uncertainty is partly subjective, and is a function of the trader's tolerance for risk. All that can be done is to insist that for a given level of uncertainty that the cost be minimized. This idea extends to a complete theory of optimal execution that includes an efficient frontier of optimal execution strategies.
8. Consistency with Expectations from Intuition: The framework of risk in execution yields several results that are consistent with the intuition. For example, it is evident

that all else equal, a trader will choose to execute a block of illiquid security less rapidly than a liquid security.

9. Models Lacking Consistency with Intuition: While this seems obvious, Almgren and Chriss (2000) demonstrate that a model that ignores risk does not have this property; without enforcing a strictly positive penalty for risk one cannot produce models that trade differently across the spectrum of liquidity.
10. Arithmetic Brownian Motion Price Dynamics: The incorporation of risk into optimal execution does not come without cost. First, in order to be able to produce tractable analytical results, Almgren and Chriss (2000) are forced to work in largely in the framework of price dynamics that are an arithmetic walk with independent increments.
11. Use of Static Optimization Procedures: They obtain results using *static optimization* procedures which they show lead to globally optimal trading trajectories. That is, optimal trading paths may be determined in advance of trading. Only the composition of the portfolio and the trader's utility function figure on the trading path.
12. Why does Static Optimization Work? The fact that the static strategy can be optimal even when the trader has the option to dynamically change his trading mid-course is a direct result of the assumptions of independence of returns and symmetry for the penalty functions for risk.
13. Using Non-Symmetric Penalty Functions: An interesting deviation from the symmetric penalty function was communicated by Ferstenberg, Karchmer, and Malamut at ITG Inc. They argue that the opportunity is a subjective quantity and is measures differently by different traders. Using a trader defined cost function g, they define opportunity costs as the expected costs of g applied to the average execution price obtained by the trader relative a benchmark price. They assume that the risk-averse traders will use a convex function g that is not symmetric in the sense that there is a strictly greater penalty for underperformance than for the same level of outperformance. They show that in this setting, the optimal strategy relative to g not only depends on the time remaining, but also on the performance of the strategy up to the present time, and the present price of the security. In particular, this means that in their setting, optimal strategies are dynamic.

14. Serial Correlations among Price Movements: As it is well known that price movements exhibit some serial correlations across various time horizons (Lo and MacKinlay (1988)), that market conditions change, and that some participants possess private information (Bertsimas and Lo (1998)), one may question the usefulness of results that obtain strictly in an independent-increment framework.
15. The Dynamic Nature of Trading: Moreover, as trading is known to be a dynamic process, the conclusion that optimal trading strategies can be statically determined calls for critical examination. Almgren and Chriss (2000) examine what quantitative gains are available that incorporate all the relevant information.
16. Impact of the Serial Correlations: First they consider short term serial correlations in price movements. They demonstrate that the marginal improvements available by explicitly incorporating this information into trading strategies is small, and more importantly, independent of the portfolio sizes; as portfolio sizes increase, the percentage gains possible decrease proportionately.
17. Combining "Correlated" and "Shifting" Strategies: The above is precisely true for linear transaction cost models, and is approximately true for more general models. The results of Bertsimas and Lo (1998) suggest that trading a strategy built to take advantage of serial correlation will essentially be a combination of a "correlation free" strategy and a "shifting strategy" that moves from one trade period to the next based on the information available in the last period's return. Therefore Almgren and Chriss (2000) argue that by ignoring serial correlation, they a) preserve the main interesting features of their analysis, and b) introduce virtually no bias away from "truly optimal" solutions.
18. Impact of Scheduled News Events: Second, Almgren and Chriss (2000) examine the impact of scheduled new events on optimal execution strategies. There is ample evidence that anticipated news announcements, depending on their outcome, can have a significant temporary impact on the parameters governing price movements.
19. Scheduled News Events - Literature Review: For a theoretical treatment see Brown, Harlow, and Tinic (1988), Kim and Verrecchia (1991), Easterwood and Nutt (1999), and Ramaswami (1999). For empirical studies concerning earnings announcements, see Patell and Wolfson (1984) for changes in mean and variance of intra-day prices,

and Lee, Mucklow, and Ready (1993) and Krinsky and Lee (1996) for changes in the bid-ask spread. For additional studies concerning news announcements, see Charest (1978), Morse (1981), and Kalay and Loewentstein (1985).

20. Model Incorporation of Scheduled Events: Almgren and Chriss (2000) work in a simple extension of their static framework by assuming that the security again follows an arithmetic random walk, but at a time known at the beginning of trading, an uncorrelated event will cause a material shift in price dynamics, e.g., an increase or decrease of volatility.
21. Combining Piece-Wise Static Strategies: In this context they show that optimal strategies are piece-wise static. To be precise, they show that an optimal strategy entails following a static strategy up to the moment of the event, followed by another static strategy that can only be determined once the outcome of the event in known.
22. Variation from the Original Static Strategy: It is interesting to note that the static strategy that one follows in the first leg is in general not the same strategy one would follow in the absence of information concerning the event.
23. Accommodating Unanticipated External “Sudden’ Events: Finally Almgren and Chriss (2000) note that any optimal execution strategy is vulnerable to *unanticipated events*. If such an event occurs during the course of trading and causes a material shift in the parameters of the price dynamics, then indeed a shift in the optimal trading trajectory must also occur.
24. Adaptation at Parameter Shift Edges: However if one makes a simplifying assumption that all events are either “scheduled” or “unanticipated” one then concludes that optimal execution is always a game of static trading punctuated by shifts in the trading strategies that adapt to material changes in the price dynamics.
25. Pre-determined vs. Active Approaches: If shifts are caused by events that are known ahead of time, then optimal execution benefits from a precise knowledge of the possible outcomes of the event. If not, the best approach is to be actively “watching” the market for such changes and react swiftly should they occur.
26. Simple Proxy for Unexpected Uncertainty: One approximate way to include such completely unexpected uncertainty into the model is to artificially raise the value of the volatility parameter.

27. Risk Averse Optimal Trading Strategies: As a first step, Almgren and Chriss (2000) obtain closed form solutions for trading strategies for any level of risk aversion.
28. Efficient Frontier of Optimal Strategies: They then show that this leads to an efficient frontier of optimal strategies, where an element of the frontier is represented by a strategy with a minimal level of cost for its level of variance of the cost.
29. Graphical Structure of the Frontier: The structure of the frontier is of some interest. It is a smooth convex function differentiable at its minimal point. The minimal point is what Bertsimas and Lo (1998) call the naïve strategy because is corresponds to trading equally sized packets using all available trading time equally.
30. Differential at the Minimum Point: The differentiability of the frontier at its minimum point indicates that one can obtain a first order reduction in the variance of the trading cost at the expense of only a second order in cost by trading a strategy slightly away from the globally minimal strategy.
31. Curvature at the Minimal Point: The curvature of the frontier at its minimum point is a measure of the liquidity of the security.
32. Half-Life of Optimal Execution: Another ramification of the Almgren and Chriss (2000) study is that for all levels of risk aversion except risk neutrality, optimal execution trades have a "half-life" that fall out of the calculations.
33. Independence form the Time to Complete Execution: A trade's half-life is independent of the actual specified time to liquidation, and is a function of the security's liquidity and volatility, and the trader's level of risk aversion.
34. Half-Life as Execution Time: As such Almgren and Chriss (2000) regard the half-life as an idealized time to execution, and perhaps a guide to the proper amount of time over which to execute a transaction.
35. Time Lesser than Half Life: If the specified time to liquidation is short relative to the trade's half-life, once can expect the cost of trading to be dominated by transaction costs.
36. Time Greater than Half Life: If the time to trade is long relative to the half-life, one can then expect most of the liquidation to take place well in advance of the limiting time.

The Definition of a Trading Strategy

1. Price Dynamics and Trade Execution: As a starting point, Almgren and Chriss (2000) define s trading strategy, and lay out the dynamics that they study. They start with a formal definition of a strategy for a sell program consisting of liquidating a single security. The definitions and results are analogous for a buy program.
2. Problem Setup - Security Liquidation: Suppose that the seller holds a block of X units of a security that they want to completely liquidate before time T. To keep the discussion, Almgren and Chriss (2000) speak of *units* of a security. Specifically they have in mind shares of stock, futures contract, and units of a foreign currency.
3. Trading Strategy - Price/Unit Strategy: The seller divides T into N units of length

$$\tau = \frac{T}{N}$$

   and defines the discrete times

$$t_k = k\tau$$

   for

$$k = 0, \cdots, N$$

   The *trading trajectory* is defined to be the list $x_0, \cdots, x_N$ where x_k is the number of units that the seller plans to hold at time t_k.
4. Outright/Re-balanced Trajectories: The initial holdings is

$$x_0 = X$$

   and liquidation at time T requires

$$x_N = 0$$

A trading trajectory can thought of as either the ex-post realized trades resulting from some process, or as a plan concerning how to trade a block of securities. In either case one may also consider *re-balancing* trajectories by requiring

$$x_0 = X$$

the initial position, and

$$x_1 = Y$$

the new position, but this is formally equivalent to studying trajectories of the form

$$x_0 = X - Y$$

and

$$x_N = 0$$

5. Outstanding Holdings/Incremental Trade Lists: Equivalently, a strategy may be specified using the "trade list" $n_1, \cdots, n_N$ where

$$n_k = x_{k-1} - x_k$$

is the number of units that the seller will sell between times t_{k-1} and t_k. Clearly, x_k and n_k are related by

$$x_k = X - \sum_{j=1}^{k} n_j = \sum_{j=k+1}^{N} n_j$$

$$k = 0, \cdots, N$$

6. Simultaneous Portfolio Buying and Selling: Almgren and Chriss (2000) also consider more general programs of buying and selling simultaneously several securities.
7. Inter-Execution Time Interval Specification: For notational simplicity they consider all the time intervals to be of equal length τ, but this restriction is not essential.
8. Behavior at N/τ Limits: Although they do not discuss it, in all their results it is easy to take the continuous-time limit of

$$N \to \infty$$

and

$$\tau \to 0$$

9. Definition of a Trading Strategy: Almgren and Chriss (2000) define a "trading strategy" to be a rule for determining n_k in terms of the information available at t_{k-1}. Broadly speaking they distinguish between two types of trading strategies – static and dynamic.
10. Static vs. Dynamic Trading Strategy: Static strategies are determined in advance of trading, that is the rule for determining each n_k depends only on information available at t_0. Dynamic strategies, conversely, depend on all information up to, and including, time t_{k-1}

Price Dynamics

1. Exogenous/Endogenous Price Move Factors: Suppose that the initial security price is S_0 so that the initial market value of the position is XS_0. The securities' price evolves

according to two exogenous factors – volatility and drift, and one endogenous factor – market impact.

2. Market Forces vs. Trading Impact: Volatility and drift are assumed to the result of market forces that occur randomly and independent of the trading.
3. Earlier Literature on Market Impact: Almgren and Chriss (2000) discussion s largely reflect the work of Kraus and Stoll (1972), and the subsequent works of Holthausen, Leftwich, and Mayers (1987, 1990) and Chan and Lakonishok (1993, 1995). See also Keim and Madhavan (1995, 1997).
4. Origin of the Market Impact: As the market participants begin to detect the volume that the seller (buyer) is selling (buying), they naturally adjust their bids (offers) downward (upward). Almgren and Chriss (2000) distinguish two kinds of market impact.
5. Definition of Temporary Market Impact: *Temporary* impact refers to the temporary imbalances in supply and demand caused by the seller's trading leading to temporary price movements away from equilibrium.
6. Definition of Permanent Market Impact: *Permanent* impact refers to the changes in the "equilibrium" price due to the seller's trading, which remain at least for the life of the liquidation.
7. Price Evolution Stochastic Difference Equation: Almgren and Chriss (2000) assume that the security price evolves according to the discrete random walk

$$S_k = S_{k-1} + \sigma\sqrt{\tau}\xi_k - \tau g\left(\frac{n_k}{\tau}\right)$$

for

$$k = 1, \cdots, N$$

8. Glossary of the Equation Terms: Here σ represents the volatility of the asset, ξ_k's are draws from independent random variables each with zero mean and unit variance, and the permanent impact function $g(v)$ is a function of the *average rate* of trading

$$v = \frac{n_k}{\tau}$$

during the interval t_{k-1} to t_k.

9. Lack of Explicit Drift Term: In the above equation there is no drift term. Almgren and Chriss (2000) indicate that this is due to the assumption that they have no information about the direction of the future price movements.
10. Trading Term Horizons under Consideration: Over long term investment time scales, or in extremely volatile markets, it is important to consider *geometric* rather than arithmetic Brownian motion – this corresponds to letting σ in

$$S_k = S_{k-1} + \sigma\sqrt{\tau}\xi_k - \tau g\left(\frac{n_k}{\tau}\right)$$

scale with S. But over short term "trading" horizons of interest, the total fractional price changes are small, and the differences between arithmetic and geometric Brownian motions are negligible.

Temporary Market Impact

1. Intuition behind the Temporary Market Impact: The intuition behind the temporary market impact is that a trader plans to sell a certain number of units n_k between times t_k and t_{k-1}, but may work the order in several smaller sizes to locate optimal points of liquidity.
2. Liquidity Reduction Impact on Price: If the total number of units n_k is sufficiently large, the execution price may steadily decrease between t_{k-1} and t_k in part due to the exhaustion of the supply of liquidity at each successive price level. This effect is assumed to be short-lived, and in particular, liquidity is assumed to return back after each period, and a new equilibrium price is established.

3. The Temporary Price Impact Function: This effect is modeled by introducing a temporary price impact function $h(v)$, the temporary drop in the average price per share caused by trading at an average rate v during one time interval.
4. Net Price Received at Execution: Give this, the actual price per share received on sale k is

$$\tilde{S}_k = S_{k-1} - h\left(\frac{n_k}{\tau}\right)$$

but the effect of $h(v)$ does not appear in the next "market" price S_k.
5. Choice of Market Microstructure: The functions $g(v)$ in

$$S_k = S_{k-1} + \sigma\sqrt{\tau}\xi_k - \tau g\left(\frac{n_k}{\tau}\right)$$

and $h(v)$ in

$$\tilde{S}_k = S_{k-1} - h\left(\frac{n_k}{\tau}\right)$$

may be chosen to reflect any preferred model of market microstructure, subject only to certain natural convexity conditions.

Capture and Cost of Trading Trajectories

1. Capture across a Trading Trajectory: Almgren and Chriss (2000) then discuss the profits resulting from trading along a certain trajectory. They define the *capture* of a trajectory to be the full trading revenue upon completion of all trades. Due to the short term horizons that they consider, they do not include any notion of carry or time value of money in their discussions.

2. Full Trading Revenue across Execution: Thus, the capture is the sum of the product of the number of units n_k sold in each time interval times the effective price per share $\tilde{S}_k$ received on that sale. It is readily computed as

$$\sum_{k=1}^{N} n_k \tilde{S}_k = X S_0 + \sum_{k=1}^{N} \left[\sigma\sqrt{\tau}\xi_k - \tau g\left(\frac{n_k}{\tau}\right)\right] x_k + \sum_{k=1}^{N} n_k h\left(\frac{n_k}{\tau}\right)$$

3. Decomposition of the Capture Components: The first term on the RHS above is the initial market value of the position; each additional term represents a gain or a loss due to a specific market factor.
4. The Volatility Price Impact Term: The first term $\sigma\sqrt{\tau}\xi_k x_k$ represents the total impact from the volatility.
5. The Permanent Market Impact Term: The permanent market impact term $-\tau x_k g\left(\frac{n_k}{\tau}\right)$ represents the loss in the value of the position caused by a permanent price drop associated with selling a small piece of the position.
6. The Temporary Market Impact Term: And the temporary market impact term $n_k h\left(\frac{n_k}{\tau}\right)$ is the price drop due to selling, acting only on the units sold during the k^{th} period.
7. The Total Cost of Trading: The *total cost of trading* is the difference $X S_0 - \sum_{k=1}^{N} n_k \tilde{S}_k$ between the initial book value and the capture. This is the standard *ex-post* measure of the performance costs used in performance evaluations, and is essentially what Perold (1988) calls *implementation shortfall*.
8. Estimation of Implementation Short-fall: In this model, prior to trading, the implementation short-fall is a random variable. Write $\mathbb{E}[X]$ for the expected short-fall and $\mathbb{V}[X]$ for the variance of the short-fall.
9. Implementation Short-fall Mean/Variance: Given the simple nature of price dynamics, Almgren and Chriss (2000) readily compute

$$\mathbb{E}[X] = \sum_{k=1}^{N} \tau x_k g\left(\frac{n_k}{\tau}\right) + \sum_{k=1}^{N} n_k h\left(\frac{n_k}{\tau}\right)$$

$$\mathbb{V}[X] = \sigma^2 \sum_{k=1}^{N} \tau {x_k}^2$$

The units of $\mathbb{E}[X]$ are in dollars, and the units of $\mathbb{V}[X]$ are dollars squared.

10. Distribution of Implementation Short-fall: The distribution of the short-fall is Gaussian if ξ_k is Gaussian, in any case if N is large, it is very nearly Gaussian.
11. Almgren and Chriss Minimizer Utility: Almgren and Chriss (2000) devote much of their paper to finding trajectories that minimize $\mathbb{E}[X] + \lambda\mathbb{V}[X]$ for various values of λ. They demonstrate that for each value of λ there corresponds a unique trading trajectory x such that $\mathbb{E}[X] + \lambda\mathbb{V}[X]$ is minimal.

Linear Impact Functions

1. Linear Temporary/Permanent Market Impact: Although Almgren and Chriss (2000) formulation does not require it, computing optimal trajectories is significantly easier if one takes the permanent and temporary impact functions to be *linear* in the rate of trading.
2. Linear Permanent Impact Market Function: For linear permanent impact, $g(v)$ has the form

$$g(v) = \gamma v$$

   in which the constant γ has units of ($/share)/share.
3. Corresponding Execution Time Security Price: With this form, each n units sold depresses the price per share by γn regardless of the time taken to sell n units.

$$S_k = S_{k-1} + \sigma\sqrt{\tau}\xi_k - \gamma g\left(\frac{n_k}{\tau}\right)$$

readily yields

$$S_k = S_0 + \sigma \sum_{j=1}^{k} \sqrt{\tau_j} \xi_j - \tau\gamma(X - x_k)$$

4. Permanent Implementation Short-fall Mean: Then summing by parts, the permanent impact term in

$$\mathbb{E}[X] = \sum_{k=1}^{N} \tau x_k g\left(\frac{n_k}{\tau}\right) + \sum_{k=1}^{N} n_k h\left(\frac{n_k}{\tau}\right)$$

becomes

$$\sum_{k=1}^{N} \tau x_k g\left(\frac{n_k}{\tau}\right) = \gamma \sum_{k=1}^{N} x_k n_k = \gamma \sum_{k=1}^{N} x_k (x_k - x_{k-1})$$
$$= \frac{1}{2}\gamma^2 \sum_{k=1}^{N} [{x_{k-1}}^2 - {x_k}^2 - (x_k - x_{k-1})^2] = \frac{1}{2}\gamma X^2 - \frac{1}{2}\gamma \sum_{k=1}^{N} {n_k}^2$$

5. Linear Temporary Impact Market Function: Similarly, for the temporary impact we take

$$h\left(\frac{n_k}{\tau}\right) = \epsilon sgn(n_k) + \frac{\eta}{\tau} n_k$$

where sgn is the sign function.

6. Estimating the Fixed Costs of Execution: The units of ϵ are \$/share, and those of η are (\$/share)/(share/time). A reasonable estimate for ϵ is the fixed cost of selling, such as half of bid-ask spread plus premium.
7. Estimating the Linear Impact Coefficient: It is more difficult to estimate η since it depends on the internal and the transient aspects of the market microstructure. It is in

this term that one would expect the on-linear terms to be most important, and the approximation

$$h\left(\frac{n_k}{\tau}\right) = \epsilon sgn(n_k) + \frac{\eta}{\tau} n_k$$

to be most doubtful.

8. Total Temporary Impact Function: The linear model

$$h\left(\frac{n_k}{\tau}\right) = \epsilon sgn(n_k) + \frac{\eta}{\tau} n_k$$

is often called a *quadratic* cost because the total costs incurred by buying or selling n units in a single unit of time is

$$nh\left(\frac{n}{\tau}\right) = \epsilon|n| + \frac{\eta}{\tau} n^2$$

9. Temporary Implementation Short-fall Mean: With both linear cost models

$$g(v) = \gamma v$$

and

$$h\left(\frac{n_k}{\tau}\right) = \epsilon sgn(n_k) + \frac{\eta}{\tau} n_k$$

the expectation of the impact costs

$$\mathbb{E}[X] = \sum_{k=1}^{N} \tau x_k g\left(\frac{n_k}{\tau}\right) + \sum_{k=1}^{N} n_k h\left(\frac{n_k}{\tau}\right)$$

becomes

$$\mathbb{E}[X] = \frac{1}{2}\gamma X^2 + \epsilon \sum_{k=1}^{N} |n_k| + \frac{\tilde{\eta}}{\tau} \sum_{k=1}^{N} {n_k}^2$$

in which

$$\tilde{\eta} = \eta - \frac{1}{2}\gamma\tau$$

10. <u>Strictly Convex Nature of $\mathbb{E}[X]$</u>: Clearly $\mathbb{E}[X]$ is a strictly convex function as long as

$$\tilde{\eta} > 0$$

Note that if n_k all have the same sign, as would be the case for a pure sell program or a pure buy program, then

$$\sum_{k=1}^{N} |n_k| = |X|$$

11. <u>$\mathbb{E}[X]$ and $\mathbb{V}[X]$ Computation Illustration</u>: To illustrate, Almgren and Chriss (2000) compute $\mathbb{E}[X]$ and $\mathbb{V}[X]$ for linear impact functions for two of trajectory schemes at the opposite extremes: sell at a constant rate, and sell to maximize variance without regard to transaction costs.
12. <u>Minimum Impact: Constant Execution Rate</u>: The most obvious trajectory is to sell at a constant rate over the entire liquidation period. Thus one takes each

$$n_k = \frac{X}{N}$$

and

$$x_k = (N - k)\frac{X}{N}$$

$$k = 1, \cdots, N$$

13. Minimum Impact $\mathbb{E}[X]$ and $\mathbb{V}[X]$: From

$$\mathbb{E}[X] = \sum_{k=1}^{N} \tau x_k g\left(\frac{n_k}{\tau}\right) + \sum_{k=1}^{N} n_k h\left(\frac{n_k}{\tau}\right)$$

and

$$\mathbb{E}[X] = \frac{1}{2}\gamma X^2 + \epsilon \sum_{k=1}^{N} |n_k| + \frac{\tilde{\eta}}{\tau} \sum_{k=1}^{N} {n_k}^2$$

one has

$$\mathbb{E}[X] = \frac{1}{2} X T g\left(\frac{X}{T}\right)\left(1 - \frac{1}{N}\right) + X h\left(\frac{X}{T}\right) = \frac{1}{2}\gamma X^2 + \epsilon X + \tilde{\eta}\frac{X^2}{T}$$

and from

$$\mathbb{V}[X] = \sigma^2 \sum_{k=1}^{N} \tau {x_k}^2$$

$$\mathbb{V}[X] = \frac{1}{3}\sigma^2 X^2 T\left(1 - \frac{1}{N}\right)\left(1 - \frac{1}{2N}\right)$$

14. Minimum Impact N/T Limits: The trajectory minimizes total expected costs, but the variance may be large if the period T is long. As the number of trading periods

$$N \to \infty$$

$$v = \frac{X}{T}$$

remains finite, and $\mathbb{E}[X]$ and $\mathbb{V}[X]$ have finite limits.

15. Minimum Variance: One Step Execution: The other extreme is to execute the entire position in the first time step. One then takes

$$n_1 = X$$

$$n_2 = \cdots = n_N = 0$$

$$x_1 = \cdots = x_N = 0$$

which results in

$$\mathbb{E}[X] = Xh\left(\frac{X}{\tau}\right) = \epsilon X + \eta\frac{X^2}{\tau}$$

and

$$\mathbb{V}[X] = 0$$

16. Minimum Variance N/T Limits: The trajectory has the smallest possible variance – equal to zero – because of the way time has been discretized in the model above. If N is large and hence τ is short, then on the full initial portfolio, one takes a price hit that can be arbitrarily large.
17. Trajectory between the Two Extremes: Almgren and Chriss (2000) show how to effectively compute trajectories that lie between the two extremes.

The Efficient Frontier of Optimal Execution

1. Computing the Optimal Execution Trajectories: Almgren and Chriss (2000) define and compute optimal execution trajectories and use that to later demonstrate a precise relationship between risk aversion and the definition of optimality.
2. Uniqueness of Optimal Execution Strategy: In particular, they show that each level of risk aversion there is a uniquely determined optimal execution strategy.

The Definition of the Frontier

1. Minimization of Expected Short-fall: The rational trader will always seek to minimize the expectation of short-fall for a given level of variance of the short-fall. Naturally a trader will prefer a strategy that provides minimum error in its estimate of expected costs.
2. Efficient Optimal Trading Strategy Definition: Thus a strategy is *efficient* or *optimal* if there is no other strategy that has lower variance for the same or a lower variance of the expected transaction costs, or, equivalently, no strategy which has no lower expected transaction costs for the same or lower level of variance.
3. Static vs. Dynamic Strategy Optimality: This definition of optimality of a strategy is the same whether the strategy is static or dynamic. It will be established later that under this definition and the price dynamics already stated, optimal strategies are in fact static.
4. Efficient Strategies - Constrained Optimization Formulation: One may construct efficient strategies by solving the constrained optimization problem

$$\min_{x:\, \mathbb{V}[x] \leq V_*} \mathbb{E}[x]$$

That is, for a given maximum level of variance

$$V_* \geq 0$$

one finds a strategy that has the minimum expected levels of transaction costs.

5. Convex Objective Function and Domain: Since $\mathbb{V}[x]$ is convex, the set

$$\{\mathbb{V}[x] \leq V_*\}$$

is convex – it is a sphere – and since $\mathbb{E}[x]$ is strictly convex, there is a unique minimizer $x_*(V_*)$.

6. Sub-Optimal Trajectory Variance Cost: Regardless of the preferred balance of risk and return, every other solution x which has

$$\mathbb{V}[x] \leq V_*$$

has higher expected costs than $x_*(V_*)$ for the same or lower variance, and can never be more efficient.

7. Efficient Frontier of Optimal Strategies: Thus the family of all possible efficient (optimal) strategies is parametrized by a single variable V_* representing all possible maximum levels of variance in transaction costs. This family is referred to as *the efficient frontier of optimal trading strategies*.

8. Introducing KKT Type Constraint Multipliers: The constrained optimization problem

$$\underset{x:\, \mathbb{V}[x] \leq V_*}{min} \mathbb{E}[x]$$

is solved by introducing a constraint multiplier λ, thereby solving the unconstrained problem

$$\underset{x}{min} \left(\mathbb{E}[x] + \lambda \mathbb{V}[x]\right)$$

9. Frontier as a Function of λ: If

$$\lambda > 0$$

$\mathbb{E}[x] + \lambda\mathbb{V}[x]$ is strictly convex, and the above minimizer has a unique solution $x^*(\lambda)$. As λ varies, $x^*(\lambda)$ sweeps out the same one parameter family, and thus traces out an efficient frontier.

10. λ as a Risk Aversion Parameter: The Parameter λ has a direct financial interpretation. It is already apparent from

$$\underset{x}{min}\left(\mathbb{E}[x] + \lambda\mathbb{V}[x]\right)$$

that λ is a measure of risk aversion, that is, how much the variance is penalized relative to the cost.

11. λ as an Efficient Frontier Curvature: In fact, λ is the curvature – second derivative – of a smooth utility function, as will be made more precise eventually.
12. Solution given $h(v)$ and $g(v)$: For given values of the parameters, problem

$$\underset{x}{min}\left(\mathbb{E}[x] + \lambda\mathbb{V}[x]\right)$$

can be solved by various numerical techniques depending on the functional forms chosen for $h(v)$ and $g(v)$. In the special case that these are *linear* functions, we may write the solution explicitly and gain a great deal of insight into the trading strategies.

Explicit Construction of Optimal Strategies

1. Optimal Solution in Trajectory Space: With $\mathbb{E}[x]$ from

$$\mathbb{E}[X] = \frac{1}{2}\gamma X^2 + \epsilon\sum_{k=1}^{N}|n_k| + \frac{\tilde{\eta}}{\tau}\sum_{k=1}^{N}{n_k}^2$$

and $\mathbb{V}[x]$ from

$$\mathbb{V}[X] = \sigma^2 \sum_{k=1}^{N} \tau {x_k}^2$$

and assuming that n_j does not change sign, the combination

$$\mathbb{U}[x] = \mathbb{E}[x] + \lambda \mathbb{V}[x]$$

is a quadratic function of the control parameters $x_1, \cdots, x_{N-1}$; it is strictly convex for

$$\lambda \geq 0$$

2. <u>Finding the Unique Global Minima</u>: Therefore one determines the unique global minimum by setting its partial derivatives to zero. One readily calculates

$$\frac{\partial \mathbb{U}[x]}{\partial x_j} = 2\tau \left(\lambda \sigma^2 x_j - \tilde{\eta} \frac{x_{j-1} - 2x_j + x_{j+1}}{\tau^2} \right)$$

for

$$j = 1, \cdots, N-1$$

3. <u>Combinations of Linear Difference Equations</u>: Then

$$\frac{\partial \mathbb{U}[x]}{\partial x_j} = 0$$

is equivalent to

$$\frac{x_{j-1} - 2x_j + x_{j+1}}{\tau^2} = \tilde{\kappa}^2 x_j$$

with

$$\tilde{\kappa}^2 = \frac{\lambda\sigma^2}{\tilde{\eta}} = \frac{\lambda\sigma^2}{\eta\left(1 - \frac{\gamma\tau}{2\eta}\right)}$$

4. τ Abstracted and Re-factored Parameter Set: Note that

$$\frac{x_{j-1} - 2x_j + x_{j+1}}{\tau^2} = \tilde{\kappa}^2 x_j$$

is a linear difference equation whose solution may be written as a combination of the exponentials $e^{\pm\kappa t_j}$ where κ satisfies

$$\frac{2}{\tau^2}[\cosh(\kappa\tau) - 1] = \tilde{\kappa}^2$$

The tilde's on $\tilde{\eta}$ and $\tilde{\kappa}$ denote an $\mathcal{O}(\tau)$ correction; as

$$\tau \to 0$$

one has

$$\tilde{\eta} \to \eta$$

and

$$\tilde{\kappa} \to \kappa$$

5. Trading Trajectory/Trade List Solutions: The specific solution with

$$x_0 = X$$

and

$$x_N = 0$$

is a trading trajectory of the form

$$x_j = \frac{\sinh\left(\kappa(T - t_j)\right)}{\sinh(\kappa T)} X$$

$$j = 0, \cdots, N$$

and the associated trade list is

$$n_j = \frac{2\sinh\left(\frac{1}{2}\kappa T\right)}{\sinh(\kappa T)} \cosh\left(\kappa\left(T - t_{j-\frac{1}{2}}\right)\right) X$$

$$j = 1, \cdots, N$$

where sinh and cosh are the hyperbolic sine and cosine functions, and

$$t_{j-\frac{1}{2}} = \left(j - \frac{1}{2}\right)\tau$$

These solutions – although not the efficient frontier – have been constructed previously by Grinold and Kahn (1999).

6. <u>Monotonicity of the Trading Trajectory</u>: One has

$$n_j > 0$$

as long as

$$X > 0$$

Thus for a program of selling a large initial long position, the solution decreases *monotonically* from its initial value to zero at the rate determined by the parameter κ.

7. Consequence of Monotonic Trading Trajectories: For example, the optimal execution of a sell program never involves buying of securities – although this ceases to be true if there is drift or serial correlation in price movements.
8. Approximation under Small Time Step: For a small time step τ one has the approximate expression

$$\kappa \sim \tilde{\kappa} + \mathcal{O}(\tau^2) \sim \sqrt{\frac{\lambda\sigma^2}{\eta\left(1 - \frac{\gamma\tau}{2\eta}\right)}} + \mathcal{O}(\tau)$$

$$\tau \to 0$$

Thus if the trading intervals are short κ^2 is essentially the ratio of the product of volatility and the risk-intolerance to the temporary transaction cost parameter.
9. Optimal Strategy Expected Cost/Variance: The expectation and the variance of the optimal strategy for a given initial portfolio size X are then

$$\mathbb{E}[X] = \frac{1}{2}\gamma X^2 + \epsilon X + \tilde{\eta} X^2 \frac{\tanh\left(\frac{1}{2}\kappa\tau\right)[\tau\sinh(2\kappa T) + 2T\sinh(\kappa\tau)]}{2\tau^2[\sinh(\kappa\tau)]^2}$$

and

$$\mathbb{V}[X] = \frac{1}{2}\sigma^2 X^2 \frac{\tau\sinh(\kappa T)\cosh\big(\kappa(T-\tau)\big) - T\sinh(\kappa\tau)}{[\sinh(\kappa T)]^2\sinh(\kappa\tau)}$$

which reduce to

$$\mathbb{E}[X] = \frac{1}{2} X T g\left(\frac{X}{T}\right)\left(1 - \frac{1}{N}\right) + X h\left(\frac{X}{T}\right) = \frac{1}{2}\gamma X^2 + \epsilon X + \tilde{\eta}\frac{X^2}{T}$$

$$\mathbb{V}[X] = \frac{1}{3}\sigma^2 X^2 T\left(1 - \frac{1}{N}\right)\left(1 - \frac{1}{2N}\right)$$

$$n_1 = X$$

$$n_2 = \cdots = n_N = 0$$

$$x_1 = \cdots = x_N = 0$$

$$\mathbb{E}[X] = X h\left(\frac{X}{\tau}\right) = \epsilon X + \eta\frac{X^2}{\tau}$$

$$\mathbb{V}[X] = 0$$

in the limits

$$\kappa \to 0, \infty$$

The Half-Life of a Trade

1. <u>Definition of the Half-Life</u>: Defining

$$\theta = \frac{1}{\kappa}$$

the trade's "half-life", and using the discussion above, it can be seen that the larger the value of κ and smaller the value of θ, the more rapidly the trade list will be depleted. The value θ is exactly the amount of time it takes to deplete the holdings by a factor of e.

2. Half-Life Different from T: The definition of θ is independent of the exogenously specified execution time T; it is determined only by the security price dynamics and the market impact factors. If the risk aversion λ is greater than zero, i.e., if the trader is risk-averse, then θ is finite and independent of T.
3. Timeless Initial Portfolio Liquidation Rate: Thus, in the absence of any external time constraint, i.e.

$$T \to \infty$$

   the trader will still liquidate his position on a time scale θ. The half-life θ is the intrinsic time scale of the trade.
4. Half Life Smaller than T: For a given T the ratio

$$\kappa T = \frac{T}{\theta}$$

   tells us what factors constrain the trade. If

$$T \gg \theta$$

   then the intrinsic half-life θ of the trade is small compared to the imposed time T; this happens because temporary costs are very small, because volatility is very large, or because of high risk aversion.
5. Impact of Small Half-Life: In this case the bulk of the trading will be done well in advance of the time T. Viewed on a time scale T the trajectory will look like a minimum variance solution

$$n_1 = X$$

$$n_2 = \cdots = n_N = 0$$

$$x_1 = \cdots = x_N = 0$$

6. Very High Half Life Limit: Conversely if

$$T \ll \theta$$

then the trade is highly constrained, and is dominated by temporary market impact costs. In the limit

$$\frac{T}{\theta} \to 0$$

one approaches the straight line minimum cost strategy

$$n_k = \frac{X}{N}$$

$$x_k = (N - k)\frac{X}{N}$$

$$k = 1, \cdots, N$$

7. Trade Size Independent Execution Strategy: A consequence of this analysis is that different sized baskets of the same security will be liquidated in exactly the same fashion, on the same scale, provided the risk aversion parameter λ is held constant.

8. <u>Basket Size Based Liquidity Dependence</u>: This may seem contrary to the expectation that large baskets are effectively less liquid, and should hence be liquidated less rapidly than smaller baskets.
9. <u>Reasons for the Counter-Intuitiveness</u>: This is a consequence of the linear market impact assumption which has the *mathematical* consequence that both variance and market impact scale quadratically with respect to the portfolio size.
10. <u>Higher Order Temporary Impact Function</u>: For large portfolios it may be more reasonable to assume that the temporary impact cost function has higher-order terms, so that such costs increase *super-linearly* with the trade size. With non-linear impact functions, the general framework used here still applies, but one does not obtain explicit exponential solutions as in the linear impact case.
11. <u>Size Dependent Temporary Impact Parameter</u>: A simple practical solution to this problem is to choose different values of η - the temporary impact parameter – depending up on the overall problem size being considered, recognizing that the model is at best only approximate.

Structure of the Frontier

1. <u>Efficient Frontier and the Corresponding Trajectories</u>: Using a specific choice for the parameters explained below, Almgren and Chriss (2000) produce a sample plot of the efficient frontier – each point on the frontier represents a distinct strategy for optimally liquidating the same basket. Their tangent line represents the optimal solution for a specified risk parameter

$$\lambda = 10^{-6}$$

   They also illustrate the trajectories corresponding to a few sample points on the frontier.
2. <u>Trajectory corresponding to Positive λ</u>: Their first trajectory has

$$\lambda = 2 \times 10^{-6}$$

– this would be chosen by a risk-averse trader who wishes to sell quickly to reduce exposure to volatility risk, despite the trading costs incurred in doing so.

3. Trajectory corresponding to Zero λ: Their second trajectory has

$$\lambda = 0$$

   They refer to this as the naïve strategy since this represents an optimal strategy corresponding to simply minimizing expected transaction costs without regard to variance.

4. Linear Reduction of the Holdings: For a security with zero drift and linear transaction costs as defined above

$$\lambda = 0$$

   corresponds to a simple linear reduction of holdings over the trading period. Since drift is generally not significant over short trading horizons, the naïve strategy is very close to the linear strategy.

5. Sub Optimality of the Strategy: As Almgren and Chriss (2000) demonstrate later, in a certain sense this is *never* an optimal strategy because one can obtain substantial reductions in variance for a relatively small increase in transaction costs.

6. Trajectory corresponding to Negative λ: Finally their trajectory C has

$$\lambda = -2 \times 10^{-6}$$

   it would only be chosen by a trader who likes risk. He postpones execution, thus incurring higher costs both due to rapid sales at the end, and higher variance during the extended period that he holds the security for.

The Utility Function

1. The Risk-Reward Trade-off: Almgren and Chriss (2000) offer an interpretation of the efficient frontier of optimal strategies in terms of the utility function of the seller. They do this in two ways – by direct analogy with modern portfolio theory employing a utility function, and by a novel approach: Value-at-risk. This eventually leads to some general observations regarding the importance of utility in forming execution strategies.
2. Utility of Risk-Averse Functions: Suppose on measure utility by a smooth convex function $u(w)$ where w is the total wealth. This function may be characterized by its risk-aversion coefficient

$$\lambda_u = -\frac{u''(w)}{u'(w)}$$

3. Approximation in Estimating the λ: If the initial portfolio is fully owned, then as the transfer of assets happens from the risky stock into the alternative riskless investment, w remains roughly constant, and one may take λ_u to be a constant throughout the trading period. If the initial portfolio is highly leveraged, then the assumption of constant λ is an approximate one.
4. Formulation of the Optimal Execution Strategy: For short time horizons and small changes in w the higher derivatives of $u(w)$ may be neglected. Thus choosing an optimal execution strategy is equivalent to minimizing the scalar function

$$\mathbb{U}_{UTIL}[x] = \lambda_u \mathbb{V}[x] + \mathbb{E}[x]$$

   The units of λ_u are $\$^{-1}$; one is willing to accept an extra square \$ of variance if it reduces the expected cost by $\$\lambda_u$.
5. Constructing Family of Optimal Paths: The combination $\lambda\mathbb{V}[x] + \mathbb{E}[x]$ is precisely the one used to construct the efficient frontier seen earlier; the parameter λ, introduced as a Lagrange multiplier, has a precise definition as a measure of aversion

to risk. Thus, the methodology above used to construct the efficient frontier likewise produces a family of optimal paths, one for each level of risk aversion.

6. <u>Static Nature of Optimal Path</u>: Returning now to an important point raised earlier, the computation of optimal strategies by minimizing $\lambda\mathbb{V}[x] + \mathbb{E}[x]$ as measured at the initial trading time is equivalent to maximizing the utility at the outset of trading. As one trades, information arrives that could potentially alter the optimal path. The following theorem eliminates that possibility.
7. <u>Time Homogenous Quadratic Utility Theorem</u>: For a fixed quadratic utility function, the static strategies computed above are "time homogenous". More precisely given a strategy that begins at a time

$$t = 0$$

and ends at a time

$$t = T$$

the optimal strategy computed at

$$t = t_k$$

is simply a continuation from

$$t = t_k$$

to

$$t = T$$

of the optimal strategy computed at time

$$t = 0$$

8. Proof Steps: General/Specific Functions: The proof may be seen in two ways – by the algebraic computations based on the specific solutions above, and by general valid for generic non-linear impact functions.
9. Proof Steps: Function Time Shift: First suppose that at time k, where

$$k = 0, \cdots, N - 1$$

one were to compute a new optimal strategy. The new strategy would precisely be

$$x_j = \frac{\sinh\left(\kappa(T - t_j)\right)}{\sinh(\kappa T)} X$$

$$j = 0, \cdots, N$$

with X replaced by x_k, T replaced by $T - t_k$, and t_j replaced by $t_j - t_k$. Using the subscript (k) to denote the strategy computed at time k one would have

$$x_j{}^{(k)} = \frac{\sinh\left(\kappa(T - t_j)\right)}{\sinh\left(\kappa(T - t_k)\right)} x_k$$

$$j = k, \cdots, N$$

and the trade lists

$$n_j{}^{(k)} = \frac{2\sinh\left(\frac{1}{2}\kappa\tau\right)}{\sinh\left(\kappa(T - t_k)\right)} \cosh\left(\kappa\left(T - t_{j-\frac{1}{2}}\right)\right) X$$

$$j = k + 1, \cdots, N$$

10. <u>Proof Step: Recovering Optimal Solutions</u>: It is then apparent that if x_k is the optimal solution from

$$x_j = \frac{\sinh\left(\kappa\left(T - t_j\right)\right)}{\sinh(\kappa T)} X$$

$$j = 0, \cdots, N$$

with

$$j \longmapsto k$$

then

$$x_j{}^{(k)} = x_j{}^0$$

and

$$n_j{}^{(k)} = n_j{}^0$$

where

$$x_j{}^0 = x_j$$

and

$$n_j{}^0 = n_j$$

are the strategies from

$$x_j = \frac{\sinh\left(\kappa\left(T - t_j\right)\right)}{\sinh(\kappa T)} X$$

$$j = 0, \cdots, N$$

and

$$n_j = \frac{2 \sinh\left(\frac{1}{2}\kappa T\right)}{\sinh(\kappa T)} \cosh\left(\kappa\left(T - t_{j-\frac{1}{2}}\right)\right) X$$

$$j = 1, \cdots, N$$

11. Proof Step: Non-linear Impact: For general non-linear impact functions $g(v)$ and $h(v)$ the optimality condition

$$\frac{x_{j-1} - 2x_j + x_{j+1}}{\tau^2} = \tilde{\kappa}^2 x_j$$

is replaced by a second-order *non-linear* difference relation. The solution ${x_j}^{(k)}$ beginning at a given time is determined by the two boundary values x_k and

$$x_N = 0$$

It is then apparent that the solution does not change if we re-evaluate it at later times.

12. Origin of Time Stable Solutions: More fundamentally, the solutions are time stable because in the absence of serial correlations in the asset price movements, there is no more information about the price changes at later times than there is at the initial time.
13. Optimality over each Sub-interval: Thus, the solution which was determined to be optimal over the entire time interval is optimal as a solution over each sub-interval.

This general phenomenon is well known in the theory of optimal control (Bertsekas (1976)).

Value at Risk

1. Motivation behind Value at Risk: The concept of value at risk is traditionally used to measure the greatest amount of money – maximum profit or loss - a portfolio will sustain over a given period of time under "normal circumstances", where "normal" is defined by a confidence level.
2. Trading Value at Risk Definition: Given a trading strategy

$$x = (x_1, \cdots, x_N)$$

the value-at-risk of x defined $Var_p[x]$ is defined to be the level of transaction costs by the trading strategy x that will not be exceeded p percent of the time. Put another way, it is the p^{th} percentile level of transaction costs for total costs of trading x.
3. Trading Value at Risk Expression: Under the arithmetic Brownian motion assumption, the total costs – the market value minus capture – are normally distributed with known mean and variance. Thus the confidence level is determined by the number of standard deviations λ_v from the mean of the inverse of the cumulative normal distribution function, and the value at risk for the strategy x is given by

$$Var_p[x] = \lambda_v \sqrt{\mathbb{V}[x]} + \mathbb{E}[x]$$

4. Relation to Implementation Short-fall: That is, with a probability p the trading strategy will not lose more than $Var_p[x]$ of its market value in trading. Borrowing from the language of Period (1988), the implementation shortfall of execution will

not exceed $Var_p[x]$ more than a fraction p of the time. A strategy x is efficient if it has the minimum possible value at risk for the confidence level p.

5. Execution Trajectory Optimized for VaR: Note that $Var_p[x]$ is a complicated non-linear function of x_j composing x; it can be easily evaluated for any given trajectory, but finding the minimizing trajectory directly is difficult.
6. Single Parameter Efficient Frontier Solution: But once the one-parameter family of solutions that form the efficient frontier is obtained, one only needs to solve a one-dimensional problem to find the optimal solutions for the value at risk model, that is, to fund the value of λ_u corresponding to a given value of λ_v. Alternatively on may characterize the solutions by a simple graphical procedure, or may read off the confidence levels corresponding to any particular point on the curve.
7. Almgren-Chriss Optimal VaR Illustration: Almgren and Chriss (2000) produce an illustration of the above, using the square root of variance in the x-axis as opposed to the variance in itself. In this co-ordinate system lines of optimal VaR have a constant slope, and for a given value of λ_v they simply find a tangent to the curve where the slope is λ_v.
8. Interim Optimal Execution Re-evaluation: The question of re-evaluation of the strategy is more complicated and subtle. If one re-evaluates the strategy half-way through the execution process, they will choose a new optimal strategy that is not the same as the original optimal one. The reason is that since λ_v is now held constant, λ_u necessarily changes.
9. General Challenges with the VaR Approach: Value at risk has many flaws from a mathematical point of view, as recognized by Artzner, Delbaen, Eber, and heath (1997). The particular issue encountered her would occur in any problem in which the time of measurement is a fixed date, rather than maintained at a fixed distance in the future. It is an open issue to formulate suitable measures of risk for general time-dependent problems.
10. Liquidity Adjusted Value at Risk: Despite this shortcoming, Almgren and Chriss (2000) the smallest possible value of $Var_p[x]$ as an informative measure of the possible loss associated with the initial position, in the presence of liquidity effects. This value, which they call L-VaR for Liquidity Adjusted Value at Risk, depends on

the time to liquidation and the confidence level chosen, in addition to the market parameters such as the impact coefficient (Almgren and Chriss (1999)).

11. Advantages of the L-VaR Approach: The optimal trajectories determined by minimizing the value at risk do *not* have the counter-intuitive scaling behavior seen earlier; even for linear impact functions, large portfolios will be traded closer to the straight line trajectory.
12. Using L-VaR for Large Portfolios: This is because the cost assigned to uncertainty scales *linearly* with the portfolio size, while the temporary impact cost scales *quadratically* as before. Thus the latter is more important for large portfolios.

The Role of Utility in Execution

1. General Observations on Optimal Execution: Almgren and Chriss (2000) use the structure of the efficient frontier in the framework that they have developed to make some general observations concerning optimal executions.
2. The Naïve Strategy Benchmark: They first restrict themselves to the situation where the trader has no directional view on the security being traded. Recall that in this case, the naïve strategy is the simple straight line strategy in which the trader breaks the blocks being executed into equal sized blocks to be sold over equal time intervals. They use this strategy as a benchmark for comparison with the other strategies used throughout here.
3. Convex $\mathbb{E}[x]$ to $\mathbb{V}[x]$ Mapping: A crucial insight is that the curve defining the efficient frontier is a smooth convex function $\mathbb{E}[\mathbb{V}]$ mapping the levels of variance $\mathbb{V}$ to the corresponding minimum mean transaction cost levels.
4. Region around the Naïve Strategy: Write $(\mathbb{E}_0, \mathbb{V}_0)$ for the mean and variance around the naïve strategy. Regarding $(\mathbb{E}_0, \mathbb{V}_0)$ as a point on the smooth curve $\mathbb{E}[\mathbb{V}]$ defined by the frontier, $\frac{\partial \mathbb{E}}{\partial \mathbb{V}}$ evaluated at $(\mathbb{E}_0, \mathbb{V}_0)$ is equal to zero. Thus for $(\mathbb{E}, \mathbb{V})$ near $(\mathbb{E}_0, \mathbb{V}_0)$ one has

$$\mathbb{E} - \mathbb{E}_0 = \frac{1}{2}(\mathbb{V} - \mathbb{V}_0)^2 \left.\frac{\partial^2 \mathbb{E}}{\partial \mathbb{V}^2}\right|_{\mathbb{V}=\mathbb{V}_0}$$

where

$$\left.\frac{\partial^2 \mathbb{E}}{\partial \mathbb{V}^2}\right|_{\mathbb{V}=\mathbb{V}_0}$$

is positive is positive by the convexity of the frontier at the naïve strategy.

5. Special Feature of the Naïve Strategy: By definition, the naïve strategy has the property that any strategy with lower variance in cost has a greater expected cost. However a special feature of the naïve strategy is that a first-order decrease in variance can be obtained – in the sense of finding a strategy with a lower variance – while only incurring a second order increase in cost.
6. Disadvantages of Risk Neutral Strategy: From the above it follows that for small increases in variance, one can obtain much larger reductions in cost. Thus unless the trader is risk-neutral it is always advantageous to execute a strategy that is at least to some degree "to the left" of the naïve strategy. Thus one concludes that, in this framework, from a theoretical standpoint, it never makes sense to trade a strictly risk-neutral strategy.
7. The Role of a Security's Liquidity: An intuitive proposition is that with all things being equal, a trader will execute a more liquid basket more rapidly than a less liquid one. In the extreme this is particularly clear. A broker given a small order to execute over the course of the day will execute the entire order almost immediately.
8. Executing the Highly Liquid Security: How does one explain this? The answer is that the market impact cost attributable to rapid trading is negligible compared with the opportunity cost incurred in breaking up the order over an entire day. Thus, even if the expected return on a security over the day is zero, the perception is that the risk of waiting is overweighed by any small cost of immediacy.
9. Absence of Risk Reduction Premium: Now if the trader were truly risk neutral, in the absence of any views, he would always use the naïve strategy and employ the allotted

time fully. This would make sense because any price to pay for trading immediately is worthless if one places no premium on risk reduction.

10. Limitation of Risk Neutral Approach: It follows that any model that proposes optimal trading behavior should predict that more liquid baskets are traded more rapidly than less liquid ones. A model that only considers the minimization of transaction costs, like that of Bertsimas and Lo (1998), is essentially a model that excludes utility.
11. Optimal Execution Independent of Liquidity: In such a model, and under Almgren and Chriss (2000) basic assumptions, traders will trade all baskets at the same rate irrespective of the liquidity, that is unless they have an explicit directional view on the security, or the security possesses extreme serial correlation in its price movements.
12. Super Linear Market Impact Functions: Almgren and Chriss (2000) do note that their model in the case of linear transaction costs does not predict a more rapid trading for smaller versus larger baskets of the same security. However, this is a consequence of choosing linear temporary impact functions and the problem goes away when one considers more realistic super-linear functions.
13. Risk Neutral Execution Half Life: Another way of looking at this is that the half-life of all black executions, under the assumption of risk-neutral preferences, is infinite.

Choice of Parameters

1. The Asset Intrinsic Dynamics Parameters: Almgren and Chriss (2000) compute some numerical examples for the purposes of exploring the qualitative properties of the efficient frontier. Throughout the examples they consider a single stock with the current market price of

$$S_0 = 50$$

and that they initially have one million shares, for an initial portfolio size of \$50 million. The stock will have 30% annual volatility, 10% expected annual rate of return, a bid-ask spread of $\frac{1}{8}$, and a median daily trading volume of 5 million shares.

2. Stock Asset Daily Return/Volatility: With a trading year of 250 days this gives a daily volatility of

$$\frac{0.3}{250} = 0.019$$

and expected fractional return of

$$\frac{0.1}{250} = 4 \times 10^{-4}$$

To obtain our absolute parameters σ and α one must scale it by the price, so

$$\sigma = 0.019 \times 50 = 0.95$$

and

$$\alpha = (4 \times 10^{-4}) \times 50 = 0.02$$

The table below summarizes the information.

3. Parameter Values for the Test Case:

Parameter Description	Parameter Symbol	Parameter Value
Initial Stock Price	S_0	\$50/share
Initial Holdings	X	10^6 shares
Liquidation Time	T	5 days
Number of Time Periods	N	5

30% Annual Volatility	σ	$0.95\ (\$/share)/day^{\frac{1}{2}}$
10% Annual Growth	α	$0.02\ (\$/share)/day$
Bid Ask Spread $\frac{1}{8}$	ϵ	$\$0.0625/share$
Daily Volume 5 million shares	γ	$2.5 \times 10^{-7} \$/share^2$
Impact at 1% of market	η	$2.5 \times 10^{-6} (\$/share) /(share /day)$
Static Holdings 11,000 shares	λ_u	$10^{-6}/\$$
VaR Confidence $p = 95\%$	λ_v	1.645

4. Incremental and Total Execution Times: Suppose that one wants to liquidate this position in one week so that

$$T = 5\ days$$

This is divided into daily trades such that τ is $1\ day$ and

$$N = 5$$

5. Standard Deviation of the Trajectory: Over this period, if one holds the original position with no trading, the fluctuations in the stock value will be Gaussian with a standard deviation of

$$\sigma\sqrt{T} = 2.12\ (\$/share)$$

and the fluctuations in this value will have an absolute standard deviation of

$$\sqrt{V} = \$2.12M$$

As expected this is precisely the value of $\sqrt{V}$ for the lowest point in the efficient frontier, since that point corresponds selling along a linear trajectory rather than holding a constant amount.

6. <u>Temporary Cost Function Parameter - ϵ</u>: One then chooses the parameters for the temporary cost function

$$h\left(\frac{n_k}{\tau}\right) = \epsilon sgn(n_k) + \frac{\eta}{\tau} n_k$$

Almgren and Chriss (2000) set

$$\epsilon = \frac{1}{16}$$

that is, the fixed part of the temporary costs will be one-half the bid-ask spread.

7. <u>Temporary Cost Function Parameter - η</u>: For η they suppose that for each 1% of the daily volume traded they incur a price impact equal to one bid-ask spread. For example trading at a rate of 5% daily volume incurs a one-time cost on each trade of $\frac{5}{8}$. Under this assumption

$$\eta = \frac{\frac{1}{8}}{0.01 \times 5 \times 10^6} = 2.5 \times 10^{-6}$$

8. <u>Permanent Cost Function Parameter - γ</u>: For permanent costs, the common rule of thumb is that price effects become significant when 10% of the daily volume is sold. Assuming that "significant" means that the price depression is one bid-ask spread, and that the effect is linear for both smaller and larger trading rates, one has

$$\gamma = \frac{\frac{1}{8}}{0.1 \times 5 \times 10^6} = 2.5 \times 10^{-7}$$

Recall that this parameter gives a fixed cost independent of the path.

9. The Risk Aversion Parameter - λ: Almgren and Chriss (2000) have chosen

$$\lambda = \lambda_u = 10^{-6}$$

For these parameters, from

$$\kappa \sim \tilde{\kappa} + \mathcal{O}(\tau^2) \sim \sqrt{\frac{\lambda\sigma^2}{\eta\left(1 - \frac{\gamma\tau}{2\eta}\right)}} + \mathcal{O}(\tau)$$

$$\tau \to 0$$

one has for the optimal strategy that

$$\kappa \approx 0.61\ day$$

so that

$$\kappa T \approx 3$$

Since this value is near 1 in magnitude, the behavior is an interesting intermediate in-between the naïve extremes.

10. λ_v at 95% Confidence Level: For the value at risk representation, as assumed 95% confidence level gives

$$\lambda_v = 1.645$$

The Value of Information

1. Zero Drift Random Walk Assumption: The discussion carried out so far assumed that the price dynamics followed an arithmetic random walk with zero drift. Since past price paths provide no extra information on future price movements, the conclusion was that the optimal trajectories can be statically determined. There are three ways by which a random walk with zero drift may fail to represent the price process.
2. Non-zero Drift in Dynamics: First the price process may have drift. For example, if the trader has a strong directional view, the trader may want to incorporate this view into the liquidation strategy.
3. Cross Period Serial Correlation Impact: Second, the price process may exhibit serial correlation. The presence of first order serial correlation for example, implies that the price moves in a given period provide non-trivial information concerning the next period movement of the asset.
4. Incorporation of the Investor's Private Information: Bertsimas and Lo (1998) study a general form of this assumption, wherein an investor possesses possibly private information of a serially correlated information vector that acts as a linear factor in the asset returns.
5. Exogenously Induced Material Parameter Shift: Lastly, at the start of trading, it may be known that at some specific point in time, an event will take place whose outcome will cause a material shift in the parameters governing the price process.
6. Literature Survey on Exogenous Events: Such event induced parameter shifts include quarterly and annual earnings announcements, dividend announcements, and share re-purchases. Event studies documenting these parameter shifts and providing theoretical grounding for their existence include Beaver (1968), Fama, Fisher, Jensen, and Roll (1969), Dann (1981), Patell, and Wolfson (1984), Kalay and Loewentstein (1985), Kim and Verrecchia (1991), Campbell, Lo, and MacKinlay (1997), Easterwood and Nutt (1999), and Ramaswami (1999).
7. Temporary Shifts on Dynamic Parameters: For example, Brown, Harlow, and Tinic (1988) show that events cause temporary shifts in both the risk and returns of individual securities, and the extent of these shifts depends on the outcome of the event. In general, securities react more strongly to bad news than good news.

8. Probabilistic Event Outcomes/Parameter Shifts: Almgren and Chriss (2000) study a stylized version of the events in which a known event at a known time – e.g., an earnings announcement – has several possible outcomes. The probability of each outcome is known, and the impact that a given outcome will have on the parameters of the price is also known. Clearly, optimal strategies must explicitly use this information, and Almgren and Chriss (2000) develop methods to incorporate event-specific information into their risk-reward framework.
9. Back-to-Back Static Strategies: The upshot is a piece-wise strategy that trades statically up to the event, and then reacts explicitly to the outcome of the event. Thus the burden is on the trader to determine which of the possible outcomes occurred and then trade accordingly.

Drift

1. Drift as a Directional View: It is convenient to regard the drift parameter in the price process as a directional view of price movements. For example, the trader charged with liquidating a single security may believe that this security is likely to rise. Intuitively it makes more sense to trade this issue more slowly to take advantage of this view.
2. Incorporating Drift into Price Dynamics: To incorporate drift into the price dynamics Almgren and Chriss (2000) modify

$$S_k = S_{k-1} + \sigma\sqrt{\tau}\xi_k - \gamma g\left(\frac{n_k}{\tau}\right)$$

to

$$S_k = S_{k-1} + \sigma\sqrt{\tau}\xi_k + \alpha\tau - \gamma g\left(\frac{n_k}{\tau}\right)$$

where α is an expected drift term. If the trading proceeds are invested in an interest bearing account, then α should be taken as the *excess* rate of return of the risky asset.

3. Price Expectation over Time Period: One can readily write the modified version of

$$\mathbb{E}[X] = \frac{1}{2}\gamma X^2 + \epsilon \sum_{k=1}^{N} |n_k| + \frac{\tilde{\eta}}{\tau} \sum_{k=1}^{N} {n_k}^2$$

as

$$\mathbb{E}[X] = \frac{1}{2}\gamma X^2 - \alpha \sum_{k=1}^{N} \tau x_k + \epsilon \sum_{k=1}^{N} |n_k| + \frac{\tilde{\eta}}{\tau} \sum_{k=1}^{N} {n_k}^2$$

4. Updated Objective Function Optimality Condition: The variance is still given by

$$\mathbb{V}[X] = \sigma^2 \sum_{k=1}^{N} \tau {x_k}^2$$

The optimality condition

$$\frac{x_{j-1} - 2x_j + x_{j+1}}{\tau^2} = \tilde{\kappa}^2 x_j$$

becomes

$$\frac{x_{j-1} - 2x_j + x_{j+1}}{\tau^2} = \tilde{\kappa}^2 \left(x_j - \bar{x}\right)$$

in which the new parameter

$$\bar{x} = \frac{\alpha}{2\lambda\sigma^2}$$

is the optimal level of security holding for a time independent portfolio optimization problem.

5. Drift Based Updated Execution Slice: For example, the parameters used in the example above give approximately

$$\bar{x} = 1{,}100\ shares$$

or 0.11% of our initial portfolio. One expects this fraction to be very small, since, by hypothesis, the eventual aim is complete liquidation.

6. Drift Based Updated Optimal Solution: The optimal solution

$$x_j = \frac{\sinh\left(\kappa\left(T - t_j\right)\right)}{\sinh(\kappa T)} X$$

$$j = 0, \cdots, N$$

becomes

$$x_j = \frac{\sinh\left(\kappa\left(T - t_j\right)\right)}{\sinh(\kappa T)} X + \left\{1 - \frac{\sinh\left(\kappa\left(T - t_j\right)\right) + \sinh\left(\kappa t_j\right)}{\sinh(\kappa T)}\right\} \bar{x}$$

for

$$j = 0, \cdots, N$$

with the associated trades

$$n_j = \frac{2\sinh\left(\frac{1}{2}\kappa\tau\right)}{\sinh(\kappa T)}\cosh\left(\kappa\left(T - t_{j-\frac{1}{2}}\right)\right)X + \frac{2\sinh\left(\frac{1}{2}\kappa\tau\right)}{\sinh(\kappa T)}\left[\cosh\left(\kappa t_{j-\frac{1}{2}}\right) - \cosh\left(\kappa\left(T - t_{j-\frac{1}{2}}\right)\right)\right]\bar{x}$$

7. Initial Position Independent Trajectory Correction: This trading trajectory is a sum of two distinct trajectories – the zero-drift solution as computed before, plus a "correction" which profits by capturing a piece of the predictable drift component. The size of this correction term is proportional to $\bar{x}$, and thus to α: it is independent of the initial portfolio size X.
8. Practical Incorporation into Program Trading: To place this in an institutional framework, consider a program trading desk that sits in front of customer flow. If this desk were to explicitly generate alphas on all securities that flow through the desk in an attempt to, say, hold securities with high alphas and sell securities with low alphas more rapidly, the profit would not scale in proportion to the average size of the programs. Rather it would only scale with the number of securities that flow through the desk. An even stronger conclusion is that since the optimal strategy disconnects into a static strategy unrelated to the drift term, and a second strategy related to the drift term, there is no particular advantage to restricting trading in securities which the desk currently holds the positions in.
9. Comparison: Highly Liquid Markets Scenario: The difference between this solution and the no-drift solution in

$$x_j = \frac{\sinh\left(\kappa\left(T - t_j\right)\right)}{\sinh(\kappa T)}X$$

$$j = 0, \cdots, N$$

may be understood by considering the case

$$\kappa T \gg 1$$

corresponding to highly liquid markets. Whereas the previous one relaxed from X to $\frac{X}{e}$ in a time scale of

$$\theta = \frac{1}{\kappa}$$

this one relaxes instead to the optimal static portfolio size $\bar{x}$. Near the end of the trading period the trader sells the remaining holdings to achieve

$$x_N = 0$$

at

$$t = T$$

10. Caveat: Buy-Sell Symmetry Breaking: In this case, one requires

$$0 \leq \bar{x} \leq X$$

in order for all trades to be in the same direction. This breaks the symmetry between a buy program and a sell program, if one wanted to consider buy programs it would be more logical to set

$$\alpha = 0$$

Gain due to Drift

1. Gain from Drift – Calculation Motivation: Now suppose that the price dynamics is given by

$$S_k = S_{k-1} + \sigma\sqrt{\tau}\xi_k + \alpha\tau - \gamma g\left(\frac{n_k}{\tau}\right)$$

with

$$\alpha > 0$$

but one chooses to determine the solution as though

$$\alpha = 0$$

The situation may arise, for example, in case where the trader is trading a security with non-zero drift, but *unknowingly* assumes that the security has no drift. Almgren and Chriss (2000) explicitly calculate the loss associated with ignoring the drift term.

2. Gain adjusted $\mathbb{E}[x]$ and $\mathbb{V}[x]$: Write ${x_j}^*$ for the optimal solution

$$x_j = \frac{\sinh\left(\kappa\left(T - t_j\right)\right)}{\sinh(\kappa T)}X + \left\{1 - \frac{\sinh\left(\kappa\left(T - t_j\right)\right) + \sinh\left(\kappa t_j\right)}{\sinh(\kappa T)}\right\}\bar{x}$$

with

$$\alpha > 0$$

${x_j}^0$ for the sub-optimal solution

$$x_j = \frac{\sinh\left(\kappa\left(T - t_j\right)\right)}{\sinh(\kappa T)}X$$

$$j = 0, \cdots, N$$

or

$$x_j = \frac{\sinh\left(\kappa\left(T - t_j\right)\right)}{\sinh(\kappa T)} X + \left\{1 - \frac{\sinh\left(\kappa\left(T - t_j\right)\right) + \sinh\left(\kappa t_j\right)}{\sinh(\kappa T)}\right\} \bar{x}$$

with

$$\alpha = 0$$

Also write $\mathbb{E}^*[x]$ and $\mathbb{V}^*[x]$ for the optimal expected cost and its variance measured by

$$\mathbb{E}[X] = \frac{1}{2}\gamma X^2 - \alpha \sum_{k=1}^{N} \tau x_k + \epsilon \sum_{k=1}^{N} |n_k| + \frac{\tilde{\eta}}{\tau} \sum_{k=1}^{N} {n_k}^2$$

and

$$\mathbb{V}[X] = \sigma^2 \sum_{k=1}^{N} \tau {x_k}^2$$

with

$$x_j = {x_j}^*$$

and write $\mathbb{E}^0[x]$ and $\mathbb{V}^0[x]$ for the sub-optimal values of

$$\mathbb{E}[X] = \frac{1}{2}\gamma X^2 - \alpha \sum_{k=1}^{N} \tau x_k + \epsilon \sum_{k=1}^{N} |n_k| + \frac{\tilde{\eta}}{\tau} \sum_{k=1}^{N} {n_k}^2$$

and

$$\mathbb{V}[X] = \sigma^2 \sum_{k=1}^{N} \tau {x_k}^2$$

evaluated with

$$x_j = {x_j}^0$$

3. Objective Function Gain from Drift: The corresponding objective functions are

$$\mathbb{U}^*[X] = \mathbb{E}^*[X] + \lambda \mathbb{V}^*[X]$$

and

$$\mathbb{U}^0[X] = \mathbb{E}^0[X] + \lambda \mathbb{V}^0[X]$$

One can then define the *gain due to drift* to be the difference $\mathbb{U}^0[X] - \mathbb{U}^*[X]$; this is the reduction in the cost and the variance by being aware of and taking into account of the drift term. Clearly

$$\mathbb{U}^0[X] - \mathbb{U}^*[X] \geq 0$$

since x^* is the unique optimal strategy for the model with

$$\alpha > 0$$

4. Upper Bound for the Gain: Now the value of the terms in $\mathbb{U}^0[X]$ that come from

$$\mathbb{E}[X] = \frac{1}{2}\gamma X^2 - \alpha \sum_{k=1}^{N} \tau x_k + \epsilon \sum_{k=1}^{N} |n_k| + \frac{\tilde{\eta}}{\tau} \sum_{k=1}^{N} {n_k}^2$$

and

$$\mathbb{V}[X] = \sigma^2 \sum_{k=1}^{N} \tau {x_k}^2$$

is only *increased* by going from x^0 to x^* since x^0 and not x^* was the optimum strategy with

$$\alpha = 0$$

Therefore an *upper bound* for the gain is

$$\mathbb{U}^0[X] - \mathbb{U}^*[X] \geq \alpha\tau \sum_{k=1}^{N} ({x_k}^* - {x_k}^0)$$

5. Adjustment Applied to the Holdings: That is, in response to positive drift, one should increase the holdings throughout the trading. This reduces the net cost by the amount of the increase in the asset price one captures, at the expense of slightly increasing the transaction costs and the volatility exposure. An upper bound for the possible benefit is the amount of increase one captures.
6. Explicit Expression for the Bound: But ${x_k}^* - {x_k}^0$ is just the term in the square brackets in

$$x_j = \frac{\sinh\left(\kappa(T - t_j)\right)}{\sinh(\kappa T)} X + \left\{1 - \frac{\sinh\left(\kappa(T - t_j)\right) + \sinh(\kappa t_j)}{\sinh(\kappa T)}\right\} \bar{x}$$

times $\bar{x}$, which is clearly independent of X. Indeed this can be explicitly evaluated to get

$$\alpha\tau\sum_{k=1}^{N}({x_k}^* - {x_k}^0) = \alpha\bar{x}T\left[1 - \frac{\tau}{T}\frac{\tanh\left(\frac{1}{2}\kappa T\right)}{\tanh\left(\frac{1}{2}\kappa\tau\right)}\right]$$

7. Gain Comparison against Execution Cost: Since $\frac{\tanh x}{x}$ is a positive decreasingly function, this quantity is positive and bounded above by $\alpha\bar{x}T$, the amount one would gain by holding $\bar{x}$ for a time T. Any reasonable estimates for the parameters show that this quantity is negligible compared to the impact costs incurred in liquidating an institutional sized portfolio over a short period.

Serial Correlation

1. Prior Period Price Increment Component: Now one supposes that the asset prices exhibit serial correlation, so that at each period one discovers a component of predictability of the asset price in the next period.
2. Methodology behind the Price Increment Estimation: In the model

$$S_k = S_{k-1} + \sigma\sqrt{\tau}\xi_k - \gamma g\left(\frac{n_k}{\tau}\right)$$

with a drift

$$\alpha = 0$$

one now supposes that the ξ_k are serially correlated with period-to-period correlation ρ

$$|\rho| < 1$$

One can determine ξ_k at time k based on the obtained $S_k - S_{k-1}$ and sale n_k

3. Optimal Strategy no more Static: With serial correlation the optimal trajectory is no longer a static trajectory determined in advance of trading; since each price movement gives some information about the immediate future price movements, the optimal trade list can be determined only one period at a time.
4. Estimation of the Realized Gain: Thus a full optimal solution requires the use of dynamic programming methods. However since the information is still roughly local in time, one can estimate the optimal gain attainable by an optimal strategy.
5. Almgren and Chriss (2000) Conclusions: Almgren and Chriss (2000) state their conclusion in advance of their estimation. The value of information contained in pure movements due to serial correlations is independent of the size of the portfolios being traded. The calculation demonstrated below lends intuition to this counter-intuitive statement.
6. Per Period Price Change Impact: Consider two consecutive periods during which the base strategy has the trader trading the same number of shares n in each period. With a linear impact price model, in each period price changes by $\left[\epsilon + \eta\frac{n}{\tau}\right]$ dollars/share. The trader pays this cost in each of the n shares, so the total cost due of market impact per period is $\left[\epsilon + \eta\frac{n}{\tau}\right]n$
7. Price Change from Serial Correlation: Suppose on has some price information due to serial correlations. If one knows ξ_k at the previous period, then the predictable component of the price change is roughly $\rho\sigma\sqrt{\tau}\Delta n$.
8. Incremental Cost of the Adapted Strategy: But this adaptation increases the impact costs. After the shift in the first period the price change is $\epsilon + \eta\frac{n-\Delta n}{\tau}$ while in the second period $\epsilon + \eta\frac{n+\Delta n}{\tau}$. These costs are paid on $n - \Delta n$ and $n + \Delta n$ shares respectively, so the market impact per period is now

$$\left[\frac{1}{2}\left(\epsilon + \eta\frac{n-\Delta n}{\tau}\right)(n - \Delta n) + \frac{1}{2}\left(\epsilon + \eta\frac{n+\Delta n}{\tau}\right)(n + \Delta n)\right] = \left[\epsilon + \eta\frac{n}{\tau}\right]n + \frac{n}{\tau}\Delta n^2$$

9. Optimal Per-Period Execution Shift: To determine has many shares one should shift, one solves the quadratic optimization problem

$$\max_{\Delta n} \left[\rho\sigma\sqrt{\tau}\Delta n - \frac{n}{\tau}\Delta n^2\right]$$

The optimal Δn is readily found as

$$\Delta n^* = \frac{\rho\sigma\tau^{\frac{3}{2}}}{2\eta}$$

and the maximum possible gain per period is $\frac{\rho^2\sigma^2\tau^2}{4\eta}$. This heuristic can be confirmed by a detailed dynamic programming computation that accounts for optimal shifts across multiple periods.

10. Limitation of Optimal Gain Execution: Almgren and Chriss (2000) also explain briefly the limitation of the above approximation. When ρ is close to zero, clearly this approximation is extremely close to accurate, because the persistence of the serial correlation effect dies down very quickly after the first period. When $|\rho|$ is too large to ignore, the approximation is too small for

$$\rho > 0$$

That is, $\frac{\rho^2\sigma^2\tau^2}{4\eta}$ understates the possible gains over ignoring serial correlation. Conversely when

$$\rho < 0$$

$\frac{\rho^2\sigma^2\tau^2}{4\eta}$ overstates the possible gains due to serial correlation. As

$$\rho > 0$$

is more frequently the case Almgren and Chriss (2000) assert that $\frac{\rho^2\sigma^2\tau^2}{4\eta}$ is useful for bounding the possible gains in most situations available from serial correlations.

11. Position Independence of Gain/Cost: Note that both the size of the adaptation, and the resulting gain, are independent of the amount of shares n that would be sold under an unadapted strategy. That is they are also independent of the size of the initial portfolio.
12. Gain/Cost Liquidity/Correlation Dependence: Instead the binding constraint is the liquidity of the security being traded, and the magnitude of the correlation coefficient. The more information available due to correlation and the more liquid the security, the more overall gain that is available due to adapting the strategy to the correlations.
13. Higher Order Impact Function Optimality: The results above are especially simple because of the assumption of linear impact functions. Almgren and Chriss (2000) also show briefly what happens in the more general case of nonlinear market impact functions

$$h(v) = h\left(\frac{n}{\tau}\right)$$

The cost per period due to market impact is

$$\left[\frac{1}{2}h\left(\frac{n-\Delta n}{\tau}\right)(n-\Delta n)+\frac{1}{2}h\left(\frac{n+\Delta n}{\tau}\right)(n+\Delta n)\right]$$
$$\approx\ h\left(\frac{n}{\tau}\right)n+\left[\frac{1}{2}h''\left(\frac{n}{\tau}\right)\frac{n}{\tau}+h'\left(\frac{n}{\tau}\right)\right]\frac{\Delta n^2}{\tau}$$

for small Δn. Now the optimal shift and the maximal gain are given by

$$\Delta n^* = \frac{\rho\sigma\tau^{\frac{3}{2}}}{vh''+2h'}$$

and $\frac{\rho^2\sigma^2\tau^2}{2(vh''+2h')}$ respectively, where h' and h'' are evaluated at the base execution rate of

$$v = \frac{n}{\tau}$$

The linear case is recovered by setting

$$h(v) = \epsilon + \eta v$$

This has the special property that h' is independent of v and

$$h'' = 0$$

14. <u>Optimality Dependence on Impact Exponent</u>: In general suppose

$$h(v) \sim \mathcal{O}(v^{\varpi})$$

as

$$v \to \infty$$

$$\varpi > 0$$

is required so that $h(v)$ is increasing; selling the share always pushes the price down more. The marginal cost is

$$h'(v) \sim \mathcal{O}(v^{\varpi-1})$$

$$\varpi > 1$$

corresponds to an increasing marginal impact, and

$$\varpi < 1$$

corresponds to a decreasing marginal impact. Then the per-period cost one pays on the base strategy is

$$\sim \mathcal{O}(v^{\varpi+1})$$

for large initial portfolios, and hence large rates of execution. The marginal gain from adapting to evolution is

$$\sim \mathcal{O}(v^{\varpi-1})$$

in the same limit.

Parameter Shifts

1. Price Dynamics Parameter Set Shift: Almgren and Chriss (2000) discuss the impact on optimal execution of scheduled news earnings such as earnings and dividend announcements. Such events have two features that make them an important object of study. First the outcome of the event determines the shift in the parameters governing the price dynamics – see Brown, Harlow, and Tinic (1988), Easterwood and Nutt (1999), and Ramaswami (1999).
2. Determining an Event' Full Impact: Second, the fact that they are scheduled increases the likelihood that one can detect what the true outcome of the event is. This situation is formalized below, and explicit formulas are given for price trajectories before and after the event takes place.

3. Scheduled Event Occurrence Time T_*: Suppose at some time T_* between now and the specified final time T an event will occur, the outcome of which may or may not cause a shift in the parameters of price dynamics.
4. New Regime Shifted Parameter Set: The term *regime set* or *parameter set* refers to the collection

$$R = \{\sigma, \eta, \cdots\}$$

   of the parameters that govern the dynamics at any particular time, and the events of interest are those that have the possibility of causing *parameter shifts*.
5. Initial to Final Regime Shift: Let

$$R_0 = \{\sigma_0, \eta_0, \cdots\}$$

   be the parameters of price dynamics at the time the execution begins. Suppose the market can shift to one of possible new sets of parameters p so that $R_1, \cdots, R_p$ is characterized by parameters $\sigma_j, \eta_j, \cdots$ for

$$j = 1, \cdots, p$$

6. Probability of a Regime Switch: One also supposes that probabilities can be assigned to these possible new states, so that p_j is the probability that regime R_j occurs. The probabilities are *independent* of the short term market fluctuations represented by ξ_k. Of course it is possible that some R_j has the same values as R_0 in which case p_j is the probability that no change occurs.
7. Globally Optimal Dynamic Trading Strategy: Almgren and Chriss (2000) consider a dynamic trading strategy the yields globally optimal strategies in the presence of a parameter shift at time T_*. Taking

$$T_* = t_s = s\tau$$

one pre-computes an initial trajectory

$$x^0 = \{{x_0}^0, \cdots, {x_s}^0\}$$

with

$${x_0}^0 = X$$

Denote

$$X_* = {x_s}^0$$

8. Landscape of Switchable Trajectories: They also compute a family of trajectories

$$x^j = \left\{{x_0}^j, \cdots, {x_s}^j\right\}$$

for

$$j = 1, \cdots, p$$

all of which have

$${x_s}^j = X_*$$

and

$${x_N}^j = 0$$

They follow the trajectory x^0 until the time of the shift. Once the shift occurs they assume they can quickly identify the outcome of the event and the new set of parameters governing the price dynamics. With this settled, the complete trading using the corresponding trajectory x^j is determined.

9. Key Almgren and Chriss (2000) Results: Almgren and Chriss (2000) show that it is possible to determine each trajectory using static optimization; although one cannot choose which one to use until the event occurs. Also the starting trajectory x^0 will *not be the same* as the trajectory one would use if they believed the regime R_0 would hold through the entire time T.
10. Trajectory Conditional on Fixed X_*: To determine the trajectories $x^0, x^1, \cdots, x^p$ they reason as follows. Suppose that the common value of

$$X_* = {x_s}^0 = {x_s}^j$$

is fixed. Then by virtue of the independence of the regime shift in itself from the security motions, the optimal trajectories conditional on the values of X_* are simply those that have already been computed with a small modification to include the given non-zero final value.
11. Sequential Pair of Static Strategies: One can immediately write

$${x_k}^0 = \frac{\sinh\left(\kappa_0(T_* - t_k)\right)}{\sinh(\kappa_0 T_*)} X + \frac{\sinh(\kappa_0 t_k)}{\sinh(\kappa_0 T_*)} X_*$$

$$k = 0, \cdots, s$$

where κ_0 is determined from $\sigma_0, \eta_0, \cdots$. The trajectory is determined the same way as seen before; it is the unique combination of the exponentials $x^{\pm\kappa_0 t}$ that has

$${x_0}^0 = X$$

and

$${x_s}^0 = X_*$$

Similarly

$$x_k{}^j = \frac{\sinh\left(\kappa_j(T - t_k)\right)}{\sinh\left(\kappa_j(T - T_*)\right)} X_*$$

$$k = s, \cdots, N$$

$$j = 1, \cdots, p$$

Thus one only needs to determine X_*.

12. Principle behind the Estimation of X_*: To determine X_* one needs to determine the expected loss and the variance of the combined strategy. Let $\mathbb{E}_0$ and $\mathbb{V}_0$ denote the expectation and the loss incurred by the trajectory x^0 on the first segment

$$k = 0, \cdots, s$$

The quantities can be determined readily using

$$\mathbb{E}[X] = \frac{1}{2}\gamma X^2 + \epsilon \sum_{k=1}^{N} |n_k| + \frac{\tilde{\eta}}{\tau} \sum_{k=1}^{N} n_k{}^2$$

and

$$\mathbb{V}[X] = \sigma^2 \sum_{k=1}^{N} \tau x_k{}^2$$

13. Mean Variance of the Compound Strategy: Then by virtue of the regime shift and the security motion's independence, the expected loss of the compound strategy is

$$\mathbb{E} = \mathbb{E}_0 + \mathbb{P}_1\mathbb{E}_1 + \cdots + \mathbb{P}_p\mathbb{E}_P$$

and its variance is

$$\mathbb{V} = \mathbb{V}_0 + \mathbb{P}_1\mathbb{V}_1 + \cdots + \mathbb{P}_p\mathbb{V}_P + \frac{1}{2}\sum_{i,j=1}^{p0} \mathbb{P}_i\mathbb{P}_j\left(\mathbb{E}_i - \mathbb{E}_j\right)^2$$

One can now do a one-variable optimization in X_* to maximize $\mathbb{E} + \lambda\mathbb{V}$. Almgren and Chriss (2000) provide a pictorial representation of the above.

Conclusions and Further Extensions

1. Efficient Frontier of Transaction Costs: The central feature of the Almgren and Chriss (2000) analysis has been to construct an *efficient frontier* in a two-dimensional plane whose axes are the expectation of the total cost and its variance.
2. Linear Impact Functions Analytical Solutions: Regardless of an individual's tolerance to risk, the only strategies which are candidates for being optimal solutions are found in this one-parameter set. For linear impact functions, they give complete analytical solutions for the strategies in this set.
3. Efficient Frontier Optimal Operating Characteristic: Then, considering the details of risk aversion, they have shown how to select an optimal point on the frontier either by classical mean-variance optimization, or by the concept of value at risk. These solutions are easily constructed numerically, and interpreted graphically by examining the frontier.
4. First Conclusion: Sub-optimal Strategies: Because the set of attainable strategies, and hence the efficient frontier, are generally *smooth* and *convex,* a trader who is at all risk-averse should never trade according to the naïve strategy of minimizing expected cost. This is because in the neighborhood of that strategy, the first order reduction in

the variance is attained at the expense of only a second order increase in the expected cost.

5. Second Conclusion: Custom Risk Optimization: Almgren and Chriss (2000) also observe that this careful analysis of the costs and risks of liquidation can be used to give a more precise characterization of the risk of holding the initial portfolio. As an example, they define a Liquidity-Adjusted VaR (L-VaR) to be, for as given time horizon, the minimum VaR of any static liquidation strategy.
6. Actual Gains of Dynamic Trading: Although it may seem counter-intuitive that the optimal strategies can be determined in advance of trading, Almgren and Chriss (2000) argue that only very small gains can be realized by adapting the strategy to the information as it is needed.
7. First Extension: Continuous Time Trading: The limit

$$\tau \to 0$$

is immediate in all of their solutions. Their trading strategy is characterized by a holdings function $x(t)$ and a *trading rate*

$$x(t) = \underset{\tau \to 0}{Limit} \frac{n_k}{\tau}$$

Almgren and Chriss (2000) minimum variance strategy has infinite cost, but the optimal strategies for finite λ have finite cost and variance. However, this limit is at best a mathematical convenience, as the market model is implicitly a "coarse-grained" description of the real dynamics.
8. Second Extension: Nonlinear Cost: The conceptual framework outlined by Almgren and Chriss (2000) is not limited to the linear permanent and temporary impact functions

$$g(v) = \gamma v$$

and

$$h(v) = \epsilon sgn(n_k) + \frac{\eta}{\tau} n_k$$

though the exact exponential/hyperbolic solutions are specific to that case. For nonlinear functions $g(v)$ and $h(v)$ that satisfy suitable convexity conditions, optimal risk-averse trajectories are found by solving a non-quadratic optimization problem; the difficulty of the problem depends on the specific functional forms chosen.

9. Third Extension Time Varying Coefficients: Almgren and Chriss (2000) framework also covers the case in which the volatility, the market impact parameters, and perhaps the expected drift are all time-dependent; finding the optimal strategy entails solving a linear system of size equal to the number of time periods (times the number of assets, for a portfolio problem). One example in which this is useful is if the price is expected to jump up or down on a known future date – say, an earnings announcement – as long as one has a good estimate of the expected *size* of this jump.

Numerical Optimal Trajectory Generation

1. Varying Time Interval Cost Distribution:

$$\mathbb{E}[x] = \sum_{k=1}^{N} \tau_k x_k g\left(\frac{n_k}{\tau_k}\right) + \sum_{k=1}^{N} n_k h\left(\frac{n_k}{\tau_k}\right)$$

$$\mathbb{V}[x] = \sum_{k=1}^{N} \tau_k {\sigma_k}^2 {x_k}^2$$

$$\tau_k = t_k - t_{k-1}$$

$$n_k = x_k - x_{k-1}$$

2. <u>Time Varying Interval Linear Impact</u>:

$$\mathbb{E}[x] = \frac{1}{2}\gamma X^2 + \epsilon\sum_{k=1}^{N}|n_k| + \eta\sum_{k=1}^{N}\frac{{n_k}^2}{\tau_k} - \frac{1}{2}\gamma\sum_{k=1}^{N}{n_k}^2$$

$$\mathbb{V}[x] = \sum_{k=1}^{N}\tau_k{\sigma_k}^2{x_k}^2$$

$$\tau_k = t_k - t_{k-1}$$

$$n_k = x_k - x_{k-1}$$

3. <u>Varying Interval Linear Impact Objective</u>:

$$\mathbb{U}[x] = \mathbb{E}[x] + \lambda\,\mathbb{V}[x]$$

implies

$$\mathbb{U}[x] = \frac{1}{2}\gamma X^2 + \epsilon\sum_{k=1}^{N}|n_k| + \eta\sum_{k=1}^{N}\frac{{n_k}^2}{\tau_k} - \frac{1}{2}\gamma\sum_{k=1}^{N}{n_k}^2 + \lambda\sum_{k=1}^{N}\tau_k{\sigma_k}^2{x_k}^2$$

4. <u>Varying Time Interval Linear Impact Jacobian</u>:

$$\frac{\partial\mathbb{U}[x]}{\partial x_j} = 2\left\{\lambda\tau_j{\sigma_j}^2x_j - \tilde{\eta}\left[\frac{x_{j+1}}{\tau_{j+1}} - \frac{x_{j-1}}{\tau_j} - x_j\left(\frac{1}{\tau_j} + \frac{1}{\tau_{j+1}}\right)\right]\right\} + \frac{1}{2}\gamma\left[x_{j-1} - 2x_j + x_{j+1}\right]$$

Extrema requires that

$$\frac{\partial\mathbb{U}[x]}{\partial x_j} = 0\ \forall\ j = 1,\cdots,N-1$$

5. Estimation Quantities for Numerical Optimization: To carry out the numerical optimization, one needs the Jacobian (i.e., gradient) and the Hessian of the optimizer objective function in terms of

$$x_j \; \forall \; j = 1, \cdots, N - 1$$

This has to be computed for each of the following quantities:

a. Trajectory Slice Permanent Impact Function Expectation Left Holdings
b. Trajectory Slice Permanent Impact Function Expectation Right Holdings
c. Trajectory Slice Permanent Impact Function Expectation Cross Holdings Jacobian
d. Trajectory Slice Temporary Impact Function Expectation Left Holdings
e. Trajectory Slice Temporary Impact Function Expectation Right Holdings
f. Trajectory Slice Temporary Impact Function Expectation Cross Holdings Jacobian
g. Trajectory Slice Permanent Impact Function Variance Left Holdings
h. Trajectory Slice Permanent Impact Function Variance Right Holdings
i. Trajectory Slice Permanent Impact Function Variance Cross Holdings Jacobian
j. Trajectory Slice Temporary Impact Function Variance Left Holdings
k. Trajectory Slice Temporary Impact Function Variance Right Holdings
l. Trajectory Slice Temporary Impact Function Variance Cross Holdings Jacobian
m. Trajectory Slice Core Market Function Expectation Left Holdings
n. Trajectory Slice Core Market Function Expectation Right Holdings
o. Trajectory Slice Core Market Function Expectation Cross Holdings Jacobian
p. Trajectory Slice Core Market Function Variance Left Holdings
q. Trajectory Slice Core Market Function Variance Right Holdings
r. Trajectory Slice Core Market Function Variance Cross Holdings Jacobian
s. Trajectory Permanent Impact Function Expectation Left Holdings

t. Trajectory Permanent Impact Function Expectation Right Holdings
u. Trajectory Permanent Impact Function Expectation Cross Holdings Jacobian
v. Trajectory Temporary Impact Function Expectation Left Holdings
w. Trajectory Temporary Impact Function Expectation Right Holdings
x. Trajectory Temporary Impact Function Expectation Cross Holdings Jacobian
y. Trajectory Permanent Impact Function Variance Left Holdings
z. Trajectory Permanent Impact Function Variance Right Holdings
aa. Trajectory Permanent Impact Function Variance Cross Holdings Jacobian
bb. Trajectory Temporary Impact Function Variance Left Holdings
cc. Trajectory Temporary Impact Function Variance Right Holdings
dd. Trajectory Temporary Impact Function Variance Cross Holdings Jacobian
ee. Trajectory Core Market Function Expectation Left Holdings
ff. Trajectory Core Market Function Expectation Right Holdings
gg. Trajectory Core Market Function Expectation Cross Holdings Jacobian
hh. Trajectory Core Market Function Variance Left Holdings
ii. Trajectory Core Market Function Variance Right Holdings
jj. Trajectory Core Market Function Variance Cross Holdings Jacobian
kk. Objective Utility Function Permanent Impact Function Expectation Left Holdings
ll. Objective Utility Function Permanent Impact Function Expectation Right Holdings
mm. Objective Utility Function Permanent Impact Function Expectation Cross Holdings Jacobian
nn. Objective Utility Function Temporary Impact Function Expectation Left Holdings
oo. Objective Utility Function Temporary Impact Function Expectation Right Holdings
pp. Objective Utility Function Temporary Impact Function Expectation Cross Holdings Jacobian
qq. Objective Utility Function Permanent Impact Function Variance Left Holdings

rr. Objective Utility Function Permanent Impact Function Variance Right Holdings

ss. Objective Utility Function Permanent Impact Function Variance Cross Holdings Jacobian

tt. Objective Utility Function Temporary Impact Function Variance Left Holdings

uu. Objective Utility Function Temporary Impact Function Variance Right Holdings

vv. Objective Utility Function Temporary Impact Function Variance Cross Holdings Jacobian

ww. Objective Utility Function Core Market Function Expectation Left Holdings

xx. Objective Utility Function Core Market Function Expectation Right Holdings

yy. Objective Utility Function Core Market Function Expectation Cross Holdings Jacobian

zz. Objective Utility Function Core Market Function Variance Left Holdings

aaa. Objective Utility Function Core Market Function Variance Right Holdings

bbb. Trajectory Slice Core Market Function Variance Cross Holdings Jacobian

6. <u>Permanent Impact Expectation Left Sensitivity</u>:

$$\mathbb{E}_{P,k} = s\tau_k x_k g\left(\frac{x_k - x_{k-1}}{\tau_k}\right)$$

$$\frac{\partial \mathbb{E}_{P,k}}{\partial x_{k-1}} = s\tau_k x_k \frac{\partial g\left(\frac{x_k - x_{k-1}}{\tau_k}\right)}{\partial x_{k-1}}$$

$$\frac{\partial^2 \mathbb{E}_{P,k}}{\partial {x_{k-1}}^2} = s\tau_k x_k \frac{\partial^2 g\left(\frac{x_k - x_{k-1}}{\tau_k}\right)}{\partial {x_{k-1}}^2}$$

$$s = sign\left(\frac{x_k - x_{k-1}}{\tau_k}\right)$$

7. Permanent Impact Expectation Right Sensitivity:

$$\mathbb{E}_{P,k} = s\tau_k x_k g\left(\frac{x_k - x_{k-1}}{\tau_k}\right)$$

$$\frac{\partial \mathbb{E}_{P,k}}{\partial x_k} = s\tau_k g\left(\frac{x_k - x_{k-1}}{\tau_k}\right) + s\tau_k x_k \frac{\partial g\left(\frac{x_k - x_{k-1}}{\tau_k}\right)}{\partial x_k}$$

$$\frac{\partial^2 \mathbb{E}_{P,k}}{\partial {x_k}^2} = 2s\tau_k \frac{\partial g\left(\frac{x_k - x_{k-1}}{\tau_k}\right)}{\partial x_k} + s\tau_k x_k \frac{\partial^2 g\left(\frac{x_k - x_{k-1}}{\tau_k}\right)}{\partial {x_k}^2}$$

8. Permanent Impact Expectation Cross Jacobian:

$$\mathbb{E}_{P,k} = s\tau_k x_k g\left(\frac{x_k - x_{k-1}}{\tau_k}\right)$$

$$\frac{\partial^2 \mathbb{E}_{P,k}}{\partial x_{k-1} \partial x_k} = s\tau_k x_k \frac{\partial^2 g\left(\frac{x_k - x_{k-1}}{\tau_k}\right)}{\partial x_{k-1} \partial x_k} + s\tau_k \frac{\partial g\left(\frac{x_k - x_{k-1}}{\tau_k}\right)}{\partial x_{k-1}}$$

9. Temporary Impact Expectation Left Sensitivity:

$$\mathbb{E}_{T,k} = (x_k - x_{k-1}) h\left(\frac{x_k - x_{k-1}}{\tau_k}\right)$$

$$\frac{\partial \mathbb{E}_{T,k}}{\partial x_{k-1}} = -h\left(\frac{x_k - x_{k-1}}{\tau_k}\right) + (x_k - x_{k-1}) \frac{\partial h\left(\frac{x_k - x_{k-1}}{\tau_k}\right)}{\partial x_{k-1}}$$

$$\frac{\partial^2 \mathbb{E}_{T,k}}{\partial {x_{k-1}}^2} = -2 \frac{\partial h\left(\frac{x_k - x_{k-1}}{\tau_k}\right)}{\partial x_{k-1}} + (x_k - x_{k-1}) \frac{\partial^2 h\left(\frac{x_k - x_{k-1}}{\tau_k}\right)}{\partial {x_{k-1}}^2}$$

10. <u>Temporary Impact Expectation Right Sensitivity</u>:

$$\mathbb{E}_{T,k} = (x_k - x_{k-1})h\left(\frac{x_k - x_{k-1}}{\tau_k}\right)$$

$$\frac{\partial \mathbb{E}_{T,k}}{\partial x_k} = h\left(\frac{x_k - x_{k-1}}{\tau_k}\right) + (x_k - x_{k-1})\frac{\partial h\left(\frac{x_k - x_{k-1}}{\tau_k}\right)}{\partial x_k}$$

$$\frac{\partial^2 \mathbb{E}_{T,k}}{\partial {x_k}^2} = 2\frac{\partial h\left(\frac{x_k - x_{k-1}}{\tau_k}\right)}{\partial x_k} + (x_k - x_{k-1})\frac{\partial^2 h\left(\frac{x_k - x_{k-1}}{\tau_k}\right)}{\partial {x_k}^2}$$

11. <u>Temporary Impact Expectation Cross Jacobian</u>:

$$\mathbb{E}_{T,k} = (x_k - x_{k-1})h\left(\frac{x_k - x_{k-1}}{\tau_k}\right)$$

$$\frac{\partial^2 \mathbb{E}_{T,k}}{\partial x_{k-1}\partial x_k} = -\frac{\partial h\left(\frac{x_k - x_{k-1}}{\tau_k}\right)}{\partial x_k} + \frac{\partial h\left(\frac{x_k - x_{k-1}}{\tau_k}\right)}{\partial x_{k-1}} + (x_k - x_{k-1})\frac{\partial^2 h\left(\frac{x_k - x_{k-1}}{\tau_k}\right)}{\partial x_k \partial x_{k-1}}$$

12. <u>Trajectory Jacobian and Hessian Computation</u>: In general, the trajectory Jacobian's and the Hessian's may be computed as a sequential, aggregate accumulations over the corresponding slices, with one very critical caveat. In the automated sensitivity generation schemes, all sensitivities to the left-most and the right-most nodes must be excluded, since these do not constitute the control nodes.
13. <u>Power Objective Function Rationale/Formulation</u>: A generalization of the mean-variance optimization and the Value-at-risk schemes is the power objective function formulation

$$\mathbb{U}[x] = \mathbb{E}[x] + \lambda(\mathbb{V}[x])^p$$

$$p = 1$$

corresponds to the regular mean-variance optimization scheme, and

$$p = 0.5$$

corresponds to the liquidity based VaR formulation.

14. Liquidity VaR Control Jacobian/Hessian:

$$\mathbb{U}[x] = \mathbb{E}[x] + \lambda(\mathbb{V}[x])^p$$

$$\frac{\partial \mathbb{U}[x]}{\partial x_i} = \frac{\partial \mathbb{E}[x]}{\partial x_i} + \lambda p(\mathbb{V}[x])^{p-1} \frac{\partial \mathbb{V}[x]}{\partial x_i}$$

$$\frac{\partial^2 \mathbb{U}[x]}{\partial x_i \partial x_j} = \frac{\partial^2 \mathbb{E}[x]}{\partial x_i \partial x_j} + \lambda p(p-2)(\mathbb{V}[x])^{p-2} \frac{\partial \mathbb{V}[x]}{\partial x_i} \frac{\partial \mathbb{V}[x]}{\partial x_j} + \lambda p(\mathbb{V}[x])^{p-1} \frac{\partial^2 \mathbb{V}[x]}{\partial x_i \partial x_j}$$

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Optimal Execution with Non-linear Impact Functions and Trading-Enhanced Risk

Abstract

1. Price and Market Impact Volatility: Almgren (2003) determines optimal trading strategies for the liquidation of a large single-asset portfolio to minimize a combination of volatility risks and market impact costs.
2. Power Law Market Impact Function: The market impact cost is taken to be a power law of the trading rate with an arbitrary positive exponent. This includes, for example, the square root law that has been proposed based on market microstructure theory.
3. Holdings Size Dependent Characteristic Time: In analogy with the linear model, a *characteristic time* is defined for optimal trading, which now depends on the initial portfolio size and decreases as the execution proceeds.
4. Trade Size Dependent Liquidity Volatility: Also considered is a model in which the uncertainty of the realized price is increased by demanding rapid execution; it is shown that the optimal trajectories are defined by a *critical portfolio size* above which this effect is dominant and below which this effect may be neglected.

Introduction

1. Active vs. Passive Execution Strategy: In the execution of large portfolio transactions, a trading strategy must be determined that balances the risk of delayed execution against the cost of rapid execution; the choice is roughly between an *active* and *passive* trading strategy (Hasbrouck and Schwartz (1988), Wagner and Banks (1992)).

2. Construction of Optimal Execution Strategies: Several papers have constructed optimal strategies for the problem (Almgren and Chriss (1999), Grinold and Kahn (1999), Almgren and Chriss (2000), Konishi and Makimoto (2001) under the assumption that the liquidity costs per share traded are a linear function of trading rate or block size, and that the only source of volatility in execution is the price volatility of the underlying asset.
3. Price Effect of Block Trades: There is an extensive literature studying the effects of block trades on prices – see Kraus and Stoll (1972), Holthausen, Leftwich, and Mayers (1987, 1990), Chan and Lakonishok (1993, 1995), Keim and Madhavan (1995, 1997), and Koski and Michaely (2000).
4. Assumption of Linear Trading Costs: In practice linearity of trading costs is an unrealistic assumption. Perold and Salomon Jr. (1991) have argued that the liquidity premium per share demanded by the market will either be a convex or a concave function of the block size depending on whether the market's perception is that the trader is information driven or liquidity driven, respectively.
5. Barra Market Impact Liquidity Premium: In the Barra Market Impact Model (Loeb (1983), Kahn (1993), Barra (1997), Grinold and Kahn (1999)) it is argued, based upon the detailed analysis of the risk-reward choice faced by the equity market maker that the liquidity premium per share should grow as the square root of the block size traded.
6. Block Size Dependent Liquidity Premium: Electronic trading systems such as Optimark (Rickard and Torre (1999)) have been constructed to allow traders specify precisely what liquidity premium they are willing to pay as a function of the block size, and search for clearing opportunities in the mismatch between profiles of different market participants. These effects can be captured by introducing *nonlinear impact functions* into the cost function which is minimized to determine the optimal trading strategies.
7. Approaches of Nonlinear Models: Although linear models are commonly used in empirical regression analyses for simplicity, nonlinear models can often be emulated by dividing trades into categories by size (Bessembinder and Kaufmann (1997),

Huang and Stoll (1997)). In fact, Chakravarthy (2001) argues medium-sized trades have a disproportionately large effect on prices.

8. Handling Non-Deterministic Liquidity Premiums: An additional effect not considered in the theoretical strategies considered in the previous work is that the liquidity premium demanded by the market is not deterministic. In fact the premium will depend on the presence in the market at that instant of participants who are willing to take the other side of the trade.
9. Motivation for Trading Enhanced Risk: Since the presence of these counterparties cannot be predicted in advance, it represents an additional source of risk incurred by the trading profile. That is, a more complete model should include *trading-enhanced risk* representing the increased uncertainty in the execution price incurred by demanding rapid execution of large blocks.
10. Manifestation of Stochastic Liquidity Premiums: Trading-enhanced risk is an implicit feature in the model described in Rickard and Torre (1999). Chordia, Subrahmanyam, and Anshuman (2001) and Hasbrouck and Seppi (2001) argue that liquidity fluctuates due to intrinsic variations in the market activity independent of the trade size. This effect is included in the model by Almgren (2003) via the constant term $f(0)$ but additional interest is in the *increase* in the execution price uncertainty due to larger block sizes.
11. Nonlinear Block Size Dependence: Thus Almgren (2003) extends the models of Grinold and Kahn (1999) and Almgren and Chriss (2000) in a few important ways. First, the liquidity premium, expressed as an unfavorable motion of the price per share, may be an increasing nonlinear function of the trading rate and the block size – one is considered to be a proxy for the other.
12. Power Law Premium - Closed Form: This cost is reduced by trading slowly, but it must be balanced against the volatility risk incurred by holding the initial portfolio longer than is necessary. In particular, exact solutions are provided in the case this function is a power law with an arbitrary positive exponent, which covers the range of behavior outlined above.
13. Dependence on Initial Portfolio Size: Whereas in the linear case optimal trajectories are characterized by a single *characteristic time* independent of the initial portfolio

size, in the nonlinear case the characteristic time depends upon the initial portfolio size, and scales appropriately as the remaining portfolio diminishes during trading.

14. Comparison with Price Volatility Risk: The realized price per share itself is a random variable, whose variance increases ith the increased rate of trading. This introduces an additional source of risk in addition to the volatility. In contrast to the effect of market volatility, this additional risk is *decreased* by trading slowly, submitting small blocks for execution at each time.
15. Volatile Liquidity - Closed Form Solutions: Nearly explicit optimal solutions including this effect can be constructed, and an asymptotic analysis can be used to show that the effect of trading-enhanced risk is most important for large initial portfolios. Indeed for any given set of parameters there is a characteristic portfolio size above which the optimal strategy is determined by the need to reduce trading-enhanced risk, and below which this effect may be ignored.

The Model

1. Holdings Trade and Liquidation Time: The general framework followed is that from Almgren and Chriss (2000). At time

$$t = 0$$

X shares of an asset are held, which are to be completely liquidated by the time

$$t = T$$

The initial size X is positive for a sell program and negative for a buy program; in the former case, there is a long exposure to the market until all the holdings have been eliminated, while in the latter case there is short exposure to the market until the purchase to which the trader has committed to at

$$t = 0$$

is completed. The focus here is on the case

$$X > 0$$

In the case of a portfolio trading problem X may be a vector, but the consideration here is only on a single asset.

2. The Problem Trade List Determination: $x(t)$ denotes the holdings at time t with

$$x(0) = X$$

and

$$x(T) = 0$$

The problem is to choose an optimal function $x(\cdot)$ so as to minimize a chosen cost functional. Later the limit

$$T \to \infty$$

will be taken in which the natural execution time emerges as a result of the analysis, but for now the consideration is on an exogenously imposed time horizon.

3. Origin of the *Static* Strategy: It is a rather surprising fact that in the absence of serial correlation in the asset price movements, the optimal price may be determined *statically* at the start of the trading. Unless the market parameters change, observations of price movements in the course of trading do not convey any information that would lead to a change in the strategy.
4. Evenly Spaced Discrete Time Intervals: The analysis starts with the construction of a discrete time model. Thus for a given trading interval

$$\tau > 0$$

$$t_k = k\tau$$

for

$$k = 0, \cdots, N$$

with

$$N = \frac{T}{\tau}$$

and let x_k be the holdings at time t_k with

$$x_0 = X$$

and

$$x_N = 0$$

The sales between times t_k and t_{k-1} are

$$n_k = x_k - x_{k-1}$$

corresponding to the velocity

$$v_k = \frac{n_k}{\tau} \textit{ shares per unit time}$$

Thus

$$x_k = X - \sum_{j=1}^{k} n_j = \sum_{j=k+1}^{N} n_j$$

$$k = 0, \cdots, N$$

5. Generating the Optimal Trade List: In the discrete time model there is no assumption that the shares are traded at a uniform rate *within* each interval. Rather the assumption is that the trader achieves the optimal execution possible subject to the constraint that n_k shares are to be traded in the next time interval τ. The functions introduced below are a model to describe the trader's best efforts.
6. Temporary/Permanent Market Impact Components: On a standard manner (Stoll (1985)) the impact is divided into a permanent and a temporary component. Thus S_k describes the price per share of the asset that is publically available in the market.
7. Discrete Arithmetic Permanent Impact Component: The price satisfies the arithmetic random walk

$$S_k = S_{k-1} + \sigma\sqrt{\tau_k}\xi_k - \tau_k g\left(\frac{n_k}{\tau_k}\right) = S_0 + \sigma \sum_{j=1}^{k} \sqrt{\tau_j}\xi_j - \sum_{j=1}^{k} \tau_j g(v_j)$$

   where ξ_j are independent random variables with zero mean and unit variance, σ is an *absolute* (not percentage) volatility, $g(v)$ is the *permanent impact function* representing the effect of the share price of the information conveyed by the trade. This effect is generally small, and below $g(v)$ is taken to be a linear function, in which case it will have no effect on determining the optimal strategy.
8. Nonlinear Temporary Impact Component: The price that one actually gets on the k^{th} trade is

$$\tilde{S}_k = S_{k-1} - h\left(\frac{n_k}{\tau_k}\right) + \frac{1}{\sqrt{\tau_k}} f\left(\frac{n_k}{\tau_k}\right)\tilde{\xi}_k$$

$$k = 1, \cdots, N$$

Here $h(v)$ is a nonlinear *temporary impact function* representing the price concession one must accept in order to trade

$$n_k = v_k \tau_k$$

shares in time τ_k. The random variables $\tilde{\xi}_k$ are independent of each other and of ξ_k with zero mean and unit variance. The new function $f(v)$ represents the uncertainty of the trade execution as a function of the block size.

9. Liquidity Volatility Term Time Dependence: The factor $\frac{1}{\sqrt{\tau_k}}$ in the last term of

$$\tilde{S}_k = S_{k-1} - h\left(\frac{n_k}{\tau_k}\right) + \frac{1}{\sqrt{\tau_k}} f\left(\frac{n_k}{\tau_k}\right) \tilde{\xi}_k$$

simply represents a scaling of the parameters if τ_k is fixed and finite. When τ_k varies, for example when the continuous time

$$\tau_k \to 0$$

is taken, this factor is necessary to preserve the effect of the trading-enhanced risk.

10. Liquidity Volatility Incremental Time Dependence: If the above term were not present, then breaking a block into several smaller blocks would diversify away the risk due to the uncertainty of each one, regardless of the form of the risk.
11. *Capture* of the Trade Program: The *capture* of the trade program is the total cash received

$$\sum_{k=1}^{N} n_k \tilde{S}_k = XS_0 + \sigma \sum_{k=1}^{N} \sqrt{\tau_k} x_k \xi_k - \sum_{k=1}^{N} \tau_k x_k g(v_k) + \sum_{k=1}^{N} \sqrt{\tau_k} v_k f(v_k) \tilde{\xi}_k$$
$$- \sum_{k=1}^{N} \tau_k v_k h(v_k)$$

12. <u>The Trade Program *Implementation Cost*</u>: Discounting is ignore since the trading horizon is short. The *implementation cost* is $XS_0 - \sum_{k=1}^{N} n_k \tilde{S}_k$ - a random variable due to uncertainties in price movements and realized prices.
13. <u>Components of the *Implementation Cost*</u>: Note that the implementation cost includes both the costs of finite liquidity and price uncertainty due to delayed execution. This is the *implementation shortfall* of Perold (1988) – see also Jones and Lipson (1999).
14. <u>Implementation Cost Mean and Variance</u>: Its expectation and variance at

$$t = 0$$

depend on the free parameters $x_1, \cdots, x_{N-1}$ of the trade strategy:

$$\mathbb{E}[x_1, \cdots, x_{N-1}] = \sum_{k=1}^{N} \tau_k x_k g(v_k) + \sum_{k=1}^{N} \tau_k v_k h(v_k)$$

$$\mathbb{V}[x_1, \cdots, x_{N-1}] = \sum_{k=1}^{N} \tau_k {\sigma_k}^2 {x_k}^2 + \sum_{k=1}^{N} \tau_k {v_k}^2 f^2(v_k)$$

15. <u>Mean Variance Optimal Static Strategies</u>: A rational trader will construct his or her own strategies to minimize some combination of $\mathbb{E}[x]$ and $\mathbb{V}[x]$. As t advances the values of $\mathbb{E}[x]$ and $\mathbb{V}[x]$ change, but if $\mathbb{E}[x]$ and $\mathbb{V}[x]$ are constructed using a classic mean-variance approach, the optimal strategy continues to be the one determined initially (Almgren and Chriss (2000), Huberman and Stanzl (2005)).
16. <u>Continuous Time Limit Trading Strategy</u>: Now, for analytical convenience, the continuous time limit

$$\tau \to 0$$

is taken. The trade strategy becomes a continuous path $x(t)$ and the block sizes n_k are assumed to be well-behaved so that

$$v_k \to v(\tau_k k)$$

with

$$v(t) = -\dot{x}(t)$$

17. Continuous Time Mean and Variance: The above expressions have finite limits

$$\mathbb{E}[x] = \int_0^T \left[x(t)g\big(v(t)\big) + v(t)h\big(v(t)\big)\right]dt$$

$$\mathbb{V}[x] = \int_0^T \left[\sigma^2 x^2(t) + v^2(t)f^2\big(v(t)\big)\right]dt$$

where the square brackets indicate that these are *functionals* of the entire continuous-time path $x(t)$.

18. Caveat behind Continuous Time Analytics: It needs to be emphasized that the continuous time limit is simply an analytical device for obtaining solutions when τ_k is reasonably small; in reality the discreteness of the trading intervals must be taken into account in order to correctly describe trading-enhanced risk.
19. Mean Variance Optimization Objective Function: Introducing the risk-aversion parameter λ, the combined quantity

$$\mathbb{U}[x] = \mathbb{E}[x] + \lambda\mathbb{V}[x]$$

is minimized. Whether or not mean variance optimization is appropriate for a particular case, λ may be considered to be a Lagrange/KKT multiplier for the constrained problem of minimizing $\mathbb{E}[x]$ for a given $\mathbb{V}[x]$ and used to construct an efficient frontier in the space of trading trajectories.

20. VaR Based Optimization Objective Functions: More general weightings of risk, including Value-at-Risk, present thorny conceptual problems for time-dependent strategies (Artzner, Delbaen, Eber, and Heath (1999), Basak and Shapiro (2001)).
21. The Calculus of Variations Approach: Minimizing $\mathbb{U}[x]$ is a standard problem in the calculus of variations:

$$\underset{x(t)}{min}\,\mathbb{U}[x(t)] = \underset{x(t)}{min}\int_0^T F\big(x(t), -\dot{x}(t)\big)dt$$

with

$$F(x, v) = xg(v) + vh(v) + \lambda\sigma^2x^2 + \lambda v^2f^2(v)$$

22. Perturbation Stationarity: Euler-Lagrange Equation: Stationarity to small perturbations requires that the optimal $x(t)$ solve the Euler-Lagrange equation

$$\begin{aligned} 0 &= \frac{\partial F\big(x(t), -\dot{x}(t)\big)}{\partial x(t)} + \frac{d}{dt}\left[\frac{\partial F\big(x(t), -\dot{x}(t)\big)}{\partial v(t)}\right] \\ &= \frac{\partial F\big(x(t), -\dot{x}(t)\big)}{\partial x(t)} + \dot{x}(t)\frac{\partial^2 F\big(x(t), -\dot{x}(t)\big)}{\partial x(t)\partial v(t)} \\ &\quad - \ddot{x}(t)\frac{\partial^2 F\big(x(t), -\dot{x}(t)\big)}{\partial v^2(t)} \end{aligned}$$

– a second order ordinary differential equation to be solved with respect to the given endpoints $x(0)$ and $x(T)$

23. Integration into the First Order Form: Since $F\left(x(t), -\dot{x}(t)\right)$ does not depend explicitly on t multiplying throughout by $\dot{x}(t)$ and integrating results in the first-order equation

$$F\left(x(t), -\dot{x}(t)\right) + \dot{x}(t)\frac{\partial F\left(x(t), -\dot{x}(t)\right)}{\partial v(t)} = constant$$

24. Application to Optimal Trajectory Determination: In the current case one obtains

$$P\left(-\dot{x}(t)\right) - P(v_0) = x\left[g\left(-\dot{x}(t)\right) + \dot{x}(t)g'\left(-\dot{x}(t)\right)\right] + \lambda\sigma^2 x^2$$

with

$$P(v) = v^2\frac{\partial h(v)}{\partial v} + \lambda v^2\left[f^2(v) + 2vf(v)\frac{\partial f(v)}{\partial v}\right] = v^2\frac{\partial[h(v) + \lambda v f^2(v)]}{\partial v}$$

25. Properties of the Almgren "P" Function: The constant of integration

$$v_0 = -\dot{x}(t)|_{x=0}$$

is the velocity with which $x(t)$ hits

$$x = 0$$

For a sell program with

$$X > 0$$

$$v_0 \geq 0$$

and conversely for a buy program. Note that

$$P(0) = 0$$

additional assumption is that $P(v)$ is always an *increasing* function of v and hence invertible.

26. Explicit Solutions - Key Simplifying Assumptions: Almgren (2001) makes two simplifying assumptions to obtain explicit solutions.
    a. Permanent impact is linear in the trading rate.
    b. The imposed time horizon is infinite.
27. Linear Permanent Market Impact Function: A linear cost function

$$g(v) = \gamma v$$

gives a total cost γX independent of the path $x(t)$. The first term on the right side of

$$P\left(-\dot{x}(t)\right) - P(v_0) = x\left[g\left(-\dot{x}(t)\right) + \dot{x}(t)g'\left(-\dot{x}(t)\right)\right] + \lambda\sigma^2 x^2$$

vanishes, and then since $\dot{x}(t)$ appears only on the left side and x itself appears only on the right side, the general solution can be written in the quadrature form as

$$\int_{x(t)}^{X} \frac{dt}{P^{-1}[\lambda\sigma^2 x^2 + P(v_0)]} = t$$

28. Bid Ask Permanent Linear Absent: The constant v_0 is to be chosen so that

$$x = 0$$

corresponds to

$$T = 0$$

Note also that any constant in h disappears; the bid-ask spread does not affect the optimal strategy.

29. No Extraneously Specified Liquidation Time: Since $P(\cdot)$ is an increasing function, so is $P^{-1}(\cdot)$. It is thus clear that as v_0 decreases towards zero, the liquidation time T increases.
30. Invoking Longest Possible Liquidation Time: If no time horizon is exogenously imposed, the longest possible liquidation time can be obtained by setting

$$v_0 = 0$$

which leads to the quadrature problem

$$\int_{x(t)}^{X} \frac{dt}{P^{-1}[\lambda\sigma^2 x^2]} = t$$

31. Tractability of the above Solution: Often analytic solutions to the above problem can be found when

$$\int_{x(t)}^{X} \frac{dt}{P^{-1}[\lambda\sigma^2 x^2 + P(v_0)]} = t$$

with

$$v_0 \neq 0$$

would be too intractable. These solutions will still give nearly complete liquidations in finite time determined by market parameters.

Nonlinear Cost Functions

1. Power Law Temporary Impact Functions: Restricting the attention to the sell program, with

$$v \geq 0$$

the temporary impact functions are taken to be

$$h(v) = \eta v^k$$

$$f(v) = 0$$

with

$$k > 0$$

- for a buy program the signs will be changed in an obvious way. The linear case corresponds to

$$k = 1$$

2. Temporary Impact Almgren "P" Function: As noted above a possible constant in h corresponding to the bid-ask spread has been neglected. Then

$$P(v) = \eta k v^{k+1}$$

which, for the case of a general finite time horizon with

$$v_0 \geq 0$$

leads to the quadrature problem

$$\int_{x(t)}^{X} \left(\frac{\lambda \sigma^2}{\eta k} x^2 + {v_0}^{k+1} \right)^{-\frac{1}{k+1}} dx = t$$

3. Longest Optimal Trajectory Explicit Solution: Taking

$$v_0 \geq 0$$

explicit solutions for the longest optimal trajectories can be obtained:

$$\frac{x(t)}{X} = \begin{cases} \left(1 + \frac{1-k}{1+k} \frac{t}{T_*} \right)^{-\frac{1+k}{1-k}} & 0 < k < 1 \\ e^{-\frac{t}{T_*}} & k = 1 \\ \left(1 - \frac{k-1}{k+1} \frac{t}{T_*} \right)^{\frac{k+1}{k-1}} & k > 1 \end{cases}$$

4. Characteristic Time for Optimal Execution: Here the *characteristic time* is

$$T_* = \left(\frac{k \eta X^{k-1}}{\lambda \sigma^2} \right)^{\frac{1}{k+1}}$$

This is the analog of the *half-life* in the linear case. Only in the linear case

$$k = 1$$

is T_* independent of the initial portfolio size X. For

$$k \neq 1$$

the characteristic time depends on the initial size as

$$T_* \sim X^{\frac{k-1}{k+1}}$$

5. <u>Sub Linear Power Law Exponent</u>: For

$$k < 1$$

rapid trading is *under*-penalized relative to the linear case. As the portfolio size increases, volatility risk dominates the trading costs, and the optimal trading time *decreases* since the exponent is negative.

6. <u>Supra Linear Power Law Exponent</u>: For

$$k > 1$$

rapid trading is *over*-penalized relative to the linear case. As the portfolio size increases, the trading cost dominates the volatility risk, and the optimal trading time *increases*, since the exponent is positive. For example, if

$$k = 3$$

then

$$T_* \sim \sqrt{X}$$

7. <u>Characteristic Time vs Half Life</u>: As the portfolio size decreases to zero, reconciliation of the optimal trajectory would use a different starting value X and hence a different time T_*. The meaning of T_* is thus a little less fundamental than in the linear case. However, T_* scales in exactly the right way to make $x(t)$ still a static solution.

8. Intuition behind the Characteristic Time: For more intuition, note that the initial rate of selling is

$$-\dot{x}(0) = \frac{X}{T_*}$$

and T_* is the solution to the relation

$$\lambda\sigma^2 X^2 T = k\eta \left(\frac{X}{T}\right)^k X$$

9. Characteristic Time as a Cost Balance: The left side is the risk penalty associated with holding X shares for a time T, and the right side, up to a factor k, is $Xh\left(\frac{X}{T}\right)$, the impact cost associated with selling X shares over a time T (without the constant term representing the bid-ask spread, which does not impact the optimal solution).
10. The Longest Optimal Execution Time: For

$$k > 1$$

the trajectory reaches

$$x = 0$$

with

$$v = 0$$

at a finite time

$$T_{MAX} = \frac{k+1}{k-1} T_*$$

Thus these trajectories are the solution for finite imposed time T if

$$T > T_{MAX}$$

the trajectory reaches 0 at T_{MAX} and stays there till T

11. Self-Similar Scaling Trajectory Form: Almgren (2003) contains a graphical illustration of the optimal trajectories generated from

$$\frac{x(t)}{X} = \begin{cases} \left(1 + \frac{1-k}{1+k}\frac{t}{T_*}\right)^{-\frac{1+k}{1-k}} & 0 < k < 1 \\ e^{-\frac{t}{T_*}} & k = 1 \\ \left(1 - \frac{k-1}{k+1}\frac{t}{T_*}\right)^{\frac{k+1}{k-1}} & k > 1 \end{cases}$$

The form of the portfolio is independent of a particular choice of time scale T_* and initial portfolio size X; these solutions may be easily scaled to any case.

12. Time Realization of Trajectory Differences: A sense of the differences between the solutions may be gained by noting that for short times, all optimal trajectories are fairly close to each other; but the *tail* of the trajectories are extended for small value of k which strongly penalize trading at slow rates.
13. Example: Sell-Order Exponent Dependence: For example, as shown by Almgren (2003), at

$$t = T_*$$

the optimal trajectories reduce holdings to 30%, 37%, and 42% of the initial portfolio for

$$k = 2$$

$$k = 1$$

and

$$k = \frac{1}{2}$$

respectively.

14. <u>Example Sell Time Exponent Dependence</u>: Further as demonstrated by Almgren (2003), at

$$\frac{t}{T_*} = 3$$

the trajectory for

$$k = 2$$

has reached

$$x = 0$$

and remains there, the trajectory for

$$k = 1$$

retains 5% of its initial holdings, and the trajectory for

$$k = 2$$

retains 12.5% of the initial holdings. The relative differences become even more pronounced as time continues.

Objective Function

1. Determination of $\mathbb{E}[x]$ and $\mathbb{V}[x]$: $\mathbb{E}[x]$ and $\mathbb{V}[x]$ can be explicitly computed for these solutions from

$$\mathbb{E}[x] = \int_0^T \left[x(t)g\big(v(t)\big) + v(t)h\big(v(t)\big)\right]dt$$

$$\mathbb{V}[x] = \int_0^T \left[\sigma^2 x^2(t) + v^2(t)f^2\big(v(t)\big)\right]dt$$

   and hence the frontier can be drawn. In doing this the contributions from $g(v)$ and the term ϵX in $\mathbb{E}[x]$ are neglected.

2. Closed Form for $\mathbb{E}_\lambda[x]$ and $\mathbb{V}_\lambda[x]$: Then for a general k

$$\mathbb{E}_\lambda[x] = \frac{k+1}{3k+1}\eta\left(\frac{X}{T_*}\right)^{k+1} T_* = \frac{k+1}{3k+1}\eta\left(\frac{\eta\sigma^{2k}X^{3k+1}}{k^k}\lambda^k\right)^{\frac{1}{k+1}} T_*$$

$$\mathbb{V}_\lambda[x] = \frac{k+1}{3k+1}\sigma^2 T_* X^2 = \frac{k+1}{3k+1}\left(\frac{k\eta\sigma^{2k}X^{3k+1}}{\lambda}\right)^{\frac{1}{k+1}}$$

3. The $(\mathbb{E}, \mathbb{V})$ Efficient Frontier Curve: As λ varies $(\mathbb{E}, \mathbb{V})$ moves along the hyperboloid-like curve

$$\mathbb{E}_\lambda[x](\mathbb{V}_\lambda[x])^k = \left(\frac{k+1}{3k+1}\right)^{k+1} \eta\sigma^{2k}X^{3k+1}$$

   For any positive λ there is a unique solution.

4. <u>Asymptotics of $\lambda \to 0$ and $\lambda \to \infty$</u>: As

$$\lambda \to 0$$

one gets

$$T_* \to \infty$$

$$\mathbb{E}_\lambda[x] \to 0$$

and

$$\mathbb{V}_\lambda[x] \to \infty$$

i.e., optimizing expected cost without regard to variance leads to use of all available time. As

$$\lambda \to \infty$$

it can be seen that

$$T_* \to 0$$

$$\mathbb{E}_\lambda[x] \to \infty$$

and

$$\mathbb{V}_\lambda[x] \to 0$$

i.e., uncertainty is minimized regardless of the cost.

Almgren (2003) Example

1. Reference Trading Rate/Market Depth: Almgren (2003) provides a sample methodology for the estimation of the parameters. The first is to choose a representative level of trading rate v_{REF} If a specific time period τ is chosen v_{REF} is equivalent to a certain block size

$$n_{REF} = \tau v_{REF}$$

   traded in the time period; it may be interpreted as the market *depth* in the sense of Kyle (1985) or Bondarenko (2001).
2. Choice of Reference Trading Rate: The examples below consider stocks that trade one million shares a day, and v_{REF} is taken to be 10% of that rate, or

$$v_{REF} = 100{,}000\ share/day$$

   For time period

$$\tau = 1\ hour$$

   with $6.5\ periods\ per\ day$ this rate is equivalent to trading a block of approximately $15{,}300$ shares in each hour.
3. The Corresponding Temporary Price Impact: Next the price impact h_{REF} which would be incurred by the steady trading at the reference rate v_{REF} would be chosen. The share price is assumed to be $\$50/share$ and the assumption is that trading

$$v_{REF} = 100{,}000\ shares/day$$

   incurs a price impact of 1% or $\$0.50/share$

4. Choosing the Exponent - The Rationale: Finally a choice for the value of the exponent k is made that best fits the belief of how the price impact would depend upon the trading rate for rates smaller or larger than v_{REF}.
5. Reminder Trading Rate k Dependence: The choice

$$k = 1$$

corresponds to the linear dependence of price impact on rate;

$$k > 1$$

means that *large* trading rates or block sizes have a disproportionately *large* effect on price; while

$$k < 1$$

means that large trading rates or block sizes have a relatively *smaller* impact.
6. h_{REF}/v_{REF} Based Impact Model: This impact model is then written as

$$h(v) = h_{REF}\left(\frac{v}{v_{REF}}\right)^k$$

or

$$\eta = \frac{h_{REF}}{{v_{REF}}^k}$$

7. Daily Volatility and Initial Portfolio: It is also supposed that the stock has an annual volatility of 32% for the daily price change of

$$\sigma = \$1/share \cdot \sqrt{day}$$

A portfolio of initial size

$$X = 100{,}000$$

is considered, equal to $\frac{1}{10}^{th}$ of the daily volume.

8. Construction of the Efficient Frontier: There is now enough information to construct the efficient frontier

$$\mathbb{E}_\lambda[x] = \frac{k+1}{3k+1}\eta\left(\frac{X}{T_*}\right)^{k+1} T_* = \frac{k+1}{3k+1}\eta\left(\frac{\eta\sigma^{2k}X^{3k+1}}{k^k}\lambda^k\right)^{\frac{1}{k+1}} T_*$$

$$\mathbb{V}_\lambda[x] = \frac{k+1}{3k+1}\sigma^2 T_* X^2 = \frac{k+1}{3k+1}\left(\frac{k\eta\sigma^{2k}X^{3k+1}}{\lambda}\right)^{\frac{1}{k+1}}$$

from $\mathbb{E}_\lambda[x]$ and $\mathbb{V}_\lambda[x]$ for any chosen k describing the family of solutions as the risk aversion parameter λ ranges over all the possible values

$$0 < \lambda < \infty$$

To construct particular optimal solutions a specific value for λ needs to be set.

9. Variance/Cost Dependence on λ: The results are shown numerically in the table below. For any value of k the natural liquidation time T_* increases with the *risk tolerance* parameter $\frac{1}{\lambda}$; as both increase the expected cost decreases and the variance increases.

10. λ Impact: T_*, $\mathbb{E}_\lambda[x]$, $\sqrt{\mathbb{V}_\lambda[x]}$: The table below shows that the optimal time scale T_*, the expected cost $\mathbb{E}_\lambda[x]$, and the standard deviation of the cost $\sqrt{\mathbb{V}_\lambda[x]}$ as functions of the risk tolerance parameter $\frac{1}{\lambda}$ and the temporary impact exponent k. Market and portfolio parameters are as given in the treatment above (the initial portfolio value is

\$5 $million$). As k is varied, the reference values h_{REF} and v_{REF} are held constant; thus the coefficient η varies as in

$$h(v) = h_{REF}\left(\frac{v}{v_{REF}}\right)^k$$

or

$$\eta = \frac{h_{REF}}{{v_{REF}}^k}$$

Time T_* is measured in days; $\frac{1}{\lambda}$, $\mathbb{E}_\lambda[x]$, and $\sqrt{\mathbb{V}_\lambda[x]}$ are in thousands of dollars.

11. Table of T_*, $\mathbb{E}_\lambda[x]$, $\sqrt{\mathbb{V}_\lambda[x]}$:

	$\frac{1}{\lambda}$	$k = \frac{1}{2}$	$k = 1$	$k = 2$
T_*	1	0.02	0.07	0.22
	10	0.09	0.22	0.46
	100	0.40	0.71	1.00
	1000	1.84	2.24	2.15
	10000	8.55	7.07	4.64
$\mathbb{E}_\lambda[x]$	1	221	354	462
	10	103	112	99
	100	48	35	21
	1000	22	11	5
	10000	10	4	1
$\sqrt{\mathbb{V}_\lambda[x]}$	1	11	19	30
	10	23	33	45
	100	49	59	65
	1000	105	106	96
	10000	226	188	141

12. Large λ Execution Time Dependence: For large values of λ, optimal trajectories all execute rapidly, to reduce the volatility risk associated with the portfolio. When the trading rate is larger than v_{REF} costs *increase* with increasing k so larger k leads to slower trading.
13. λ and k Combination Impact: Conversely for small λ generally trading proceeds *more slowly* than v_{REF} in order to minimize the total expected cost. In this regime *smaller* k is more expensive and leads to relatively slower trading.
14. Cross-Over λ Execution Time: In the intermediate parameter regime the trajectories cross-over from one behavior to the other; larger k suggests slower trading at the beginning when the rate is large, then relatively more rapid in the tail.
15. Characteristic Time Reference Rate Dependence: Note that

$$T_* = \left(\frac{k\eta X^{k-1}}{\lambda\sigma^2}\right)^{\frac{1}{k+1}}$$

may be re-written as

$$T_* = \left(\frac{k h_{REF}}{\lambda\sigma^2 X}\right)^{\frac{1}{k+1}} \left(\frac{X}{v_{REF}}\right)^{\frac{k}{k+1}}$$

from which it is clear that

$$T_* \to \frac{X}{v_{REF}}$$

as

$$k \to \infty$$

regardless of the values of the other parameters.

16. Trade Speed vs. Cost Balance: In this limit, trading more rapidly than the reference rate is very strongly penalized, while trading more slowly is almost without cost, so the optimal strategy is to always trade exactly at the critical rate.
17. λ Estimation from Reference Parameters: Finally since λ is a difficult parameter to select in practice, it may be observed that it can be estimated if a time scale T_* is chosen from

$$T_* = \left(\frac{k\eta X^{k-1}}{\lambda\sigma^2}\right)^{\frac{1}{k+1}}$$

and

$$h(v) = h_{REF}\left(\frac{v}{v_{REF}}\right)^k$$

or

$$\eta = \frac{h_{REF}}{{v_{REF}}^k}$$

- it can be found that

$$\lambda = k\frac{h_{REF}\left(\frac{X/T_*}{v_{REF}}\right)^k X}{\sigma^2 T_* X^2}$$

18. Trading Cost vs. Variance Balance: The numerator is the price concession per share for trading at a concession rate $\frac{X}{T_*}$ multiplied by the total number of shares X to get the expected cost; the denominator is the variance that would be incurred by holding X

shares for time T_*. This ratio is multiplied by k to correct for nonlinearities which are ignored by this simple description.

Trading-Enhanced Risk

1. Liquidity Volatility: Functional Form Considered: Now the following functional form is taken for a sell program with

$$v \geq 0$$

$$h(v) = \eta v$$

$$f(v) = \alpha + \beta v$$

The deterministic part of the temporary impact is the linear case

$$k = 1$$

of the previous section.
2. Trading Rate Independent Volatility Component: The constant term in $f(v)$, with coefficient α, represents a constant uncertainty in the realized sale price independent of the rate of selling and of the underlying process. The total risk associated with this term is minimized by splitting the sale into as many parts as possible; thus this term pushes towards the linear trajectory.
3. Trading Rate Dependent Volatility Component: The linear term, with coefficient β, represents the increase in variance caused by non-zero amounts of selling. This term can even more strongly push toward the linear trajectory.
4. Liquidity Volatility Almgren "P" Function: Then with

$$\dot{x} = -v \leq 0$$

$$P(-\dot{x}(t)) - P(v_0) = x[g(-\dot{x}(t)) + \dot{x}(t)g'(-\dot{x}(t))] + \lambda\sigma^2 x^2$$

becomes

$$P(v) = (\eta + \lambda\alpha^2)v^2 + 4\lambda\alpha\beta v^3 + 3\lambda\beta^2 v^4$$

5. Behavior of the Almgren "P" Function: The polynomial $P(v)$ has

$$P(0) = 0$$

and is increasing for

$$v \geq 0$$

so the graph of the trajectory is always convex, and the inverse of P^{-1} is well-defined. For a buy program with

$$x \geq 0$$

the sign of the odd term in $P(v)$ is reversed.

6. No Hard Maximum Execution Time: Since

$$P(v) \sim \mathcal{O}(v^2)$$

for v near zero, the integrand appearing in the quadrature formulation

$$\int_{x(t)}^{X} \frac{dt}{P^{-1}[\lambda\sigma^2 x^2 + P(v_0)]} = t$$

behaves as $\mathcal{O}(x^{-1})$ as

$$x \to 0$$

for

$$v_0 = 0$$

and there is no "hard" maximum time as was found above for

$$k > 1$$

Constant Enhanced Risk

1. Analytical Solution for $\beta = 0$ Case: Two special cases are considered for obtaining analytical solution. The first is

$$\beta = 0$$

With this assumption the price uncertainty on each trade is independent of the size of the trade.

2. Trading Trajectory and Execution Time: A solution can then be found for

$$v_0 = 0$$

$$x(t) = Xe^{-\frac{t}{T_*}}$$

$$T_* = \sqrt{\frac{\eta + \lambda\alpha^2}{\lambda\sigma^2}}$$

3. Comparison with $f(v) = 0, k = 1$ Case: This is a pure exponential solution, except that the time constant has been increased by adding the additional variance per transaction to the impact coefficient

$$\eta \longmapsto \eta + \lambda\alpha^2$$

4. Expressions for $\mathbb{E}_\lambda[x]$ and $\mathbb{V}_\lambda[x]$: The value functions are

$$\mathbb{E}_\lambda[x] = \frac{1}{2}\eta\frac{X^2}{T_*} = \frac{1}{2}X^2\sqrt{\frac{\lambda\eta^2\sigma^2}{\eta + \lambda\alpha^2}}$$

$$\mathbb{V}_\lambda[x] = \frac{1}{2}X^2\sigma^2T_*\left(1 + \frac{\alpha^2}{\sigma^2T_*}\right) = \frac{1}{2}X^2\frac{\sigma}{\sqrt{\lambda}}\sqrt{\frac{\eta + 2\lambda\alpha^2}{\eta + \lambda\alpha^2}}$$

5. $\lambda \to 0$ $\mathbb{E}_\lambda[x]$ and $\mathbb{V}_\lambda[x]$ Behavior: The optimal value functions change in a more complicated way than the trajectory. As

$$\lambda \to 0$$

   the behavior is the same as that found earlier;

$$\mathbb{E}_\lambda[x] \to 0$$

   and

$$\mathbb{V}_\lambda[x] \to \infty$$

   since there is less care about the enhanced risk.

6. $\lambda \to 0$ $\mathbb{E}_\lambda[x]$ T_* $\mathbb{V}_\lambda[x]$ Behavior: In contrast as

$$\lambda \to \infty$$

all quantities have finite limits;

$$T_* \to \frac{\sigma}{\alpha}$$

$$\mathbb{E}_\lambda[x] \to \frac{1}{2}\eta\frac{X^2}{T_*}$$

and

$$\mathbb{V}_\lambda[x] \to \alpha\sigma X^2$$

Since trading itself introduces risk, risk-aversion and cost reduction both encourage spreading the trade over several periods; the minimum variance solution takes finite time and has finite cost.

Linear Enhanced Risk

1. Analytical Solutions for the $\alpha = 0$ Case: The next special case is

$$\alpha = 0$$

and

$$P(v) = 0$$

becomes

$$P(v) = \eta v^2 + 3\lambda\beta^2 v^4$$

and thus

$$P^{-1}(\omega) = \sqrt{\frac{\sqrt{\eta^2 + 12\lambda\beta^2\omega} - \eta}{6\lambda\beta^2}}$$

2. Trading Trajectory and Characteristic Fields: This can be integrated to obtain

$$\frac{t}{T_*} = F\left(\frac{X}{X_*}\right) - F\left(\frac{x}{X_*}\right)$$

in which the characteristic time and the characteristic share level are

$$T_* = \sqrt{\frac{\eta}{\lambda\sigma^2}}$$

$$X_* = \frac{1}{\sqrt{3}}\frac{\eta}{\lambda\sigma\beta} = \frac{1}{\sqrt{3}}\frac{\sigma {T_*}^2}{\beta}$$

and the nonlinear function is

$$F(u) = 2z - \coth^{-1} z$$

where

$$z = \sqrt{\frac{1}{2}\left(1 + \sqrt{1 + 4u^2}\right)}$$

3. Intuition behind the Characteristic Size: The characteristic time is the same as in the earlier section for

$$k = 1$$

and does not depend on the new coefficient β. To understand the characteristic level X_* note that

$$\sqrt{3}\beta \frac{X_*}{T_*}\frac{1}{\sqrt{T_*}} = \sigma\sqrt{T_*}$$

4. Trading Enhanced Market Volatility Balance: In this expression the left side is the trading induced variance in the share price given by the model

$$\tilde{S}_k = S_{k-1} - h\left(\frac{n_k}{\tau_k}\right) + \frac{1}{\sqrt{\tau_k}} f\left(\frac{n_k}{\tau_k}\right)\tilde{\xi}_k$$

$$k = 1, \cdots, N$$

if an initial portfolio of size X_* were sold in a single period T_* The right side is the variance in the share price due to the volatility in the same time interval; at the characteristic share level these two quantities are of comparable size.

5. X_* much bigger than x, X: To compare with the previous results note that

$$F(u) \sim \log u + constant + \mathcal{O}(u^2)$$

$$u \to 0$$

If this limit is attained by taking a limit of the *parameters* so that

$$\frac{X_*}{X} \to \infty$$

so that

$$F(u) \sim \log u + constant + \mathcal{O}(u^2)$$

is valid uniformly over x, since

$$0 \leq x \leq X$$

- a pure exponential solution results

$$\frac{t}{T_*} = \log \frac{X}{x(t)} + \mathcal{O}\left[\left(\frac{\lambda\alpha\beta}{\eta}\right)^2\right]$$

$$\frac{\lambda\alpha\beta}{\eta} \to 0$$

which, in particular, recovers the previous result with

$$k = 1$$

in the limit

$$\beta \to 0$$

6. <u>Behavior Towards $x \to 0$; Trajectory *Tail*</u>: And for any fixed value of the parameters

$$F(u) \sim \log u + constant + \mathcal{O}(u^2)$$

$$u \to 0$$

describes the tail of the solution as

$$x \to 0$$

the time constant of the decay is not affected by the addition of β.

7. <u>X_* much smaller than x, X</u>: For

$$x \gg X_*$$

i.e., the initial behavior when

$$X \gg X_*$$

using the expansion

$$F(u) \sim 2\sqrt{u} - \mathcal{O}\left(\frac{1}{\sqrt{u}}\right)$$

$$u \to \infty$$

gives

$$x(t) \sim X_* \left(C - \frac{1}{2}\frac{t}{T_*}\right)^2$$

$$x \gg X_*$$

with

$$C = \frac{1}{2}F(1)$$

This is the same solution constructed in the earlier Section with

$$k = 3$$

with

$$\eta = \lambda\beta^2$$

8. Almgren (2003) Asymptotic Solution Illustration:

$$\frac{t}{T_*} = F\left(\frac{X}{X_*}\right) - F\left(\frac{x}{X_*}\right)$$

together with

$$\frac{t}{T_*} = \log\frac{X}{x(t)} + \mathcal{O}\left[\left(\frac{\lambda\alpha\beta}{\eta}\right)^2\right]$$

$$\frac{\lambda\alpha\beta}{\eta} \to 0$$

and

$$x(t) \sim X_*\left(C - \frac{1}{2}\frac{t}{T_*}\right)^2$$

$$x \gg X_*$$

are illustrated in elaborate figures in Almgren (2003) Figure 5.

9. Strategy Construction Approach: Starting Trajectory: Thus the optimal strategy for construction would be as follows. Assuming

$$x > X_*$$

the initial trades are done using the trajectories of the temporary impact power law with

$$k = 3$$

with

$$\eta = \lambda \beta^2$$

That is, the volatility due to trading completely dominates the intrinsic volatility σ

10. Strategy Construction Approach: Tail Trajectory: As $x(t)$ reaches the level X_* switch is done to the optimal solution in the linear case

$$k = 1$$

with the other parameters taking their market values. In the tail trading-enhanced risk is a negligible quantity compared to the volatility.

Almgren (2003) Nonlinear Example Sample

1. Working out the $\alpha = 0$ Case: In this case the focus is on the previous section in which

$$\alpha = 0$$

and

$$\beta \neq 0$$

so that trading enhanced risk increases linearly with block size with no constant term.

2. The Corresponding Discrete Price Equation: To estimate the coefficients, one starts with the discrete time model. With

$$h(v) = \eta v$$

and

$$f(v) = \beta v$$

the price model

$$\tilde{S}_k = S_{k-1} - h\left(\frac{n_k}{\tau_k}\right) + \frac{1}{\sqrt{\tau_k}} f\left(\frac{n_k}{\tau_k}\right) \tilde{\xi}_k$$

$$k = 1, \cdots, N$$

becomes

$$\tilde{S}_k = S_{k-1} - \eta \frac{n_k}{\tau_k} + \beta n_k {\tau_k}^{-\frac{3}{2}} \tilde{\xi}_k$$

3. Liquidity Risk as Price Volatility Fraction: Assuming that for a particular choice of the trading interval τ the standard deviation of price concession associated with trading-enhanced risk is a fraction ϱ of the deterministic impact – a plausible assumption since both quantities are linearly proportional to the block size.
4. The Corresponding Characteristic Price: That is

$$\beta n_k {\tau_k}^{-\frac{3}{2}} = \varrho \eta \frac{n_k}{\tau_k}$$

or

$$\beta = \varrho\eta\sqrt{\tau}$$

which gives

$$X_* = \frac{1}{\sqrt{3}\varrho}\frac{1}{\lambda\sigma\sqrt{\tau}}$$

5. X_* Dependence on Risk Aversion: At this portfolio size, the volatility risk of holding the portfolio roughly balances the risk of selling along the optimal trajectory. Al though both of these quantities are risks, X_* involves λ through its influence on trading time T_*.
6. Choice of λ, τ, ϱ: The market parameters are taken to be the same as in the earlier section, with

$$\frac{1}{\lambda} = \$10{,}000$$

corresponding to the case where the liquidation is one a day. The trading is divided into one hour time intervals, so

$$\tau = \frac{2}{13}\ days$$

and

$$\varrho = \frac{1}{2}$$

7. Estimate of β and X_*: One obtains

$$\beta = 10^{-6}\$ \cdot day^{\frac{3}{2}} \cdot share^{-2}$$

and

$$X_* = 30{,}000\ shares$$

corresponding to a portfolio size of $\$1.5m$. A liquidation problem with initial value greater than will begin in the large x regime where trading-enhanced risk is dominant and end in the small x regime where it is negligible.

Conclusions: Summary and Extensions

1. Summary: Power Law Temporary Component: The treatment seen above obtains explicit analytical solutions for certain cases of the impact model. First, it neglects the effects of trading-enhanced risk, and takes the impact function to be a simple power law. The solutions in this case are straightforward nonlinear extensions of Almgren and Chriss (2000); the exponential solutions obtained there are a particular dividing case of these power law solutions.
2. Summary: Constant Trading-Enhanced Risk: With trading-enhanced risk, two particular cases with linear impact functions were considered. If the price uncertainty per transaction is independent of the transaction size, then the optimal trajectories are given by the previous results, simply augmenting the impact coefficient by the additional variance. A risk-averse trader lengthens his trade program, diversifying some variance away by spreading the execution over more different transactions at the expense of slightly higher volatility risk.
3. Summary: Linear Trading-Enhanced Risk: If price uncertainty per transaction is linearly proportional to the transaction size, then a characteristic portfolio size emerges, above which reduction of the added variance is the dominant effect. In this regime trade trajectories are equivalent to the previous power law solutions with exponent equal to 3. For portfolios smaller than this size the new effect may be neglected compared to the deterministic impact costs and volatility.

4. Extension #1: Optimal Numerical Trajectories: Throughout this treatment, the focus has been on obtaining explicit solutions for the sake of analytical insight. Numerical solutions would be quite straightforward, and allow lifting the restrictions described above, and consideration of a more general class of models.
5. Extension #2: Linear Impact Portfolios: Portfolios of assets are an interesting extension. Already in the linear case (Almgren and Chriss (2000)), to obtain explicit solutions it is necessary to make simplifying assumptions about cross-impacts, for example, that trading in each asset affects only the price of that asset.
6. Extension #3: Nonlinear Impact Portfolios: Even with that assumption, the nonlinear formulation opens a wide class of possible models; for example, should the exponent be the same for each asset? Determination and characterization of optimal trajectories in this case is a topic for future work.

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Algorithmic Market Making

Symbology Glossary

1. Tight Skew: α_T
2. Loose Skew: α_L
3. Tight Width: ω_T
4. Loose Width: ω_L
5. Algorithmically generated Ideal Mid Cash Price: Π_{Ideal}
6. Position: P (expressed in cumulative net position per unit under consideration – firm/desk/trader)
7. Position Pivot: P_{Pivot}. Dimensionless ontological view of the scaling position metric – roughly equivalent to the Reynolds' number of market making position units. Expressed in currency units.
8. Risk: R (expressed in cumulative net risk per unit under consideration – firm/desk/trader)
9. Risk Pivot: R_{Pivot}. Dimensionless ontological view of the scaling risk metric – roughly equivalent to the Reynolds' number of market making risk units. Expressed in PV01 currency units.

Framework Glossary

1. Equilibrium quantity: Quantity that only changes with the macro drivers/factors, and not the technical factors. Typically stable, but jumpy and undergoes changes when drivers shift – and introduces perturbations on the disequilibrium quantities.
2. Disequilibrium quantity: Quantity that changes with the technical, transient factors.

Width/Skew/Size Estimation Models

1. <u>Tight Models</u>:
   - Tight models estimate the market making quantities on a trader/firm/desk independent manner.
   - They estimate the "secular" market making parameters – width, skew, and size for either the Market Making Outputs or the Axe Outputs – estimate them based on classes of input parameters.
   - For each input parameter class, the following are needed:
     a. A proxy that serves as a quantitative estimate of the desired parameter class.
     b. Segmentation of the proxy over the sub-classified parameter set.
2. <u>Input Class => Risk Profile</u>:
   - Captures all the cumulative risk components => the credit/solvency, market, and liquidity risk behind the issue.
   - Proxy => CDS Spread, rating, bond basis
   - Sub-classification => Issue, issuer, and sector.
3. <u>Input Class => Liquidity</u>:
   - Captures the frequency and volume of the trade flow of a given issue, and the ease of getting in and getting out at the given side.
   - Proxy:
     i. Aggregated periodic (e.g., daily) volume for each side (buy/sell).
     ii. Aggregated periodic (e.g., daily) notional for each side (buy/sell).
   - Sub-classification => Issue, issuer, sector, and the instrument universe.
4. <u>Firm/Desk/Trader level parameters</u>: These provide aggregated controls for trading.
   - Net Position => vital metric for inventory control.
   - Risk limits => to control/manage exposure to specific granules – issue, issuer, tenor, sector, unit etc.

5. Monitor Mobility: Certain measures such as PV01 based risk, inventory, etc. are more easily human-monitored, so they are done daily. Others (such as tenor 01s) are less easily monitored, so they are done infrequently.

Market Making System SKU

1. Intra day Curve Generation Scheme
2. Mid Price Estimation Models
    i. Accommodate different mid price estimation models, and their respective parameters
3. Algorithmic Quote Construction => used for generating venue/ECN independent width/skew/size [composed of tight/loose components]. Broadly speaking achieves the following:
    i. Specific parameters to control skew for targeted alpha generation strategies
    ii. Accommodate different width and size estimation models, and their respective parameters
    iii. Venue-independent base quote synthesis/construction
    iv. Circuit breaker heuristics
    v. Policy driven/policy enforcement/policy control applied at this level
4. Quote Management: Publishing/tailoring the constructed quote towards specific venues (possibly with order routing applied at this stage).
    i. Venue specific rules (and thereby external vendor incorporations, like Broadway etc. at this stage.

Market Making Parameter Types

1. Model Parameters: Parameters for generation of algorithmic generation of width, skew, and size.

2. Quote Generation Control Parameters
3. Quote Heuristics Control
4. Quote Management Control

Intra-day Pricing Curve Generation Schemes

1. Issue Benchmark Bonds: The following set of threshold criteria are used to determine the issuer specific benchmark bonds:

    ii. Threshold of daily TRACE volume/number of trades

    iii. Threshold of outstanding notional

    iv. Only senior obligations

    v. Some combination of the following threshold of the ratios:

    - $\dfrac{CUMULATIVEDAILYISSUETRACEVOLUME}{OUTSTANDINGNOTIONAL}$
    - $\dfrac{CUMULATIVEDAILYISSUETRACEVOLUME}{CUMULATIVEDAILYISSUERTRACEVOLUME}$

2. Benchmark bonds basis tracking: Track the bid side and ask side credit basis of the benchmark bonds from each TRACE print, using EMA VWAP/TWAP from the intra-day rates/credit curves. This will be the attempt to estimate the mid credit basis for the, and it is generally well behaved.

- Need to find a way to accommodate the institutional closing CDS mid marks and the benchmark bonds into the credit curve construction – these are highly valid points.

3. Liquid vs. illiquid: Typical liquid securities' quote may be proxied out of print (or at least EMA'd). Intra-day quote generation, however, is materially important for illiquid securities.
4. Intra-day credit curve generation inputs: Need a way to generate the credit curve from

    i. The CDS marks

    ii. The basis-adjusted benchmark bonds

iii. It always needs to be used in conjunction with tension splines.
iv. Also need intra-day TRACE series to update the basis (direct or EMA) – will use this to establish the intra-day relationship between the CDS nodes and the TRACE cut-off threshold).

5. Intra-day credit curve updating:
    a. Use the relationship grid between CDS 5Y, the off-tenors, and the benchmark bonds
    b. Any change in any of them automatically re-adjusts using the set relationships.
    c. CDS Curves are trader set; bond basis are EMA'd from the TRACE series using the prior credit curve
    d. Relationships are either reviewed daily EOD
6. Live updating of bond prices: Use the live curve (either pure CDS, or a mixture of CDS/bond instruments) to extract the basis of each print, and then EMA that to generate the bond live prices.

Mid Price Models

i. Definition: Computed theoretical mid-price, as to where the next print should be – assuming zero transaction costs, zero position/risk constraints, and infinite liquidity. Mid Price is an ***Equilibrium Quantity***.
ii. Estimation parameters: Typical mid price estimation parameters are: the IR curve, the survival curve, and the recovery curve. The other possible drivers are: funding curve – typically for long position, and repo curve – typically for shorts.

Width Models

1. Tight Width: Computed theoretical width, after accounting for the issue liquidity and the issue riskiness. Tight width is the first in the set of disequilibrium quantities. Tight width is:
    a. Proportional to issue risk (combination of credit and market risk – not counter party risk).
    b. Inversely proportional to liquidity

Skew Models

1. Tight Skew: This measure how far the last print has been OFF from the theoretical mid price. Thus Tight Skew is representative of the alpha potential – for a theoretical mid price that chases the print in a sequence, the tight skew is zero.
2. Tight Bid Skew and Tight Ask Skew: This is an alternative SKU – instead of tight width and tight skew cognitive view, tight bid/ask skew parameters are determined only from their corresponding liquidity and flow metrics (i.e., bid/ask liquidity metrics).
3. Loose Skew: Simply put, loose skew is:

$$\alpha_L = \max\left(\frac{P}{P_{Pivot}}, \frac{R}{R_{Pivot}}\right)$$

4. Heuristic Checks on Loose Skew: Following checks applied to round out quoting:
    1. Ceiling/floor applied
    2. Maximum cutoff for width
    3. Best right skew – bid becomes ask.
    4. Best left skew – ask becomes bid.

Size Models

1. Tight bid size/tight ask size: Basically, tight size is inversely proportional to tight width, to within normalized bounds.

Heuristics Control

1. Can Buy/Can Short: Can But/Can Short => whether the bid/ask stays within the LONGABLE/SHORTABLE cutoff.
2. ECN Threshold Cross: Check to see if there is a cross between the published bid/ask and a given ECN's bid/ask.

Published Market Quote Picture

1. Bid/Ask Sizes: Truncated to their appropriate rounding.
2. Bid Price: $\Pi_{Ideal} - \frac{1}{2}\omega_L \alpha_L$
3. Ask Price: $\Pi_{Ideal} + \frac{1}{2}\omega_L \alpha_L$
4. Bid/Ask Prices rounded downward/upward to their appropriate increments.

Flow Analysis

1. Dimensionless flow classifier: If the metric (ADV etc.) is greater than a specific threshold, then the flow becomes "turbulent", else it is "laminar".
2. Flow Potential: Skew of all kinds is related to the flow driver/equilibration strength.