Splines Under Tension

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Outline

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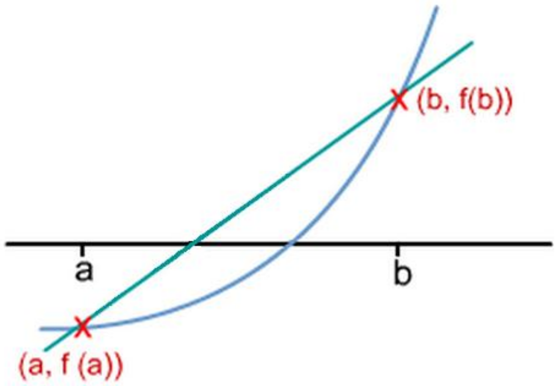
- Cubic/Linear Splines
- Breakdown of Cubic splines
- Splines Under Tension
- Examples
- Multi-dimensions
- Modern applications
- Conclusion

Problem

- Given a set of $\{x_i\}_{i=1}^{n}, \{y_i\}_{i=1}^{n}$ we seek to find a continuous function ***f*** which fits the data.
- Why?
  - Integrate the function
  - Calculate intermediate points
  - Etc

Cubic/Linear Splines

- The simplest solution is to fit your data linearly between known points.



- Often if you desire a good fit, linear interpolation is not very accurate.
- Generally the most widely used solution is a cubic spline.

Cubic Splines

- Uses higher order polynomials to achieve continuity across intervals.
- Required to have information about the derivatives of the function at the boundaries.

$$y = Ay_i + By_{i+1} + Cy_i'' + Dy_{i+1}''$$

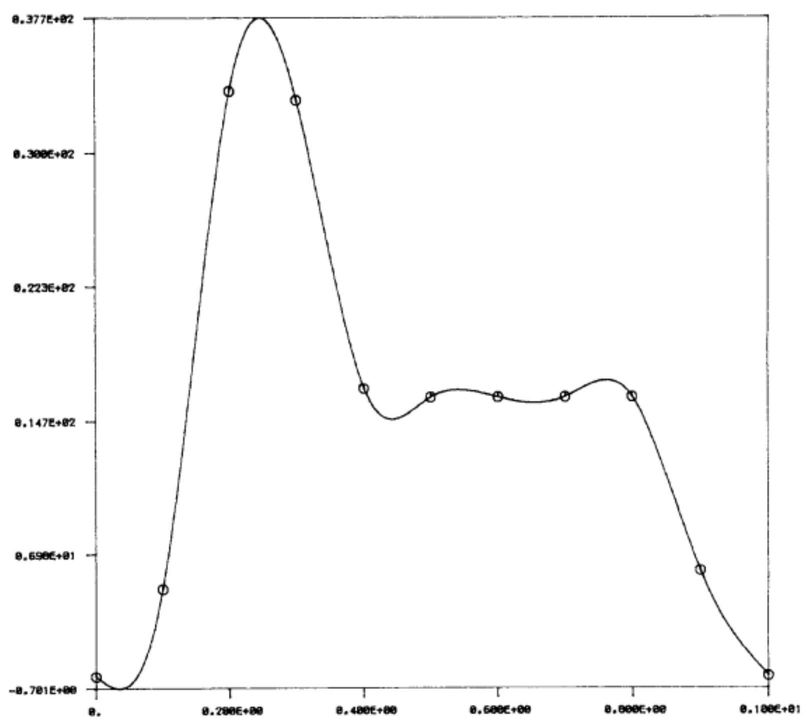
- Need to find second order derivatives at the knots.

$$\underbrace{\frac{x_j - x_{j-1}}{6}}_{a_j} y_{j-1}'' + \underbrace{\frac{x_{j+1} - x_{j-1}}{3}}_{b_j} y_j'' + \underbrace{\frac{x_{j+1} - x_j}{6}}_{c_j} y_{j+1}'' = \underbrace{\frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}}}_{F_j}$$

$$\begin{bmatrix} b_1 & c_1 & 0 & & \cdots & & 0 \\ a_2 & b_2 & c_2 & & & & \\ 0 & \ddots & \ddots & \ddots & & & \vdots \\ & & a_j & b_j & c_j & & \\ \vdots & & & \ddots & \ddots & \ddots & 0 \\ & & & & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & & \cdots & & 0 & a_n & b_n \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ \vdots \\ y_j'' \\ \vdots \\ y_{n-1}'' \\ y_n'' \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_j \\ \vdots \\ F_{n-1} \\ F_n \end{bmatrix}$$

Cubic Splines

- What are the potential problems with cubic splines?
  - Ringing!



Natural cubic spline

Cubic/Linear Splines

- What happens when one of the data points is a point of inflection (local max or min)?
  - Cubic spline interpolation breaks down i.e. cannot have continuous derivatives across a knot.
- One option is to break up the function into two intervals.
- Another option is to use splines under tension.

Splines Under Tension

- Requires two continuous derivatives at the boundaries, and $f''(x) - \sigma^2 f(x)$ to vary linearly on each of the intervals $x \in [x_i, x_{i+1}]$
- Where σ is a constant tension factor.

$$f(x_i) = y_i \quad i = 1, \dots, n$$

$$f''(x) - \sigma^2 f(x) = [f''(x_i) - \sigma^2 y_i]\frac{(x_{i+1} - x)}{h_i} + [f''(x_{i+1}) - \sigma^2 y_{i+1}]\frac{(x - x_i)}{h_i}$$

$$h_i = x_{i+1} - x_i, for\ i = 1, \dots, n-1$$

Splines Under Tension….

- This may look similar to the linear interpolation scheme.

$$y = Ay_i + By_{i+1}$$

$$y = \frac{(x_{i+1} - x)}{(x_{i+1} - x_i)} y_i + \frac{(x - x_i)}{(x_{i+1} - x_i)} y_{i+1}$$

$$y = \frac{(x_{i+1} - x)}{h_i} y_i + \frac{(x - x_i)}{h_i} y_{i+1}$$

$$f''(x) - \sigma^2 f(x) = [f''(x_i) - \sigma^2 y_i] \frac{(x_{i+1} - x)}{h_i} + [f''(x_{i+1}) - \sigma^2 y_{i+1}] \frac{(x - x_i)}{h_i}$$

Splines Under Tension....

- How do we solve the differential equation?
  - We start with a linear combination of the following basis functions

$$\{1, x, e^{\sigma x}, e^{-\sigma x}\}$$

  - In various software packages it is often a different set of basis functions used to maintain numerical stability.

$$\left\{1, x, \cosh\left(\sigma\frac{(x_i - x)}{\Delta x}\right) - 1, \sinh\left(\sigma\frac{(x_i - x)}{\Delta x}\right) - \sigma\frac{(x_i - x)}{\Delta x}\right\}$$

Splines Under Tension....

- Solving gives the following solution

$$f(x) = \left[\frac{f''(x_i)}{\sigma^2}\right] \cdot \frac{\sinh\left(\sigma(x_{i+1} - x)\right)}{\sinh\,(\sigma h_i)} + \left[y_i - \frac{f''(x_i)}{\sigma^2}\right] \cdot \frac{(x_{i+1} - x)}{h_i} + \left[\frac{f''(x_{i+1})}{\sigma^2}\right] \cdot \frac{sinh\left(\sigma(x - x_i)\right)}{\sinh\,(\sigma h_i)}$$
$$+ \left[y_{i+1} - \frac{f''(x_{i+1})}{\sigma^2}\right] \cdot \frac{(x - x_i)}{h_i} \qquad for\; x \in [x_i, x_{i+1}]$$

- This was achieved by solving the previous equation and using the fact that $f(x_i) = y_i \quad i = 1, \ldots, n$
- In order for this solution to work we need to know

$$f''(x_i) \qquad for\; i = 1, \ldots, n$$

Splines Under Tension....

- By differentiating ***f*** and equating the right and left hand derivatives at $x_i, for\ i = 2, \dots, n-1$ we obtain

$$\left[\frac{1}{h_i} - \frac{\sigma}{\sinh(\sigma h_{i-1})}\right] \cdot \frac{f''(x_{i-1})}{\sigma^2} + \left[\frac{\sigma\cosh(\sigma h_{i-1})}{\sinh(\sigma h_{i-1})} - \frac{1}{h_{i-1}} + \frac{\sigma\cosh(\sigma h_i)}{\sinh(\sigma h_i)} - \frac{1}{h_i}\right] \cdot \frac{f''(x_i)}{\sigma^2}$$
$$+ \left[\frac{1}{h_i} - \frac{\sigma}{\sinh(\sigma h_i)}\right] \cdot \frac{f''(x_{i+1})}{\sigma^2} = \frac{(y_{i+1} - y_i)}{h_i} - \frac{(y_i - y_{i-1})}{h_{i-1}}$$

- The last two equations can be obtained in two different ways depending on the behaviour desired for the function ***f***.

Splines Under Tension….

- The first set of conditions can be obtained given we know the first derivatives at the end points and that the first derivatives are equal at the knots.

$$\left[\frac{\sigma\cosh(\sigma h_1)}{\sinh(\sigma h_1)} - \frac{1}{h_1}\right] \cdot \frac{f''(x_1)}{\sigma^2} + \left[\frac{1}{h_1} - \frac{\sigma}{\sinh(\sigma h_1)}\right] \cdot \frac{f''(x_2)}{\sigma^2} = \frac{(y_2 - y_1)}{h_1} - y_1'$$

$$\left[\frac{1}{h_{n-1}} - \frac{\sigma}{\sinh(\sigma h_{n-1})}\right] \cdot \frac{f''(x_{n-1})}{\sigma^2} + \left[\frac{\sigma\cosh(\sigma h_{n-1})}{\sinh(\sigma h_{n-1})} - \frac{1}{h_{n-1}}\right] \cdot \frac{f''(x_n)}{\sigma^2} = y_n' - \frac{(y_n - y_{n-1})}{h_{n-1}}$$

Splines Under Tension....

- The second conditions are for a periodic function where

$some\ x_{n+1} > x_n,\ y_1 = f(x_{n+1}), f'(x_1) = f'(x_{n+1}), and\ f''(x_1) = f''(x_{n+1})$

- This results in the following two additional conditions

$$\left[\frac{1}{h_n} - \frac{\sigma}{\sinh(\sigma h_n)}\right] \cdot \frac{f''(x_n)}{\sigma^2} + \left[\frac{\sigma\cosh(\sigma h_n)}{\sinh(\sigma h_n)} - \frac{1}{h_n} + \frac{\sigma\cosh(\sigma h_1)}{\sinh(\sigma h_1)} - \frac{1}{h_1}\right] \cdot \frac{f''(x_1)}{\sigma^2} + \left[\frac{1}{h_1} - \frac{\sigma}{\sinh(\sigma h_1)}\right] \cdot \frac{f''(x_2)}{\sigma^2} = \frac{(y_2 - y_1)}{h_1} - \frac{(y_1 - y_n)}{h_n}$$

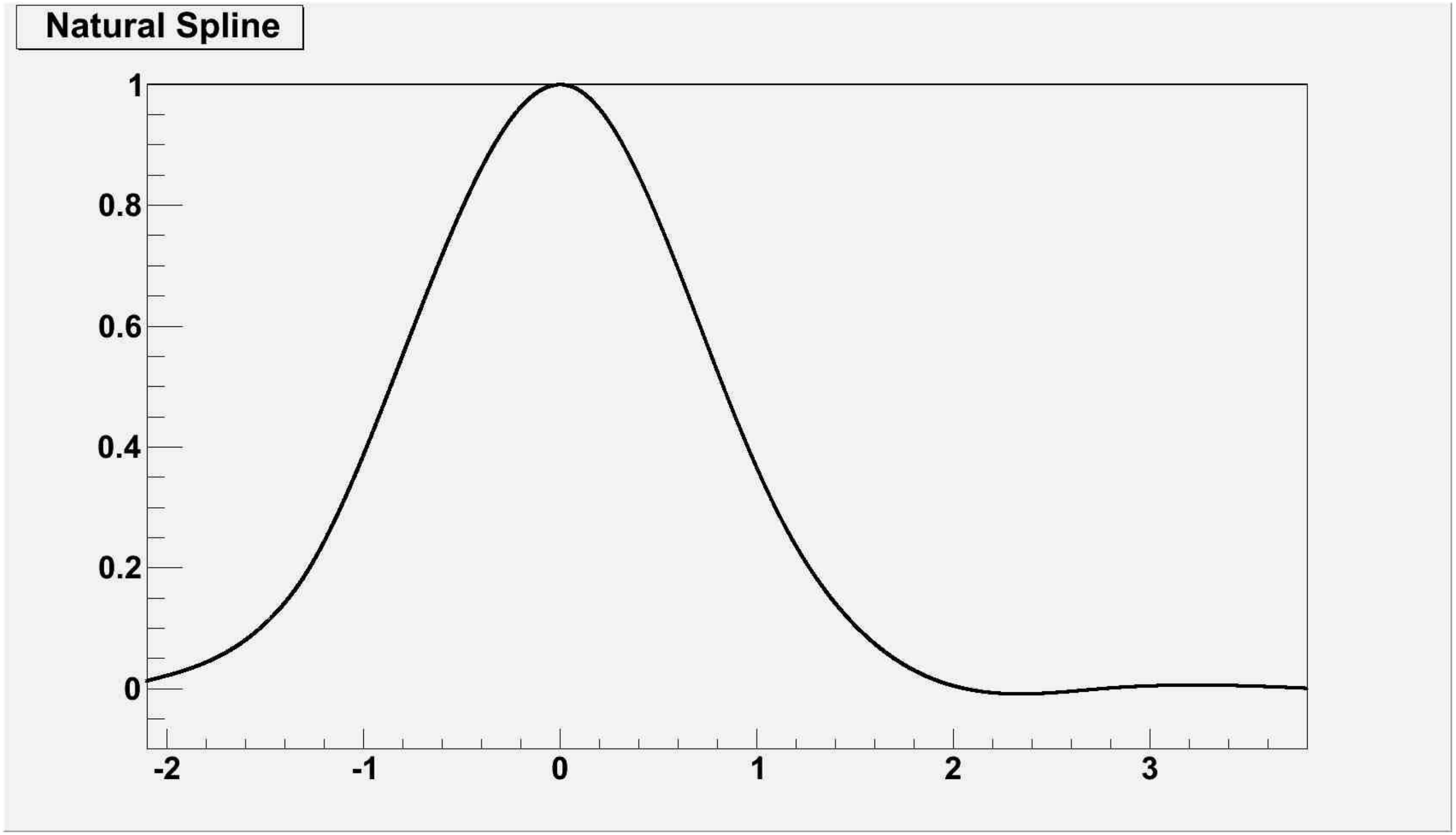
$$\left[\frac{1}{h_{n-1}} - \frac{\sigma}{\sinh(\sigma h_{n-1})}\right] \cdot \frac{f''(x_{n-1})}{\sigma^2} + \left[\frac{\sigma\cosh(\sigma h_{n-1})}{\sinh(\sigma h_{n-1})} - \frac{1}{h_{n-1}} + \frac{\sigma\cosh(\sigma h_n)}{\sinh(\sigma h_n)} - \frac{1}{h_n}\right] \cdot \frac{f''(x_n)}{\sigma^2} + \left[\frac{1}{h_n} - \frac{\sigma}{\sinh(\sigma h_n)}\right] \cdot \frac{f''(x_1)}{\sigma^2} = \frac{(y_1 - y_n)}{h_n} - \frac{(y_n - y_{n-1})}{h_{n-1}}$$

Splines Under Tension....

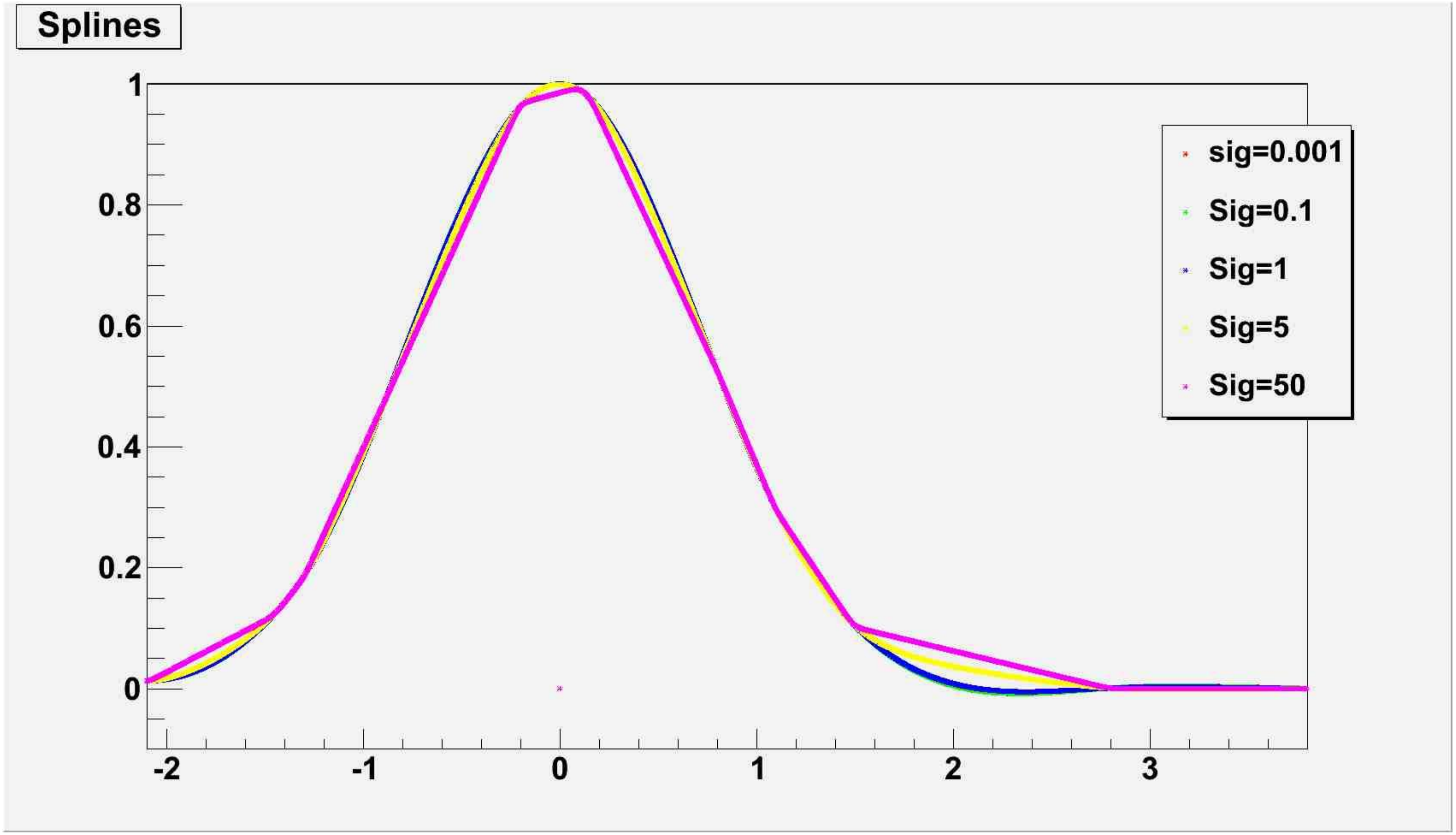
- Now that we have all the necessary conditions for ***f*** we can solve the linear tridiagonal matrix for the $f''(x_i)$ ‘s using Gaussian elimination or any other preferred method.
- The tension factor has two extreme points worth noting.
  - When $\sigma \to 0$ the curve is nearly identical to the cubic spline.
  - When $\sigma \to \infty$ the curve is almost identical to the linear spline.

Examples: Non Periodic

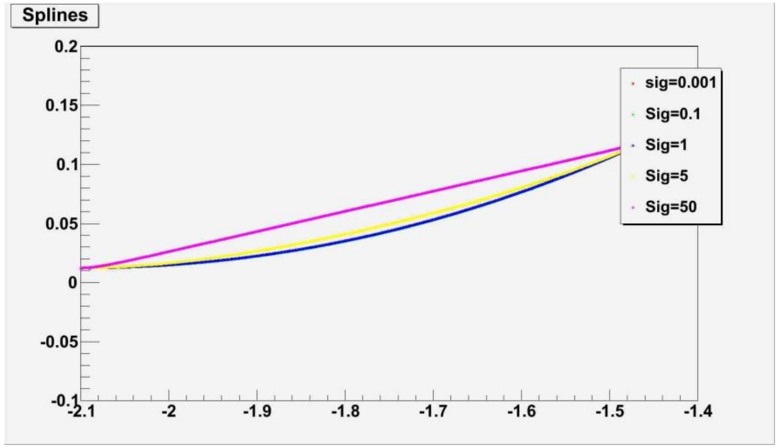
- If we recall our old friend from assignment #1

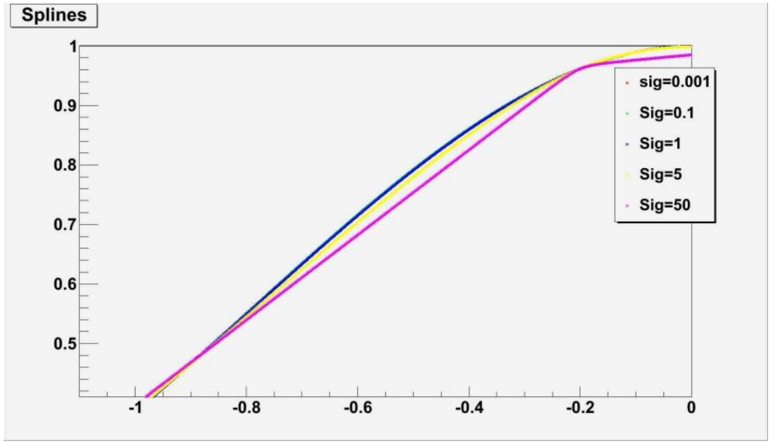


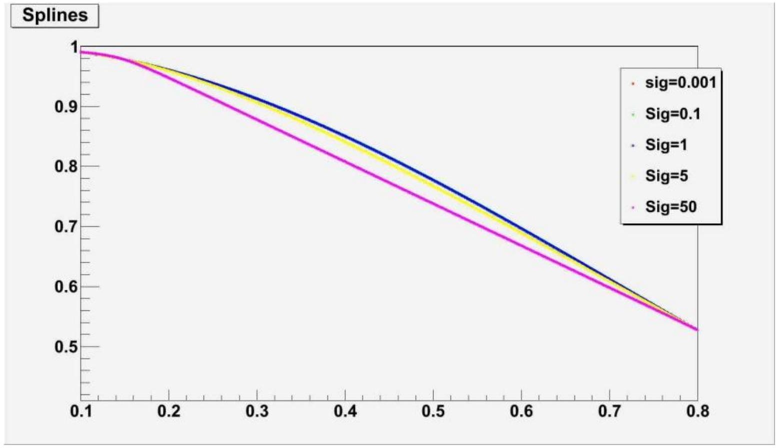
Examples: Non Periodic

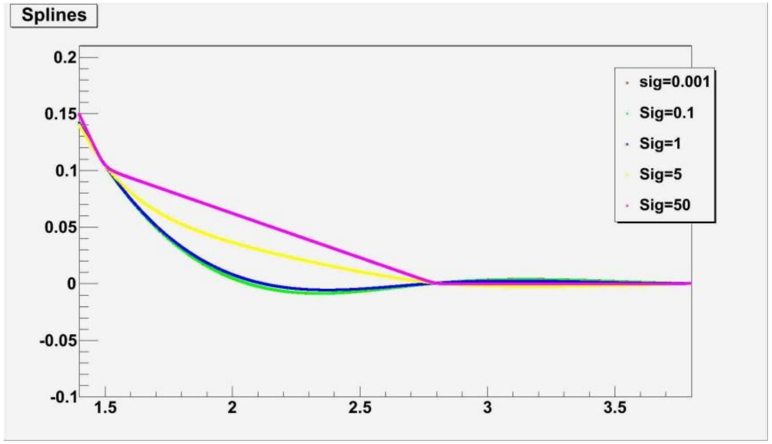


Examples: Non Periodic

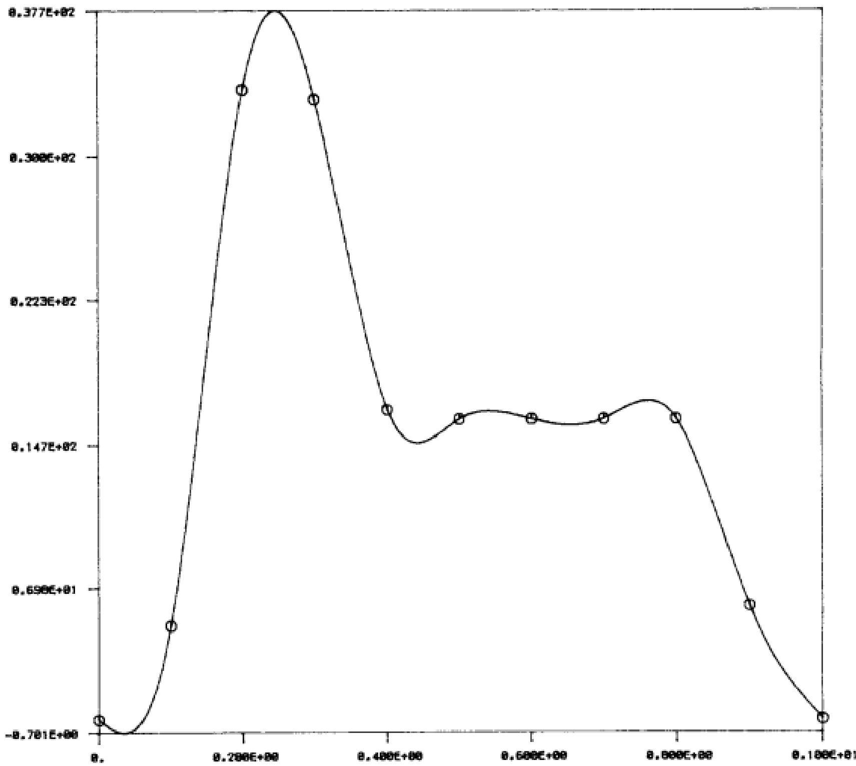




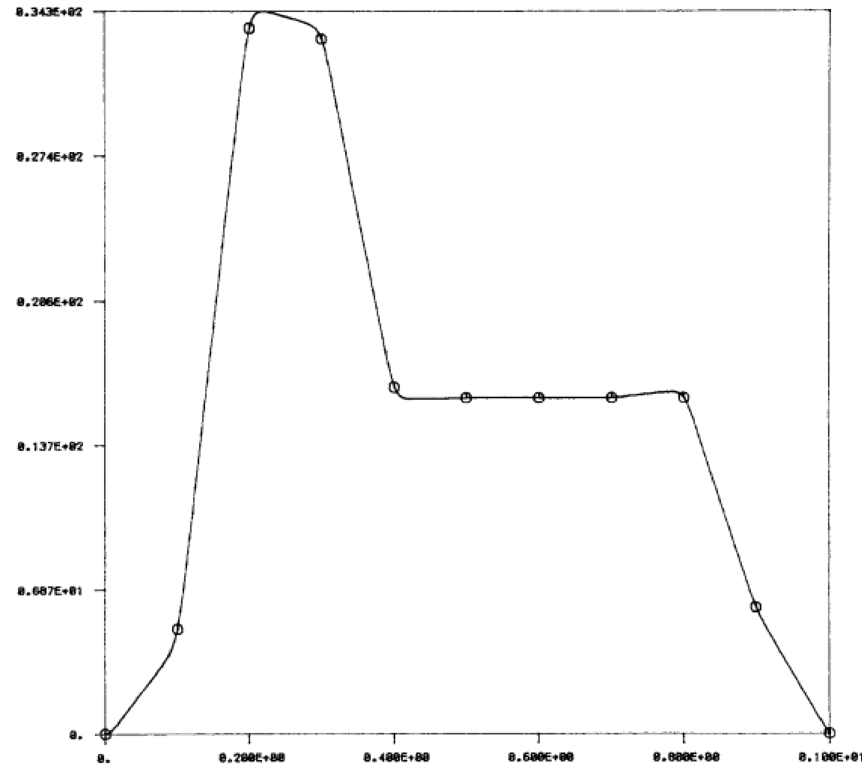




Examples: Ringing



Natural cubic spline



Spline with uniform tension

Splines With Uniform Tension

- Fits data better than linear interpolation.
- Handles inflection points.
- Can get rid of ringing however, can lose the functions shape.

- What can we do to improve this?

Splines: Varying Tension

- Tension factor options: constant, normalized in x, or varying over each interval.
- Constant tension factor is achieved simply σ = *k*
- Normalized tension factor given by $$\sigma_i = \frac{\sigma}{(x_{i+1} - x_i)}$$
- Varying tension factors can be set using one of the three following criteria
  - By bound values of the function at given points
  - By bound values of the functions derivatives at given points
  - By concavity

Splines: Varying Tension

- Finding σ when *f* is bound.
  - To get the minimum tension factor required we need to find the zeros of *f'*.
  - The form of σ can be found in [4]
- Finding σ when *f'* is bound.
  - To get the minimum tension factor required we need to find the zeros of *f''*.
  - The form of σ can be found in [4]

- The equations for σ are complicated and not very enlightening to see and therefore are not shown.

Splines: Varying Tension

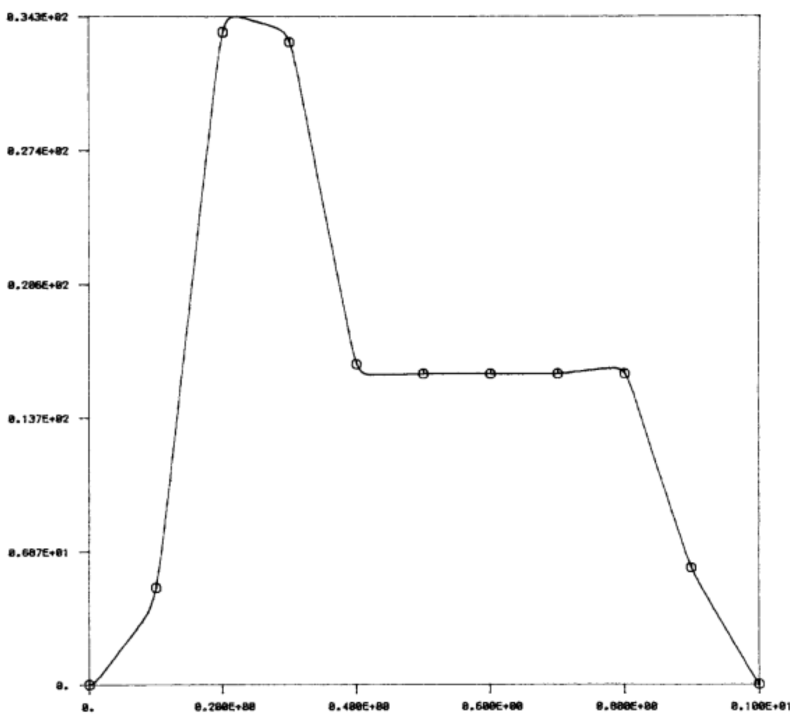
- In order to keep the convexity of ***f***, if $d_1 d_2 > 0$ then f is convex if $g(\sigma) > 0$.

$$d_1 = \frac{(y_2 - y_1)}{h_1} - y'_1 \qquad d_2 = y'_2 - \frac{(y_2 - y_1)}{h_1}$$

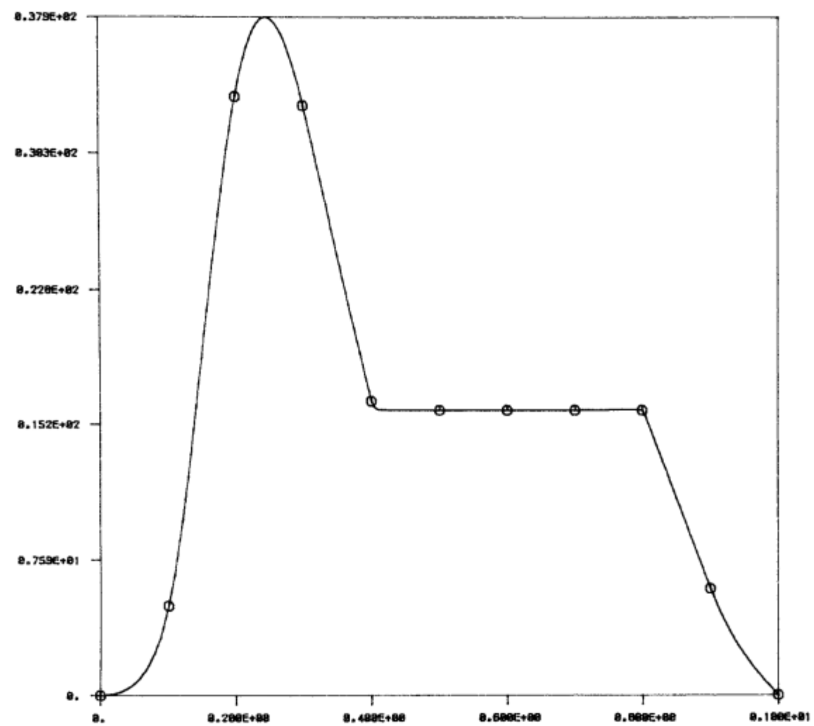
$$g(\sigma) = \begin{cases} \sigma * \operatorname{coshm}(\sigma)/\operatorname{sinhm}(\sigma) - 1 - \max\{d_1/d_2,\, d_2/d_1\} & \text{if } \sigma > 0, \\ 2 - \max\{d_1/d_2,\, d_2/d_1\} & \text{if } \sigma = 0. \end{cases}$$

- A proof can be found in [4]
- In order to minimize $g(\sigma)$ Newton's Method is used and results in an approximation of $\sigma = \sqrt{-10g(0)}$

Splines: Varying Tension



Spline with uniform tension



Variable tension spline

Splines: Varying Tension

- Preserves shape.
- Handles inflection points well.
- Relatively expensive to compute.

Multi-Dimensional Splines

- In general splines in multi-dimensions can become very complicated.
- There are several schemes out there and so I will briefly talk about one which I find useful.
- Thin Plate Splines (TPS) are a nice way of doing higher dimensions and has a nice pictorial analogy in 3D.

Multi-Dimensional Splines

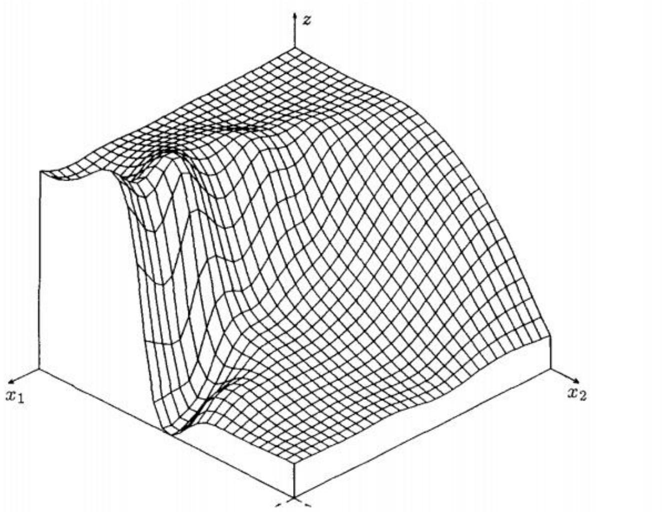
- Given $\{w_{ij} = (x_i, y_j), i = 1, \ldots, N, j = 1, \ldots, M\}$ we can use a combination of radial basis function to specify where any point in between the given points.

$$f(\vec{x}) = \sum_{i=1}^{N} \sum_{j=1}^{M} c_{ij} \varphi(\vec{x} - w_{ij}) \qquad \varphi(r) = r^2 \log r$$

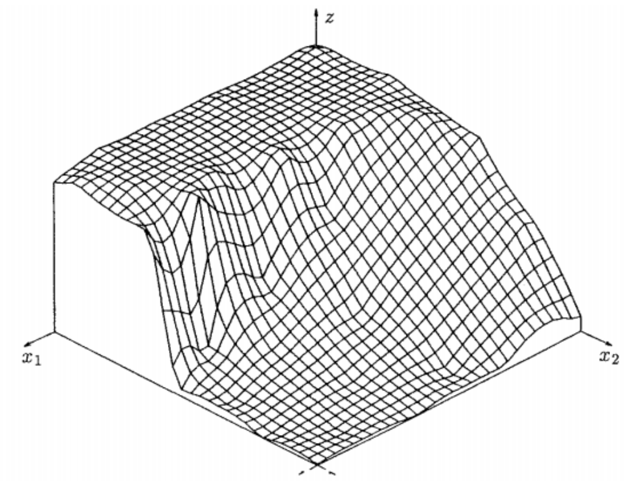
- Minimize the "Energy", where λ is the tension parameter

$$E_{tps} = \sum_{i=1}^{N} \sum_{j=1}^{M} \left\| x_{ij} - f(x_i, y_j) \right\|$$
$$+ \lambda \iiint \left[\left(\frac{\partial^2 f}{\partial x^2} \right)^2 + \left(\frac{\partial^2 f}{\partial y^2} \right)^2 + \left(\frac{\partial^2 f}{\partial z^2} \right)^2 + 2 \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 + 2 \left(\frac{\partial^2 f}{\partial x \partial z} \right)^2 + 2 \left(\frac{\partial^2 f}{\partial y \partial z} \right)^2 \right] dx dy dz$$

Multi-Dimensional Splines



TPS of 33 points with no tension

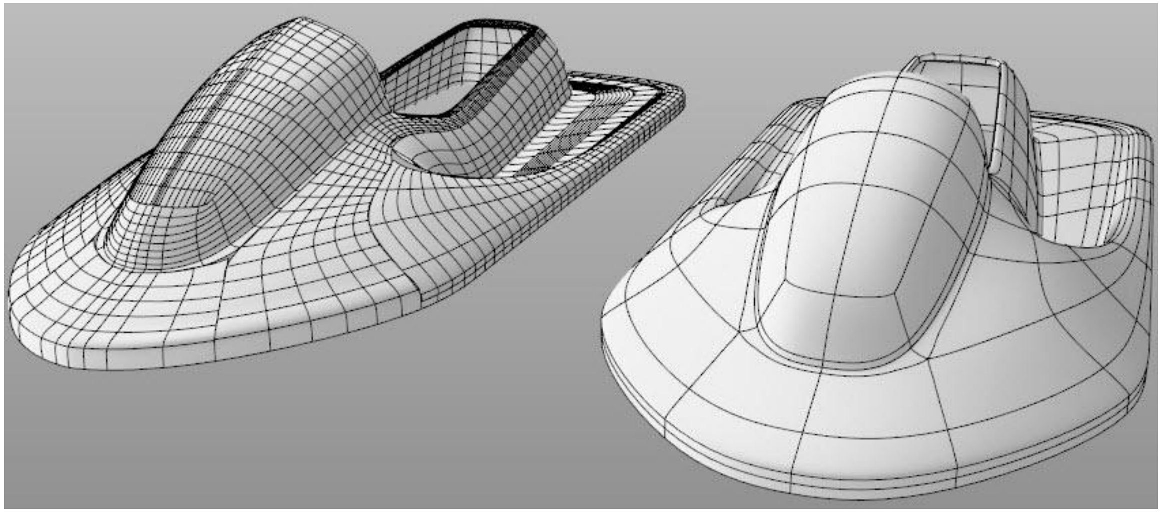


TPS of 33 points with no λ=45

- Benefits: smooth with derivatives of any order and has no free parameters to be tuned manually.

Modern Applications

- Designing of boat hulls.



- Splines are used to model multiple complex shapes into bigger single watertight surfaces.

Modern Applications

- Image processing is an area that makes use of many different spline techniques.
  - Recovering images
  - Rendering low resolution images
- Architecture
  - Gives empirical form to free hand designs and organic structures
- Engineering
  - To do geometric designs as well as analysis such as fitting stress-strain curves.

Conclusion

- Splines under tension can handle points of inflection in the data set.
- Can avoid having a solution with ringing without the addition of more data points.
- Multi-Dimensional splines are only your friend when all else fails.
- Splines are heavily used in industry.

References

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[7] R.J. RENKA, Algorithm 716 TSPACK: Tension Spline Curve-Fitting Package, ACM Transactions on Mathematical Software, Vol. 19, No. 1, March 1993. Pages 81-94.
[8] http://www.tsplines.com/, March 26th, 2011.

Follow up

- 1. Show how the interpolating function $f(x)$ reduces to a cubic spline in the limit $\sigma \rightarrow 0$
- 2. Give some more details on the function $g(\sigma)$, ie does it have roots? If so, under what conditions? Does it have a local minimum? Or is it monotonic in sigma? Discuss why/whether the optimal sigma should be found by minimising or finding the roots of $g(\sigma)$.
- 3. Why do we choose the condition $f''(x) - \sigma f(x)$ to be the thing that varies linearly? What are the advantages and disadvantages of this choice vs other possible choices?

Limit of Cubic Spline

$$f(x)=\left[\frac{f''(x_i)}{\sigma^2}\right]\cdot\frac{\sinh\left(\sigma(x_{i+1}-x)\right)}{\sinh\left(\sigma h_i\right)}+\left[y_i-\frac{f''(x_i)}{\sigma^2}\right]\cdot\frac{(x_{i+1}-x)}{h_i}+\left[\frac{f''(x_{i+1})}{\sigma^2}\right]\cdot\frac{sinh\left(\sigma(x-x_i)\right)}{\sinh\left(\sigma h_i\right)}$$
$$+\left[y_{i+1}-\frac{f''(x_{i+1})}{\sigma^2}\right]\cdot\frac{(x-x_i)}{h_i}$$

$$f(x)=\left[\frac{f''(x_i)}{\sigma^2}\right]\cdot\left[\frac{\left(\sigma(x_{i+1}-x)+\frac{\left(\sigma(x_{i+1}-x)\right)^3}{3!}+\cdots\right)}{\left(\sigma h_i+\frac{(\sigma h_i)^3}{3!}+\cdots\right)}\right]+\left[y_i-\frac{f''(x_i)}{\sigma^2}\right]\cdot\frac{(x_{i+1}-x)}{h_i}+\left[\frac{f''(x_{i+1})}{\sigma^2}\right]$$
$$\cdot\left[\frac{\left(\sigma(x-x_i)+\frac{\left(\sigma(x-x_i)\right)^3}{3!}+\cdots\right)}{\left(\sigma h_i+\frac{(\sigma h_i)^3}{3!}+\cdots\right)}\right]+\left[y_{i+1}-\frac{f''(x_{i+1})}{\sigma^2}\right]\cdot\frac{(x-x_i)}{h_i}$$

Limit of Cubic Spline

$$f(x) = \left[\frac{f''(x_i)}{\sigma^2}\right] \cdot \left[\frac{\left(\sigma(x_{i+1} - x) + \frac{(\sigma(x_{i+1} - x))^3}{3!} + \cdots\right)}{\left(\sigma h_i + \frac{(\sigma h_i)^3}{3!} + \cdots\right)}\right] + \left[y_i - \frac{f''(x_i)}{\sigma^2}\right] \cdot \frac{(x_{i+1} - x)}{h_i} + \left[\frac{f''(x_{i+1})}{\sigma^2}\right]$$

$$\cdot \left[\frac{\left(\sigma(x - x_i) + \frac{(\sigma(x - x_i))^3}{3!} + \cdots\right)}{\left(\sigma h_i + \frac{(\sigma h_i)^3}{3!} + \cdots\right)}\right] + \left[y_{i+1} - \frac{f''(x_{i+1})}{\sigma^2}\right] \cdot \frac{(x - x_i)}{h_i}$$

$$\left[\frac{f''(x_i)}{\sigma^2}\right] \cdot \left[\frac{\left(\sigma(x_{i+1} - x) + \frac{(\sigma(x_{i+1} - x))^3}{3!}\right)}{\left(\sigma h_i + \frac{(\sigma h_i)^3}{3!}\right)} - \frac{(x_{i+1} - x)}{\sigma^2 h_i}\right]$$

Limit of Cubic Spline

$$[f''(x_i)]\cdot\left[\frac{\left(\sigma(x_{i+1}-x)+\frac{(\sigma(x_{i+1}-x))^3}{3!}\right)}{\sigma^2\left(\sigma h_i+\frac{(\sigma h_i)^3}{3!}\right)}-\frac{(x_{i+1}-x)}{\sigma^2 h_i}\right]$$

$$[f''(x_i)]\cdot\left[\frac{(\sigma(x_{i+1}-x))}{\left(\sigma^3 h_i+\frac{\sigma^5 h_i{}^3}{3!}\right)}+\frac{(\sigma(x_{i+1}-x))^3}{6\left(\sigma^3 h_i+\frac{\sigma^5 h_i{}^3}{3!}\right)}-\frac{(x_{i+1}-x)}{\sigma^2 h_i}\right]$$

$$[f''(x_i)]\cdot\left[\frac{(\sigma(x_{i+1}-x))}{\left(\sigma^3 h_i+\frac{\sigma^5 h_i{}^3}{3!}\right)}+\frac{(\sigma(x_{i+1}-x))^3}{6\left(\sigma^3 h_i+\frac{\sigma^5 h_i{}^3}{3!}\right)}-\frac{(x_{i+1}-x)}{\sigma^2 h_i}\right]$$

Limit of Cubic Spline

$$[f''(x_i)] \cdot \left[\frac{(\sigma(x_{i+1} - x))}{(\sigma^3 h_i + 0)} + \frac{(\sigma(x_{i+1} - x))^3}{6(\sigma^3 h_i + 0)} - \frac{(x_{i+1} - x)}{\sigma^2 h_i} \right]$$

$$[f''(x_i)] \cdot \left[\frac{(x_{i+1} - x)}{\sigma^2 h_i} + \frac{(\sigma(x_{i+1} - x))^3}{6\sigma^3 h_i} - \frac{(x_{i+1} - x)}{\sigma^2 h_i} \right]$$

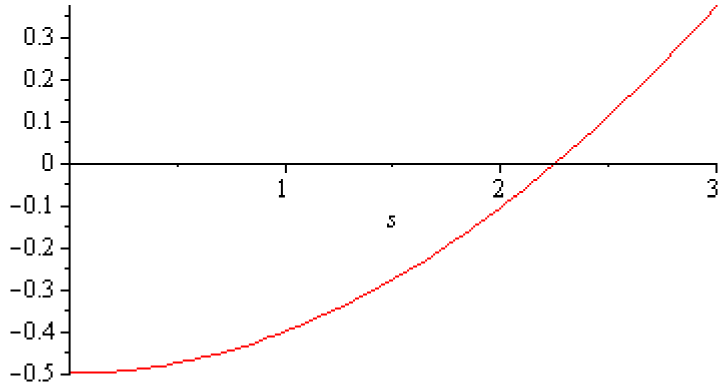
$$[f''(x_i)] \cdot \left[\frac{(x_{i+1} - x)^3}{6h_i} \right]$$

$$f(x) = \frac{(x_{i+1} - x)}{h_i} f(x_i) + \frac{(x - x_i)}{h_i} f(x_{i+1}) + \left[\left(\frac{(x_{i+1} - x)}{h_i} \right)^3 - \frac{(x_{i+1} - x)}{h_i} \right] h_i^2 \frac{f''(x_i)}{6}$$
$$+ \left[\left(\frac{(x - x_i)}{h_i} \right)^3 - \frac{(x - x_i)}{h_i} \right] h_i^2 \frac{f''(x_{i+1})}{6}$$

g(σ)

$$g(\sigma)=\begin{cases}\sigma * \operatorname{coshm}(\sigma)/\operatorname{sinhm}(\sigma)-1-\max\{d_1/d_2,\, d_2/d_1\} & \text{if } \sigma>0,\\ 2-\max\{d_1/d_2,\, d_2/d_1\} & \text{if } \sigma=0.\end{cases}$$

- “ A straightforward approach to proving this statement is extremely tedious and, since no alternative approach has been found, it remains an open question”



Alternative Derivations

$$f''(x) - \sigma^2 f(x) = 0$$

$$h^4(x) - \sigma^2 h''(x) = 0$$

$$f''''(x) - \sigma^2 f''(x) = 0$$