Funding Costs, Funding Strategies

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Abstract

The economic value of derivatives depends on their funding costs, because they can result in windfalls or shortfalls to bondholders on their firm's default. But this depends on not just who is funding them, but how - so the resulting adjustments depend on the funding strategy deployed. It is another layer of complexity to derivatives pricing, argue Christoph Burgard and Mats Kjaer.

1 Introduction

Incorporating the effects of funding derivatives into their pricing has become a hot topic in the last couple of years. Following on from earlier works of Piterbarg [13], our last paper in Risk, Burgard and Kjaer [4], established how funding costs, funding benefits and counterparty risk could be treated within one framework extending the approach of Black-Scholes-Merton. This showed that if a derivative's issuer is able to perfectly hedge the risk of its own default, the only adjustment to the classical price is the bilateral credit valuation adjustment (BL CVA) inclusive of the debit valuation adjustment (DVA). However this would require the issuer to freely dynamically trade spread positions of different seniorities of its own bonds, and so is unrealistic. Assuming that the issuer can hedge own default only when its derivative position is in-the-money and so provides funding is more realistic. When it is out-of-the-money and requires funding, a post-default windfall to the issuer's estate is generated. In that case, a funding cost adjustment (FCA) is added in to compensate. There have been a flurry of papers proposing alternative approaches from different authors, including Morini and Prampolini [12], Brigo, Pallavicini and Perini [1], Crepey [6],[7], and perhaps most famously Hull and White [9], that use risk neutral valuation principles to examine the question. Such an approach, discounting all expected cash flows at the risk free rate results in the classical price with bilateral CVA. But this disregards the preferences of different stake-holders regarding the value of pre- and post-own default cash flows. This is justified if all risks - including own default - are hedgable so that net post-default cashflows are zero. But, as mentioned above they are not. Shareholders are primarily interested in the pre-default cashflows of derivatives and their hedges, but post-default cashflows matter for bond holders, as they contribute to the recovery realised. Shareholders only care about the latter through balance sheet effects that are in practice hard to realise and account for. Some authors have considered cases where the post-default cash flows on the funding leg are disregarded but not the ones on the derivative. But it is not clear why some post default cashflows should be disregarded but not others, and without specifying funding strategies the resulting recursive relations cannot be solved. Crepey's generalisation of the original Markovian framework to non-Markov processes by using backward stochastic equations is elegant but difficult to solve explicitly. As far as we are aware explicit solutions are only available for the original cases we presented.

This article looks at funding strategies in terms of holding or issuing own bonds. The strategies hedge out some but not all cashflows at own default. The economic value of a derivative to shareholders is then given by assuming they disregard any remaining post default cashflows and pre-default balance sheet effects. The funding cost adjustment is then given by the discounted expected value of the post-default cashflows. The strategies previously considered are special cases of those considered here, and so this article generalises that work. Dealers can consider their own funding strategies, and decide which adjustments represent the economic funding costs they expect to charge while in business. We discuss boundary conditions covering practical cases such as one or two-way credit support annexes (CSAs) governing collateral agreements and so-called set-offs. Set-offs are particularly interesting as they mitigate the need for funding cost adjustments. Explicit examples are calculated, and numerical results show that different funding strategies can yield quite different valuation adjustments and asymmetries between counterparties in the absence of set-off provisions.

2 General semi-replication and pricing PDE

We consider a generic derivative contract, possibly collateralized, between an issuer B and a counterparty C, with an economic value $\hat{V}$ that incorporates the risk of default of counterparty and issuer and any net funding costs the issuer may encounter prior to own default. We will describe a general semi-replication strategy that the issuer can deploy to perfectly hedge out market factors and counterparty default but which may not provide a perfect hedge in the event of the issuer's "own" default. The tradable instruments used in this strategy are a counterparty zero-coupon zero-recovery bond P_C [1] , two issuer "own" bonds P_1 and P_2 of different seniorities, i.e. different recoveries R_1 and R_2, respectively, and a market instrument S that can be used to hedge out the market factor underlying the derivative contract (e.g. a stock). The set-up can easily be extended to several market factors. We assume the following standard dynamics for these instruments ($i = 1, 2$):

$$\begin{aligned} dS &= \mu S dt + \sigma S dW && (1) \\ dP_C &= r_C P_C^- dt - P_C^- dJ_C && (2) \\ dP_i &= r_i P_i^- dt - (1 - R_i) P_i^- dJ_B && (3) \end{aligned}$$

where J_B and J_C are default indicators for B and C, respectively, and $P_{i/C}^- = P_{i/C}(t^-)$ are the pre-default bond prices. Without loss of generality we let P_1 be the junior bond, i.e. $R_1 < R_2$ and $r_1 > r_2$. In the case of zero basis between bonds of different seniority it is trivial to show that

$$r_i - r = (1 - R_i)\lambda_B, \quad (4)$$

where r is the risk-free rate and λ_B corresponds to the spread of a (potentially hypothetical) zero-recovery zero-coupon bond of the issuer. For the remainder of this paper we assume such zero basis between the bonds of different seniority in order to obtain more readable adjustment formulas expressed in terms of λ_B rather than r_1 and r_2.

Let $\hat{V}(t, S, J_B, J_C)$ be the total economic value of the derivative to the issuer. Like in Kjaer [11], we want to consider general boundary conditions at default of the issuer or counterparty that are given by

$$\begin{aligned} \hat{V}(t, S, 1, 0) &= g_B(M_B, X) \quad \text{B defaults first,} \\ \hat{V}(t, S, 0, 1) &= g_C(M_C, X) \quad \text{C defaults first,} \end{aligned} \quad (5)$$

with general close-out amounts M_B and M_C and collateral X. We shall call these boundary conditions "regular", if $M_B = M_C = V$, where V is the classic Black-Scholes price of the derivative, i.e. without counterparty and own default risks and no funding costs. For example, the regular bilateral boundary conditions with collateral are defined as

$$\begin{aligned} g_B &= (V - X)^+ + R_B (V - X)^- + X \\ g_C &= R_C (V - X)^+ + (V - X)^- + X. && (6) \end{aligned}$$

[1] The setup could equivalently be formulated with a counterparty bond with recovery. For ease of notation we choose P_C to be a zero-recovery bond.

Burgard and Kjaer [3] consider alternative close-out cases for $M_B = M_C$ and Brigo and Morini [2] extend this to cases when $M_B \neq M_C$ and include e.g. cost of funding in the close-out amounts. In a separate paper, Burgard and Kjaer [5] apply the present framework to such funding aware close-outs.

An example for the close-out functions are regular bilateral close-outs without collateral, which are described by $g_B = V^+ + R_B V^-$ and $g_C = R_C V^+ + V^-$. We will examine this and other examples like one-way CSAs and set-offs later.

2.1 Semi-replication

For the semi-replication strategy we set up a hedge portfolio Π as

$$\Pi(t) = \delta(t)S(t) + \alpha_1(t)P_1(t) + \alpha_2(t)P_2(t) + \alpha_C(t)P_C(t) + \beta_S(t) + \beta_C(t) - X(t), \tag{7}$$

with $\delta(t)$ units of S, $\alpha_{1/2}(t)$ and $\alpha_C(t)$ units of own and counterparty bonds, respectively, cash accounts $\beta_S(t)$ and $\beta_C(t)$ and a collateral account $X(t)$. For the remainder of this paper we suppress the time dependence to conserve space. The cash accounts β_S and β_C are used to finance the S and P_C-positions, i.e. $\alpha_C P_C + \beta_C = 0$ and $\delta S + \beta_S = 0$, and are assumed to pay net rates of $(q_S - \gamma_S)$ and q_C respectively, where γ_S may be a dividend income. The hedge positions may be collateralized or repo-ed, so q_S and q_C may be the collateral or repo-rates, respectively. The derivative collateral balances X are assumed to be fully re-hypothecable and pay the rate r_X, and $X > 0$ correspond to the counterparty having posted the amount X with the issuer. The strategy shall be designed such that $\hat{V} + \Pi = 0$, except, possibly, at issuer default. The issuer bond positions $\alpha_1 P_1$ and $\alpha_2 P_2$ are used to finance/invest any remaining cash that is not funded via the collateral, which yields the following *funding constraint*:

$$\hat{V} - X + \alpha_1 P_1 + \alpha_2 P_2 = 0. \tag{8}$$

Using this constraint, Ito's lemma, the boundary conditions (5) and hedge ratios δ and α_C, chosen such that the market and counterparty default risks are hedged out, we find that the evolution of the total portfolio over time step dt is given by

$$\begin{aligned} d\hat{V} + d\bar{\Pi} \quad = \quad & \Big(\partial_t \hat{V} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} - s_X X + \lambda_C g_C + \lambda_B g_B - \epsilon_h \lambda_B\Big) dt \\ & + \epsilon_h dJ_B, \end{aligned} \tag{9}$$

where $\epsilon_h \equiv g_B + P_D - X$, with P_D the value of the issuer own bond portfolio post issuer default, i.e. $P_D = \alpha_1 R_1 P_1 + \alpha_2 R_2 P_2$, λ_C is the effective financing rate of the counterparty default hedge position, i.e. the yield of P_C over the repo rate q_C, and $\mathcal{A}_t \equiv \frac{1}{2}\sigma^2 S^2 \partial_S^2 + (q_S - \gamma_S) S \partial_S$, with $q_S - \gamma_S$ the effective financing rate of the market factor hedge S. The derivation of Equation (9) is given in Appendix A.

It can be seen from Equation (9), that the combination of the derivative $\hat{V}$ and hedge portfolio Π is risk-free as long as the issuer is alive. At issuer default, the jump term $\epsilon_h dJ_B$ gives rise to a hedge error of size ϵ_h. This hedge error can be a windfall or a shortfall and its sign and size depends on the post-default value of the own bond portfolio and thus the funding strategy employed. While alive, on the other hand, the issuer will accrue a corresponding cost/gain of size $-\epsilon_h \lambda_B$ per unit of time. This can be seen as the running spread to pay for the potential windfall/shortfall upon issuer default.

We assume that the issuer wants the strategy described in Equation (9) to evolve in a self-financed fashion while he is alive. This implies that the issuer requires the total drift term of (9) to be zero. This yields the following PDE for the risky economic value $\hat{V}$ of the derivative

$$\begin{aligned} \partial_t \hat{V} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} &= s_X X - \lambda_C g_C - \lambda_B g_B + \lambda_B \epsilon_h \\ \hat{V}(T,S) &= H(S), \end{aligned} \tag{10}$$

where $H(S)$ is the payout of the derivative at maturity and $s_X \equiv r_X - r$.

We will be interested in the correction $U = \hat{V} - V$ to the risk-free Black-Scholes price V. Using the Black-Scholes PDE for V we find the PDE for U to be

$$\begin{aligned} \partial_t U + \mathcal{A}_t U - (r + \lambda_B + \lambda_C)U &= s_X X - \lambda_C(g_C - V) - \lambda_B(g_B - V) + \lambda_B \epsilon_h \\ U(T,S) &= 0. \end{aligned} \tag{11}$$

Applying Feynman-Kac to this PDE gives $U \equiv CVA + DVA + FCA + COLVA$ with

$$CVA = -\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t \left[V(u) - g_C(V(u), X(u)) \right] du \tag{12}$$

$$DVA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t \left[V(u) - g_B(V(u), X(u)) \right] du \tag{13}$$

$$FCA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t \left[\epsilon_h(u) \right] du \tag{14}$$

$$COLVA = -\int_t^T s_X(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t \left[X(u) \right] du, \tag{15}$$

where $D_{\hat{r}}(t,u) \equiv \exp(-\int_t^u \hat{r}(v)dv)$ is the discount factor between t and u for a rate $\hat{r}$. The measure of the expectations in these equations is such that S drifts at rate $(q_S - \gamma_S)$. The sum of DVA and FCA we sometimes refer to as FVA.

Here, the sum of the CVA, DVA and COLVA terms is symmetric in that it is identical (with opposite sign) when computed by the issuer and counterparty respectively. The FCA (14) on the other hand is the discounted survival probability weighted expected value of the hedge error ϵ_h implied by the semi-replication strategy chosen. Because the hedge error upon own default is different for the issuer and counterparty, the FCA is not symmetric. This is a generalisation of the result presented in Burgard and Kjaer [4] for regular bilateral close-outs and a particular choice of issuer bonds, that states that the FCA is the cost for generating a windfall to the issuer bondholders in case of default. If the issuer wants to break even while being alive, this cost has to be included in the derivative price charged to the counterparty.

We shall note that we have assumed that the funding rates r_1, r_2 are unaffected by the addition of the derivative and funding positions. In Burgard and Kjaer [4] we have noted that the presence of a potential windfall has a positive balance sheet effect: it improves the recovery rate to the bondholders and therefore the funding spread of the issuer should go down. Hull and White [9] have used a similar argument. For a simple balance sheet model with floating funding costs we have demonstrated in Burgard and Kjaer [4] that this effect can result in an effective marginal funding rate that corresponds to the risk free rate. However, as discussed there, in practice this balance sheet effect on the funding costs is rather indirect, fraught with accounting issues and will in general only feed through over time. For the purpose of this paper we will therefore assume that the issuer disregards this potential effect.

3 Examples of different bond portfolios

Using the general framework developed in the previous section, we will give three different examples of semi-replication strategies that generate different hedge errors ϵ_h and consequently different valuation adjustments.

The first strategy, if employable, allows perfect replication and generates zero FCA. The second one is equivalent to the setup used in Burgard and Kjaer [3] for the bilateral close-outs and ensures that there is never a shortfall at issuer default, only a potential windfall. The third strategy assumes hedging with a single issuer bond. It generates both, potential windfalls and potential losses post-default, and is an extension of the model derived in Piterbarg [13]. The different strategies generate different economic values (and therefore different adjustments) to the issuer while he is alive. They demonstrate the assumptions implicitly being made when using different adjustment formulas in practise.

Throughout this section we will assume that the closeout value is V, i.e. that $M_B = M_C = V$, and we discuss funding aware close-outs separately in Burgard and Kjaer [5].

3.1 Perfect replication: the FCA vanishes

First we want to consider the case where the issuer is able to perfectly hedge out the windfall/shortfall at own default. This corresponds to the case discussed in Section *Balance-sheet management to mitigate funding costs* in Burgard and Kjaer [4] and also covers the risk neutral approach discussed in Hull and While [9].

Perfect hedging is equivalent to the hedge error ϵ_h to be zero, i.e.

$$g_B + P_D - X = 0. \tag{16}$$

The valuation PDE (10) then becomes

$$\partial_t \hat{V} + \mathcal{A}_t \hat{V} - (r + \lambda_C + \lambda_B)\hat{V} = s_X X - \lambda_B g_B - \lambda_C g_C. \tag{17}$$

Correspondingly, the FCA as given in Equation (14) vanishes.

The hedge ratios α_1 and α_2 that achieve this perfect replication are determined by the no-windfall condition (16) and the funding constraint (8). These two conditions provide two equations that can be solved to find

$$\begin{aligned}\alpha_1 &= \frac{R_2\hat{V} - g_B + (1-R_2)X}{(R_1-R_2)P_1},\\ \alpha_2 &= \frac{R_1\hat{V} - g_B + (1-R_1)X}{(R_2-R_1)P_2}.\end{aligned} \tag{18}$$

As an example, for the regular bilateral boundary conditions (6) and these adjustments specialise to

$$CVA = -(1-R_C)\int_t^T \lambda_C(u)D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[(V(u)-X(u))^+\right]du \tag{19}$$

$$DVA = -(1-R_B)\int_t^T \lambda_B(u)D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[(V(u)-X(u))^-\right]du \tag{20}$$

with FCA and $COLVA$ being zero (assuming $r_X = r$).

If the derivatives are not collateralised, i.e. $X = 0$, then these adjustments correspond to the classical bilateral CVA. Thus, the classical bilateral CVA can be achieved when perfect replication is possible. As mentioned, in practice such a dynamic balance sheet management via actively traded spread positions between own junior versus senior bonds is in general not a viable option.

3.2 Strategy I: Semi-replication with no shortfall at own default

In this section we demonstrate a bond portfolio and hedging strategy that constitute an equivalent setup to the one presented in Burgard and Kjaer [3], extended to more general boundary conditions including the possibility of collateral. It still involves dynamic trading of two bonds but it is more conservative than the perfect replication strategy of the previous section as it does not aim to monetizing the potential windfall upon own default by entering into a offsetting spread position between the two own bonds. While the generates potential windfalls at own default, it does not generate shortfalls, and has the additional advantage that for regular bilateral close-outs without collateral it results in the usual bilateral CVA adjustment plus a funding cost adjustment, so presents a simple extension to the existing framework, where the derivatives dealer does not think he can monetize the windfall.

The strategy involves a zero recovery bond P_1 with $R_1 = 0$ and a recovery bond with $R_2 = R_B$. The issuer runs the following bond positions:

- Invest or fund the difference between $\hat{V}$ and V by buying or issuing P_1-bonds.
- Hold the number of P_2-bonds given by the funding constraint (8).

This strategy is thus defined by the following values of α_1 and α_2:

$$\begin{aligned}\alpha_1 P_1 &= -(\hat{V} - V) = -U,\\ \alpha_2 P_2 &= -\alpha_1 P_1 - \hat{V} + X = -(V - X).\end{aligned} \tag{21}$$

The strategy is thus symmetric between positive and negative funding, and the risk-free value V not covered by the collateral X is funded/invested via standard unsecured own bonds with recovery R_B. Only the adjustment U, which falls away upon default, is funded/invested via a zero recovery bond. As such this strategy looks more palatable from a regulatory and accounting perspective than the perfect replication strategy, which tries to actively extract the funding spread from the balance sheet by issuing own senior bonds to buy back own junior bonds.

This bond portfolio is equivalent to the one described in Burgard and Kjaer [3], [4] for the bilateral close-outs considered there. [2]

Let us analyse this set-up in more detail for the case of the regular bilateral close-out as given in equation (6).

For this case the hedge error ϵ_h specialises to

$$\epsilon_h = (1 - R_B)(V - X)^+, \tag{22}$$

which is always a windfall (possibly zero) to the bondholders of the issuer.

We shall call this setup Strategy I. It is characterised by the ability of the issuer to perfectly hedge out the difference U between the risky value of the derivative before own default and the close-out amount after own default by means of trading own bonds of zero recovery. The remainder, i.e. the difference between the close-out amount and the collateral, is invested/funded using own bonds with recovery R_B. This part generates the windfall ϵ_h to the issuer's bondholders when $(V - X)$ is in the money and the issuer defaults.

The adjustments (12) to (15) specialise to

$$\begin{aligned}
CVA &= -(1 - R_C) \int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t \left[(V(u) - X(u))^+ \right] du && (23)\\
DVA &= -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t \left[(V(u) - X(u))^- \right] du && (24)\\
FCA &= -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t \left[(V(u) - X(u))^+ \right] du && (25)\\
COLVA &= - \int_t^T s_X(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t \left[X(u) \right] du. && (26)
\end{aligned}$$

It is possible to combine the DVA (the funding benefit) and FCA (the funding cost) into a funding value adjustment FVA[3] with

$$FVA \equiv DVA + FCA = -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t \left[V(u) - X(u) \right] du. \tag{27}$$

[2]To see this keep in mind that the linear combination of investing -(1-R)*V in a zero recovery bond and -R*V in a risk free investment, as given in Burgard and Kjaer [3] for negative values of V, [4], corresponds to an investment of -V into a bond with recovery R.

[3]Hull and White [9], [10] refer to what we have named FCA as FVA. Our notation here is consistent with our previous papers and Gregory [8].

For un-collateralized derivatives, where $X = 0$, the COLVA term (26) vanishes. The CVA and DVA term correspond to the classical bilateral CVA. The FCA term gives a funding cost adjustment on top.

For gold-plated two-way CSAs, where $X = V$, the CVA, DVA and FCA terms are all zero. If the collateral rate is the risk free rate, then the COLVA term, which represents the spread earned by the issuer on the collateral, vanishes as well. In this case, the risky value $\hat{V}$ of a derivative with gold-plated two-way CSA, becomes the risk free price V. The intuition is that the collateral cash is exactly what is needed to fund the hedge and to eliminate all counterparty risk and as a consequence $\alpha_1 = \alpha_2 = 0$. This corresponds to the result for fully collateralised trades of Piterbarg [13].

Another special case worthwhile considering is that of a one-way CSA, whereby only the issuer posts collateral when the risk-free value of the trade is out-of-the-money, i.e. $X = V^-$. One-way CSAs are common when the issuer trades with sovereign counterparties or other sovereign-like public entities. In this case the adjustments specialize to

$$
\begin{aligned}
CVA &= -(1-R_C)\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+(u)\right] du \qquad (28)\\
FCA &= -(1-R_B)\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+(u)\right] du \qquad (29)\\
COLVA &= -\int_t^T s_X(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^-(u)\right] du. \qquad (30)
\end{aligned}
$$

Unsurprisingly, the introduction of a 1-way CSA makes the DVA vanish while leaving the CVA and FCA terms unchanged. Unlike the un-collateralized case any cash available when the derivative is out-of-the money must now be handed over as collateral and can thus not be used to generate a funding benefit. And unlike the 2-way CSA case there is no influx of collateral cash that can be used to fund the hedge while the derivative is in-the money. The issuer is thus faced with the worst of the un-collateralized and 2-way CSA cases, and to compensate for that needs to charge a higher price to the counterparty in order to break even.

3.2.1 Set-offs

It is also instructive to study the case of so-called set-offs. A set-off is a legal agreement that allows the surviving party to settle outstanding derivative claims of the defaulting party by means of supplying bonds of the defaulting party at nominal value rather than cash. Since post default, these bonds trade at their recovery rate, this type of settlement is valuable to the surviving party. Explicitly, for regular bilateral set-offs without collateral, the boundary conditions are given by $g_B = R_B V$ and $g_C = R_C V$. Inserting this into Equation (21) implies that hedge error ϵ_h disappears. The CVA and DVA adjustments in this case are given by

$$
\begin{aligned}
CVA &= -(1-R_C)\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V(u)\right] du \qquad (31)\\
DVA &= -(1-R_B)\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V(u)\right] du. \qquad (32)
\end{aligned}
$$

Significantly, with $\epsilon_h = 0$ the FCA term vanishes, resulting in symmetric prices between the issuer and counterparty. Thus, adoption of such close-out provisions would mitigate the need for economic funding cost adjustments.

3.3 Strategy II: Semi-replication with single bond

In this section we describe a very simple strategy, where the issuer uses a single own bond with a recovery R_F, i.e.

$$\begin{aligned}\alpha_1 P_1 &= 0, \\ \alpha_2 P_2 &= -(\hat{V} - X) = -(V + U - X),\end{aligned} \tag{33}$$

where the second line follows from the funding constraint (8). For aesthetic reasons we relabel the remaining bond P_F and its yield $r_F = r + s_F$. Equation (33) implies that the issuer raises all necessary (net) cash by issuing P_F-bonds and invests any surplus (net) cash into repurchasing the same bonds.

With a single bond, once the funding constraint is fulfilled, there are no degrees of freedom left for the issuer to hedge out his own default. This is in contrast to the previous setup of Strategy 1, where the issuer is able to hedge out his own default risk when the trade is out-of-the-money at least.

For the own bond portfolio of equation (33) the hedge error ϵ_h amounts to

$$\epsilon_h = g_B + P_D - X = g_B - \tilde{R}_F \hat{V} - (1 - \tilde{R}_F) X, \tag{34}$$

where we have used that the post-default value of the own bond portfolio is given by $P_D = -\tilde{R}_F(\hat{V} - X)$.

Using this hedge error in the general adjustment formula (14) for the FCA yields a recursive relationship. This is because the $\hat{V} = V + U$ appearing on the rhs of equation (34) includes the contribution from U and thus FCA itself.

The best way to deal with this case is by actually going back to the PDE in (10) and inserting the hedging error (34) to obtain

$$\begin{aligned}\partial_t \hat{V} + \mathcal{A}_t \hat{V} - (r_F + \lambda_C)\hat{V} &= -\lambda_C g_C(V, X) - (r_F - r_X)X \\ \hat{V}(T, S) &= H(S)\end{aligned} \tag{35}$$

The boundary condition g_B does not enter - this is because in this strategy there is no attempt to hedge own default. It is also worth noting that for un-collateralized trades (i.e. X=0) and zero counterparty risk ($\lambda_C = 0$) the PDE specializes to a simple discounting-with-funding approach as in Piterbarg [13]. So simple discounting-with-funding is a special case of strategy II and assumes that the issuer deals with any funding requirement or surplus by using a single funding instrument and is happy to generate a windfall or shortfall upon own default.

Similarly, inserting the hedging error (34) into the PDE (11) gives the adjustment U for this strategy:

$$\begin{aligned} \partial_t U + \mathcal{A}_t U - (r_F + \lambda_C) U &= -\lambda_C (g_C(V, X) - V) + s_F (V - X) + s_X X \\ U(T, S) &= 0 \end{aligned} \tag{36}$$

As before, let's analyze the case of regular bilateral close-outs with collateral as given in equation (6). For these $g_C(V, X) - V = -(1 - R_C)(V - X)^+$. Applying Feynman-Kac theorem we obtain $U = CVA_F + DVA_F + FCA_F + COLVA_F$ with

$$CVA_F = -(1 - R_C) \int_t^T \lambda_C(u) D_{r_F + \lambda_C}(t, u) \mathbb{E}_t \left[(V(u) - X)^+ \right] du \tag{37}$$

$$DVA_F = - \int_t^T s_F(u) D_{r_F + \lambda_C}(t, u) \mathbb{E}_t \left[(V(u) - X(u))^- \right] du \tag{38}$$

$$FCA_F = - \int_t^T s_F(u) D_{r_F + \lambda_C}(t, u) \mathbb{E}_t \left[(V(u) - X(u))^+ \right] du \tag{39}$$

$$COLVA_F = - \int_t^T s_X(u) D_{r_F + \lambda_C}(t, u) \mathbb{E}_t \left[X(u) \right] du. \tag{40}$$

Combining DVA_F and FCA_F into a FVA_F gives:

$$FVA_F = DVA_F + FCA_F = - \int_t^T s_F(u) D_{r_F + \lambda_C}(t, u) \mathbb{E}_t \left[V(u) - X(u) \right] du. \tag{41}$$

These adjustments are very similar to the ones from Strategy I in Section 3.2, except that the discounting used is $D_{r_F + \lambda_C}$ rather than $D_{r + \lambda_B + \lambda_C}$. There is no reference to λ_B, only to the funding rate r_F of the P_F bond used in the own-bond portfolio of the semi-replication strategy.

It should be noted, that the adjustments CVA_F, DVA_F, FCA_F and $COLVA_F$ are not direct specialisations of the general adjustments defined in Equations (12) to (15). In particular, FCA_F does not correspond anymore to the discounted expectation of the hedge error ϵ_h upon issuer default. Obviously, this expectation is still equal to the difference between $CVA_F + FVA_F + COLVA_F$ and the classical bilateral CVA with collateral. We can thus calculate the FCA of equation (14) as

$$FCA = FCA_F + (CVA_F - CVA) + (DVA_F - DVA) + (COLVA_F - COLVA). \tag{42}$$

The adjustments for special cases of un-collateralized, gold-plated two-way CSA and one-way CSAs can then easily derived equivalently to section 3.2.

The strategy is very simple to implement and understand. It is of practical relevance, not the least because dealers who simply discount by the funding rate implicitly assume this strategy (for zero counterparty risk), including the potential windfalls and shortfalls to his estate upon own default.

4 Results

In Tables 1 to 3 below we give, for illustration purposes, the generalised bilateral CVA (the sum of the CVA (12) and DVA (13)), generalised FCA (14) and valuation asymmetries for Strategy I and II computed from the perspective of the issuer and the counterparty (i.e. with counterparty's FCA), respectively, together with the issuer's hedge error $\epsilon_h(t_0)$ in case of an immediate default of the issuer for three sample trades and some dummy market data. For Strategy II the FCA is calculated as per equation (42). The example trades are 10Y swaps of 100M USD notional, where issuer pays fixed and receives 6M Libor floating. We consider different close-out provisions. The three fixed rates considered are 3.093% (OTM), 2.693% (ATM) and 2.293% (ITM). The adjustments computed from the perspective of the counterparty has been negated. One can see that if both sides include their funding costs their economic value can be far apart and the two parties may not agree on the deal. If the counterparty does not include funding costs, it will in general deal with the issuer that has the lowest funding costs. As can be seen, the difference between the adjustments of strategy I and strategy II are in general not particular big (increasing with the funding rate), but, as expected, strategy II has potential significant shortfalls upon issuer default, whereas strategy I only generates windfalls. As discussed in chapter 3.2.1, when using set-off close outs the impact of funding costs are mitigated. When following strategy I the FCA vanishes completely and symmetric prices are obtained. Even when implementing strategy II the FCA is pretty small and prices are close to symmetric. Set-off close-outs are thus an attractive way of mitigating the need for funding cost adjustments.

Tables 1 to 3 show how asymmetric valuation and hedging error can vary under the funding strategies I and II outlined above, and how their interact with CSAs and set-offs. Each in turn prices an out-of-the-money (OTM), at-the-money (ATM) and in-the-money (ITM) 10 year 100M USD notional vanilla swap, with the issuer paying fixed and receiving floating six-month Libor. Three fixed rates are considered - the ATM level is 2.693%, based on 18 July 2013 data, with the OTM and ITM strikes 40bps either side, at 3.093% and 2.293% respectively.

Issuer bond spreads are considered at 100bps and 500bps, while the counterparty spread is constant at 300bps. The total adjustments are calculated for issuer and counterparty, as well as the bilateral CVA and hedging error. Uncollateralised, one-way CSA in counterparty's favour, and cases including a set-off are considered.

The adjustments differ between the two counterparties, creating an asymmetry in the derivative's valuation. The degree of this and the resultant hedging error depends on funding strategy and collateralisation. In the 100 bp case, the uncollateralised swap has a valuation asymmetry of the order of 50bps of notional, with the magnitude decreasing OTM through ATM to ITM. The introduction of a one-way CSA increases the size of the issuer's adjustment, but reduces the asymmetry to 20 - 30bps, with the amount of reduction skewed towards the opposite direction, from ITM to OTM. This is because under the one-way CSA the issuer has to post more collateral for an OTM swap, thus reducing the funding benefit.

Strategy I has zero hedge error for OTM and ATM, but produces a roughly 2% of notional hedging error in the ITM case regardless of the existence of a CSA. In the presence of a set-off, the FCA is eliminated, and so the total adjustment is equal to the Bilateral CVA. The strategy II case is more complex, but the valuation asymmetry is dramatically reduced -

by roughly a factor ten in our examples. The hedging error is also reduced, with the biggest reduction coming for the ITM case. When the issuer is ITM and defaults, the set-off implies that the counterparty can pay back the full present value of the trade using issuer bond notional rather than cash, which reduces the post default bond holder windfall.

The interesting point is that these qualitative findings are independent of the issuer's bond spread - only the magnitudes change, with a greater proportional reduction in the valuation asymmetry and hedging error of strategy II in the presence of a set-off, for instance - as can be seen from the results for the 500bps case.

Setup (λ_B=100bp/500bp)	No CSA	One-way CSA	Set-off	No CSA	One-way CSA	Set-off
BL CVA	-467,283	-594,543	-283,227	61,393	-499,142	-435,111
Strategy I FCA (Issuer)	-198,024	-198,024	0	-832,560	-832,560	0
Strategy I FCA (Cparty)	382,081	0	0	336,056	0	0
Strategy I valuation asymmetry	580,105	198,024	0	1,168,616	832,560	0
Strategy I Hedge error	0	0	0	0	0	0
Strategy II FCA (Issuer)	-210,622	-212,309	-7,468	-916,177	-952,191	-61,743
Strategy II FCA (Cparty)	370,178	-32,650	-22,876	343,988	-26,229	-36,233
Strategy II valuation asymmetry	580,800	179,659	15,408	1,260,165	925,962	25,510
Strategy II Hedge error	271,162	322,741	116,278	341,913	580,533	198,742

Table 1: The FCA, hedge errors and valuation asymmetries for the OTM swap for the case of no-CSA, one-way CSA (issuer posts) and a set-off. In all the examples $\lambda_C = 300$bp.

Setup (λ_B=100bp/500bp)	No CSA	One-way CSA	Set-off	No CSA	One-way CSA	Set-off
BL CVA	-672,876	-759,963	-664,532	-264,611	-643,727	-1,111,046
Strategy I FCA (Issuer)	-253,122	-253,121	0	-1,073,725	-1,073,724	0
Strategy I FCA (Cparty)	261,465	0	0	227,290	0	0
Strategy I valuation asymmetry	514,587	253,121	0	1,301,015	1,073,724	0
Strategy I Hedge error	0	0	0	0	0	0
Strategy II FCA (Issuer)	-269,228	-270,486	-12,286	-1,192,869	-1,219,596	-102,928
Strategy II FCA (Cparty)	237,282	-39,661	-37,520	220,619	-31,990	-60,563
Strategy II valuation asymmetry	506,510	230,825	25,234	1,413,488	1,187,606	42,365
Strategy II Hedge error	376,842	412,180	270,727	582,992	745,329	485,590

Table 2: The FCA, hedge errors and valuation asymmetries for the ATM swap for the case of no-CSA, one-way CSA (issuer posts) and a set-off. In all the examples $\lambda_C = 300$bp.

Setup (λ_B=100bp/500bp)	No CSA	One-way CSA	Set-off	No CSA	One-way CSA	Set-off
BL CVA	-900,726	-958,754	-1,045,837	-569,999	-819,176	-1,786,980
Strategy I FCA (Issuer)	-319,333	-319,333	0	-1,366,370	-1,366,369	0
Strategy I FCA (Cparty)	174,222	0	0	149,389	0	0
Strategy I valuation asymmetry	493,555	319,333	0	1,515,759	1,366,369	0
Strategy I Hedge error	2,164,198	2,164,198	0	2,164,198	2,164,198	0
Strategy II FCA (Issuer)	-339,222	-340,141	-17,103	-1,522,245	-1,541,651	-144,114
Strategy II FCA (Cparty)	138,035	-47,490	-52,163	129,300	-38,446	-84,895
Strategy II valuation asymmetry	477,257	292,651	35,060	1,651,545	1,503,205	59,219
Strategy II Hedge error	2,660,177	2,683,756	425,176	3,001,096	3,108,529	772,438

Table 3: The FCA, hedge errors and valuation asymmetries for the ITM swap for the case of no-CSA, one-way CSA (issuer posts) and a set-off. In all the examples $\lambda_C = 300$bp.

5 Conclusion

In conclusion, different derivatives funding strategies generate different funding costs to the issuer while he is alive against different windfalls/shortfalls when he defaults. We have derived a general relationship between the two for an issuer who requires the hedged derivatives position to be self-financing while being alive. This give rise to extensions and generalisations of the classical bilateral CVA adjustments that depend on the specific strategies employed and the details of the close-out conditions. Different issuers with different funding strategies and spreads may attach different economic values to the derivative positions, for which we have given specific examples. Allowing for active balance sheet management as in section 3.1, better accounting for balance sheet effects as described in Burgard and Kjaer [4], ability to repo derivative positions as mentioned in Burgard and Kjaer [4], or the use of set-off close outs as described in section 3.2.1 are ways to mitigate these funding cost effects. In the absence of such mitigation, dealers can use the funding strategies and assumptions presented in this paper to determine the economic value of their derivative positions default risk of the counterparty and himself and the future funding costs encountered while being alive.

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A Technical Appendix: Portfolio evolution under semi-replication

In this appendix we will give the derivation of the SDE given in equation (9), which describes the evolution of the derivative and hedge portfolio.

The evolution of the hedge portfolio Π defined in equation (7) is given by

$$d\bar{\Pi} = \delta dS + \alpha_1 dP_1 + \alpha_2 dP_2 + \alpha_C dP_C(t) + d\bar{\beta}_S + d\bar{\beta}_C - d\bar{X}, \tag{43}$$

where dS, dP_1, dP_2 and dP_C are given in equations (1) to (3) and $d\bar{\beta}_S$, $d\bar{\beta}_C$ and $d\bar{X}$ are the changes in the cash and collateral accounts excluding re-balancing.

As in Burgard and Kjaer [3] we assume that the hedge account β_S is collateralized with financing rate q_S and income (e.g. dividend) yield γ_S. Likewise, the counterparty bond position is assumed to be set up via a repo transaction costing a repo rate q_C. The derivatives collateral account is assumed to cost a collateral rate r_X. Excluding re-balancing, this yields the following increments in the accounts

$$\begin{aligned} d\bar{\beta}_S &= \delta(\gamma_S - q_S)S dt, & (44)\\ d\bar{\beta}_C &= -\alpha_C q_C P_C dt, & (45)\\ d\bar{X} &= -r_X X dt. & (46) \end{aligned}$$

With the pre- and post-default values of the issuer bond position given by $P \equiv \alpha_1 P_1 + \alpha_2 P_2$ and $P_D \equiv \tilde{R}_1 \alpha_1 P_1 + \tilde{R}_2 \alpha_2 P_2$, respectively, and by inserting (1) to (3) and (44) to (46) into (43) we find that

$$
\begin{aligned}
d\bar{\Pi} &= (r_1\alpha_1 P_1 + r_2\alpha_2 P_2 + \lambda_C\alpha_C P_C + (\gamma - q)\delta S - r_X X)\,dt \\
&\quad + (P_D - P)dJ_B - \alpha_C P_C dJ_C + \delta dS \qquad (47)
\end{aligned}
$$

where $\lambda_C \equiv r_C - q_C$ is the spread of the yield of the zero-coupon bond P_C over its repo rate, i.e. the financing rate of the counterparty default hedge position.

The evolution $d\hat{V}$ of the derivative value, on the other hand, is given by Itô's Lemma for jump diffusions

$$
d\hat{V} = \partial_t\hat{V}dt + \partial_S\hat{V}dS + \frac{1}{2}\sigma^2S^2\partial_S^2\hat{V}dt + \Delta\hat{V}_B dJ_B + \Delta\hat{V}_C dJ_C, \qquad (48)
$$

with

$$
\begin{aligned}
\Delta\hat{V}_B &= \hat{V}(t, S, 1, 0) - \hat{V}(t, S, 0, 0) \\
&= g_B - \hat{V}, \qquad (49) \\
\Delta\hat{V}_C &= \hat{V}(t, S, 0, 1) - \hat{V}(t, S, 0, 0) \\
&= g_C - \hat{V}, \qquad (50)
\end{aligned}
$$

Combining the evolution of the derivative portfolio in equation (48) and one of the hedge portfolio in equation (47) gives

$$
\begin{aligned}
d\hat{V} + d\bar{\Pi} &= \Big(\partial_t\hat{V} + \frac{1}{2}\sigma^2S^2\partial_S^2\hat{V} + r_1\alpha_1P_1 + r_2\alpha_2P_2 \\
&\quad + \lambda_C\alpha_CP_C + (\gamma - q)\delta S - r_XX\Big)dt \\
&\quad + (g_B + P_D - X)dJ_B + (\Delta\hat{V}_C - \alpha_CP_C)dJ_C + (\delta + \partial_S\hat{V})dS, \qquad (51)
\end{aligned}
$$

where the term in front of dJ_B follows from the fact that $\hat{V} + P - X = 0$ by the funding constraint (8). From (51) we see that we can eliminate the stock price and counterparty risks by choosing

$$
\begin{aligned}
\alpha_C P_C &= \Delta V_C \qquad (52) \\
\delta &= -\partial_S\hat{V} \qquad (53)
\end{aligned}
$$

which yields

$$
\begin{aligned}
d\hat{V} + d\bar{\Pi} &= \Big(\partial_t\hat{V} + \mathcal{A}_t\hat{V} - r_XX + r_1\alpha_1P_1 + r_2\alpha_2P_2 + \lambda_C\Delta\hat{V}_C\Big)dt \\
&\quad + (g_B + P_D - X)dJ_B, \qquad (54)
\end{aligned}
$$

where $\mathcal{A}_t \equiv \frac{1}{2}\sigma^2S^2\partial_S^2V + (q_S - \gamma_S)S\partial_SV$.

Next we use the zero bond basis relation (4), the funding constraint (8) and the definition of $\Delta\hat{V}_C = g_C - \hat{V}$ to write (54) as

$$\begin{aligned} d\hat{V} + d\bar{\Pi} &= \Big(\partial_t \hat{V} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} - s_X X + \lambda_C g_C + \lambda_B g_B - \epsilon_h \lambda_B\Big)dt \\ &\quad + \epsilon_h dJ_B. \end{aligned} \tag{55}$$

where $s_X \equiv r_X - r$.

From the jump term in equation (55) it follows that upon issuer default there is a hedge error of size ϵ_h. While alive, on the other hand, the issuer will incur a corresponding cost/gain of size $-\epsilon_h \lambda_B$ per unit of time.

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