Cooking with collateral

*In the wake of the crisis, the traditional assumption of a risk-free counterparty and rate has been shown to be false, yet it still underpins finance theory. **Vladimir Piterbarg** develops theoretical foundations for a model of an economy without a risk-free rate and with all assets traded on a collateralised basis. A cross-currency extension is considered, with a view to develop a model of multi-currency collateral choice*

An economy without a risk-free rate has been considered in the past (see Black, 1972) but traditional derivatives pricing theory (see, for example, Duffie, 2001) assumed the existence of such a rate as a matter of course. Until the crisis, this assumption worked well, but now even government bonds cannot be considered credit risk-free. Hence, using a risk-free money-market account or a zero-coupon bond as a foundation for asset pricing theory needs revisiting. While some of the standard constructions in asset pricing theory could be reinterpreted in a way consistent with the developments of this article, there is significant value in going through the steps of derivations to show how they should be adapted to the prevailing market practice. This is the programme we carry out here.

What comes closest to a credit risk-free asset in a modern economy, in our view, is an asset fully collateralised on a continuous basis. Of course, possible jumps in asset values and practicalities of collateral monitoring and posting do not allow for full elimination of credit risk, but we will neglect this here.

A collateralised asset is fundamentally different from the money-market account that serves the role of risk-free asset traditionally. Whereas with a money-market account one can deposit money now and withdraw it, credit risk-free, in 10 years, a collateralised asset produces a continuous stream of payments from the changes in mark-to-market value. In this article, we show how to develop a model of the economy from these non-traditional ingredients. We show that a risk-neutral measure can still be defined and hence much of the pricing technology developed in the traditional setting can, fortunately, still be used. A similar argument is independently developed in Macey (2011).

Once the basic building blocks are developed, we apply our approach in a cross-currency setting. This allows us to rigorously develop a model of assets collateralised in different currencies. A model with these features has recently been presented in Fujii & Takahashi (2011). However, the authors start from assuming the existence of a risk-free rate and measure. The lack of financial meaning of this risk-free rate is the main weakness of the argument of Fujii & Takahashi (2011), although, as shall be clear later, much of what they do can be justified from the developments of this article.

Collateralised processes

A collateralised derivative has quite a different set of cashflows from an uncollateralised 'traditional' one. At the inception of the collateralised trade there is no exchange of cashflows – the price paid for the derivative is immediately returned as collateral. During the life of a collateralised trade, there is a continuous stream of payments based on the changes in the trade's mark-to-market. A collateralised trade can be terminated at any moment at zero additional cost. So the notion of a price of a collateralised asset is actually somewhat misleading – as the trade can be terminated at zero additional cost, the value of this transaction is always zero. What we would call a price is nothing but a process that defines the level of the collateral holding. Or, in the language of the classic asset pricing theory, a collateralised transaction is an asset with a zero price process and given cumulative-dividend process (see Duffie, 2001). A moment's thought shows that this is very much the same as for a futures contract. In fact, a futures contract is just a particular type of collateralised forward contract, with the collateral rate set to zero. Keeping this in mind helps set the right picture.

Still, given the standard terminology, we would still use the word 'price' for a collateralised trade, but the meaning should always be clear – this is the level of holding of a collateral at any given time.

Let us start by introducing some notation. Let $V(t)$ be the price of a collateralised asset between parties A and B. If $V(t)$ is positive from the point of view of A, party B will post $V(t)$ to A. Party A will then pay party B a contractually specified collateral rate $c(t)$ on $V(t)$. Throughout we expressly do not assume that $c(t)$ is deterministic.

Suppose two parties agree to enter into a collateralised transaction at time t, and in particular A buys some collateralised asset from B. Let us consider the cashflows.

- Purchase of the asset. The amount of $V(t)$ is paid by A to B.
- Collateral at t. Since A's mark-to-market is $V(t)$, the amount $V(t)$ of collateral is posted by B to A.
- Return of collateral. At time $t + dt$, A returns collateral $V(t)$ to B.
- Interest. At time $t + dt$, A also pays $V(t)c(t)\,dt$ interest to B.
- New collateral. The new mark-to-market is $V(t + dt)$. Party B pays $V(t + dt)$ in collateral to A.

Note that there is no actual cash exchange at time t. At time $t + dt$, the net cashflow to A is given by:

$$V(t+dt)-V(t)(1+c(t)dt)=dV(t)-c(t)V(t)dt$$

As already noted, at time $t + dt$ the sum of the mark-to-market and the collateral for each party is zero, meaning they can terminate the contract at no cost, keeping the collateral.

Two collateralised assets

Now assume there are two assets both collateralised at rate $c(t)$. Assume that in the real-world measure the asset prices follow:

$$dV_i(t)=\mu_i(t)V_i(t)dt+\sigma_i(t)V_i(t)dW(t),\quad i=1,2 \tag{1}$$

with both assets driven by the same Brownian motion. This is the case when, for example, we have a stock[1] and an option on that stock. At time t, we can enter into a portfolio of two collateralised transactions to hedge the effect of the randomness of $dW(t)$ on the cash exchanged at time $t + dt$. To do that, we go long a notional of $\sigma_2(t)V_2(t)$ in asset 1 and short a notional of $\sigma_1(t)V_1(t)$ in asset 2. The cash exchange at time $t + dt$ is then equal to:

$$\sigma_2(t)V_2(t)\big(dV_1(t)-c(t)V_1(t)dt\big)$$
$$-\sigma_1(t)V_1(t)\big(dV_2(t)-c(t)V_2(t)dt\big)$$

which, after some manipulation, gives us:

$$\sigma_2(t)V_1(t)V_2(t)\big(\mu_1(t)-c(t)\big)dt$$
$$-\sigma_1(t)V_1(t)V_2(t)\big(\mu_2(t)-c(t)\big)dt$$

This amount is known at time t and, moreover, the contract can be terminated at $t + dt$ after the cashflow is paid at zero cost. Hence, the only way both parties agree to transact on this portfolio – in other words, for the economy to have no arbitrage – this cashflow must actually be zero, which gives us:

$$\sigma_2(t)\big(\mu_1(t)-c(t)\big)=\sigma_1(t)\big(\mu_2(t)-c(t)\big)$$

and, in particular:

$$\frac{\mu_1(t)-c(t)}{\sigma_1(t)}=\frac{\mu_2(t)-c(t)}{\sigma_2(t)}$$

Let us now define:

$$d\tilde{W}(t)\triangleq dW(t)+\frac{\mu_1(t)-c(t)}{\sigma_1(t)}dt$$

By the previous result, we also have:

$$d\tilde{W}(t)=dW(t)+\frac{\mu_2(t)-c(t)}{\sigma_2(t)}dt$$

Hence, we can rewrite (1) using the newly defined $d\tilde{W}$ as:

$$dV_i(t)=c(t)V_i(t)dt+\sigma_i(t)V_i(t)d\tilde{W}(t),\quad i=1,2 \tag{2}$$

Now, looking at (2) we see that there exists a measure, equivalent to the real-world one, in which both assets grow at rate $c(t)$. This is the analogue to the traditional risk-neutral measure. In this measure Q, the price process for each asset is given by:

$$V_i(t)=E_t^Q\left(e^{-\int_t^T c(s)ds}V_i(T)\right),\quad i=1,2 \tag{3}$$

Different collateral rates

Note that the two assets can be collateralised at different rates, c_1 and c_2, and the same result would apply. In particular, we would still have the condition:

$$\sigma_2(t)\big(\mu_1(t)-c_1(t)\big)=\sigma_1(t)\big(\mu_2(t)-c_2(t)\big)$$

from the cashflow analysis. Hence, the change of measure is still possible, and (3) still holds with $c(t)$ replaced by the appropriate c_1 or c_2:

$$V_i(t)=E_t^Q\left(e^{-\int_t^T c_i(s)ds}V_i(T)\right),\quad i=1,2 \tag{4}$$

In the stock option example of the previous section, the stock will grow at its repo rate and the option will grow at its collateral rate in the risk-neutral measure, consistent with the analysis of Piterbarg (2010).

Many collateralised assets

Let us now consider a general economy where we have more assets than sources of noise. In particular, assume that $N + 1$ collateralised (with the same collateral rate c) assets are traded, and their real-world dynamics are given by:

$$dV=\mu V\,dt+\Sigma dW$$

where dW is an N-dimensional Brownian motion. Here μ and $dV = (dV_1, \dots, dV_{N+1})^\top$ are column vectors of dimension $N + 1$, μV is understood as a vector with elements $\mu_i V_i$, $i = 1, \dots, N + 1$, and Σ is an $(N + 1) \times N$ matrix of full rank N. We can find a column vector of weights w of dimension $N + 1$ such that:

$$w^\top\Sigma=0 \tag{5}$$

Then the cash in the portfolio $w^\top V$ has no randomness and hence, by the no-arbitrage arguments used previously, we must have that:

$$w^\top(\mu V-cV)=0$$

Therefore the vector $\mu V - cV$ belongs to the N-dimensional subspace of vectors orthogonal to w. This sub-space also contains all columns of the matrix Σ by (5) and since they are linearly independent by the full rank assumption, the vector $\mu V - cV$ is spanned by the N columns of Σ. Hence, there exists an N-dimensional vector λ such that:

$$\mu V-cV=\Sigma\lambda$$

So we can write:

$$dV=cV\,dt+\Sigma\big(dW+\lambda\,dt\big)$$

and define the risk-neutral measure by the condition that $dW + \lambda\, dt$ is a driftless Brownian motion. In this measure, all processes V have drift c.

If we now consider the assets to be collateralised zero-coupon bonds, we obtain a model of interest rates that looks exactly like the standard Heath-Jarrow-Morton (HJM) model, except each collateralised zero-coupon bond grows at a – possibly maturity-dependent, though this is unusual – collateral rate $c(t)$. A zero-coupon bond is then given, in analogy with the corresponding HJM formula, by:

$$P(t,T)=E_t^Q\left(e^{-\int_t^T c(s)ds}\right)$$

Counterparty-specific collateral rates

The same asset can be collateralised with different rates when, for example, traded with different counterparties. Clearly, it will have two different price processes if the collateral rates are different, and so it should be actually thought of as two different assets.

Given that the price processes are different, this case is no different from that considered above. So (4) would still hold. For example, if the asset is a zero-coupon bond with maturity T collateralised with either rate c_1 or c_2, then the ratio of their prices under different collateral mechanisms would be given by:

$$\frac{P_1(t,T)}{P_2(t,T)}=\frac{E_t^Q\left(e^{-\int_t^T c_1(s)ds}\right)}{E_t^Q\left(e^{-\int_t^T c_2(s)ds}\right)}$$

Switching to the measure in which $P_2(t, T)$ is a martingale (which

[1] *Collateralised stock sale is actually a repo transaction. Here we assume that the repo rate is the same as the collateral rate. We consider different rates later*

we denote by $Q^{2,T}$), this ratio is given by:

$$E^{2,T}\left(e^{\int_t^T (c_2(s)-c_1(s))ds}\right)$$

Cross-currency model

The previous section gives a flavour of the results one gets for an asset collateralised with different rates, but probably the main example when this situation occurs is in cross-currency markets. According to London-based clearing house LCH.Clearnet's collateral rules, single-currency swaps are collateralised in the currency of the trade, while cross-currency swaps, when they start to be cleared, are likely to be collateralised in dollars. Clearly, having both types of swap leads to an economy where we must consider zero-coupon bonds collateralised in the domestic, as well as in some foreign currency. Collateralised zero-coupon bonds are not traded by themselves, but provide convenient fundamental building blocks for swaps collateralised in different currencies. We carefully develop such a model in this section.

■ **Domestic and foreign collateral.** Consider an economy with domestic and foreign assets and a forex rate $X(t)$ expressed as a number of domestic (denoted $\mathcal{D}$) units per one foreign (denoted $\mathcal{F}$). Suppose the only possible collateral types are the domestic currency with a given unique domestic collateral rate $c_d(t)$, and the foreign currency with a given unique foreign rate $c_f(t)$. Denote a domestic zero-coupon bond collateralised in domestic currency by $P_{d,d}(t, T)$. This bond generates the following cashflow at time $t + dt$:

$$dP_{d,d}(t,T) - c_d(t)P_{d,d}(t,T)dt \qquad (6)$$

Now consider a foreign zero-coupon bond collateralised with the domestic rate. Let its price, in foreign currency, be $P_{f,d}(t, T)$. We consider the cashflows to determine its price process from no-arbitrage arguments.

■ Purchase of the asset. The amount of $P_{f,d}(t, T)$ is paid (in foreign currency $\mathcal{F}$) by party A to B.

■ Collateral at t. Since A's mark-to-market is $P_{f,d}(t, T)$ in foreign currency, the amount $P_{f,d}(t, T)X(t)$ of collateral is posted in domestic currency $\mathcal{D}$ by B to A.

■ Return of collateral. At time $t + dt$, A returns collateral $P_{f,d}(t, T)X(t)\mathcal{D}$ to B.

■ Interest. At time $t + dt$, A also pays $c_d(t)P_{f,d}(t, T)X(t)dt$ interest to B in $\mathcal{D}$.

■ New collateral. The new mark-to-market is $P_{f,d}(t + dt, T)$. Party B pays $P_{f,d}(t + dt, T)X(t + dt)$ collateral to A in $\mathcal{D}$.

The cashflow, in $\mathcal{D}$, at $t + dt$ is:

$$d\left(P_{f,d}(t,T)X(t)\right) - c_d(t)P_{f,d}(t,T)X(t)dt \qquad (7)$$

■ **Drift of FX rate.** Equations (6) and (7) are insufficient to determine the drift of the forex rate $X(\cdot)$. From (7) we can only deduce the drift of the combined quantity $XP_{f,d}$ and the drift of $P_{f,d}$ is in general not c_f (nor it is c_d, for that matter). To understand the drift of $X(\cdot)$, we need to understand what kind of domestic cashflow we can generate from holding a unit of foreign currency. So suppose we have $1\mathcal{F}$. If it was a unit of stock, we could repo it out (that is, borrow money secured by the stock) and pay a repo rate on the stock. What is the equivalent transaction in the forex markets? Having $1\mathcal{F}$, we can give it to another dealer and receive its price in domestic currency, $X(t)\mathcal{D}$. The next instant $t + dt$ we would get back $1\mathcal{F}$, and pay back $X(t) + r_{d,f}(t)X(t)dt$, where $r_{d,f}(t)$ is a rate agreed on this domestic loan collateralised by foreign currency. As we can sell our $1\mathcal{F}$ for $X(t + dt)\mathcal{D}$ at time $t + dt$ the cashflow at $t + dt$ would be:

$$dX(t) - r_{d,f}(t)X(t)dt$$

It is not hard to see that the transaction is in fact an 'instantaneous' forex swap, with a real-life equivalent an overnight (or what is known as tom/next) forex swap.

Is there any relationship between the rate $r_{d,f}(t)$ and collateralisation rates in two different currencies? We contend that no, and the rate is independent of collateral rates in either currency.

■ **Cross-currency model under domestic collateral.** Let us summarise the instruments we have discussed so far and the cashflows they generate at time $t + dt$.

■ The market in instantaneous forex swaps allows us to generate cashflow $dX(t) - r_{d,f}(t)X(t)dt$.

■ The market in $P_{d,d}$ allows us to generate cashflow $dP_{d,d}(t, T) - c_d(t)P_{d,d}(t, T)dt$.

■ The market in $P_{f,d}$ allows us to generate cashflow $d(P_{f,d}(t, T)X(t)) - c_d(t)P_{f,d}(t, T)X(t)dt$.

Assuming real-world measure dynamics (μ, dW are vectors and Σ is a matrix):

$$\begin{pmatrix} dX/X \\ dP_{d,d}/P_{d,d} \\ d(P_{f,d}X)/(P_{f,d}X) \end{pmatrix} = \mu\, dt + \Sigma dW$$

by the same arguments as above (see *Different collateral rates*), we can find a measure ('domestic risk-neutral') Q^d under which the dynamics are:

$$\begin{pmatrix} dX/X \\ dP_{d,d}/P_{d,d} \\ d(P_{f,d}X)/(P_{f,d}X) \end{pmatrix} = \begin{pmatrix} r_{d,f} \\ c_d \\ c_d \end{pmatrix} dt + \Sigma dW^d \qquad (8)$$

In particular, we have:

$$X(t) = E_t^d\left(e^{-\int_t^T r_{d,f}(s)ds} X(T)\right)$$

$$P_{d,d}(t,T) = E_t^d\left(e^{-\int_t^T c_d(s)ds}\right) \qquad (9)$$

$$P_{f,d}(t,T) = \frac{1}{X(t)} E_t^d\left(e^{-\int_t^T c_d(s)ds} X(T)\right)$$

■ **Cross-currency model under foreign collateral.** We can consider the same model under foreign collateralisation. We would look at foreign bonds $P_{f,f}$ and domestic bonds collateralised in foreign currency $P_{d,f}$. By repeating the arguments above, we can find a measure Q^f under which:

$$\begin{pmatrix} d(1/X)/(1/X) \\ dP_{f,f}/P_{f,f} \\ d(P_{d,f}/X)/(P_{d,f}/X) \end{pmatrix} = \begin{pmatrix} -r_{d,f} \\ c_f \\ c_f \end{pmatrix} dt + \tilde{\Sigma} dW^f \qquad (10)$$

Note the drift of the first component is the rate $-r_{d,f}$, which is the rate on the instantaneous forex swap from the point of view of the foreign party. In particular:

$$P_{d,f}(t,T) = X(t) E_t^f\left(e^{-\int_t^T c_f(s)ds} \frac{1}{X(T)}\right) \qquad (11)$$

It is not hard to see the connection between Q^f and Q^d. In particular:

$$\left.\frac{dQ^f}{dQ^d}\right|_{F_t} = M(t) \triangleq e^{-\int_0^t r_{d,f}(s)ds} \frac{X(t)}{X(0)} \tag{12}$$

where the quantity:

$$M(t) = e^{-\int_0^t r_{d,f}(s)ds} \frac{X(t)}{X(0)}$$

is a normalised positive martingale under the domestic measure. Having this connection allows us to find the dynamics of $P_{f,f}$ under Q^d, for example.

Not all processes in (8) and (10) can be specified independently. In fact, with the addition of the dynamics of $P_{f,f}$ to (8), the model is fully specified, as the dynamics of $P_{d,f}$ can then be derived.

Comparing our setup with that of Fujii & Takahashi (2011), we can clarify the roles of risk-free rates that were introduced there. While by themselves they are superfluous for the development of a cross-currency model with collateralisation, their spreads have a concrete market interpretation – they are given by the rates quoted for instantaneous forex swaps and define the rate of growth of the forex rate.

■ **Forward forex.** Forward forex contracts are traded among dealers and, as such, are subject to collateralisation rules. A forward forex contract pays $X(T) - K$ at T in domestic currency. The price process of the domestic-currency-collateralised forward contract is given by:

$$E_t^d\left(e^{-\int_t^T c_d(s)ds}\left(X(T)-K\right)\right) = X(t)P_{f,d}(t,T) - KP_{d,d}(t,T)$$

and so the forward forex rate, that is, K that makes the price process have value zero is given by:

$$X_d(t,T) = \frac{X(t)P_{f,d}(t,T)}{P_{d,d}(t,T)} \tag{13}$$

Note that by switching to the measure associated with $P_{d,d}(\cdot, T)$ we get:

$$X_d(t,T) = E_t^{d,T}\left(X(T)\right)$$

so $X_d(\cdot, T)$ is a martingale under this measure.

We can also view a forward forex contract as paying $1 - K/X(T)$ in foreign currency. Then, with foreign collateralisation, the value would be:

$$E_t^f\left(e^{-\int_t^T c_f(s)ds}\left(1-K/X(T)\right)\right) = P_{f,f}(t,T) - KP_{d,f}(t,T)/X(t)$$

and the forward forex rate collateralised in c_f is given by:

$$X_f(t,T) = \frac{X(t)P_{f,f}(t,T)}{P_{d,f}(t,T)} \tag{14}$$

so that:

$$1/X_f(t,T) = E_t^{f,T}\left(1/X(T)\right)$$

In a general model, there is no reason why $X_f(t, T)$ would be equal to $X_d(t, T)$, and the forward forex rate would depend on the collateral used. It appears, however, that in current market practice forex forwards are quoted without regard for the collateral arrangements, essentially assuming that the cross-currency spread $q_{d,f}(t)$, defined in the next section, is deterministic or that its volatility is small enough to make no practical difference at liquidly observed maturities of forex forwards.

Forward forex rates are fundamental market inputs, so the formulas (13) and (14) are not, strictly speaking, required for pricing them. They are needed, however, for calibrating a model such as developed in the next section, as they are the source of information on the initial term structures $P_{f,d}$ and $P_{d,f}$ and, ultimately, on the expected values of the spot forex drift $r_{d,f}$.

A simple model for collateral choice

■ **Collateral choice.** Let us consider a domestic asset, with price process $V(t)$, that can be collateralised either in the domestic (rate c_d) or the foreign (rate c_f) currency. What is the price process of such an asset? From the analysis in the previous section, it follows that the foreign-collateralised domestic zero-coupon bond grows (in the domestic currency) at the rate $c_f + r_{d,f}$. It can be shown rigorously, through the type of cashflow analysis we have performed a few times in this article, that the same is true for any domestic asset. When one can choose the collateral, one would maximise the rate received on it, so the collateral choice rate is equal to:

$$\begin{aligned} &\max\left(c_d(t), c_f(t) + r_{d,f}(t)\right) \\ &= c_d(t) + \max\left(c_f(t) + r_{d,f}(t) - c_d(t), 0\right) \end{aligned} \tag{15}$$

The simplest extension of the traditional cross-currency model that accounts for different collateralisation would keep the spread:

$$q_{d,f}(t) \triangleq c_f(t) + r_{d,f}(t) - c_d(t)$$

deterministic. In this case, the collateral choice will not generate any optionality, although the discounting curve for the choice collateral rate will be modified (see Fujii & Takahashi, 2011). Anecdotal evidence at the time of writing suggests that at least some dealers do assign some value to the option to switch collateral in the future. So let us build a simple model that would give some value to the collateral choice option. The most technically straightforward extension of the standard cross-currency model would then involve specifying volatilities for the following objects: $P_{d,d}$, $P_{f,d}$, $P_{f,f}$ and X, and then proceeding to derive relevant drifts through the HJM-type calculations. In our view this is not particularly convenient as it would make it difficult to choose parameters in a way that would keep the spread $q_{d,f}$ deterministic, which is an important boundary case. So, instead, we will specify the dynamics of $P_{d,d}$, $P_{f,f}$, X and, importantly, the spread $q_{d,f}$ directly.

■ **Zero-coupon curves.** Before we start, let us discuss time-zero market data that the model needs to recover. We have the domestic-collateral, domestic-currency zero-coupon bonds $P_{d,d}(0, T)$ that can be obtained from the market on linear instruments in a single currency. We denote corresponding instantaneous forward rates by $p_{d,d}(0, T) = -\partial \log P_{d,d}(0, T)/\partial T$. Similarly we can build the 'pure foreign' discounting curve $P_{f,f}(0, T)$, $p_{f,f}(0, T)$. From the cross-currency swaps collateralised in the foreign currency (or from the forex forward market via (14)), we can obtain the foreign-collateral domestic zero-coupon bonds $P_{d,f}(0, T)$ and corresponding forward rates $p_{d,f}(0, T)$.

Note that we have from (11) and the measure change (12) that:

$$
\begin{aligned}
&P_{d,f}(t,T)\\
&= X(t)E_t^f\left(e^{-\int_t^T c_f(s)ds}\frac{1}{X(T)}\right) = E_t^d\left(e^{-\int_t^T (c_f(s)+r_{d,f}(s))ds}\right)\\
&= E_t^d\left(e^{-\int_t^T c_d(s)ds}e^{-\int_t^T q_{d,f}(s)ds}\right) = P_{d,d}(t,T)E_t^{d,T}\left(e^{-\int_t^T q_{d,f}(s)ds}\right)
\end{aligned}
$$

where the T-forward measure $Q_t^{d,T}$ corresponds to $P_{d,d}(t, T)$ being a numeraire. Hence, in the deterministic spread case, the time-zero curve $q_{d,f}(\cdot)$ will be given by the forward rate difference $p_{d,f}(0, T) - p_{d,d}(0, T)$.

■ **Dynamics.** We work under the domestic measure. Let the dynamics for $P_{d,d}(t, T)$ be given by:

$$dP_{d,d}(t,T)/P_{d,d}(t,T) = c_d(t)dt - \Sigma_d(t,T)dW_d(t) \quad (16)$$

Standard HJM machinery can be employed to obtain the dynamics of $c_d(t)$, which, in the simplest case, can be taken to be of the Hull-White form.

Let the forex rate dynamics be given by:

$$dX(t)/X(t) = r_{d,f}(t)dt + \Sigma_X(t)dW_X(t) \quad (17)$$

We will eventually be able to derive the dynamics for $r_{d,f}$ through that of c_d, c_f and $q_{d,f}$.

The dynamics of $P_{f,f}$ have the same form as (16) but under the foreign measure. Changing measure per (12) and using (17), we obtain:

$$
\begin{aligned}
&dP_{f,f}(t,T)/P_{f,f}(t,T)\\
&= c_f(t)dt - \Sigma_f(t,T)\left(dW_f(t) - \rho_{X,f}\Sigma_X(t)dt\right)
\end{aligned}
$$

where $\rho_{X,f}$ is the correlation between dW_X and dW_f. Again, the dynamics for c_f will follow from the standard HJM arguments.

Now let us decide on the dynamics of $q_{d,f}$. Recall:

$$Q_{d,f}(t,T) = E_t^{d,T}\left(e^{-\int_t^T q_{d,f}(s)ds}\right)$$

where $Q_{d,f}(t, T) \triangleq P_{d,f}(t, T)/P_{d,d}(t, T)$. Denoting the volatility of $Q_{d,f}(t, T)$ by $\Sigma_q(t, T)$ and using $W_q(t)$ as a driving Brownian motion under Q^d we can write down the dynamics of $Q_{d,f}(t, T)$ as:

$$
\begin{aligned}
&dQ_{d,f}(t,T)/Q_{d,f}(t,T)\\
&= q_{d,f}(t)dt - \Sigma_q(t,T)\left(dW_q(t) + \rho_{q,d}\Sigma_d(t,T)dt\right)
\end{aligned}
\quad (18)
$$

By standard calculations along the lines of Section 10.1 of Andersen & Piterbarg (2010), we obtain that $q_{d,f}(t)$ is given by:

$$
\begin{aligned}
q_{d,f}(t) &= p_{d,f}(0,t) - p_{d,d}(0,t)\\
&+\frac{1}{2}\int_0^t \frac{\partial}{\partial t}\left(\Sigma_q(s,t)^2\right)ds + \rho_{q,d}\int_0^t \frac{\partial}{\partial t}\left(\Sigma_q(s,t)\Sigma_d(s,t)\right)ds\\
&+\int_0^t \frac{\partial}{\partial t}\Sigma_q(s,t)dW_q(s)
\end{aligned}
$$

Choosing $\Sigma_q(t, T)$ to be of the standard deterministic mean-reverting form, $\Sigma_q(t, T) = \sigma_q(t)(1 - e^{-a_q(T-t)})/a_q$, we obtain the following model dynamics under the domestic risk-neutral measure Q^d:

$$
\begin{aligned}
dP_{d,d}(t,T)/P_{d,d}(t,T) &= c_d(t)dt - \Sigma_d(t,T)dW_d(t)\\
dP_{f,f}(t,T)/P_{f,f}(t,T)&\\
&= c_f(t)dt - \Sigma_f(t,T)\left(dW_f(t) - \rho_{X,f}\Sigma_X(t)dt\right)\\
dX(t)/X(t) &= \left(c_d(t) - c_f(t) + q_{d,f}(t)\right)dt + \Sigma_X(t)dW_X(t)\\
q_{d,f}(t) &= p_{d,f}(0,t) - p_{d,d}(0,t)\\
&+\frac{1}{2}\int_0^t \frac{\partial}{\partial t}\left(\Sigma_q(s,t)^2\right)ds\\
&+\rho_{q,d}\int_0^t \frac{\partial}{\partial t}\left(\Sigma_q(s,t)\Sigma_d(s,t)\right)\\
&+\int_0^t e^{-a_q(t-s)}\sigma_q(s)dW_q(s)
\end{aligned}
\quad (19)
$$

By setting $\sigma_q(\cdot) = 0$, we recover the deterministic-spread model. What is also convenient about this formulation is that it is symmetric, that is, all quantities change in an expected way when switching from the domestic to the foreign point of view, that is, $q_{f,d} = -q_{d,f}$, $r_{f,d} = -r_{d,f}$.

■ **Observations.** We make the following observations regarding the model (19). While writing the dynamics is relatively straightforward, using such a model in practice presents considerable challenges. There are a number of parameters that are simply not observable in the market, such as those for the process $q_{d,f}$ and various correlations. Even if statistical estimates could be used, hedging these parameters would be very difficult. Moreover, the option to switch collateral – which is ultimately the application of this model – could disappear for reasons unrelated to the model, such as a move to central clearing or a standard credit support annex. Moreover, there are doubts about whether an ability to instantaneously switch collateral from one currency to another is a good reflection of reality. On the other hand, it does give a way of getting some estimate for the option to switch collateral, and is derived in a rigorous way.

Another point worthy of note is that a model (19) is a model of discounting only. If one were to use it to price interest rate derivatives beyond those depending on discounting rates only, additional dynamics would need to be specified for forecasting curves. This can be done, for example, either for Libor forwards or for short rates that drive the forecasting curves. These can be specified as deterministic spreads to the collateral rates or, in full generality, modelled with their own stochastic drivers, further increasing the number of unobservable parameters. In the latter case, not only the discounting curves will depend on the collateralisation used, but also forecasting curves such as forward Libor curves will as well, in close analogy to the quanto-type adjustments obtained in Piterbarg (2010).

■ **Valuing a collateral choice option.** We proceed to look at the problem of collateral choice option valuation. Given an asset subject to collateral choice, its value at $t = 0$ is given by:

$$P_{d,d}(0,T)E^{d,T}\left(e^{-\int_0^T \max(q_{d,f}(s),0)ds}V(T)\right) \quad (20)$$

For an interest rate swap, say, $V(T)$ here will be either a constant (fixed leg cashflow) or a Libor rate fixing (floating leg cashflow). Let us consider the fixed leg first. Here we need to calculate:

$$E^{d,T}\left(e^{-\int_0^T \max(q_{d,f}(s),0)ds}\right) \quad (21)$$

There appears to be no closed-form expression for an option like this. However, given stringent computational requirements for a typical swaps trading system, using say a Monte Carlo or a partial differential equation method is rarely feasible, and a fast analytic approximation is required. By Jensen's inequality:

$$E^{d,T}\left(e^{-\int_0^T \max(q_{d,f}(s),0)ds}\right) \geq e^{-\int_0^T E^{d,T}\left[\max(q_{d,f}(s),0)\right]ds}$$

The integrand in the exponent on the right-hand side is T-dependent (through $E^{d,T}$ expectation), which means we have to re-evaluate the integrand terms for each T, slowing down calculations. It seems sensible to replace the expectations with $E^{d,s}[\max(q_{d,f}(s), 0)]$, allowing them to be calculated once for all T. Given small differences between the two, other more significant approximations involved and the uncertainty in market parameters, the trade-off seems justified. So we use a simple first-order approximation:

$$E^{d,T}\left(e^{-\int_0^T \max(q_{d,f}(s),0)ds}\right) \approx e^{-\int_0^T E^{d,s}\left[\max(q_{d,f}(s),0)\right]ds} \qquad (22)$$

Given that in the model (19) $q_{d,f}(s)$ is Gaussian, the required $E^{d,s}[\max(q_{d,f}(s), 0)]$ can be readily calculated in closed form. In practice, we would calculate it for a number of points s_i and interpolate in between. While (22) is only an approximation, at least for some values of market parameters it appears to be a good one (see below).

For the floating leg, a pragmatic choice would be to move the Libor fixing outside of the expected value, that is, replace (20) with:

$$P_{d,d}(0,T)E^{d,T}\left(e^{-\int_0^T \max(q_{d,f}(s),0)ds}\right)E^{d,T}\left(V(T)\right)$$

and proceed with (22).

Example. Here, we present a numerical example for collateral choice option valuation. We use data from November 2011 from Barclays. The domestic currency is the euro and the foreign currency is sterling. We use the following parameters for the process $q_{d,f}(\cdot)$, estimated historically: $\sigma_q = 0.50\%$ and $a_q = 40\%$. In figure 1, we plot a number of forward curves against time t in years. The curve labelled option forward is $p_{d,f}(0, t) - p_{d,d}(0, t)$, that is, the forward curve for the spread process $q_{d,f}(\cdot)$. The curve labelled option value (intrinsic) is the curve $\max(p_{d,f}(0, t) - p_{d,d}(0, t), 0)$. This would be the value of the collateral choice option assuming deterministic evolution of the spread $q_{d,f}(\cdot)$. The curve labelled option value (exp) is the true value calculated from (21) by Monte Carlo simulation, expressed in instantaneous forward rate terms:

$$\text{Option value}(\exp) = -\frac{\partial}{\partial t}\log E^{d,t}\left(e^{-\int_0^t \max(q_{d,f}(s),0)ds}\right)$$

Finally, the curve labelled option value (first order) is the first-order approximation from (22), $E^{d,t}[\max(q_{d,f}(t), 0)]$.

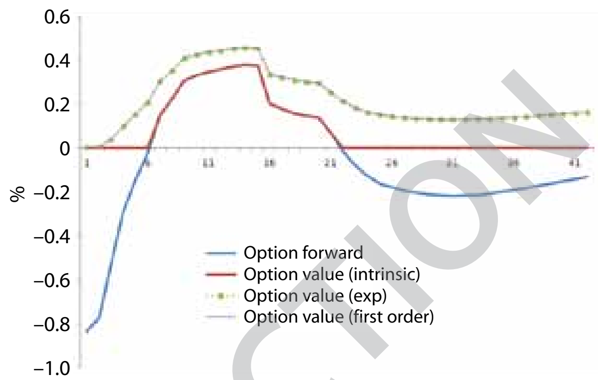
We see that the option value is not insignificant. We also see that the first-order approximation matches the true value of the option closely, at least for the values of the parameters used.

Conclusion

We have developed a framework for asset pricing in an economy where there is no risk-free rate and all transactions are collateralised. It turns out that much of the machinery of standard risk-neutral pricing theory can be reused, with a few changes. In the risk-neutral measure, each collateralised asset grows at the rate at which it is collateralised. The forex rate drift is not given by the difference of the risk-free rates in two currencies (as they do not exist in such an economy), but is given by a rate on an instantaneous forex swap, which is essentially an overnight repo rate on the sale of one unit of foreign currency for domestic price. Consequently, the forex rate drift is not dependent on the collateral rates in the two economies (domestic and foreign), but the forward forex rates are.

Furthermore, we demonstrated a simple model with stochastic dynamics for the difference between the forex-adjusted foreign collateral rate and the domestic collateral rate in which the option to switch collateral has time value, commented on the practical use of such a model and presented a numerical example. ■

1 Collateral choice forward, intrinsic and option values



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Correction

In *Funding beyond discounting: collateral agreements and derivatives pricing* (*Risk* February 2010, pages 97–102, *www.risk.net/1589992*) the author used $d\gamma(t)$, the total change in cash accounts, where he meant to use the growth of those accounts only, that is, exclusive of rebalancing. All instances of $d\gamma(t)$ should in fact refer to the total growth in cash positions $g(t)dt$, and the self-financing condition should read $g(t)dt = dV(t) - \Delta(t)dS(t)$. No further equation or any of the article's conclusions are affected. The article's online version has been adjusted accordingly. The technical editor and author are grateful to Damiano Brigo for pointing this out.