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LIKELIHOOD INFERENCE FOR NON-LINEAR, MULTIVARIATE JUMP DIFFUSIONS WITH STATE DEPENDENT INTENSITY

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AbstractJump diffusion processes provide a means of modelling both small and large deviations in continuously evolving processes. Unfortunately, the calculus of jump diffusion processes makes it difficult to analyse non-linear models. This paper develops a method for approximating the transition densities of time-inhomogeneous multivariate jump diffusions with state-dependent and/or stochastic intensity. By deriving a system of equations that govern the evolution of the moments of the process we are able to approximate the transitional density through a density factorisation that contrasts the dynamics of the jump diffusion with that of its purely diffuse counterpart. Within this framework we develop a class of quadratic jump diffusions for which we can calculate accurate approximations to the likelihood function. Subsequently, we analyse a number of non-linear jump diffusion models for Google equity volatility, alternating between various drift, diffusion and jump mechanism specifications. In doing so we find evidence of both cyclical drift and state dependent jump intensity.

Keywords: Jump Diffusion Process, Markov Chain Monte-Carlo, Kolmogorov Equation, Inference, Jump Stochastic Differential Equation.

1. INTRODUCTION

Allowing for discontinuous jump innovations in the paths of a diffusion process makes for extremely useful generalisation of diffusion processes. Indeed, many naturally occurring processes exhibit multifarious random behaviour which can be difficult to explain using a single source of randomness. Since the primary stochastic component to any diffusion model is governed by Brownian motion, diffusion models often fail to explain some salient features of real-world processes. The generalisation of diffusion processes to include randomly occurring "jump" innovations have been primarily motivated in financial contexts where diffusion models used to describe the dynamics of price/asset processes fail to explain seemingly spontaneous yet frequent large deviations in observed time series. For example, Merton (1976) proposed the inclusion of jumps in the diffusion trajectory in order to create a more realistic model of asset price returns than was predicted by the continuous paths of geometric Brownian motion – a jump free diffusion process that formed the basis of option pricing theory at the time. Although the inclusion of alternate sources of randomness in the path of a diffusion process serves to improve model flexibility and allows for the formulation of more realistic models of observed processes, this flexibility comes at the cost of magnifying the already significant difficulties associated with the calculus of diffusion

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processes. Consequently, the space of analytically tractable jump diffusion models is even more sparse than that of the jump free diffusion processes. Furthermore, where analytical solutions to quantities such as the transitional density are available, they are often precluded by simplifying assumptions on the specification of both the diffuse part of the process as well as the distributional properties of the jump mechanism of the process. That said, a number of different jump mechanisms have been proposed for jump diffusion models that appear in the literature: Ball and Torous (1985) propose log-normally distributed jumps under geometric Brownian motion as a model for stock price returns, whilst Ramezani and Zeng (1998), Kou (2002) and Kou and Wang (2004) assumed that jump innovations follow a double-exponential distribution under geometric Brownian motion. Although the choice of distribution is usually based on some *a priori* information of the process to be modelled, choosing a valid jump distribution can be a subtle process. For example, in the case of Ball and Torous (1985) it is actually meant that the log of the underlying process has normally distributed jumps (i.e., Brownian motion with drift and normally distributed jumps), implying both the diffuse and jump dynamics are based on the Normal distribution. As such Honore (1998) note that, depending on the quality of the data, it can be difficult to distinguish which source of randomness is responsible for a random innovation in the underlying process, thus making it difficult to calculate reliable parameter estimates for such a model.

Despite the limited set of analytically tractable jump diffusion models, numerous estimation techniques have been proposed for jump diffusion models with analytically intractable dynamics. Craine et al. (2000) apply indirect inference procedures to estimate the parameters of multivariate jump diffusion models. Eraker (2001) apply Monte Carlo techniques, replacing missing sample paths with simulated trajectories (see also Eraker et al. (2003) and Eraker et al. (2003)) in order to estimate the likelihood, thus circumventing the analytical threshold for non-linear jump diffusions. Other notable approaches include the efficient method of moments (EMM) scheme of Gallant et al. (1997), and the empirical characteristic function estimation schemes of Jiang and Knight (2002) and Rockinger and Semenova (2005). Yu (2007) extended the popular Hermite series approximations for jump free diffusions (Ait-Sahalia, 2002, 2008) to include closed-form likelihood approximations for multivariate jump diffusions. Although most of the literature on the estimation of jump diffusion models are concerned with parametric inference, non-parametric techniques have been developed by Johannes (1999); Bandi and Nguyen (2003) and Ait-Sahalia et al. (2009). Although the structure of jump diffusion models remain more or less consistent throughout the literature, Aıt-Sahalia (2004); Aït-Sahalia et al. (2009) and Aït-Sahalia and Jacod (2011) explore technical aspects regarding the nature of jump mechanisms in the context of inference.

In the present paper we develop a procedure for performing likelihood based inference on a class of non-linear, multivariate jump diffusion processes with state dependent intensity. Using this it is possible to create a rich ecosystem of jump diffusion models that generalize many well-known diffusion models such as the Cox-Ingersoll-Ross (CIR) process (Cox et al., 1985) and Ornstein-Uhlenbeck process (Uhlenbeck and Ornstein, 1930) to the jump diffusion class. Furthermore, the methodology readily allows for the specification of any jump distribution with a known moment structure (e.g. the higher order moments of a normal distribution). The paper is organised as follows: Section 2 outlines theoretical concepts which precede the methodology to follow. Section 3 develops the core methodology of the paper, in which we detail a scheme for approximating the transitional density of a jump diffusion process based on its moment trajectories. Section 5 demonstrates how the methodology can be used to conduct inference on jump diffusion models with non-linear dynamics. In Section 6 we apply the methodology to a real-world dataset by fitting various jump diffusion models to Google equity volatility time series. Finally we give some concluding remarks in Section 7.

2. MULTIVARIATE JUMP DIFFUSIONS WITH STATE DEPENDENT JUMP INTENSITY

Let $\{S \subseteq \mathbb{R}^k, \mathcal{X}, f\}$, $\{\Psi_j, \mathcal{Z}_j, \phi_j\}_j$ and $\{\Omega_i, \mathcal{L}_i, \pi_i\}_i$ be probability spaces for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, q$ then a multivariate, k-dimensional jump process can be written in differential form as:

$$d\mathbf{P}_t = \mathbf{J}(\mathbf{P}_t, \dot{\mathbf{z}}_t, t)d\mathbf{N}_t, \tag{2.1}$$

where $\mathbf{J}(\mathbf{P}_t, \dot{\mathbf{z}}_t, t) = (\epsilon_{ij}(\mathbf{P}_t, \dot{\mathbf{z}}_t, t))_{k\times q}$ denotes a jump matrix, $\dot{\mathbf{z}}_t$ is a $k \times q$ matrix of random variables and $\mathbf{N}_t = (N_t^{(j)})_{q\times 1}$ is a q-dimensional counting process with intensity vector $\boldsymbol{\lambda}(\mathbf{P}_t, \dot{\mathbf{r}}_t, t) = (\lambda_i(\mathbf{P}_t, \dot{\mathbf{r}}_t, t))_{k\times 1}$. Under this formulation, the jump matrix relates discrete increments in the process $\mathbf{N}_t$ into continuous innovations in the process $\mathbf{P}_t$. These are in turn in turn determined by the realisations of the random variables $\dot{\mathbf{z}}_t$, and may also depend on the state of the process at the instant that the innovation occurs. Furthermore, the rate at which these innovations occur is determined by the intensity vector, which is further allowed to depend on an external process $\dot{\mathbf{r}}_t$ and the level of the process. For example, if the first element of the process $\mathbf{N}_t$ increments at time τ, then every element of $\mathbf{P}_\tau$ changes in accordance to the outcome of the first column of the jump-matrix, which in turn is determined by the functional relation between the first column of jump variables in $\mathbf{z}_\tau$ and $\mathbf{P}_\tau$. To see this, it is useful to write the process $\mathbf{P}_t = \{P_t^{(1)}, P_t^{(2)}, \ldots, P_t^{(k)}\}'$ in terms of its individual components as:

$$P_t^{(i)} = \sum_{j=1}^{q} \sum_{l=0}^{N_j(t)} \epsilon_{ij}(\mathbf{P}_t, \dot{\mathbf{z}}_t^{(.j),l}, t) \quad \text{for} \quad i = 1, 2, \ldots, k. \tag{2.2}$$

where $\dot{\mathbf{z}}_t^{(.j),l}$ denotes the l-th realisation of the j-th column of the jump variable matrix where $\dot{\mathbf{z}}_t^{(.j)} \in \dot{\mathbf{z}}_t : \{\dot{z}_{1j}(t), \dot{z}_{2j}(t), \ldots, \dot{z}_{kj}(t)\}' \sim \phi_j(t)$ and $\mathbf{P}_t$ evolves