Project Report

Price Bond Options by Simulating Short Rates
Based on CIR and G2++ Models with GPU

Ping Fan

pf905@nyu.edu

Advisor: Louis Scott

New York University

1 Objective

I would like to simulate instantaneous-short-rate under a CIR Model and two-factor additive Gaussian model separately, and then price a European call option written on a zero-coupon bond. Specifically, consider a G2++ short rate model, which is equivalent to 2-factor Hull and White model (1994). Firstly, calibrate the parameters approximately, comparing the discount factors provided by Bloomberg with analytical formulas for discount factors. Then, perform the simulation by Monte Carlo Simulation under risk neutral measure with CudaC. Lastly, obtain the average European bond options prices. I run my code with GPU in the Bloomberg lab.

2 Introduction

If simulate future interest rates, an interest-rate model must be used. One choice is a one-factor model and another choice is two-factor model.
Denote $P(t,T)$, the price at time t of a zero-coupon with maturity T with unit face value, derived from the model as

$$P(t,T) = \mathbf{E}_t\left[D(t,T)\right] = \mathbf{E}_t\left[exp(-\int_t^T r_s ds)\right]$$

where $D(t,T)$ is the discount factor. We know that $D(t,T) = exp(-\int_t^T r_s ds)$, and when r_s is deterministic, $D(t,T) = P(t,T)$.

The payoff of European call option is $[P(T,S) - K]^+$, where S and T are the maturity of bond and call respectively. And K is the strike price. Therefore, the option value at time t is

$$\begin{aligned} V(t) &= \mathbf{E}^Q\left[D(t,T)(P(T,S)-K)^+|\mathcal{F}_t\right] \\ &= P(t,T)\mathbf{E}^T\left[(P(T,S)-K)^+|\mathcal{F}_t\right] \end{aligned}$$

When pricing with Monte Carlo, it is advisable to use a single measure to price all the cash flows of a given derivative. Therefore, I use risk neutral measure to price the option rather than T-forward measure. If under T-forward measure, I need to have different SDE to simulate paths and then different formulas to price options.

2.1 CIR Model

Cox, Ingersoll, and Ross(1985) introduced a "square-root" term in the dynamics of r under the risk-adjusted measure Q. The CIR Model is:

$$dr(t) = k(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t), r(0) = r_0$$

where W is a Brownian motion, and r_0, k, θ, σ are positive constants. r_t is the short rate, a is mean reversion constants, σ are volatility constants. This short-rate process is always positive if $2k\theta > \sigma^2$.

The analytical expression of price at time t of a zero-coupon bond with maturity T is

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$
$$A(t,T) = \left[\frac{2h\exp\{(k+h)(T-t)/2\}}{2h+(k+h)(\exp\{(T-t)h\}-1)}\right]^{2k\theta/\sigma^2}$$
$$B(t,T) = \frac{2(\exp\{(T-t)h\}-1)}{2h+(k+h)(\exp\{(T-t)h\}-1)}, \quad h = \sqrt{k^2+2\sigma^2}$$

The payoff of European call option is $[P(T,S)-K]^+$, where S and T are the maturity of bond and call respectively. And K is the strike price. Therefore, the option value at time t is

$$\begin{aligned}&\mathrm{ZBC}(t,T,S,X)\\ &= P(t,S)\,\chi^2\left(2\bar{r}[\rho+\psi+B(T,S)]; \frac{4k\theta}{\sigma^2}, \frac{2\rho^2 r(t)\exp\{h(T-t)\}}{\rho+\psi+B(T,S)}\right)\\ &\quad - XP(t,T)\,\chi^2\left(2\bar{r}[\rho+\psi]; \frac{4k\theta}{\sigma^2}, \frac{2\rho^2 r(t)\exp\{h(T-t)\}}{\rho+\psi}\right)\end{aligned}$$

where X is strike price and

$$\rho = \rho(T-t) := \frac{2h}{\sigma^2(\exp[h(T-t)]-1)}$$
$$\psi = \frac{k+h}{\sigma^2}, \quad \bar{r} = \bar{r}(S-T) := \frac{\ln(A(T,S)/X)}{B(T,S)}$$

2.2 G2++ Model

A two-factor additive Gaussian model is referred to as G2++ by Brigo and Mercurio[2]. The G2++ Interest Rate Model under the risk-adjusted measure Q is:

$$dr_t = \varphi_t + x_t + y_t, r(0) = r_0$$

where x_t and y_t are defined by

$$dx_t = -ax_t dt + \sigma dW_t^1$$
$$dy_t = -by_t dt + \eta dW_t^2$$

where (W1,W2) is a two-dimensional Brownian motion with instantaneous correlation ρ as from $dW_t^1 dW_t^2 = \rho dt$, and r_0, a, b, σ, η are positive constants. r_t is the short rate, a and b are mean reversion constants, σ, η are volatility constants. The function φ deterministic and well defined in the time interval $[0, T^*]$, with T^* a given time horizon, typically 10, 30 or 50 (years). In particular, $\varphi(0) = r0$. When fitting the initial zero-coupon curve, φ_T is given by

$$\varphi_T = f^M(0,T) + \frac{\sigma^2}{2a^2}(1-e^{-aT})^2 + \frac{\eta^2}{2b^2}(1-e^{-bT})^2 + \rho\frac{\sigma\eta}{ab}(1-e^{-aT})(1-e^{-bT})$$

where $f(0,T)^M$ is the market forward rate, or the forward rate observed on the Settle date.

The dynamics of the processes x and y can be also expressed in terms of two independent Brownian motions $\widetilde{W}^1$ and $\widetilde{W}^2$ as follows:

$$dx_t = -ax_t dt + \sigma d\widetilde{W}_t^1$$
$$dy_t = -by_t dt + \eta\rho d\widetilde{W}_t^1 + \eta\sqrt{1-\rho^2}\,d\widetilde{W}_t^2$$

The analytical expression of price at time t of a zero-coupon bond with maturity T is

$$P(t,T) = A(t,T)exp\{-B(a,t,T)x(t) - B(b,t,T)y(t)\}$$
$$A(t,T) = \frac{P^M(0,T)}{P^M(0,t)}exp\left\{\frac{(t,T-V(0,T)+V(0,t))}{2}\right\}$$
$$B(t,T) = \frac{\left[1-e^{-z(T-t)}\right]}{z}$$

where

$$\begin{aligned} V(t,T) = &\frac{\sigma^2}{a^2}\left[T-t+\frac{2}{a}e^{-a(T-t)}-\frac{1}{2a}e^{-2a(T-t)}-\frac{3}{2a}\right] \\ &+\frac{\eta^2}{b^2}\left[T-t+\frac{2}{b}e^{-b(T-t)}-\frac{1}{2b}e^{-2b(T-t)}-\frac{3}{2a}\right] \\ &+2\rho\frac{\sigma\eta}{ab(a+b)}\left[T-t+\frac{e^{-a(T-t)}-1}{a}+\frac{e^{-b(T-t)}-1}{b}-\frac{e^{-(a+b)(T-t)}-1}{a+b}\right] \end{aligned}$$

The analytical expression of price at time t of a European call option with maturity T and strike K, written on a zero-coupon bond with unit notional and maturity S is given by

$$ZBC(t,T,S,K) = P(t,S)\varphi(\frac{ln\frac{P(t,S)}{KP(t,T)}}{\sum(t,T,S)}+\frac{1}{2}\sum(t,T,S)) - P(t,T)K\varphi(\frac{ln\frac{P(t,S)}{KP(t,T)}}{\sum(t,T,S)}-\frac{1}{2}\sum(t,T,S))$$

where

$$\begin{aligned} \sum(t,T,S)^2 = &\frac{\sigma^2}{2a^3}\left[1-e^{-a(S-T)}\right]^2\left[1-e^{-2a(T-t)}\right] \\ &+\frac{\eta^2}{2b^3}\left[1-e^{-b(S-T)}\right]^2\left[1-e^{-2b(T-t)}\right] \\ &+2\rho\frac{\sigma\eta}{ab(a+b)}\left[1-e^{-a(S-T)}\right]\left[1-e^{-b(S-T)}\right]\left[1-e^{-(a+b)(T-t)}\right] \end{aligned}$$

One limitation to two-factor Gaussian models like G2++ is that it does permit negative interest rates. And We may lose some insights and intuitions on the nature and the interpretation of the two factors.

3 Experiment and Algorithm

I calculate the price of a 2 year European option on zero-coupon bond that will mature in 3 years. The principal of the bond is 100 and the strike price of the option is 96. That is, $T = 2, S = 3, K = 96$.

3.1 Data

I pull zero rates and discount rates on May 11, 2018 from Bloomberg. The rates are calculated from swap rates by Bloomberg based on a time point. I use discount rates directly as a proxy of zero-coupon bond with principal of 1.

3.2 Parameter Calibration

Based on the analytical formula of zero-coupon bond, I get the theoretical values. Compare theoretical values and market prices of zero-coupon bond using least square method, and then obtain the reasonable parameters. The estimated parameters might have collinearity. This is done in Python.

For the CIR Model, set $a = 0.5, b = 0.35, \sigma = 0.09, \rho = 0.8, \eta = 0.08$. Let the initial short-term interest rate $r0 = 0.01705$ according to the overnight Libor rate on May 11, 2018.

For the G2++ Model, I tried to calibrate parameters but failed. So here I just use some reasonable parameters. Set $x0 = 0.01, y0 = 0.00705, a = 5, b = 0.35, \sigma = 0.15, \rho = -0.9, \eta = 0.08$.

3.3 Prediction with Monte Carlo Simulation

I implement Monte Carlo Simulation to generate paths in CudaC. The steps are as follows. For CIR Model, check if the parameters meet $2k\theta > \sigma^2$ in advance. In addition, I assume 365 days in each year. At first, I tried 252 days, but got very different results from 365 days.

First of all, generate many interest rate paths based on the model. Regards to G2++ Model, for every point in a path, we need to generate 2 independent normal random variables. Meanwhile, for CIR Model, generate 1 normal random variable for each time point in a path. Actually, generating random normal variables can be done within each path or in main function. Here I generate a list of normal variables in main function using curand. Then, pass normal variables into the kernel function.

Second, for each path, compute the corresponding discount factor $D(t,T)$ and zero-coupon bond price $P(T,S)$. Compute $P(T,S)$ based on the corresponding analytical formula regards to the model. $D(t,T)$ is the integration from current time to the maturities of option. Use left Riemann sum to integrate r_s as

$$D(t,T) = exp(-\int_t^{T_k} r_s ds) = exp(-\sum_{i=1}^{n} r_{t_{i-1}}(t_i - t_{i-1}))$$

Third, for each path, calculate the European option price as $D(t,T)(P(T,S) - K)^+$.

Last, in main function, sum over European option prices of all paths and then average them. The option value at time t is obtained.

4 Performance Evaluation

The accuracy of the simulation will be addressed through the prices using Monte Carlo simulation and the prices computed through analytical formulas. When the two prices of European bond options are close, the results of Monte Carlo methods are good and converge to its analytical result.

The simulated call price of CIR model is 1.78, and put price is 0.078. As the analytical formula of call price contains the chi squared distribution, I failed to find references about whether it is cdf or pdf of chi square. In this way, I am suspicious that I did not get the correct analytical price. What I get is 0.956.

The simulated call price of G2++ model is 2.87, and put price is 2.15. The analytical call price of G2++ model is 2.89, and thus the error between analytical price and simulated price is 0.005.

The results of 2 models are very different as the parameters of G2++ is just estimated. At least, the call price calculated by G2++ is good based on the estimated parameters.

5 Conclusion

I can also check whether the simulation results are reasonable by simulating under another measure like the T-forward measure. More work related to calibration of G2++ model and the calculation of analytical formula for CIR Model is expected in future.

6 Reference

[1] John C. Hull. (2014) *Options, Futures and Other Derivatives.* 9th Edition.
[2] Brigo D. & Mercurio F. (2001) *Interest Rate Models: Theory and Practice.* 2nd edition.