Maximum Entropy Prediction Markets

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In this paper, we draw connections between the aggregation performed by learning algorithms and the information aggregation done in prediction markets. We show that, under reasonable conditions, the behavior of rational traders can be understood as the result of performing a learning algorithm on their private data. Similarly, the market state can be interpreted as a distribution over the outcome space. In particular, we show that a proper scoring rule can be derived from maximum entropy distributions. This scoring rule can be used as a general form of LMSR in prediction markets with over continuous outcome spaces. In order to provide insight on the behavior of rational traders in the market, we use the concept of exponential utility. We show that the traders' behavior can be understood as updating his belief using a Bayesian process and updating the market state in accordance with this utility function. These maxent prediction markets can also be used to design markets that are robust against adversarial traders. In fact, when traders are required to report their budgets and their beliefs, we can show that an informative trader eventually makes money and damaging traders eventually have limited influence in the market. Using ideas from convex analysis and the properties of the prediction market, we analyze the properties of the maxent market maker thus providing insight into the information content of the prediction market.

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1. INTRODUCTION

We have dual goals in this paper. On the one hand, we highlight a structural similarity between prediction markets and exponential families. We see this as the syntax of our prediction market mechanism. On the other, this formulation has rich semantics as well: it allows for analysis of market behavior under various environments. For

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instance, we analyze market behavior with budget limited traders, traders that are exponential-utility-maximizers.

2. NOTATION AND DEFINITIONS

convex conjugates, bregman divergences, scoring rules, sufficient statitics, exp families, Bayesian updates, conjugate prior.

3. MAXIMUM LIKELIHOOD SCORING RULES

3.1. Preliminaries

We consider a state space $\mathcal{X}$ which may be discrete or continuous, and a measure ν over sets of states. We will typically take the state space to be $\mathbf{R}$ or $\mathbf{R}^d$ together with the Lebesgue measure (perhaps restricted to some subset of states), or some finite set together with the counting measure.[1] A *probability density* is a ν-measurable function $p : \mathcal{X} \to \mathbf{R}_+$ that integrates to 1 over $\mathcal{X}$.

Let $\mathcal{P}$ denote the set of all probability densities over the given measure space. We are interested in eliciting information about some unknown density $p \in \mathcal{P}$ which corresponds to the beliefs of an expert. The expert may be familiar with p in its entirety, or only with certain properties of p. Given a set of densities $\mathcal{D} \subseteq \mathcal{P}$, a *property* is a real (possibly vector-valued) function $\Gamma : \mathcal{D} \to \mathbf{R}^k$. With a slight abuse of terminology we call elements of its image $\Theta = \Gamma(\mathcal{D})$ *properties* and we say that p has property θ when $\Gamma(p) = \theta$. If Γ is one-to-one, we instead call Γ a *parametrization* and refer to the elements of its image Θ as *parameters*.

To be concrete, our outcome space could for instance be $\mathbf{R}$ with ν the Lebesgue measure restricted to $[0, +\infty)$. The set $\mathcal{P}$ then consists of all probability densities with support contained in the non-negative reals, and $\mathcal{D} \subset \mathcal{P}$ might consist of the densities of exponential distributions. An example of a property over $\mathcal{P}$ is the mapping from a density to its mean.[2] When restricted to the exponential densities $\mathcal{D}$, this mapping is a parametrization. In much of this work, as in this example, the purpose of the measure ν is to implicitly encode bounds on the support of possible densities. The reason we use this approach, rather than explicitly stating bounds, is that it allows for clearer connections with maximum entropy estimation later on.

Given the set of relevant densities $\mathcal{D} \subseteq \mathcal{P}$, which contains the expert's belief p, and a property Γ over this set, the expert is asked to report $\theta = \Gamma(p)$. The expert is rewarded according to a *scoring rule* $S : \Theta \times \mathcal{X} \to \mathbf{R} \cup \{-\infty\}$ which pays it $S(\hat{\theta}, x)$ according to how well its report $\hat{\theta} \in \Theta$ agrees with the eventual outcome $x \in \mathcal{X}$. The idea is to design the scoring rule so that the expert is incentivized to report its true belief, setting $\hat{\theta} = \theta$.

Definition 3.1. Given a property $\Gamma : \mathcal{D} \to \mathbf{R}^k$, a scoring rule S is *proper* over domain $\mathcal{D}$ if for each $\theta \in \Theta$ and $p \in \mathcal{D}$ with property θ, we have

$$\mathbf{E}_p[S(\theta, x)] > \mathbf{E}_p[S(\hat{\theta}, x)] \tag{1}$$

[1]We will not need to make reference to the underlying σ-algebra, but to complete the specification of the measure space we take it to be the collection of Borel sets in the case of $\mathbf{R}$ and $\mathbf{R}^d$, and the power set in the case of a discrete set.

[2]The property may not be defined for all $p \in \mathcal{P}$. In this case we implicitly restrict it to the subset of $\mathcal{P}$ for which it is defined whenever its domain $\mathcal{D}$ is not made explicit.

for all $\hat{\theta} \neq \theta$. If the domain is $\mathcal{D} = \mathcal{P}$, the set of all possible densities, then we say the scoring rule is *strongly proper*.

— Note that the inequality is strict, in the literature these rules are called *strictly* proper, but we only consider strictly proper rules here so don't make this qualification.

— Lambert et al.'s rules are actually strongly proper, however we consider scoring rules over vector valued properties. Lambert et al only develop these by adding together different scoring rules for each property. We go further in our constructions and develop rules that are not necessarily separable.

— It is useful to consider rules that are not strongly proper because we might have constructions for these for properties where no strongly proper rule is known.

— It is important to understand the informational requirements placed on the expert. Under a proper scoring rule, the expert does not need to be aware of the entire density p to be incentivized to report its property θ, since θ is an optimal report no matter what the density might be. The agent only needs to agree that the density indeed lies in $\mathcal{D}$.

— For instance, if we are considering densities with support on $[0, +\infty)$, and $\mathcal{D}$ is the set of lognormal densities, then a proper scoring rule over $\mathcal{D}$ for the mean does not require the agent to know anything beyond the mean (e.g., it does not need to know the variance).

— A strongly proper scoring rule goes further; with such a rule we do not require the expert to believe the density is drawn from some subset $\mathcal{D}$, only that it agrees with the support implied by the base measure ν. Therefore, a strongly proper scoring rule for the mean here would elicit the mean no matter what kind of density over the positive reals might apply (e.g., exponential, Pareto, lognormal).

— In practical scenarios this seems like an important extension, because while it might be clear which states are possible, it might be difficult to ensure the expert agrees that the probability density for states falls within some restricted family $\mathcal{D}$.

3.2. Maximum Likelihood

THEOREM 3.2. *The logarithmic scoring rule defined by*

$$S(\theta, x) = \log p(x; \theta)$$

is weakly proper for $\mathcal{F}$ over Θ if and only if the maximum likelihood estimate for θ is consistent. Furthermore, if $\mathcal{M}$ is an alternative parameter space such that there exists a bijection $t : \mathcal{M} \to \Theta$, then $\bar{S}(\mu, x) = S(t(\mu), x)$ is weakly proper for $\mathcal{F}$ over $\mathcal{M}$.

Example 3.3. Suppose X is supported on $[0, +\infty)$ and follows an exponential distribution with density $f(x; \lambda) = \lambda e^{-\lambda x}$ parametrized by a rate $\lambda > 0$. The mean μ is related to the rate via the one-to-one mapping $\mu = 1/\lambda$, so the density can alternatively be parametrized by the mean. According to Theorem **??** the following is a weakly proper scoring rule for the mean of an exponential distribution:

$$S(\mu, x) = -\frac{x}{\mu} - \log \mu. \tag{2}$$

To verify this, suppose the agent believes X follows an exponential distribution with rate λ_0, equivalently mean $\mu_0 = 1/\lambda_0$. By the one-to-one relation between rate and mean, choosing μ to maximize the expected score (**??**) is equivalent to choosing λ to maximize $\mathbf{E}-\lambda x + \log \lambda = -\lambda\mu_0 + \log \lambda$. The latter is strictly concave in λ, and using basic calculus its unique global maximum is $\lambda^* = 1/\mu_0$, which therefore leads the agent to report $\mu = \mu_0 = 1/\lambda^*$ under scoring rule (**??**).

Example 3.4. Suppose that X is supported on $[1, +\infty)$ and follows a Pareto distribution with density $f(x; \alpha) = \alpha/x^{\alpha+1}$ parametrized by an index $\alpha > 0$. The mean μ is related to the index via the one-to-one mapping $\mu = \frac{\alpha}{\alpha-1}$, so the density can alternatively be parametrized by the mean. Theorem **??** gives the following weakly proper scoring rule for the mean of a Pareto distribution with support on $[1, +\infty)$:

$$S(\mu, x) = \log \frac{\mu}{\mu - 1} - \left(\frac{\mu}{\mu - 1} + 1 \right) \log x. \tag{3}$$

To verify this, suppose the agent believes X follows a Pareto distribution with index α_0, equivalently mean $\mu_0 = \frac{\alpha_0}{\alpha_0-1}$. By the one-to-one relation between the index and mean, choosing μ to maximize the expected score (**??**) is equivalent to choosing α to maximize $\mathbf{E}\log \alpha - (\alpha + 1) \log x = \log \alpha - (\alpha + 1)\mathbf{E}\log x$. The latter is strictly concave in α, and using basic calculus its unique global maximum is $\alpha^* = 1/\mathbf{E}\log X = \alpha_0$, where the last equality follows from the fact that $\log X$ has an exponential distribution with rate α when X is Pareto with rate α. Therefore (**??**) leads the agent to report $\mu = \mu_0 = \frac{\alpha^*}{\alpha^*-1}$.

— Explain that the scoring rule from Example 2 does not elicit the mean when the agent's distribution is exponential (with change of variable so that it lies in $[1, +\infty)$.
— Observe that scoring rule from Example does elicit the mean when the agent's distribution is exponential, Pareto, or in fact any other distribution with support on $[0, +\infty)$ such as the lognormal.
— This occurs because the scoring rule is linear in x, so its expectation is the same for any distribution with mean μ.
— Another feature to note is that under an appropriate parametrization (rate instead of mean), the scoring rule (1) is convex in the parameter being elicited.
— These two observations taken together lay the groundwork for the characterization of strongly proper scoring rules for distribution statistics.
— Before we proceed should note that the arguments behind Theorem **??** apply not just to maximum likelihood but to any *m-estimator*, which are notions from robust statistics. Explain what an m-estimator looks like and how there are general consistency results for those as well.
— The point of these alternative estimators is to be more robust to outliers in the sample, and so these kinds of estimators could translate into scoring rule more robust to inaccuracies in the agent's assessment of the underlying probability distribution or its parameters. (Just a hunch.)

4. MAXIMUM ENTROPY MARKET MAKING

The purpose of a prediction market is to elicit and aggregate the beliefs (i.e., subjective probabilities) of agents over a space of *outcomes*. In a single-agent setting, a scoring rule is used to elicit the agent's beliefs. In a multi-agent setting, an information market is used to aggregate the agent's beliefs. Hanson [] introduced the idea of a market scoring rule, which inherits the appealing elicitation and aggregation properties of both so that they can perform well in both thin and thick markets. In this work we derive a wide-ranging generalization of the logarithmic market scoring rule. Using a maximum entropy approach, we explain how a market scoring rule can be developed for outcome spaces both discrete and continuous, and for generic properties of the underlying distribution (e.g., mean and variance), rather than just the probabilities of individual outcomes.

4.1. The Model

Let $\mathcal{X}$ be the outcome space, which may be discrete or continuous. Let $\mathcal{P}$ denote the set of probability distributions over the outcome space. We represent a probability distribution as a density p absolutely continuous with respect to some base measure ν (e.g., counting for discrete outcomes, or Lebesgue for continuous outcomes).

There is an unknown distribution p over outcomes. We are interested in estimating the expected value of different outcome *statistics* under this distribution by aggregating the beliefs of agents. A statistic is a real-valued function $\phi_s : \mathcal{X} \to \mathbf{R}$ over outcomes, independent of p; here s belongs to a finite index set $\mathcal{S}$ of size d. The collection of functions $\phi = (\phi_s,\ s \in \mathcal{S})$ can be viewed as a vector-valued statistic mapping outcomes into $\mathbf{R}^d$. Formally, the aim is to estimate

$$\mathbf{E}_p[\phi(x)] = \mu. \tag{4}$$

We stress that we are only seek to elicit $\mu \in \mathbf{R}^d$, not the full distribution p. As an example, suppose that $\mathcal{X} = \mathbf{R}$. If we were interested in eliciting the first two uncentered moments of p, we would include $\phi_1(x) = x$ and $\phi_2(x) = x^2$ as statistics. Note that the variance cannot be directly obtained through any single statistic: it corresponds to the expectation of $(x - \mathbf{E}_p[x])$, which depends on p, violating our definition of a statistic. Of course, the variance can be indirectly obtained by eliciting the first two moments separately.[3]

There are two approaches to eliciting an agent's estimated μ in (**??**). The first approach is to incentivize the agent to directly report μ via a *scoring rule*. The second approach is to form an *information market* by issuing securities for each statistic $s \in \mathcal{S}$, with payoff depending on the realized value of the statistic. (Such securities are known as contingent claims.) The amount of shares of each security acquired by the agent indirectly reveals it estimate μ. In a market scoring rule [**?**], these two approaches are dual to each other in a formal sense.

4.2. Scoring Rule

We first consider how to directly elicit μ via a scoring rule. The space of feasible reports from an agent corresponds to

$$\mathcal{M} = \left\{ \mu \in \mathbf{R}^d : \mathbf{E}_p[\phi(x)] = \mu, \text{ for some } p \in \mathcal{P} \right\}. \tag{5}$$

A scoring rule is a function $S : \mathcal{M} \times \mathcal{X} \to \mathbf{R}$ that rewards an agent with $S(\mu, x)$ based on how its report μ agreed with the eventual outcome x. A scoring rule is *proper* if for all $\mu \in \mathcal{M}$, and $p \in \mathcal{P}$ such that $\mathbf{E}_p[\phi(x)] = \mu$, we have

$$\mathbf{E}_p[S(\mu, x)] \geq \mathbf{E}_p[S(\hat{\mu}, x)] \tag{6}$$

for all $\hat{\mu} \neq \mu$; the rule is *strictly proper* if the inequality is strict. Note that in the literature, scoring rules are defined over entire probability distributions, not just properties

[3]This is related to the notion of *elicitation complexity* of statistical properties introduced by Lambert et al. [**?**], but we do not pursue this connection here.

of those distributions. Our definition imposes a minimum of informational requirements on agents, because an agent does not need to assess the full distribution p that might have lead to its estimates μ in order to realize that truthful reporting is an optimal strategy.

Our scoring rule is implicitly defined via the solution of a mathematical program. The program computes the maximum entropy distribution consistent with the agent's reported beliefs μ.

$$\max_{p \geq 0} \; -\int_{x \in \mathcal{X}} p(x) \log p(x) \nu(dx) \tag{7}$$

$$\text{s.t.} \; \int_{x \in \mathcal{X}} \phi(x) p(x) \nu(dx) = \mu \tag{8}$$

$$\int_{x \in \mathcal{X}} p(x) \nu(dx) = 1 \tag{9}$$

The non-negativity constraints together with (**??**) ensure that the solution is a probability distribution. Constraints (**??**), which correspond to d separate constraints, ensure consistency with the agent's report. The objective (**??**) is the entropy of the solution p with respect to the measure ν. The program has a convex objective and linear constraints.

THEOREM 4.1. *Let $p(x; \mu)$ be the optimal solution to the maximum entropy program, given report $\mu \in \mathbf{R}^d$. The scoring rule defined by*

$$S(\mu, x) = a_x + b \log p(x; \mu), \tag{10}$$

where $b, a_x \in \mathbf{R}$ and $b > 0$, is strictly proper.

Recall that a proper scoring rule $\mathbf{S}$ satisfies

$$\mathbf{E}_{x \sim p}[\mathbf{S}(p, x)] \geq \mathbf{E}_{x \sim p}[\mathbf{S}(p', x)]$$

Note that the maximum entropy distribution is an exponential family distribution. Let $p(x) = \exp\{\beta \cdot \phi(x) - \psi(\beta)\}$ be the true distribution and $p'(x) = \exp\{\beta' \cdot \phi(x) - \psi(\beta')\}$ be the predicted distribution on the event x.

We need to show that

$$\mathbf{E}_{x \sim p}[\mathbf{S}(p, x)] \overset{?}{\geq} \mathbf{E}_{x \sim p}[\mathbf{S}(p', x)]$$

In other words, we need

$$\mathbf{E}_{x \sim p}[\log p(x)] \overset{?}{\geq} \mathbf{E}_{x \sim p}[\log p'(x)]$$

Equivalently,

$$\mathbf{E}_{x \sim p}[\beta \cdot \phi(x) - \psi(\beta)] \overset{?}{\geq} \mathbf{E}_{x \sim p}[\beta' \cdot \phi(x) - \psi(\beta')]$$

Or

$$\beta \cdot \mathbf{E}_{x \sim p}[\phi(x)] - \psi(\beta) \overset{?}{\geq} \beta' \cdot \mathbf{E}_{x \sim p}[\phi(x)] - \psi(\beta') \tag{11}$$

For an exponential family distribution, $\mathbf{E}_{x \sim p}[\phi(x)] = \nabla \psi(\beta)$. Substituting in (**??**), we need to show

$$\beta \cdot \nabla \psi(\beta) - \psi(\beta) \overset{?}{\geq} \beta' \cdot \nabla \psi(\beta) - \psi(\beta')$$

Rearranging the terms,

$$(\beta - \beta') \cdot \nabla\psi(\beta) \overset{?}{\geq} \psi(\beta) - \psi(\beta')$$

which is the first order condition on the convexity of $\psi(\cdot)$. □

Remark. The distance between the scores of the true and any predictive exponential family distribution is, in fact, the Bregman divergence (based on the log partition function) between the corresponding natural parameters. That is,

$$\mathbf{E}_{x\sim p}[\mathbf{S}(p,x)] - \mathbf{E}_{x\sim p}[\mathbf{S}(p',x)] = D_\psi(\beta,\beta')$$

This rule represents a generalization of the logarithmic scoring rule for probability distributions to any properties of distributions that can be captured as expectations of statistics. As explained in the next section, the solution to the maximum entropy program takes the form of an *exponential family* distribution. For many common properties of interest, these distributions are familiar; for example, the maximum entropy distribution with a given mean is an exponential distribution, and the maximum entropy distribution with a given mean and variance is a normal distribution []. We provide here two examples in more depth.

Multinomial Distribution. As a first example, consider the problem of estimating the individual probabilities of a finite set of outcomes. We have $\mathcal{X} = \{1,\ldots,k\}$. The relevant statistics indicate which outcome actually occurs, so the index set is $\mathcal{S} = \mathcal{X}$. Statistic $\phi_x(x')$ is 1 if $x' = x$ and 0 otherwise. Observe that if p is the distribution over outcomes, then $\mathbf{E}_p[\phi(x)] = p$. A feasible report from an agent is any non-negative $\pi = (\pi_1,\ldots,\pi_k)$ such that $\sum_{i=1}^k \pi_i = 1$. Given a report of $\mu = \pi$, the unique solution to the maximum entropy program is $p(x;\pi) = \pi$, and (**??**) corresponds to the classic logarithmic scoring rule of $S(\pi,x) = \log \pi_x$. □

Normal Distribution. As a second example, suppose the outcome space is $\mathcal{X} = \mathbf{R}$ and that we are interested in estimating the mean μ and variance σ^2 of the underlying distribution. We choose the first and second (uncentered) moments, $\kappa_1 = \mu$ and $\kappa_2 = \mu^2 + \sigma^2$ as our statistics. It is well-known that the maximum entropy distribution with a given mean and variance is the normal distribution, so the solution to the maximum entropy program (using Lebesgue as the base measure ν) is

$$p(x;\kappa) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right].$$

Applying Theorem **??** and choosing appropriate constants, we find that the following scoring rule is strictly proper in this context.

$$S(\kappa,x) = -\frac{(x-\mu)^2}{\sigma^2} - \log\sigma^2.$$

Note that the fact that this scoring rule is proper for the mean amounts to the well-known fact that reporting the mean is a Bayes act under quadratic loss [**?**]. □

A brief observation on parametrizations is in order. Note that the maximum entropy distribution here is parametrized by the uncentered moments $\kappa = (\kappa_1,\kappa_2)$ rather than (μ,σ^2) because the variance is not a valid statistic. However, it is clear that the agent does not have to directly report κ: the market maker itself can do the translation from reported mean and variance to uncentered moments. This is relevant because the κ parametrization may be unintuitive for an agent, especially given the constraints $\kappa_2 - \kappa_1^2 \geq 0$ that must be enforced. In the (μ, σ^2) parametrization each parameter is unrestricted, and the parameters have intuitive interpretations.

Our approach so far has been to identify relevant properties of the unknown distribution, and elicit them via a scoring rule computed through a maximum entropy program. However, it is the case that a distribution is a maximum entropy distribution if and only if it is an exponential family []. Therefore, we see the following alternative approach. The market designer can begin by considering the outcome space $\mathcal{X}$, and assessing what family of distributions over $\mathcal{X}$ the unknown p might come from. If this family is an exponential family, then Theorem **??** immediately gives a proper scoring rule to elicit the parameters of the distribution. Many of the most familiar distributions are exponential families, including the normal, exponential, gamma, beta, binomial, Poisson, Weibull, and Dirichlet. To illustrate, we consider two examples in more depth.

Exponential. Suppose the designer would like an estimate of the rate at which servers fail in a data center. An outcome is the time between failures, so $\mathcal{X} = [0, +\infty)$. The usual distribution for failure times is the exponential distribution $p(x; \lambda) = \lambda e^{-\lambda x}$, which is parametrized by the rate λ and has mean $\mu = 1/\lambda$. The exponential distribution is the maximum entropy distribution given a fixed mean μ. From Theorem **??** we obtain the following proper scoring rule for this setting:

$$S(\mu, x) = \log \lambda - \lambda x.$$

It is simple to check using basic calculus that an agent maximizes the expected score by reporting $\lambda = 1/\hat{\mu}$, where here $\hat{\mu}$ is its estimate of the mean. In this case, it is unclear a priori which of the mean or rate parametrizations is most intuitive, but the maximum entropy approach easily allows for either. □

Beta. Suppose the designer would like an estimate of the click-through rate of an online advertisement. The outcome space here is $\mathcal{X} = (0, 1)$. A typical prior over a probability like this is the beta distribution, which has two parameters $\alpha, \beta > 0$ and density function $p(x; \alpha, \beta) = x^{\alpha-1}(1-x)^{\beta-1}/B(\alpha, \beta)$, where B is the beta function. By theorem **??** this leads to the proper scoring rule

$$S((\alpha, \beta), x) = (\alpha - 1) \log x + (\beta - 1) \log(1 - x) - \log B(\alpha, \beta).$$

Although this parametrization is standard for the beta, it is perhaps more intuitive to use the parametrization $\mu = \alpha/(\alpha + \beta) \in (0, 1)$ and $n = \alpha + \beta > 0$. Here μ is the mean of the distribution—corresponding to the agent's actual estimate of the click-through rate—and n is a measure of the agent's confidence in the estimate. □

4.3. Information Market

We next consider how to indirectly elicit μ by setting up a market of contingent claim securities. Under this approach, the elements of the index set $\mathcal{S}$ are interpreted not as statistics but as securities. The payoff from one share of security s when outcome x occurs is given by the mapping ϕ_s. Thus if the vector of shares held by the agent is $\theta \in \mathbf{R}^d$, where entry θ_s corresponds the number of shares of security s, then the payoff to the agent when x occurs is evaluated by taking the inner product $\langle \theta, \phi(x) \rangle$. As a concrete example, recall our multinomial distribution example from the previous section, where $\phi_x(x') = 1$ if $x' = x$ and 0 otherwise. This means that security $x \in \mathcal{S}$ pays 1 dollar if outcome $x \in \mathcal{X}$ occurs, and nothing otherwise. (Such securities are known as Arrow-Debreu securities.) In our normal distribution example, the statistic for the mean had the mapping $\phi_1(x) = x$. Therefore a share of the corresponding security has a payoff that is linear in the outcome. (Such securities amount to futures contracts.)

To extract useful information from such a securities market, we set a pricing scheme and examine the number of shares the agent chooses to acquire. Let Ω be the set of all portfolios (i.e., vectors of shares) that the agent can feasibly hold; we will see how this domain is determined in an instant. Each security s has an associated price function $c_s : \Omega \to \mathbf{R}$, which gives the marginal price $c_s(\theta)$ of security s when the agent holds portfolio θ. (Note that the marginal price depends on the entire portfolio θ, not just the number of shares θ_s.) A risk-neutral agent will choose to acquire shares up to the point where, for each share, expected payoff equals marginal price. Formally, if the agent acquires portfolio θ, then for each $s \in \mathcal{S}$ we must have

$$\mathbf{E}_p[\phi_s(x)] = c_s(\theta). \tag{12}$$

In this way, by its choice of θ, the agent reveals that its belief is $\mu = c(\theta)$. There are several important properties that the price function c should have. To simplify the agent's portfolio acquisition process, it should not be the case that the total cost of acquiring θ depends on the order in which shares are bought. This means that there should exist a *cost function* $C : \Omega \to \mathbf{R}$ such that $c = \nabla C$. The gradient ∇C must be onto $\mathcal{M}$ to ensure that **(??)** always has a solution—this is the most important but technical condition to realize. To ensure that the solution is unique, it would also be convenient if the cost function were strictly convex.

PROPOSITION 4.2. *The following cost function is monotone, convex, and onto $\mathcal{M}$:*

$$C(\theta) = \log \int_{x \in \mathcal{X}} \exp\left[\langle \theta, \phi(x) \rangle\right] \nu(dx). \tag{13}$$

Observe that in the context of our earlier multinomial example, cost function **(??)** is exactly the cost function for Hanson's logarithmic market scoring rule. Our approach here provides a generalization of this market scoring rule to markets with contingent claim securities with arbitrary payoffs, designed to elicit specific properties of distributions, beyond just the probabilities of different outcomes.

Because an agent would never select a portfolio with infinite cost, the effective domain of C is $\Omega = \{\theta \mid C(\theta) < +\infty\}$. In the context of the multinomial distribution example, $\Omega = \mathbf{R}^d$, so that shares of each security can always be bought or sold short. In the context of the normal distribution example the effective domain is $\Omega = \{(\theta_1, \theta_2) \in \mathbf{R}^2 \mid \theta_2 > 0\}$ if we re-define the second statistic to be $\phi_2(x) = -x^2$; this means that the security has a negative payoff for each share, and consequently the agent must be compensated to acquire such shares. Evaluating **(??)**, the cost function takes the form $C(\theta) = \frac{\theta_1^2}{4\theta_2} - \frac{1}{2}\log(2\theta_2)$.

Now, in our previous elicitation approach that used a scoring rule, the process of computing the scoring rule also provided a complete distribution $p(x; \mu)$ over outcomes, which could be used to infer other properties beyond just the agent beliefs μ. In the current market-based approach, the agent's chosen portfolio θ can also form the basis of a distribution over outcomes. Consider the following distribution, represented as a density with respect to a base measure ν:

$$p(x; \theta) = \exp\left[\langle \theta, \phi(x) \rangle - C(\theta)\right]. \tag{14}$$

Meaning, the probability that a subset $X \subseteq \mathcal{X}$ of the outcomes occurs is $\int_{x \in X} p(x;\theta)\nu(dx)$. Observe that by definition (**??**), this probability density indeed integrates to 1 over $\mathcal{X}$, and it is clear that the density is non-negative as required.

In the statistical literature a distribution that takes the form (**??**) is known as an *exponential family*. The mapping ϕ is known as the *sufficient statistic*, θ is the *natural parameter*, and C is the log-partition or *cumulant* function.

4.4. Duality

There is a well-known duality between the maximum entropy approach and exponential families, which translates into a duality between the scoring rule and information market just developed. The duality implies that the approach leads to a *market scoring rule*, applicable to both thin and thick multi-agent settings.

It is known that a distribution is a maximum entropy distribution if and only if it is an exponential family []. To see this, let $\theta(\mu) \in \mathbf{R}^d$ be the Lagrange multiplier corresponding to constraints (**??**) when solving the maximum entropy program given the agent report μ. (Our choice of notation is deliberately suggestive.) Let $A(\mu)$ be the Lagrange multiplier corresponding to (**??**). From the first order necessary conditions for optimality, we find that

$$p(x;\mu) = \exp\left[\langle\theta(\mu), \phi(x)\rangle - A(\mu) - 1\right],$$

so that the solution is indeed in exponential family form. Conversely, given a cumulant C from an exponential family and a parameter θ, we see that $p(x;\theta)$ defined according to (**??**) is the maximum entropy distribution under constraints $\mathbf{E}_p[\phi(x)] = \mu$ where $\mu = \nabla C(\theta)$. We obtain the following.

PROPOSITION 4.3. *Let C be the cumulant (i.e, cost function) for the exponential family corresponding to ϕ, and let $\theta(\mu)$ be the optimal Lagrange multiplier for the mean constraints given an agent report of $\mu \in \mathcal{M}$. Then the following scoring rule is equivalent to (**??**):*

$$S(\mu, x) = \langle\theta(\mu), \phi(x)\rangle - C(\theta(\mu)). \tag{15}$$

The scoring rule (**??**) decomposes neatly into payoff and cost functions. The first term $\langle\theta(\mu), \phi(x)\rangle$ defines the agent's outcome-contingent reward, while the second term $C(\theta(\mu))$ is the cost of acquiring a portfolio $\theta(\mu)$. We see here the duality between the approach of directly reporting μ, or acquiring shares θ: an agent reporting beliefs μ under scoring rule (**??**) would choose to acquire shares $\theta(\mu)$ in the information market with cost function (**??**). Since (**??**) is a proper scoring rule, the information market with cost function (**??**) is based on a proper market scoring rule.

5. THE EXPONENTIAL FAMILY MARKET MECHANISM: A MAXIMUM LIKELIHOOD APPROACH

In this section, we will set up a prediction market that aggregates beliefs from traders where the outcome is a continuous random variable. In particular, we assume that the outcome is drawn from an exponential family distribution. Each of the traders has access to a series of points drawn from this distribution. In other words, the traders have access an empirical mean of the sufficient statistics of the exponential family. Every trader has infinite budget so that the current market price after a trader has traded in the prediction market, exactly reflects his beliefs.

For $x_i \sim P_{\boldsymbol{\beta}}$, recall that the likelihood function for independently drawn data $x_1, \ldots, x_n$ is given by $\prod_{i=1}^{n} P_{\boldsymbol{\beta}}(x_i)$. The maximum likelihood estimate of the natural parameters $\boldsymbol{\beta}$ is the value of the natural parameters that maximizes the likelihood function.

We will now set up a prediction market with log market scoring rule (LMSR) and infinite budget traders such that the market state represents the maximum likelihood estimate (MLE) of the natural parameters of an exponential family distribution.

For a given exponential family distribution, the prediction market is defined as follows:

Traders. The prediction market will simulate a trader i corresponding to expert i. This trader processes all information samples available to her directly or inferred from previous trades, and forms a belief distribution such that the believed means of the sufficient statistics matches the empirical means of the sufficient statistics from the information samples. The trader trades in the market to maximize her expected payoff under her believed distribution.

Securities and their payoff. For each $i = 1, 2, \ldots, k$, we define a security s_i with payoff $\phi_i(x)$ where x is the ultimate outcome and $\Phi()$ defines the vector of sufficient statistics of the exponential family distribution according to which x is drawn. We define an additional security s_0 with payoff $\phi_0(x) := a - \sum_{i=1}^{k} \phi_i(x)$ where a is an appropriately chosen constant dependent on the range of Φ so that the payoff of s_0 is non-negative.

We also note that since we want non-negative payoffs, we restrict the exponential families under consideration so that the sufficient statistics are lower bounded. If we have a constant lower bound on the sufficient statistics, then without loss of generality we can add a constant to each sufficient statistic without changing the exponential family in any way, as the constants will be absorbed in the log-partition function.

Arbitrage-free Property. We can easily show that the prediction market we have defined is arbitrage-free: there is no sequence of trades that guarantees the trader a profit under all conditions. First, note that the incremental cost and payoff function are additive over multiple trades, so the net profit of a sequence of trades depends only on the initial and final market position, and is independent of the actual path along which the trade takes place.

Now, consider any trade (or sequence of trades) that moves the market from an initial state of $\mathbf{q}$ to a final state of $\mathbf{q} + \mathbf{q}'$. The cost of this trade is $C(\mathbf{q} + \mathbf{q}') - C(\mathbf{q})$. The cost function of our market is the log-partition function of an exponential family, and thus, $C(.)$ is a strictly convex function **[?]**. Thus, we have the inequality:

$$C(\mathbf{q} + \mathbf{q}') - C(\mathbf{q}) > \mathbf{q}' \nabla C|_{\mathbf{q}} = \mathbf{q}' \mu,$$

where μ is the mean sufficient statistics vector for the distribution with parameters $\mathbf{q}$. The payoff due to this trade, with outcome x, is $\mathbf{q}'.\phi(x)$. Now, we observe that under the distribution with parameters $\mathbf{q}$, the expected payoff is $\mathbf{q}' E[\Phi(x)] = \mathbf{q}' \mu$. Thus, under this distribution over outcomes x, the expected profit is strictly less than 0. This is only possible if the realized profit is less than 0 for at least one outcome x. As this statement is true for every $\mathbf{q}$ and $\mathbf{q}'$, and the profits are path-independent, the market is arbitrage-free.

Market State and Natural Parameters. We define the interpretation function

$$\begin{aligned} I(\mathbf{q}) &= (q_1 - q_0, q_2 - q_0, \ldots, q_k - q_0) \\ &= (\beta_1, \beta_2, \ldots, \beta_k) = \boldsymbol{\beta} \end{aligned} \tag{16}$$

This allows us to interpret the state of the market in terms of a prediction on the natural parameters of the distribution.

THEOREM 5.1. *The natural parameter vector corresponding to the interpretation of the market state given by Equation* **??** *is the vector of their maximum likelihood estimates. Further, this interpretation is unique.*

PROOF. We observe that for $i = 1, \dots, n$,

$$\begin{aligned}
\mu_i &= \left(\frac{\partial C(\mathbf{q})}{\partial q_i}\right)_{\mathbf{q}=\mathbf{q}^*} \\
&= \left(\frac{\partial \log \int \exp(\mathbf{q}^T \Phi(x)) dx}{\partial q_i}\right)_{\mathbf{q}=\mathbf{q}^*} \\
&= \left(\frac{\partial \log \int e^{q_0 \phi_0(x) + \sum_{i=1}^n q_i \phi_i(x)} dx}{\partial q_i}\right)_{\mathbf{q}=\mathbf{q}^*} \\
&= \left(\frac{\partial \log \int e^{q_0 (a - \sum_{i=1}^n \phi_i(x)) + \sum_{i=1}^n q_i \phi_i(x)} dx}{\partial q_i}\right)_{\mathbf{q}=\mathbf{q}^*} \\
&= \left(\frac{\partial \log \int e^{q_0 a} e^{\sum_{i=1}^n (q_i - q_0) \phi_i(x)} dx}{\partial q_i}\right)_{\mathbf{q}=\mathbf{q}^*} \\
&= \left(\frac{\partial \log (e^{q_0 a} \int e^{\sum_{i=1}^n (q_i - q_0) \phi_i(x)} dx)}{\partial q_i}\right)_{\mathbf{q}=\mathbf{q}^*} \\
&= \left(\frac{\partial (q_0 a)}{\partial q_i} \frac{\partial \log (\int e^{\sum_{i=1}^n (q_i - q_0) \phi_i(x)} dx)}{\partial q_i}\right)_{\mathbf{q}=\mathbf{q}^*} \\
&= \left(\frac{\partial \psi(\boldsymbol{\beta})}{\partial q_i}\right)_{\mathbf{q}=\mathbf{q}^*} \\
&= \left(\frac{\partial \psi(\boldsymbol{\beta})}{\partial \beta_i} \frac{\partial \beta_i}{\partial q_i}\right)_{\mathbf{q}=\mathbf{q}^*} \\
&= \left(\frac{\partial \psi(\boldsymbol{\beta})}{\partial \beta_i}\right)_{\boldsymbol{\beta}=\boldsymbol{\beta}^*}
\end{aligned}$$

Thus, this choice of parameters satisfies $\frac{\partial \psi(\boldsymbol{\beta})}{\partial \beta_i} = \mu_i$. We will now argue that this vector is unique.

If the exponential family is represented so that there is a unique parameter vector associated with each distribution, the representation is said to be minimal. The Bernoulli, Gaussian, and Poisson distributions all have minimal representations. Now, for an exponential family whose representation is minimal, the gradient is an injection [**?**, Prop. 3.2]. Thus, there is a unique parameter vector $\boldsymbol{\beta}$ that satisfies $\frac{\partial \psi(\boldsymbol{\beta})}{\partial \beta_i} = \mu_i$, for each $i \in \{1, \dots, n\}$. Thus, if the μ_i's correspond to the empirical means, since our choice of $\boldsymbol{\beta}$ satisfies the equality, it must also be the vector corresponding to the MLE. □

Thus, we have shown that this prediction market (along with its strategic traders) aggregates trader information in a way that the market state can be interpreted as a predictive distribution (the maximum likelihood distribution) over an infinite outcome space.

It is worth noting here that if we can bound the parameter space and the sufficient statistics of the distribution, the market maker defined here has bounded worst case loss: The probability density, and hence log loss, would then be bounded for all market states that could be reached and all outcomes. In fact, for many practical applications, a distribution with unbounded parameter space is an approximation for a true space which itself may be bounded. In this case, worst case loss would thus be bounded.

5.1. The Exponential Family Market Mechanism in Adversarial Markets

In the previous section, we saw that we can define a cost-function based prediction market so that the aggregated belief of the traders represents the maximum likelihood estimate of the natural parameters of the true exponential family distribution.

In this section, we consider the the prediction market setup with traders that may be either informative or malicious. The malicious traders may want to inject faulty information into the market. The informative traders on the other hand receive points drawn from the true distribution on which they base their beliefs.

We will show that if we are able to impose finite initial budgets on the traders and control the market prices based on these budgets, then it possible to set up the market so that it is prohibitive for damaging traders to participate in the market. Further, the informative traders can be shown to have expected growth in budget so that they are eventually able to move the market prices without restriction.

5.1.1. Budget-limited Aggregation. Imposing budget limits on the traders will allow us to control the amount of influence any one trader can have on moving the market prices. We will also satisfy an additional requirement that no trader has negative budget at any point of participation in the market. This is achieved by restricting the movement of the market and hence influencing the cost incurred by the trader. Recall that the payoff in this market is non-negative and hence the only adverse influence on a trader's budget is the cost of movement of the market state.

In this section, we assume that the budget of each trader is known to the market maker, and that the market maker can directly limit the allowed trades based on a trader's budget. Let α be the budget of a trader in the market. Suppose with infinite budget, the trader would have moved the market state from $\mathbf{q}_{init}$ to $\mathbf{q}$, where $\mathbf{q}$ represents his true beliefs. Suppose further that $\alpha < C(\mathbf{q}) - C(\mathbf{q}_{init})$. In this case, we want to budget-limit the trader's influence on the market state.

We define the budget-limited final market state as $\tilde{\mathbf{q}}$. Here, we consider a specific functional form of $\tilde{\mathbf{q}}$:

$$\tilde{\mathbf{q}} = \lambda\mathbf{q} + (1-\lambda)\mathbf{q}_{init}$$

where

$$\lambda = \min\left(1, \frac{\alpha}{C(\mathbf{q}) - C(\mathbf{q}_{init})}\right)$$

We first show that this trade is feasible given the trader's budget:

THEOREM 5.2. *Let the current market state be given by the vector $\mathbf{q}_{init}$ and $\tilde{\mathbf{q}} = \lambda\mathbf{q} + (1-\lambda)\mathbf{q}_{init}$. For $\lambda = \min\left(1, \frac{\alpha}{C(\mathbf{q})-C(\mathbf{q}_{init})}\right)$, the cost to the trader to move the market state from $\mathbf{q}_{init}$ to $\tilde{\mathbf{q}}$ is at most his budget α.*

PROOF. From the convexity of C, we have

$$C(\tilde{\mathbf{q}}) \leq (1-\lambda)C(\mathbf{q}_{init}) + \lambda C(\mathbf{q})$$

Now

$$\begin{aligned} C(\tilde{\mathbf{q}}) - C(\mathbf{q}_{init}) &\leq (1-\lambda)C(\mathbf{q}_{init}) \\ &\quad + \lambda C(\mathbf{q}) - C(\mathbf{q}_{init}) \\ &= \lambda\left(C(\mathbf{q}) - C(\mathbf{q}_{init})\right) \end{aligned}$$

Thus, $C(\tilde{\mathbf{q}}) - C(\mathbf{q}_{init}) \leq \alpha$. □

We note that moving to $\tilde{\mathbf{q}}$ as defined may not be the optimal trade for a rational trader maximizing her expected profit. In general, the inequality above is strict, and so a trader does not fully exhaust her budget by moving to $\tilde{\mathbf{q}}$. Our results below will continue to hold in the case that strategic informative traders move to a position closer to their beliefs $\mathbf{q}$.

5.1.2. Damage Bound. Given this restriction on the initial budget of traders, we can now show that the total loss that can be induced by malicious traders is bounded. We define the loss function for $\mathbf{q}$ shares held as:

$$L(\mathbf{q}, x) = -\log(p_{\boldsymbol{\beta}}(x)) = \log \frac{\int e^{\mathbf{q}^T \Phi(x)}\, dx}{e^{\mathbf{q}^T \Phi(x)}}$$

with the correspondence between $\boldsymbol{\beta}$ and $\mathbf{q}$ as defined earlier.

Suppose the prediction market runs over multiple rounds t. Let $\mathbf{q}_0^t$ be the initial number of shares of each security that are held. Let $\tilde{\mathbf{q}}_k^t$ be the final values corresponding to the market state after the traders have made their reports. Let us assume that at this point we receive the value of the random variable x^t.

Over multiple instances of the prediction market, we can track the change in budget of each trader. The change in budget for trader i is

$$\begin{aligned} \alpha_i^t - \alpha_i^{t-1} &= C(\tilde{\mathbf{q}}_{i-1}^t) - C(\tilde{\mathbf{q}}_i^t) - (\tilde{\mathbf{q}}_{i-1}^t - \tilde{\mathbf{q}}_i^t)^T \Phi(x^t) \\ &= L(\tilde{\mathbf{q}}_{i-1}^t, x^t) - L(\tilde{\mathbf{q}}_i^t, x^t) \end{aligned}$$

Define the myopic impact of a trader i in segment t as

$$\Delta_i^t := L(\tilde{\mathbf{q}}_{i-1}^t, x^t) - L(\tilde{\mathbf{q}}_i^t, x^t)$$

Thus, the myopic impact captures incremental gain due to the trader in a round. Note that the myopic impact caused by trader i at round t is equal to the change in his budget in that round.

The total myopic impact due to all k active traders is given by

$$\Delta^t = L(\mathbf{q}_0^t, x^t) - L(\tilde{\mathbf{q}}_k^t, x^t)$$

Thus $-\Delta^t$ captures the incremental loss due to the predictive probability after aggregation of all k traders.

THEOREM 5.3. *A coalition of b malicious traders can at most cause loss bounded by their initial budgets.*

PROOF. Consider the myopic impact of a single trader i after participating in the market T times. Since the market evolves so that the budget of any trader never falls below zero, the total myopic impact in T rounds caused due to trader i is:

$$\Delta_i := \sum_{t=1}^{T} \Delta_i^t = \sum_{t=1}^{T} (\alpha_i^t - \alpha_i^{t-1}) = \alpha_i^T - \alpha_i^0 \geq -\alpha_i^0$$

Thus, any coalition of b adversaries $\{1, \dots, b\}$ can cause at most $\sum_{i=1}^{b} \alpha_i^0$ damage. □

This means that if it can be made prohibitively expensive for an attacker to generate clones, we can set up the prediction market with mostly informative traders.

In Section **??** we show that for an informative trader in every round, his budget increases in expectation. We provide an information-theoretic justification for this claim in Section **??**. The intuition behind this claim is that his prediction moves the input moves the market probability closer to the true probability distribution resulting in net expected profit.

5.1.3. An Information-Theoretic Interpretation. We observe a useful alternative view of the market scoring rule prediction market for exponential family learning. We connect the cost, payoff and profit function to information-theoretic quantities associated with the exponential family.

The following result has been previously pointed out by Amari [**?**].

THEOREM 5.4. **(Profit Decomposition)***: Consider an exponential family $\mathcal{F}$ of distributions over some set of statistics $\boldsymbol{\phi}(x)$, with natural parameters $\boldsymbol{\beta}$. Let $\pi, \rho \in \mathcal{F}$ be any two probability distributions in the family. We use $\boldsymbol{\beta}_\pi$ to denote the natural parameters of π and $\boldsymbol{\mu}_\pi$ to denote the expected value of the sufficient statistics under π. Likewise, we can define $\boldsymbol{\beta}_\rho$ and $\boldsymbol{\mu}_\rho$. We abuse notation slightly and let $\psi(\rho)$ indicate the log partition function of ρ which technically depends on its natural parameters. Let $H(\pi)$ denote the entropy of the distribution π, and $K(\pi||\rho)$ denote the KL-divergence of ρ relative to π. Then, the following equality holds:*

$$K(\pi||\rho) + H(\pi) = \psi(\rho) - \boldsymbol{\beta}_\rho \cdot \boldsymbol{\mu}_\pi \tag{17}$$

PROOF.

$$\begin{aligned} K(\pi||\rho) + H(\pi) &= \int_x \pi(x) \log \frac{\pi(x)}{\rho(x)} dx \\ &\quad - \int_x \pi(x) \log \pi(x) dx \\ &= -\int_x \pi(x) \log \rho(x) dx \\ &= -\int_x \pi(x) \left[\boldsymbol{\beta}_\rho \cdot \boldsymbol{\phi}(x) - \psi(\rho)\right] dx \\ &= -\boldsymbol{\beta}_\rho \cdot \int_x \pi(x) \boldsymbol{\phi}(x) dx \\ &\quad + \psi(\rho) \int_x \pi(x) dx \\ &= \psi(\rho) - \boldsymbol{\beta}_\rho \cdot \boldsymbol{\mu}_\pi \end{aligned}$$

□

Equation **??** gives us an alternative view of the market scoring rule construction. Assume that π is the true distribution, and consider two arbitrary distributions $\rho_1, \rho_2 \in \mathcal{F}$. Note that $\psi(\rho)$ is independent of π, and $\boldsymbol{\beta}_\rho \cdot \boldsymbol{\mu}_\pi$ is linear in the probabilities $\pi(x)$. If we want to measure loss by the KL-divergence, we can do so (in expectation) by setting a cost function that captures the first term, and defining security quantities ($\boldsymbol{\beta}$) and payoffs ($\boldsymbol{\phi}$) to capture the second term. In particular, in a market with cost function ψ, if the market price initially implies a distribution ρ_1, and a trader moves the market to price than implies a distribution ρ_2, then the cost she incurs is $\psi(\rho_2) - \psi(\rho_1)$. The number of securities bought to make this trade is given by the vector ($\boldsymbol{\beta}_{\rho_2} - \boldsymbol{\beta}_{\rho_1}$), and the expected payoff of the securities are given by $\boldsymbol{\mu}_\pi$. Thus, by equation **??**, the *net profit* of the trader is equal to $K(\pi||\rho_1) - K(\pi||\rho_2)$, *i.e.*, the reduction in KL-divergence with respect to the true distribution. We note one useful property of this construction: For a fixed vector $\boldsymbol{\beta}$ of purchased securities, the cost is independent of the outcome (and outcome distribution π), while the payoff is independent of the initial market state in which these securities were purchased.

5.1.4. Budget of Informative Traders. Given the information-theoretic interpretation of the cost-function based prediction market, we now show that the informative trader in the prediction market defined above increases his budget in a round in expectation *under his own belief distribution*.

We now characterize the expected change in budget for an informative trader. The following result holds for any round t; for simplicity, we have therefore dropped the superscript from the notation.

THEOREM 5.5. *Suppose that each informative trader gets a random sample of data, resulting in a sequence of trader beliefs $\mathbf{q}_i$, and hence a sequence of budget-limited market positions $\tilde{\mathbf{q}}_i$, before a final outcome x. Then, the expectation, over trader i's belief distribution $\mathbf{q}_i$, of trader i's realized profit is greater than zero whenever her budget is positive and her belief differs from the previous market position $\tilde{\mathbf{q}}_{i-1}$.*

PROOF. Trader i's believed distribution is the distribution parametrized by $\mathbf{q}_i$. Therefore, the expected profit, over possible outcome values x, for a trader i in a given round is given by

$$\begin{aligned} & E[C(\tilde{\mathbf{q}}_{i-1}) - C(\tilde{\mathbf{q}}_i) - (\tilde{\mathbf{q}}_{i-1} - \tilde{\mathbf{q}}_i)\Phi(x)] \\ = \; & K(\mathbf{q}_i||\tilde{\mathbf{q}}_{i-1})] + H(\mathbf{q}_i) - [K(\mathbf{q}_i||\tilde{\mathbf{q}}_i) + H(\pi)]] \\ = \; & K(\mathbf{q}_i||\tilde{\mathbf{q}}_{i-1}) - K(\mathbf{q}_i||\tilde{\mathbf{q}}_i) \end{aligned}$$

We recall that $\tilde{\mathbf{q}}_i$ can be expanded as:

$$\tilde{\mathbf{q}}_i = \lambda \mathbf{q}_i + (1-\lambda)\tilde{\mathbf{q}}_{i-1}$$

where λ is strictly greater than 0, but no more than 1. It is a standard result from information theory that $K(.,.)$ is a convex function, and therefore, we have:

$$K(\mathbf{q}_i||\tilde{\mathbf{q}}_{i-1}) \leq \lambda K(\mathbf{q}_i||\mathbf{q}_i) + (1-\lambda)K(\mathbf{q}_i||\tilde{\mathbf{q}}_{i-1})$$

The first term on the right hand side is zero, and thus, we get:

$$K(\mathbf{q}_i||\tilde{\mathbf{q}}_{i-1}) - K(\mathbf{q}_i||\tilde{\mathbf{q}}_i) \geq \lambda K(\mathbf{q}_i||\tilde{\mathbf{q}}_{i-1})$$

□

For continuous distributions with a density, the probability that a trader with private information will form exactly the same beliefs as the current market position is 0, and thus, each trader will have positive expected profit on almost all sequences of observed samples and beliefs. This result suggests that, eventually, every informative trader will have the ability to influence the market state in accordance with his beliefs, without being budget limited.

We note one important aspect of Theorem **??**: The expectation is taken with respect to each trader's belief at the time of trade, rather than with respect to the true distribution. This is needed because we have made no assumptions about the optimality of the traders' belief updating procedure; depending on the distribution family and the prior distribution over parameter values, maximum likelihood estimation might not optimize the true expected score. If we assume that the traders' belief formation is optimal, then this growth result will extend to the true distribution as well.

6. THE MAXENT MARKET MECHANISM: A BAYESIAN VIEW

6.0.5. Conjugate Priors. Let $p(x;\theta)$ denote a probability density drawn from an exponential family with sufficient statistic $\phi : \mathcal{X} \to \mathbf{R}^d$, where θ is the natural parameter:

$$p(x;\theta) = \exp\left[\langle \theta, \phi(x)\rangle\right],$$

and

$$g(\theta) = \log \int_{\mathcal{X}} \exp\langle \phi(x), \theta\rangle dx.$$

Recall that $\nabla g(\theta) = \mathbf{E}[\phi(x)]$ and $\nabla^2 g(\theta) = \mathrm{Var}[\phi(x)]$. The family of conjugate priors is also an exponential family and takes the form

$$p(\theta; n, \nu) = \exp\left[\langle n\nu, \theta\rangle - ng(\theta) - h(\nu, n)\right].$$

Here the feature map is $\psi(\theta) = (\theta, -g(\theta))$, the natural parameter is $(n\nu, n)$ where $n \in \mathbf{R}$ and $\nu \in \mathbf{R}^d$. The normalizer $h(\nu, n)$ is convex in $(n\nu, n)$. It is helpful to think of the prior as being based on a 'phantom' sample of size n and mean ν. The justification for this is that

$$\mathbf{E}_\theta\left[\mathbf{E}_x\left[\phi(x)|\theta\right]\right] = \mathbf{E}_\theta\left[\nabla g(\theta)\right] = \nu.$$

Suppose we draw a sample $X = (x_1, \ldots, x_m)$ of size m, and denote the empirical mean by $\mu[X] = \sum_{i=1}^m \phi(x_i)$. The posterior distribution is then

$$p(\theta|X) \propto p(X|\theta)p(\theta|n,\nu) \propto \exp\left[\langle \mu[X] + n\nu, \theta\rangle - (m+n)g(\theta)\right],$$

and so the posterior mean is

$$\frac{m\mu[X] + n\nu}{m+n}. \tag{18}$$

Thus the posterior mean is a convex combination of the prior and posterior means, and their relative weights depend on the phantom and empirical sample sizes.

In the context of prediction markets, one could imagine an agent who uses the current market estimate as its prior, and draws a empirical sample of size m. Its belief then takes the form (**??**), where n depends on the importance the agent places on the market estimate (perhaps based on how long the market has been running). However, note that the *mean parameter* becomes a convex combination of market state and empirical belief, and this does not translate to the *natural parameter*, which is what we would have liked. The new natural parameter is

$$\nabla g^{-1}\left(\frac{m\mu[X] + n\nu}{m+n}\right) = \nabla g^{*}\left(\frac{m\mu[X] + n\nu}{m+n}\right).$$

where g^* is the convex conjugate of g.

6.0.6. Exponential Utility. Assume the agent's belief distribution p belongs to an exponential family, so it takes the form

$$p(x;\theta) = \exp[\theta x - T(\theta)]$$

where θ is the natural parameter and T is the log-partition function. (I'm assuming that the sufficient statistic is $\phi(x) = x$ just for simplicity.) Assume also that the agent has an exponential utility for money w:

$$U(w) = -\frac{1}{a}\exp(-aw).$$

Here a is the coefficient of risk aversion (higher means more risk averse, and the utility function is more concave).

PROPOSITION 6.1. *An agent with exponential family belief with natural parameter $\hat{\theta}$, and exponential utility with coefficient a, makes a trade that moves the current market share vector θ to the convex combination $\frac{1}{1+a}\hat{\theta} + \frac{a}{1+a}\theta$ assuming an LMSR cost function.*

PROOF. Let δ be the vector of shares the agent trades. The payoff given eventual outcome x is then $\delta x - C(\delta+\theta) + C(\theta)$. The utility for this payoff is as follows (recall that $\hat{\theta}$ is the agent's believed natural parameter).

$$U(\delta x - C(\delta+\theta) + C(\theta)) = -\frac{1}{a}\exp(-a\delta x + aC(\delta+\theta) - aC(\theta)).$$

Taking the expected utility, we obtain

$$\begin{aligned}
&\mathbf{E}\left[U(\delta x - C(\delta+\theta) + C(\theta))\right] \\
=\ & \int_{\mathcal{X}} -\frac{1}{a}\exp(-a\delta x + aC(\delta+\theta) - aC(\theta))\exp[\hat{\theta}x - T(\hat{\theta})]\,dx \\
=\ & -\frac{1}{a}\int_{\mathcal{X}} \exp[(\hat{\theta} - a\delta)x + aC(\delta+\theta) - aC(\theta)) - T(\hat{\theta})]\,dx \\
=\ & -\frac{1}{a}\exp[aC(\delta+\theta) - aC(\theta)) + T(\hat{\theta} - a\delta) - T(\hat{\theta})]\int_{\mathcal{X}} \exp[(\hat{\theta} - a\delta)x - T(\hat{\theta} - a\delta)]\,dx \\
=\ & -\frac{1}{a}\exp[aC(\delta+\theta) - aC(\theta)) + T(\hat{\theta} - a\delta) - T(\hat{\theta})]\int_{\mathcal{X}} p(x;\hat{\theta} - a\delta)\,dx \\
=\ & -\frac{1}{a}\exp[aC(\delta+\theta) - aC(\theta)) + T(\hat{\theta} - a\delta) - T(\hat{\theta})] \\
=\ & U\left(-C(\delta+\theta) + C(\theta)) - \frac{1}{a}T(\hat{\theta} - a\delta) + \frac{1}{a}T(\hat{\theta})\right)
\end{aligned}$$

The second-last equality follows from the fact that $\int_{\mathcal{X}} p(x;\hat{\theta} - a\delta)\,dx = 1$. Since utility U is monotone increasing, it is maximized by maximizing its argument, which is a concave function of δ by convexity of C and T. The optimality condition for the argument is

$$\nabla C(\delta^* + \theta) = \nabla T(\hat{\theta} - a\delta^*) \tag{19}$$

Now if the market maker is using LMSR, then C is the log-partition function of the corresponding exponential family and $C = T$. Then **(??)** can be solved by equating the arguments. This leads to $\delta^* = (\hat{\theta} - \theta)/(1+a)$, which moves the share vector to $\theta + \delta^* = \frac{1}{1+a}\hat{\theta} + \frac{a}{1+a}\theta$. $\square$

In the statement of the result I use the term "LMSR cost function" somewhat loosely, because we are not necessarily dealing with a market over exhaustive, mutually exclusive outcomes. What is meant is the cost function that arises by taking the dual to entropy of the maxent distribution with given mean parameter μ. As we've discussed this seems like the right generalization of LMSR to arbitrary mean parameter spaces.

Note that as $a \to 0$, we approach risk neutrality and the agent moves the share vector all the way to its private estimate $\hat{\theta}$. As a grows (agent grows more risk averse) the agent makes smaller and smaller trades that keep it closer to the current estimate θ.

So there is a kind of congruence between exponential family beliefs and exponential utility (as the names would suggest).

6.1. Properties of the Maximum Entropy Market

Consider the dual of the cost function, $C^*(\mu)$ defined as

$$C^*(\mu) = \sup_{\beta} \beta \cdot \mu - C(\beta)$$

This supremum is obtained at the value of β for which $\mu = \nabla C(\beta)$; that is the natural parameter β for which $\mu = \mathbf{E}_{p_\beta}[\phi(x)]$ is the mean parameter. Rewriting,

$$C^*(\mathbf{E}_{p_\beta}[\phi(x)]) = \beta \cdot \mathbf{E}_{p_\beta}[\phi(x)] - C(\beta)$$

Thus, the mean parameters are the dual variables to the natural parameters.

THEOREM 6.1. *The value of the dual of the cost function is the negative entropy of the exponential family distribution obtained from backward mapping the mean parameters.*

PROOF. To see this, we note that for $p(x) = \exp\{\beta \cdot \phi(x) - C(\beta)\}$

$$\begin{aligned} -H(p) &= \int_x p(x) \log p(x) \\ &= \int_x p(x)[\beta \cdot \phi(x) - C(\beta)] \\ &= \beta \cdot \int_x p(x)[\phi(x)] - \int_x p(x) C(\beta) \\ &= \beta \cdot \mathbf{E}_p[\phi(x)] - C(\beta) \end{aligned}$$

□

Remark. This result shows a nice parallel to LMSR, since the dual of the cost function of the LMSR is the negative entropy.

Remark. The LMSR is essentially a special case applied to a multinomial distribution. The LMSR is known to have bounded ($\log n$) market maker loss. Thus, while in general the exponential family LMSR does not guarantee bounded market maker loss, for some special cases it can.

THEOREM 6.2. *Negative differential entropy* $-H(p) \stackrel{\text{def}}{=} \int_x p(x) \log p(x) dx$ *is unbounded from above for an exponential family distribution.*

PROOF. Note that, for an exponential family distribution

$$\begin{aligned} H(p) &= -\int_x p(x) \log p(x) dx \\ &= -\int_x p(x)[\beta \cdot \phi(x) - \psi(\beta)] dx \\ &= -\beta \cdot \mathbf{E}[\phi(x)] + \psi(\beta) \end{aligned}$$

If the range of β is unbounded, the negative differential entropy is unbounded as well. □

Loss of the market maker. First we derive an expression for the conjugate dual C^* in terms of the primal variables.

Recall the definition of the conjugate dual:

$$C^*(\mu) = \sup_q q \cdot \mu - C(q)$$

The supremum is achieved at q' such that $\nabla C(q') = \mu$. So we may rewrite:

$$C^*(\nabla C(q')) = \sup_q q \cdot \nabla C(q') - C(q)$$

This supremum is achieved at q such that $\nabla C(q') = \nabla C(q)$. One such value of q is q'. So we have

$$C^*(\nabla C(q')) = q' \cdot \nabla C(q') - C(q')$$

Also, recall that $\nabla C^*(\nabla C(q)) = q$ for the exponential family market. (I think there are some restrictions on q.)

Let q_0 be the initial and q_f the final market state. Then if μ is the expected value of the outcome sufficient statistics (*i.e.* the mean parameter) under the true distribution, the loss of the exponential family market maker can be written as:

$$\begin{aligned} \phi(x)(q_f - q_0) - C(q_f) + C(q_0) &= \phi(x)(q_f - q_0) - q_f \nabla C(q_f) + C^*(\nabla C(q_f)) \\ &\quad + q_0 \nabla C(q_0) - C^*(\nabla C(q_0)) \\ &= q_f(\phi(x) - \nabla C(q_f)) + C^*(\nabla C(q_f)) \\ &\quad - q_0(\phi(x) - \nabla C(q_0)) - C^*(\nabla C(q_0)) \\ &= -C^*(\nabla C(q_0)) + C^*(\phi(x)) - q_0(\phi(x) - \nabla C(q_0)) \\ &\quad + C^*(\nabla C(q_f)) - C^*(\phi(x)) + q_f(\phi(x) - \nabla C(q_f)) \\ &= C^*(\phi(x)) - C^*(\nabla C(q_0)) - \nabla C^*(\nabla C(q_0))(\phi(x) - \nabla C(q_0)) \\ &\quad - [C^*(\phi(x)) - C^*(\nabla C(q_f)) - \nabla C^*(\nabla C(q_f))(\phi(x) - \nabla C(q_f))] \\ &= D_{C^*}[\phi(x), \nabla C(q_0))] - D_{C^*}[\phi(x), \nabla C(q_f))] \end{aligned}$$

6.1.1. Example: Gaussian Markets

EXAMPLE 6.3. *We will now derive an expression for the dual of the log partition function for the Gaussian distribution. In this case,* β *is a* 2*-dimensional vector. Let*

$\beta = \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}$ *and* $\mu = \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}$*. So*

$$\begin{aligned}\boldsymbol{\mu} &= \nabla\psi(\boldsymbol{\beta})\\ &= \nabla\left(-\frac{\beta_1^2}{4\beta_2} - \frac{1}{2}\log(-2\beta_2)\right)\\ &= \begin{pmatrix}-\frac{\beta_1}{2\beta_2}\\ \frac{\beta_1^2}{4\beta_2^2} - \frac{1}{2\beta_2}\end{pmatrix}\end{aligned}$$

Thus, $\boldsymbol{\beta}$ can be written in terms of $\boldsymbol{\mu}$ as follows: $\beta_1 = \frac{\mu_1}{\mu_2-\mu_1^2}$ *and* $\beta_2 = \frac{1}{\mu_1^2-\mu_2}$*. This leads to the following closed form expression for the dual of the log partition function:*

$$\psi^*(\mu) = -\frac{1}{2} - \frac{1}{2}\log(\mu_2 - \mu_1^2)$$

EXAMPLE 6.4. *We will now derive the expression for the differential entropy for the normal distribution. Recall that for the normal distribution*

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\}$$

$$\begin{aligned}H(p) &= -\int_x p(x)\log p(x)dx\\ &= -\int_x p(x)\left[-\log(2\pi) - \log\sigma + \frac{-(x-\mu)^2}{2\sigma^2}\right]dx\\ &= \log(2\pi) + \log\sigma + \frac{1}{2\sigma^2}\int_x p(x)(x-\mu)^2]dx\\ &= \log(2\pi) + \log\sigma + \frac{1}{2\sigma^2}\mathbf{E}_{p(x)}[(x-\mu)^2]dx\\ &= \log(2\pi) + \log\sigma + \frac{1}{2\sigma^2}\sigma^2 dx\\ &= \log(2\pi) + \frac{1}{2} + \log\sigma\end{aligned}$$

Recall that the value of the dual of the log partition function is the value of the negative entropy of the distribution for an exponential family distribution. Thus when the variance of the normal distribution $\sigma^2 = 0$, the value of the dual of the cost function goes to ∞ and is thus unbounded.

EXAMPLE 6.5. *For the normal distribution, the natural parameters $\boldsymbol{\beta} = (\beta_1, \beta_2)$ can be written in terms of μ and σ as*

$$\beta_1 = \frac{\mu}{\sigma^2}, \quad \beta_2 = \frac{-1}{2\sigma^2}$$

and the log partition function is

$$\psi(\boldsymbol{\beta}) = -\frac{\beta_1^2}{4\beta_2} - \frac{1}{2}\log(-2\beta_2)$$

Alternately, the log partition function can be written in terms of its variance and mean as $\frac{\mu^2}{2\sigma^2} + \log\sigma$. Also note that the mean parameters are μ and $\mu^2 + \sigma^2$. Let's now work out the loss for the Gaussian market maker. First note that for a Gaussian market for mean parameters $\begin{pmatrix} \mu_1 \\ \mu_1^2 + \sigma_1^2 \end{pmatrix}$ and $\begin{pmatrix} \mu_2 \\ \mu_2^2 + \sigma_2^2 \end{pmatrix}$, we have

$$\begin{aligned}
D_{C^*}\left(\begin{pmatrix} \mu_1 \\ \mu_1^2 + \sigma_1^2 \end{pmatrix}, \begin{pmatrix} \mu_2 \\ \mu_2^2 + \sigma_2^2 \end{pmatrix}\right) &= C^*\left(\begin{pmatrix} \mu_1 \\ \mu_1^2 + \sigma_1^2 \end{pmatrix}\right) - C^*\left(\begin{pmatrix} \mu_1 \\ \mu_1^2 + \sigma_1^2 \end{pmatrix}\right) \\
&\quad - \nabla C^*\left(\begin{pmatrix} \mu_2 \\ \mu_2^2 + \sigma_2^2 \end{pmatrix}\right)\begin{pmatrix} \mu_1 - \mu_2 \\ \mu_1^2 + \sigma_1^2 - \mu_2^2 - \sigma_2^2 \end{pmatrix} \\
&= -\log\frac{\sigma_1}{\sigma_2} - \begin{pmatrix} \frac{\mu_2}{\sigma_2^2} \\ -\frac{1}{2\sigma_2^2} \end{pmatrix} \cdot \begin{pmatrix} \mu_1 - \mu_2 \\ \mu_1^2 + \sigma_1^2 - \mu_2^2 - \sigma_2^2 \end{pmatrix} \\
&= -\log\frac{\sigma_1}{\sigma_2} - \frac{\mu_2(\mu_1 - \mu_2) - \frac{1}{2}(\mu_1^2 - \mu_2^2 + \sigma_1^2 - \sigma_2^2)}{\sigma_2^2} \\
&= \log\frac{\sigma_2}{\sigma_1} + \frac{\sigma_2^2 - \sigma_1^2}{2\sigma_2^2} + \frac{(\mu_1 - \mu_2)^2}{2\sigma_2^2}
\end{aligned}$$

So the Gaussian market maker loss is

$$\begin{aligned}
D_{C^*}[\phi(x), \nabla C(q_0))] - D_{C^*}[\phi(x), \nabla C(q_f))] &= \log\frac{\sigma_0}{\sigma_x} + \frac{\sigma_0^2 - \sigma_x^2}{2\sigma_0^2} + \frac{(\mu_x - \mu_0)^2}{2\sigma_0^2} \\
&\quad - \left[\log\frac{\sigma_f}{\sigma_x} + \frac{\sigma_f^2 - \sigma_x^2}{2\sigma_f^2} + \frac{(\mu_x - \mu_f)^2}{2\sigma_f^2}\right] \\
&= \log\frac{\sigma_0}{\sigma_f} + \frac{(x - \mu_0)^2}{2\sigma_0^2} - \frac{(x - \mu_f)^2}{2\sigma_f^2}
\end{aligned}$$

Here we have used the fact that $\mu_x = x$ and $\sigma_x = 0$.

Remark. In statistics, the standard score (aka z-score), $(x-\mu)/\sigma$, is the (signed) number of standard deviations an observation is above the mean.

7. GENERALIZED LOG SCORING RULE

Suppose that an agent holds a belief in the expected value of the moments of a random variable x. The maximum entropy distribution that is consistent with these beliefs is an exponential family distribution.

Definition. For an outcome x with predicted probability p, the generalized log scoring rule is defined as

$$S(p, x) \stackrel{\text{def}}{=} a + b \log p(x)$$

where a and b are constants that we will set to 0 and 1 respectively in these notes.

THEOREM 7.1. *The generalized log scoring rule for the maximum entropy distribution is proper.*

PROOF. Recall that a proper scoring rule $\mathbf{S}$ satisfies

$$\mathbf{E}_{x\sim p}[\mathbf{S}(p,x)] \geq \mathbf{E}_{x\sim p}[\mathbf{S}(p',x)]$$

Note that the maximum entropy distribution is an exponential family distribution. Let $p(x) = \exp\{\beta \cdot \phi(x) - \psi(\beta)\}$ be the true distribution and $p'(x) = \exp\{\beta' \cdot \phi(x) - \psi(\beta')\}$ be the predicted distribution on the event x.

We need to show that

$$\mathbf{E}_{x\sim p}[\mathbf{S}(p,x)] \stackrel{?}{\geq} \mathbf{E}_{x\sim p}[\mathbf{S}(p',x)]$$

In other words, we need

$$\mathbf{E}_{x\sim p}[\log p(x)] \stackrel{?}{\geq} \mathbf{E}_{x\sim p}[\log p'(x)]$$

Equivalently,

$$\mathbf{E}_{x\sim p}[\beta \cdot \phi(x) - \psi(\beta)] \stackrel{?}{\geq} \mathbf{E}_{x\sim p}[\beta' \cdot \phi(x) - \psi(\beta')]$$

Or

$$\beta \cdot \mathbf{E}_{x\sim p}[\phi(x)] - \psi(\beta) \stackrel{?}{\geq} \beta' \cdot \mathbf{E}_{x\sim p}[\phi(x)] - \psi(\beta') \tag{20}$$

For an exponential family distribution, $\mathbf{E}_{x\sim p}[\phi(x)] = \nabla\psi(\beta)$. Substituting in (**??**), we need to show

$$\beta \cdot \nabla\psi(\beta) - \psi(\beta) \stackrel{?}{\geq} \beta' \cdot \nabla\psi(\beta) - \psi(\beta')$$

Rearranging the terms,

$$(\beta - \beta') \cdot \nabla\psi(\beta) \stackrel{?}{\geq} \psi(\beta) - \psi(\beta')$$

which is the first order condition on the convexity of $\psi(\cdot)$. □

Remark. The distance between the scores of the true and any predictive exponential family distribution is, in fact, the Bregman divergence (based on the log partition function) between the corresponding natural parameters. That is,

$$\mathbf{E}_{x\sim p}[\mathbf{S}(p,x)] - \mathbf{E}_{x\sim p}[\mathbf{S}(p',x)] = D_\psi(\beta,\beta')$$

7.1. Exponential Smoothing

One way to model the exponential family prediction market might be through exponential smoothing.

— A *time series* is a set of data points sampled at periodic instances.

— *Smoothing* is the process of creating an approximation function based on time series that while capturing patterns in the data is relatively insensitive to noise.

— Natural first step is to use moving average. But average on how many samples (say k)? What to do until the first k samples have been received?

— Next attempt might be to use weighted moving average. Advantage is that you can give higher weight to more recent terms. But still same disadvantage as simple moving average technique.

— *Exponential Smoothing* attempts to remove this disadvantage.

Exponential smoothing is defined iteratively for $\alpha \in (0,1)$ as

$$\begin{aligned} s_1 &= x_0 \\ s_t &= \alpha x_{t-1} + (1-\alpha) s_{t-1} \end{aligned}$$

Smaller values of α means greater smoothing e.g., $\alpha = 0$ gives a constant function $(= x_0)$. Note that the selection of x_0 is important – and increases in importance as α. Some advantages over moving average: all data points matter (decreasing importance with time) and computationally only last data point need be stored.

8. BAYESIAN TRADERS

Suppose data is drawn from an exponential family distribution. Further suppose, following a Bayesian model, that each trader has some prior belief distribution over the parameters of the data distribution. Assuming a conjugate prior, this prior distribution is also an exponential family distribution.

The data distribution is given by $\exp\{\beta \cdot \phi(x) - \psi(\beta)\}$ where $\psi()$ is the log partition function and $\phi()$ are the sufficient statistics. Then the conjugate prior parametrized by $b_0 = (n\nu, n)$ is given by $\exp\{b_0 \cdot (\beta, \psi(\beta)) - \Psi(b_0)\}$where $\Psi()$ is the corresponding log partition function. The posterior distribution is given by $b = (n\nu + m\hat{\mu}, n+m)$ where $\hat{\mu}$ is the empirical mean of the sufficient statistics of the m data points drawn from the data distribution, $\hat{\mu} = \sum_{i=1}^{m} \phi(x_i)$. We also have $\mathbf{E}_{\beta \sim b_0} \mathbf{E}_{x \sim \beta}[\phi(x)] = \nu$.

8.1. Linear Utility

Suppose the trader wishes to maximize his expected payoff. Then the number of shares δ that he purchases when the current market state is θ is given by

$$\arg\max_{\delta} \mathbf{E}_{\beta \sim b} \mathbf{E}_{x \sim \beta} \left[\delta \phi(x) - C(\delta + \theta) + C(\theta)\right]$$

But, $\mathbf{E}_{\beta \sim b} \mathbf{E}_{x \sim \beta}[\phi(x)] = \frac{n\nu + m\hat{\mu}}{n+m}$. To obtain the maximum, we set the gradient to 0. Thus, we have

$$\nabla C(\delta + \theta) = \frac{n\nu + m\hat{\mu}}{n+m}$$

For an exponential family prediction market, the final market state is given by

$$\nabla C^* \left(\frac{n\nu + m\hat{\mu}}{n+m} \right)$$

8.2. Exponential Utility

Now suppose the trader has exponential utility, given by

$$U(w) = -\frac{1}{a} \exp(-aw)$$

where a is the coefficient of risk aversion. Suppose the trader wishes to maximize his expected utility. Then the number of shares δ that he purchases when the current market state is θ is given by

$$\begin{aligned} & \arg\max_{\delta} \mathbf{E}_{\beta\sim b}\mathbf{E}_{x\sim\beta} U\left[\delta\phi(x) - C(\delta+\theta) + C(\theta)\right] \\ = \; & \arg\max_{\delta} \mathbf{E}_{\beta\sim b}\mathbf{E}_{x\sim\beta} - \frac{1}{a}\exp\left[-a\delta\phi(x) + aC(\delta+\theta) - aC(\theta)\right] \end{aligned}$$

$$\begin{aligned} \mathbf{E}_{\beta\sim b}\mathbf{E}_{x\sim\beta}\exp\left[-a\delta\phi(x)\right] \; = \; & \int_{\mathcal{B}}\int_{\mathcal{X}} \exp[-a\delta\phi(x)] \; \exp\{\beta\cdot\phi(x) - \psi(\beta)\}\, dx \\ & \quad \exp\{(n\nu + m\hat{\mu})\cdot\beta - (n+m)\psi(\beta)) - \Psi((n\nu + m\hat{\mu}, n+m))\}\, d\beta \\ = \; & \int_{\mathcal{B}} \exp\{(n\nu + m\hat{\mu})\cdot\beta - (n+m)\psi(\beta)) - \Psi((n\nu + m\hat{\mu}, n+m))\} \\ & \quad \exp\{\psi(\beta - a\delta) - \psi(\beta)\}\; d\beta \int_{\mathcal{X}} \exp[(\beta - a\delta)\cdot\phi(x) - \psi(\beta - a\delta)]\, dx \\ = \; & \int_{\mathcal{B}} \exp\{(n\nu + m\hat{\mu})\cdot\beta - (n+m)\psi(\beta)) - \Psi((n\nu + m\hat{\mu}, n+m))\} \\ & \quad \exp\{\psi(\beta - a\delta) - \psi(\beta)\}\; d\beta \end{aligned}$$

9. REINTERPRETING REPEATED TRADES IN A MARKET

Let us suppose that traders in this market have exponential utility

$$U(w) = -\frac{1}{a}\exp(-aw).$$

Here a is the coefficient of risk aversion (higher means more risk averse, and the utility function is more concave).

Let δ_1^* be the optimal vector of shares the agent decides to trade on first entering the market with exponential family belief parametrized by natural parameter $\hat{\theta}$. Thus, his belief distribution is given by the pdf

$$\exp\{\hat{\theta}\phi(x) - T(\hat{\theta})\}]$$

where $T(\hat{\theta}) = \int_{\mathcal{X}} \exp\{\hat{\theta}\phi(x)\}\, dx$ is the log partition function and $\phi(x)$ are the sufficient statistics. On a subsequent entry into this market when the market state is θ', his optimal purchase δ_2^* is given by the solution of

$$\arg\max_{\delta_2} \mathbf{E}_{x\sim\hat{\theta}} U[\text{payoff}]$$

Here, the payoff given eventual outcome x is $[\delta_1^* + \delta_2^*]\phi(x) - C(\delta_1^* + \theta) + C(\theta) - C(\delta_2^* + \theta') + C(\theta')$. Here $\theta' = \theta + \delta_1^* + \delta'$ where δ' captures the movement of the share vector by other traders in the market. The expected utility for this payoff is as follows.

$$
\begin{aligned}
& \arg\max_{\delta_2} \mathbf{E}\left[(\delta_1^* + \delta_2)\phi(x) - C(\delta_1^* + \theta) + C(\theta) - C(\delta_2 + \theta') + C(\theta')\right] \\
= \; & \arg\max_{\delta_2} \int_{\mathcal{X}} -\frac{1}{a}[\exp(-a(\delta_1^* + \delta_2)\phi(x) \\
& +aC(\delta_1^* + \theta) - aC(\theta) + aC(\delta_2 + \theta') - aC(\theta'))\exp\{\hat{\theta}\phi(x) - T(\hat{\theta})\}]\, dx \\
= \; & \arg\max_{\delta_2} \int_{\mathcal{X}} -\frac{1}{a}[\exp\{-a\delta_2\phi(x) + aC(\delta_2 + \theta') - aC(\theta') \\
& +aC(\delta_1^* + \theta) - aC(\theta)\}\exp\{(\hat{\theta} - a\delta_1^*)\phi(x) - T(\hat{\theta})\}]\, dx \\
= \; & \arg\max_{\delta_2} \int_{\mathcal{X}} U[N(\delta_2, \theta')]\exp\{(\hat{\theta} - a\delta_1^*)\phi(x) - T(\hat{\theta}) + aC(\delta_1^* + \theta) - aC(\theta)\}]\, dx
\end{aligned}
$$

Here the first equality follows from the fact that we are taking expectation over the traders belief parameter $\hat{\theta}$ and the second equality follows simply from rearranging the factors of $\phi(x)$. And lastly we have written $-\frac{1}{a}[\exp\{-a\delta_2\phi(x) + aC(\delta_2 + \theta') - aC(\theta')]$ as $U[N(\delta_2, \theta')]$ where $N(\delta_2, \theta')$ is the net payoff when δ_2 shares are purchased when the current market state is θ'.

This is equivalent to maximizing expected utility of $N(\delta_2, \theta')$, where expectation is taken with respect to an exponential family distribution over $\mathcal{X}$ parametrized by $\hat{\theta} - a\delta_1^*$, so long as

$$
\int_{\mathcal{X}} \exp\{(\hat{\theta} - a\delta_1^*)\phi(x) - T(\hat{\theta}) + aC(\delta_1^* + \theta) - aC(\theta)\}]\, dx = c \times \int_{\mathcal{X}} \exp\{(\hat{\theta} - a\delta_1^*)\phi(x) - T(\hat{\theta} - a\delta_1^*)\}\, dx
$$

In other words we want the following statement to hold.

$$
\begin{aligned}
\exists c : c \times \exp\{-T(\hat{\theta}) + aC(\delta_1^* + \theta) - aC(\theta)\} \; &= \; T(\hat{\theta} - a\delta_1^*) \\
&= \int_{\mathcal{X}} \exp\{(\hat{\theta} - a\delta_1^*)\phi(x)\}\; dx
\end{aligned}
$$

where c is a constant with respect to δ_2 and x (but could potentially depend on $\hat{\theta}$, a and δ_1^*). This is true because the right hand side of the equation is a function of only $\hat{\theta}$, a and δ_1^*. Let $\Theta \stackrel{\text{def}}{=} \hat{\theta} - a\delta_1^*$ be the effective belief. Thus we have that the trader chooses his share vector as follows.

$$
\begin{aligned}
& \arg\max_{\delta_2} \int_{\mathcal{X}} U[N(\delta_2, \theta')]\exp\{(\hat{\theta} - a\delta_1^*)\phi(x) - T(\hat{\theta}) + aC(\delta_1^* + \theta) - aC(\theta)\}\, dx \\
= \; & \arg\max_{\delta_2} \int_{\mathcal{X}} U[N(\delta_2, \theta')]\exp\{(\hat{\theta} - a\delta_1^*)\phi(x) - T(\hat{\theta} - a\delta_1^*)\}]\, dx \\
= \; & \arg\max_{\delta_2} \int_{\mathcal{X}} -\frac{1}{a}\exp\{-a\delta_2\phi(x) + aC(\delta_2 + \theta') - aC(\theta')\}\exp\{\Theta \cdot \phi(x) - T(\Theta)\}\, dx \\
= \; & \arg\max_{\delta_2} U\left[-C(\delta_2 + \theta') + C(\theta') - \frac{1}{a}T(\Theta - a\delta_2) + \frac{1}{a}T(\Theta)\right]
\end{aligned}
$$

This follows from the argument detailed in in exp-util and simple substitution. Further, the maximizer is $\delta_2^* = (\Theta - \theta')/(1 + a)$, which moves the share vector to

$\theta' + \delta_2^* = \frac{1}{1+a}\Theta + \frac{a}{1+a}\theta'$ which is a convex combination of the effective belief and the current market state.

In other words, an exponential utility maximizing trader who has belief $\hat{\theta}$ with prior exposure δ in a market will behave identically as an exponential utility maximizing trader with belief $\hat{\theta} - a\delta$ and no prior exposure in the market. Here a is the utility parameter. This means that financial exposure can be equivalently understood as changing the privately held beliefs.

Note that as the trader becomes more risk averse, the belief is updated more aggressively. As $a \to 0$, the trader becomes more risk neutral and his effective belief is closer to his true belief.

9.1. Equilibrium in a market with multiple traders

We have shown that every exponential-utility maximizing trader picks the share vector δ_i so as to maximize the utility of $-C(\delta_i + \theta) + C(\theta) - \frac{1}{a_i}T(\beta_i - a_i\delta_i) + \frac{1}{a_i}T(\beta_i)$ where θ is the initial market state, a_i the utility parameter and β_i the belief parameter of trader i. To maximize utility, it suffices to maximize the argument, assuming that $C() = T() = \psi()$.

Now consider a market with multiple exponential-utility maximizing traders. We wish to compute the equilibrium state. Let us assume that equilibrium is reached at a market state $\theta + \sum_i \delta_i$ where θ is the initial market state before trading begins and δ_i is each trader's purchase.

Recall the following result from game theory.

THEOREM 9.1. *Let $U_i(\vec{\delta})$ be the utility function of the i^{th} trader given strategies $\vec{\delta} \stackrel{\text{def}}{=} \delta_1, \ldots, \delta_i, \ldots, \delta_n$. If there exists a potential function $\phi(\vec{\delta})$ such that*

$$U_i(\vec{\delta}) - U_i(\vec{\delta}_{-i}, \delta_i') = \phi(\vec{\delta}) - \phi(\vec{\delta}_{-i}, \delta_i')$$

then when $\phi(\vec{\delta})$ is maximized, $\vec{\delta}$ is an equilibrium.

Let

$$\phi(\vec{\delta}) \stackrel{\text{def}}{=} \psi(\theta + \sum_i \delta_i) + \sum_i \frac{1}{a_i}\psi(\beta_i - a_i\delta_i)$$

Now $U_i(\vec{\delta}) = -\psi(\theta + \sum_j \delta_j) + \psi(\theta + \sum_{j\neq i} \delta_j) - \frac{1}{a_i}\psi(\beta_i - a_i\delta_i) + \frac{1}{a_i}\psi(\beta_i)$. Thus, Theorem **??** applies and we can find the equilibrium market state by maximizing $\phi(\vec{\delta})$ for each δ_i.

$$\begin{aligned} \nabla_{\delta_i}\phi(\vec{\delta}) &= \nabla\psi(\theta + \sum_{j=1}^{n} \delta_i) - \nabla\psi(\beta_i - a_i\delta_i) \\ &= 0 \end{aligned}$$

This can be achieved by equating the arguments. That is, for each trader i,

$$\beta_i - a_i\delta_i = \theta + \sum_{j=1}^{n} \delta_j \tag{21}$$

Rewriting, we have for each trader i,

$$\frac{\beta_i}{a_i} - \delta_i = \frac{1}{a_i}(\theta + \sum_{j=1}^{n} \delta_j)$$

Thus,

$$\sum_{i=1}^{n} \left(\frac{\beta_i}{a_i}\right) - \sum_{i=1}^{n} \delta_i = \left(\theta + \sum_{j=1}^{n} \delta_j\right) \sum_{i=1}^{n} \frac{1}{a_i}$$

And

$$\sum_{j=1}^{n} \delta_j = \frac{\sum_{i=1}^{n} \left(\frac{\beta_i}{a_i}\right) - \theta \sum_{i=1}^{n} \left(\frac{1}{a_i}\right)}{1 + \sum_{i=1}^{n} \frac{1}{a_i}}$$

Substituting in Equation **??** we have the following expression for the final market state.

$$\theta + \sum_{j=1}^{n} \delta_j = \frac{\theta + \sum_{i=1}^{n} \left(\frac{\beta_i}{a_i}\right)}{1 + \sum_{i=1}^{n} \frac{1}{a_i}}$$

Let us modify the cost function so that it includes a liquidity parameter:

$$C_\lambda(q) \stackrel{\text{def}}{=} \lambda\psi(q/\lambda)$$

9.1.1. Lower Bound on Expected Payoff for Exponential Utility Traders. Let θ be the current market state. We have shown that an exponential utility trader with belief distribution parametrized by β will move the market state to $\theta' = \frac{1}{1+a}\beta + \frac{a}{1+a}\theta$. Therefore, the trader's expected net payoff is given by

$$\begin{aligned}
& \mathbf{E}_{x\sim P_\beta}[C(\theta) - C(\theta') - (\theta - \theta')\phi(x)] \\
= \; & \psi(\theta) - \theta\mathbf{E}_{x\sim P_\beta}[\phi(x)] - (\psi(\theta') - \theta'\mathbf{E}_{x\sim P_\beta}[\phi(x)]) \\
= \; & \psi(\theta) - \theta\nabla\psi(\beta) - (\psi(\theta') - \theta'\nabla\psi(\beta)) \\
= \; & \psi(\theta) - \psi(\beta) - \nabla\psi(\beta)(\theta - \beta) - (\psi(\theta') - \psi(\beta) - \nabla\psi(\beta)(\theta' - \beta)) \\
= \; & D_\psi(\theta, \beta) - D_\psi(\theta', \beta) \\
\geq \; & \frac{1}{a} D_\psi(\theta', \beta) \geq 0
\end{aligned}$$

The second to last inequality holds since $D_\psi(\theta', \beta)$ is convex in θ' and we have:

$$\begin{aligned}
D_\psi(\theta', \beta) \; &= \; D_\psi\left(\frac{1}{1+a}\beta + \frac{a}{1+a}\theta, \beta\right) \\
&\leq \; \frac{1}{1+a} D_\psi(\beta, \beta) + \frac{a}{1+a} D_\psi(\theta, \beta) \\
&= \; \frac{a}{1+a} D_\psi(\theta, \beta)
\end{aligned}$$

Thus, a trader who moves the market state can expect his profit to be positive and at least $\frac{1}{a} D_\psi(\theta', \beta)$.

10. CONCLUSIONS