Reinterpreting repeated trades in a market

Let us suppose that traders in this market have exponential utility

$$U(w) = -\frac{1}{a}\exp(-aw).$$

Here a is the coefficient of risk aversion (higher means more risk averse, and the utility function is more concave).

Let δ_1^* be the optimal vector of shares the agent decides to trade on first entering the market with exponential family belief parametrized by natural parameter $\hat{\theta}$. Thus, his belief distribution is given by the pdf

$$\exp\{\hat{\theta}\phi(x) - T(\hat{\theta})\}]$$

where $T(\hat{\theta}) = \int_{\mathcal{X}} \exp\{\hat{\theta}\phi(x)\}\ dx$ is the log partition function and $\phi(x)$ are the sufficient statistics. On a subsequent entry into this market when the market state is θ', his optimal purchase δ_2^* is given by the solution of

$$\arg\max_{\delta_2} \mathbb{E}_{x\sim\hat{\theta}} U[\text{payoff}]$$

Here, the payoff given eventual outcome x is $[\delta_1^* + \delta_2^*]\phi(x) - C(\delta_1^* + \theta) + C(\theta) - C(\delta_2^* + \theta') + C(\theta')$. Here $\theta' = \theta + \delta_1^* + \delta'$ where δ' captures the movement of the share vector by other traders in the market. The expected utility for this payoff is as follows.

$$\begin{aligned}
& \arg\max_{\delta_2} \mathbb{E}\left[(\delta_1^* + \delta_2)\phi(x) - C(\delta_1^* + \theta) + C(\theta) - C(\delta_2 + \theta') + C(\theta')\right] \\
= \;& \arg\max_{\delta_2} \int_{\mathcal{X}} -\frac{1}{a}[\exp(-a(\delta_1^* + \delta_2)\phi(x) \\
& +aC(\delta_1^* + \theta) - aC(\theta) + aC(\delta_2 + \theta') - aC(\theta'))\exp\{\hat{\theta}\phi(x) - T(\hat{\theta})\}]\,dx \\
= \;& \arg\max_{\delta_2} \int_{\mathcal{X}} -\frac{1}{a}[\exp\{-a\delta_2\phi(x) + aC(\delta_2 + \theta') - aC(\theta') \\
& +aC(\delta_1^* + \theta) - aC(\theta)\}\exp\{(\hat{\theta} - a\delta_1^*)\phi(x) - T(\hat{\theta})\}]\,dx \\
= \;& \arg\max_{\delta_2} \int_{\mathcal{X}} U[N(\delta_2, \theta')]\exp\{(\hat{\theta} - a\delta_1^*)\phi(x) - T(\hat{\theta}) + aC(\delta_1^* + \theta) - aC(\theta)\}]\,dx
\end{aligned}$$

where we have written $-\frac{1}{a}[\exp\{-a\delta_2\phi(x) + aC(\delta_2 + \theta') - aC(\theta')]$ as $U[N(\delta_2, \theta')]$ where $N(\delta_2, \theta')$ is the net payoff when δ_2 shares are purchased when the current market state is θ'. This is equivalent to maximizing expected utility of $N(\delta_2, \theta')$ so long as

$$\int_{\mathcal{X}} \exp\{(\hat{\theta} - a\delta_1^*)\phi(x) - T(\hat{\theta}) + aC(\delta_1^* + \theta) - aC(\theta)\}]\,dx = c \times \int_{\mathcal{X}} \exp\{(\hat{\theta} - a\delta_1^*)\phi(x) - T(\hat{\theta} - a\delta_1^*)\}\,dx$$

In other words we want the following statement to hold.

$$\begin{aligned}
\exists c : c \times \exp\{-T(\hat{\theta}) + aC(\delta_1^* + \theta) - aC(\theta)\} &= T(\hat{\theta} - a\delta_1^*) \\
&= \int_{\mathcal{X}} \exp\{(\hat{\theta} - a\delta_1^*)\phi(x)\}\, dx
\end{aligned}$$

where c is a constant with respect to δ_2 and x (but could potentially depend on $\hat{\theta}$, a and δ_1^*). This is true because the right hand side of the equation is a function of only $\hat{\theta}$, a and δ_1^*.

Thus, optimizing a trade in a market with prior exposure δ_1^* and belief parameter $\hat{\theta}$ is equivalent to optimizing with a belief parameter $\hat{\theta} - a\delta_1^*$. Thus trades in the exponential family market can be understood as updates on the traders' beliefs.