1. COST FUNCTION WITH A LIQUIDITY PARAMETER

We may introduce a liquidity parameter to the cost function as $C_\lambda(\theta) = \lambda C\left(\frac{\theta}{\lambda}\right)$. We would like to find an expression for a lower bound on expected payoff. We would also like to find an expression for final market state for exponential utility traders with exponential family beliefs.

1.1. lower bound on expected payoff

The prediction market is set up with cost function $C_\lambda(\theta)$ and payoff for a unit share vector as $\phi(x)$. To find a lower bound on expected payoff, we first note that

$$\begin{aligned}
\mathbf{E}_{x\sim p(x;\beta)}[C_\lambda(\theta) - \theta\phi(x)] &= C_\lambda(\theta) - \theta\nabla T(\beta) \\
&= \lambda T(\theta/\lambda) - \theta\nabla T(\beta) \\
&= \lambda[T(\theta/\lambda) - \theta/\lambda\nabla T(\beta) - T(\beta) + \beta\nabla T(\beta)] + \lambda T(\beta) - \theta\lambda\nabla T(\beta) \\
&= \lambda D_T(\theta/\lambda, \beta) + \lambda[T(\beta) - \theta\nabla T(\beta)]
\end{aligned}$$

where $D_T(x_1, x_2) = T(x_1) - T(x_2) - \nabla T(x_1 - x_2)$ is the Bregman Divergence between x_1 and x_2 based on T.

Thus, the expected payoff for a trader with exponential family belief parameter β who moves the market state from θ to θ' can be written as

$$\mathbf{E}_{x\sim p(x;\beta)}\left[C_\lambda(\theta) - \theta\phi(x) - (C_\lambda(\theta') - \theta'\phi(x))\right] = \lambda[D_T(\theta/\lambda, \beta) - D_T(\theta'/\lambda, \beta)]$$

If $\theta' = a\beta + (1-a)\theta$ for some $a \in [0,1]$ we can write

$$\begin{aligned}
D_T(\theta'/\lambda, \beta) &\leq aD_T(\beta/\lambda, \beta) + (1-a)D_T(\theta/\lambda, \beta) \\
D_T(\theta/\lambda, \beta) - D(\theta'/\lambda, \beta) &\geq aD_T(\theta/\lambda, \beta) - aD_T(\beta/\lambda, \beta)
\end{aligned}$$

1.2. exponential utility traders with exponential family beliefs

As before the prediction market is set up with cost function $C_\lambda(\theta)$ and payoff for a unit share vector as $\phi(x)$. Thus, the payoff of a trader who moves the market state from θ to θ' is given by $C_\lambda(\theta) - C_\lambda(\theta') + \delta\phi(x)$ where $\delta = \theta' - \theta$ and the expected utility of the payoff for a trader with exponential belief parameter β is given by

$$\begin{aligned}
&\mathbf{E}_{x\sim p(x;\beta)}U[C_\lambda(\theta) - C_\lambda(\theta') + \delta\phi(x)] \\
&= U[C_\lambda(\theta) - C_\lambda(\theta+\delta) + \frac{1}{a}(T(\beta) - T(\beta - a\delta))] \\
&= U[\lambda C(\theta/\lambda) - \lambda C\left(\frac{\theta+\delta}{\lambda}\right) + \frac{1}{a}(T(\beta) - T(\beta - a\delta))] \\
&= U[\lambda T(\theta/\lambda) - \lambda T\left(\frac{\theta+\delta}{\lambda}\right) + \frac{1}{a}(T(\beta) - T(\beta - a\delta))]
\end{aligned}$$

Here T is the log partition function; for the exponential family prediction market $C = T$. To maximize the utility, we set the gradient of the argument with respect to δ to 0.

$$\nabla T\left(\frac{\theta+\delta}{\lambda}\right) = \nabla T(\beta - a\delta)$$

So we have for an optimal choice of δ,

$$\begin{aligned}\frac{\theta+\delta}{\lambda} &= \beta - a\delta \\ \implies \delta &= \frac{\beta\lambda-\theta}{1+a\lambda} \\ \implies \theta+\delta &= \lambda\frac{\beta+a\delta}{1+a\lambda}\end{aligned}$$