Black-Scholes-Merton and the Greeks

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Abstract

The purpose of this vignette is to demonstrate the Black-Scholes-Merton pricing formulas and the "Greeks" as outlined in Chapter 4 and Chapter 5 of Valuation and Risk Models.

Contents

1 Black-Scholes-Merton Pricing Formulas

This section focuses on the application of the Black-Scholes-Merton pricing formulas for European call and put options. For derivation and theoretical background, the reader is encouraged to to study Chapter 4 of Valuation and Risk Models.

The Black-Scholes-Merton pricing formulas for European call and put options are

Call

$$c = S_0N(d_1) - Ke^{-rT}N(d_2)$$

Put

$$p = Ke^{-rT}N(-d_2) - S_0N(-d_1)$$

where

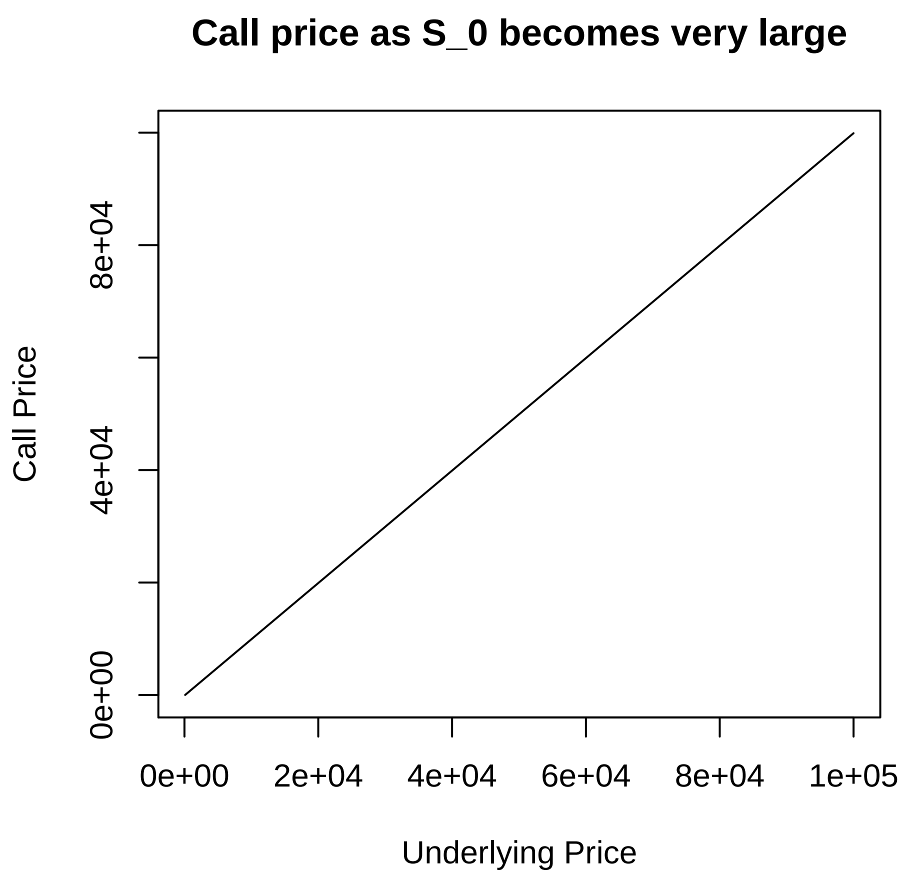
$$\begin{aligned}
d_1 &= \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\
d_2 &= \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \\
S_0 &\quad \text{is the underlying stock price at } t = 0 \\
K &\quad \text{is the strike price} \\
r &\quad \text{is the risk free rate} \\
T &\quad \text{is the time to maturity in years} \\
\sigma &\quad \text{is the volatility of the stock} \\
N(.) &\quad \text{is the cumulative distribution function for a standard normal distribution}
\end{aligned}$$

1.1 Properties

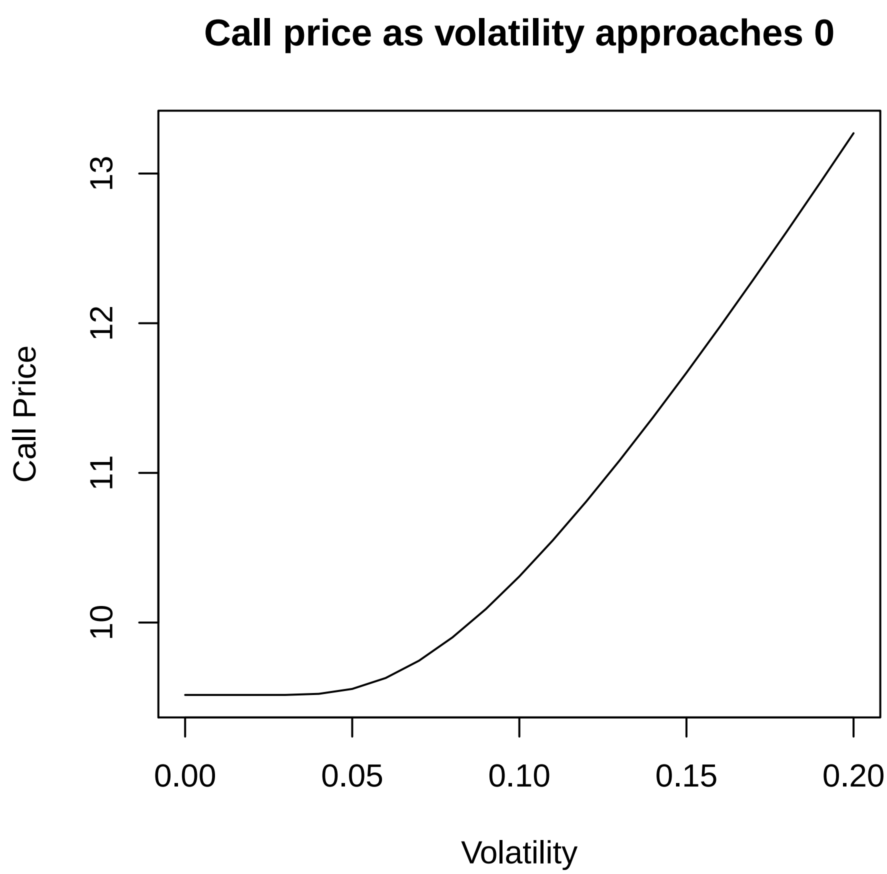
For a call option, as the underlying price, S_0, becomes very large, the option will almost surely be exercised. The price of the call then becomes

$$S_0 - Ke^{-rT}$$

```r
# Demonstrate the property of the Black-Scholes-Merton formula for a call
# option as the underlying price becomes very large
S0 <- seq(100, 100000, 100)
eu.call <- optionSpec(style = "european",
                      type = "call",
                      S0 = S0,
                      K = 100,
                      maturity = 1,
                      r = 0.1,
                      volatility = 0.2)
call.price <- optionValue(option = eu.call, method = "Black-Scholes")
plot(S0, call.price, xlab="Underlying Price", ylab="Call Price",
     main="Call price as S_0 becomes very large", type="l")
```




```r
# Demonstrate the property of the Black-Scholes-Merton formula for a call
# option as the volatility approaches 0
sigma <- seq(0.2, 0, -0.01)
eu.call <- optionSpec(style = "european",
                      type = "call",
                      S0 = 100,
                      K = 100,
                      maturity = 1,
                      r = 0.1,
                      volatility = sigma)
call <- optionValue(option = eu.call, method = "Black-Scholes")
# S_0 - K * e^{-r T}
100 - 100 * exp(-0.1 * 1)

## [1] 9.516

plot(sigma, call, ylab="Call Price", xlab="Volatility",
     main="Call price as volatility approaches 0", type = "l")
```



Example 4.6: The stock price 6 months for the expiration of an option is \$42, the exercise price of the option is \$40, the risk-free interest rate is 10% per annum, and the volatility is 20% per annum.

```
# The stock price 6 months for the expiration of an option is £42,
# the exercise price of the option is £40, the risk-free interest
# rate is 10% per annum, and the volatility is 20% per annum.

# Price the European call option
eu.call <- optionSpec(style = "european",
                      type = "call",
                      S0 = 42,
                      K = 40,
                      maturity = 0.5,
                      r = 0.1,
                      volatility = 0.2)
call <- optionValue(option = eu.call, method = "Black-Scholes")
call
```

```
## [1] 4.759

# Price the European put option
eu.put <- eu.call
eu.put$type <- "put"
put <- optionValue(option = eu.put, method = "Black-Scholes")
put

## [1] 0.8086
```

2 Implied Volatilities

The volatility the stock price is not directly observable and must be implied by option prices in the market. Here we calculate the implied volatility of a European call option with a price of \$1.875 and $S_0 = 21$, $K = 20$, $r = 0.1$, and $T = 0.25$.

```
# Compute the implied volatility of a European call option with a
# price of £1.875 and S_0 = 21, K = 20, r = 0.1, and T = 0.25
eu.call <- optionSpec(style = "european",
                      type = "call",
                      S0 = 21,
                      K = 20,
                      maturity = 0.25,
                      r = 0.1)
impliedVolatility(eu.call, 1.875)

## [1] 0.2345
```

3 Dividends

Example 4.9: Consider a European call option on a stock when there are ex-dividend dates in two months and five months. The dividend on each ex-dividend date is expected to be \$0.50. The current share price is \$40, the exercise price is \$40, and the stock price volatility is 30% per annum, the risk-free rate of interest is 9% per annum, and the time to maturity is 6 months. Here we compute the value of a European call option with known dividends.

```
# Consider a European call option on a stock when there are
# ex-dividend dates in two months and five months. The dividend
# on each ex-dividend date is expected to be \£0.50. The current
# share price is \£40, the exercise price is \£40, and the stock
# price volatility is 30\% per annum, the risk-free rate of
# interest is 9\% per annum, and the time to maturity is 6 months.

# Subtract the present value of the dividends from the underlying price
S0 <- 40 - (0.5 * exp(-0.09 * 2 / 12) + 0.5 * exp(-0.09 * 5 / 12))
```

```r
eu.call <- optionSpec(style = "european",
                      type = "call",
                      S0 = S0,
                      K = 40,
                      maturity = 0.5,
                      r = 0.09,
                      volatility = 0.3)
optionValue(eu.call, "Black-Scholes")

## [1] 3.671
```

4 The Greek Letters

This section introduces what is referred to as the "Greeks" for European options using the Black-Scholes-Merton formulas. The Greeks measure risk in a position in an option or portfolio of options.

Here we create option specifications for call and put options that will be used in the following sections.

```r
# Specify European call and put options where the current stock price is
# £49, the strike price is £50, the risk-free rate is 5%, the time to
# maturity is 20 weeks, and the volatility is 20%.

eu.call <- optionSpec(style = "european",
                      type = "call",
                      S0 = 49,
                      K = 50,
                      maturity = 20/52,
                      r = 0.05,
                      volatility = 0.2)

eu.put <- optionSpec(style = "european",
                      type = "put",
                      S0 = 49,
                      K = 50,
                      maturity = 20/52,
                      r = 0.05,
                     volatility = 0.2)
```

4.1 Delta

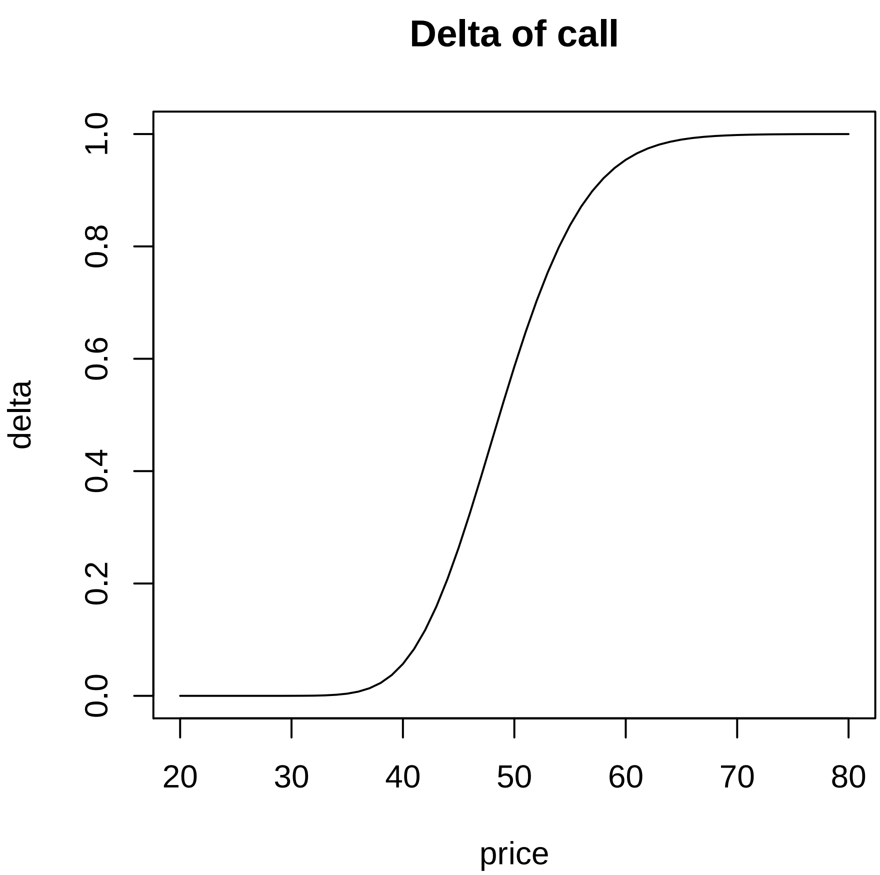
The delta (Δ) of an option is defined as the rate of change of the option price with respect to the price of te underlying asset.

The delta of a European option is given as

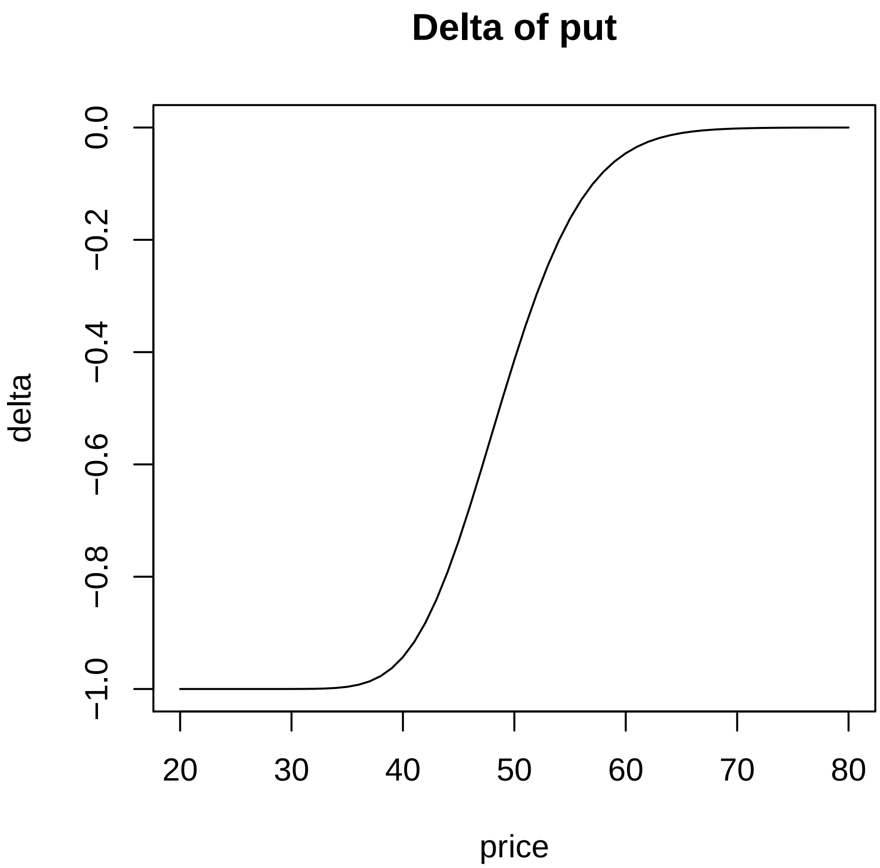
$$\Delta(call) = N(d_1)$$
$$\Delta(put) = N(d_1) - 1$$

```r
# Compute the delta of the European call option
computeGreeks(eu.call, "delta")

## [1] 0.5216
```

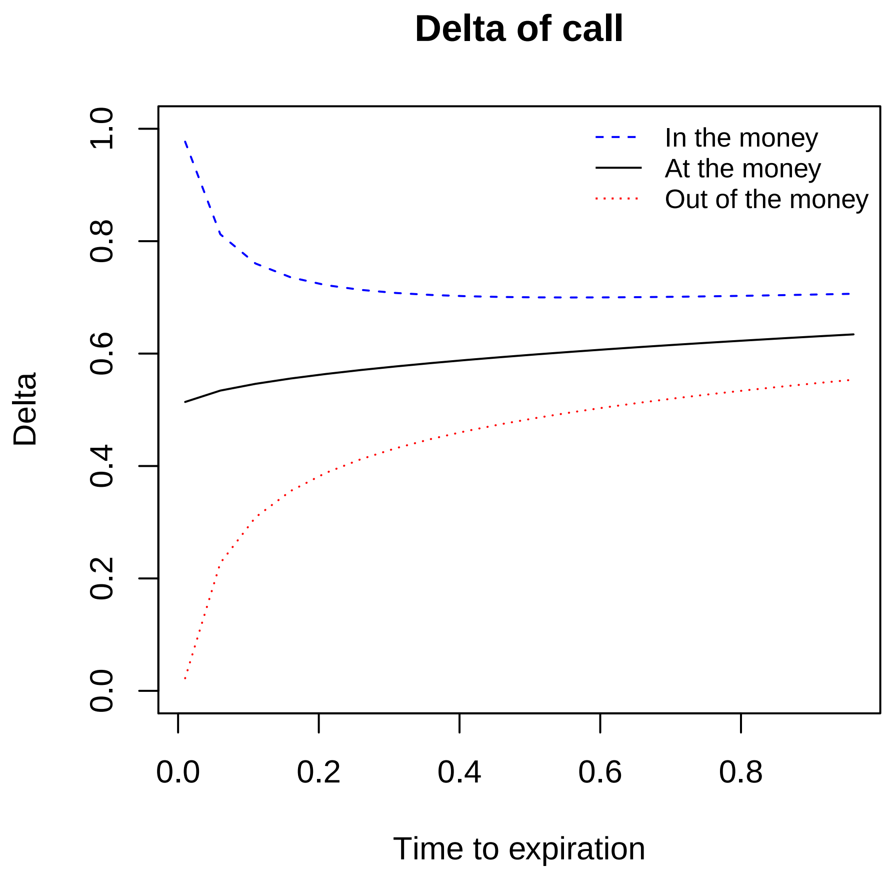
```r
# Delta as the underlying price varies
computeGreeks(eu.call, "delta", prices = seq(20, 80, 1),
              plot = TRUE, main="Delta of call")
```



```r
computeGreeks(eu.put, "delta", prices = seq(20, 80, 1),
              plot = TRUE, main="Delta of put")
```




```r
# Delta as the time to maturity varies for in the money, at the money, and
# out of the money call options
maturity <- seq(0.01, 1, 0.05)
plot(maturity, xlim = range(maturity), ylim = c(0,1),
     type="n", xlab="Time to expiration", ylab="Delta")
lines(x = maturity, y = deltaBS(52, 50, 0.05, 0, 0.2, maturity, "call"),
      lty=2, col="blue")
lines(x = maturity, y = deltaBS(50, 50, 0.05, 0, 0.2, maturity, "call"),
      lty=1, col="black")
lines(x = maturity, y = deltaBS(48, 50, 0.05, 0, 0.2, maturity, "call"),
      lty=3, col="red")
title("Delta of call")
legend("topright", legend=c("In the money", "At the money", "Out of the money"),
       lty=c(2,1,3), col=c("blue", "black", "red"), bty="n", cex=0.8)
```



The delta of a portfolio of options is simply the sum of the delta's of the individual options.

$$\Delta_P = \sum_{i=1}^{n} w_i \Delta_i$$

Suppose a financial instituion has the following three positions in options on a stock.

1. A long position in 100,000 call options with the strike price of \$55 and an expiration date in 3 months. The delta of each option is 0.533.
2. A short position in 200,000 call options with stick price of \$56 and an expiration date in 2 months. The delta of each option is 0.468.
3. A short position in 50,000 options with strike price of \$56 and an expiration date in 2 months. The delta of each option is -0.508.

The delta of the portfolio is

```
100000 * 0.533 - 200000 * 0.468 - 50000 * -0.508

## [1] -14900
```

This means that the portfolio can be made delta neutral by purchasing 14,900 shares of the underlying stock.

4.2 Theta

The theta (Θ) of an option is defined as the rate of change of the value of the option with respect to the passage of time with all else remaining equal.

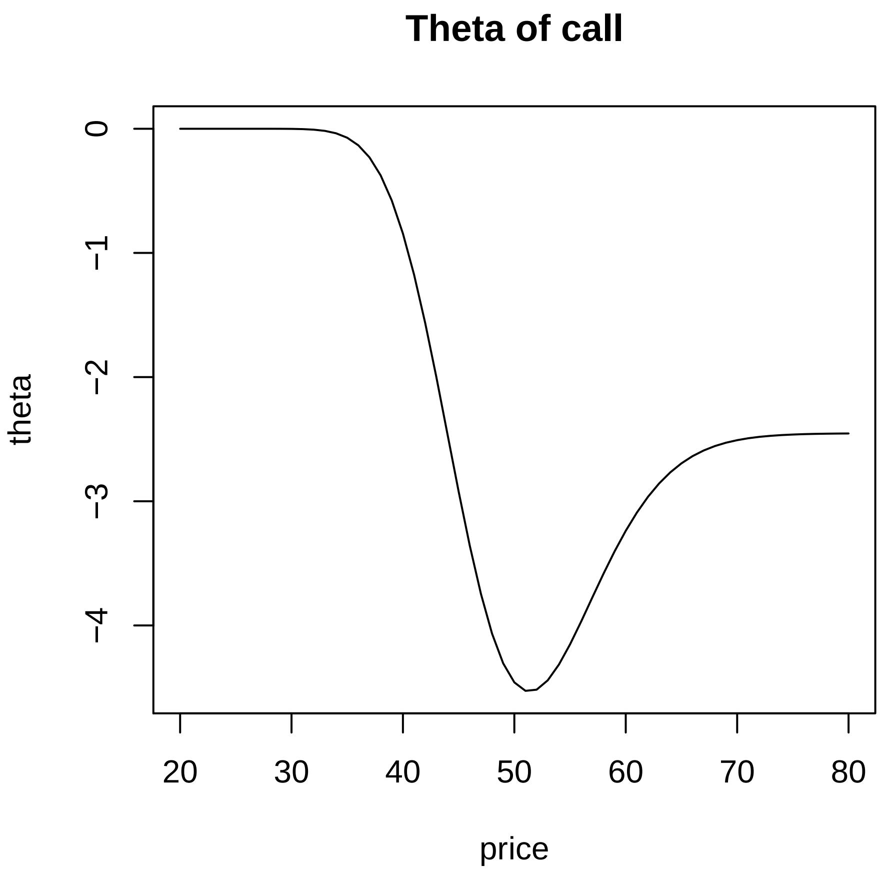
The theta of a European option is given as

$$\Theta(call) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$
$$\Theta(put) = \frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$$

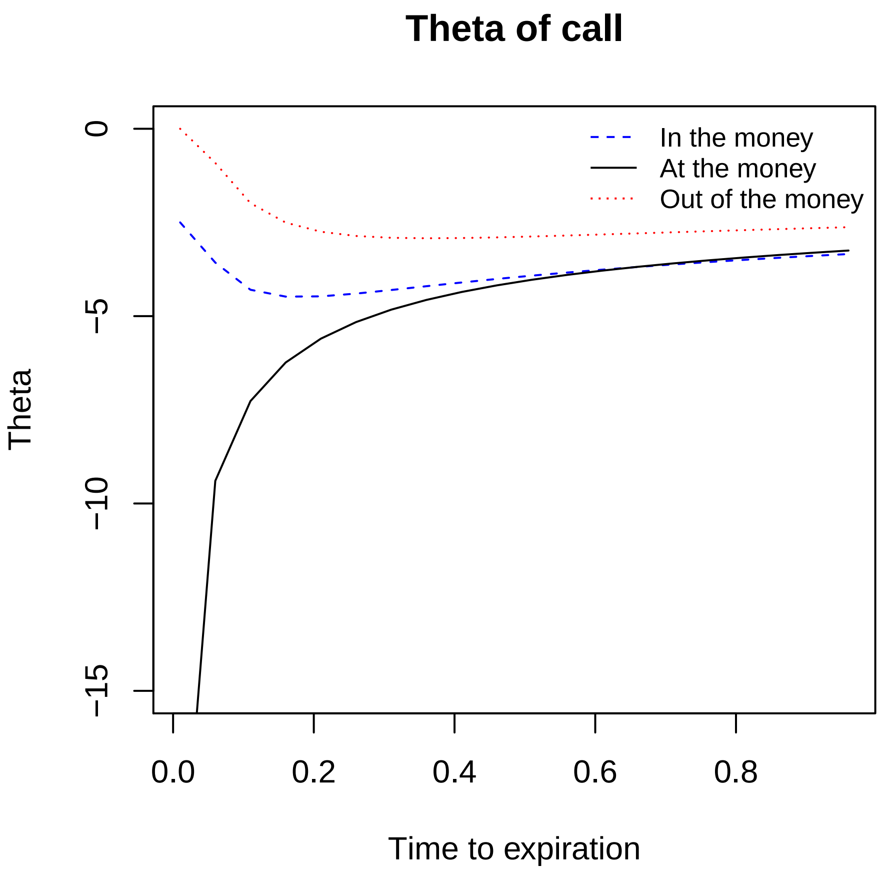
where $N'(.)$ is the probability density function for a standard normal distribution.

```
# Compute the theta of the European call option
computeGreeks(eu.call, "theta")

## [1] -4.305
```

```
# Theta as the underlying price varies
computeGreeks(eu.call, "theta", prices = seq(20, 80, 1),
              plot = TRUE, main="Theta of call")
```




```
# Theta as the time to maturity varies for in the money, at the money, and
# out of the money call options
maturity <- seq(0.01, 1, 0.05)
plot(maturity, xlim = range(maturity), ylim = c(-15,0),
     type="n", xlab="Time to expiration", ylab="Theta")
lines(x = maturity, y = thetaBS(55, 50, 0.05, 0, 0.2, maturity, "call"),
      lty=2, col="blue")
lines(x = maturity, y = thetaBS(50, 50, 0.05, 0, 0.2, maturity, "call"),
      lty=1, col="black")
lines(x = maturity, y = thetaBS(45, 50, 0.05, 0, 0.2, maturity, "call"),
      lty=3, col="red")
title("Theta of call")
legend("topright", legend=c("In the money", "At the money", "Out of the money"),
       lty=c(2,1,3), col=c("blue", "black", "red"), bty="n", cex=0.8)
```



4.3 Gamma

The gamma (Γ) of an option is defined as the rate of change of the delta of the option with respect to the price of the underlying asset.

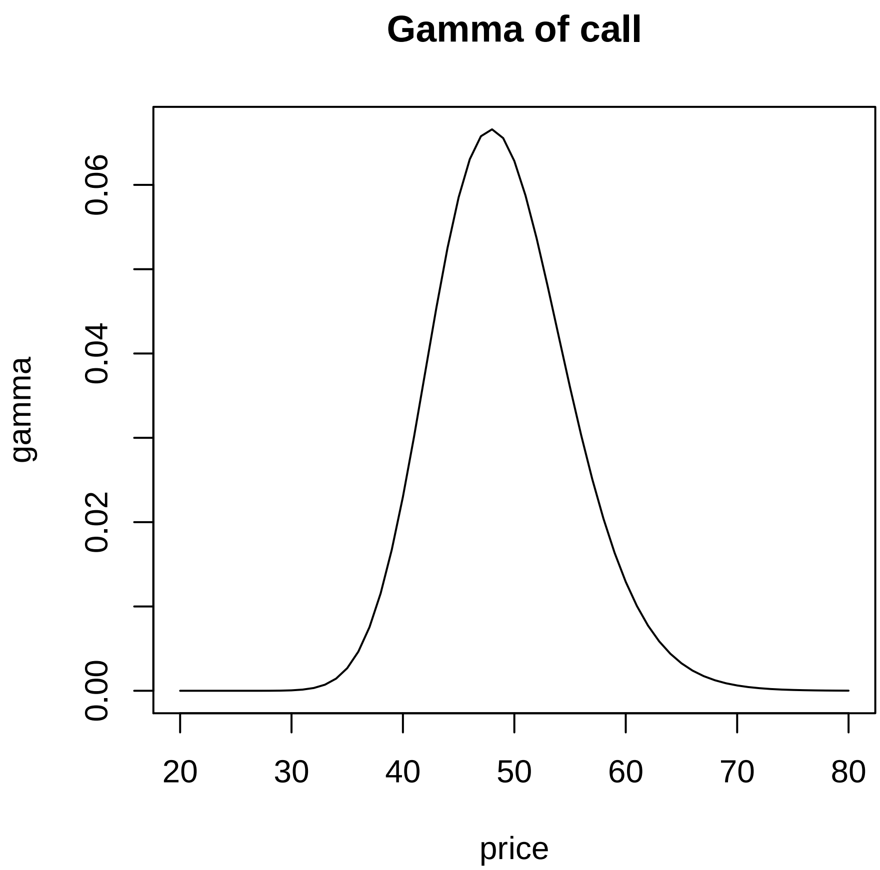
The gamma of a European option is given as

$$\Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}}$$

Note that the gamma for a European put option is equal to the gamma of a European call option.

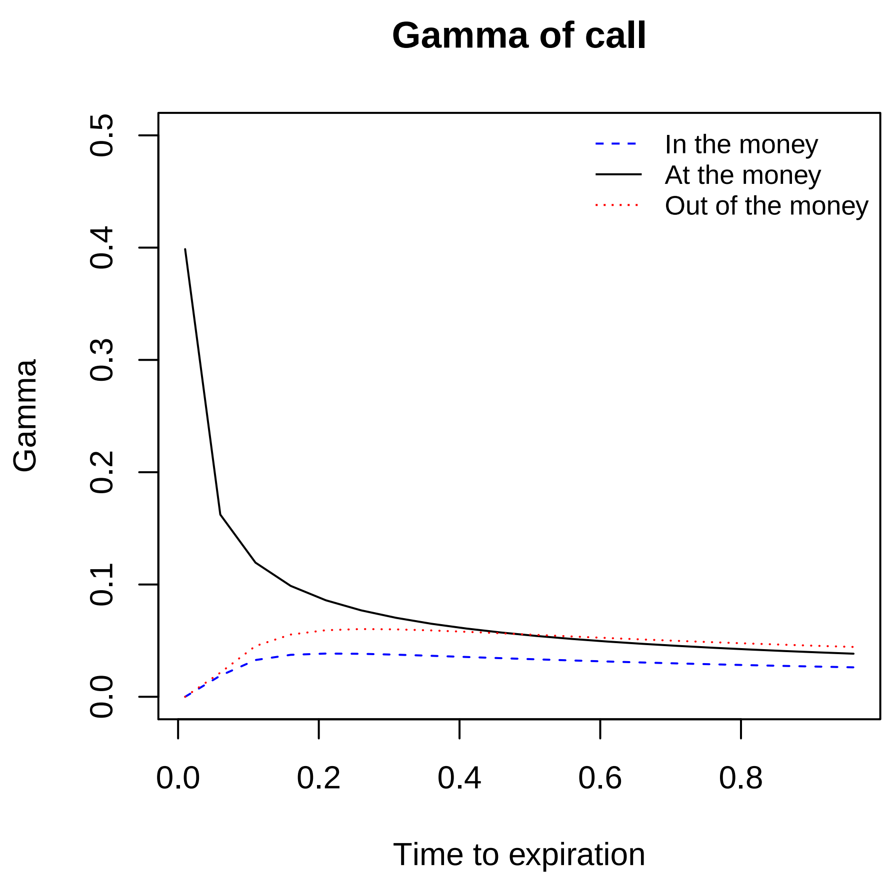
```
# Compute the gamma of the European call option
computeGreeks(eu.call, "gamma")

## [1] 0.06554
```

```r
# Gamma as the underlying price varies
computeGreeks(eu.call, "gamma", prices = seq(20, 80, 1),
              plot = TRUE, main="Gamma of call")
```




```r
# Gamma as the time to maturity varies for in the money, at the money, and
# out of the money call options
maturity <- seq(0.01, 1, 0.05)
plot(maturity, xlim = range(maturity), ylim = c(0,0.5),
     type="n", xlab="Time to expiration", ylab="Gamma")
lines(x = maturity, y = gammaBS(55, 50, 0.05, 0, 0.2, maturity, "call"),
      lty=2, col="blue")
lines(x = maturity, y = gammaBS(50, 50, 0.05, 0, 0.2, maturity, "call"),
      lty=1, col="black")
lines(x = maturity, y = gammaBS(45, 50, 0.05, 0, 0.2, maturity, "call"),
      lty=3, col="red")
title("Gamma of call")
legend("topright", legend=c("In the money", "At the money", "Out of the money"),
```

```
        lty=c(2,1,3), col=c("blue", "black", "red"), bty="n", cex=0.8)
```



4.4 Vega

The vega (ν) of an option is defined as the rate of change of the value of the option with respect to the volatility of the underlying asset.

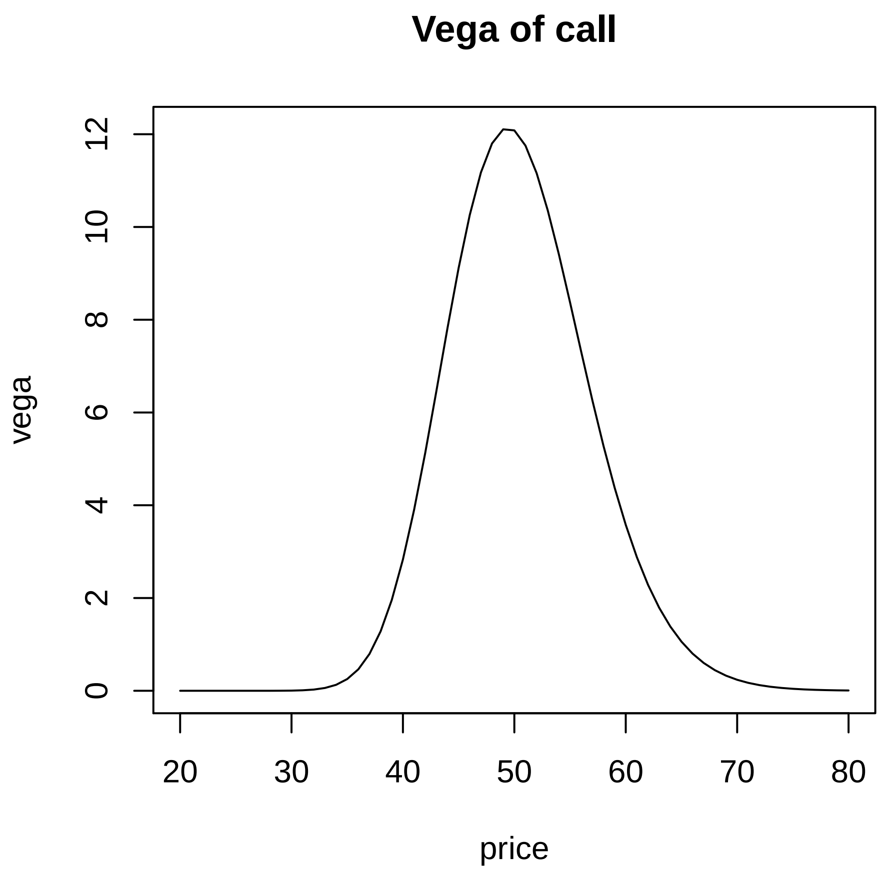
The vega of a European option is given as

$$\nu = S_0\sqrt{T}N'(d_1)$$

Note that the vega for a European put option is equal to the vega of a European call option.

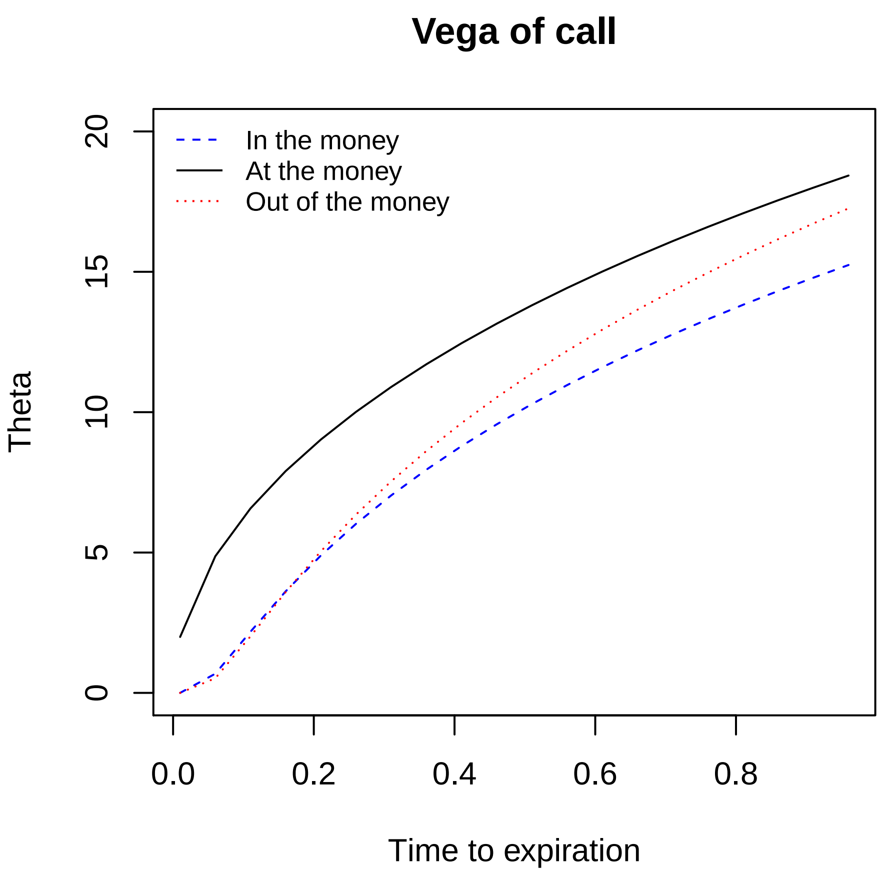
```
# Compute the vega of a European call option
computeGreeks(eu.call, "vega")

## [1] 12.11
```

```r
# Vega as the underlying price varies
computeGreeks(eu.call, "vega", prices = seq(20, 80, 1),
              plot = TRUE, main="Vega of call")
```




```r
# Vega as the time to maturity varies for in the money, at the money, and
# out of the money call options
maturity <- seq(0.01, 1, 0.05)
plot(maturity, xlim = range(maturity), ylim = c(0,20),
     type="n", xlab="Time to expiration", ylab="Theta")
lines(x = maturity, y = vegaBS(55, 50, 0.05, 0, 0.2, maturity, "call"),
      lty=2, col="blue")
lines(x = maturity, y = vegaBS(50, 50, 0.05, 0, 0.2, maturity, "call"),
      lty=1, col="black")
lines(x = maturity, y = vegaBS(45, 50, 0.05, 0, 0.2, maturity, "call"),
      lty=3, col="red")
title("Vega of call")
legend("topleft", legend=c("In the money", "At the money", "Out of the money"),
```

```r
      lty=c(2,1,3), col=c("blue", "black", "red"), bty="n", cex=0.8)
```



4.5 Rho

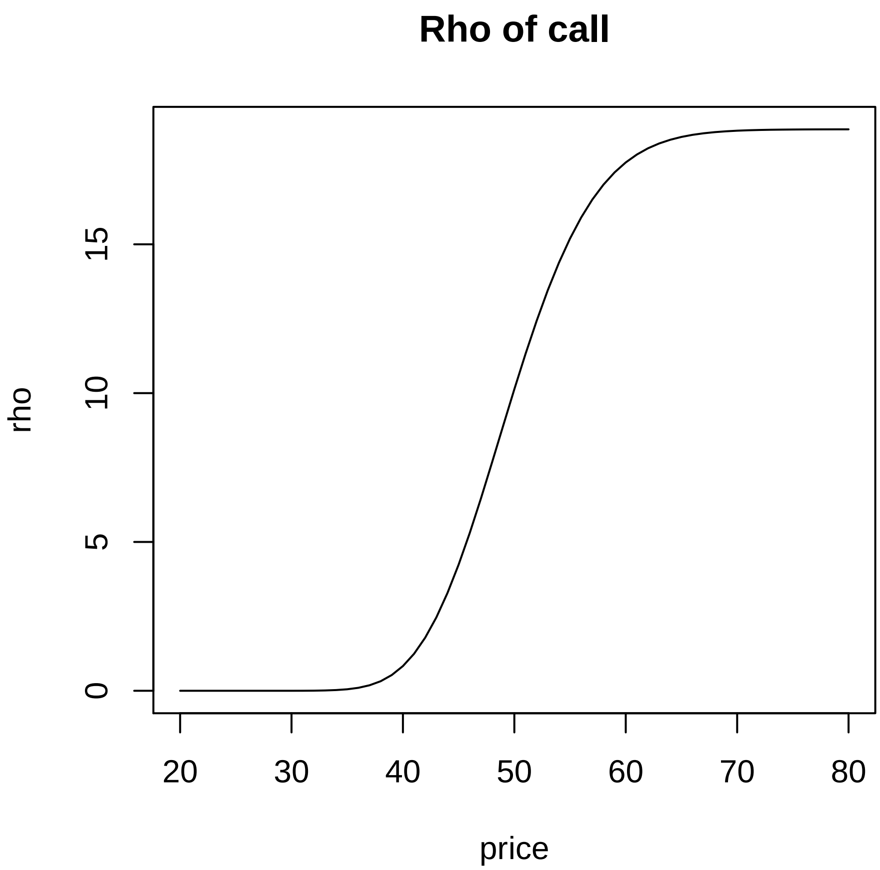
The rho (ρ) of an option is defined as the rate of change of the value of the option with respect to the risk-free interest rate.

The rho of a European option is given as

$$\rho(call) = KTe^{-rT}N(d_2)$$
$$\rho(put) = -KTe^{-rT}N(-d_2)$$

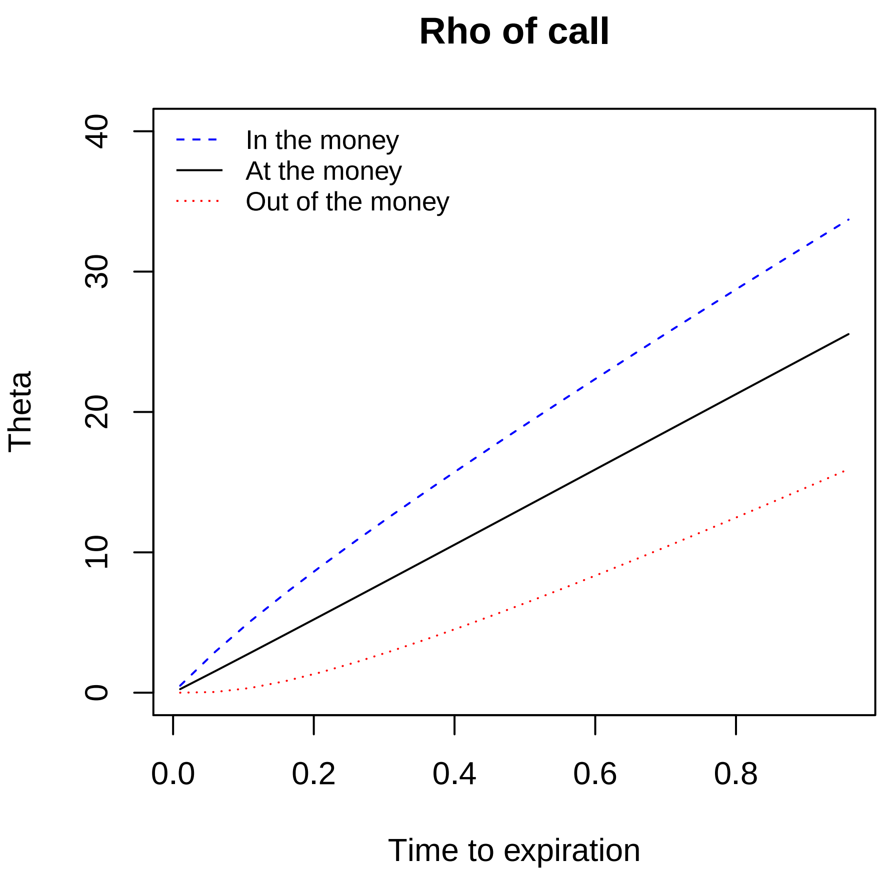
```r
# Compute the rho of the European call option
computeGreeks(eu.call, "rho")

## [1] 8.907
```

```r
# Rho as the unerlying price varies
computeGreeks(eu.call, "rho", prices = seq(20, 80, 1),
              plot = TRUE, main="Rho of call")
```




```r
# Rho as the time to maturity varies for in the money, at the money, and
# out of the money call options
maturity <- seq(0.01, 1, 0.05)
plot(maturity, xlim = range(maturity), ylim = c(0,40),
     type="n", xlab="Time to expiration", ylab="Theta")
lines(x = maturity, y = rhoBS(55, 50, 0.05, 0, 0.2, maturity, "call"),
      lty=2, col="blue")
lines(x = maturity, y = rhoBS(50, 50, 0.05, 0, 0.2, maturity, "call"),
      lty=1, col="black")
lines(x = maturity, y = rhoBS(45, 50, 0.05, 0, 0.2, maturity, "call"),
      lty=3, col="red")
title("Rho of call")
legend("topleft", legend=c("In the money", "At the money", "Out of the money"),
```

```
    lty=c(2,1,3), col=c("blue", "black", "red"), bty="n", cex=0.8)
```



4.6 Portfolio Insurance

Example 5.9: A portfolio with worth \$90 million. To protect against market downturns the managers of the portfolio require a 6-month European put option on the portfolio with a strike price of \$87 million. The risk-free rate is 9% per annum, the dividend yield is 3% per annum, and the volatility of the portfolio is 20% per annum. The S&P 500 index stands at 900. As the portfolio is considered to mimic the S&P 500 fairly closely, one alternative is to buy 1000 put options on the S&P 500 with a strike price of 870. Another option is to create the option synthetically. In this case, $S_0 = 90$ million, $K = 87$ million, $r = 0.09$, $q = 0.03$, $\sigma = 0.25$, and $T = 0.5$.

```
eu.put <- optionSpec(style = "european",
                     type = "put",
                     S0 = 90,
                     K = 87,
```

```
                     maturity = 0.5,
                     r = 0.09,
                     volatility = 0.25,
                     q = 0.03)
computeGreeks(eu.put, "delta")

## [1] -0.3215
```

The delta of the synthetic option is -0.3215. This means that 32.15% of the portfolio should be sold to match the delta of the synthetic option.