Valuation of Convertible Bonds With Credit Risk

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Convertible bonds can be difficult to value, given that they incorporate elements of both debt and equity. Further complications arise with the presence of additional options such as callability and puttability, and contractual complexities such as trigger prices and soft call provisions, when the ability of the issuing firm to exercise its option to call depends on the history of its stock price.

This article explores the valuation of convertible bonds subject to credit risk using an approach based on the numerical solution of linear complementarity problems. Models that do not explicitly specify what happens in the event of a default by the issuer can lead to internal inconsistencies, such as a call by the issuer just before expiration rendering the convertible value independent of the credit risk of the issuer, or implied hedging strategies that are not self-financing. A general and consistent framework for valuing convertible bonds assuming a Poisson default process allows various models for stock price behavior, recovery, and action by holders of the bonds in the event of a default.

The numerical algorithm uses a partially implicit method to decouple the system of linear complementarity problems at each time step. Numerical examples illustrating the convergence properties of the algorithm are provided.

The market for convertible bonds has been expanding rapidly (see Schultz [2001]). In the U.S., over $105 billion of new convertibles were issued in 2001, compared to just over $60 billion in 2000. As of early 2002, there were about $270 billion of convertibles outstanding, more than double the level of five years previously, and the global market for convertibles exceeded $500 billion.

In the last few decades there has been considerable innovation in the contractual features of convertibles. Examples include liquid yield option notes (McConnell and Schwartz [1986]), mandatory convertibles (Arzac [1997]), "death spiral" convertibles (Hillion and Vermaelen [2001]), and cross-currency convertibles (Yigitbasioglu [2001]). It is now common for convertibles to feature exotic and complicated features, such as trigger prices and soft call provisions. These preclude the issuer from exercising its call option unless the firm's stock price is either above some specified level, or has remained above a level for a specified period of time (e.g., 30 days), or has been above a level for some specified fraction of time (e.g., 20 of the last 30 days).

The modern academic literature on the valuation of convertibles begins with Ingersoll [1977] and Brennan and Schwartz [1977, 1980]. These authors build on a structural approach for valuing risky non-convertible debt (e.g., Merton [1974]; Black and Cox [1976]; Longstaff and Schwartz [1995]). In this approach, the basic underlying state variable is the value of the issuing firm. The firm's debt and equity are claims contingent on the firm's value, and options on its debt and equity are compound options on this variable. In general terms, default occurs when the firm's value becomes so low that it is unable to meet its financial obligations. An overview of this type of model is provided in Nyborg [1996].[1]

While in principle this is an attractive framework, it is subject to the same criticisms that have been applied to the valuation of risky debt by Jarrow and Turnbull [1995]. That is, because the value of the firm is not a traded asset, parameter estimation is difficult. Also, any other liabilities that are more senior than the convertible must be valued simultaneously.

To circumvent these problems, some authors have proposed models of convertible bonds whose basic underlying factor is the issuing firm's stock price (augmented in some cases with additional random variables such as an interest rate). As the stock is a traded asset, parameter estimation is simplified (compared to the structural approach). Moreover, there is no need to estimate the values of all other more senior claims.

An early example of this approach is McConnell and Schwartz [1986]. The basic problem is that the model ignores the possibility of bankruptcy. McConnell and Schwartz address this in an ad hoc manner by simply using a risky discount rate rather than the risk-free rate in their valuation equation. More recent authors who similarly include a risky discount rate in a somewhat arbitrary fashion are Cheung and Nelken [1994] and Ho and Pfeffer [1996].

An additional complication that arises in the case of a convertible bond (as opposed to risky debt) is that different components of the instrument are subject to different default risks. This is noted by Tsiveriotis and Fernandes, who argue that:

> the equity upside has zero default risk since the issuer can always deliver its own stock [while] coupon and principal payments and any put provisions… depend on the issuer's timely access to the required cash amounts, and thus introduce credit risk [1998, p. 95].

To handle this, Tsiveriotis and Fernandes propose splitting convertible bonds into two components: a cash-only part, which is subject to credit risk, and an equity part, which is not. This leads to a pair of coupled partial differential equations that can be solved to value convertibles.

A simple description of this model in the binomial context may be found in Hull [2003]. Yigitbasioglu [2001] extends this framework by adding an interest rate factor and, in the case of cross-currency convertibles, a foreign exchange risk factor.

Recently, an alternative to the structural approach has emerged. This is known as the reduced-form approach. It is based on developments in the literature on the pricing of risky debt (see, e.g., Jarrow and Turnbull [1995]; Duffie and Singleton [1999]; Madan and Unal [2000]). Unlike the structural approach, in this setting default is exogenous, the "consequence of a single jump loss event that drives the equity value to zero and requires cash outlays that cannot be externally financed" (Madan and Unal [2000, p. 44]).

The probability of default over the next short time interval is determined by a specified hazard rate. When default occurs, some portion of the bond is assumed to be recovered (either its market value immediately prior to default, or its par value, or the market value of a default-free bond with the same terms).

Authors who have used this approach in the convertible bond context include Davis and Lischka [1999], Takahashi, Kobayashi, and Nakagawa [2001], Hung and Wang [2002], and Andersen and Buffum [2003]. As in models such as that of Tsiveriotis and Fernandes [1998], the basic underlying state variable is the firm's stock price (although some of these authors also consider additional factors such as stochastic interest rates or hazard rates).

While this approach is quite appealing, the assumption that the stock price instantly jumps to zero in the event of a default is highly questionable. It may be a reasonable approximation in some circumstances, but it is clearly not in others.

For instance, Clark and Weinstein [1983] report that shares in firms filing for bankruptcy in the U.S. had average cumulative abnormal returns of –65% during the three years prior to a bankruptcy announcement, and had abnormal returns of about –30% around the announcement. Beneish and Press [1995] find average cumulative abnormal returns of –62% for the 300 trading days prior to a Chapter 11 filing, and a drop of 30% upon the filing announcement. The corresponding figures for a debt service default are –39% leading up to the announcement and –10% at the announcement.

This clearly indicates that the assumption of an instantaneous jump to zero is extreme. In most cases, default is better characterized as involving a gradual erosion of the stock price prior to the event, followed by a significant (but much less than 100%) decline upon the announcement, even in the most severe case of a bankruptcy filing.

As we shall see below, in some models it is at least implicitly assumed that a default has no impact on the firm's stock price. This may also be viewed as unsatisfactory. To address this, our model proposes that the firm's stock price drops by a specified percentage (between 0% and 100%) upon a default. This effectively extends the reduced-form approach, which in the case of risky debt specifies a fractional loss in market value for a bond, to convertibles by similarly specifying a fractional decline in the issuing firm's stock price.

Our main contributions are as follows:

- We provide a general single-factor framework for valuing risky convertible bonds, assuming a Poisson default process.
- We consider precisely what happens on default, assuming optimal action by the holder of the convertible. Our framework permits a wide variety of assumptions concerning the behavior of the stock of the issuing company on default, and also allows various assumptions concerning recovery on default.
- We demonstrate that the widely used convertible bond model of Tsiveriotis and Fernandes [1998] is internally inconsistent.
- We develop numerical methods for determining prices and hedge parameters for convertible bonds under the framework developed here.

As our main interest is the modeling of default risk, we confine our attention to models that assume the interest rate is a known function of time, and the stock price is stochastic. We can easily extend the models to handle a stochastic risk-free rate or hazard rate, but this would be a side excursion from the primary goal—determining how to incorporate the hazard rate in a basic convertible pricing model. We also note that practitioners often regard a convertible bond primarily as an equity instrument, where the main risk factor is the stock price, and the random nature of the risk-free rate is of second-order importance.[2]

For ease of exposition, we ignore various contractual complications such as call notice periods, soft call provisions, trigger prices, and dilution. An appendix describes the numerical methods.

I. CONVERTIBLE BONDS: NO CREDIT RISK

We begin by reviewing the valuation of convertible bonds under the assumption that there is no default risk. We assume that interest rates are known functions of time, and that the stock price is stochastic. We assume that:

$$dS = \mu S dt + \sigma S dz. \tag{1}$$

where S is the stock price, μ is its drift rate, σ is its volatility, and dz is the increment of a Wiener process.

Following the usual arguments, the no-arbitrage value $V(S, t)$ of any claim contingent on S is given by:

$$V_t + \left(\frac{\sigma^2}{2}S^2 V_{SS} + (r(t) - q)SV_S - r(t)V\right) = 0 \tag{2}$$

where $r(t)$ is the known interest rate, and q is the dividend rate.

We assume that a convertible bond has contractual features as follows:

- A continuous (time-dependent) put provision (with an exercise price of B_p).
- A continuous (time-dependent) conversion provision. At any time, the bond can be converted to κ shares.
- A continuous (time-dependent) call provision. At any time, the issuer can call the bond for price $B_c > B_p$. The holder can convert the bond if it is called.

Note that option features that are exercisable only at certain times(rather than continuously) can easily be handled by simply enforcing the relevant constraints only at those times.

Let

$$\mathcal{L}V \equiv -V_t - \left(\frac{\sigma^2}{2}S^2 V_{SS} + (r(t) - q)SV_S - r(t)V\right) \tag{3}$$

We will consider the points in the solution domain where $\kappa S > B_c$ and $\kappa S \leq B_c$ separately:

- $B_c > \kappa S$. In this case, we can write the convertible bond pricing problem as a linear complementarity problem:

$$\begin{pmatrix} \mathcal{L}V = 0 \\ (V - \max(B_p, \kappa S)) \geq 0 \\ (V - B_c) \leq 0 \end{pmatrix} \vee$$

$$\begin{pmatrix} \mathcal{L}V \geq 0 \\ (V - \max(B_p, \kappa S)) = 0 \\ (V - B_c) \leq 0 \end{pmatrix} \vee$$

$$\begin{pmatrix} \mathcal{L}V \leq 0 \\ (V - \max(B_p, \kappa S)) \geq 0 \\ (V - B_c) = 0 \end{pmatrix} \tag{4}$$

where the notation $(x = 0) \vee (y = 0) \vee (z = 0)$ is to be interpreted as that at least one of $x = 0$, $y = 0$, $z = 0$ holds at each point in the solution domain.

- $B_c \leq \kappa S$. In this case, the convertible value is simply

$$V = \kappa S \tag{5}$$

since the holder would convert immediately.

Equation (4) is a precise mathematical formulation of intuition as follows. The value of the convertible bond is given by the solution to $\mathcal{L}V = 0$, subject to the constraints

$$\begin{aligned} V &\geq \max(B_p, \kappa S) \\ V &\leq \max(B_c, \kappa S) \end{aligned} \tag{6}$$

More specifically, either we are in the continuation region where $\mathcal{L}V = 0$ and neither the call constraint nor the put constraint is binding [left-hand side term in (4)], or the put constraint is binding [middle term in (4)], or the call constraint is binding [right-hand side term in (4)].

As far as boundary conditions are concerned, we merely alter the operator $\mathcal{L}V$ at $S = 0$ and as $S \to \infty$. At $S = 0$, $\mathcal{L}V$ becomes

$$\mathcal{L}V \equiv -(V_t - r(t)V); \quad S \to 0 \tag{7}$$

while as $S \to \infty$ we assume that the unconstrained solution is linear in S:

$$\mathcal{L}V \equiv V_{SS} \qquad S \to \infty \tag{8}$$

The terminal condition is given by

$$V(S, t = T) = \max(F, \kappa S) \tag{9}$$

where F is the face value of the bond.

Equation (4) has been derived by many authors (although not using the precise linear complementarity formulation). In practice, however, corporate bonds are not risk-free. To highlight the modeling issues, we consider a simplified model of risky corporate debt.

II. A RISKY BOND

To motivate our discussion of credit risk, consider the valuation of a simple coupon-bearing bond issued by a corporation having non-zero default risk. The ideas are quite similar to some of those presented in Duffie and Singleton [1999]. We rely only on simple hedging arguments, and we assume that the risk-free rate is a known deterministic function.

For ease of exposition, we assume here (and generally throughout) that default risk is diversifiable, so that real-world and risk-neutral default probabilities will be equal.[3] With this in mind, let the probability of default in the time period t to $t + dt$, conditional on no-default in $[0, t]$, be $p(S, t)dt$, where $p(S, t)$ is a deterministic hazard rate.

Let $B(S, t)$ denote the price of a risky corporate bond. Construct the standard hedging portfolio:

$$\Pi = B - \beta S \tag{10}$$

In the absence of default, if we choose $\beta = B_S$, the usual arguments give:

$$d\Pi = \left[B_t + \frac{\sigma^2 S^2}{2} B_{SS}\right] dt + o(dt) \tag{11}$$

where $o(dt)$ denotes terms that go to zero faster than dt. Assume that:

- The probability of default in $t \to t + dt$ is $p\, dt$.
- The value of the bond immediately after default is RX where $0 \leq R \leq 1$ is the recovery factor. It is possible to make various assumptions about X. For example, for coupon-bearing bonds, it is often assumed that X is the face value. For zero-coupon bonds, X can be the accreted value of the issue price, or we could assume that $X = B$, the pre-default value.
- The stock price S is unchanged on default.

Then Equation (11) becomes

$$\begin{aligned} d\Pi &= (1 - p\, dt) \left[B_t + \frac{\sigma^2 S^2}{2} B_{SS}\right] dt - p\, dt(B - RX) + o(dt) \\ &= \left[B_t + \frac{\sigma^2 S^2}{2} B_{SS}\right] dt - p\, dt(B - RX) + o(dt) \end{aligned} \tag{12}$$

The assumption that default risk is diversifiable implies:

$$E(d\Pi) = r(t)\Pi\, dt \tag{13}$$

where E is the expectations operator. Combining Equations (12) and (13) gives

$$B_t + r(t)SB_S + \frac{\sigma^2 S^2}{2} B_{SS} - (r(t) + p)B + pRX = 0 \tag{14}$$

Note that if $p = p(t)$, and we assume that $X = B$, the solution to Equation (14) for a zero-coupon bond with face value F payable at $t = T$ is:

$$B = F \exp\left[-\int_t^T (r(u) + p(u)(1 - R))\, du\right] \tag{15}$$

which corresponds to the intuitive idea of a spread $s = p(1 - R)$.[4]

We can change the assumptions about the stock price in the event of default. If we assume that the stock price S jumps to zero in the case of default, Equation (12) becomes

$$\begin{aligned} d\Pi &= (1 - p\, dt)\left[B_t + \frac{\sigma^2 S^2}{2} B_{SS}\right] dt - \\ &\quad p\, dt(B - RX - \beta S) + o(dt) \\ &= \left[B_t + \frac{\sigma^2 S^2}{2} B_{SS}\right] dt - p\, dt(B - RX - \beta S) + o(dt) \end{aligned} \tag{16}$$

Following the steps as above with $\beta = B_S$, we obtain

$$\begin{aligned} &B_t + (r(t) + p)SB_S + \frac{\sigma^2 S^2}{2} B_{SS} - \\ &(r(t) + p)B + pRX = 0 \end{aligned} \tag{17}$$

Note that in this case p appears in the drift term as well as in the discounting term. Even in this relatively simple case of a risky corporate bond, different assumptions about the behavior of the stock price in the event of default will change our valuation. While this is perhaps an obvious point, it is worth remembering that in some popular models for convertible bonds no explicit assumptions are made regarding what happens to the stock price upon default.

III. BONDS WITH CREDIT RISK: THE HEDGE MODEL

To add credit risk to the convertible bond model, we follow the general line of reasoning described in Ayache, Vetzal, and Forsyth [2002]. Let the value of the convertible bond be denoted by $V(S, t)$. To avoid complications at this stage, we assume there are no put or call features and that conversion is allowed only at the terminal time or in the event of default. Let S^+ be the stock price immediately after default, and S^- be the stock price right before default. We will assume that:

$$S^+ = S^-(1 - \eta) \tag{18}$$

where $0 \leq \eta \leq 1$. We call the case where $\eta = 1$ the *total default* case (the stock price jumps to zero), and the case where $\eta = 0$ the *partial default* case (the issuing firm defaults but the stock price does not jump anywhere).

As usual, we construct the hedging portfolio:

$$\Pi = V - \beta S \tag{19}$$

If there is no credit risk, i.e., $p = 0$, then choosing $\beta = V_S$ and applying standard arguments gives

$$d\Pi = \left[V_t + \frac{\sigma^2 S^2}{2} V_{SS}\right] dt + o(dt) \tag{20}$$

Now, consider the case where the hazard rate p is non-zero. We make the assumptions:

- Upon default, the stock price jumps according to Equation (18).
- Upon default, the convertible bondholders have the option of receiving:
    - The amount RX, where $0 ″ R ″ 1$ is the recovery factor (as in the case of a simple risky bond, there are several possible assumptions that can be made about X (e.g., face value, pre-default value of bond portion of the convertible), but for now, we will not make any specific assumptions), or:
    - Shares worth $\kappa S^-(1 - \eta)$.

Under these assumptions, the change in value of the hedging portfolio during $t \rightarrow t + dt$ is:

$$d\Pi = (1 - p\,dt)\left[V_t + \frac{\sigma^2 S^2}{2} V_{SS}\right]\,dt - p\,dt[V - \beta S\eta)] +$$

$$p\,dt \max(\kappa S(1-\eta), RX) + o(dt)$$

$$= \left[V_t + \frac{\sigma^2 S^2}{2} V_{SS}\right]\,dt - p\,dt(V - V_S S\eta) + \\ p\,dt \max(\kappa S(1-\eta), RX) + o(dt) \qquad (21)$$

Assuming the expected return on the portfolio is the risk-free rate, as given by Equation (13), and equating this with the expectation of Equation (21), we obtain:

$$r\,[V - SV_S]\;dt = \left[V_t + \frac{\sigma^2 S^2}{2} V_{SS}\right]\,dt - p\,[V - V_S S\eta]\;dt + \\ p\,[\max(\kappa S(1-\eta), RX)]\;dt + o(dt) \qquad (22)$$

This implies

$$V_t + (r(t) + p\eta)SV_S + \frac{\sigma^2 S^2}{2} V_{SS} - \\ (r(t) + p)V + p\max(\kappa S(1-\eta), RX) = 0 \qquad (23)$$

Note that $r(t) + p\eta$ appears in the drift term and $r(t) + p$ appears in the discounting term in Equation (23). In the case that $R = 0$, $\eta = 1$, which is the total default model with no recovery, the final result is especially simple. We simply solve the full convertible bond problem in Equation (4), replacing $r(t)$ by $r(t) + p$. There is no need to solve an additional equation. This has been noted by Takahashi et al. [2001] and Andersen and Buffum [2003].

Defining:

$$\mathcal{M}V \equiv -V_t - \\ \left(\frac{\sigma^2}{2} S^2 V_{SS} + (r(t) + p\eta - q)SV_S - (r(t) + p)V\right) \qquad (24)$$

we can write Equation (23) for the stock paying a proportional dividend q as

$$\mathcal{M}V - p\max(\kappa S(1-\eta), RX) = 0. \qquad (25)$$

We are now in a position to consider the complete problem for convertible bonds with risky debt. We can generalize Equation (4) using Equation (25):

- $B_c > \kappa S$:

$$\begin{pmatrix} \mathcal{M}V - p\max(\kappa S(1-\eta), RX) = 0 \\ [V - \max(B_p, \kappa S)] \geq 0 \\ (V - B_c) \leq 0 \end{pmatrix} \quad \vee \\ \begin{pmatrix} \mathcal{M}V - p\max(\kappa S(1-\eta), RX) \geq 0 \\ [V - \max(B_p, \kappa S)] = 0 \\ (V - B_c) \leq 0 \end{pmatrix} \quad \vee \\ \begin{pmatrix} \mathcal{M}V - p\max(\kappa S(1-\eta), RX) \leq 0 \\ [V - \max(B_p, \kappa S] \geq 0 \\ (V - B_c) = 0 \end{pmatrix} \qquad (26)$$

- $B_c \leq \kappa S$:

$$V = \kappa S. \qquad (27)$$

Although Equations (26)-(27) appear formidable, the basic concept is easy to understand. The value of the convertible bond is given by:

$$\mathcal{M}V - p\max(\kappa S(1-\eta), RX) = 0 \qquad (28)$$

subject to the constraints

$$V \geq \max(B_p, \kappa S) \\ V \leq \max(B_c, \kappa S) \qquad (29)$$

Again, as with Equation (4), Equation (26) simply says that either we are in the continuation region or one of the two constraints (call or put) is binding. Henceforth, we refer to the basic model in Equations (26) and (27) as the *hedge model*, since this model is based on hedging the Brownian motion risk, in conjunction with precise assumptions about what occurs on default.

Recovery Under the Hedge Model

If we recover RX on default, and X is simply the face value of the convertible, or perhaps the discounted cash flows of an equivalent corporate bond (with the same face value), then X can be computed independently of the value of V, and V can be calculated using Equations (26)-(27). Note that in this case there is only a single equation to solve for the value of the convertible V.

This decoupling does not occur if we assume that X represents the *bond* component of the convertible. In this case, the bond component value should be affected by put/call provisions that are applied to the convertible bond as a whole. Under this recovery model, we need to solve another equation for the bond component B, which must be coupled with the total value V.

We emphasize here that this complication arises only for specific assumptions about what happens on default. In particular, if $R = 0$, then Equations (26)–(27) are independent of X.

Hedge Model: Recovery is a Fraction of the Bond Component

Assume that the total convertible bond value is given by Equations (26) and (27). We will make the assumption that, upon default, we recover RB, where B is the pre-default bond component of the convertible. We now split the convertible bond into two components, so that $V = B + C$, where B is the bond component and C is the equity component. The bond component, when there are no put/call provisions, should satisfy an equation similar to Equation (17).

We emphasize here that this splitting is required only if we assume that upon default the holder recovers RB, where B is the bond component, and C, the equity component, is simply $V - B$. There are many possible ways to split the convertible into two components so that $V = B + C$, but we determine the split so that B can reasonably be taken (e.g., in a bankruptcy court) to be the bond portion of the convertible to which the holder is entitled to receive a portion RB on default.

The actual specification of what is recovered on default is a controversial issue. We include this case in detail since it serves as an example to show that our framework can accept a wide variety of assumptions. If $B_p = -\infty$ (i.e., there is no put provision), the bond component should satisfy Equation (17), with initial condition $B = F$, and $X = B$. Under this circumstance, B is simply the value of risky debt with face value F.

Consequently, when the holder recovers RB on default, we propose a decomposition for the hedge model as follows:

$$\begin{pmatrix} \mathcal{M}C - p\max(\kappa S(1-\eta) - RB, 0) = 0 \\ [C - (\max(B_c, \kappa S) - B)] \leq 0 \\ (C - (\kappa S - B) \geq 0 \end{pmatrix} \vee \begin{pmatrix} \mathcal{M}C - p\max(\kappa S(1-\eta) - RB, 0) \leq 0 \\ C = \max(B_c, \kappa S) - B \end{pmatrix} \vee \begin{pmatrix} \mathcal{M}C - p\max(\kappa S(1-\eta) - RB, 0) \geq 0 \\ C = \kappa S - B \end{pmatrix} \tag{30}$$

$$\begin{pmatrix} \mathcal{M}B - RpB = 0 \\ B - B_c \leq 0 \\ B - (B_p - C) \geq 0 \end{pmatrix} \vee \begin{pmatrix} \mathcal{M}B - RpB \leq 0 \\ B = B_c \end{pmatrix} \vee \begin{pmatrix} \mathcal{M}B - RpB \geq 0 \\ B = B_p - C \end{pmatrix} \tag{31}$$

Adding Equations (30 and (31), and recalling that $V = B + C$, it is easy to see that Equations (26) and (27) are satisfied. We informally rewrite Equation (30) as:

$$\mathcal{M}C - p\max(\kappa S(1-\eta) - RB, 0) = 0 \tag{32}$$

subject to the constraints:

$$\begin{aligned} B + C &\leq \max(B_c, \kappa S) \\ B + C &\geq \kappa S \end{aligned} \tag{33}$$

Similarly, we can also rewrite Equation (31) as:

$$\mathcal{M}B - RpB = 0 \tag{34}$$

subject to the constraints:

$$\begin{aligned} B &\leq B_c \\ B + C &\geq B_p \end{aligned} \tag{35}$$

Note that the constraints (33) and (35) embody only the facts that $B + C = V$, and that V has constraints, and the requirement that B ″ B_c. No other assumptions are made regarding the behavior of the individual B and C components.

We can write the payoff of the convertible as

$$V(S,T) = F + \max(\kappa S - F, 0) \tag{36}$$

which suggests terminal conditions of

$$\begin{aligned} C(S,T) &= \max(\kappa S - F, 0) \\ B(S,T) &= F \end{aligned} \tag{37}$$

Consider the case of a zero-coupon bond where $p = p(t)$, $B < B_c$, and $B_p = 0$. In this case, the solution for B is

$$B = F\exp\left[-\int_t^T [r(u) + p(u)(1-R)]\,du\right] \tag{38}$$

independent of S. While we have made specific assumptions about what is recovered on default, the framework in Equations (26) and (27) can accommodate many other assumptions.

Some Special Cases

If we assume that $\eta = 0$ (i.e., partial default when the stock price does not jump if a default occurs), the recovery rate $R = 0$, and the bond is continuously convertible, Equations (30) and (31) become

$$\mathcal{M}V + p(V - \kappa S) = 0 \tag{39}$$

in the continuation region. This has a simple intuitive interpretation. The convertible is discounted at the risk-free rate plus spread when $V >> \kappa S$ and at the risk-free rate when $V \simeq \kappa S$, with smooth interpolation between these values. Equation (39) is suggested in Ayache [2001]. Note that in this case we need only solve a single linear complementarity problem for the total convertible value V.

Making the assumptions that $\eta = 1$ (i.e., the total default case where the stock price jumps to zero upon default) and that the recovery rate $R = 0$, Equations (30) and (31) reduce to

$$\mathcal{M}V = 0 \tag{40}$$

in the continuation region, which agrees with Takahashi et al. [2001]. In this case, there is no need to split the convertible bond into equity and bond components.

If the recovery rate is non-zero, our model is slightly different from that in Takahashi et al. There it is assumed that upon default the holder recovers RV, compared to Equations (30) and (31), where the holder recovers RB. Consequently, for non-zero R, approach (30)-(31) requires the solution of the coupled set of linear complementarity problems, while the assumption in Takahashi et al. requires only the solution of a single linear complementarity problem.

Since the total convertible bond value V includes a fixed-income component and an option component, it seems more reasonable to us that in the event of total default (the assumption made in Takahashi et al. [2001), the option component is by definition worthless, and only a fraction of the bond component can be recovered. The total default case also appears to be similar to the model suggested in Davis and Lischka [1999]. A similar total default model is also suggested in Andersen and Buffum [2003], for $R = 0$, $\eta = 1$.

It is worth observing that if we assume the stock price of a firm jumps to zero on default, we can use these arguments to deduce the partial differential equation satisfied by plain vanilla puts and calls on the issuer's equity. If the price of an option is denoted by $U(S, t)$, then U is given by the solution to

$$\begin{aligned} &U_t + (r+p)SU_S + \frac{\sigma^2 S^2}{2} U_{SS} - \\ &\quad (r+p)U + pU(0,t) = 0 \end{aligned} \tag{41}$$

This suggests that information about the hazard rate is incorporated in the market prices of plain vanilla options.

IV. COMPARISON WITH PREVIOUS WORK

There have been various attempts to value convertibles by splitting the total value of a convertible into bond and equity components, and then valuing each component separately. An early effort along these lines is described in Goldman Sachs research note, "Valuing Convertible Bonds as Derivatives" [1994]. The probability of conversion is estimated, and the discount rate is a weighted average of the risk-free rate and the risk-free rate plus spread, where the weighting factor is the probability of conversion.

More recently, the model described in Tsiveriotis and Fernandes [1998] has become popular. This model is outlined in Hull [2003] and has been adopted by several software vendors.

The Tsiveriotis-Fernandes Model

The basic idea of the TF model is that the *equity* component of the convertible should be discounted at the risk-free rate (as in any other contingent claim), and the *bond* component should be discounted at a risky rate. This leads to the equation for the convertible value V:

$$\begin{aligned} &V_t + \frac{\sigma^2}{2} S^2 V_{SS} + (r_g - q)SV_S - \\ &\quad r(V - B) - (r+s)B = 0 \end{aligned} \tag{42}$$

subject to the constraints

$$\begin{aligned} &V \geq \max(B_p, \kappa S) \\ &V \leq \max(B_c, \kappa S) \end{aligned} \tag{43}$$

In Equation (42), r_g is the growth rate of the stock, s is the spread, and B is the bond component of the convertible. Following the description of this model in Hull [2003], we will assume here that the growth rate of the stock is the risk-free rate, i.e., $r_g = r$. The bond component satisfies

$$B_t + rSB_S + \frac{\sigma^2 S^2}{2} B_{SS} - (r+s)B = 0. \tag{44}$$

Comparing Equations (14) and (44), setting $X = B$, and assuming that s and p are constant, we can see that the spread can be interpreted as $s = p(1 - R)$.

Although it is not stated in Tsiveriotis and Fernandes [1998], we deduce that their model is a partial default model (the stock price does not jump upon default) since the equity part of the convertible is discounted at the risk-free rate. Of course, we can extend their model to handle other assumptions about the behavior of the stock price upon default, while keeping the same decomposition into bond and equity components.

We can write the equation satisfied by the total convertible value V in the TF model as a linear complementarity problem:

- $B_c > \kappa S$:

$$\begin{pmatrix} \mathcal{L}V + p(1-R)B = 0 \\ (V - \max(B_p, \kappa S)) \geq 0 \\ (V - B_c) \leq 0 \end{pmatrix} \vee$$

$$\begin{pmatrix} \mathcal{L}V + p(1-R)B \geq 0 \\ (V - \max(B_p, \kappa S)) = 0 \\ (V - B_c) \leq 0 \end{pmatrix} \vee$$

$$\begin{pmatrix} \mathcal{L}V + p(1-R)B \leq 0 \\ (V - \max(B_p, \kappa S)) \geq 0 \\ (V - B_c) = 0 \end{pmatrix} \tag{45}$$

- B_c ″ κS:

$$V = \kappa S \tag{46}$$

It is convenient to describe the decomposition of the total convertible price as $V = B + C$, where B is the bond component, and C is the equity component. In general, we can express the solution for $\{V, B, C\}$ in terms of a coupled set of equations. Assuming that Equations (45) and (46) are also being solved for V, we can specify $\{B, C\}$. In the TF model, the decomposition is suggested as:

- $B_p > \kappa S$:

$$\begin{aligned} &\mathcal{L}C = 0; \quad \mathcal{L}B + p(1-R)B = 0 \qquad \text{if } V \neq B_p \text{ and } V \neq B_c \\ &B = B_p; \quad C = 0 \qquad \text{if } V = B_p \\ &B = 0; \quad C = B_c \qquad \text{if } V = B_c \end{aligned} \tag{47}$$

- $B_p \leq \kappa S$:

$$\begin{aligned} &\mathcal{L}C = 0; \quad \mathcal{L}B + p(1-R)B = 0 \qquad \text{if } V \neq \max(\kappa S, B_c) \\ &C = \max(\kappa S, B_c); \quad B = 0 \qquad \text{if } V = \max(\kappa S, B_c) \end{aligned} \tag{48}$$

It is easy to verify that the sum of Equations (47) and (48) gives Equations (45) and (46), noting that $V = B + C$.

The terminal conditions for the TF decomposition are

$$\begin{aligned} &C(S, t = T) = H(\kappa S - F)(\max(\kappa S - F, 0) + F) \\ &B(S, t = T) = H(F - \kappa S)F \end{aligned} \tag{49}$$

where

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \tag{50}$$

The splitting in Equations (47) and (48), however, does not seem to be based on theoretical arguments that require specifying precisely what happens in the case of default. Tsiveriotis and Fernandes [1998] provide no discussion of the actual events in the case of default and how this would affect the hedging portfolio. There is no clear statement in their article as to what happens to the stock price in the event of default.

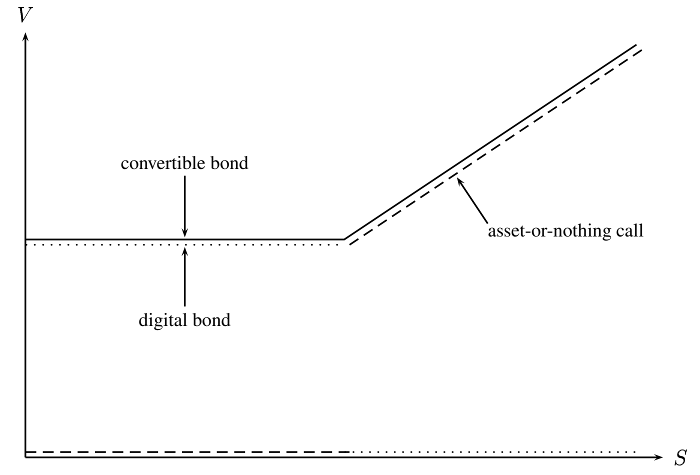
Exhibit 1 illustrates the decomposition of the convertible bond using Equation (49). Note that the convertible bond payoff is split into two discontinuous components, a digital bond and an asset-or-nothing call. The splitting occurs at the conversion boundary. This can be expected to cause some difficulties for a numerical scheme, as we have to solve for a problem with a discontinuity that moves over time (as the conversion boundary moves).

TF Splitting: Call Just Before Expiration

There are some inconsistencies in the TF model. As a first example, consider that there are no put provi-

EXHIBIT 1

TF Method for Decomposing Convertible Bond into Digital Bond Plus Asset-or-Nothing Call



sions; there are no coupons; $\kappa = 1$; conversion is allowed only at the terminal time (or at the call time); and the bond can be called only the instant before maturity, at $t = T^-$. The call price $B_c = F - \epsilon$, $\epsilon > 0$, $\epsilon << 1$.

Suppose the bond is called at $t = T^-$. From Equations (48) and (49), we conclude that we end up effectively solving the original problem with the altered payoff at $t = T^-$:

$$
\begin{aligned}
&\mathcal{L}V + pB = 0 \\
&V(S, T^-) = \max(S, F - \epsilon) \\
&\mathcal{L}B + pB = 0 \\
&B(S, T^-) = 0
\end{aligned} \tag{51}
$$

Note that the condition on B at $t = T^-$ is due to the boundary condition (48).

Now, since the solution of Equation (51) for B (with $B = 0$ initially) is $B \equiv 0$ for all $t < T^-$, the equation for the convertible bond is simply:

$$
\begin{aligned}
&\mathcal{L}V = 0 \\
&V(S, T^-) = \max(S, F - \epsilon)
\end{aligned} \tag{52}
$$

In other words, the hazard rate has no effect in this case. This peculiar situation occurs because the TF model requires that the bond value be zero if $V = B_c$, even if the effect of the call on the total convertible bond value at the instant of the call is infinitesimally small. This result indicates that calling the bond the instant before expiration with $B_c = F - \epsilon$ makes the convertible bond value independent of the credit risk of the issuer, which is clearly inappropriate.

Hedging

As a second example of an inconsistency in the TF framework, consider what happens if we attempt to hedge the convertible bond dynamically. Since there are two sources of risk (Brownian risk and default risk), we expect we will need to hedge with the underlying stock and another contingent claim, which we denote by I. This second claim could be, for instance, another bond issued by the same firm. Given the presence of this second hedging instrument, in this context we drop the assumption that default risk is diversifiable. Thus, in the following λdt is the actual probability of default during $[t, t + dt]$, while pdt is its risk-adjusted value.

Consider the hedging portfolio:

$$\Pi = V - \beta S - \beta' I + A \tag{53}$$

where A is the cash component, which has value $A = -(V - \beta S - \beta' I)$. Assume a real-world process of the form:

$$dS = (\mu + \lambda\eta)S dt + \sigma S dz - \eta S dq \tag{54}$$

where μ is the drift rate, and the Poisson default process:

$$dq = \begin{cases} 1 & \text{with probability } \lambda\, dt \\ 0 & \text{with probability } 1 - \lambda\, dt \end{cases}$$

Suppose we choose:

$$V_S - \beta' I_S - \beta = 0 \tag{55}$$

Using Itô's lemma, we obtain [from Equations (53) and (55)]:

$$
\begin{aligned}
d\Pi = &\left[\frac{\sigma^2 S^2}{2} V_{SS} + V_t - \beta' \left(\frac{\sigma^2 S^2}{2} I_{SS} + I_t\right)\right] dt \\
&+ (\beta S + \beta' I - V)\, r\, dt \\
&+ [\text{ change in } \Pi \text{ on default }]\, dq
\end{aligned} \tag{56}
$$

We implicitly assume in Equation (56) that the second contingent claim I defaults at precisely the same time as the convertible V.

To avoid tedious algebra, we will assume that the recovery rate of the bond component $R = 0$. If the contingent claims are not called, put, converted, or defaulted in $[t, t + dt]$, then:

- Hedge model [from Equation (23)]:

$$
\begin{aligned}
V_t + \frac{\sigma^2 S^2}{2} V_{SS} &= -\left[(r + p\eta) S V_S - (r + p) V + p\kappa S(1-\eta)\right] \\
I_t + \frac{\sigma^2 S^2}{2} V_{SS} &= -\left[(r + p\eta) S I_S - (r + p) I + p\kappa' S(1-\eta)\right]
\end{aligned}
\tag{57}
$$

- TF model [from Equation (42)]:

$$
\begin{aligned}
V_t + \frac{\sigma^2 S^2}{2} V_{SS} &= -\left[r S V_S - r V - pB\right] \\
I_t + \frac{\sigma^2 S^2}{2} V_{SS} &= -\left[r S I_S - r I - pB'\right]
\end{aligned}
\tag{58}
$$

Note that κ' is the number of shares a holder of the second claim I would receive in the event of a default, and B' is the bond component of I. We assume in all cases (noting that $\beta = V_S - \beta' I_S$ that:

$$
\begin{aligned}
&[\text{Change in } \Pi \text{ on Default }] \\
&= \kappa S(1-\eta) - \beta S(1-\eta) - \beta'\kappa' S(1-\eta) - (V - \beta' I - \beta S) \\
&= \kappa S(1-\eta) + (V_S - \beta' I_S) S\eta - \beta'\kappa' S(1-\eta) - V + \beta' I
\end{aligned}
\tag{59}
$$

Consequently, for both the hedge model and the TF model, we obtain [from Equations (56) and (59)]:

$$
\begin{aligned}
d\Pi = &\left[\frac{\sigma^2 S^2}{2} V_{SS} + V_t - \beta'\left(\frac{\sigma^2 S^2}{2} I_{SS} + I_t\right)\right] dt \\
&+ \left[(V_S - \beta' I_S) S + \beta' I - V\right] r\, dt \\
&+ \left[\kappa S(1-\eta) + (V_S - \beta' I_S) S\eta - \beta'\kappa' S(1-\eta) - \right. \\
&\quad \left. V + \beta' I\right] dq
\end{aligned}
\tag{60}
$$

For the hedge model, using Equation (57) in Equation (60) gives:

$$
\begin{aligned}
d\Pi = &-p\, dt \left[S V_S \eta - V + \kappa S(1-\eta) - \right. \\
&\quad \left. \beta'\left(\eta S I_S - I + \kappa' S(1-\eta)\right)\right] + \\
&\quad dq \left[\kappa S(1-\eta) + (V_S - \beta' I_S) S\eta - \right. \\
&\quad \left. \beta'\kappa' S(1-\eta) - V + \beta' I\right] \\
= &-p\, dt \left[S V_S \eta - V + \kappa S(1-\eta) - \right. \\
&\quad \left. \beta'\eta S I_S - I + \kappa' S(1-\eta)\right] + \\
&\quad dq \left[\kappa S(1-\eta) + V_S S\eta - V + \beta' I - \right. \\
&\quad \left. I_S S\eta - \kappa'(1-\eta) S\right]
\end{aligned}
\tag{61}
$$

Choosing

$$
\beta' = \frac{S V_S \eta - V + \kappa S(1-\eta)}{\eta S I_S - I + \kappa'(1-\eta) S}
\tag{62}
$$

and substituting Equation (62) into Equation (61) gives:

$$
d\Pi = 0
\tag{63}
$$

so that the hedging portfolio is risk-free and self-financing under the real-world measure.

On the other hand, in the case of the TF model, substituting Equation (58) into Equation (60) gives:

$$
\begin{aligned}
d\Pi = &\, p\, dt (B - \beta' B') + \\
&\quad dq \left[\kappa S(1-\eta) + V_S S\eta - V + \right. \\
&\qquad \left. \beta' \left(I - I_S S\eta - \kappa'(1-\eta) S\right)\right]
\end{aligned}
\tag{64}
$$

If we choose β' as in Equation (62), and substitute in Equation (64), we obtain:

$$
d\Pi = \left[B - \beta' B'\right] p\, dt.
\tag{65}
$$

This means that the hedging portfolio is no longer self-financing.

Another possibility is to require

$$
E\left[d\Pi\right] = 0
\tag{66}
$$

Using Equations (55), (64), and (66) gives:

$$
\beta' = \frac{-\lambda\left(\kappa S(1-\eta) + V_S S\eta - V\right) - pB}{\lambda\left(I - I_S S\eta - \kappa'(1-\eta) S\right) - pB'}
\tag{67}
$$

Note that in this case β' depends in general on λ.

With this choice of β', the variance in the hedging portfolio in $[t, t + dt]$ is:

$$
\text{Var}\ [d\Pi] = E\left[(d\Pi)^2\right]
\tag{68}
$$

which in general is non-zero, so that the hedging portfolio is not risk-free.

Consequently, the hedge model can be used to generate a self-financing hedging zero-risk portfolio under the real probability measure. The TF model, however, will not generate a hedging portfolio that is both risk-free and self-financing. This is simply because in the hedge model we have specified what happens on default, so that the PDE is consistent with the default model.

EXHIBIT 2
Data for Numerical Example

T	5 years
Clean call price	110 from year 2 to year 5
Clean put price	105 at 3 years
r	.05
p	.02
σ	.20
Conversion ratio	1.0
Recovery factor R	0.0
Face value of bond	100
Coupon dates	$.5, 1.0, 1.5, \ldots, 5.0$
Coupon payments	4.0
Total default	$\eta = 1.0$
Partial default	$\eta = 0.0$

V. NUMERICAL EXAMPLES

We provide some convergence tests of the numerical methods for some easily reproducible cases, as well as some more realistic examples. A detailed description of the numerical algorithms is provided in the appendix.

In order to be precise about the way put and call provisions are handled, we describe the method used to calculate the effects of accrued interest and the coupon payments in some detail. The payoff condition for the convertible bond is (at $t = T$):

$$V(S,T) = \max(\kappa S, F + K_{last}) \tag{69}$$

where K_{last} is the last coupon payment. Let t be the current time in the forward direction, t_p the time of the previous coupon payment, and t_n the time of the next pending coupon payment, i.e., $t_p \leq t \leq t_n$. Then, define the accrued interest on the pending coupon payment as:

$$\mathrm{AccI}(t) = K_n \frac{t - t_p}{t_n - t_p} \tag{70}$$

where K_n is the coupon payment at $t = t_n$.

The dirty call price B_c and the dirty put price B_p, which are used in Equations (30) and (31) and Equations (47) and (48), are given by:

$$\begin{aligned} B_c(t) &= B_c^{cl}(t) + \mathrm{AccI}(t) \\ B_p(t) &= B_p^{cl}(t) + \mathrm{AccI}(t) \end{aligned} \tag{71}$$

where B_c^{cl} and B_p^{cl} are the clean prices.

Let t_i^+ be the forward time the instant after a coupon payment, and t_i^- the forward time the instant before a coupon payment. If K_i is the coupon payment at $t = t_i$, then the discrete coupon payments are handled by setting:

$$\begin{aligned} V(S,t_i^-) &= V(S,t_i^+) + K_i \\ B(S,t_i^-) &= B(S,t_i^+) + K_i \\ C(S,t_i^-) &= C(S,t_i^+) \end{aligned} \tag{72}$$

where V is the total convertible value, and B is the bond component. The coupon payments are modeled in the same way for both the TF and the hedge models.

The data used for the numerical examples are given in Exhibit 2. They are similar to the data used in Tsiveriotis and Fernandes [1998] (except that some data, such as the volatility of the stock price, are not provided there). We confine the numerical examples to the two limiting assumptions of total default ($\eta = 1.0$) or partial default ($\eta = 0.0$) [see Equation (18)].

Exhibit 3 demonstrates the convergence of the numerical methods for both models. It is interesting to note that the hedge model partial and total default models appear to give solutions correct to \$0.01 with coarse grids or time steps, while considerably finer grids or time steps are required to achieve this level of accuracy for the TF model. This reflects our observation that the bond component of the TF model effectively involves a time-dependent knock-out barrier, which is difficult to solve accurately. Note that the partial default hedge model gives a price about \$1.00 higher than the TF price. The total default hedge model price is about \$1.00 less than the TF price.

Compare the results in Exhibit 3 with those in Exhibit 4, where the hazard rate p is set to zero. In this case, the value of the convertible bond is about \$2.00 more than for the TF model, and about \$1.00 more than for the partial default hedge model.

The appendix suggests a technique that decouples the coupled PDEs for B and C for the TF model. Unlike Tsiveriotis and Fernandes [1998], we include an extra implicit step at each time step. Exhibit 5 shows the convergence of the TF model, where the last fully implicit solution of the total bond value in Equations (A-10)–(A-12) is included/omitted. In this case, compare the results in Exhibit 5 to those in Exhibit 3. We observe that the extra implicit solution steps [Equations (A-10)–(A-12) does indeed speed up convergence as the grid is refined and the time step size is reduced.

Exhibit 6 provides a plot of no default, the TF model, and the two hedge models (partial and total default). At high enough levels of the underlying stock

EXHIBIT 3

Comparison of Hedge and TF Models

Nodes	Time Steps	Hedge Model (Partial Default)	Hedge Model (Total Default)	TF
200	200	124.9158	122.7341	124.0025
400	400	124.9175	122.7333	123.9916
800	800	124.9178	122.7325	123.9821
1600	1600	124.9178	122.7319	123.9754
3200	3200	124.9178	122.7316	123.9714

Value at t = 0, S = 100. For the TF model, partially implicit application of constraints. Total default (η = 1.0) and partial default (η = 0.0) defined in Equation (18).

EXHIBIT 4

Value of Convertible Bond at $t = 0$, $S = 100$

Nodes	Time Steps	Value ($p = 0$)
200	200	125.9500
400	400	125.9523
800	800	125.9528
1600	1600	125.9529
3200	3200	125.9529

Data given in Exhibit 2, except that hazard rate p = 0. In this case both the TF model and the hedge model give the same result.

EXHIBIT 5

TF Model Value at $t = 0$, $S = 100$

Nodes	Time Steps	TF (Exhibit 3) (partially implicit constraints)	TF (explicit constraints)
200	200	124.00249	124.09519
400	400	123.99160	124.05384
800	800	123.98210	124.02508
1600	1600	123.97538	124.00798
3200	3200	123.97141	123.99433
6400	6400	123.97050	123.98531

Comparisons of partially implicit constraints [use Equations (A-10)-(A-12) and explicit application of constraints [omit Equations (A-10)-(A-12)].

price, the bond will be converted, and all the models converge to the same value. Similarly, although it is not shown in the figure, as $S \to 0$, all the models (except for the no-default case) converge to the same value, as the valuation equation becomes an ordinary differential equation that is independent of η (although not of p).

Between these two extremes, the graph reflects the behavior shown in Exhibit 3, where the hedge partial default value is higher than the TF model, which is in turn higher than the hedge total default value. Exhibit 6 also shows the additional intuitive feature (not documented in Exhibit 3) that the case of no-default yields higher values than any of the models with default.

It is interesting to see the behavior of the TF bond component and the TF total convertible value an instant before $t = 3$ years. Recall from Exhibit 2 that the bond is puttable at $t = 3$, and there is a pending coupon payment as well. Exhibit 7 shows the discontinuous behavior of the bond component near the put price for the TF model. Since $V = B + C$, the call component also has a discontinuity.

Exhibit 8 shows results for the total default hedge model with different recovery factors R [Equation (26)]. We also show the no-default risk case ($p = 0$) for comparison. Note the rather curious fact that for the admittedly unrealistic case of $R = 100\%$, the value of the convertible bond is higher than the value with no default risk. This can be explained with reference to the hedging portfolio described by Equation (19). Note that the portfolio is long the bond and short the stock. If there is a default, and the chance of recovery is high, the hedger obtains a windfall profit, since there is a gain on the short position, and a very small loss on the bond position.

The examples so far have used a constant hazard rate (as specified in Exhibit 2). It is more realistic to model the hazard rate as increasing as the stock price declines. A parsimonious model of the hazard rate is given by

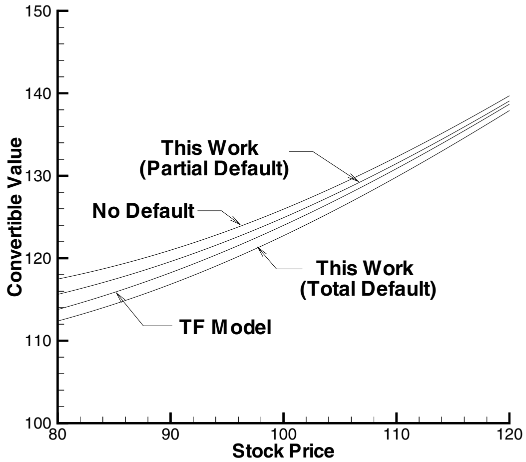
$$p(S) = p_0 \left(\frac{S}{S_0} \right)^{\alpha} \tag{73}$$

where p_0 is the estimated hazard rate at $S = S_0$. In Muromachi [1999], a function of the form of Equation (73) is observed to be a reasonable fit to bonds rated BB+ and below in the Japanese market. Typical values for α are in the range from –1.2 to –2.0.

In Exhibit 9 we compare the value of the total default hedge model for constant $p(S)$ as well as for $p(S)$ given by Equation (73). The data are as in Exhibit 2, except that for the non-constant $p(S)$ cases, we use Equation (73) with $p_0 = 0.02$ and $S_0 = 100$. Exhibits 10 and 11 show the corresponding delta and gamma values.

EXHIBIT 6

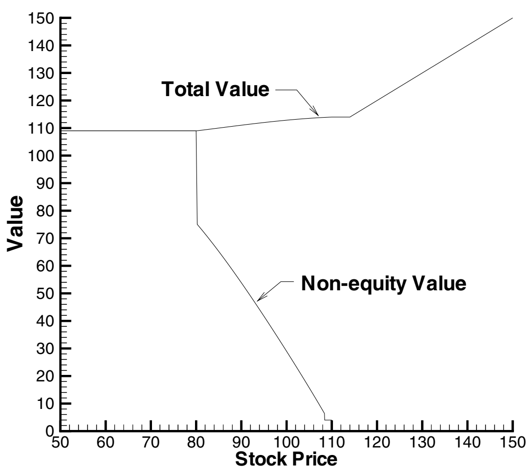
Convertible Bond Values at $t = 0$



Results for no default, TF model, and hedge (partial default (η= 0.0) and total default (η = 1.0) models.

EXHIBIT 7

TF Model—Total and Non-Equity (bond) Component at $t = 3$ Years



VI. RISK-NEUTRAL HEDGING SIMULATIONS

We can gain further insight into the difference between the TF model and the hedge model by considering the hedging performance of these models, but in a risk-neutral setting. Consider the hedging portfolio:

$$\Pi_{tot} = V - \beta S + A. \tag{74}$$

where the total portfolio Π_{tot} also includes the amount in the risk-free bank account that is required to finance the portfolio. Note that $A = \beta S - V$ in cash. Let dG be the gain in the portfolio if no default occurs, and dL be the losses due to default, in the interval $[t, t + dt]$. By definition:

$$d\Pi_{tot} - dG - dL = 0$$

Assume that no default has occurred in $[0, t]$, and that no default occurs in $[t, t + dt]$. Then for $\beta = V_S$ we obtain:

$$\begin{aligned} &d\Pi_{tot} - dG = 0 \\ &= \left[V_t + \frac{\sigma^2 S^2}{2} V_{SS}\right] dt + [\beta S - V]\, r\, dt - dG + o(dt) \end{aligned} \tag{75}$$

Equation (75) holds for both the TF model and the hedge models.

For simplicity, we now assume the recovery rate $R = 0$. With the further assumption that the convertible bond is not called, put, converted, or defaulted in $[t, t + dt]$, it follows that:

- Hedge model [from Equation (23)]:

$$\begin{aligned} &V_t + \frac{\sigma^2 S^2}{2} V_{SS} = \\ &\quad -[(r + p\eta) S V_S - (r + p) V + p\kappa S(1 - \eta)] \end{aligned} \tag{76}$$

- TF model [from Equation (42)]:

$$V_t + \frac{\sigma^2 S^2}{2} V_{SS} = -[rSV_S - rV - pB] \tag{77}$$

If $p = 0$, then Equations (75), (76), and (77) give $dG = 0$. This is to be expected, since setting $\beta = V_S$ eliminates the Brownian risk. If $p \uparrow 0$, then the expected gain in value of the portfolio assuming no default in $[t, t + dt]$ is:

- Hedge model [from Equations (75) and (76)]:

$$dG = [-p\eta S V_S + pV - p\kappa S(1 - \eta)]\ dt \tag{78}$$

- TF model [from Equations (75) and (77)]:

$$dG = p\, B\, dt. \tag{79}$$

Let $E[dG(t)]$ be the expected value of the excess amount in the portfolio if no default occurs in $[t, t + dt]$. Then, given that the probability of no default occurring in $[t, t + dt]$ is $1 - p\, dt$, it follows that:

- Hedge model [from Equation (78)]:

$$E[dG] = [-p\eta SV_S + pV - p\kappa S(1-\eta)]\ dt + o(dt) \qquad (80)$$

- TF model [from Equation (79)]:

$$E[dG] = p\,B\,dt + o(dt) \qquad (81)$$

In the risk-neutral measure, expected gains in value of the hedging portfolio must compensate for expected losses due to default. Let S_t^i be the value of S at time t on the i-th realized path of the underlying stock price process. Let $\chi(S_t^i, t)$ be the probability of no default in $[0, t]$, along path S_t^i. Then, the discounted value of the expected no-default gain is:

- Hedge model [from Equation (80)]:

$$E[G]_d = E_{<S_t^i>} \int_0^T \chi(S_t^i, t) e^{-rt} \times [-p\eta SV_S + pV - p\kappa S(1-\eta)]_{S_t^i}\ dt \qquad (82)$$

- TF model [from Equation (81)]:

$$E[G]_d = E_{<S_t^i>} \int_0^T \chi(S_t^i, t) e^{-rt}\, [pB]_{S_t^i}\ dt \qquad (83)$$

Now consider the losses due to default. Given:

$$\Pi_{tot} = V - \beta S + A \qquad (84)$$

where $A = (\beta S - V)$ in cash, we assume that no default has occurred in $[0, t]$, but that default occurs in $[t, t + dt]$.Consequently, on default we have (assuming $R = 0$, and that conversion is possible):

$$\begin{aligned} &V \to \kappa S(1-\eta) \\ &S \to S(1-\eta) \\ &A \to A \end{aligned} \qquad (85)$$

Thus:

$$\begin{aligned} d\Pi_{tot} &= \Pi_{after} - \Pi_{before} \\ &= \Pi_{after} \\ &= \kappa S(1-\eta) + V_S S\eta - V \end{aligned} \qquad (86)$$

which gives:

$$dL = \kappa S(1-\eta) + V_S S\eta - V \qquad (87)$$

Now, default occurs in $[t, t + dt]$ with probability $p\,dt$, so that the expected discounted losses due to default are:

$$E[L]_d = E_{<S_t^i>} \int_0^T \chi(S_t^i, t) e^{-rt} p \times [\kappa S(1-\eta) + V_S S\eta - V]_{S_t^i}\ dt \qquad (88)$$

Equation (88) is valid for both the hedge and the TF models. Moreover, from Equations (82) and (88), we have:

$$E[G]_d + E[L]_d = 0 \qquad (89)$$

for the hedge model.

EXHIBIT 8
Total Default Hedge Model at Different Recovery Rates

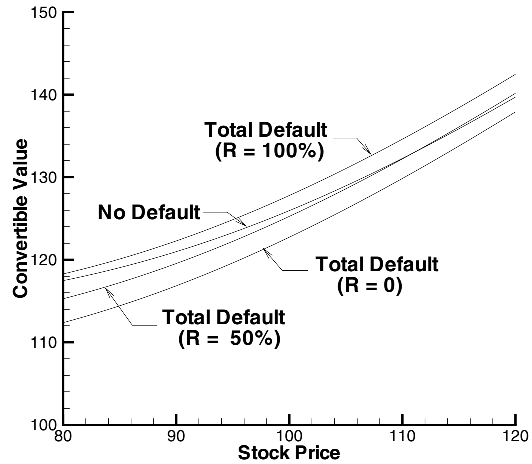
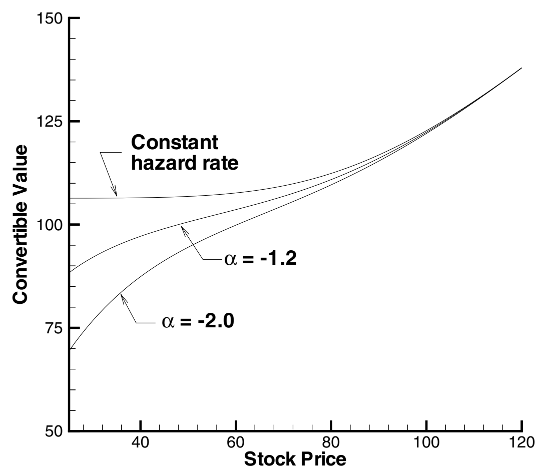


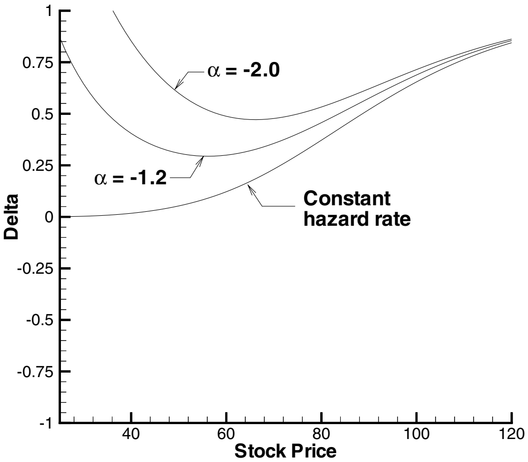
EXHIBIT 9
Total Default Hedge Model



Constant and non-constant hazard rates with different exponents in Equation (73).

EXHIBIT 10

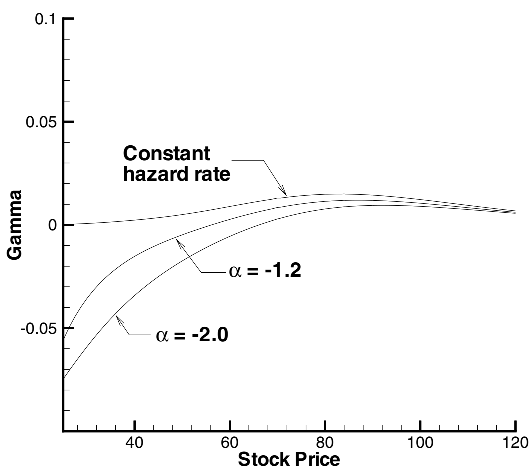
Delta for Total Default Hedge Models



Constant and non-constant hazard rates with different exponents in Equation (73).

EXHIBIT 11

Gamma for Total Default Hedge Models



Constant and non-constant hazard rates with different exponents in Equation (73).

In other words, the expected no-default gains exactly offset the expected default losses for a delta-hedged portfolio under the hedge model. Of course, the Brownian motion risk is identically zero along all paths for this model as well. From Equations (83) and (86), however, we see that in general Equation (89) may not hold for the TF model.

We can verify these results using Monte Carlo simulations. First, we compute and store the discrete PDE linear complementarity solutions for both TF and hedge models. The discrete values of V and V_S are stored at each grid point and time step. We also store flags to indicate whether the convertible bond has been called, put, or converted at every grid node and discrete time t_j. We then compute a realized path S_t^i, assuming a process of the form (54), but in a risk-neutral setting (i.e., replacing μ by r and λ by its risk-neutral counterpart p).

At each discrete time $t_j = j\Delta t$, $S = S(t_j)$, we carry out steps as follows:

- If the convertible has been called, converted, or put, the simulation along this path ends.
- A random draw is made to determine if default occurs in $[t, t + dt]$. If default occurs, add up the losses using Equation (87). The simulation ends.
- If the convertible bond is not called, put, converted, or defaulted, we can compute the gain from Equation (82) for the hedge model, or from Equation (83) for the TF model.
- Repeat for $S(t_j + 1)$ until $t_j = T$.

We repeat this process for many realized paths to obtain an estimate of Equations (82), (83), and (88).

The Monte Carlo hedging simulations are carried out using the data in Exhibit 2 except that we use the variable hazard rate [Equation (73)],with $p_0 = 0.02$, $S_0 = 100$, and $\alpha = -1.2$. Various values of η are used.

Exhibit 12 shows a convergence study of the hedging simulation for $\eta = 1.0$. The expected discounted net value is shown as:

$$E[\text{Net}] = E[G]_d + E[L]_d \tag{90}$$

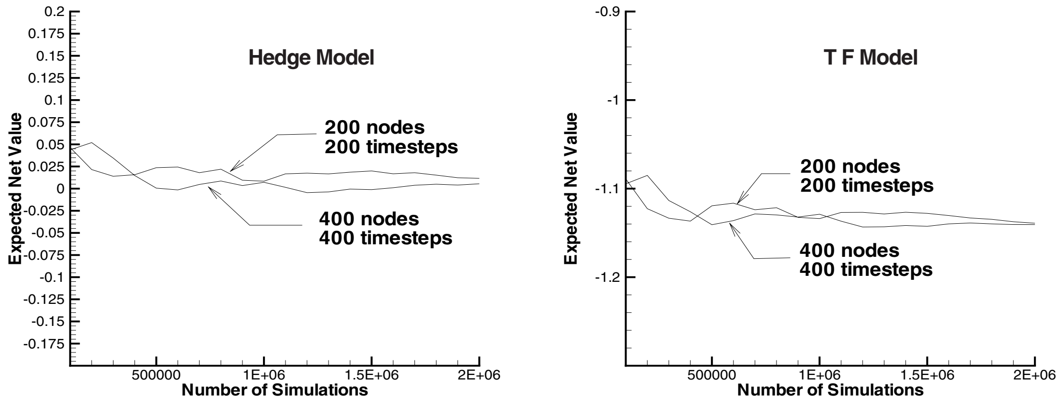
Each time step of the PDE solution is divided into five substeps for the Monte Carlo simulation. According to the results in Exhibit 12, it appears that a PDE solution with 400 nodes and time steps and 2×10^6 Monte Carlo trials is accurate to within a cent.

Exhibit 13 shows that the hedge model has expected gains (no-default) that exactly compensate for expected losses due to default (within the accuracy of the Monte Carlo simulations). In general, this is not true for the TF model, except for a particular choice for the stock jump parameter η.

VII. SUMMARY

Even when the single risk factor is the stock price (interest rates being deterministic), there have been several models proposed for default risk involving convertible bonds. To value convertible bonds with credit

EXHIBIT 12
Convergence Test for Hedging in the Risk-Neutral Measure



Expected net value from Equation (90). Data given in Exhibit 2, but with variable hazard rate [Equation (73)], with $p_0 = 0.02$, $S_0 = 100$, $\alpha = -1.2$, and $\eta = 1$.

EXHIBIT 13
Hedging Simulations

η	Expected Gain	Expected Loss	Expected Net
Hedge Model			
0.0	1.19521	-1.19575	-0.00054
0.5	2.42083	-2.42162	-0.00079
1.0	3.38514	-3.37966	0.00548
TF Model			
0.0	2.32902	-1.11340	1.21562
0.5	2.32902	-2.29152	0.03750
1.0	2.32902	-3.46964	-1.14062

Data given in Exhibit 2, with hazard rate [Equation (73)], but with $p_0 = 0.02$, $S_0 = 100$, $\alpha = -1.2$, and $\eta = 1$. 400 nodes and time steps in PDE solution, 2×10^6 Monte Carlo trials.

risk, it is necessary to specify precisely what happens to the components of the hedging portfolio in the event of a default.

We consider a continuum of possibilities for the value of the stock price after default. Various assumptions can also be made about what is recovered on default. The two special cases we have examined in detail are:

- Partial default: The stock price is un-changed upon default. The holder of the convertible bond can elect to
  1. Receive a recovery factor times the bond component value, or
  2. Convert the bond to shares.
- Total default: The stock price jumps to zero upon default. The equity component of the convertible bond is, by definition, zero. A fraction of the bond value of the convertible is recovered.

In the case of total default with a recovery factor of zero, this model agrees with that in Takahashi et al. [2001]. In this case, there is no need to split the convertible bond into equity and bond components. In the case of non-zero recovery, our model is slightly different from that in Takahashi et al. [2001]. This would appear to be due to a different definition of the term *recovery factor*.

In the partial default case, the model we develop uses a different splitting of bond and equity components than that used in Tsiveriotis and Fernandes [1998). We have presented several arguments as to why we think their model is somewhat inconsistent.

Both the TF model and our model hedge the Brownian risk. In the risk-neutral measure, though, our model ensures that the expected value of the net gains and losses due to default is zero. This is not the case for the TF model. Monte Carlo simulations (in a risk-neutral setting) demonstrate that the net gain/loss of the TF model due to defaults is significant.

It is also possible (using an additional contingent claim) to construct a hedging portfolio that is self-financing and eliminates risk for the hedge model in a real-world default process. This is not possible for the TF model. The impact of model assumptions on real-world hedging is also discussed.

It is possible to make other assumptions about the behavior of the stock price on default. There may also be limits on conversion rights on default, and other assumptions can be made about recovery on default.

The convertible pricing equation is developed through steps as follows:

- The usual hedging portfolio is constructed.
- A Poisson default process is specified.
- Specific assumptions are made about the behavior of the stock price on default, and recovery after default.

It is then straightforward to derive a risk-neutral pricing equation. There are no ad hoc decisions required about what part of the convertible is discounted at the risky rate, and what part at the risky rate. The framework we develop can accommodate many different assumptions.

Convertible bond pricing generally results in a complex coupled system of linear complementarity problems. We have used a partially implicit method to decouple the system of linear complementarity problems at each time step. The final value of the convertible bond is computed by solving a full linear complementarity problem (but with explicitly computed source terms), which gives good convergence as the mesh and time step are reduced, and also results in smooth delta and gamma values.

It is clear that the value of a convertible bond depends on the precise behavior assumed when the issuer goes into default. Given any particular assumption, it is straightforward to model these effects using our framework. A decision on the appropriate assumptions requires an extensive empirical study for different classes of corporate debt.

APPENDIX

Numerical Method

Define $\tau = T - t$, so that the operator $\mathcal{L}V$ becomes

$$\mathcal{L}V = V_\tau - \left(\frac{\sigma^2}{2}S^2V_{SS} + (r(t) - q)SV_S - r(t)V\right) \tag{A-1}$$

and

$$\mathcal{M}V = V_\tau - \left(\frac{\sigma^2}{2}S^2V_{SS} + (r(t) + p\eta - q)SV_S - (r(t) + p)V\right) \tag{A-2}$$

It is also convenient to define:

$$\mathcal{H}V \equiv \left(\frac{\sigma^2}{2}S^2V_{SS} + (r(t) - q)SV_S\right) \tag{A-3}$$

and

$$\mathcal{P}V \equiv \left(\frac{\sigma^2}{2}S^2V_{SS} + (r(t) + p\eta - q)SV_S\right) \tag{A-4}$$

so that Equation (A-1) can be written

$$\mathcal{L}V = V_\tau - (\mathcal{H}V - r(t)V) \tag{A-5}$$

and Equation (A-2) becomes

$$\mathcal{M}V = V_\tau - (\mathcal{P}V - (r(t) + p)V) \tag{A-6}$$

The terms $\mathcal{H}V$ and $\mathcal{P}V$ are discretized using standard methods (see Zvan, Forsyth, and Vetzal [2001]; and Forsyth and Vetzal [2001, 2002]). Let $V_i^n \equiv V(S_i, \tau^n)$, and denote the discrete form of $\mathcal{H}V$ at (S_i, τ^n) by $(\mathcal{H}V)_i^n$, and the discrete form of $\mathcal{M}V$ by $(\mathcal{M}V)_i^n$.

For ease of exposition, we describe the time step method for a fully implicit discretization of Equation (A-1). In actual practice, we use Crank-Nicolson time-stepping with the modification suggested in Rannacher [1984] to handle non-smooth initial conditions (which generally occur at each coupon payment). The reader should have no difficulty generalizing the equations to the Crank-Nicolson or BDF (Becker [1998]) cases. We also suppress the dependence of r on time for notational convenience.

TF Model: Numerical Method

Here is a method that can be used to solve Equations (47) and (48). We denote the total value of the convertible bond computed using explicit constraints by V^E. A corrected total convertible value, obtained by applying estimates for the constraints in implicit fashion, is denoted by V^I. Given initial values of $(V^E)_i^n$, $(V^I)_i^n$, and B_i^n, the time-stepping proceeds as follows.

First, the value of B_i^{n+1} is estimated, ignoring any constraints. We denote this estimate by $(B^*)_i^{n+1}$:

$$\frac{(B^*)_i^{n+1} - B_i^n}{\Delta\tau} = (\mathcal{H}B^*)_i^{n+1} - (r + p)_i^{n+1}(B^*)_i^{n+1} \tag{A-7}$$

This value of $(B^*)_i^{n+1}$ is then used to compute $(V^E)_i^{n+1}$ from:

$$\frac{(V^E)_i^{n+1} - (V^E)_i^n}{\Delta\tau} = (\mathcal{H}V^E)_i^{n+1} - (rV^E)_i^{n+1} - (pB^*)_i^{n+1} \quad \text{(A-8)}$$

Then, we check the minimum-value constraints:

For $i = 1, \dots,$

$B_i^{n+1} = (B^*)_i^{n+1}$
  If $(B_p > \kappa S)$ then
    If $((V^E)_i^{n+1} < B_p)$ then
      $(B)_i^{n+1} = B_p; \quad (V^E)_i^{n+1} = B_p$
    Endif
  Else
    If $((V^E)_i^{n+1} < \kappa S)$ then
      $(B)_i^{n+1} = 0.0; \quad (V^E)_i^{n+1} = \kappa S$
    Endif
  Endif

Endfor

Next, the maximum-value constraints are applied:

For $i = 1, \dots,$
  If $((V^E)_i^{n+1} > \max(B_c, \kappa S))$ then
    $(B)_i^{n+1} = 0; \quad (V^E)_i^{n+1} = \max(B_c, \kappa S)$
  Endif
Endfor

In principle, we could simply go on to the next time step at this point using B_i^{n+1} and $(V^E)_i^{n+1}$, but we have found that convergence (as the time step size is reduced) is enhanced and the delta and gamma values are smoother if we add additional steps. Let

$$(\mathcal{Q}V)_i^{n+1} \equiv \left(\frac{(V^I)_i^{n+1} - (V^I)_i^n}{\Delta\tau}\right) - \big((\mathcal{H}V^I)_i^{n+1} - (rV^I)_i^{n+1} - (pB)_i^{n+1}\big) \quad \text{(A-9)}$$

Then, if $B_c > \kappa S$, $(V^I)_i^{n+1}$ is determined by solving the discrete linear complementary problem:

$$\begin{pmatrix} (\mathcal{Q}V^I)^{n+1} = 0 \\ ((V^I)^{n+1} - \max(B_p, \kappa S)) \geq 0 \\ ((V^I)^{n+1} - B_c)) \leq 0 \end{pmatrix} \vee \begin{pmatrix} (\mathcal{Q}V^I)^{n+1} \geq 0 \\ ((V^I)^{n+1} - \max(B_p, \kappa S)) = 0 \\ ((V^I)^{n+1} - B_c) < 0 \end{pmatrix} \vee \begin{pmatrix} (\mathcal{Q}V^I)^{n+1} \leq 0 \\ ((V^I)^{n+1} - \max(B_p, \kappa S)) \geq 0 \\ ((V^I)^{n+1} - B_c) = 0 \end{pmatrix} \quad \text{(A-10)}$$

while if B_c ″ κS, we apply the Dirichlet conditions:

$$(V^I)_i^{n+1} = \kappa S_i \quad \text{(A-11)}$$

A penalty method is used to solve the discrete complementarity problem (A-12) (see Forsyth and Vetzal [2002]).

Finally we set:

$$\begin{aligned} (V^E)^{n+1} &= (V^I)^{n+1} \\ B_i^{n+1} &= \min(B_i^{n+1}, (V^I)^{n+1}) \end{aligned} \quad \text{(A-12)}$$

This algorithm essentially decouples the system of linear complementarity problems for B and V by applying the constraints in a partially explicit fashion. We apply the constraints as implicitly as possible, without having to solve the fully coupled linear complementarity problem. Consequently, we can expect only first-order convergence (in the time step size $\Delta\tau$), even if Crank-Nicolson time-stepping is used. This approach makes it comparatively straightforward to experiment with different convertible bond models. As well, it is unlikely that the overhead of the fully coupled approach will result in lower computational cost than the decoupled method (at least for practical convergence tolerances).

Hedge Model: Numerical Method

We describe the numerical method used to solve discrete forms of (26) and (27) and (30) and (31). Given initial values of C_i^n and B_i^n, and the total value V_i^{n+1}, the time-stepping proceeds as follows. First, the value of B_i^{n+1} is estimated, ignoring any constraints. We denote this estimate by $(B^*)_i^{n+1}$:

$$\frac{(B^*)_i^{n+1} - B_i^n}{\Delta\tau} = (\mathcal{P}B^*)_i^{n+1} - (r+p)_i^{n+1}(B^*)_i^{n+1} + (pRB^*)_i^{n+1} \quad \text{(A-13)}$$

Then, C_i^{n+1} is estimated, also ignoring constraints. We denote this estimate by $(C^*)_i^{n+1}$:

$$\frac{(C^*)_i^{n+1} - C_i^n}{\Delta\tau} = (\mathcal{P}C)_i^{n+1} - ((r+p)C^*)_i^{n+1} + p\max(\kappa S(1-\eta) - R(B^*)_i^{n+1}, 0) \quad \text{(A-14)}$$

Then, we check the minimum-value constraints:

For $i = 1, \dots,$
$B_i^{n+1} = \min(B_c, (B^*)_i^{n+1})$ // overriding maximum constraint
$C_i^{n+1} = (C^*)_i^{n+1}$
  If $(B_p > \kappa S_i)$ then
    $B_i^{n+1} = \max(B_i^{n+1}, B_p - C_i^{n+1})$
  Else
    $C_i^{n+1} = \max(\kappa S_i - B_i^{n+1}, C_i^{n+1})$
  Endif
Endfor

Then, the maximum-value constraints are applied:

For $i = 1, \ldots,$
$$C_i^{n+1} = \min(C_i^{n+1}, \max(\kappa S_i, B_c) - B_i^{n+1})$$
Endfor

In principle, we could simply continue on to the next time step, setting $V_i^{n+1} = B_i^{n+1} + C_i^{n+1}$. As in the TF splitting, convergence is enhanced if we follow additional steps. Let:

$$(\mathcal{T}V)_i^{n+1} \equiv \left(\frac{V_i^{n+1} - V_i^n}{\Delta t}\right) - \big((\mathcal{P}V)_i^{n+1} - ((r+p)V)_i^{n+1} + p\max(\kappa S_i(1-\eta), (RB)_i^{n+1})\big) \tag{A-15}$$

Then V_i^{n+1} (the total convertible value) is determined by solving the discrete linear complementary problem:

$$\begin{pmatrix} (\mathcal{T}V)^{n+1} = 0 \\ ((V)^{n+1} - \max(B_p, \kappa S)) \geq 0 \\ ((V)^{n+1} - B_c) \leq 0 \end{pmatrix} \vee \begin{pmatrix} (\mathcal{T}V^I)^{n+1} \geq 0 \\ (V^{n+1} - \max(B_p, \kappa S)) = 0 \\ (V^{n+1} - B_c) < 0 \end{pmatrix} \vee \begin{pmatrix} (\mathcal{T}V^I)^{n+1} \leq 0 \\ (V^{n+1} - \max(B_p, \kappa S)) \geq 0 \\ (V^{n+1} - B_c) = 0 \end{pmatrix} \tag{A-16}$$

if $B_c > \kappa S$, while if B_c″ κS we apply the Dirichlet conditions:

$$V_i^{n+1} = \kappa S_i \tag{A-17}$$

As in the TF case, a penalty method is used to solve the discrete complementarity problem (A-10). Finally, we set:

$$\begin{aligned} B_i^{n+1} &= \min(B_i^{n+1}, V_i^{n+1}) \\ C_i^{n+1} &= V_i^{n+1} - B_i^{n+1} \end{aligned} \tag{A-18}$$

which ensures that

$$V_i^{n+1} = C_i^{n+1} + B_i^{n+1} \tag{A-19}$$

ENDNOTES

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[1]There are some variations across these models in terms of the precise specification of default. For example, Merton [1974] considers zero-coupon debt and assumes that default occurs if the value of the firm is lower than the face value of the debt at its maturity. Longstaff and Schwartz [1995] assume that default occurs when the firm value first reaches a specified default level, much like a barrier option.

[2]This is consistent with the results of Brennan and Schwartz, who conclude that "for a reasonable range of interest rates the errors from the [non-stochastic] interest rate model are likely to be slight" [1989, p. 926].

[3]Of course, in practice this is not the case (see, for instance, the discussion in Chapter 26 of Hull [2003]). More complex economic equilibrium arguments can be made, but these lead to pricing equations of the same form as we obtain here, albeit with risk-adjusted parameters.

[4]This is analogous to the results of Duffie and Singleton [1999] in the stochastic interest rate context.

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