# Pricing CoCos with stochastic interest rate via an operator splitting method

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# Contents

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**Abstract**

## Acknowledgments

CHAPTER 1

# Introduction

Suppose a newly established company needs to borrow money. It can seek angel investors but after an initial growth period will inevitably want to issue bonds or equity. Each possesses problems for a small, private company. An initial public offering (IPO) is a complex affair and will result in perhaps the owner(s) losing a great deal of control over the company. The alternative is to issue bonds. But a company in immediate need of money will find bond issuance not only takes a great deal of time, but such a company will inevitably have a low credit rating and find they must entice investors with higher yields.

One solution is to issue convertible bonds. These are faster to issue than regular bonds, offer lower yields than is typical, and appeal to investors who believe in the company's potential growth. A convertible bond is distinguished by one key feature: the holder is able to convert at will, with some constraints, the bond into a specified number of shares of the issuer's stock. This gives the convertible bond properties of equity but also a bond. The yield is typically lower due to this additional feature.

A convertible is clearly a hybrid instrument. It can be fruitfully considered as a bond with a call option to buy shares with the bond, or as stock with a put option to sell the stock in exchange for bond payments.

When the stock price is very high, the call option is "in the money" and worth more than the bond itself. Due to optionality, the option is worth at least a bit more than the conversion value of the bond. When the stock price is very low, the convertible bond behaves like a regular bond and interest rates have the most impact. When interest rates are extremely high, this pushes down the value of the bond. While if the interest rate is very low, this raises the value of the bond. The value of the convertible bond as a pure bond is called the bond floor.

CHAPTER 2

# Bonds

Since the Black–Scholes revolution, it has become common to price bonds by specifying a risk-neutral model where the short-rate is modeled stochastically with one or more sources of uncertainty. Then bonds of any maturity and cash flows, such as zero-coupon bonds, are priced by the risk-neutral paradigm, i.e. the bond at any time is the risk-neutral expected value of the discounted payoff of the bond.

Particularly well-known short-rate models are Rendleman-Bartter, Vasicek, and Cox–Ingersoll–Ross (CIR). We will discuss the CIR model here, since that is the model we used to price the convertible bond with stochastic interest rate.

The CIR model for the interest rate $r(t)$ is given by the following stochastic differential equation (SDE):

$$dr(t) = a(b - r(t))dt + \sigma_r \sqrt{r(t)} d\tilde{W}(t)$$

where $a$, $b$, $\sigma_r$ are all positive.

The redeeming feature of this slightly complicated looking SDE is that $r$ can never be negative. The closer $r$ gets to 0, the more the volatility is reduced, and the drift-term will dominate. The drift, on the other hand, will be negative when $r(t) > b$, which pushes $r$ downwards; when $r(t) < b$, the drift is positive and $r$ gets pushed upwards. Thus we see that that the process for $r$ is mean-reverting with mean $b$. The constant $a$ is called the mean-reversion speed and controls the rate at which the mean-reversion occurs. Finally, note that the volatility is greatest when $r$ is large, and the least when $r$ is small; this fits empirical behavior of short-term interest rates. (For a general reference on the CIR and other short-rate models, see [**MCF06**])

There have been two main approaches to the valuation of defaultable bonds. The first is called the *structural approach*, and was pioneered by Merton in [**Mer74**]. In this approach, the firm's value is modeled. When the firm value reaches a critical lower threshold, default occurs. In such structural models, a firm with an extremely high value has bonds that are considered perfectly safe; thus these models under-estimate the risk. The primary difficulty with structural models is that the firm value is not observable in the markets, and indeed, companies may purposefully obscure meaningful information that would help market participants understand the company value.

In the second approach, the *reduced-form approach*, rather than modeling the firm value, one models the default intensity. Pioneering works in this direction include [**DS99**] and [**JT95**]. This approach is broadly applicable, e.g. it can be used to model sovereign default, and is tractable in practice. Default time is modeled by a Cox process, i.e. a Poisson process with varying intensity. This intensity is typically defined in some way to reflect the risk premium associated with the firm, e.g. a variable may take into account the ratio of credit spread to equity. Since the intensity only depends on the risk and in reality default can involve other factors, in this type of pricing model, the bond's default risk is over-estimated.

For a reference on the mathematics of modeling default time in such a way, see the introduction to [**Duf05**] (also see [**DL01**]).

## 2.1. Pricing a bond with default risk

Since our goal is to price a convertible bond (CB) with default risk, we begin by pricing a zero-coupon bond with default risk; the conversion options and other features will be discussed in Chapter 3, and CB pricing will be explained in Chapter **??**.

As mentioned, structural models have the firm value as a state variable, but this is difficult to observe. So we will take the reduced-form approach, treating the stock value as a state value and incorporating all default risk information into the default intensity function. The main idea here is due to Duffie and Singleton [**DS99**], who showed that the default could be handled by using a risk-adjusted interest rate for the drift of the bond and applying the usual risk-neutral pricing formula.

To be precise, suppose we are given a risk-neutral model with stochastic short-rate $r(t)$. Suppose that the probability of default in an infinitesimal time period is $\lambda(t, s)\ dt$, where $\lambda$ is a deterministic function of equity and time. Let $B(t, s)$ denote the price of a bond with risk of default and suppose upon default we recover $R_v$, which is a fixed percentage of the face value of the bond.

Now, as pointed out in [**AFV03**] (in particular section 3), it is crucial what we assume happens to the stock price upon default. The resulting PDE depends on our assumption. In [**AFV03**], different PDEs are derived assuming that: a) the stock price is unaffected by default b) the stock price is driven to 0 by default.

Their derivations are done via a form of delta-hedging. We will derive a PDE by using a martingale approach suggested in [**AB02**]. Suppose $J$ is the size of the downward jump of the stock value at default time $\tau$, e.g. $S(\tau) = (1 - J)S(\tau-)$ or $\frac{\Delta S}{S(\tau-)} = -J$.

In order to incorporate such jump phenomenon into the evolution of the stock price, it is usual to define an SDE such as the following:

$$\frac{dS(t)}{S(t-)} = \alpha(t)\ dt + \sigma(t, S(t-))\ dW(t) - J\ dN(t)$$

where $W$ is Brownian motion and $N$ is a Cox process with intensity $\lambda(S, t)$ (this can be imagined as a Poisson process with time-varying intensity), where $W$ and $N$ are independent.

Note we can rewrite the above as:

$$\frac{dS(t)}{S(t-)} = r(t)\ dt + \sigma(t, S(t-)) \left[\frac{\alpha(t) - r(t)\ dt - J\lambda(t, S(t))}{\sigma(t, S(t-)}\ dt + \ dW(t)\right] + [J\lambda(t, S(t))\ dt - J\ dN(t)]$$

Note that the second bracketed term is a compensated Cox process and thus a martingale. Girsanov's theorem states there is a change of measure such that the first bracketed term becomes a martingale but the Cox process is unaffected. Then the expected drift of $\frac{dS(t)}{S(t-}$ becomes $r(t)\ dt$ and so this new measure must be the risk-neutral measure.

So under the risk-neutral measure, we have:

$$\frac{dS(t)}{S(t-)} = (r(t) + J\lambda(t, S(t)))\ dt + \sigma(t, S(t-))\ d\tilde{W}(t) - J\ dN(t)$$

where $\tilde{W}$ is Brownian motion. Also, as usual, if the stock gives out dividends continuously with yield $q(t)$, we adjust the drift term by subtracting $q(t)\ dt$.

Now apply Ito's rule for jump-diffusion processes on $D(t)B(t, S(t))$, where $D(t)$ is the discount process associated to $r(t)$. We get that $d\left(D(t)B(t, S(t))\right)$ is equal to (suppressing arguments for ease-of-reading):

$$D\left(B_t + (r - q + J\lambda)SB_s + \frac{1}{2}\sigma^2 S^2 V_{ss} - rB\right) dt + D\Delta B\ dN + d\tilde{W} \text{ term}$$

Note this looks like the usual Black–Scholes PDE but with an additional jump term. The Brownian motion term is already a martingale, so we need only show the other terms together are a martingale.

We can compensate the jump term to be a martingale by using its intensity. Note that $\Delta B(t, S(t)) = R_v(t, S(t-)) - B(t, S(t-))$. Making the substitution for the jump term:

$$D\left(B_t + (r - q + J\lambda)SB_s + \frac{1}{2}\sigma^2 S^2 V_{ss} - rB\right) dt + D\left[R_v - B\right]\ dN$$

Now after some rearrangement, we obtain:

$$D\left(B_t + (r - q + J\lambda)SB_s + \frac{1}{2}\sigma^2 S^2 V_{ss} - (r + \lambda)B + \lambda R_v\right) dt + D(R_v - B)J\left[dN(t) - \lambda\ dt\right]$$

Since the bracketed term is a compensated Cox process, it is a martingale, so in order for the discounted bond process to be a martingale, we need the $dt$ term to be 0, which gives the PDE:

$$B_t + (r - q + J\lambda)sB_s + \frac{1}{2}\sigma^2 s^2 V_{ss} - (r + \lambda)B + \lambda R_v = 0$$

Actually, note there was nothing special about $B$ in this argument. Any derivative security $V(t, S)$ which has the same default behavior would satisfy this PDE. Now given a payoff function for $V$ and appropriate boundary conditions, we can price as usual by solving this PDE numerically. We will do this for a convertible bond in the following chapters.

CHAPTER 3

# Convertible bonds

A convertible bond is challenging to price, as at least one, and typically more than one, option is embedded in it. The interaction between the pure bond and option components can be challenging, especially if parameters such as interest rate and volatilities are modeled stochastically for added realism and (hopefully) more accurate pricing.

In actual convertible bond markets, the issued convertible bonds exhibit derivative features with a wide range of conditions and restrictions. Here we list the main features that are common to convertible bonds and that are implemented in our code:

**Call option:** – convertible bonds are always issued as callable bonds. This means the issuer has a call option on the bond, i.e. the right, but not the obligation, to pay the stated strike to buy the bond back.

**Put option:** – convertible bonds also sometimes have a put feature. The holder has the option to forcibly sell the bond to the issuer for the stated strike value.

**Conversion option:** – This is the primary source of optionality that lends the convertible bond its name. Modulo some restrictions, the holder has the option to convert the bond to shares of common stock of the issuing company. The number of shares per bond, the *conversion ratio*, is specified in the provisions of the bond. Assuming optimal exercise, the bond holder should only convert the bond when the bond value equals the conversion payoff value. Before this happens, the bond value will be more than the conversion payoff, reflecting the price premium for the optionality.

## 3.1. Restrictions on the options

**3.1.1. Hard and soft call/put protection.** A *hard* call is a call option on the bond that cannot be exercised before a certain waiting period expires, e.g. the issuer cannot call the bond for a year. A *soft* call is a call option that knocks-in when certain conditions are met.

Usually the contract specifies a condition of the following variety:

- on any $n$ of the last $m$ days, stock must have exceeded the hurdle
- stock must have crossed the hurdle at least $n$ times
- stock price must have stayed above the hurdle for the last $n$ days

Similar remarks hold for puts, except one should note that the holder of a convertible bond typically have much fewer opportunities to exercise the put than the issuer has to exercise the call.

Both call and put features are usually given strict schedules, but we have chosen to ignore this for the paper. Note that it is easy to modify our code to accommodate the schedules if the allowable exercise times fall on the grid points; simply set the call hurdle and put values to "$\infty$" or 0 resp. when exercise is not allowed.

**3.1.2. Contingent conversion.** The conversion option is also contingent upon the stock price exceeding a hurdle (barrier). Usually the contract specifies a condition like mentioned previously, where a hurdle is met for at least $n$ of the last $m$ days. There is also normally a waiting period before any such conditions are applicable, so that the bond does not get converted too early.

These path-dependencies of the state variables upon which the bond pricing depends, introduces immense complications, theoretical and practical. So in actual practice, the convertible bond is often priced by using an approximation. Most commonly, a "one touch" condition is used: the conversion option *knocks-in* when the hurdle is first met. Thus the convertible is being approximated as a callable bond plus a knock-in call option whose payoff has a strike equal to the price of the regular bond.

The plain callable bond price and the knock-in price are both path-dependent also, but these pricing problems are well-understood textbook problems and easy to implement in code, particularly using a PDE approach.

CHAPTER 4

# Operator splitting method

For well-posed linear PDEs, the Lax equivalence theorem states that for solutions of consistent numerical methods to converge to the solution of the PDE, it is necessary and sufficient that the method be *stable*, i.e. the result at each time step is bounded by a function of the original data at the initial time. "Explicit" schemes such as forward Euler-stepping suffer from *conditional* stability – the time step must be sufficiently small in relation to the spatial steps for the scheme to be stable (see, for example, [**Wil07**] Chap. 77 or [**Duf06**] Chap. 18).

In one spatial dimension, the (fully) implicit method is unconditionally stable, as is the famed Crank–Nicolson method, which can be considered an "average" of the explicit and implicit method. Crank–Nicolson has order two convergence in the time direction, while the implicit method is only order one in time ([**Wil07**] Chapt. 78, [**Duf06**] Chapt. 18); however, the implicit method is known to be much better at Crank–Nicolson at smoothing out higher-order oscillations that result from nasty discontinuities in the payoff function. Thus the method of Rannacher time-marching is commonly used. This consists of using the fully implicit method for several steps (four is recommended) to handle wild oscillatory behavior and then applying Crank–Nicolson for the better convergence properties [**GC05**].

With two or more spatial dimensions, it becomes increasingly harder to solve the equations resulting from the implicit or Crank–Nicolson methods. So typically, in the financial literature, the alternating direction implicit (ADI) method is used. In ADI, a multi-spatial dimensional problem is broken down into one-dimensional problems, one for each space dimension, requiring only the solution of a tridiagonal system. Nonetheless, ADI has trouble with mixed derivative terms ([**Duf06**] Chap. 20).

The PDE we need to solve for our convertible bond pricing problem has two spatial variables, stock and interest rate, and a mixed derivative. So instead of ADI, we use an operating splitting method (sometimes referred to as "Soviet splitting" or fractional stepping).

The particular approach we take is due to Yanenko, so is called “Yanenko splitting” ([**Duf06**] Chap. 20, and particularly, [**She07**]).

## 4.1. Yanenko splitting

A good reference for a fully worked-out example of Yanenko splitting is R. Sheppard's thesis [**She07**]. We rely on it for much of our explanation that follows.

Recall the PDE we obtained in Chapter2 for a bond with default risk:

$$V_t + (r(t) - q + J\lambda)sV_s + \frac{1}{2}\sigma^2 s^2 V_{ss} - (r+\lambda)V + \lambda R_v = 0$$

We derived the PDE under fairly general conditions, but for our implementation we chose to limit some variables to constants.

We make the following assumptions:

(1) Volatilities of the stock, $\sigma$, and interest rate, $\sigma_r$, are constant and have a constant correlation $\rho$.
(2) The interest rate $r(t)$ is CIR.
(3) The stock evolves under geometric Brownian motion with constant dividend yield $q$.
(4) Upon default, the stock value drops to 0.
(5) Upon default, a fixed percentage of the par value is recovered; this value is denoted $R_v$.

Recall the CIR model:

$$dr(t) = a(b - r(t)) + \sigma_r d\tilde{Z}(t)$$

where $a$, $b$, and $\sigma_r$ are positive constants. Let $W$ be the Brownian motion driving $S$. Then by assumption $dW \cdot dZ = \rho dt$. Using these facts, we can simplify the above PDE:

$$\begin{aligned} V_t + (r(t) - q + \lambda(s) - \frac{1}{2}\sigma^2)sV_s + a(b - r(t))V_r + \frac{1}{2}\sigma^2 s^2 V_{ss} \\ + \frac{1}{2} r\sigma_r^2 V_{rr} + \rho\sqrt{r}\sigma_r \sigma s V_{rs} - (r(t) + \lambda(s))V + \lambda(s)R_v = 0 \end{aligned}$$

Note the correlation term has a mixed derivative.

The idea behind operator splitting methods is to solve several PDEs, each at a fictitious time step we imagine to be between the two actual times.

With the above PDE, when we solve from time step $n$ to $n+1$, going backwards in time from the terminal payoff function, we can insert a fictitious time step $n+1/2$ and solve a "piece" of the original PDE. Then we continue from time step $n+1/2$ to $n+1$, solving another piece of the PDE and arriving at a solution the original PDE.

There is not a unique way to do this, even were we to stick with Yanenko splitting. Although not necessary, we have chosen to group derivatives of one spatial dimension together. So the two PDEs we must solve are:

$$W_t + (r(t) - q + \lambda(s) - \frac{1}{2}\sigma^2)sW_s + \frac{1}{2}\sigma^2 s^2 W_{ss} + \frac{1}{2}\rho\sqrt{r}\sigma_r\sigma s U_{rs} = 0$$
$$V_t + a(b - r(t))V_r + \frac{1}{2}r\sigma_r^2 V_{rr} + \frac{1}{2}\rho\sqrt{r}\sigma_r\sigma s W_{rs} - (r(t) + \lambda(s))V + \lambda(s)R_v = 0$$

The function $U$ in the first equation is considered given. First, we solve for $W$. Next using that as the given $W$ in the second equation, we solve for $V$, which we expect to be close to the solution of the original PDE. Then for the next iteration from step $n+1$ to $n+2$, we set $U = V$ and begin anew.

**4.1.1. The discretization.** We will discretize the two PDEs by using central differences in the spatial variables and one-sided difference for the time derivative. Let the equity variable be indexed by $i$ and interest rate by $j$. When discretizing, it is helpful to think of $U$ as the value at the previous time step. Thus we see that there is a natural way to make the method fully implicit.

$$\frac{W_{i,j} - U_{i,j}}{\Delta t} + \left(r_j - q + \lambda(s_i) - \frac{1}{2}\sigma^2\right) s_i \left(\frac{W_{i+1,j} - W_{i-1,j}}{2\Delta s}\right) + \frac{1}{2}\sigma^2 s_i^2 \left(\frac{W_{i+1,j} - 2W_{i,j} + W_{i-1,j}}{\Delta t^2}\right)$$
$$+ \frac{1}{2}\rho\sqrt{r_i}\sigma_r\sigma s_i \left(\frac{U_{i+1,j+1} + U_{i-1,j-1} - U_{i+1,j-1} - U_{i-1,j+1}}{4\Delta s\Delta r}\right) = 0$$

$$\frac{V_{i,j} - W_{i,j}}{\Delta t} + (a(b - r_j)) \left(\frac{V_{i+1,j} - V_{i-1,j}}{2\Delta r}\right) + \frac{1}{2}r\sigma_r^2 \left(\frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{\Delta t^2}\right)$$
$$+ \frac{1}{2}\rho\sqrt{r_i}\sigma_r\sigma s_i \left(\frac{W_{i+1,j+1} + W_{i-1,j-1} - W_{i+1,j-1} - W_{i-1,j+1}}{4\Delta s\Delta r}\right) - (r_j + \lambda(s_i))\, V_{i,j} + \lambda(s_i)R_v = 0$$

Recall that $s_i = i\Delta s$ and $r_j = r\Delta j$. We can clearly write each equation as a linear system where we are solving for values of $W$ and $V$, respectively, at the grid points.

If we wanted to solve the systems by a method such as LU decomposition, this is what we should do. However, it is simpler to write each equation so that $W_{i,j}$ and $V_{i,j}$ resp. are on one side and everything is on the other. That form suggests an iterative approach to solving the linear systems.

$$\begin{aligned}\left(\frac{1}{\Delta t}+i^2\sigma^2\right)W_{i,j} &= \frac{1}{2}\left(i^2\sigma^2+i^2\left(j\Delta r-q+\lambda(i\Delta s)-\frac{1}{2}\sigma^2\right)\right)W_{i+1,j}\\ &+\frac{1}{2}\left(i^2\sigma^2-i^2\left(j\Delta r-q+\lambda(i\Delta s)-\frac{1}{2}\sigma^2\right)\right)W_{i-1,j}\\ &+\frac{1}{\Delta t}U_{i,j}+\frac{1}{8}\rho\sqrt{(}j\Delta R)\sigma_r^2\frac{i}{\Delta r}(U_{i+1,j+1}+U_{i-1,j-1}-U_{i-1,j+1}-U_{i+1,j-1})\end{aligned}$$

$$\begin{aligned}\left(\frac{1}{\Delta t}+\frac{j}{\Delta r}\sigma_r^2+j\Delta r+\lambda(i\Delta s)\right)V_{i,j} &= \frac{1}{2\Delta r}\left(j\sigma_r^2+a(b-j\Delta r)\right)V_{i,j+1}\\ &+\frac{1}{2\Delta r}\left(j\sigma_r^2-a(b-j\Delta r)\right)V_{i,j-1}+\frac{1}{\Delta t}W_{i,j}+\lambda(i\Delta s)R_v\\ &+\frac{1}{8}\rho\sqrt{(}j\Delta R)\sigma_r^2\frac{i}{\Delta r}(W_{i+1,j+1}+W_{i-1,j-1}-W_{i-1,j+1}-W_{i+1,j-1})\end{aligned}$$

**4.1.2. PSOR.** To use the equations in the iterative approach, start with a guess for $W$. Then as we increase the indices $i$ and $j$ we use the first equation to set the value of $W_{i,j}$. Note that the computation of a single $W_{i,j}$ will use some of the values of $W$ from the guess only (the ones with at least one greater index), while other values used are the result of previous computations at this step of the iteration (the ones with both smaller indices). Then we repeat this procedure using the result as the new guess. After we have done this enough so that the maximal difference between the old and new guess is small enough, we finish the iteration and move on to solving the next system.

This particular iterative method is called *Gauss–Seidel.* If we didn't update the values during each level of iteration, the method is called *Jacobi* and is known to converge slower than Gauss–Seidel.

For equations that come from parabolic PDEs, it is known that Gauss–Seidel will, at any given grid point, push the guesses in one direction only, without oscillating about the true solution. Knowing that, it makes sense to give an extra push in that direction. Thus if $U_{i,j}$ is the old value and $W_{i,j}$ is the new value from Gauss–Seidel, our new guess should be a weighted combination $(1-\omega)U_{i,j} + \omega W_{i,j}$ where $1 < \omega < 2$. This is called successive over-relaxation (SOR) (sometimes Gauss–Seidel itself is called a SOR method). There is often an optimal $\omega$ for a given problem.

In our code, we use projective SOR (technically, projective Gauss–Seidel). This is a fancy name for simply doing SOR at a grid point and then possibly replacing that value with another, e.g. taking a max of the option value and payoff for an American option.

**4.1.3. Boundary conditions.** In the previous section, we gave a tridiagonal linear system that must be solved at each fractional time-step. But this is incomplete, as boundary conditions were not specified, so it is not clear we have a unique or even well-behaved solution.

Particularly worrisome is that although the original problem presumably had well-defined boundary conditions, there are seemingly not obvious boundary conditions to use at a fictitious time step. While this can be a challenging issue to resolve, our situation is very pleasant. At each fractional step, we are using a one-dimensional fully implicit method on a parabolic PDE. Thus our method is unconditionally stable at each stage and will preserve the order of accuracy. Not only that, no corrections to the original boundary conditions are needed! See ([**She07**], section 5.6) for further details and references.

**4.1.4. Consistency.** Finally, note that the Yanenko splitting is consistent to order one in time and order two in space. See ([**She07**], section 5.7) for an involved calculation of the local truncation error.

CHAPTER 5

# Implementation and testing

Now that we have a well-defined PDE with boundary conditions, we can create a grid and price using the finite difference method explained in Chapter 4. What remains is to price the given financial instrument, in this case a contingent convertible bond, the so-called "CoCo", and see if our results are reasonable.

The following are default parameters unless we state otherwise.

| Parameter | Value |
|---|---|
| mean reversion speed $a$ | 2.0 |
| mean interest rate $b$ | 0.05 |
| interest rate vol $\sigma_r$ | 0.04 |
| dividend yield $q$ | 0.01 |
| equity vol $\sigma$ | 0.3 |
| vol correlation $\rho$ | -0.5 |
| maturity $T$ | 1 |
| par $F$ | 1000 |
| recovery value $R_v$ | 0.400 |
| max equity $s_max$ | 300 |
| max interest $r_max$ | 1 |
| Equity steps $I$ | 50 |
| Interest steps $J$ | 50 |
| Time steps $K$ | 252 |

We model the contingent convertible partly as a convertible bond (CB) where the holder *always* has the option to convert the bond to stock. We call this the "no barrier" CB component. A contingent CB should be have a lower price, all other features being equal, than the no-barrier CB, as it can only be converted once the stock price exceeds the given hurdle; in other words, a contingent CB has conversion option that *knocks in* once the stock price meets a barrier (hurdle).

So to price the contingent convertible, we price the no-barrier CB component, and then we price the knock-in conversion option. Note that the underlying of the knock-in option is the no-barrier CB. It turns out to be trickier to price a knock-in rather than a knock-*out*, so we price the knock-out conversion option. This option allows the holder to exercise the conversion until the stock price meets the hurdle. Then the contingent convertible price is given by the no-barrier CB price minus the knock-out price.

## 5.1. Code correctness

First we did some tests to test the correctness of our Yanenko code. We implemented the fully implicit method to price the no-barrier CB and compared the two implementations for the above parameters. The sample portion of the data below is typical (full table is in the appendix). There is no discernible difference to at least $10^{-4}$ for most of the grid; however there were differences of about $10^{-1}$ for equity space steps of near $i = 15$ with the range of such $i$'s diminishing as $j$ increases.

One would expect most of the differences to be near the boundary of the grid. So we did another test. Since most of the 0s may be resulting from the max conditions in our algorithms (due to the put and call features), we set the call hurdle to $\infty$, put value to 0, and $\kappa = 0$. Now as we expected, we see most errors near the boundary of the grid, particularly near $i = 0$, although errors are of magnitude $10^{-1}$.

## 5.2.

Next, we tested to see whether we are getting financially sensible results.

Looking at the prices for the no-barrier CB, we did a check of the $t = 0$ prices for the range of $r$ and $s$ shown above to see that:

- As $r \to \infty$ (fixing $i$), the price decreased.
- As $s \to \infty$ (fixing $j$), the price increased.

Also, since our parameters for the embedded options are integers, we can visibly see where we get non-integers, which are the places where the options are exercising. As we would expect, for larger $r$, the conversion and call options activate at a lower equity value, while the put option is exercised at a higher spot.

Having done several heuristic tests for reasonability, we then moved to more rigorous testing. We priced a defaultable (but not convertible) CIR-based bond with recovery value $R_v$. Then we changed the parameters for our knock-in CB so that the call, put, and conversion features were turned off on the CB component (call and conversion hurdles set to infinity and put value set to 0) and moved the barrier from the knock-out option component to the top of the grid. Thus we expected that the difference of the two components would cancel out the conversion option, leaving just a CIR bond that has default risk.

Comparing the results we got in Tables , we see that the values are very close.

Next we checked the defaultable CIR bond against a risk-free CIR zero-coupon bond. First, we priced a CIR bond using a closed-form formula. Then we set $R_v = F$, the par value, for $r = 0$, since a risk-free zero-coupon bond will be worth par when interest rate is zero.

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