Pricing CoCos under stochastic interest rate via an operator splitting method

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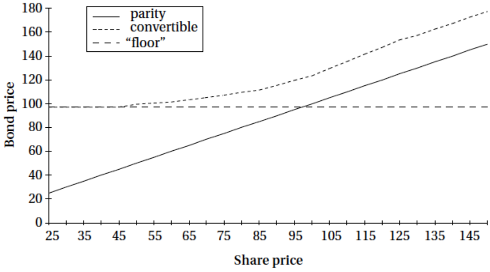
Convertible bonds

Convertible bonds (CB) are a hybrid of bond and equity. Besides having basic bond features, the holder has the option to exchange (convert) the bond for a specified number of shares of stock of the issuing company.

Some reasons companies issue convertible bonds:

1. Avoid immediate stock dilution
2. Lower yields (typically 1-3 % less) despite lower credit rating
3. Upon conversion, debt vanishes (appealing for companies with high growth potential)

CB price



- Terminal payoff function is $\max(F, \kappa S)$ (F is par value, κ is conversion ratio).
- When S is small, price is close to that of a straight bond.
- When S is large, price is close to that of an American call option with the above payoff.t

Bond with default risk

Suppose default time τ is the first event time of a Cox process with intensity $\lambda(t, s)$ and R_v is recovered upon bond default.

Consider the bond as a derivative on the stock, which jumps downward J % upon default, i.e. $\frac{\Delta S}{S(\tau-)} = -J$. The risk-neutral SDE for the stock is:

$$\frac{dS(t)}{S(t-)} = (r(t) + J\lambda(t, S(t)))\ dt + \sigma(t, S(t-))\ d\tilde{W}(t) - J\ dN(t)$$

Then the PDE for a bond with default risk is:

$$B_t + (r - q + J\lambda)sB_s + \frac{1}{2}\sigma^2 s^2 B_{ss} - (r + \lambda)B + \lambda R_v = 0$$

Now we consider stochastic interest rate $r(t)$. We will use the CIR model:

$$dr(t) = a(b - r(t)) + \sigma_r d\tilde{Z}(t)$$

We can derive the following PDE by considering our bond V to be a function of t, r, and s:

$$V_t + (r(t) - q + \lambda(s) - \frac{1}{2}\sigma^2)sV_s + a(b - r(t))V_r + \frac{1}{2}\sigma^2 s^2 V_{ss} + \frac{1}{2}r\sigma_r^2 V_{rr} + \rho\sqrt{r}\sigma_r\sigma s V_{rs} - (r(t) + \lambda(s))V + \lambda(s)R_v = 0$$

where $d\tilde{Z} \cdot d\tilde{W} = \rho\ dt$. We need appropriate boundary conditions to solve this PDE. We have the terminal payoff condition.

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Yanenko splitting

Fractional-stepping method: we will insert a fictitious half time-step between actual grid point times. The equations to solve at a given (real) time step are:

$$W_t + (r(t) - q + \lambda(s) - \frac{1}{2}\sigma^2)sW_s + \frac{1}{2}\sigma^2 s^2 W_{ss} + \frac{1}{2}\rho\sqrt{r}\sigma_r \sigma s U_{rs} = 0$$

$$V_t + a(b - r(t))V_r + \frac{1}{2}r\sigma_r^2 V_{rr} + \frac{1}{2}\rho\sqrt{r}\sigma_r \sigma s W_{rs} - (r(t) + \lambda(s))V + \lambda(s)R_v = 0$$

The function U in the first equation is considered given at time step n. First, we solve for W. Next using that as the given W in the second equation, we solve for V, the solution to the original PDE at time step $n+1$. Then for the next iteration from step $n+1$ to $n+2$, we set $U = V$ and begin anew.

We will discretize the two PDEs by using central differences in the spatial variables and one-sided difference for the time derivative. Let the equity variable be indexed by i and interest rate by j and let $s_i = i\Delta s$ and $r_j = r\Delta j$.

After discretization, we rearrange to obtain:

$$
\begin{aligned}
\left(\frac{1}{\Delta t}+i^2\sigma^2\right)W_{i,j} &= \frac{1}{2}\left(i^2\sigma^2+i^2\left(j\Delta r-q+\lambda(i\Delta s)-\frac{1}{2}\sigma^2\right)\right)W_{i+1,j}\\
&+\frac{1}{2}\left(i^2\sigma^2-i^2\left(j\Delta r-q+\lambda(i\Delta s)-\frac{1}{2}\sigma^2\right)\right)W_{i-1,j}\\
&+\frac{1}{\Delta t}U_{i,j}+\frac{1}{8}\rho\sqrt(j\Delta r)\sigma_r^2\frac{i}{\Delta r}(U_{i+1,j+1}+U_{i-1,j-1}-U_{i-1,j+1}-U_{i+1,j-1})
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{1}{\Delta t}+\frac{j}{\Delta r}\sigma_r^2+j\Delta r+\lambda(i\Delta s)\right)V_{i,j} &= \frac{1}{2\Delta r}\left(j\sigma_r^2+a(b-j\Delta r)\right)V_{i,j+1}\\
&+\frac{1}{2\Delta r}\left(j\sigma_r^2-a(b-j\Delta r)\right)V_{i,j-1}+\frac{1}{\Delta t}W_{i,j}+\lambda(i\Delta s)R_v\\
&+\frac{1}{8}\rho\sqrt(j\Delta r)\sigma_r^2\frac{i}{\Delta r}(W_{i+1,j+1}+W_{i-1,j-1}-W_{i-1,j+1}-W_{i+1,j-1})
\end{aligned}
$$

PSOR

The equations can be solved iteratively using Gauss–Seidel or SOR and the boundary conditions from before. This method is faster but equivalent in accuracy to the fully implicit method (which we checked).

A note on boundary conditions: the same boundary conditions work at a fictitious time step, as each step is unconditionally stable (we are using the one-dimensional fully implicit method).

Call and put features are easy to implement using PSOR.

CB features

Call option – convertible bonds are always issued as callable bonds. This means the issuer has a call option on the bond, i.e. the right, but not the obligation, to pay the stated strike to buy the bond back.

Put option – convertible bonds also sometimes have a put feature. The holder has the option to forcibly sell the bond to the issuer for the stated strike value.

Conversion option – Modulo some restrictions, the holder has the option to convert the bond to shares of common stock of the issuing company. The number of shares per bond, the *conversion ratio*, is specified in the provisions of the bond. Assuming optimal exercise, the bond holder should only convert the bond when the bond value equals the conversion payoff value.

Contingent conversion

A contingent convertible ("CoCo") only allows the holder to convert after some conditions are met, typically that the stock has exceeded a hurdle for at least 20 of the last 30 days. This path-dependency makes CoCos much harder to price.

We have chosen to implement the contingent conversion as a knock-in barrier option. So we price V_{nb}, a CB with no contingent condition and V_b, an American call option that knocks-*out* when the stock exceeds the hurdle.

The exercise value of V_b is $\max(\kappa s - V_{nb}, 0)$.

Then the price of the contingent convertible is $V_{nb} - V_b$.

Sanity checks

- Compared pricing of V_{nb} with fully implicit method with and without max conditions in PSOR (most errors near boundary of grid and of at most 10^{-1}).
- We checked that:
  - As $r \to \infty$ (fixing i), the price decreased.
  - As $s \to \infty$ (fixing j), the price increased.
  - For larger r, conversion and call options exercised at lower s
  - For larger r, put is exercised at higher s

Checking implementation against pure bonds

- Priced a plain CIR bond with default risk; compared against $V_{nb} - V_b$ pricing with all optionality turned off and barrier at infinity.
- Priced CIR bond using closed-form formula; compared against plain CIR bond with default risk and full recovery value. (defaultable bond was sometimes slightly higher-priced, but only a few cents more for $F = 1000$)