A DYNAMIC GENERAL EQUILIBRIUM MODEL OF THE ASSET MARKET AND ITS APPLICATION TO THE PRICING OF THE CAPITAL STRUCTURE OF THE FIRM

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1. Introduction.

In earlier papers, [4] and [5], the problem of lifetime consumption-portfolio decisions for an individual investor was examined in the context of a continuous-time model. The current paper uses a similar approach to derive general equilibrium relationships among securities in the asset market. Under the assumption that the value of the firm is independent of its capital structure, an explicit equation for pricing the individual securities within the capital structure is presented.

In sections two and three, a partial equilibrium model for pricing the capital structure is developed to aid in the understanding of the approach and for comparison with the general equilibrium solution in later sections. In sections four, five, six, and seven, the general model is derived and its implications for security pricing, the term structure of interest rates, and the capital structure of the firm under various assumptions, are discussed.

Although the paper concentrates on the asset markets, the model can be generalized to examine equilibrium behavior in other sectors of the economy as well.

2. A Partial Equilibrium Period Model.

In an earlier paper with P. A. Samuelson [8], we derived a theory of warrant pricing based on expected utility maximization when the individual has a portfolio choice among three assets: the warrant, the stock of the firm, and a riskless asset. The model presented in this section follows the approach used in that paper.

Consider an economy made up of one firm with current value $V(t)$. Further, assume that there exists a "representative man" for the economy who acts so as to maximize the expected utility of wealth at the end of a period of length τ.[1] That is, since the firm is the only asset in the economy, he acts so as to

$$\text{(1)} \qquad \text{Max } E_t U[V(t+\tau)]$$

where "E_t" is the conditional expectation, conditional on knowing that $V(t) = V$, and U is assumed to be strictly concave and monotonically increasing, i.e. $U' > 0$ and $U'' < 0$.

We further postulate a known probability distribution, $P(Z,\tau)$, for the value of the firm at the end of period where the random variable Z is defined by

$$\text{(2)} \qquad Z \equiv V(t+\tau)/V(t).$$

The random variable Z reflects both the uncertainty about the cash flow (or earnings) of the firm and the changes in value of the firm's capital stock or earning assets over the period. A crucial assumption is that P(Z) is independent of the particular capital structure of the firm, i.e. P is determined solely by the characteristics of the asset side of the balance sheet and is not affected by the particular instruments used by the firm to finance these assets. This assumption is consistent with the Modigliani-Miller theorem, and as such, we implicitly assume perfect capital markets and tax effects are not considered.

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[1] Since, in this section, we are using a period model, τ could be set equal to one. However, it will be useful for later development to carry the general symbol τ.

Consider that the firm chooses a particular set of financial instruments (debt, equity, etc.) defined by their terminal conditions, find the current equilibrium value of each of these future claims on the terminal (random) value of the firm. Define $F_i(V,\tau)$[2] as the current value of the i^{th} type of security $i = 1, \ldots, n$ with terminal date τ from now issued by the firm. The different types of securities are distinguishable by their terminal value, $F_i(VZ,0)$, contingent on the terminal value of the firm, $V(t+\tau) = VZ$. For example, if one of the securities is a debt issue $(i = 1)$, senior to all other claims on the firm, with a terminal claim of B dollars on the firm, then

$$F_1(VZ,0) = \min(B,VZ) \tag{3}$$

i.e. the debt holders will receive B dollars at the end of the period if the firm can pay, or in the event that the firm cannot pay (default), they are entitled to all the assets of the firm which will have value VZ.

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[2] Strictly, F_i will be a function of the current values of all securities senior to it, the capitalization rate, etc. in addition to V. However, in equilibrium, the F_i are perfectly positively correlated with changes in the value of the firm and so, these other arguments of the function will enter only as parameters.

To determine the equilibrium values of each of the securities, note that since each of the securities appears separately in the market place, they must be priced so that when examined by the representative man, he will choose his portfolio so as to hold the amount supplied, i.e.,

$$V = \sum_{1}^{n} F_i(V,\tau) \tag{4}$$

and, of course, $VZ = \sum_{1}^{n} F_i(VZ,0)$. Define $w_i \equiv F_i(V,\tau)/V$ = percentage of the firm's assets financed by the i^{th} security. Then, because the firm is the only asset in the economy, w_i will also be equal to the percent of the representative man's initial wealth invested in the i^{th} security. We re-write (1) as a maximization under constraint problem

$$\max_{\{w_i\}} E_t U[V \sum_{1}^{n} w_i \frac{F_i(VZ,0)}{F_i(V,\tau)}] + \lambda[1 - \sum_{1}^{n} w_i]. \tag{5}$$

The first order conditions[3] derived from (5) are

$$E_t\left\{\frac{F_i(VZ,0)}{F_i(V,\tau)} U'[V \sum_{1}^{n} w_i \frac{F_i(VZ,0)}{F_i(V,\tau)}]\right\} = \lambda,\ i = 1, \ldots, n. \tag{6}$$

(6) can be re-written in terms of util-prob distributions,[4] Q, as

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[3] The assumption of strict concavity of U is sufficient to ensure an unique interior maximization which rules out any need for inequalities in the first order conditions.

[4] See [8], p. 19-20 for further discussion of the util-prob concept.

$$(7)\quad \int_0^\infty \frac{F_i(VZ,0)}{F_i(V,\tau)}\,dQ = \int_0^\infty \frac{F_j(VZ,0)}{F_j(V,\tau)}\,dQ \equiv e^{\eta\tau},\ \text{for all } i, j = 1, \ldots n,$$

where $dQ \equiv \dfrac{U'(ZV)dP(Z,\tau)}{\int_0^\infty U'(ZV)dP(Z,\tau)}$ and $e^{\eta\tau}$ is a new multiplier related to the original λ multiplier. Note the important substitution of VZ for $V\sum_1^n w_i \frac{F_i(VZ,0)}{F_i(V,\tau)}$ in the definition of dQ. By the assumption that the value of the firm is independent of its capital structure, we have that dQ is independent of the functions F_i, $i = 1, \ldots, n$. Therefore, (7) is a set of integral equations, linear in the F_i.[5] Hence, we can meaningfully re-write (7) as

$$(8)\qquad F_i(V,\tau) = e^{-\eta\tau}\int_0^\infty F_i(VZ,0)\,dQ(Z;\tau),\ i = 1, \ldots n.$$

Since the $F_i(VZ,0)$ are known functions determined by the type of security and U and $P(Z,\tau)$ are assumed known, (8) would be sufficient to determine the current equilibrium values of the i^th security if we knew η.

From examination of (7) and noting again that dQ is independent of the particular capital structure chosen, we find that $e^{\eta\tau}$ (and hence η) is independent of the particular capital structure. Since (7) holds for all capital structures, it must hold for the trivial capital structure, namely, when the firm issues just one type of security, equity, and n = 1. In this case, it is obvious that $F_1(V,\tau) = V$ and $F_1(VZ,0) = VZ$. Substituting in (7), we have that

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[5] Thus, the assumption that the firm's value is independent of its capital structure provides the same mathematical simplification that the assumption of the "incipient" case for warrant pricing did in [8], p. 26.

$$e^{\eta\tau} = \int_0^{\infty} Z\,dQ(Z;\tau) \tag{9}$$

i.e. $e^{\eta\tau}$ is the expected rate of return of the firm in util-prob space. Equation (7) says that the expected return on all securities in util-prob space must be equated. If U was linear (i.e. the representative man was "risk-neutral"), then dQ = dP and (7) would imply the well-known result for risk-neutrality that expected returns (in the ordinary sense) be equated. Hence, the util-prob distribution is the distribution of returns adjusted for risk.

3. Some Examples. Using equation (8), we can derive the equilibrium pricing for various capital structures of the firm. Example one assumes that there are two types of securities: debt and equity. Suppose that the amount of debt issued by the firm represents a terminal claim of B dollars on the firm. Let $F_1(V,\tau)$ be the current value of the debt outstanding and $F_2(V,\tau)$ be the current value of the (residual) equity. Then, from previous discussion and equation (3), the terminal value of the debt will be $F_1(VZ,0) = \min(B,VZ)$. From equations (8) and (3), the current value of the debt will be

$$F_1(V,\tau) = e^{-\eta\tau}\left[\int_0^{B/V} ZV\,dQ(Z,\tau) + \int_{B/V}^{\infty} B\,dQ(Z,\tau)\right]. \tag{10}$$

We can re-write (10) as

$$(10') \quad F_1(V,\tau) = e^{-\eta\tau}B - e^{-\eta\tau}\int_0^{B/V}(B - ZV)dQ(Z,\tau).$$

Suppose that the terminal claim of the debt holders is very small relative to the (current) total value of the firm (i.e. $0 < B << V$) or alternatively, $dQ(Z,\tau) = 0$ for $0 \leq Z \leq B/V$, then

$$(11) \quad F_1(V,\tau) \rightarrow e^{-\eta\tau}B \text{ as } B/V \rightarrow 0.$$

In the limit, the debt becomes risk-less and so, from (11), we have that η must be the risk-less rate of return per unit time (in both util-prob and ordinary returns space) for the period of length τ. Hence, from now on, η will be replaced by r, the usual notation for the risk-less rate. Examining (10'), the second term is the discounted expected loss in util-prob space due to default on the debt,[6] and as such, is a risk-premium charged over the risk-less rate. A second useful form of (10) is

$$(10'') \quad F_1(V,\tau) = e^{-r\tau}\left[\int_0^{\infty} ZVdQ(Z,\tau) - \int_{B/V}^{\infty}(ZV - B)dQ(Z,\tau)\right]$$

$$= V - e^{-r\tau}\int_{B/V}^{\infty}(ZV - B)dQ(Z,\tau).$$

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[6] Throughout the paper, all debt is assumed to be of the "discounted-loan" type with no payments prior to maturity. Similarly, it is assumed that no dividends are paid on the equity. It is not difficult to rewrite interest-paying bonds as a sum of discounted loans, and hence the analysis of the paper is easily adapted to the examination of these types of securities.

Since in equilibrium, $V = F_1(V,\tau) + F_2(V,\tau)$, the current value of equity, $F_2(V,\tau)$, must satisfy

$$(12) \qquad F_2(V,\tau) = e^{-r\tau}\int_{B/V}^{\infty} (ZV - B)dQ(Z,\tau).$$

(12) is identical to the warrant pricing equation derived in [8] on pages 27-29 for a warrant with exercise price B. Because we are pricing the securities as functions of the total value of the firm, the nature of equity as a "residual" security makes it a "warrant" on the firm. Equation (12) could have been derived directly in a similar manner to the derivation of the debt equation (10) by starting with the terminal value of equity which is

$$(13) \qquad F_2(VZ,0) = \text{Max}[0, VZ - B].^7$$

In example two, consider a firm with a capital structure made up from three types of securities: debt with a terminal claim of B dollars on the firm; equity of which there are N shares outstanding with current price per share of S (i.e. $F_2(V,\tau) \equiv NS$.); warrants which terminate at the end of the period and each warrant gives the holder the right to purchase one share of stock at $\overline{S}$ dollars per share. Assume there are n warrants outstanding with current market

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[7] Equation (13) immediately suggests a warrant "interpretation" of equity since it is the standard terminal condition for a warrant.

value per warrant of W (i.e. $F_3(V,\tau) = nW$). Because the warrant is a junior security to the debt, the current value of the debt for this firm will be the same as in example one, namely, equation (10). The current value of the equity will be

$$(14) \qquad F_2(V,\tau) = e^{-r\tau}\left[\int_{B/V}^{\gamma/V}(ZV - B)\,dQ(Z,\tau) + \frac{N}{n+N}\int_{\gamma/V}^{\infty}(ZV + n\bar{S} - B)\,dQ(Z,\tau)\right],$$

where γ is the maximum value of VZ such that the price per share of equity is less than or equal to $\bar{S}$ (i.e. the maximum value of the firm at the end of the period such that the warrants are not exercised.). Thus, for $ZV \leq \gamma$, as in example one, the equity owners receive the residual value of the firm, $ZV - B$. However, for $ZV > \gamma$, the warrant holders will exercise their warrants by turning in the warrants plus a total of $n\bar{S}$ dollars to the firm in return for n shares of equity. In this event, the original equity holders ownership will be diluted and they will be entitled to $N/n+N$ percent of the residual value of the firm which will be $ZV + n\bar{S} - B$.

To determine γ, let S' be the terminal price per share of equity and suppose that the warrants are not exercised, i.e. $S' \leq \bar{S}$, but that Z is such that $ZV > B$. Then, $VZ - B = NS'$ or $VZ = NS' + B$. But γ is defined as the maximum value of VZ such that $S' \leq \bar{S}$. Hence

$$(15) \qquad \gamma = N\bar{S} + B.$$

To determine the current value of the warrants, re-write (14) as

$$(16)\qquad F_2(V,\tau) = e^{-r\tau}\Big[\int_{B/V}^{\infty}(ZV - B)dQ(Z,\tau) + \frac{N}{n+N}\int_{\gamma/V}^{\infty}(n\bar{S} + (1 - \frac{(n+N)}{N})(ZV - B))dQ(Z,\tau)\Big]$$

$$= V - F_1(V,\tau) + \frac{N e^{-r\tau}}{n+N}\Big[\int_{\gamma/V}^{\infty}(n\bar{S} - \frac{n}{N}(ZV - B))dQ(Z,\tau)\Big].$$

But, in equilibrium, $F_3(V,\tau) = V - F_1(V,\tau) - F_2(V,\tau)$, and so, from (16), we have, after re-arranging terms, that

$$(17)\qquad F_3(V,\tau) = \frac{n\, e^{-r\tau}}{n+N}\int_{\gamma/V}^{\infty}(ZV - \gamma)dQ(Z,\tau).$$

To compare (17) with the warrant pricing formula derived in [8], (17) is re-written in a "normalized"[9] price per warrant form. Let the normalized price of the firm be defined as

$$(18)\qquad y \equiv V/\gamma = V/n+N/(N\bar{S}+B)/n+N$$

and the normalized price of a warrant be defined as

$$(19)\qquad w \equiv F_3(V,\tau)/n(\frac{\gamma}{n+N}).$$

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[9] By "normalized" price, we mean instead of dollars as the unit of price, use the exercise price as the unit so that when the "normalized" price of the stock is one, the dollar price of the stock is $\bar{S}$. See [7] for a complete description of this useful standardization process.

Then (17) can be re-written as

$$(20) \qquad w(y,\tau) = e^{-\lambda\tau}\int_{1/y}^{\infty}(Zy - 1)dQ(Z,\tau)$$

which is of the same form as (24) in [8]. However, there is a difference between the two equations: namely, in [8], we used the exercise price of the common, $\bar{S}$, as the normalizing price while to derive (20), the "exercise" price of the firm, $\gamma/n+N$, was used. From the definition of γ in (15), the "exercise" price of the firm is $(N\bar{S}+B)/n+N$. If the firm holds no debt (which is implicitly assumed in [8], since we work directly with the stock price distribution as exogeneous instead of with the distribution of the firm's value) and, as in [8], one concentrates on the "incipient" case (i.e. $n = 0$), then $\gamma/n+N = \bar{S}$, and (17) and (24) in [8] are identical. The advantage of the present analysis is that it explicitly takes into account in current valuation the future dilution possiblity of a large number of warrants outstanding.

For the third example, consider a firm whose capital structure contains a convertible bond issue with a total terminal claim on the firm of B dollars or alternatively the bonds can be exchanged for a total of n shares of equity, and N shares of equity with current price per share of S dollars. The terminal value of the original N shares of equity (i.e., $F_2(VZ,0)$) will be zero if $VZ < B$, equal to $VZ - B$ if $VZ > B$ and the bonds are not converted, or equal to $NVZ/n+N$, if the bonds are converted. The bond holders have the option of conversion, and they will convert or not depending on which option gives the larger value.

Because $VZ = F_1(VZ,0) + F_2(VZ,0)$, this implies that

$$(21) \qquad F_2(VZ,0) = \mathrm{Max}[0, \mathrm{Min}(VZ - B, NVZ/n + N)].$$

Hence from (8) and (21), we have that the current value of equity is

$$(22) \qquad F_2(V,\tau) = e^{-r\tau}\left[\int_{B/V}^{\gamma/V}(VZ - B)dQ(Z,\tau) + \frac{N}{n+N}\int_{\gamma/V}^{\infty} VZdQ(Z,\tau)\right]$$

where by a similar procedure to the previous example, γ is determined to be $(n+N)B/n$. By using the equilibrium condition that $F_1(V,\tau) = V - F_2(V,\tau)$, the current value of the convertible bond issue is derived as

$$(23) \quad F_1(V,\tau) = e^{-r\tau}\left[\int_0^{B/V} VZdQ(Z,\tau) + \int_{B/V}^{\infty} BdQ(Z,\tau) + \frac{n}{n+N}\int_{\gamma/V}^{\infty}(VZ-\gamma)dQ(Z,\tau)\right].$$

By comparing (23) with (10) and (17), we have the well-known result that the value of a convertible is equal to its value as a straight bond plus a warrant with exercise price $\overline{S} = B/n$.[9]

The fourth and final example is to price the capital structure of a "dual" fund. A dual fund is the same as an ordinary closed-end mutual fund on the asset side (i.e., both type funds hold marketable securities as their only assets). However, unlike the usual closed-end fund, the dual fund issues two types of securities to finance these assets: namely, capital shares (equity) and income shares (a type of bond). The difference between the income shares and the bond of example one is that, in addition to a fixed terminal claim, the income shares are entitled to all ordinary income (dividends, interest, etc.) of the fund while the capital shares are entitled to all capital gains (over the fixed terminal claim). To protect the income shareholders, the fund managers may be required to invest the fund's portfolio in securities which will earn some fixed proportion of the total asset value in the form of dividends or interest. Let γ be the instantaneous fixed proportion of total asset value earned as ordinary income. Further, if the fund managers act to maximize the capital shares return subject to the above constraints, then they will choose a portfolio which just meets the γ requirement. Let V be the current asset value of the fund and Z the (random variable) total return on the fund including dividends, interest, and capital gains. Clearly, the distribution of asset returns is independent of the capital structure since all assets are marketable securities. Let $F_1(V,\tau)$ be the current value of the income shares with terminal claim on the fund of B dollars plus all interest and dividends earned. Let $F_2(V,\tau)$ be the current value of the capital shares. From the definition of Z and γ, the capital gains part of the total return on the assets is $e^{-\gamma\tau}Z$. Hence, the terminal value of the capital shares will be

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[9] This result holds provided the bond is non-callable. Because the model is of the one-period type, we have implicitly assumed all securities to be non-callable. Few convertible bonds actually sell at their bond plus warrant value because the call feature on the bond means that *both* the bond and the warrant are callable which reduces the value of the package.

$$(24) \qquad F_2(VZ,0) = \text{Max}[0, e^{-\rho\tau} VZ - B].$$

From equation (8), we have that the current value of the capital shares is

$$(25) \qquad F_2(V,\tau) = e^{-(r+\rho)\tau} \int_{\gamma/V}^{\infty} (VZ - \gamma) dQ(Z,\tau)$$

where $\gamma \equiv Be^{\rho\tau}$. The current value of the income shares is

$$(26) \quad F_1(V,\tau) = e^{-r\tau}\left[\int_0^{\gamma/V} VZ dQ(Z,\tau) + \int_{\gamma/V}^{\infty} B dQ(Z,\tau) + \int_{\gamma/V}^{\infty} VZ(1 - e^{-\rho\tau}) dQ(Z,\tau)\right].$$

From (25), one can show that the current value of the capital shares can be less than the current net asset value of the capital shares defined to be V - B.[10] From (25), we have that

$$(27) \qquad F_2(V,\tau) < e^{-(\rho+r)\tau} \int_{\gamma/V}^{\infty} VZ dQ(Z,\tau)$$

$$< e^{-(\rho+r)\tau} \int_0^{\infty} VZ dQ(Z,\tau)$$

$$< e^{-\rho\tau} V.$$

Hence, if $e^{-\rho\tau} V < V - B$, then $F_2(V,\tau)$ will be less than V - B. So for $V > B/(1 - e^{-\rho\tau})$, $F_2(V,\tau) < V - B$.

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[10] This is the definition generally used by the Wall Street Journal, for example, to determine whether the capital shares are selling at a premium or discount.

This concludes the examples of capital structure pricing based on the model of section two. One could extend the theory to include multi-period analysis by the use of the iterated integral technique employed on pages 28-29 of [8]. However, rather than extend the present partial equilibrium model further, an intertemporal general equilibrium model will be developed in the following sections which will include the model of section two as a special case.

4. A General Equilibrium Intertemporal Model of the Asset Market. Consider an economy with K consumers-investors and n firms with current value V_i, $i = 1, \ldots, n$. Each consumer acts so as to

$$(28) \qquad \text{Max } E_o\left[\int_0^{T^k} U^k[C^k(s),s]ds + B^k[W(T^k),T^k]\right]$$

where "E_o" is the conditional expectation operator, conditional on the value of current wealth, $W^k(0) = W^k$, of the k^{th} consumer and on the current value of the firms, $V_i(0) = V_i$, $i = 1, \ldots, n$. $C^k(s)$ is his instantaneous consumption at time s. U^k is a strictly concave von Neumann-Morgenstern utility function; B is a strictly concave "bequest" or utility-of-terminal wealth function; and T^k is the date of death of the k^{th} consumer. Define $N_i(t)P_i(t) \equiv V_i(t)$ where $N_i(t)$ is the number of shares[11] of firm i outstanding at time t and $P_i(t)$ is the price per share at time t. It is assumed that expectations about the dynamics of the prices per share in the future are the same for all investors and these dynamics can be described by the stochastic differential equation[12]

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[11] In this section, the particular capital structure of the firm will not be discussed, and hence, one can think of each firm as having the trivial capital structure, namely, all equity. However, the assumption that the value of the firm is independent of its capital structure is retained throughout the paper.

$$\frac{dP_i}{P_i} = \alpha_i dt + \sigma_i dZ_i, \quad i = 1, \ldots, n. \tag{29}$$

where the instantaneous expected rate of return, α_i, and the instantaneous standard deviation of return, σ_i, may change stochastically over time, but only in a way which is instantaneously uncorrelated with price changes (i.e. $d\alpha_i dZ_j = d\sigma_i dZ_j = 0$ for $i, j = 1, \ldots, n$). The dZ_i represent a simple Gauss-Wiener process with zero mean and standard deviation one (often referred to as Gaussian "white noise"). (29) includes returns from both capital gains and dividends, and reflects both the uncertainties about future cash flows and changes in the "capitalized" value of the firm's earning assets. Notice that if α_i and σ_i were constant, then the $P_i(t)$ would be log-normally distributed. Further assume that one of the n assets (by convention the n^th one) is an "instantaneously" risk-less asset[13] with instantaneous return r(t) and that the dynamics of this rate are described by

$$dr = f(r,t)dt + g(r,t)dq \tag{30}$$

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[12] For a discussion of and further references to stochastic differential equations of the type in (29), see [5].

[13] What is meant by an "instantaneously" risk-less asset is that, at each instant of time, each investor knows with certainty that he can earn return r(t) over the next instant by holding the asset (i.e. $\sigma_n = 0$ and $\alpha_n = r$). However, the future values of r (t) are not known with certainty. It is assumed here that one of the firms is characterized by this asset. Alternatively, one could postulate a government which issues (very) short bonds, or that r(t) is the instantaneous private sector borrowing (and lending) rate.

where (30) is the same type of equation as (29) and dq is a simple Gauss-Wiener process. For computational simplicity, it is further assumed that α_i and σ_i, in (29), are functions only of $r(t)$[14], i.e. investors only anticipate revising their expectations about returns if the interest rate changes.

From the definition of N_i and P_i, we have that the change in the value of the i^{th} firm over time is $dV = N_i dP_i + dN_i(P_i + dP_i)$. The first term is that part of the changed value of the firm due to cash flow and changes in the value of its assets. The second term is that part of the changed value of the firm due to the issue (or purchase) of new shares at the new price per share, $P_i + dP_i$.[15] Substituting from (29) for dP_i/P_i and writing everything in percentage terms, we have that

$$\text{(31)} \quad \frac{dV_i}{V_i} = \alpha_i dt + \sigma_i dZ_i + \frac{dN_i}{N_i}(1 + \alpha_i dt + \sigma_i dZ_i), \; i = 1, \ldots, n$$

The accumulation equation for the k^{th} investor can be written as[16]

$$\text{(32)} \quad dW^k = \sum_1^n w_i^k W^k dP_i/P_i + (y^k - C^k)dt,$$

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[14] Since $dqdZ_i$ will not be zero in general, the changes in α_i and σ_i are correlated with price changes. Hence, we modify the earlier assumption of no correlation to include this particular "indirect" correlation caused by interest rate changes.

[15] For symmetry, it is assumed that firms do not pay dividends, but adjust their total size by issuing or purchasing their shares in the market.

where y^k is his wage income and w_i^k is the percentage of his wealth invested in the i^{th} security (hence, $\sum_1^n w_i^k \equiv 1$). Therefore, his demand for the i^{th} security, d_i^k, can be written as

$$d_i^k = w_i^k W_i^k = N_i^k P_i \tag{33}$$

where N_i^k is the number of shares of the i^{th} security demanded. Substituting for dP_i/P_i from (29) (and noting that the n^{th} asset is risk-less), we can re-write (32) as

$$dW^k = [\sum_1^m w_i^k (\alpha_i - r) + r]dt + \sum_1^m w_i^k \sigma_i dZ_i + (y^k - C^k)dt \tag{34}$$

where $m \equiv n - 1$ and the $w_1^k, \ldots, w_m^k$ are unconstrained[17] because w_n can always be chosen to satisfy the constraint $\sum_1^n w_i^k = 1$.

From the budget constraint, $W^k = \sum_1^n N_i^k P_i$, and the accumulation equation, (32), we have that

$$(y^k - C^k)dt = \sum_1^n dN_i^k(P_i + dP_i), \tag{35}$$

i.e. the net value of shares purchased must equal the value of savings from wage income.

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[16] See [5] for a derivation of (32). Although taken here to be deterministic, there are no particular problems with letting wage income be stochastic.

[17] Hence, we allow borrowing and short-selling by all investors.

I have shown elsewhere[18] that the necessary optimality conditions for an individual who acts according to (28) in choosing his consumption-investment program are[19]

$$
(36) \quad 0 = \max_{\{C^k, w^k\}} [U^k(C^k,t) + J_3^k(W^k,r,t) + J_2^k f + J_1^k[(\sum_1^m w_i^k(\alpha_i - r) + r)W^k
+ (y^k - C^k)] + 1/2 J_{22}^k g^2 + 1/2 J_{11}^k \sum_1^m \sum_1^m w_i^k w_j^k \sigma_{ij} (W^k)^2 +
J_{12}^k \sum_1^m \sigma_{ir} w_i^k W^k]
$$

subject to $J^k(W^k, r, T^k) = B^k(W^k, T^k)$ and where subscripts on the $J^k(W^k, r, t)$ function denote partial derivatives. The σ_{ij} are the instantaneous covariances between the returns on the i^{th} and j^{th} assets (i.e. $dP_i/P_i dP_j/P_j \equiv \sigma_{ij} dt$), and σ_{ir} is the instantaneous covariance between the return on the i^{th} asset and the change in the rate of interest (i.e. $dr\, dP_i/P_i \equiv \sigma_{ir} dt$). The m+1 first-order conditions derived from (36) are

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[18] See [4] and [5].

[19] $J^k(W^k,r,t) \equiv \max E_t \left\{ \int_t^{T^k} U(C^k,s)ds + B^k(W^k,T^k) \right\}$ and is called the "derived" utility of wealth function. Substituting from (37) and (38) to eliminate w^k and C^k in (36), makes (36) a partial differential equation for J^k, subject to the boundary condition

$J^k(W^k,r,T^k) = B^k(W^k,T^k)$. Having solved for J^k, we then substitute for J^k and its derivatives in (37) and (38) to find the optimal rules (w^k,C^k).

$$(37) \qquad 0 = U_1^k(C^k,t) - J_1^k(W^k,r,t)$$[20]

and

$$(38) \qquad 0 = J_1^k(\alpha_i - r) + J_{11}^k \sum_1^m w_j^k W^k \sigma_{ij} + J_{12}^k \sigma_{ir}, \; i = 1, \ldots, m.$$

(38) can be solved explicitly for the demand functions for each risky security as

$$(39) \qquad d_i^k = A^k \sum_1^m V_{ij}(\alpha_j - r) + H^k \sum_1^m V_{ij} \sigma_{jr}, \; i = 1, \ldots, m,$$

where the V_{ij} are the elements of the inverse of the instantaneous variance-covariance matrix of returns, $\Omega = [\sigma_{ij}]$, $A^k \equiv -J_1^k/J_{11}^k$, and $H^k \equiv -J_{12}^k/J_{11}^k$. Applying the implicit function theorem to (37), we have that

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[20] Because the paper is primarily interested in finding equilibrium conditions for the asset markets, the model assumes a single consumption good. However, C^k could have easily been taken as a vector of h different consumption goods in which case "C^k" in (34) would be replaced by

$$\text{"}\sum_1^n X_i C_i^k\text{"}$$

where X_i is the price of the i^{th} good. There would then be an additional (h - 1) equations in (37) of the type derived in ordinary consumer demand theory. One can see how the model could be extended to examine the dynamics of consumption good demand by incorporating expectations about future commodity (relative) prices.

$$(40) \qquad A^k = -U_1/(U_{11}\frac{\partial C^k}{\partial W^k}) > 0$$

$$H^k = -\frac{\partial C^k}{\partial r} \Big/ \frac{\partial C^k}{\partial W^k} \gtreqless 0.$$

The aggregate demands for the risky securities can be derived from (39) by summing over all investors as follows:

$$(41) \qquad D_i \equiv \sum_1^K d_i^k$$

$$= A\sum_1^m V_{ij}(\alpha_j - r) + H\sum_1^m V_{ij}\,\sigma_{jr}, \; i = 1, \ldots, m$$

where $A \equiv \sum_1^K A^k$ and $H \equiv \sum_1^K H^k$. (41) can be re-written in matrix-vector form, as

$$(41a) \qquad D = A\Omega^{-1}(\alpha - r) + H\Omega^{-1}\sigma_r.$$

If it is assumed that the asset market is always in equilibrium, then $N_i = \sum_1^K N_i^k$ and $dN_i = \sum_1^K dN_i^k$ for i = 1, . . ., n. Furthermore, $\sum_1^n N_i P_i = \sum_1^n D_i \equiv M$ where M is the total value of all assets, i.e. the value of the "market," in equilibrium. From the definition of M, we have that

$$(42) \qquad dM = \sum_1^n N_i dP_i + \sum_1^n dN_i(P_i + dP_i)$$

$$= \sum_1^K dW^k \quad , \text{ in equilibrium,}$$

$$= \sum_1^n D_i \frac{dP_i}{P_i} + \sum_1^K (y^k - C^k)dt, \text{ from (32).}$$

Changes in the value of the market come about by capital gains on current shares outstanding (the first term in (42)) and by expansion of the total number of shares outstanding (the second term in (42)). To separate these two effects, let P_M be the price per "share" of the market (portfolio) and N be the number of "shares," i.e. $M \equiv NP_M$. Then, $dM = NdP_M + dN(P_M + dP_M)$, and P_M and N are defined by

$$(43) \qquad NdP_M \equiv \sum_1^n N_i dP_i$$

$$dN(P_M + dP_M) \equiv \sum_1^n dN_i(P_i + dP_i).$$

If we combine the equilibrium condition, $dN_i = \sum_1^K dN_i^k$, with equation (35), then

$$(44) \qquad dN(P_M + dP_M) = \sum_1^K (y^k - C^k)dt$$

and hence, from (42),

$$(45) \qquad NdP_M = \sum_1^n D_i \frac{dP_i}{P_i} .$$

Define $w_i \equiv N_iP_i/M = D_i/M$, the percentage contribution of the i^{th} firm to total market value. By dividing equation (45) by M and substituting for dP_i/P_i from (29), we can re-write (45) in terms of the instantaneous rates of return as

$$(46)\qquad \frac{dP_M}{P_M} = [\sum_1^m w_j(\alpha_j - r) + r]dt + \sum_1^m w_j \sigma_j dZ_j .$$

The instantaneous expected rate of return, α_M; the variance of the return, σ_M^2; the covariance with the return on the i^{th} asset, σ_{iM}; and the covariance of the return with a change in the interest rate, σ_{Mr}, of the market portfolio can be determined from (46) as follows:

$$(47)\qquad \alpha_M \equiv \sum_1^m w_j(\alpha_j - r) + r$$

$$\sigma_{iM}dt \equiv \frac{dP_M}{P_M}\frac{dP_i}{P_i} = \sum_1^m w_j \sigma_{ij}dt,\ i = 1, \ldots, m$$

$$\sigma_M^2 dt \equiv \frac{dP_M}{P_M}\frac{dP_M}{P_M} = \sum_1^m w_j \sigma_{jM}dt$$

$$\sigma_{Mr}dt \equiv \frac{dP_M}{P_M}dr = \sum_1^m w_j \sigma_{jr}dt .$$

By manipulating (41a), one can solve for the yields on individual risky assets: namely, in matrix-vector form

$$\alpha - r = \frac{1}{A}\Omega D - \frac{H}{A}\sigma_r. \tag{48}$$

In equilibrium, $D_i = w_i M$, and hence, (48) can be re-written in scalar form as

$$\begin{aligned} \alpha_i - r &= \frac{M}{A}\sum_1^m w_j \sigma_{ij} - \frac{H}{A}\sigma_{ir}. \\ &= \frac{M}{A}\sigma_{iM} - \frac{H}{A}\sigma_{ir}\ , \quad i = 1, \ldots, m \end{aligned} \tag{49}$$

By multiplying (49) by w_i and summing from one to m, we have that

$$\alpha_M - r = \frac{M}{A}\sigma_M^2 - \frac{H}{A}\sigma_{Mr}. \tag{50}$$

In summary, given the distribution of returns, individual preferences, and endowments, equation (41a) can be used to determine the equilibrium (relative to the risk-less asset) prices of the m risky assets. Given the consumption level and (37), the price of the risk-less asset relative to the price of the consumption good can be determined, closing out the system.

Alternatively, one can assume that security prices are "correct" (i.e., equilibrium prices) and then, use (49) to determine the equilibrium (relative to the risk-less asset) expected yields of the risky assets. Then, given the consumption level, the equilibrium interest rate can be determined from (37).

Since prices are observable and expected yields are not, the second formulation will be used. Because the emphasis is on finding equilibrium (relative) relationships among assets, the consumption equation, (37), will be ignored, and the interest rate treated as exogeneous to the asset market.

5. Model I.: A Constant Interest Rate Assumption. Consider the particular case of the general model of the previous section when the interest rate is assumed to be constant (i.e. $f = g = 0$ in (30)). By examining (39), one can see that the ratios of an investor's demands for risky assets are the same for all investors (i.e. independent of preferences, wealth, etc.). Hence, the "mutual fund" or Separation Theorem holds,[21] and all optimal portfolios can be represented as a linear combination of any two distinct efficient portfolios (mutual funds).[22] In equilibrium, the market portfolio must be efficient, and so, one can choose the two efficient funds to be the market portfolio and the risk-less aset. From (49) with $\sigma_{ir} = 0$, we see that the ratios of equilibrium relative expected yields are independent of preferences. Further, by combining (49) and (50), the term depending on preferences can be eliminated, and the equilibrium expected return on an individual security can be written as a function of the expected market return and the interest rate as

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[21] See [5] for a discussion, further references, and a proof of the Separation theorem for this model.

[22] See [6] for a proof and further discussion.

$$\alpha_i - r = \frac{\sigma_{iM}}{\sigma_M^2}(\alpha_M - r),\ k = 1, \ldots, m. \tag{51}$$

With a slightly different interpretation of the variables, (51) is the equation for the Security Market Line (p. 89, [9]) of the Sharpe-Lintner-Mossin capital asset pricing model, and all the implications of their model will be implied by Model I as well. The S-L-M model is a "period" model and implicitly must assume quadratic utility functions or Gaussion-distributed prices to be consistent with the expected utility maxim. Model I. is an intertemporal model which assumes that trading takes place continuously and that price changes are continuous (although not differentiable in the usual sense). If α_i and σ_i are constant, then prices are log-normally distributed (which is reasonable because "limited liability" is ensured and by the central limit theorem, it is the only regular solution to any independent, multiplicative, finite-moment, continuous space, infinitely-divisible process in time). The model as presented is consistent over time in the sense that the implications of the assumed price behavior is not a priori refutable. The assumption of normally-distributed prices is bothersome because, no matter how compact the distribution, given enough time, one would expect to observe some negative prices. Similarly, the assumption of quadratic utility, given enough time, leads to the problem of wealth satiation or negative marginal utility.

The models are empirically distinguishable since over time samples drawn from log-normal distributions will differ from those drawn from normal distributions. If it is assumed that the w_i are constant over time, then from (46), it can be shown that P_M is log-normally distributed.[23] We can integrate (29) to get, conditional on $P_i(t) = P_i$

$$(52) \qquad P_i(t + \tau) = P_i \exp[(\alpha_i - 1/2\ \sigma_i^2)\tau + \sigma_i Z_i(t;\tau)]$$

where $Z_i(t;\tau) \equiv \int_t^{t+\tau} dZ_i$ is a normal variate with zero mean and variance τ. Similarly, we can integrate (46) to get

$$(53) \qquad P_M(t + \tau) = P_M \exp\ [(\alpha_M - 1/2\,\sigma_M^2) + \sigma_M X(t;\tau)]$$

where $X(t;\tau) \equiv \int_t^{t+\tau} \sum_1^m w_j \sigma_j dZ_j / \sigma_M$ is a normal variate with zero mean and variance τ. Define the variables

$$(54) \quad \rho_i(t + \tau) \equiv \log[P_i(t + \tau)/P_i(t)] = (\alpha_i - 1/2\,\sigma_i^2)\tau + \sigma_i Z_i(t;\tau)$$

$$\rho_M(t + \tau) \equiv \log[P_M(t + \tau)/P_M(t)] = (\alpha_M - 1/2\,\sigma_M^2)\tau + \sigma_M X(t;\tau).$$

Consider the ordinary least squares regression

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[23] See [5], p. 14-15.

$$(55) \qquad \mathcal{P}_i(t) - r = \beta_i(\mathcal{P}_M(t) - r) + \gamma_i + \varepsilon_t^i .$$

After making the usual assumptions about the differences between ex-ante expectations and ex-post outcomes, if Model I. is the "true" specification, then from (51) and (55) the following must hold:

$$(56) \qquad \begin{aligned} \beta_i &= \sigma_{iM}/\sigma_M^2 \\ \gamma_i &= 1/2(\beta_i\sigma_M^2 - \sigma_i^2)\tau \\ \varepsilon_t^i &= \sqrt{1 - \rho_{iM}^2}\ \sigma_i Y_i(t;\tau) \end{aligned}$$

where τ is the length of the time period between observations; ρ_{iM} is the instantaneous correlation coefficient between the return on the i^{th} security and the market; $Y_i(t;\tau)$ is a normal variate with zero mean, variance τ, and a covariance with the market return of zero. In the context of Model I., the correct specification is logarithmic changes in prices, and notice that the constant, γ_i, will not in general be zero.[24] Notice that for (55) to be a correct specification, r must be constant over time,[25] and since r does vary, our general model shows that the specification will be incorrect. In section seven, this more general case is discussed.

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[24] This result has implications for various tests of portfolio performance (e.g. see [3]) which have used a regression model similar to (55) and have assumed that the correct "benchmark" is $\gamma_i = 0$. Unfortunately, γ_i as derived in (56) is ambiguous in sign.

[25] It is sufficient to assume that $\sigma_{ir} = 0$ for $i = 1, ..., m$ to have Model I. be the correct specification. However, this assumption seems to be no more reasonable than r equal a constant. Particularly, when an asset which is correlated with changes in r could (and would) easily be created if r did vary.

Equation (51) describes the equilibrium expected yield relationship among firms. Using the model of this section, we return to the problem posed in sections one and two, namely pricing the capital structure of the firm.

When the firm issued one type of security, we could define a relationship $V_i(t) \equiv N_i(t)P_i(t)$. Now, $V_i(t) = \sum_1^k N_{ij}(t)P_{ij}(t)$ where N_{ij} is the number of units of the jth type security issued by the i^{th} firm and P_{ij} is the price per unit. Since only one firm will be considered at a time, the i subscript will be dropped and the unsubscripted variables will be for any firm, e.g. $V(t) = \sum_1^k N_j(t)P_j(t)$. For simplicity, consider the case of example one, section two, where the firm's capital structure consists of two securities: equity and debt. It is also assumed that the firm is enjoined from the issue or purchase of securities prior to the redemption date of the debt (τ "years" from now).[26] Hence, from (31), we have that

$$\frac{dV}{V} = \frac{dP}{P} = \alpha\, dt + \sigma dZ, \tag{57}$$

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[26] This assumption is stronger than necessary. It is sufficient that investors have no expectations about future issues, or alternatively, that any new issues have the same terms as the current capital structure and that they be issued in the same proportions of units (not values) as the current structure. A more general model using the same approach as in the text could be formulated to include the expectations of future issues.

The assumption that the debt is of the discounted-loan type is not completely innocent because of the possibility of default on interim interest payments. Although the resulting mathematics is more complicated, the basic approach used here could be modified to include the case of interim payments as well.

where α and σ are constants.

Let $D(t;\tau)$ be the current value of the debt with τ years until maturity and with redemption value at that time of B. Then, $D(t+\tau;0) = \text{Min}(V_{(t+\tau)},B)$, and therefore, it is reasonable to assume that $D(t;\tau)$ will depend on the interest rate and the probability of default which will be a function of the current value of the firm. Because the current value of equity is $V_t - D(t;\tau)$, equity will only depend on the current value of the firm and the interest rate. Let $F(V,\tau)$ be the current value of equity where the variable, r has been suppressed because, in this model, it is constant. The dynamics of the return on equity can be written as

$$\frac{dF}{F} = \alpha_e dt + \sigma_e dZ \tag{58}$$

where α_e is the instantaneous expected rate of return, σ_e is the instantaneous standard deviation of return, and dZ is the same standard Wiener process as in (57). α_e and σ_e are not constants, but functions of V and τ. Like every security in the economy, the equity of the firm must satisfy (51) in equilibrium, and hence,

$$\alpha_e - r = \frac{\rho\sigma_e\sigma_M}{\sigma_M^2}(\alpha_M - r) \tag{59}$$

where ρ is the instantaneous correlation coefficient between dZ and the market return. Further, by Ito's Lemma (see [5]), we have that

(60) $$dF = F_V dV + F_\tau d\tau + 1/2 F_{VV}(dV)^2$$

where subscripts denote partial derivatives. Since τ is the length of time until maturity, $d\tau = -dt$. Substituting for dV from (57), we re-write (60) as

(61) $$dF = [\frac{\sigma^2 V^2}{2} F_{VV} + \alpha V F_V - F_\tau]dt + \sigma V F_V dZ$$

where $(dV)^2 \equiv \sigma^2 V^2 dt$. Comparing (58) and (61), it must be that

(62) $$\alpha_e F \equiv \frac{\sigma^2 V^2}{2} F_{VV} + \alpha V F_V - F_\tau$$

and

(63) $$\sigma_e F \equiv \sigma V F_V \text{ .}$$

As previously shown, the return on holding the firm itself must satisfy equation (51) in equilibrium. Hence,

(64) $$\alpha - r = \frac{\rho \sigma \sigma_M}{\sigma_M^2} (\alpha_M - r).$$

Substituting for α and α_e from (59) and (64) into (62), we have the fundamental partial differential equation of security pricing

$$(65) \qquad 0 = 1/2\sigma^2V^2F_{VV} + rVF_V - F_\tau - rF$$

subject to the boundary condition $F(V,0) = \text{Max}[0, V - B]$. The solution[27] to (65) is

$$(66) \qquad F(V,\tau) = e^{-r\tau}\int_{B/V}^{\infty}(VZ - B)\,d\Lambda(Z;\tau)$$

where Z is a log-normally distributed random variable with mean $r\tau$ and variance $\sigma^2\tau$, and $d\Lambda$ is the log-normal density function. I call (65) the fundamental partial differential equation of asset pricing because all the securities in the firm's capital structure must satisfy it. As was true of the model in section one, securities are distinguished by their terminal claims (boundary conditions). For example, the value of the debt of the firm satisfies (65) subject to the boundary condition $F(V,0) = \min[V,B]$. A comparison of (66) with (12) shows that they are the same for $dQ = d\Lambda$. (66) can be re-written in general form as

$$(67) \qquad F(V,\tau) = e^{-r\tau}\int_{0}^{\infty}F(VZ,0)\,d\Lambda(Z;\tau)$$

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[27] See Samuelson, [7], p. 22, the "$\alpha = \beta$" case.

where F(V,0) is the terminal claim of the security on the firm. Note: (67) depends only on the rate of interest, r, which is an observable and σ^2 which can be estimated from past data reasonably accurately; and not on α, which would be difficult to estimate. The actual value of F can be computed by using standard error function tables. Hence, (67) is subject to rigorous empirical investigation.

Although (67) is a kind of discounted expected value formula, one should not infer that the expected return on F is r. From (59), (63) and (64), the expected return on F can be written as

$$\alpha_e = r + \frac{F_V V}{F}(\alpha - r), \tag{68}$$

which will vary with changes in V and τ, although it too can be computed from the error function tables, given an estimate of α.

Equation (65) was derived previously by F. Black and M. Scholes [2] as a method for pricing option contracts. Their derivation shows that (65) holds without the assumption of market equilibrium used here. Because of its elegance, I present their fundamental approach in a fashion which makes use of Ito's Lemma and the associated theory of stochastic differential equations. Consider a two-asset portfolio constructed so as to contain the firm as one security and any one of the securities in the firm's capital structure as the other. Let P be the price per unit of this portfolio and δ = percentage of the total portfolio's value invested in the firm and $(1 - \delta)$ = percentage in the particular security chosen from the firm's capital structure. Then, from (57) and (58),

$$\frac{dP}{P} = \delta \frac{dV}{V} + (1 - \delta)\frac{dF}{F} \tag{69}$$

$$= [\delta(\alpha - \alpha_e) + \alpha_e]dt + [\delta(\sigma - \sigma_e) + \sigma_e]dZ.$$

Suppose δ is chosen such that $[\delta(\sigma - \sigma_e) + \sigma_e] = 0$. Then, the portfolio will be "perfectly hedged" and the instantaneous return on the portfolio will be $[\delta(\alpha - \alpha_e) + \alpha_e]$ with certainty. By arbitrage[28] conditions, $[\delta(\alpha - \alpha_e) + \alpha_e] = r$, the instantaneous risk-less rate of return. Combining these two conditions, we have that

$$\alpha_e - r = \frac{\sigma_e}{\sigma}(\alpha - r). \tag{70}$$

Then, as done previously, we use Ito's Lemma to derive (62) and (63). By combining (62), (63), and (70), we arrive at (65). Nowhere was the market equilibrium assumption needed.

Two further remarks before leaving this section to examine asset pricing in more complex models. Although the value of the firm follows a simple dynamic process with constant parameters as described in (57), the individual component securities follow more complex process with changing expected returns and variances. Thus, in empirical examinations using a regression such as (55), if one were to use equity instead of firm values, systematic biases will be introduced. One can find cases where the debt of one firm is more camparable to the equity of another firm than the comparison of the two firms' equities.

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[28] The meaning of "arbitrage" here is not as strong as the usual definition since differences of opinion among investors about the value of σ^2 or the belief that F is a function of other variables beside the value of the firm, time, and interest rates, would lead to different values for F without infinite profits. However, given homogeneous expectations and agreement that F is only a function of the stated variables, then (66) is the valuation function which all investors would agree upon and not be proved wrong at some time in the future.

One possibly practical application of the equations of this section is to provide a systematic method of measuring the riskiness of debt of various firms. Hence, by using equation (67), one could derive a risk-structure of interest rates as a function of the percentage of the total capital structure subordinated to the issue and the overall riskiness of the firm. It would be interesting to see how such a method of rating debt would compare with the classical methods of Moody's and Standard and Poor's.

6. Model II." The "No Risk-less Asset" Case. In the previous two sections, one of the assets available to investors was risk-less. In this section, it is assumed that no such asset exists. The rationale for this assumption is inflation. Because consumers are interested in investing only as a means to a higher(real) consumption level, a security which is "risk-less" in money terms is not risk-less in real terms.[29]

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[29] It is possible for a "risk-less" money asset to be risk-less in real terms. If the rate of inflation is sufficiently smooth (stochastically), then by re-contracting loans sufficiently often and adjusting the interest rate, one can eliminate any risk of the loss of real purchasing power. If the changes in the price level are very fast (i.e. at Brownian motion " $\sqrt{dt}$ " speed similar to postulated asset price behavior), then the re-contracting approach is not a solution.

Thus, if there are no futures markets in consumption goods or other guaranteed "purchasing power" securities available, there will be no perfect hedge against future (consumption) price changes.

In an earlier paper,[30] I derived the analogous equations to (36), (37), and (38) and further, showed that the separation or mutual fund theorem obtains in this case as well.[31]

Following the same procedure as in section four, we can derive analogous equilibrium conditions to (49) and (50): Namely,

$$(71) \qquad \alpha_i = \frac{M}{A}\sigma_{iM} + G, \quad i = 1, \ldots, n$$

and

$$(72) \qquad \alpha_M = \frac{M}{A}\sigma_M^2 + G$$

where the interest rate, r, is no longer a variable, and hence H, σ_{ir}, σ_{Mr}, etc. are not relevant. However, a new term (dependent on preferences), G, comes in reflecting the constraint that the sum of the proportions of the risky assets in each investor's portfolio must be one.

The n^{th} security[32] must satisfy (71) in equilibrium, i.e.

$$(73) \qquad \alpha_n = \frac{M}{A}\sigma_{Mn} + G .$$

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[30] [5], p. 12-13. Although derived for a single consumer, one need only add the superscript k for each consumer to make the equations identical.

[31] See [5] and [6] for further discussion of this theorem when none of the assets is risk-less.

[32] Alternatively, one could use any other security or portfolio of securities whose rate of return is not perfectly correlated with the market portfolio.

(72) and (73) can be solved for M/A and G and the results substituted into (71) to determine the equilibrium relationships among all securities in terms of the parameters of the market portfolio and the nth security:

$$(74)\quad \alpha_i = \frac{[\sigma_{Mi} - \sigma_{Mn}]}{\sigma_M^2 - \sigma_{Mn}} \alpha_M + \frac{[\sigma_M^2 - \sigma_{Mi}]}{\sigma_M^2 - \sigma_{Mn}} \alpha_n, \quad i = 1, \ldots, m.$$

(74) reduces to (51) when $\sigma_n = 0$ and $\alpha_n = r$, and so, Model II. contains Model I. as a particular case.

In a similar fashion to the previous section, the fundamental partial differential equation for security pricing for Model II. can be shown to be

$$(75)\qquad \tfrac{1}{2}\sigma^2 V^2 F_{VV} + \mu V F_V - F_\tau - \mu F = 0$$

where $\mu \equiv [\sigma_M^2 \alpha_n - \sigma_{Mn}\alpha_M]/(\sigma_M^2 - \sigma_{Mn})$, provided that μ is non-stochastic. If security n is such that $\sigma_{Mn} = 0$, the "zero-beta" case (see Black, [1]), then $\mu = \alpha_n$ and (74) can be re-written as

$$(74')\qquad \alpha_i = \beta_i \alpha_M + (1 - \beta_i)\alpha_n, \quad i = 1, \ldots, m$$

where $\beta_i \equiv \sigma_{Mi}/\sigma_M^2$. If μ varies stochastically over time, then the fundamental equation of security pricing will be more complicated than (75), and this case will be discussed in the following section.

7. Model III.: The General Model. We now return to the general model of section four where the interest rate varies stochastically over time. The equilibrium conditions are (49) and (50), and we note that the Separation theorem does not obtain. However, by an approach similar to the previous section, preferences can be removed from (49). The m^{th} security must satisfy (49) in equilibrium, i.e.

$$(76) \qquad \alpha_m - r = \frac{M}{A}\sigma_{mM} - \frac{H}{A}\sigma_{mr}.$$

Hence, (50) and (76) can be solved for M/A and H/A and the results substituted into (49) to determine the equilibrium relationships among all securities in terms of the parameters of the market portfolio, the m^{th} security,[33] and the interest rate:

$$(77) \quad \alpha_k - r = \frac{[\sigma_{mr}\sigma_{Mk} - \sigma_{Mm}\sigma_{kr}]}{Q}(\alpha_M - r) + \frac{[\sigma_M^2\sigma_{kr} - \sigma_{Mr}\sigma_{Mk}]}{Q}(\alpha_m - r)$$

where $Q \equiv \sigma_M^2\sigma_{mr} - \sigma_{Mr}\sigma_{Mm}$ and $k = 1, \ldots, m - 1$. The same method of proof used to prove the Separation theorem in [5] and [6] can be applied to prove the following more-general Separation theorem:

Theorem I. (Three "fund" theorem).[34] Given n assets satisfying

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[33] It is assumed that the m^{th} security is not perfectly correlated with the market. However, it is either correlated with the market and/or changes in the interest rate, i.e. σ_{Mm} and σ_{mr} are not both zero.

[34] The theorem can be generalized to the k-fund case when other variables such as inflation, wage income, etc. are stochastic, and investors want to hedge against unfavorable outcomes by purchasing securities correlated with these variables. This would certainly be the case with many consumption goods whose relative prices are changing over time.

> the conditions of the model in section four, then there exist three portfolios ("mutual funds") constructed from these n assets, such that all risk-averse individuals, who behave according to (28), will be indifferent between choosing portfolios from among the original n assets or from these three funds. Further, a possible choice for the three funds is: the market portfolio, the risk-less asset, and a portfolio which is (instantaneously) perfectly correlated with changes in the interest rate.

Equation (77) can be derived directly from Theorem I. in the same way (51) can be derived from the usual Separation theorem. Notice that if the k^{th} security has no "market risk" in the usual sense (i.e. $\sigma_{Mk} = 0$), its expected return will not be equal to the risk-less rate, r. Further, even if the market is not correlated with changes in the interest rate (i.e. $\sigma_{Mr} = 0$), this statement still holds. Hence, we have a result which differs fundamentally with the results of the static capital asset pricing model. This strictly intertemporal effect is caused by investors' attempts to hedge against possible unfavorable future investment opportunities (i.e. yields) caused by the change in the rate of interest.

Suppose there exists a security (or portfolio) whose return is perfectly correlated (instantaneously) with changes in the interest rate. Then, if this security is taken as the m^{th} security, its dynamics are described by

$$\frac{dP_m}{P_m} = \alpha_m dt + \sigma_m dq, \tag{78}$$

and from (30), $\sigma_{Mm}' \equiv \sigma_m \sigma_{Mr}/g$ and $\sigma_{mr} \equiv \sigma_m g$. Both from a theoretical and empirical standpoint, it makes sense to choose, as the third ("interest-rate hedging") mutual fund, a portfolio which is strongly correlated with interest rate changes. Throughout the rest of the section, it is assumed that the m^{th} asset satisfies (78).

In the previous models, the term-structure of interest rates was either trivial ("flat" as in Model I.) or non-existent (as in Model II.). Model III. is rich in this respect because: (1) it provides an explanation for the existence of "long," default-free bonds as an efficient means of hedging against interest rate changes[35] (existence of a term structure), (2) it gives insight into how to price these bonds (determination of the shape of the term-structure), (3) it is sufficiently flexible to be consistent with many existing theories. Nowhere in the model is it necessary to introduce concepts such as liquidity, transactions costs, time horizon, or habitat to explain the existence of a term structure.

Consider as a possible set of securities, bonds guaranteed against default, which pay one dollar at various maturity dates.[36] It is assumed that the price of these bonds is a function of the (short) interest rate and the length of time until maturity.[37] Therefore, let $P(r,\tau)$ be the

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[35] Although any security whose return is correlated with interest rate changes would be sufficient in theory, securities which are perfectly correlated with interest rate changes are more effective as was mentioned in the text. Because of the risk of default, to use ordinary corporate bonds instead of guaranteed ("Government") bonds would require significant diversification to eliminate that risk.

[36] These bonds are discounted loans. Because the payments are risk-less, once these bonds are priced, it is straightforward to derive the price of coupon bonds by weighting each maturity by the coupon payment and adding.

[37] If each investor's expectations include other variables such as the

(continued on next page)

price of a discounted loan which pays a dollar at time τ in the future when the current interest rate is r. Then, the dynamics of P can be written as

$$\text{(79)} \qquad \frac{dP}{P} = \alpha_{\tau} dt + \sigma_{\tau} dq$$

where α_{τ} is the instantaneous expected rate of return on a τ year bond and σ_{τ} is the instantaneous standard deviation. From the equilibrium condition (77) and the assumptions leading to (78), α_{τ} must satisfy

$$\text{(80)} \qquad \alpha_{\tau} - r = \frac{\sigma_{\tau}}{\sigma_m} (\alpha_m - r).$$

By Ito's Lemma, α_{τ} and σ_{τ} must satisfy

$$\text{(81)} \qquad 0 = \frac{1}{2} g^2 P_{rr} + f P_r - P_{\tau} - \alpha_{\tau} P$$

and

$$\text{(82)} \qquad \sigma_{\tau} = \frac{P_r g}{P}$$

(footnote 37 continued from previous page)
relative supplies of each maturity, etc., then the valuation formula presented in the text is incorrect. However, the assumption that investors believe that the only (anticipated) risk in holding government bonds is due to changing interest rates, seems reasonable. Further, if each investor does have such a belief, then the "correct" price will be agreed upon by all and will not be refutable at any time in the future.

where subscripts on the P denote partial derivatives with respect to r and τ. Equation (81) is similar to the fundamental equation of security pricing previously discussed. Given α_τ, (81) could be solved, subject to the boundary condition $P(r,0) = 1$, to determine $P(r,\tau)$ and hence, the term structure of interest rates. However, without some independent knowledge of α_m (and hence, α_τ), we cannot determine an explicit solution for the term structure.

Suppose that one knew that the Expectations Hypothesis held. Then $\alpha_\tau \underset{\tau}{\equiv} r$ and the term structure is completely determined by

$$(83) \qquad 0 = \tfrac{1}{2}g^2 P_{rr} + fP_r - P_\tau - rP$$

subject to $P(r,0) = 1$. Further, from (80), it must be that in equilibrium, $\alpha_m = r$. In this case, the equilibrium condition (77) simplifies to

$$(84) \qquad \alpha_k - r = \frac{\sigma_k(\rho_{kM} - \rho_{Mr}\rho_{kr})}{\sigma_M(1 - \rho^2_{Mr})}(\alpha_M - r),\ k = 1, \ldots, m - 1$$

where the ρ's are the instantaneous correlation coefficients defined by $\rho_{kM} \equiv \sigma_{kM}/\sigma_k\sigma_M$, $\rho_{kr} \equiv \sigma_{kr}/g\sigma_k$, and $\rho_{Mr} \equiv \sigma_{Mr}/g\sigma_M$. Hence, the individual expected returns are proportional to the market expected return as was the case in Model I. However, the proportionality factor is not σ_{Mk}/σ^2_M. If the m^{th} security is chosen to be a portfolio of government bonds, then, given specific knowledge of the term structure, the rest of the equilibrium relationships work out in a determined fashion.

(83) cannot be solved in closed form for arbitrary f and g. However, if it is assumed that f and g are constants (i.e. r follows a gaussian random walk), then, under the Expectations Hypothesis, we do have the explicit solution,

$$(85) \qquad P(r,\tau) = \exp[-r\tau - \frac{f}{2}\tau^2 + \frac{g^2}{6}\tau^3].$$

Note that in (85) as $\tau \to \infty$, $P \to \infty$ which is not at all reasonable. Certainly, the current value of a discounted loan which will never be paid should be zero for any realistic assumption about interest rates. The reason that (85) gives such nonsensical results is that by the assumption that r is gaussian, there is a positive probability of r becoming negative. In fact, as $\tau \to \infty$, r will be negative for an arbitrary amount of time with positive probability. This result illustrates how the assumption of the normal distribution for variables which are constrained to be non-negative can lead to absurd implications. However, equation (83) with reasonable assumptions about f and g can be solved numerically and further research is planned in this area.

By arguments similar to those used in section five, the fundamental equation of security pricing for the capital structure of the firm in Model III. can be derived as

$$(86) \qquad 0 = \frac{1}{2}\sigma^2 V^2 F_{VV} + \frac{1}{2}g^2 F_{rr} + \rho g \sigma V F_{rV} + rVF_V + fF_r - F_\tau - rF$$

subject to the given boundary condition F(V,r,0), where subscripts denote partial derivatives and ρ is the instantaneous correlation coefficient of the return on the firm with interest rate changes. The basic difference between (86) and (65) of Model I. is the explicit dependence of F on r which must be taken into account. Under most conditions, (86) will not be solvable in closed form. However, numerical solution seems quite reasonable which implies many possibilities for empirical testing both by direct statistical methods and by simulation.

8. Conclusion. A general equilibrium intertemporal model of the asset market has been derived for arbitrary preferences, time horizon, and wealth distribution. The equilibrium relationships among securities were shown to depend only on certain "observable" market aggregates and hence, are subject to empirical investigation. Under the additional assumption of a constant rate of interest, these equilibrium relationships are essentially the same as those of the static capital asset pricing model of Sharpe-Lintner-Mossin. However, these results were derived without the assumption of gaussian distributions for security prices or quadratic utility functions. When interest rates vary, some of the intuition about "market risk" provided by the capital asset pricing model was shown to be incorrect. In addition, the model clearly differentiates between the trading period horizon (dt, an infinitesimal) and the planning or time horizon (T^k, which is arbitrary).

Under the assumption that the value of the firm is independent of the composition of its capital structure, we have shown how to price any security in the capital structure by means of the fundamental partial differential equation of security pricing. This relationship depends only on observables and therefore, is subject to empirical study. As a particular use of the model, one can derive a "risk-structure" of interest rates for ranking debt and explaining differential yields on bonds. Further applications would include the examination of the effects of interest rate changes, dilution, etc. on the prices of different types of securities.

The existence of a term structure of interest rates is a direct result of the model. Further, from the fundamental partial differential equation of security pricing, a method for determining the term-structure was presented.

The model does not allow for non-homogeneous expectations, non-serially independent preferences, or transactions costs (all are areas for further research). Although not done here, the analysis of demands for consumption goods when future prices are uncertain could be made along the lines suggested in footnote twenty. Similarly, given a theory of the firm, the supply dynamics of new shares can be brought explicitly into the model rather than treating such changes as exogeneous. Further research along the lines of the model presented here is aimed at including these additions as well as other sectors of the economy. It is believed that this research will lead to a better understanding of the mechanism by which government actions in the securities markets affect security prices and firms' investment decision.

The fundamental assumption which allows the model to be so general and yet yield strong results, is the continuous-time assumption. If the model were formulated in discrete time with time-spacing of length h between trading periods, then the results derived in the paper no longer hold.[38] Since the option to trade continuously includes the option to trade at discrete intervals, all investors would prefer this option (at no cost). Hence, the assumption seems legitimate under the usual perfect market assumptions of no transactions costs, no indivisibilities, cost-less information, etc. . .

The usual reason given for the discrete-time formulation is that such transactions costs exist. However, the approach is to take equal time spacings of non-specified length. If one wants to include transactions costs, it seems logical to incorporate them in the continuous time model and derive the "h" (which almost certainly will not be equally spaced in calendar time, but will depend on the size of price changes of securities in the portfolio among other things). In many empirical studies, the trading period spacing, h, is implicitly assumed to coincide with the observation spacing (e.g. one year) which seems quite unreasonable. Further, even if each investor had the same trading period spacing, different investors would most likely begin on different days, and hence, the resulting "smear" of the aggregate may be more closely approximated by the continuous-trading assumption. The continuous-time assumption buys a lot of results. So, until the existence of a fundamental minimum-quantum of time in economics is proved it will be a helpful assumption to make.

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[38] However, the continuous-time solution is not singular in the sense that the limit, as $h \to 0$, of the discrete-time solution is the continuous-time solution.

References

[1] F. Black, "Capital Market Equilibrium with No Riskless Borrowing or Lending," Financial Note 15A, unpublished, August, 1970.

[2] ________ and M. Scholes, "A Theoretical Valuation Formula for Options, Warrants, and Other Securities," unpublished and undated.

[3] M. Jensen, "Risk, the Pricing of Capital Assets and the Evaluation of Investment Portfolios," Journal of Business, Vol. 42, April, 1969.

[4] R. Merton, "Lifetime Portfolio Selection Under Uncertainty: The Continuous-time Case," Review of Economics and Statistics, LI, August, 1969.

[5] ________, "Optimum Consumption and Portfolio Rules in a Continuous-time Model," Working paper #58, Department of Economics, Massachusetts Institute of Technology, August, 1970.

[6] ________, "An Analytic Derivation of the Efficient Portfolio Frontier," Working paper #493-70, Alfred P. Sloan School of Management, Massachusetts Institute of Technology, October, 1970.

[7] P. Samuelson, "Rational Theory of Warrant Pricing," Industrial Management Review, Spring, 1965.

[8] ____________ and R. Merton, "A Complete Model of Warrant Pricing that Maximizes Utility," Industrial Management Review, Winter, 1969.

[9] W. Sharpe, Portfolio Theory and Capital Markets, McGraw-Hill, 1970.