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APPENDIX: CONTINUOUS-TIME SPECULATIVE PROCESSES

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Let the dynamics of stock price x be described by the stochastic differential equation of the Itô-type^{A1}

$$(A.1) \quad dx = \alpha x dt + \sigma x dz,$$

where α is the instantaneous expected rate of return, σ is the instantaneous standard deviation of that return, and dz is a standard Gauss-Wiener process with mean zero and standard deviation one. It is assumed that α and σ are constants, and hence, the return on the stock over any finite time interval is log-normal.

Suppose we are in the world of the Samuelson 1965 theory [90] where investors require an instantaneous expected return β to hold the warrant and β is constant with $\beta \geq \alpha$. Let $W = F(x, \tau; \sigma^2, a, \alpha, \beta)$ be the price of a warrant with exercise price a and length of time until expiration τ . Using Itô's lemma,^{A2} the dynamics of the warrant price can be described by the stochastic differential equation

$$(A.2) \quad dW = F_1 dx + F_2 d\tau + \frac{1}{2} F_{11} (dx)^2,$$

where subscripts denote partial derivatives. Substituting for dx from (A.1) and

^{A1} For a complete discussion of Itô processes, see the seminal paper of Itô [40], Itô and McKean [41] and McKean [69].

^{A2} See McKean [69, pp. 32-35 and 44] for proofs of the lemma in 1 and n dimensions. For applications of Itô processes and Itô's lemma to a variety of portfolio and option pricing problems, see Merton [70], [71] and [73].

noting that $d\tau = -dt$ and $(dx)^2 = \sigma^2 x^2 dt$, we can rewrite (A.2) as

$$(A.3) \quad dW = [\frac{1}{2}\sigma^2 x^2 F_{11} + \alpha x F_1 - F_2] dt + \sigma x F_1 dz,$$

where $[\frac{1}{2}\sigma^2 x^2 F_{11} + \alpha x F_1 - F_2]/F$ is the instantaneous expected rate of return on the warrant and $\sigma x F_1/F$ is the instantaneous standard deviation. Applying the condition that the required expected return on the warrant is β to (A.3), we derive a linear partial differential equation of the parabolic type for the warrant price, namely.

$$(A.4) \quad 0 = \frac{1}{2}\sigma^2 x^2 F_{11} + \alpha x F_1 - \beta F - F_2$$

subject to the boundary conditions for a "European" warrant:

$$(a) F(0, \tau; \sigma^2, a, \alpha, \beta) = 0,$$

$$(b) F(x, 0; \sigma^2, a, \alpha, \beta) = \max [0, x - a].$$

Make the change of variables $T \equiv \sigma^2 \tau$, $S \equiv x e^{\alpha \tau}/a$, $f \equiv F e^{\beta \tau}/a$, and substitute into (A.4) to obtain the new equation for f ,

$$(A.5) \quad 0 = \frac{1}{2}S^2 f_{11} - f_2$$

subject to

$$(a) f(0, T) = 0,$$

$$(b) f(S, 0) = \max [0, S - 1].$$

By inspection, f is the value of a "European" warrant with unit exercise price and time to expiration T , on a common stock with zero expected return and unit instantaneous variance, when investors require a zero return on the warrant, i.e.,

$$(A.6) \quad f(S, T) = F(S, T; 1, 1, 0, 0)$$

which verifies the homogeneity properties described in (6.11). To solve (A.5), we put it in standard form by the change in variables $y \equiv \log S + \frac{1}{2}T$ and $\phi(y, T) \equiv f(S, T)/S$ to arrive at

$$(A.7) \quad 0 = \frac{1}{2}\phi_{11} - \phi_2$$

subject to

$$(a) |\phi| \leq 1,$$

$$(b) \phi(y, 0) = \max [0, 1 - e^{-y}].$$

Equation (A.7) is a standard free-boundary problem to be solved by separation of variables or Fourier transforms.^{A3} Hence, the solution to (A.4) is

$$\begin{aligned}
 F &= \frac{e^{-\beta\tau}}{\sqrt{2\pi\sigma^2\tau}} \int_{\log(a/x)}^{\infty} (xe^Z - a) \exp \left[-\frac{1}{2} \frac{(Z - (\alpha - \frac{1}{2}\sigma^2)\tau)^2}{\sigma^2\tau} \right] dZ \\
 (A.8) \quad &= e^{-(\beta-\alpha)\tau} x N \left[\frac{\log(x/a) + (\alpha + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \right] \\
 &\quad - a e^{-\beta\tau} N \left[\frac{\log(x/a) + (\alpha - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \right]
 \end{aligned}$$

which reduces to (6.11)–(6.12) when $\beta = \alpha$.

The analysis leading to solution (A.8) assumed that the warrant was of the “European” type. If the warrant is of the “American” type, we must append to (A.4) the arbitrage boundary condition that

$$(A.4.c) \quad F(x, \tau; \sigma^2, a, \alpha, \beta) \geq F(x, 0; \sigma^2, a, \alpha, \beta).$$

It has been shown^{A4} that for $\beta = \alpha$, (A.4.c) is never binding, and the European and American warrants have the same value with (A.8) or (6.11)–(6.12) the correct formula. It has also been shown that for $\beta > \alpha$, for every τ , there exists a level of stock price, $C[\tau]$, such that for all $x > C[\tau]$, the warrant would be worth more if exercised than if one continued to hold it (i.e., the equality form of (A.4.c) will hold at $x = C[\tau]$). In this case, the equation for the warrant price is (A.4) with the boundary condition

$$(A.4.c') \quad F(C[\tau], \tau; \sigma^2, a, \alpha, \beta) = C[\tau] - a \quad \text{appended} \quad \text{and} \quad 0 \leq x \leq C[\tau].$$

If $C[\tau]$ were a known function, then, after the appropriate change of variables, (A.4) with (A.4.c') appended, would be a semi-infinite boundary value problem with a time-dependent boundary. However, $C[\tau]$ is not known, and must be determined as part of the solution. Therefore, an additional boundary condition is required for the problem to be well-posed.

Fortunately, the economics of the problem are sufficiently rich to provide this extra condition. Because the warrant holder is not contractually obliged to exercise his warrant prematurely, he chooses to do so only in his own best interest (i.e., when the warrant is worth more “dead” than “alive”). Hence, the only rational choice for $C[\tau]$ is that time-pattern which maximizes the value of the warrant. Further, the structure of the problem makes it clear that the optimal $C[\cdot]$ will be independent of the current level of the stock price.

^{A3} For the separation of variables solution, see Churchill [13, pp. 154–156], and for the Fourier transform solution, see Dettman [22, p. 390].

^{A4} Samuelson [90] gives a heuristic economic argument. Samuelson and Merton [95] prove it under more general conditions than those in the text. An alternative proof, based on mere arbitrage, is given in Merton [73].

In attacking the difficult $\beta > \alpha$ case, Samuelson [90] postulated that the extra condition was "high-contact" at the boundary, i.e.,

$$(A.9) \quad F_1(C[\tau], \tau; \sigma^2, a, \alpha, \beta) = 1.$$

It can be shown that (A.9) is implied by the maximizing behavior described in the previous paragraph. In an appendix to the Samuelson paper, McKean [68, p. 38-39] solved (A.4) with conditions (A.4.c') and (A.9) appended, to the point of obtaining an infinite set of integral equations, but was unable to find a closed-form solution. The problem remains unsolved.

In their important paper, Black and Scholes [8] use a hedging argument to derive their warrant pricing formula. Unlike Samuelson [90], they do not postulate a required expected return on the warrant, β , but implicitly derive as part of the solution the warrant's expected return. However, the mathematical analysis and resulting needed tables are identical to Samuelson [90].

Assume that the stock price dynamics are described by (A.1).^{A5} Further, assume that there are no transactions costs; short-sales are allowed; borrowing and lending are possible at the same riskless interest rate, r , which is constant through time.

Consider constructing a portfolio containing the common stock, the warrant and the riskless security with w_1 = number of dollars invested in the stock, w_2 = number of dollars invested in the warrant, and w_3 = number of dollars invested in the riskless asset. Suppose, by short sales, or borrowing, we constrain the portfolio to require net zero investment, i.e., $\sum_1^3 w_i = 0$. If trading takes place continuously, it can be shown^{A6} that the instantaneous change in the portfolio value can be written as

$$(A.10) \quad w_1 \left(\frac{dx}{x} - r dt \right) + w_2 \left(\frac{dW}{W} - r dt \right),$$

where the constraint has been eliminated from (A.10) by substituting $w_3 = -(w_1 + w_2)$, and so, any choice of w_1 and w_2 is allowed. We can substitute for dx/x and dW/W from (A.1) and (A.3), and rearrange terms, to rewrite (A.10) as

$$(A.11) \quad [w_1(\alpha - r) + w_2(\frac{1}{2}\sigma^2 x^2 F_{11} + \alpha x F_1 - F_2 - rF)/F] dt \\ + [w_1 \sigma + w_2 \sigma x F_1/F] dz.$$

Note that w_1 and w_2 can be chosen so as to eliminate all randomness from the return; i.e., we can choose $w_1 = w_1^*$ and $w_2 = w_2^*$, where

$$(A.12) \quad w_1^*/w_2^* = -x F_1/F.$$

^{A5} The assumptions and method of derivation presented here are not those of Black and Scholes [8]. However, the method is in the spirit of their analysis and it leads to the same formula. For a complete discussion of the Black and Scholes model and extensions to more general option pricing problems, see Merton [73].

^{A6} See Merton [70, pp. 247-248] or Merton [73, § 3].

Then, for this particular portfolio, the expected return will be the realized return and since no net investment was required, to avoid positive "arbitrage" profits this return must be zero. Substituting for w_1^* and w_2^* in (A.11), combining terms and setting the return equal to zero, we have that

$$(A.13) \quad 0 = \frac{1}{2}\sigma^2 x^2 F_{11} + rxF_1 - F_2 - rF.$$

Equation (A.13) is the partial differential equation to be satisfied by the equilibrium warrant price. Formally, it is identical to (A.4) with " $\beta = \alpha = r$," and is subject to the same boundary conditions. It is important to note that this formal equivalence does not imply that the expected returns on the warrant and on the stock are equal to the interest rate. Even if the expected return on the stock is constant through time, the expected return on the warrant will not be,^{A7} i.e.,

$$(A.14) \quad \beta(x, \tau) = r + \frac{xF_1}{F}(\alpha - r).$$

Further, the Black-Scholes formula for the warrant price is completely independent of the expected return on the stock price. Hence, two investors with different assessments of the expected return on the common stock will still agree on the "correct" warrant price for a given stock price level. Similarly, we could have postulated a more general stochastic process for the stock price with α itself random, and the analysis still goes through.

The key to the Black-Scholes analysis is the continuous-trading assumption since only in the instantaneous limit are the warrant price and stock price perfectly correlated, which is what is required to form the "perfect" hedge in (A.11).

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^{A7} In this respect, the Black-Scholes result is closer to the Samuelson and Merton [95] case, where $\beta = \beta(x, \tau) \geq \alpha$ (and where no premature conversion takes place), than to the case of Samuelson [90].

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