

# Notes on Feng & Palomar 2016: *Portfolio Optimization with Asset Selection and Risk Control*

Zé Vinícius

August 26, 2018

## Notation

Vectors are represented by bold, small-case letters, *e.g.*,  $\mathbf{x}$ . All vectors are column vectors. Given a vector  $\mathbf{x}$ , both  $(\mathbf{x})_i$  and  $x_i$  represent the  $i$ -th component of  $\mathbf{x}$ .

## 1 Background

Risky parity is a portfolio design technique which aims to promote diversification of risk contributions amongst assets. This approach often leads to a dense portfolio, *i.e.*, a portfolio which has contributions from all its assets. However, investing in all assets is impractical because of, *e.g.*, high transaction costs. The problem of jointly designing a sparse risk-parity portfolio is precisely the subject of study of Feng & Palomar.

Consider a collection of  $n$  assets with random returns  $\mathbf{r} \in \mathbb{R}^n$  such that  $\mathbb{E}[\mathbf{r}] \triangleq \boldsymbol{\mu}$  and  $\mathbb{E}[(\mathbf{r} - \boldsymbol{\mu})(\mathbf{r} - \boldsymbol{\mu})^T]$  are its mean vector and its (positive definite) covariance matrix. Also, let  $\mathbf{w} \in \mathbb{R}^n$  denote the normalized portfolio (*e.g.*  $\mathbf{w}^T \mathbf{1} = 1$ ), which represents the distribution of capital budget allocated over the assets.

Then, for every normalized portfolio  $\mathbf{w}$ , define the portfolio volatility as  $\sigma(\mathbf{w}) \triangleq \sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}$ . Intuitively, the portfolio volatility is a measure of the risk contributions, *i.e.*, the loss contributions from each asset. Besides, the proper definition of a measure of risk contribution is a paramount step before actually advancing on the study of risk parity portfolio.

Note that the portfolio volatility is a positively homogenous function, which implies that it can be expressed as

$$\sigma(\mathbf{w}) = \sum_{i=1}^n w_i \frac{\partial \sigma(\mathbf{w})}{\partial w_i}, \quad (1)$$

in fact, the RHS of (??) is

$$\sum_{i=1}^n w_i \frac{\partial \sigma(\mathbf{w})}{\partial w_i} = \sum_{i=1}^n w_i \frac{(\boldsymbol{\Sigma} \mathbf{w})_i}{\sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}}. \quad (2)$$

From (??),  $w_i \frac{\partial \sigma(\mathbf{w})}{\partial w_i}$  can be thought as the risk contribution of the  $i$ -th asset. Additionally, the risk contributions of every asset in a risk-parity portfolio are the same, therefore

$$w_i \frac{(\boldsymbol{\Sigma} \mathbf{w})_i}{\sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}} = w_j \frac{(\boldsymbol{\Sigma} \mathbf{w})_j}{\sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}} \quad \forall i, j. \quad (3)$$

## 2 Problem formulae

The problem of designing risk-parity portfolios with asset selection, as formulated by Feng & Palomar, is given as

$$\begin{aligned} & \underset{\mathbf{w}, \theta}{\text{minimize}} && F(\mathbf{w}) + \lambda_1 \|\mathbf{w}\|_0 + \lambda_2 R(\mathbf{w}, \theta) \\ & \text{subject to} && \mathbf{w}^T \mathbf{1} = 1, \mathbf{w} \in \mathcal{W}, \end{aligned} \quad (4)$$

where

- $F(\mathbf{w}) \triangleq \mathbf{w}^T (-\nu \boldsymbol{\mu} + \boldsymbol{\Sigma} \mathbf{w})$
- $R(\mathbf{w}, \theta) \triangleq \sum_{i=1}^n (g_i(\mathbf{w}) - \theta)^2 \mathbb{I}_{\{w_i \neq 0\}}$
- $g_i(\mathbf{w}) \triangleq w_i (\boldsymbol{\Sigma} \mathbf{w})_i$

### 2.1 $\theta$ - update

For fixed  $\mathbf{w}$ , say  $\mathbf{w}^k$ , the objective function reduces to

$$\underset{\theta}{\text{minimize}} \quad \sum_{i=1}^n [(g_i(\mathbf{w}^k) - \theta) \rho_p^\epsilon(w_i^k)]^2, \quad (5)$$

which is the classical univariate weighted least squares problem whose solution is given as

$$\hat{\theta} = \sum_{i=1}^n x_i^k g_i(\mathbf{w}^k), \quad (6)$$

where  $x_i^k = \frac{(\rho_p^\epsilon(w_i^k))^2}{\sum_{j=1}^n (\rho_p^\epsilon(w_j^k))^2}.$

## 2.2 w - update

For a fixed  $\theta$ :

$$\begin{aligned} & \underset{\mathbf{w}}{\text{minimize}} && F(\mathbf{w}) + \lambda_1 \|D_o^k \mathbf{w}\|_o + \lambda_2 P(\mathbf{w}, \theta^k) + \tau \|\mathbf{w} - \mathbf{w}^k\|_2^2 \\ & \text{subject to} && \mathbf{w}^T \mathbf{1} = 1, \mathbf{w} \in \mathcal{W}, \end{aligned} \quad (7)$$

where

$$P(\mathbf{w}, \theta) \triangleq \sum_{i=1}^n \left\{ \tilde{g}_i(\mathbf{w}^k, \theta) + (\nabla \tilde{g}_i(\mathbf{w}^k, \theta))^T (\mathbf{w} - \mathbf{w}^k) \right\}^2 \quad (8)$$

- $\tilde{g}_i(\mathbf{w}^k, \theta) \triangleq (g_i(\mathbf{w}^k) - \theta) \rho_p^\epsilon(w_i^k)$
- $\nabla_{\mathbf{w}} \tilde{g}_i(\mathbf{w}^k, \theta) = \rho_p^\epsilon(w_i^k) \cdot \nabla_{\mathbf{w}} g_i(\mathbf{w}^k) + [(g_i(\mathbf{w}^k) - \theta) \cdot \nabla_{\mathbf{w}} \rho_p^\epsilon(w_i^k)] \cdot \mathbf{e}_i$