Risk-Parity Portfolio

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Outline

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Motivation

- The Markowitz portfolio has never been embraced by practitioners, among other reasons because

  1. variance is not a good measure of risk in practice since it penalizes both the unwanted high losses and the desired low losses: the solution is to use **alternative measures for risk, e.g., VaR and CVaR**,
  2. it is highly sensitive to parameter estimation errors (i.e., to the covariance matrix $\boldsymbol{\Sigma}$ and especially to the mean vector $\boldsymbol{\mu}$): solution is **robust optimization**,
  3. it only considers the risk of the portfolio as a whole and ignores the risk diversification (i.e., concentrates risk too much in few assets, this was observed in the 2008 financial crisis): solution is the **risk-parity portfolio**.

👉 ***We will here address the risk diversification among the assets by totally changing the portfolio formulation (and somehow address the sensitivity w.r.t. parameters).***

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Returns

- Let us denote the log-returns of N assets at time t with the vector $\mathbf{r}_t \in \mathbb{R}^N$.
- The time index t can denote any arbitrary period such as days, weeks, months, 5-min intervals, etc.
- $\mathcal{F}_{t-1}$ denotes the previous historical data.
- Econometrics aims at modeling $\mathbf{r}_t$ conditional on $\mathcal{F}_{t-1}$.
- $\mathbf{r}_t$ is a multivariate stochastic process with conditional mean and covariance matrix denoted as[1]

$$\begin{aligned}
\boldsymbol{\mu}_t &\triangleq \mathsf{E}\left[\mathbf{r}_t \mid \mathcal{F}_{t-1}\right] \\
\boldsymbol{\Sigma}_t &\triangleq \mathrm{Cov}\left[\mathbf{r}_t \mid \mathcal{F}_{t-1}\right] = \mathsf{E}\left[(\mathbf{r}_t - \boldsymbol{\mu}_t)(\mathbf{r}_t - \boldsymbol{\mu}_t)^T \mid \mathcal{F}_{t-1}\right].
\end{aligned}$$

[1] Y. Feng and D. P. Palomar, *A Signal Processing Perspective on Financial Engineering*. Foundations and Trends in Signal Processing, Now Publishers, 2016.

I.I.D. Model

- For simplicity we will assume that $\mathbf{r}_t$ follows an i.i.d. distribution (which is not very innacurate in general).
- That is, both the conditional mean and conditional covariance are constant

$$\begin{aligned} \boldsymbol{\mu}_t &= \boldsymbol{\mu}, \\ \boldsymbol{\Sigma}_t &= \boldsymbol{\Sigma}. \end{aligned}$$

- Very simple model, however, it is one of the most fundamental assumptions for many important works, e.g., the Nobel prize-winning Markowitz portfolio theory[2].

[2] H. Markowitz, "Portfolio selection," *J. Financ.*, vol. 7, no. 1, pp. 77–91, 1952.

Parameter Estimation

- Consider the i.i.d. model:

$$\mathbf{r}_t = \boldsymbol{\mu} + \mathbf{w}_t,$$

  where $\boldsymbol{\mu} \in \mathbb{R}^N$ is the mean and $\mathbf{w}_t \in \mathbb{R}^N$ is an i.i.d. process with zero mean and constant covariance matrix $\boldsymbol{\Sigma}$.
- The mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ have to be estimated in practice based on T observations.
- The simplest estimator is the sample estimator:
  - sample mean estimator: $\hat{\boldsymbol{\mu}} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{r}_t$
  - sample covariance matrix: $\hat{\boldsymbol{\Sigma}} = \frac{1}{T-1}\sum_{t=1}^{T}(\mathbf{r}_t - \hat{\boldsymbol{\mu}})(\mathbf{r}_t - \hat{\boldsymbol{\mu}})^T$.
- Many more sophisticated estimators exist, namely: shrinkage estimators, Black-Litterman estimators, etc.

Parameter Estimation

- The parameter estimates $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$ are only good for large T, otherwise the estimation error is unacceptable.
- For instance, the sample mean is particularly a very inefficient estimator, with very noisy estimates.[3]
- In practice, T cannot be large enough due to either:
  - unavailability of data or
  - lack of stationarity of data.
- As a consequence, the estimates contain too much estimation error and a portfolio design (e.g., Markowitz mean-variance) based on those estimates can be fatal.
- Indeed, this is why Markowitz portfolio and other extensions are rarely used by practitioners.

[3] A. Meucci, *Risk and Asset Allocation*. Springer, 2005.

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Portfolio Return

- Suppose the budget is B dollars.
- The portfolio $\mathbf{w} \in \mathbb{R}^N$ denotes the normalized weights of the assets such that $\mathbf{1}^T\mathbf{w} = 1$ (then $B\mathbf{w}$ denotes dollars invested in the assets).
- For each asset, the initial wealth is Bw_i and the end wealth is

$$Bw_i\left(p_{i,t}/p_{i,t-1}\right) = Bw_i\left(R_{it} + 1\right).$$

- Then the portfolio return is

$$R_t^p = \frac{\sum_{i=1}^N Bw_i\left(R_{it} + 1\right) - B}{B} = \sum_{i=1}^N w_i R_{it} \approx \sum_{i=1}^N w_i r_{it} = \mathbf{w}^T\mathbf{r}_t$$

- The portfolio expected return and variance are $\mathbf{w}^T\boldsymbol{\mu}$ and $\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}$, respectively.[4]

[4] G. Cornuejols and R. Tütüncü, *Optimization Methods in Finance*. Cambridge University Press, 2006.

Performance Measures

- Expected return: $\mathbf{w}^T\boldsymbol{\mu}$
- Volatility: $\sqrt{\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}}$
- Sharpe Ratio (SR): expected return per unit of risk

$$\mathrm{SR} = \frac{\mathbf{w}^T\boldsymbol{\mu} - r_f}{\sqrt{\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}}}$$

  where r_f is the risk-free rate (e.g., interest rate on a three-month U.S. Treasury bill).
- Information Ratio (IR):

$$\mathrm{IR} = \frac{\mathbf{w}^T\boldsymbol{\mu}}{\sqrt{\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}}}$$

- Drawdown: decline from a historical peak of the cumulative profit $X(t)$: $D(T) = \max\left\{0, \max_{t\in(0,T)} X(t) - X(T)\right\}$
- VaR (Value at Risk)
- ES (Expected Shortfall) or CVaR (Conditional Value at Risk)

Practical Constraints

- Capital budget constraint:

$$\mathbf{w}^T\mathbf{1} = 1.$$

- Long-only constraint:

$$\mathbf{w} \geq 0.$$

- Market-neutral constraint:

$$\mathbf{w}^T\mathbf{1} = 0.$$

- Turnover constraint:

$$\left\|\mathbf{w} - \mathbf{w}_0\right\|_1 \leq u$$

  where $\mathbf{w}_0$ is the currently held portfolio.

Practical Constraints

- Holding constraint:
$$\mathbf{l} \leq \mathbf{w} \leq \mathbf{u}$$
where $\mathbf{l} \in \mathbb{R}^N$ and $\mathbf{u} \in \mathbb{R}^N$ are lower and upper bounds of the asset positions respectively.

- Cardinality constraint:
$$\|\mathbf{w}\|_0 \leq K.$$

- Leverage constraint:
$$\|\mathbf{w}\|_1 \leq 2.$$

Risk Control

- In finance, the expected return $\mathbf{w}^T\boldsymbol{\mu}$ is very relevant as it quantifies the average benefit.
- However, in practice, the average performance is not enough to characterize an investment and one needs to control the probability of going bankrupt.
- Risk measures control how risky an investment strategy is.
- The most basic measure of risk is given by the variance[5]: a higher variance means that there are large peaks in the distribution which may cause a big loss.
- There are more sophisticated risk measures such as downside risk, VaR, ES, etc.

[5] H. Markowitz, "Portfolio selection," *J. Financ.*, vol. 7, no. 1, pp. 77–91, 1952.

Mean-Variance Tradeoff

- The mean return $\mathbf{w}^T\boldsymbol{\mu}$ and the variance (risk) $\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}$ constitute two important performance measures.
- Usually, the higher the mean return the higher the variance and vice-versa.
- Thus, we are faced with two objectives to be optimized: it is a multi-objective optimization problem.
- They define a fundamental mean-variance tradeoff curve (Pareto curve).
- The choice of a specific point in this tradeoff curve depends on how agressive or risk-averse the investor is.

Markowitz mean-variance portfolio (1952)

- The idea of the Markowitz framework[6] is to find a trade-off between the expected return $\mathbf{w}^T\boldsymbol{\mu}$ and the risk of the portfolio measured by the variance $\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}$:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{maximize}} & \mathbf{w}^T\boldsymbol{\mu} - \lambda\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w} \\ \text{subject to} & \mathbf{1}^T\mathbf{w} = 1 \end{array}$$

  where $\mathbf{w}^T\mathbf{1} = 1$ is the capital budget constraint and λ is a parameter that controls how risk-averse the investor is.

- This is a convex QP with only one linear constraint which admits a closed-form solution:

$$\mathbf{w}^{\star} = \frac{1}{2\lambda}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} + \nu^{\star}\mathbf{1}\right),$$

  where $\nu^{\star}$ is the optimal dual variable $\nu^{\star} = \frac{2\lambda - \mathbf{1}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}}$.

[6] H. Markowitz, "Portfolio selection," *J. Financ.*, vol. 7, no. 1, pp. 77–91, 1952.

Global Minimum Variance Portfolio (GMVP)

- The global minimum variance portfolio (GMVP) ignores the expected return and focuses on the risk only:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w} \\ \text{subject to} & \mathbf{1}^T \mathbf{w} = 1. \end{array}$$

- It is a simple convex QP with solution

$$\mathbf{w}_{\text{GMVP}} = \frac{1}{\mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}} \boldsymbol{\Sigma}^{-1} \mathbf{1}.$$

- It is widely used in academic papers for simplicity of evaluation and comparison of different estimators of the covariance matrix $\boldsymbol{\Sigma}$ (while ignoring the estimation of $\boldsymbol{\mu}$).

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Drawbacks of Markowitz's formulation

- The Markowitz portfolio has never been embraced by practitioners, among other reasons because
  1. variance is not a good measure of risk in practice since it penalizes both the unwanted high losses and the desired low losses: the solution is to use **alternative measures for risk, e.g., VaR and CVaR**,
  2. it is highly sensitive to parameter estimation errors (i.e., to the covariance matrix $\boldsymbol{\Sigma}$ and especially to the mean vector $\boldsymbol{\mu}$): solution is **robust optimization**,
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1. Introduction
2. Warm-Up: Markowitz Portfolio
   - Signal model
   - Markowitz formulation
   - Drawbacks of Markowitz portfolio
3. **Risk-Parity Portfolio**
   - Problem formulation
   - Algorithms via SCA
4. Conclusions

Motivation

- The Markowitz portfolio has never been embraced by practitioners, among other reasons because
  - it only considers the risk of the portfolio as a whole and ignores the risk diversification (i.e., concentrates risk too much in few assets, this was observed in the 2008 financial crisis)
  - it is highly sensitive to the estimation errors in the parameters (i.e., small estimation errors in the parameters may change completely the designed portfolio)

- 👉 Recently, the alternative **risk parity portfolio design** has been receiving significant attention from both the theoretical and practical sides because
  - diversifies the risk, instead of the capital, among the assets
  - less sensitive to parameter estimation errors.

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Risk Contribution

- Given a portfolio $\mathbf{w} \in \mathbb{R}^N$ and the return covariance matrix $\boldsymbol{\Sigma}$, the portfolio volatility is:
$$\sigma\left(\mathbf{w}\right) = \sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}.$$
- Following Euler's theorem, the volatility can be decomposed as follows:
$$\sigma\left(\mathbf{w}\right) = \sum_{i=1}^{N} w_i \frac{\partial \sigma}{\partial w_i} = \sum_{i=1}^{N} \frac{w_i \left(\boldsymbol{\Sigma}\mathbf{w}\right)_i}{\sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}}$$
  - Risk contribution from asset i to the total risk $\sigma\left(\mathbf{w}\right)$:
$$w_i \frac{\partial \sigma}{\partial w_i} = \frac{w_i \left(\boldsymbol{\Sigma}\mathbf{w}\right)_i}{\sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}}$$
  - Normalized risk contribution: $w_i \left(\boldsymbol{\Sigma}\mathbf{w}\right)_i / \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}$
- Other measures of risk $f(\mathbf{w})$ like VAR and CVaR can also be decomposed following Euler's theorem.

Risk-Parity Portfolio

- Idea: equalize the risks contributions:

$$w_i \frac{\partial f(\mathbf{w})}{\partial w_i} = w_j \frac{\partial f(\mathbf{w})}{\partial w_j} \qquad \forall i, j.$$

- Risk budgeting is a more general concept. Given a risk budget vector $\mathbf{b} = [b_1, \ldots, b_N]^T > \mathbf{0}, \mathbf{1}^T\mathbf{b} = 1$, the risk budgeting portfolio should satisfy

$$w_i \frac{\partial f(\mathbf{w})}{\partial w_i} = b_i f(\mathbf{w}) \qquad \forall i.$$

- Risk parity portfolio is a special case of the risk budgeting portfolio with $\mathbf{b} = \mathbf{1}/N$.

Risk-Parity Portfolio

- For the volatility we can write
  - risk parity: $w_i\left(\boldsymbol{\Sigma}\mathbf{w}\right)_i = w_j\left(\boldsymbol{\Sigma}\mathbf{w}\right)_j$
  - risk budgeting: $w_i\left(\boldsymbol{\Sigma}\mathbf{w}\right)_i = b_i\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}$
- Assuming that $\boldsymbol{\Sigma}$ is diagonal and with the constraints $\mathbf{1}^T\mathbf{w} = 1$ and $\mathbf{w} \geq \mathbf{0}$, the risk budgeting portfolio is

$$w_i = \frac{\sqrt{b_i}/\sqrt{\Sigma_{ii}}}{\sum_{k=1}^{N}\sqrt{b_k}/\sqrt{\Sigma_{kk}}}, \qquad i = 1, \ldots, N.$$

- However, for non-diagonal $\boldsymbol{\Sigma}$ or with other additional constraints, a closed-form solution does not exist in general and some optimization procedures have to be constructed.
- The previous diagonal solution can always be used and is called *naive risk budgeting portfolio*.

Risk-Parity Portfolio

- Consider the risk budgeting equations

$$w_i \left(\boldsymbol{\Sigma}\mathbf{w}\right)_i = b_i \, \mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}, \qquad i = 1, \ldots, N$$

  with $\mathbf{1}^T\mathbf{w} = 1$ and $\mathbf{w} \geq \mathbf{0}$.

- If we define $\mathbf{x} = \mathbf{w}/\sqrt{\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}}$, then we can rewrite the risk budgeting equations compactly as

$$\boldsymbol{\Sigma}\mathbf{x} = \mathbf{b}/\mathbf{x}$$

  with $\mathbf{x} \geq \mathbf{0}$ and we can always recover the portfolio by normalizing: $\mathbf{w} = \mathbf{x}/(\mathbf{1}^T\mathbf{x})$.

- Spinu[7] realized that precisely that equation corresponds to the gradient of the function $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\boldsymbol{\Sigma}\mathbf{x} - \mathbf{b}^T\log(\mathbf{x})$ set to zero, which is the optimality condition.

- Indeed, $\nabla f(\mathbf{x}) = \boldsymbol{\Sigma}\mathbf{x} - \mathbf{b}/\mathbf{x}$.

[7] F. Spinu, "An algorithm for computing risk parity weights," *SSRN*, 2013. [Online]. Available: https://ssrn.com/abstract=2297383.

Risk-Parity Portfolio

- So we can finally formulate the risk budgeting problem as the following convex optimization problem:

$$\underset{\mathbf{x}\geq\mathbf{0}}{\text{minimize}} \quad \frac{1}{2}\mathbf{x}^T\boldsymbol{\Sigma}\mathbf{x} - \mathbf{b}^T\log(\mathbf{x}).$$

- Roncalli et al.[8] proposed a slightly different formulation (also convex):

$$\underset{\mathbf{x}\geq\mathbf{0}}{\text{minimize}} \quad \sqrt{\mathbf{x}^T\boldsymbol{\Sigma}\mathbf{x}} - \mathbf{b}^T\log(\mathbf{x}).$$

- Unfortunately, even though these two problems are convex, they do not conform with the typical classes that most solvers embrace (i.e., LP, QP, QCQP, SOCP, SDP, GP, etc.).

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[8] T. Griveau-Billion, J.-C. Richard, and T. Roncalli, "A fast algorithm for computing high-dimensional risk parity portfolios," *SSRN*, 2013. [Online]. Available: https://ssrn.com/abstract=2325255.

Risk-Parity Portfolio

- Nevertheless, there are several simple iterative algorithms that can be used, like the cyclical coordinate descent algorithm and the Newton algorithm.
- **Newton method**: The Newton method obtains the iterates based on the gradient ∇f and the Hessian H of the objective function $f(\mathbf{x})$ as follows:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k+1)} - \mathsf{H}^{-1}(\mathbf{x}^{(k+1)})\nabla f(\mathbf{x}^{(k+1)})$$

  - in practice, one may need to use the backtracking method to properly adjust the step size of each iteration[9]
  - for the function $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{\Sigma}\mathbf{x} - \mathbf{b}^T\log(\mathbf{x})$, the gradient and Hessian are given by

$$\begin{aligned} \nabla f(\mathbf{x}) &= \mathbf{\Sigma}\mathbf{x} - \mathbf{b}/\mathbf{x} \\ \mathsf{H}(\mathbf{x}) &= \mathbf{\Sigma} + \mathrm{Diag}(\mathbf{b}/\mathbf{x}^2). \end{aligned}$$

[9] S. P. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

Risk-Parity Portfolio

- **Cyclical coordinate descent algorithm**: This method simply minimizes in a cyclical manner with respect to each element of the variable $\mathbf{x}$.
  - the minimization w.r.t. x_i is

$$\underset{x_i \geq \mathbf{0}}{\text{minimize}} \quad \frac{1}{2} x_i^2 \sigma_i^2 + x_i (\mathbf{x}_{-i}^T \boldsymbol{\Sigma}_{:,i}) - b_i \log x_i$$

  - with gradient

$$\nabla_i f = x_i \sigma_i^2 + (\mathbf{x}_{-i}^T \boldsymbol{\Sigma}_{:,i}) - b_i / x_i$$

  - setting the gradient to zero gives us the second order equation

$$x_i^2 \sigma_i^2 + x_i (\mathbf{x}_{-i}^T \boldsymbol{\Sigma}_{:,i}) - b_i = 0$$

    with positive solution given by

$$x_i^\star = \frac{-(\mathbf{x}_{-i}^T \boldsymbol{\Sigma}_{:,i}) + \sqrt{(\mathbf{x}_{-i}^T \boldsymbol{\Sigma}_{:,i})^2 + 4 \sigma_i^2 b_i}}{2 \sigma_i^2}.$$

Risk-Parity Portfolio

- These methods are very nice and they will converge to the optimal risk budgeting solution (because the problem formulated was convex).
- However, they can only be employed for the simplest risk budgeting formulation with a simplex constraint set (i.e., $\mathbf{1}^T\mathbf{w} = 1$ and $\mathbf{w} \geq \mathbf{0}$).
- They cannot be used if
  - we have other constraints like allowing shortselling or box constraints: $l_i \leq w_i \leq i$
  - on top of the risk budgeting constraints $w_i\left(\boldsymbol{\Sigma}\mathbf{w}\right)_i = b_i\,\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}$ we have other objectives like maximizing the expected return $\mathbf{w}^T\boldsymbol{\mu}$ or minimizing the overall variance $\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}$.
- In those more general cases, we need more sophisticated formulations, which unfortunately will not be convex.

Risk-Parity Formulation

- Maillard et al.[10] aimed at solving:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \sum_{i,j=1}^{N} \left(w_i \left(\boldsymbol{\Sigma} \mathbf{w} \right)_i - w_j \left(\boldsymbol{\Sigma} \mathbf{w} \right)_j \right)^2 \\ \text{subject to} & \mathbf{1}^T \mathbf{w} = 1. \end{array}$$

- The idea is to try to achieve equal risk contributions $\frac{w_i (\boldsymbol{\Sigma}\mathbf{w})_i}{\sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}}$ by penalizing the differences between the terms $w_i \left(\boldsymbol{\Sigma} \mathbf{w} \right)_i$.
- This is a simplified formulation:

$$\begin{array}{ll} \underset{\mathbf{w},\theta}{\text{minimize}} & \sum_{i=1}^{N} \left(w_i \left(\boldsymbol{\Sigma} \mathbf{w} \right)_i - \theta \right)^2 \\ \text{subject to} & \mathbf{1}^T \mathbf{w} = 1. \end{array}$$

[10] S. Maillard, T. Roncalli, and J. Teïletche, "The properties of equally weighted risk contribution portfolios," *Journal of Portfolio Management*, vol. 36, no. 4, pp. 60–70, 2010.

More Risk-Parity Formulations

- Bruder and Roncalli[11] proposed to solve:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \sum_{i=1}^{N} \left(\frac{w_i(\boldsymbol{\Sigma}\mathbf{w})_i}{\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}} - b_i \right)^2 \\ \text{subject to} & \mathbf{w}^T\mathbf{1} = 1, \end{array}$$

  where $b_i = \frac{1}{N}$.

- More generally, one can equalize the risk contribution by setting arbitrary proportions (as opposed to equal contributions):

$$\mathbf{b} = [b_1, \ldots, b_N]^T > \mathbf{0}, \quad \mathbf{1}^T\mathbf{b} = 1.$$

- One more formulation:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \sum_{i,j=1}^{N} \left(\frac{w_i(\boldsymbol{\Sigma}\mathbf{w})_i}{b_i} - \frac{w_j(\boldsymbol{\Sigma}\mathbf{w})_j}{b_j} \right)^2 \\ \text{subject to} & \mathbf{1}^T\mathbf{w} = 1. \end{array}$$

[11] B. Bruder and T. Roncalli, "Managing risk exposures using the risk budgeting approach," University Library of Munich, Germany, Tech. Rep., 2012.

And Yet More Risk-Parity Formulations

- One more formulation:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \sum_{i=1}^{N}\left(w_i\left(\boldsymbol{\Sigma}\mathbf{w}\right)_i - b_i\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}\right)^2 \\ \text{subject to} & \mathbf{1}^T\mathbf{w} = 1. \end{array}$$

- And one more:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \sum_{i=1}^{N}\left(\frac{w_i(\boldsymbol{\Sigma}\mathbf{w})_i}{\sqrt{\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}}} - b_i\sqrt{\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}}\right)^2 \\ \text{subject to} & \mathbf{1}^T\mathbf{w} = 1. \end{array}$$

Yet Even More Risk-Parity Formulations

- What about this one:

$$\begin{array}{ll} \underset{\mathbf{w},\theta}{\text{minimize}} & \sum_{i=1}^{N}\left(\frac{w_i(\boldsymbol{\Sigma}\mathbf{w})_i}{b_i}-\theta\right)^2 \\ \text{subject to} & \mathbf{1}^T\mathbf{w}=1. \end{array}$$

- More formulations can be found in the book:
T. Roncalli, *Introduction to Risk Parity and Budgeting*. CRC Press, 2013.

General Problem Formulation

A more general risk parity formulation is[12]:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & U(\mathbf{w}) \triangleq \sum_{i=1}^{N} (g_i(\mathbf{w}))^2 + \lambda F(\mathbf{w}) \\ \text{subject to} & \mathbf{1}^T\mathbf{w} = 1, \quad \mathbf{w} \in \mathcal{W} \end{array}$$

where

- $\sum_{i=1}^{N} (g_i(\mathbf{w}))^2$: risk concentration measurement, e.g.,
$$g_i(\mathbf{w}) \triangleq \frac{w_i (\boldsymbol{\Sigma}\mathbf{w})_i}{\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}} - \frac{1}{N},$$
- $F(\mathbf{w})$: preference, e.g., 0, $-\boldsymbol{\mu}^T\mathbf{w}$, $-\boldsymbol{\mu}^T\mathbf{w} + \nu\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}$,
- $\lambda \geq 0$: trade-off parameter,
- $\mathbf{1}^T\mathbf{w} = 1, \mathbf{w} \in \mathcal{W}$: capital budget & other convex constraints.

Challenge: the problem is highly nonconvex due to $\sum_{i=1}^{N} (g_i(\mathbf{w}))^2$.

[12] Y. Feng and D. P. Palomar, "SCRIP: Successive convex optimization methods for risk parity portfolios design," *IEEE Trans. Signal Process.*, vol. 63, no. 19, pp. 5285–5300, 2015.

Unified Problem Formulation

- The previous general formulation contains the risk term $R(\mathbf{w}) = \sum_{i=1}^{N} (g_i(\mathbf{w}))^2$, which can be written in a compact way to represent the many formulations presented before.
- Define $\mathbf{M}_i \in \mathbb{R}^{N \times N}$ as a sparse matrix with its i-th row equal to that of the covariance matrix $\boldsymbol{\Sigma}$.
- Examples:
  - $R(\mathbf{w}) = \sum_{i,j=1}^{N} \left(w_i (\boldsymbol{\Sigma}\mathbf{w})_i - w_j (\boldsymbol{\Sigma}\mathbf{w})_j \right)^2$ corresponds to
$$g_{i,j}(\mathbf{w}) = \mathbf{w}^T (\mathbf{M}_i - \mathbf{M}_j) \mathbf{w}$$
  - $R(\mathbf{w}) = \sum_{i=1}^{N} (w_i (\boldsymbol{\Sigma}\mathbf{w})_i - \theta)^2$ corresponds to
$$g_i(\mathbf{w}) = \mathbf{w}^T \mathbf{M}_i \mathbf{w} - \theta$$
  - $R(\mathbf{w}) = \sum_{i=1}^{N} \left(\frac{w_i (\boldsymbol{\Sigma}\mathbf{w})_i}{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}} - b_i \right)^2$ corresponds to
$$g_i(\mathbf{w}) = \frac{\mathbf{w}^T \mathbf{M}_i \mathbf{w}}{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}} - b_i$$

Unified Problem Formulation

- More examples:
  - $R(\mathbf{w}) = \sum_{i,j=1}^{N} \left(\frac{w_i (\boldsymbol{\Sigma}\mathbf{w})_i}{b_i} - \frac{w_j (\boldsymbol{\Sigma}\mathbf{w})_j}{b_j} \right)^2$ corresponds to
$$g_{i,j}(\mathbf{w}) = \mathbf{w}^T(\mathbf{M}_i/b_i - \mathbf{M}_j/b_j)\mathbf{w}$$
  - $R(\mathbf{w}) = \sum_{i=1}^{N} \left(w_i \left(\boldsymbol{\Sigma}\mathbf{w}\right)_i - b_i \mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w} \right)^2$ corresponds to
$$g_i(\mathbf{w}) = \mathbf{w}^T(\mathbf{M}_i - b_i\boldsymbol{\Sigma})\mathbf{w}$$
  - $R(\mathbf{w}) = \sum_{i=1}^{N} \left(\frac{w_i (\boldsymbol{\Sigma}\mathbf{w})_i}{\sqrt{\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}}} - b_i\sqrt{\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}} \right)^2$ corresponds to
$$g_i(\mathbf{w}) = \frac{\mathbf{w}^T\mathbf{M}_i\mathbf{w}}{\sqrt{\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}}} - b_i\sqrt{\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}}$$
  - $R(\mathbf{w}) = \sum_{i=1}^{N} \left(\frac{w_i (\boldsymbol{\Sigma}\mathbf{w})_i}{b_i} - \theta \right)^2$ corresponds to
$$g_i(\mathbf{w}) = \mathbf{w}^T\mathbf{M}_i\mathbf{w}/b_i - \theta$$

Outline

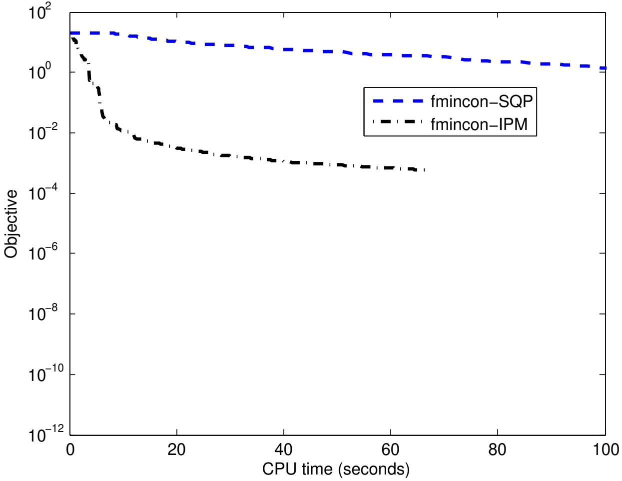
Numerical Solving Approach

- Some off-the-shelf nonlinear numerical optimization methods[13] are typically used, e.g.,
  - Sequential Quadratic Programming (SQP)
  - Interior Point Methods (IPM).
- For such risk-parity portfolio problems, they
  - may be very slow, and
  - get stuck at some unsatisfactory points.
- Because the **structure of the objective is not explored**.

[13] J. Nocedal and S. J. Wright., *Numerical Optimization*, Second. Springer Verlag, 2006.

Numerical Example: Slow Convergence

Off-the-shelf nonlinear solvers have slow convergence for the risk-parity portfolio problem:



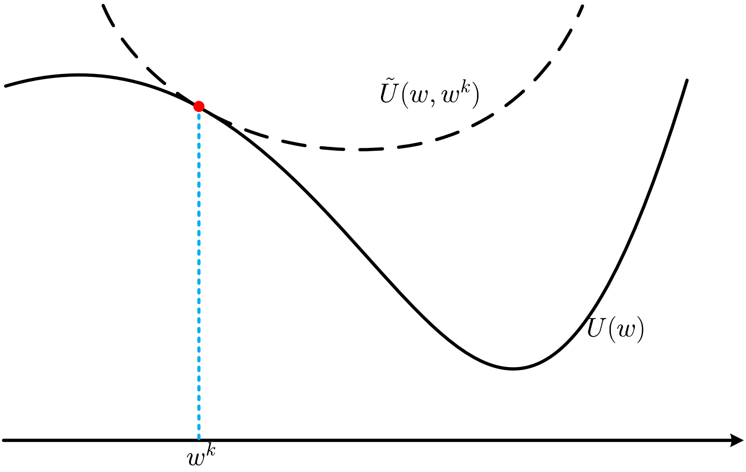
Successive Convex Approximation (SCA)

- **Basic idea**: solving a difficult problem via solving a sequence of simpler problems.
- Minimize $U(\mathbf{w})$ over $\mathbf{w} \in \overline{\mathcal{W}}$ via SCA method[14]:
  - **Construction of Approximation**: finding $\tilde{U}\left(\mathbf{w};\mathbf{w}^k\right)$ that approximates the function $U(\mathbf{w})$ at the point $\mathbf{w}^k$ and
    - $\tilde{U}\left(\mathbf{w};\mathbf{w}^k\right)$: uniformly strongly convex & cont. differentiable
    - $\nabla\tilde{U}\left(\mathbf{w};\mathbf{w}^k\right)$: Lipschitz continuous on $\overline{\mathcal{W}}$
    - $\nabla\tilde{U}\left(\mathbf{w};\mathbf{w}^k\right)|_{\mathbf{w}=\mathbf{w}^k} = \nabla U(\mathbf{w})|_{\mathbf{w}=\mathbf{w}^k}$
  - **Minimization**: minimizing $\tilde{U}\left(\mathbf{w};\mathbf{w}^k\right)$ to get the update

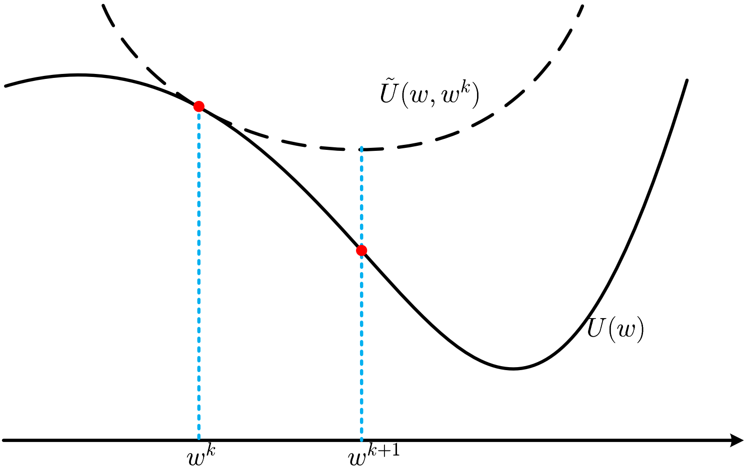
$$\mathbf{w}^{k+1} \triangleq \arg\min_{\mathbf{w}\in\overline{\mathcal{W}}} \tilde{U}\left(\mathbf{w};\mathbf{w}^k\right).$$

[14] G. Scutari, F. Facchinei, P. Song, D. P. Palomar, and J.-S. Pang, "Decomposition by partial linearization: Parallel optimization of multi-agent systems," *IEEE Trans. Signal Process.*, vol. 62, no. 3, pp. 641–656, 2014.

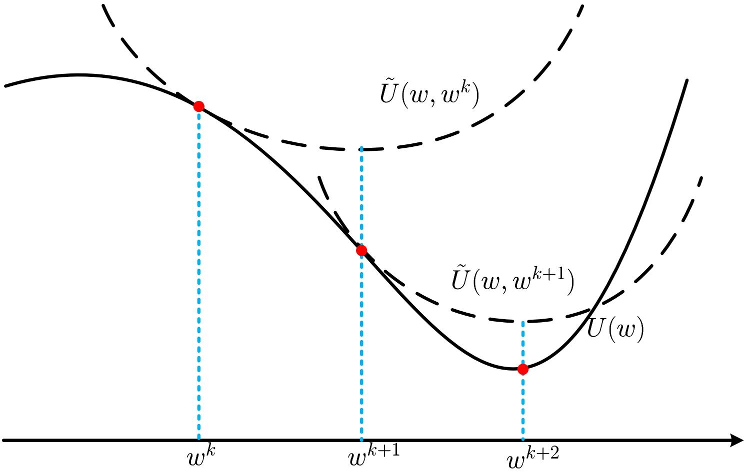
Construction of Approximation



Minimization



One More Iteration



Classical Methods as SCA

- **(Unconstrained) gradient descent**: Set

$$\tilde{U}\left(\mathbf{w};\mathbf{w}^k\right) = U\left(\mathbf{w}^k\right) + \nabla U\left(\mathbf{w}^k\right)^T\left(\mathbf{w}-\mathbf{w}^k\right) + \frac{1}{2\alpha^k}\left\|\mathbf{w}-\mathbf{w}^k\right\|_2^2.$$

  Setting the derivative w.r.t. $\mathbf{w}$ to zero yields:

$$\mathbf{w}^{k+1} = \mathbf{w}^k - \alpha^k \nabla U\left(\mathbf{w}^k\right).$$

- **(Unconstrained) Newton's method**: Set

$$\begin{aligned}\tilde{U}\left(\mathbf{w};\mathbf{w}^k\right) = U\left(\mathbf{w}^k\right) &+ \nabla U\left(\mathbf{w}^k\right)^T\left(\mathbf{w}-\mathbf{w}^k\right) \\ &+ \frac{1}{2\alpha^k}\left(\mathbf{w}-\mathbf{w}^k\right)^T\nabla^2 U\left(\mathbf{w}^k\right)\left(\mathbf{w}-\mathbf{w}^k\right).\end{aligned}$$

  Setting the derivative w.r.t. $\mathbf{w}$ to zero yields:

$$\mathbf{w}^{k+1} = \mathbf{w}^k - \alpha^k\left(\nabla^2 U\left(\mathbf{w}^k\right)\right)^{-1}\nabla U\left(\mathbf{w}^k\right).$$

SCA for Risk Parity Portfolio Design

- Recall the objective

$$U(\mathbf{w}) = \sum_{i=1}^{N} (g_i(\mathbf{w}))^2 + \lambda F(\mathbf{w}).$$

- At the k-th iteration $\mathbf{w}^k$, set $\tau > 0$ and construct

$$\tilde{U}\left(\mathbf{w}, \mathbf{w}^k\right) = \overbrace{\sum_{i=1}^{N} \left(g_i\left(\mathbf{w}^k\right) + \left(\nabla g_i\left(\mathbf{w}^k\right)\right)^T \left(\mathbf{w} - \mathbf{w}^k\right)\right)^2}^{P\left(\mathbf{w};\mathbf{w}^k\right) \triangleq} + \frac{\tau}{2}\left\|\mathbf{w} - \mathbf{w}^k\right\|_2^2 + \lambda F(\mathbf{w})$$

- IDEA: linearizing nonconvex functions $g_i(\mathbf{w})$ inside the least square $\Longrightarrow$ quadratic convex $P\left(\mathbf{w};\mathbf{w}^k\right)$ approximates $R(\mathbf{w}) = \sum_{i=1}^{N} (g_i(\mathbf{w}))^2$, with $\nabla P\left(\mathbf{w}, \mathbf{w}^k\right)|_{\mathbf{w}=\mathbf{w}^k} = \nabla R(\mathbf{w})|_{\mathbf{w}=\mathbf{w}^k}$.

Problem Reformulation

- $P\left(\mathbf{w};\mathbf{w}^k\right)$ can be rewritten more compactly as

$$P\left(\mathbf{w};\mathbf{w}^k\right)=\|\mathbf{A}^k\left(\mathbf{w}-\mathbf{w}^k\right)+\mathbf{g}\left(\mathbf{w}^k\right)\|^2$$

where

$$\begin{aligned}\mathbf{A}^k &\triangleq \left[\nabla g_1\left(\mathbf{w}^k\right),\ldots,\nabla g_N\left(\mathbf{w}^k\right)\right]^T,\\ \mathbf{g}\left(\mathbf{w}^k\right) &\triangleq \left[g_1\left(\mathbf{w}^k\right),\ldots,g_N\left(\mathbf{w}^k\right)\right]^T.\end{aligned}$$

- We can further expand $P\left(\mathbf{w};\mathbf{w}^k\right)$ as

$$\begin{aligned}P\left(\mathbf{w};\mathbf{w}^k\right)=&\left(\mathbf{w}-\mathbf{w}^k\right)^T\left(\mathbf{A}^k\right)^T\mathbf{A}^k\left(\mathbf{w}-\mathbf{w}^k\right)+\mathbf{g}\left(\mathbf{w}^k\right)^T\mathbf{g}\left(\mathbf{w}^k\right)\\ &+2\mathbf{g}\left(\mathbf{w}^k\right)^T\mathbf{A}^k\left(\mathbf{w}-\mathbf{w}^k\right)\end{aligned}$$

Problem Reformulation

- The QP approximation problem at the k-th iteration is

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \tilde{U}\left(\mathbf{w}, \mathbf{w}^k\right) = \frac{1}{2}\mathbf{w}^T\mathbf{Q}^k\mathbf{w} + \mathbf{w}^T\mathbf{q}^k + \lambda F(\mathbf{w}) \\ \text{subject to} & \mathbf{1}^T\mathbf{w} = 1, \quad \mathbf{w} \in \mathcal{W}. \end{array} \tag{1}$$

where

$$\mathbf{Q}^k \triangleq 2\left(\mathbf{A}^k\right)^T\mathbf{A}^k + \tau\mathbf{I},$$

$$\mathbf{q}^k \triangleq 2\left(\mathbf{A}^k\right)^T\mathbf{g}\left(\mathbf{w}^k\right) - \mathbf{Q}^k\mathbf{w}^k,$$

- This problem can be solved direclty with a solver or, depending on the constraints in $\mathcal{W}$, one may derive simpler closed-form solutions.
- For example, if we only have equality constraints in the form $\mathbf{C}\mathbf{w} = \mathbf{c}$, then from the KKT optimality conditions the optimal solution is found as $\hat{\mathbf{w}}^k = -(\mathbf{Q}^k)^{-1}(\mathbf{q}^k + \mathbf{C}^T\boldsymbol{\lambda}^k)$ where $\boldsymbol{\lambda}^k = -\left(\mathbf{C}(\mathbf{Q}^k)^{-1}\mathbf{C}^T\right)^{-1}\left(\mathbf{C}(\mathbf{Q}^k)^{-1}\mathbf{q}^k + \mathbf{c}\right)$.

Sequential Numerical Algorithm

Algorithm 1: Successive Convex optimization for RIsk Parity portfolio (SCRIP).

Set $k = 0$, $\mathbf{w}^0 \in \overline{\mathcal{W}}$, $\tau > 0$, $\{\gamma^k\} \in (0, 1]$

repeat

Solve QP problem (1) to get the optimal solution $\hat{\mathbf{w}}^k$ (global minimum)

$\mathbf{w}^{k+1} = \mathbf{w}^k + \gamma^k \left(\hat{\mathbf{w}}^k - \mathbf{w}^k\right)$

$k \leftarrow k + 1$

until convergence

return $\mathbf{w}^k$

- More advanced algorithms can be found in
  Y. Feng and D. P. Palomar, "SCRIP: Successive convex optimization methods for risk parity portfolios design," *IEEE Trans. Signal Process.*, vol. 63, no. 19, pp. 5285–5300, 2015.

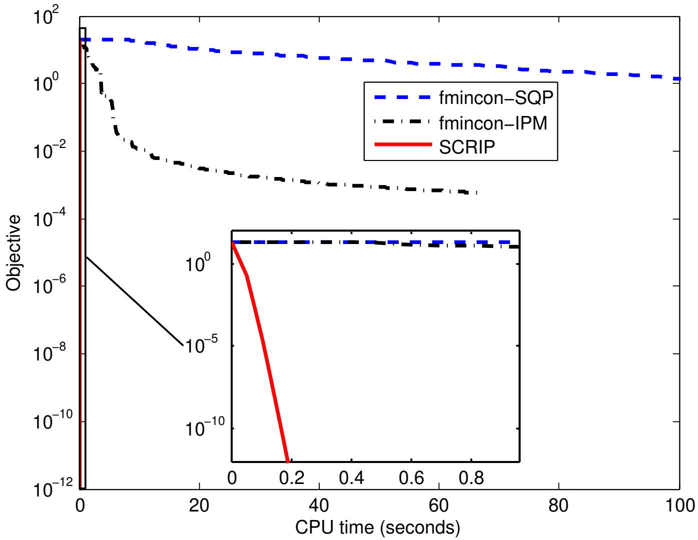
Convergence Analysis

Proposition 1

Under some technical conditions, suppose $\tau > 0$, $\gamma^k \in (0,1]$, $\gamma^k \rightarrow 0$, $\sum_k \gamma^k = +\infty$ *and* $\sum_k \left(\gamma^k\right)^2 < +\infty$, *and let* $\left\{\mathbf{w}^k\right\}$ *be the sequence generated by Algorithm 1. Then, either Algorithm 1 converges in a finite number of iterations to a stationary point or every limit of* $\left\{\mathbf{w}^k\right\}$ *(at least one such point exists) is a stationary point.*

Numerical example

Fast algorithms based on successive convex approximation (SCA):



Outline

Conclusions

- We have reviewed the Markowitz portfolio formulation and understood that it has many practical flaws that make it impractical. Indeed, it is not used by practitioners.
- We have learned about the risk-parity portfolio formulation.
- We have explored the numerical resolution of such problems via successive convex approximation (SCA) methods.
- The performance of risk-parity portfolio versus Markowitz portfolio is much improved.
- Side result: we have learned how to develop efficient numerical algorithms based on SCA.

Thanks

For more information visit:

https://www.danielppalomar.com