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Risk parity portfolios with risk factors

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Portfolio construction and risk budgeting are the focus of many studies by academics and practitioners. In particular, diversification has spawned much interest and has been defined very differently. In this paper, we analyse a method to achieve portfolio diversification based on the decomposition of the portfolio's risk into risk factor contributions. First, we expose the relationship between risk factor and asset contributions. Secondly, we formulate the diversification problem in terms of risk factors as an optimization program. Finally, we illustrate our methodology with a real example of building a strategic asset allocation based on economic factors for a pension fund facing liability constraints.

Keywords: Risk parity; Risk budgeting; Factor model; ERC portfolio; Diversification; Concentration; Strategic asset allocation

JEL Classification: G11, C58, C60

1. Introduction

While Markowitz's insights on diversification live on, practical limitations to direct implementations of his original approach have recently led to the rise of heuristic approaches. Approaches such as equally weighted, minimum variance, most diversified portfolio, equally weighted risk contributions (or ERC), risk budgeting or diversified risk parity strategies have become attractive to academics and practitioners alike (see e.g. Meucci 2007, Choueifaty and Coignard 2008, Meucci 2009, Maillard *et al*. 2010, Bruder and Roncalli 2012, Lohre *et al*. 2012) for they provide elegant and systematic methodologies to tackle the construction of diversified portfolios.

While explicitly pursuing diversification, these methodologies may lead to solutions with hidden risk concentration. Meucci (2009)'s work on diversification across principal component factors provided a clue to resolving this unfortunate problem by focusing on underlying risk factors. In this paper, we build on this idea and combine it with the risk-budgeting approach of Bruder and Roncalli (2012) to develop a risk-budgeting methodology focused on risk factors. When used with the objective of maximizing risk diversification, our approach is tantamount to diversifying across the 'true' sources of risk and often leads to a solution with equally weighted risk factor contributions. Hence, we dubbed it the 'risk factor parity' approach.

Further motivation behind this work stems from the need to resolve what Choueifaty *et al*. (2013) called the duplication invariance problem affecting in particular the ERC approach. We show in this paper that by focusing on the 'true' sources of risk, we avoid the inherent instability of the ERC solution created when introducing a duplicated asset. Furthermore, we show that this is a particular consequence of the risk factor parity approach delivering a solution that verifies the desirable invariance property (Choueifaty *et al*. 2013).

Our methodology is however more versatile. In this paper, we make explicit the relations between any risk factor and asset contributions to portfolio risk. In particular, we show that the equally weighted risk factor contributions approach is equivalent to a risk-budgeting solution on the assets of the portfolio with a specific risk budget profile. We also examine different optimization programs to obtain the desired outcome.

The plan of this paper is thus as follows. In section 2, we derive the relations between asset and risk factors contributions to overall risk, and provide some illustrations. In section 3, we consider two types of portfolio construction methodologies, first by matching risk contributions and second by minimizing a concentration index. Finally, section 4 provides two applications of our portfolio construction methodologies: one on portfolios of hedge funds and a second in the context of strategic asset allocation.

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2. Risk decomposition with risk factors

2.1. The linear factor model

We consider a set of n assets $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$ and a set of m risk factors $\{\mathcal{F}_1, \ldots, \mathcal{F}_m\}$. We denote by $\mathbf{R}_t$ the $(n \times 1)$ vector of asset returns at time step t and $\boldsymbol{\Sigma}$ its associated covariance matrix. We assume the following linear factor model

$$\mathbf{R}_t = \mathbf{A}\mathbf{F}_t + \boldsymbol{\varepsilon}_t \tag{1}$$

with $\mathbf{F}_t$ the $(m \times 1)$ vector of factor returns at t and Ω its associated covariance matrix. $\boldsymbol{\varepsilon}_t$ is an i.i.d. $(n \times 1)$ centred random vector of variance $\mathbf{D}$. $\mathbf{F}_t$ and $\boldsymbol{\varepsilon}_t$ are two uncorrelated random vectors. $\mathbf{A}$ is the $(n \times m)$ 'loadings' matrix. We denote the portfolio's asset exposures by the $(n \times 1)$ vector $\mathbf{x}$ and the portfolio's risk factor exposures by the $(m \times 1)$ vector $\mathbf{y}$. These vectors are related through the P&L function Π_t of the portfolio at time t

$$\Pi_t = \mathbf{x}^\top \mathbf{R}_t = \mathbf{x}^\top \mathbf{A}\mathbf{F}_t + \mathbf{x}^\top \boldsymbol{\varepsilon}_t = \mathbf{y}^\top \mathbf{F}_t + \eta_t$$

with† $\mathbf{y} = \mathbf{A}^\top \mathbf{x}$ and $\eta_t = \mathbf{x}^\top \boldsymbol{\varepsilon}_t$. Let us denote $\mathbf{B} = \mathbf{A}^\top$ and $\mathbf{B}^+$ the Moore–Penrose inverse of $\mathbf{B}$. We write the following decomposition (Meucci 2007)

$$\mathbf{x} = \begin{pmatrix} \mathbf{B}^+ & \tilde{\mathbf{B}}^+ \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \tilde{\mathbf{y}} \end{pmatrix} = \bar{\mathbf{B}}^\top \bar{\mathbf{y}} \tag{2}$$

where $\tilde{\mathbf{B}}^+$ is any $n \times (n-m)$ matrix that spans the left nullspace of $\mathbf{B}^+$. The vector $\tilde{\mathbf{y}}$ represents the portfolio's exposures to an arbitrary set of $n-m$ factors $\tilde{\mathbf{F}}_t$ spanning the space of the assets' idiosyncratic risks $\boldsymbol{\varepsilon}_t$.

Remark 1 The factor exposures $\mathbf{y}$ are in fact the beta exposures of the portfolio to the common risk factors $\mathbf{F}_t$.

2.2. Defining the marginal risk contribution of factors

Let us consider a convex risk measure $\mathcal{R}$. Thus, $\mathcal{R}(\mathbf{x}) = \mathcal{R}(\mathbf{y}, \tilde{\mathbf{y}})$ and we have the following theorem.

THEOREM 1 *The marginal risk contribution of* assets *are related to the marginal risk contributions of* factors *in the following way*

$$\frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,\mathbf{x}} = \frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,\mathbf{y}}\mathbf{B} + \frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,\tilde{\mathbf{y}}}\tilde{\mathbf{B}} \tag{3}$$

Hence, the marginal risk contribution of the jth factor exposure is given by

$$\frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,y_j} = \left(\mathbf{A}^+ \frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,\mathbf{x}^\top}\right)_j \tag{4}$$

For the additional factors, we have

$$\frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,\tilde{y}_j} = \left(\tilde{\mathbf{B}} \frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,\mathbf{x}^\top}\right)_j \tag{5}$$

Proof See appendix 1. □

2.3. Euler decomposition of the risk measure

It is easy to verify that for any convex risk measure that verifies the Euler decomposition ($\mathcal{R} = \partial_{\mathbf{x}}\,\mathcal{R}(\mathbf{x}) \cdot \mathbf{x} = \sum_{i=1}^n x_i \cdot \partial_{x_i}\,\mathcal{R}(\mathbf{x})$), the Euler decomposition is verified in the new coordinate system $(\mathbf{y}, \tilde{\mathbf{y}})$:

$$\mathcal{R}(\mathbf{x}) = \frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,\mathbf{y}}\mathbf{y} + \frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,\tilde{\mathbf{y}}}\tilde{\mathbf{y}}. \tag{6}$$

Let us note $\mathcal{RC}(\mathcal{F}_j) = y_j \cdot \partial_{y_j}\,\mathcal{R}(\mathbf{x})$ the risk contribution of factor j. By theorem 1, the sum of the risk contributions of the factor exposures is simply expressed as

$$\sum_{j=1}^m \mathcal{RC}(\mathcal{F}_j) = \mathbf{y}^\top \cdot \frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,\mathbf{y}^\top} = \mathbf{x}^\top(\mathbf{A}\mathbf{A}^+)\frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,\mathbf{x}^\top} \tag{7}$$

where $\mathbf{A}\mathbf{A}^+ = \left(\mathbf{B}^+\mathbf{B}\right)^\top$ is the orthogonal projector onto the range of $\mathbf{B} = \mathbf{A}^\top$. Hence, we do not retrieve the complete risk measure when considering only the risk contributions of the systematic factors. The difference is due to the risk contributions of the additional factors spanning the space of idiosyncratic risks

$$\sum_{j=1}^{n-m} \mathcal{RC}\left(\tilde{\mathcal{F}}_j\right) = \mathcal{R}(\mathbf{x}) - \sum_{j=1}^m \mathcal{RC}(\mathcal{F}_j) = \mathbf{x}^\top(\mathbf{I}_n - \mathbf{A}\mathbf{A}^+)\frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,\mathbf{x}^\top} \tag{8}$$

If we consider the volatility of the portfolio's P&L as the risk measure, the factor risk contributions are given by the following theorem.

COROLLARY 1 *When the risk measure $\mathcal{R}(\mathbf{x})$ is the volatility of the portfolio's P&L ($\mathcal{R}(\mathbf{x}) = \sigma(\mathbf{x}) = \sqrt{\mathbf{x}^\top\boldsymbol{\Sigma}\mathbf{x}}$), the risk contribution of the jth factor is*

$$\mathcal{RC}(\mathcal{F}_j) = \frac{\left(\mathbf{A}^\top\mathbf{x}\right)_j \cdot \left(\mathbf{A}^+\boldsymbol{\Sigma}\mathbf{x}\right)_j}{\sigma(\mathbf{x})}. \tag{9}$$

For the additional risk factors $\tilde{\mathbf{F}}_t$, the results become

$$\mathcal{RC}\left(\tilde{\mathcal{F}}_j\right) = \frac{\left(\tilde{\mathbf{B}}\mathbf{x}\right)_j \cdot \left(\tilde{\mathbf{B}}\boldsymbol{\Sigma}\mathbf{x}\right)_j}{\sigma(\mathbf{x})}. \tag{10}$$

Proof See appendix 2. □

Theorem 1 and corollary 1 are valid if the number of assets n is larger than the number of factors m. In the case where $n \le m$, we obtain the same results, but the additional factors $\tilde{\mathcal{F}}_j$ vanish. Using the previous results, we can define the risk contribution of the asset i with respect to the risk factors as‡

$$\begin{aligned}\mathcal{RC}(\mathcal{A}_i) = {} & \frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,\mathbf{y}}\mathbf{B}\mathbf{e}_i \left(\mathbf{e}_i^\top\mathbf{B}^+\mathbf{y} + \mathbf{e}_i^\top\tilde{\mathbf{B}}^+\tilde{\mathbf{y}}\right) \\ & + \frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,\tilde{\mathbf{y}}}\tilde{\mathbf{B}}\mathbf{e}_i \left(\mathbf{e}_i^\top\mathbf{B}^+\mathbf{y} + \mathbf{e}_i^\top\tilde{\mathbf{B}}^+\tilde{\mathbf{y}}\right)\end{aligned} \tag{11}$$

†By definition, we have $\operatorname{cov}(\mathbf{F}_t, \eta_t) = \mathbf{0}$.

‡By definition, the risk contribution of the asset i is written as:

$$\mathcal{RC}(\mathcal{A}_i) = x_i \frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,x_i}$$

Noting that both terms of this product are scalars, we commute the terms and rewrite each term to obtain equation (11) using the following facts

$$x_i = \mathbf{e}_i^\top\mathbf{x} = \mathbf{e}_i^\top\left(\mathbf{B}^+\mathbf{y} + \tilde{\mathbf{B}}^+\tilde{\mathbf{y}}\right)$$

$$\frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,x_i} = \frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,\mathbf{x}} \cdot \mathbf{e}_i = \left(\frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,\mathbf{y}}\mathbf{B} + \frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,\tilde{\mathbf{y}}}\tilde{\mathbf{B}}\right) \cdot \mathbf{e}_i$$

In the case where $\tilde{\mathbf{y}} = \mathbf{0}$, this formula reduces to

$$\mathcal{RC}(\mathcal{A}_i) = \left(\frac{\partial \mathcal{R}(\mathbf{x})}{\partial \mathbf{y}}\mathbf{B} + \frac{\partial \mathcal{R}(\mathbf{x})}{\partial \tilde{\mathbf{y}}}\tilde{\mathbf{B}}\right)\mathbf{E}_i\mathbf{B}^+\mathbf{y} \tag{12}$$

where $\mathbf{E}_i = \mathbf{e}_i\mathbf{e}_i^\top$ is a null matrix of dimension $(n \times n)$ except for the entry (i, i) which takes value one.

3. Portfolio construction with factor risk budgeting

The decomposition of a portfolio's sources of risk into factor marginal risk contributions and factor risk contributions can be used in solving the portfolio allocation problem where the objective is to build a specific risk profile. In particular, one allocation problem of interest with an intuitive appeal to most practitioners consists in matching some risk budgets with respect to risk factors. In this section, we show however that the solution may not exist. We therefore consider a second approach where the objective is to increase the diversification among the 'true' sources of risk by minimizing a measure of risk concentration between the factors.

3.1. *Matching the risk budgets*

We want to build a risk budgeting (RB) portfolio such that the risk contributions match a set $\mathbf{b}$ of given risk budgets $\{b_1, \ldots, b_m\}$

$$\mathcal{RC}(\mathcal{F}_j) = b_j\mathcal{R}(\mathbf{x}) \tag{13}$$

This general problem can be formulated as a quadratic problem in a fashion similar to Bruder and Roncalli (2012)

$$(\mathbf{y}^\star, \hat{\mathbf{y}}^\star) = \arg\min \sum_{j=1}^{m}\left(\mathcal{RC}(\mathcal{F}_j) - b_j\mathcal{R}(\mathbf{y}, \tilde{\mathbf{y}})\right)^2$$
$$\text{u.c.}\begin{cases}\mathbf{1}^\top\mathbf{B}^+\mathbf{y} + \mathbf{1}^\top\tilde{\mathbf{B}}^+\tilde{\mathbf{y}} = \mathbf{1} \\ \mathbf{0} \preceq \mathbf{B}^+\mathbf{y} + \tilde{\mathbf{B}}^+\tilde{\mathbf{y}} \preceq \mathbf{1}\end{cases} \tag{14}$$

where $\preceq$ denotes element-wise inequalities. The first constraint is a budget constraint implying that the portfolio's wealth is totally invested, while the second constraint implies that short selling is not allowed. If there is a solution to the optimization problem (14) and if the objective function is equal to zero at the optimum, it implies that it is also the solution of the matching problem (13). Problem (14) is however difficult to solve analytically as it involves PDE as its first-order conditions.† In the following paragraphs, we examine a related set of problems in which cases one can formulate an equivalent optimization problem in a more convenient form.

3.1.1. Non-negative risk factor allocation constraint.

Bruder and Roncalli (2012) show that a problem of the form

$$\mathbf{x}^\star = \arg\min \mathcal{R}(\mathbf{x})$$
$$\text{u.c.}\begin{cases}\mathcal{RC}(\mathcal{A}_i) = b_i\mathcal{R}(\mathbf{x}) \\ \mathbf{1}^\top\mathbf{x} = 1 \\ \mathbf{x} \succeq \mathbf{0}\end{cases} \tag{15}$$

can be solved by considering the alternative problem

$$\mathbf{z}^\star = \arg\min \mathcal{R}(\mathbf{z})$$
$$\text{u.c.}\begin{cases}\sum_{i=1}^{n} b_i \ln z_i \geq c \\ \mathbf{z} \succeq \mathbf{0}\end{cases} \tag{16}$$

†Nevertheless, we could solve it numerically using sequential quadratic programming.

There exists then a *unique* unnormalized solution $\mathbf{z}^\star$ and posing $\mathbf{x}^\star = \mathbf{z}^\star / \left(\mathbf{1}^\top\mathbf{z}^\star\right)$ provides the optimal portfolio's asset exposures.

Similarly, using the decomposition of the portfolio asset exposures $\mathbf{x}$ into factor risk exposures $\mathbf{y}$ given by equation (2), the optimization problem with risk factor budgeting constraints can be formulated as

$$\left(\mathbf{y}^\star, \tilde{\mathbf{y}}^\star\right) = \arg\min \mathcal{R}(\mathbf{y}, \tilde{\mathbf{y}})$$
$$\text{u.c.}\begin{cases}\sum_{j=1}^{m} b_j \ln y_j \geq c \\ \mathbf{y} \succeq \mathbf{0}\end{cases} \tag{17}$$

Notice that the first constraint in this problem induces a positivity constraint‡ on the risk factor exposures $\mathbf{y}$. Hence, in the remainder of this paper, we drop the positivity constraint on the risk factor exposures $\mathbf{y}$.

Separation principle Since we focus in this paper on portfolio building controlling for the risk budgets associated with the risk factors $\{\mathcal{F}_1, \ldots, \mathcal{F}_m\}$, we consider that we have an unconstrained minimization problem on the variable $\tilde{\mathbf{y}}$ (representing the idiosyncratic risk exposures), and focus instead on solving the constrained optimization problem on $\mathbf{y}$. Furthermore, it is always possible to solve a problem of the form (17) in two steps as long as the constraints on $\mathbf{y}$ and $\tilde{\mathbf{y}}$ are separate. The objective function thus becomes $\tilde{\mathcal{R}}(\mathbf{y}) = \inf_{\tilde{\mathbf{y}}} \mathcal{R}(\mathbf{y}, \tilde{\mathbf{y}})$. The solution will be of the form $(\mathbf{y}^\star, \varphi(\mathbf{y}^\star))$ where $\tilde{\mathbf{y}}^\star = \varphi(\mathbf{y}^\star)$ is solution to the first step of the optimization problem. The optimal portfolio allocation $\mathbf{x}^\star$ is then recovered simply as $\mathbf{x}^\star = \mathbf{B}^+\mathbf{y}^\star + \tilde{\mathbf{B}}^+\varphi(\mathbf{y}^\star)$. By construction, we also have§

$$\frac{\partial \tilde{\mathcal{R}}}{\partial \mathbf{y}}(\mathbf{y}) = \frac{\partial \mathcal{R}}{\partial \mathbf{y}}(\mathbf{y}, \varphi(\mathbf{y})) \qquad \text{and} \qquad \frac{\partial \mathcal{R}}{\partial \tilde{\mathbf{y}}}(\mathbf{y}, \varphi(\mathbf{y})) = 0$$

Application to the volatility risk measure For instance, if we measure the risk using the P&L volatility $\mathcal{R}(\mathbf{y}, \tilde{\mathbf{y}}) = \sigma(\mathbf{y}, \tilde{\mathbf{y}})$, the problem is now equivalent to a quadratic optimization problem with constraints on $\mathbf{y}$ only (cf. appendix 3). The optimal portfolio asset allocation is given by

$$\mathbf{x}^\star = \mathbf{B}^+\mathbf{y}^\star + \tilde{\mathbf{B}}^+\varphi\left(\mathbf{y}^\star\right) = \left(\mathbf{B}^+ - \tilde{\mathbf{B}}^+\tilde{\boldsymbol{\Omega}}^{-1}\left(\mathbf{B}^+\right)^\top\boldsymbol{\Sigma}\tilde{\mathbf{B}}^+\right)\mathbf{y}^\star \tag{18}$$

So long as the factors $\mathbf{F}_t$ and $\tilde{\mathbf{F}}_t$ are uncorrelated ($\boldsymbol{\Gamma} = \mathbf{0}$), a solution of the form $(\mathbf{y}^\star, \mathbf{0})$ exists. In this case, the optimization problem could also be solved separately on each of the orthogonal spaces: col $\mathbf{A}$, as a constrained problem in $\mathbf{y}$ of the form (16), and ker $\mathbf{A}$, as an unconstrained minimization problem. The (unnormalized) asset allocation is given by $\mathbf{x}^\star = \mathbf{B}^+\mathbf{y}^\star$. In the case of a single risk factor, we find that $\mathbf{x}^\star$ depends on the assets' $\boldsymbol{\beta}$ and the risk budget b in a non-trivial way (cf. appendix 4)

$$\mathbf{x}^\star = \left(\frac{\lambda_c b}{\sigma_F^2}\right)^{1/2}\frac{\boldsymbol{\beta}}{\sum_{i=1}^{n}\beta_i^2} \tag{19}$$

‡If $y_j \leq 0$, we could replace the factor $\mathcal{F}_j$ by a new factor $\mathcal{F}'_j = -\mathcal{F}_j$ meaning that $y'_j \geq 0$. Thus, we can impose that $y_j \geq 0$ without loss of generality.

§The second equality results from the definition of φ as an infimum. The key to the first equality is being able to apply the chain rule, i.e. φ being regular enough to be differentiable. If φ is a strict minimum and if $\mathcal{R}$ is differentiable, then an application of the implicit function theorem to $\partial_{\tilde{\mathbf{y}}}\mathcal{R}(\mathbf{y}, \tilde{\mathbf{y}}) = 0$ shows that φ is differentiable.

Table 1. Risk decomposition of the equally weighted portfolio.

Along assets $\mathcal{A}_1, \ldots, \mathcal{A}_n$				
$\sigma(\mathbf{x}) = 21.40\%$	x_i (%)	$\mathcal{MR}(\mathcal{A}_i)$ (%)	$\mathcal{RC}(\mathcal{A}_i)$ (%)	$\mathcal{RC}^\star(\mathcal{A}_i)$ (%)
$\mathcal{A}_1$	25.00	18.81	4.70	21.97
$\mathcal{A}_2$	25.00	23.72	5.93	27.71
$\mathcal{A}_3$	25.00	24.24	6.06	28.32
$\mathcal{A}_4$	25.00	18.83	4.71	22.00
Along factors $\mathcal{F}_1, \ldots, \mathcal{F}_m$ and $\tilde{\mathcal{F}}_1, \ldots, \tilde{\mathcal{F}}_{n-m}$				
$\sigma(\mathbf{y}) = 21.40\%$	y_i (%)	$\mathcal{MR}(\mathcal{F}_i)$ (%)	$\mathcal{RC}(\mathcal{F}_i)$ (%)	$\mathcal{RC}^\star(\mathcal{F}_i)$ (%)
$\mathcal{F}_1$	100.00	17.22	17.22	80.49
$\mathcal{F}_2$	22.50	9.07	2.04	9.53
$\mathcal{F}_3$	35.00	6.06	2.12	9.91
$\tilde{\mathcal{F}}_1$	2.75	0.52	0.01	0.07

Notes: $\sigma(\mathbf{x})$ (resp. $\sigma(\mathbf{y})$) represents the P&L volatility; x_i (resp. y_i) represents the asset weight (resp. factor exposure); $\mathcal{MR}(\mathcal{A}_i)$ (resp. $\mathcal{MR}(\mathcal{F}_i)$) represents the marginal risk contribution of asset $\mathcal{A}_i$ (resp. factor $\mathcal{F}_i$); $\mathcal{RC}(\mathcal{A}_i)$ (resp. $\mathcal{RC}(\mathcal{F}_i)$) represents the risk contribution of asset $\mathcal{A}_i$ (resp. factor $\mathcal{F}_i$); $\mathcal{RC}^\star(\mathcal{A}_i)$ (resp. $\mathcal{RC}^\star(\mathcal{F}_i)$) represents the relative risk contribution of asset $\mathcal{A}_i$ (resp. factor $\mathcal{F}_i$) as a proportion of overall portfolio risk.

3.1.2. Adding general bound constraints on asset allocation. Additionally, we are interested in solving the risk budgeting optimization problem under general bound constraints on the asset allocation, i.e.

$$\mathbf{x}_d \preceq \mathbf{x} \preceq \mathbf{x}_u \quad \text{or} \quad \mathbf{x}_d \preceq \mathbf{B}^+\mathbf{y} + \tilde{\mathbf{B}}^+\tilde{\mathbf{y}} \preceq \mathbf{x}_u \tag{20}$$

for some $\mathbf{x}_d, \mathbf{x}_u \in \mathbb{R}^n$. We can again apply the separation principle† as in section 3.1.1 and obtain the new formulation by replacing the long-only constraint on $\mathbf{y}$ by (20). It is a convex optimization problem with a unique solution (if it exists) if, and only if, φ is convex. This condition is verified if we choose $\mathcal{R}$ to be the P&L volatility. It is also true if $\mathcal{R}$ is additively separable between $\mathbf{y}$ and $\tilde{\mathbf{y}}$, in which case $\varphi(y)$ is constant. This later situation should normally arise for some classic risk measures‡ if the factors $\mathbf{F}_t$ and $\tilde{\mathbf{F}}_t$ are uncorrelated§ (which depends only on the choice of $\tilde{\mathbf{B}}^+$).

Remark 2 The solution may not exist even if φ is convex.

Remark 3 As a special case, long-only constraints are obtained naturally by posing $x_{d,i} = 0$ and $x_{u,i} = +\infty$ for all $i \in \{1, \ldots, n\}$.

In the case of the volatility risk measure and a single risk factor, the existence of a solution depends on the bounds and on the betas of the assets to the risk factor. In particular, existence of a solution is ensured if

$$\min_{i=1,\ldots,n} \left\{ \frac{x_{u,i}}{\beta_i} \right\} > \max_{i=1,\ldots,n} \left\{ \frac{x_{d,i}}{\beta_i} \right\}.$$

The unconstrained solution $\mathbf{x}^\star$ will be increased (resp. decreased) if one of the lower bound constraints (resp. one of the upper bound constraints) is active (cf. appendix 4). In the case of long-only constraints, however, if one asset has a negative beta to the risk factor, then there is no solution.

Remark 4 The existence problem of long-only portfolios could be easily illustrated with a portfolio of bonds. It is commonly argued that the three factors of the yield curve are the general level of interest rates, the slope of the yield curve and its convexity. Building a bond portfolio where the slope and convexity factors have the same magnitude than the level factor is not possible if we impose that the weights are positive. In the long-only case, risk contributions of the slope and convexity factors are then bounded. We do not face this problem if the long-only constraint vanishes.

3.1.3. An illustration. Let us consider an example with 4 assets and 3 factors. The loadings matrix is

$$\mathbf{A} = \begin{pmatrix} 0.9 & 0 & 0.5 \\ 1.1 & 0.5 & 0 \\ 1.2 & 0.3 & 0.2 \\ 0.8 & 0.1 & 0.7 \end{pmatrix}$$

The three factors are uncorrelated and their volatilities are equal to 20, 10 and 10%. We consider a diagonal matrix $\mathbf{D}$ with specific volatilities 10, 15, 10 and 15%. The corresponding correlation matrix of asset returns is (in %)

$$\boldsymbol{\rho} = \begin{pmatrix} 100 & & & \\ 69.0 & 100 & & \\ 79.5 & 76.4 & 100 & \\ 66.2 & 57.2 & 66.3 & 100 \end{pmatrix}$$

and their volatilities are, respectively, 21.19, 27.09, 26.25 and 23.04%.

The risk decomposition of the equally weighted portfolio is given in table 1. We have here a paradoxical situation. Depending on the analysis, this portfolio is either well diversified

†In general, the separation principle cannot be applied in cases where the constraints on $(\mathbf{y}, \tilde{\mathbf{y}})$ are not separate. However, in this case, if φ is a strict minimum of $\mathcal{R}$, which is the case for the P&L volatility, then the full optimization problem (FOP) and the 'separated' optimization problem (SOP) are equivalent. Indeed, if $\mathbf{y}^\star$ is solution to (SOP), then $(\mathbf{y}^\star, \varphi(\mathbf{y}^\star))$ is obviously a feasible point for (FOP). Conversely, if $(\mathbf{y}^{\star\star}, \tilde{\mathbf{y}}^{\star\star})$ is solution to (FOP), then $\mathcal{R}$ being convex, any local optimum has to be a global optimum. Therefore, we must have

$$\tilde{\mathcal{R}}(\mathbf{y}^{\star\star}) = \mathcal{R}(\mathbf{y}^{\star\star}, \tilde{\mathbf{y}}^{\star\star}) = \mathcal{R}(\mathbf{y}^{\star\star}, \varphi(\mathbf{y}^{\star\star})).$$

And, since φ is a strict minimum, $(\mathbf{y}^{\star\star}, \tilde{\mathbf{y}}^{\star\star}) = (\mathbf{y}^{\star\star}, \varphi(\mathbf{y}^{\star\star}))$ and thus, $\mathbf{B}^+\mathbf{y}^{\star\star} + \tilde{\mathbf{B}}^+\varphi(\mathbf{y}^{\star\star}) \succeq \mathbf{0}$.

‡For example, it is the case for value-at-risk and expected shortfall considering elliptic distributions or kernel estimation.

§See appendix A.2 in Meucci (2007).

(cf. table 1, top panel), or is a risk concentrated portfolio (cf. table 1, bottom panel). The equally weighted portfolio produces a risk-balanced portfolio in terms of assets' risk contributions, but not in terms of factors' risk contributions. Indeed, the first factor represents more than 80% of the risk of the portfolio.

Let us examine what happens if we want to build a portfolio with more balanced risk across the factors. For instance, if $\mathbf{b} = (49\%, 25\%, 25\%)$, we obtain the results given in table 2. The corresponding portfolio solution presents positive weights, but this is not always the case. Indeed, another risk profile, e.g. $\mathbf{b} = (19\%, 40\%, 40\%)$ reported in table 3, leads to undesirable results. The solution theretofore obtained is a long/short portfolio with a short position in the first asset. In table 4, we also report the solution to the optimization problem (14) obtained by imposing the asset weights of the portfolio are positive. At the optimum, the objective function is not equal to zero, meaning that there is no solution to the matching problem (13).

3.2. *Minimizing the risk concentration between the risk factors*

In the previous section, we saw that the solution to the matching problem (13) does not necessarily exist. Of course, we could always solve optimization problem (14) and consider the optimal solution without verifying that the objective function is equal to zero. In this case, we would obtain the portfolio for which the risk contributions are the closest to the risk budgets in the sense of the $\mathcal{L}_2$ absolute norm.† There is no reason however for this approximate solution to present any desirable property such as positive weights or even present a factor risk profile resembling the budget objectives.

If a goal of the portfolio building exercise is to seek diversification of the risk sources, we could instead consider a related problem

$$\mathcal{RC}\left(\mathcal{F}_j\right) \simeq \mathcal{RC}\left(\mathcal{F}_k\right)$$

A natural approach in the previous framework is to set $b_j = b_k$ and solve this problem as a specific case of a risk budget with equal risk factor contributions. We run the risk however of encountering the type of difficulties described above. We propose instead to minimize the concentration between the risk contributions $(\mathcal{RC}(\mathcal{F}_1), \ldots, \mathcal{RC}(\mathcal{F}_m))$.

3.2.1. Concentration indexes. Let $p \in \mathbf{R}^n_+$ such that $\mathbf{1}^\top p = 1$. p is then a probability distribution. A probability distribution p^+ is perfectly concentrated if there exists one observation i_0 such that $p^+_{i_0} = 1$ and $p^+_i = 0$ if $i \neq i_0$. Conversely, a probability distribution p^- such that $p^-_i = 1/n$ for all $i = 1, \ldots, n$ has no concentration. A concentration index is a mapping function $\mathcal{C}(p)$ such that $\mathcal{C}(p)$ increases with concentration and verifies $\mathcal{C}\left(p^-\right) \le \mathcal{C}(p) \le \mathcal{C}\left(p^+\right)$. Eventually, this index could be normalized such that $\mathcal{C}\left(p^-\right) = 0$ and $\mathcal{C}\left(p^+\right) = 1$.

In our context, the vector p represents the risk contributions of the portfolio. $\mathcal{C}(p)$ will then measure the risk concentration of the portfolio with respect to the risk factors. The most popular methods to measure the concentration are the Herfindahl index and the Gini index. Another interesting statistic is the Shannon entropy which measures diversity, the opposite of concentration. Their definition is given below.

The Herfindahl index The Herfindahl index associated to p is defined as

$$\mathcal{H}(p) = \sum_{i=1}^{n} p_i^2$$

This index takes the value 1 for a probability distribution p^+ and $1/n$ for a distribution with uniform probabilities. To scale the statistics onto [0, 1], we consider the normalized index $\mathcal{H}^\star(p)$ defined as follows

$$\mathcal{H}^\star(p) = \frac{n\mathcal{H}(p) - 1}{n - 1}$$

The Gini index The Gini index measures the distance between the Lorenz curve of p and the Lorenz curve of p^-. Its analytical expression is

$$\mathcal{G}(p) = \frac{2\sum_{i=1}^{n} i p_{i:n}}{n\sum_{i=1}^{n} p_{i:n}} - \frac{n+1}{n}$$

with $\{p_{1:n}, \ldots, p_{n:n}\}$ the ordered statistics of $\{p_1, \ldots, p_n\}$. We verify that $\mathcal{G}\left(p^-\right) = 0$ and $\mathcal{G}\left(p^+\right) = 1 - 1/n$.

The Shannon entropy It is defined as follows

$$\mathcal{I}(p) = -\sum_{i=1}^{n} p_i \ln p_i$$

When the Shannon entropy is used to measure the diversity, we prefer to consider the statistic $\mathcal{I}^\star(p) = \exp(\mathcal{I}(p))$. We notice that $\mathcal{I}^\star\left(p^-\right) = n$ and $\mathcal{I}^\star\left(p^+\right) = 1$.

Remark 5 The entropy measure $\mathcal{I}^\star$ is sometimes interpreted as the degree of diversification, because it represents the true number of bets of the portfolio (Meucci 2009). Another equivalent measure is the inverse of the Herfindal index.

3.2.2. Comparison of the concentration index minimization approach to factor ERC solution. We call *factor ERC solution* the solution to the sequential programming problem (SQP)

$$\min_{\mathbf{y}} \quad \sum_{j=1}^{m}\left(\mathcal{RC}\left(\mathcal{F}_j\right) - \frac{1}{m}\tilde{\mathcal{R}}(\mathbf{y})\right)^2. \tag{21}$$

The objective of this SQP can be rewritten in terms of the total risk and the Herfindahl index applied to the relative risk contribution of the portfolio:

$$\sum_{j=1}^{m}\left(\mathcal{RC}\left(\mathcal{F}_j\right) - \frac{1}{m}\tilde{\mathcal{R}}(\mathbf{y})\right)^2 = \tilde{\mathcal{R}}^2((\mathbf{y})\left[\mathcal{H}\left(\mathcal{RC}^\star(\mathcal{F})\right) - \frac{1}{m}\right] \tag{22}$$

where the vector $\mathcal{RC}^\star(\mathcal{F})$ is simply $\left(\mathcal{RC}^\star(\mathcal{F}_1), \ldots, \mathcal{RC}^\star(\mathcal{F}_m)\right)^\top$ and $\mathcal{RC}^\star\left(\mathcal{F}_j\right) = \mathcal{RC}\left(\mathcal{F}_j\right)/\tilde{\mathcal{R}}(\mathbf{y}))$. Solving the unconstrained SQP (21) leads to the same solution as $\min_{\mathbf{y}} \mathcal{H}\left(\mathcal{RC}^\star(\mathcal{F})\right)$, $\min_{\mathbf{y}} \mathcal{G}\left(\mathcal{RC}^\star(\mathcal{F})\right)$ or $\max_{\mathbf{y}} \mathcal{I}^\star(\mathcal{RC}^\star(\mathcal{F}))$. However, when one introduces the general bound constraints (20),

†If we prefer to use a $\mathcal{L}_2$ relative norm, we could replace the objective function as follows

$$\left(\mathbf{y}^\star, \tilde{\mathbf{y}}^\star\right) = \arg\min_{\mathbf{y}} \sum_{j=1}^{m}\sum_{k=1}^{m}\left(\frac{\mathcal{RC}\left(\mathcal{F}_j\right)}{b_j} - \frac{\mathcal{RC}\left(\mathcal{F}_k\right)}{b_k}\right)^2$$

Table 2. Matching the risk budgets (49%, 25%, 25%).

Optimal solution $\left(y^{\star}, \tilde{y}^{\star}\right)$				
$\sigma(y) = 21.27\%$	y_i (%)	$\mathcal{MR}(\mathcal{F}_i)$ (%)	$\mathcal{RC}(\mathcal{F}_i)$ (%)	$\mathcal{RC}^{\star}(\mathcal{F}_i)$ (%)
$\mathcal{F}_1$	93.38	11.16	10.42	49.00
$\mathcal{F}_2$	24.02	22.14	5.32	25.00
$\mathcal{F}_3$	39.67	13.41	5.32	25.00
$\tilde{\mathcal{F}}_1$	16.39	1.30	0.21	1.00
Corresponding portfolio $x^{\star}$				
$\sigma(x) = 21.27\%$	x_i (%)	$\mathcal{MR}(\mathcal{A}_i)$ (%)	$\mathcal{RC}(\mathcal{A}_i)$ (%)	$\mathcal{RC}^{\star}(\mathcal{A}_i)$ (%)
$\mathcal{A}_1$	15.08	17.44	2.63	12.36
$\mathcal{A}_2$	38.38	23.94	9.19	43.18
$\mathcal{A}_3$	0.89	21.82	0.20	0.92
$\mathcal{A}_4$	45.65	20.29	9.26	43.54

Note: Statistics are described in table 1.

Table 3. Matching the risk budgets (19%, 40%, 40%).

Optimal solution $\left(y^{\star}, \tilde{y}^{\star}\right)$				
$\sigma(y) = 23.41\%$	y_i (%)	$\mathcal{MR}(\mathcal{F}_i)$ (%)	$\mathcal{RC}(\mathcal{F}_i)$ (%)	$\mathcal{RC}^{\star}(\mathcal{F}_i)$ (%)
$\mathcal{F}_1$	92.90	4.79	4.45	19.00
$\mathcal{F}_2$	28.55	32.79	9.36	40.00
$\mathcal{F}_3$	45.21	20.71	9.36	40.00
$\tilde{\mathcal{F}}_1$	−23.57	−0.99	0.23	1.00
Corresponding portfolio $x^{\star}$				
$\sigma(x) = 23.41\%$	x_i (%)	$\mathcal{MR}(\mathcal{A}_i)$ (%)	$\mathcal{RC}(\mathcal{A}_i)$ (%)	$\mathcal{RC}^{\star}(\mathcal{A}_i)$ (%)
$\mathcal{A}_1$	−26.19	14.13	−3.70	−15.81
$\mathcal{A}_2$	32.69	21.21	6.94	29.63
$\mathcal{A}_3$	14.28	20.41	2.91	12.45
$\mathcal{A}_4$	79.22	21.79	17.26	73.73

Note: Statistics are described in table 1.

Table 4. Imposing the long-only constraint with $b = (19\%, 40\%, 40\%)$.

Optimal solution $\left(y^{\star}, \tilde{y}^{\star}\right)$				
$\sigma(y) = 21.82\%$	y_i (%)	$\mathcal{MR}(\mathcal{F}_i)$ (%)	$\mathcal{RC}(\mathcal{F}_i)$ (%)	$\mathcal{RC}^{\star}(\mathcal{F}_i)$ (%)
$\mathcal{F}_1$	89.85	6.89	6.19	28.37
$\mathcal{F}_2$	23.13	28.67	6.63	30.40
$\mathcal{F}_3$	47.02	19.12	8.99	41.20
$\tilde{\mathcal{F}}_1$	2.53	0.26	0.01	0.03
Corresponding portfolio $x^{\star}$				
$\sigma(x) = 21.82\%$	x_i (%)	$\mathcal{MR}(\mathcal{A}_i)$ (%)	$\mathcal{RC}(\mathcal{A}_i)$ (%)	$\mathcal{RC}^{\star}(\mathcal{A}_i)$ (%)
$\mathcal{A}_1$	0.00	15.90	0.00	0.00
$\mathcal{A}_2$	32.83	22.03	7.23	33.15
$\mathcal{A}_3$	0.00	20.51	0.00	0.00
$\mathcal{A}_4$	67.17	21.72	14.59	66.85

Note: Statistics are described in table 1.

the solution to (21) will now differ from using concentration indexes. Without explicitly solving these constrained optimization problems, on the one hand, one can see that minimizing the $\mathcal{L}_2$-norm of the deviation from the factor ERC solution (21) is a competition between two imperatives:

- minimizing the overall risk (through the term $\tilde{\mathcal{R}}^2(\mathbf{y})$);
- minimizing a 'distance' to the factor ERC solution (through the term $\mathcal{H}\left(\mathcal{RC}^{\star}(\mathcal{F})\right) - \frac{1}{m} \geq 0$).

On the other hand, minimizing the Herfindahl index simply leads to minimizing the 'distance' to the factor ERC solution (as the $\mathcal{RC}^{\star}(\mathcal{F})$ vector with the smallest $\mathcal{L}_2$-norm). Depending on the original risk measure $\mathcal{R}$, and hence on φ as well,

Table 5. The lowest risk concentrated portfolio.

Optimal solution $(y^\star, \tilde{y}^\star)$				
$\sigma(y) = 21.88\%$	y_i (%)	$\mathcal{MR}(\mathcal{F}_i)$ (%)	$\mathcal{RC}(\mathcal{F}_i)$ (%)	$\mathcal{RC}^\star(\mathcal{F}_i)$ (%)
$\mathcal{F}_1$	91.97	7.91	7.28	33.26
$\mathcal{F}_2$	25.78	28.23	7.28	33.26
$\mathcal{F}_3$	42.22	17.24	7.28	33.26
$\tilde{\mathcal{F}}_1$	6.74	0.70	0.05	0.21
Corresponding portfolio $x^\star$				
$\sigma(x) = 21.88\%$	x_i (%)	$\mathcal{MR}(\mathcal{A}_i)$ (%)	$\mathcal{RC}(\mathcal{A}_i)$ (%)	$\mathcal{RC}^\star(\mathcal{A}_i)$ (%)
$\mathcal{A}_1$	0.30	16.11	0.05	0.22
$\mathcal{A}_2$	39.37	23.13	9.11	41.63
$\mathcal{A}_3$	0.31	20.93	0.07	0.30
$\mathcal{A}_4$	60.01	21.09	12.66	57.85

Note: Statistics are described in table 1.

this constrained portfolio choice problem could have several solutions.† There is no guarantee in particular that a numerical optimization algorithm would select a solution with minimum overall risk.

Similarly, one can rewrite the Gini index in terms of the overall risk $\tilde{\mathcal{R}}(\mathbf{y})$ and the relative risk factor contributions as $\mathcal{G}\left(\mathcal{RC}^\star(\mathcal{F})\right) = \frac{2}{m\tilde{\mathcal{R}}(\mathbf{y})}\sum_{j=1}^m j\cdot\left(y\frac{\partial\tilde{\mathcal{R}}}{\partial y}\right)_{j:m} - \frac{m+1}{m}$ where we used the Euler decomposition‡ of $\tilde{\mathcal{R}}$. Under the general bounds constraints (20), minimizing the Gini concentration index is a competition between a risk maximization imperative (the term in $\tilde{\mathcal{R}}(\mathbf{y})^{-1}$) and minimization of a 'distance' term $\sum_{j=1}^m j\cdot\left(y\frac{\partial\tilde{\mathcal{R}}}{\partial y}\right)_{j:m}$.

Instead of minimizing the relative risk concentration, we could prefer to maximize diversification using the Shannon entropy which can be written $\mathcal{I}^\star(\mathcal{RC}^\star(\mathcal{F})) = \tilde{\mathcal{R}}(\mathbf{y})\left(\mathcal{I}^\star\left(\mathcal{RC}(\mathcal{F})\right)\right)^{\frac{1}{\tilde{\mathcal{R}}(\mathbf{y})}}$. Maximizing the entropy criterion has two competing imperatives: a maximization of overall risk and a maximization of the diversification of the portfolio through the Shannon entropy criterion applied to the unnormalized risk contributions $\mathcal{I}^\star(\mathcal{RC}(\mathcal{F}))$ (albeit dampened by taking the $\tilde{\mathcal{R}}(\mathbf{y})$-root of the Shannon entropy).

3.2.3. An illustration. We continue our example above by minimizing the risk concentration between the three risk factors. Results are given in table 5. The solution portfolio satisfies $\mathcal{H}^\star = 0$, $\mathcal{G} = 0$ and $\mathcal{I}^\star = 3$, i.e. minimum concentration or maximum diversification, if we consider the risk contributions of the three factors as the statistics of interest p. In this case, these three criteria are equivalent. If we impose now some constraints, the optimal portfolios will differ. For instance, if we assume that the weights are larger than 10%, we obtain the optimal portfolios given in table 6. In this case, the weights depend on the criterion and the three optimization problems are not equivalent. Therefore, the Gini index puts more weight on the fourth asset than the Herfindahl index or the entropy measure.

†There could be an infinity of possible solutions.

‡The Euler decomposition is valid for $\tilde{\mathcal{R}}$ if φ is differentiable. If φ is not differentiable, the general bounds constraints implies that the problem is not separable in terms of $\mathbf{y}$ and $\tilde{\mathbf{y}}$. However, one can choose $\tilde{\mathbf{B}}^+$ such that the factors $\{\mathcal{F}\}_{j=1}^m$ are uncorrelated with $\{\tilde{\mathcal{F}}\}_{j=1}^{n-m}$. The expressions for the Gini index and the Shannon entropy will then include, respectively, an additive constant and a positive multiplicative constant which will not change the outcome of the optimization programs.

3.3. *Solving some invariance problems*

Choueifaty *et al.* (2013) suggest that a diversified portfolio should verify a desirable property called the polico invariance property:

> The addition of a positive linear combination of assets already belonging to the universe should not impact the portfolio's weights to the original assets, as they were already available in the original universe. We abbreviate 'positive linear combination' to read Polico.

We use the previous framework and introduce an asset $n+1$ which is a linear and normalized§ combination $\boldsymbol{\alpha}$ of the first n assets. In this case, the covariance matrix of the $n+1$ assets is

$$\boldsymbol{\Sigma}^{(n+1)} = \begin{pmatrix} \boldsymbol{\Sigma}^{(n)} & \boldsymbol{\Sigma}^{(n)}\boldsymbol{\alpha} \\ \boldsymbol{\alpha}^\top\boldsymbol{\Sigma}^{(n)} & \boldsymbol{\alpha}^\top\boldsymbol{\Sigma}^{(n)}\boldsymbol{\alpha} \end{pmatrix}$$

We associate the factor model with $\boldsymbol{\Omega} = \boldsymbol{\Sigma}^{(n)}$, $\mathbf{D} = \mathbf{0}$ and

$$\mathbf{A}^{(n+1)} = \begin{bmatrix} \mathbf{I}_n \\ \boldsymbol{\alpha}^\top \end{bmatrix}$$

We consider the portfolio $\mathbf{x}^{(n+1)}$ such that the risk contribution of the factors match the risk budgets $\mathbf{b}^{(n)}$. The factor weights of the matching portfolio then satisfy $\mathbf{y}^\star = \mathbf{x}^{(n)}$. We have $\mathbf{A}^{(n+1)\top}\mathbf{x}^{(n+1)} = \mathbf{y}^\star$, which implies that $x_i^{(n)} = x_i^{(n+1)} + \alpha_i x_{n+1}^{(n+1)}$ if $i \leq n$. As the overall weights of the original n assets are preserved, a RB portfolio (and so an ERC portfolio) verifies the polico invariance property if the risk budgets are expressed with respect to factors and not to assets.

Remark 6 The duplication invariance property means that the portfolio's weights do not change if we duplicate one asset.

§Implying that $\mathbf{1}^\top\boldsymbol{\alpha} = 1$.

We can easily show that this property is verified for the RB portfolio if we consider the previous risk factor framework.†

4. Applications

We consider two different applications. In the first application, we demonstrate in the hedge fund universe how the risk factor parity approach lead to portfolios with a better diversification between the principal component factors than asset-weighted or ERC allocations. In our second application, we compare the risk parity approach based on risk factors and asset classes in a strategic asset allocation perspective.

4.1. Diversifying a portfolio of hedge funds

We consider the Dow Jones Credit Suisse AllHedge index universe and more specifically its ten subindexes: (1) convertible arbitrage, (2) dedicated short bias, (3) emerging markets, (4) equity market neutral, (5) event driven, (6) fixed income arbitrage, (7) global macro, (8) long/short equity, (9) managed futures and (10) multi-strategy as the proxies for the types of asset returns one obtains when investing in hedge funds. Data on the monthly Net Asset Value (NAV) for the global index and all subindexes was downloaded from the web site http://www.hedgeindex.com.

We consider our risk factor parity approach using a volatility risk measure and statistical factors based on the principal component analysis (PCA) of the two-year covariance matrix of asset returns. While PCA factors are difficult to interpret, and one could potentially find more pertinent risk factors, they are nonetheless frequently used to classify dynamic strategies (Fung and Hsieh 1997). Furthermore, PCA factors are independent by and allow for easy characterization of the degree of diversification of the portfolio using concentration indexes.

We compare three portfolios: the asset-weighted portfolio,‡ the ERC portfolio and the factor-weighted portfolio. Each portfolio is rebalanced at the end of each month, based on the information available during the previous month. For instance, the weights of the ERC portfolio are estimated at month t using the empirical covariance matrix of the period $[t-25, t-1]$. In this case, the weights of the ERC portfolio are computed such that the risk budget assigned to each strategy is equal to 10%. For the factor-weighted portfolio, we proceed in a similar fashion. The linear factor model at month t is estimated using the empirical covariance matrix of the period $[t-25, t-1]$ and the weights are computed such that the risk budget assigned to each of the first four PCA factors is equal to 25%.

The risk decomposition with respect to the PCA factors is given in figure 1. Most of the risk is concentrated on the first PCA factor in the asset-weighted portfolio. The ERC portfolio provides a more diversified allocation than the asset-weighted portfolio. However, the ERC weights' range for a specific factor is large implying that the actual risk factor contribution varies greatly during the period of test. This is not the case with the factor-weighted portfolio. Most of the time, we succeed in targeting the assigned budgets. The corresponding asset weights are provided in figure 1(b). The ERC-weighted portfolio or factor-weighted portfolio induce naturally more turnover than the asset-weighted portfolio (since its weights are fixed), and the factor-weighted portfolio has again a higher turnover than the ERC approach.

Simulation of the performance is reported in figure 1(c) and table 7. Table 7 also reports the performance statistics and risk and measures of concentration using the risk contributions with respect to PCA factors. The ERC-weighted portfolio and factor-weighted portfolio have smaller risks (volatility, drawdown and kurtosis) than the asset-weighted portfolio, with a slight advantage for the factor-weighted portfolio overall. Moreover, the factor-weighted portfolio improves significantly the risk diversification of the ERC portfolio: according to the $\mathcal{N}^\star$ statistic, the effective number of independent factors, the ERC portfolio plays less than three independent factors, whereas the factor-weighted portfolio is exposed to four independent factors. However, the factor-weighted portfolio induces a higher monthly turnover than for the ERC portfolio (respectively 43% and 7%). We explain this result by the relatively low number of observations used to estimate the PCA factors (24 months). Using a larger window would reduce turnover at the expense of the portfolio's reactivity. This application demonstrates the trade-off between the stability of the risk factor model and the dynamic character of the allocation.

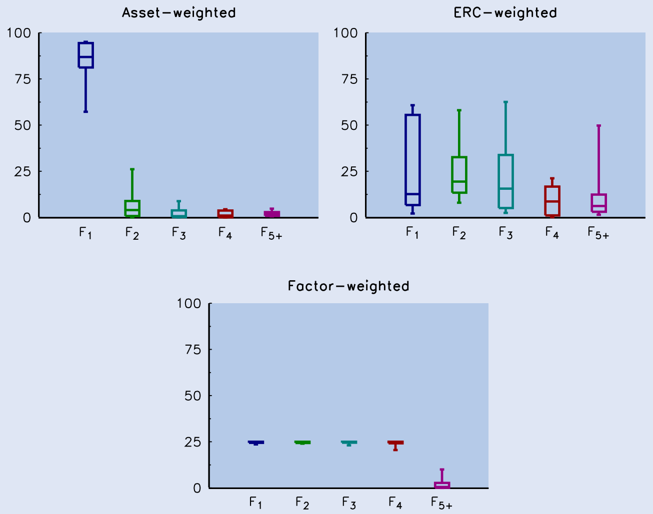
Table 6. Optimally risk diversified portfolios with investment constraint $x_i \geq 10\%$.

Criterion	$\mathcal{H}(x)$	$\mathcal{G}(x)$	$\mathcal{I}(x)$
x_1	10.00%	10.00%	10.00%
x_2	22.08%	18.24%	24.91%
x_3	10.00%	10.00%	10.00%
x_4	57.92%	61.76%	55.09%
$\mathcal{H}^\star$	0.0436	0.0490	0.0453
$\mathcal{G}$	0.1570	0.1476	0.1639
$\mathcal{I}^\star$	2.8636	2.8416	2.8643

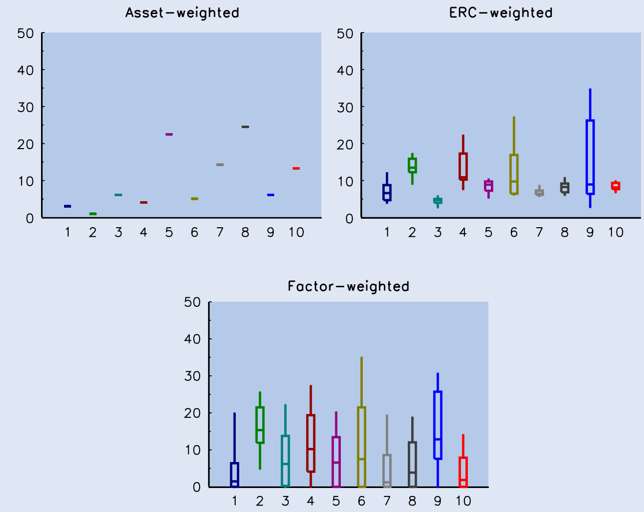
4.2. Building a strategic asset allocation

Strategic asset allocation (SAA) is the choice of equities, bonds and alternative assets that an investor wishes to hold in the long-run, usually from 10 to 50 years. Combined with tactical asset allocation and constraints on liabilities, it defines the long-term investment policy of pension funds. Typically, SAA involves long-term assumptions involving asset risk/return characteristics based on macroeconomic models and forecasts of structural factors such as population growth, productivity and inflation, as a key input to mean-variance optimization procedure. However, because of the uncertainty of these inputs and the instability of mean-variance portfolios, many institutional investors prefer to use these long-run figures as a selection

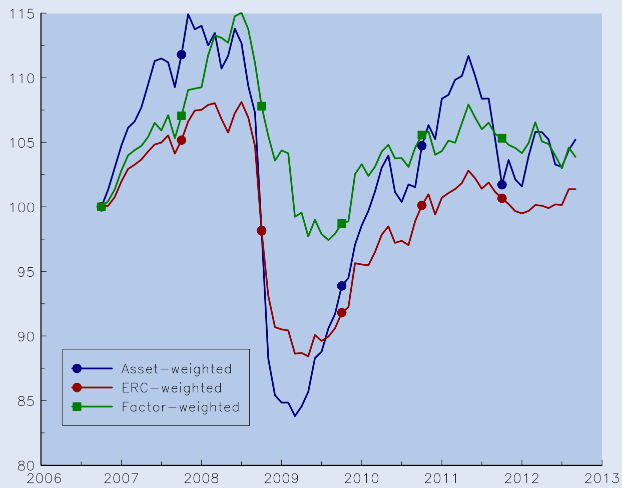
†Indeed, the duplication invariance property corresponds to a special case of the Polico invariance property, since we have $\alpha_i = 1$ for the duplicated asset.
‡The data for assets are no longer publicly available from the web site http://www.hedgeindex.com. Hence, we consider them as constant and use an estimate of the average holdings.



(a) PCA factors Risk Contributions.



(b) Asset Weights of the portfolios.



(c) Performance simulation of the portfolios.

Figure 1. Diversifying a portfolio of hedge funds using the DJ Credit Suisse AllHedge 10 subindexes as proxies for asset returns in each hedge fund strategy.

Table 7. Statistics of hedge fund portfolios (Sep. 2006–Aug. 2012).

	Asset-weighted	ERC-weighted	Factor-weighted
$\hat{\mu}_{1Y}$ (in %)	0.86	0.23	0.64
$\hat{\sigma}_{1Y}$ (in %)	7.93	4.85	4.58
$\mathcal{MDD}$ (in %)	−27.08	−18.22	−15.30
γ_1	−2.04	−1.84	−0.60
γ_2	6.24	6.88	1.68
$\mathcal{H}^\star$	0.72	0.30	0.14
$\mathcal{N}^\star$	1.40	2.96	4.16
$\mathcal{G}$	0.83	0.67	0.52
$\mathcal{I}^\star$	1.75	3.81	4.34
τ	0.00	7.45	43.34

Notes: $\hat{\mu}_{1Y}$ is the annualized performance, $\hat{\sigma}_{1Y}$ is the yearly volatility and $\mathcal{MDD}$ is the maximum drawdown observed for the entire period. These statistics are expressed in %. Skewness and excess kurtosis correspond to γ_1 and γ_2. The concentration measures are computed using the risk contributions of the first four PCA factors: $\mathcal{H}^\star$ is the normalized Herfindahl index, $\mathcal{N}^\star = \mathcal{H}^{-1}$ is the effective number of independent factors, $\mathcal{G}$ is the Gini index and $\mathcal{I}^\star$ is the diversity measure based on the Shannon entropy. The last statistic τ is the monthly turnover expressed in %.

Table 8. Weights of the four SAA portfolios.

	Equity		Sovereign Bonds		Corp. Bonds	High Yield	TIPS
	US (%)	Euro (%)	US (%)	Euro (%)	US (%)	US (%)	US (%)
#1	25	25	15	10	10	10	5
#2	10	10	25	25	10	10	10
#3	40	40	10	10	0	0	0
#4[a]	29.7	12.6	4.3	4.6	25.0	23.6	0.2

[a]Weights are estimated using the risk budgets of factors.

Table 9. Risk contributions of SAA portfolios with respect to economic factors.

Factor	#1	#2	#3	#4
Activity	68.4	48.6	73.8	60.00
Inflation	12.3	13.0	8.9	20.00
Interest rate	19.2	38.5	17.3	20.00
Specific factors	0.2	−0.1	0.1	0.00

criterion for the asset classes they would like to have in their strategic portfolio and define corresponding risk budgets.

This approach has been largely studied by Bruder and Roncalli (2012), who present, for instance, an example of the risk-budgeting policy of a pension fund. Another example of such approach is the SAA policy adopted by Danish pension fund ATP. Indeed, the fund ATP defines its SAA using a risk parity approach. According† to Henrik Gade Jepsen, CIO of ATP:

> Like many risk practitioners, ATP follows a portfolio construction methodology that focuses on fundamental economic risks, and on the relative volatility contribution from its five risk classes. [...] The strategic risk allocation is 35% equity risk, 25% inflation risk, 20% interest rate risk, 10% credit risk and 10% commodity risk.

These risk budgets are then transformed into asset classes' weights. At the end of Q1 2012, the asset allocation of ATP was also 52% in fixed income, 15% in credit, 15% in equities, 16% in inflation and 3% in commodities.‡

In this section, we combine the risk-budgeting approach to define the asset allocation and the economic approach to define the factors. Kaya *et al*. (2012) propose a similar approach and use two economic factors: growth and inflation. As explained by Eychenne *et al*. (2011), these factors are the two main pillars of SAA models. Using their long-run path, we then define the long-run path for short rates, bonds, equities, high yield, etc. This approach is adequately suited for pension funds with liabilities which are indexed on some economic factors like inflation.

Following Eychenne *et al*. (2011), we consider five economic factors grouped into three categories§

(i) activity: gdp;
(ii) inflation: consumer prices & commodity prices;
(iii) interest rate: real interest rate & slope of the yield curve.

Using quarterly data from Datastream, we estimate the factor model using YoY relative variations for the study period Q1 1999–Q4 2013 and consider seven asset classes as proxies for the asset returns ¶: equity (US and Euro), sovereign bonds (US and Euro), corporate bonds (US), High yield (US) and TIPS (US). We build four portfolios (cf. table 8). The first portfolio is a balanced 50/50 stock/bond asset mix, the second portfolio represents a defensive 20/80 allocation with only 20% invested in equities, and the third portfolio represents an aggressive 80/20 allocation with 80% invested in equities. The fourth portfolio is calibrated such that factors activity, inflation and interest rates represent, respectively, 60, 20 and 20%. In this scenario,‖ the overall weight on equities sums to 42%, while the weight on bonds sums to 68% with a large position on corporate and high yield bonds.

†*Source*: Investment & Pensions Europe, June 2012, Special Report Risk Parity.
‡*Source*: FTfm, 10 June 2012.
§We use the following variables to build the risk factors: US GDP (Overall, Total, Constant Prices, AR, SA), US Consumer Prices Index (All urban consumers, US city average, Prices, All items, SA), US Federal Funds rate (Monthly average), US Interbank three Months, US Benchmark 10 Year Datastream Government Index, and Crude Oil-Brent US Dollar per Barrel. The corresponding Datastream codes are: USGDP...D, USCONPRCE, USFDFUND, BBUSD3M, BMUS10Y(RY) and OILBREN. The yearly inflation $\mathrm{INF}(t)$ is computed using consumer prices: $\mathrm{INF}(t) = \mathrm{CPI}(t)/\mathrm{CPI}(t-4) - 1$. We can then deduce the real interest rate $\mathrm{RR}(t)$ which is equal to $(\mathrm{NR}(t) - \mathrm{INF}(t))/(1 + \mathrm{INF}(t))$ where $\mathrm{NR}(t)$ is the nominal interest rate (Fed Funds rate). The slope of the yield curve corresponds to the difference between the 10Y US Treasury interest rate and the 3M US Interbank interest rate.
¶We use the following (total return) indexes: S&P 500 Index (S&PCOMP), EURO STOXX 50 Index (DJES50I), United States Benchmark 10 Year Datastream Government Index (BMUS10Y), BD Benchmark 10 Year Datastream Government Index (BMBD10Y), Barclays US Corporate IG Index (LHCCORP), Bank of America Merrill Lynch US HY Master II Index (MLHMAU$), and Bank of America Merrill Lynch US Treasury Inflation Linked Index (MLUSGIL). All the indexes are hedged against the US Dollar and asset returns are computed using a quarterly frequency.
‖The portfolio is calibrated such that $\mathcal{RC}(\mathcal{F}_1) = 60\%$, $\mathcal{RC}(\mathcal{F}_2) + \mathcal{RC}(\mathcal{F}_3) = 20\%$ and $\mathcal{RC}(\mathcal{F}_4) + \mathcal{RC}(\mathcal{F}_5) = 20\%$. Moreover, we impose the long-only constraint $x_i \geq 0$.

In table 9, we report the risk contributions of these allocations with respect to our three categories and an additional grouping representing specific risk not explained by the economic factors. We obtain results coherent with financial and economic theories. For example, activity explains a large part of the risk of the aggressive portfolio (#3). The defensive portfolio (#2) concentrates most of the risk on interest rates. Holding a portfolio more exposed to inflation risk implies deleveraging the exposure on sovereign bonds and TIPS (cf. Portfolio #4). We believe these results to be very appealing to pension funds. And, it explains why some are exploring this route in order to aline their SAA with their liability constraints.

Remark 7 The results are highly dependent on the study period because of the time-varying relationships between economic risk factors and asset returns. Suppose, for instance, that we limit the study period to Q1 1999–Q4 2007, before the financial crisis of 2008. The allocation of Portfolio #4 becomes 11.1, 9.9, 31.3, 6.3, 30.1, 1.1 and 10.2%. Equities now represent only 21% whereas the weight of sovereign bonds is now larger than 35%. We can explain this change of allocation by a weaker relationship between equity market, activity and inflation during the 1999–2007 period due to the Internet bubble and the subprime credit expansion than over the rest of the study period.

Remark 8 The previous framework can be used with market risk factors instead of economic risk factors, for example the Fama–French–Carhart factors (value, size and momentum risk factors). In this case, the framework is closely related to the factor investing approach (Ang *et al*. 2009, Ang 2014).

5. Conclusion

This paper generalizes the risk parity approach of Bruder and Roncalli (2012) to consider risk factors instead of assets. It appears that the problem becomes trickier as multiple solutions can exist, and the existence of the RB portfolio is not guaranteed when we impose general bound constraints. We propose therefore to formulate the diversification problem in terms of risk factors as an optimization program.

We illustrate our methodology with a real-life example where we study the SAA problem of pension funds that face liability constraints. The underlying idea is to build a portfolio in order to hedge some risk factors, like activity, inflation, interest rate or currency. Our results are promising. They open a door toward rethinking the long-term investment policy of pension funds.

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Appendix 1. Decomposition of the marginal risk contribution

We consider the decomposition† given in (2). Following Meucci (2007), $\bar{\mathbf{B}}^\top$ is invertible by construction and we have

$$\begin{pmatrix} \mathbf{F}_t \\ \tilde{\mathbf{F}}_t \end{pmatrix} = \bar{\mathbf{B}}\mathbf{R}_t$$

We have therefore $\Pi_t = \mathbf{y}^\top \mathbf{F}_t + \tilde{\mathbf{y}}^\top \tilde{\mathbf{F}}_t$. Using (2) and the chain rule of calculus, it comes that

$$\begin{aligned}\frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,\mathbf{x}} &= \frac{\partial\,\mathcal{R}(\mathbf{y},\tilde{\mathbf{y}})}{\partial\,\mathbf{y}}\frac{\partial\mathbf{y}}{\partial\mathbf{x}} + \frac{\partial\,\mathcal{R}(\mathbf{y},\tilde{\mathbf{y}})}{\partial\,\tilde{\mathbf{y}}}\frac{\partial\,\tilde{\mathbf{y}}}{\partial\,\mathbf{x}} \\ &= \frac{\partial\,\mathcal{R}(\mathbf{y},\tilde{\mathbf{y}})}{\partial\,\mathbf{y}}\mathbf{B} + \frac{\partial\,\mathcal{R}(\mathbf{y},\tilde{\mathbf{y}})}{\partial\,\tilde{\mathbf{y}}}\tilde{\mathbf{B}} = \frac{\partial\,\mathcal{R}(\bar{\mathbf{y}})}{\partial\,\bar{\mathbf{y}}}\left(\bar{\mathbf{B}}^\top\right)^{-1} \qquad \text{(A1)}\end{aligned}$$

with

$$\left(\bar{\mathbf{B}}^\top\right)^{-1} = \begin{pmatrix} \mathbf{B} \\ \tilde{\mathbf{B}} \end{pmatrix} \qquad \text{(A2)}$$

for some $(n-m)\times n$ matrix $\tilde{\mathbf{B}}$. Thus, we deduce equation (3). Using (A1), it comes that

$$\frac{\partial\,\mathcal{R}(\bar{\mathbf{y}})}{\partial\,\bar{\mathbf{y}}} = \frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,\mathbf{x}}\bar{\mathbf{B}}^\top, \quad \frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,\mathbf{y}} = \frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,\mathbf{x}}\mathbf{B}^+$$

$$\text{and} \quad \frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,\tilde{\mathbf{y}}} = \frac{\partial\,\mathcal{R}(\mathbf{x})}{\partial\,\mathbf{x}}\tilde{\mathbf{B}}^+.$$

Since $\left(\mathbf{B}^+\right)^\top = \left(\mathbf{B}^\top\right)^+ = \mathbf{A}^+$ by property of the Moore–Penrose inverse, we finally obtain (4) and (5).

†We assume implicitly that the number of assets is larger than the number of factors ($n > m$).

Appendix 2. Computing the risk contribution of factors

We assume that the risk measure $\mathcal{R}(\mathbf{x})$ is the volatility of the portfolio $\sigma(\mathbf{x}) = \sqrt{\mathbf{x}^\top \boldsymbol{\Sigma} \mathbf{x}}$. A direct application of theorem 1 leads to

$$\mathcal{RC}(\mathcal{F}_j) = \frac{\left(\mathbf{A}^\top \mathbf{x}\right)_j \cdot \left(\mathbf{A}^+ \boldsymbol{\Sigma} \mathbf{x}\right)_j}{\sigma(\mathbf{x})}$$

and

$$\mathcal{RC}\left(\tilde{\mathcal{F}}_j\right) = \frac{\left(\tilde{\mathbf{B}}\mathbf{x}\right)_j \cdot \left(\tilde{\mathbf{B}}\boldsymbol{\Sigma}\mathbf{x}\right)_j}{\sigma(\mathbf{x})}$$

Appendix 3. Analysis of the budgeting optimization problem when $\mathcal{R}$ is the volatility risk measure

Let us denote $\bar{\boldsymbol{\Omega}}$ the covariance matrix between factors

$$\begin{aligned}\bar{\Omega} &= \operatorname{cov}\left(\mathbf{F}_t, \tilde{\mathbf{F}}_t\right) \\ &= \begin{pmatrix} \left(\mathbf{B}^+\right)^\top \boldsymbol{\Sigma}\mathbf{B}^+ & \left(\mathbf{B}^+\right)^\top \boldsymbol{\Sigma}\tilde{\mathbf{B}}^+ \\ \left(\tilde{\mathbf{B}}^+\right)^\top \boldsymbol{\Sigma}\mathbf{B}^+ & \left(\tilde{\mathbf{B}}^+\right)^\top \boldsymbol{\Sigma}\tilde{\mathbf{B}}^+ \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Omega} & \boldsymbol{\Gamma}^\top \\ \boldsymbol{\Gamma} & \tilde{\boldsymbol{\Omega}} \end{pmatrix} \end{aligned} \tag{C3}$$

Hence, we solve the augmented quadratic problem

$$\begin{aligned} &\min \mathbf{y}^\top \boldsymbol{\Omega}\mathbf{y} + \tilde{\mathbf{y}}^\top \tilde{\boldsymbol{\Omega}}\tilde{\mathbf{y}} + 2\,\tilde{\mathbf{y}}^\top \boldsymbol{\Gamma}\mathbf{y} \\ &\text{u.c.} \sum_{j=1}^m b_j \ln y_j \geq c \end{aligned}$$

in two steps, by first minimizing with respect to $\tilde{\mathbf{y}}$, and then with respect to $\mathbf{y}$. The solution of the first step is given by $\tilde{\mathbf{y}} = \varphi(\mathbf{y}) = -\tilde{\boldsymbol{\Omega}}^{-1}\boldsymbol{\Gamma}\mathbf{y}$. The problem is thus reduced to

$$\begin{aligned} &\min \mathbf{y}^\top \mathbf{S}\mathbf{y} \\ &\text{u.c.} \sum_{j=1}^m b_j \ln y_j \geq c \end{aligned}$$

with $\mathbf{S} = \boldsymbol{\Omega} - \boldsymbol{\Gamma}\tilde{\boldsymbol{\Omega}}^{-1}\boldsymbol{\Gamma}^\top$ the Schur complement of $\tilde{\boldsymbol{\Omega}}$. Solving this last problem leads to the general solution in terms of the asset weights

$$\mathbf{x}^\star = \mathbf{B}^+\mathbf{y}^\star + \tilde{\mathbf{B}}^+\varphi(\mathbf{y}^\star) = \left(\mathbf{B}^+ - \tilde{\mathbf{B}}^+\tilde{\boldsymbol{\Omega}}^{-1}\left(\mathbf{B}^+\right)^\top \boldsymbol{\Sigma}\tilde{\mathbf{B}}^+\right)\mathbf{y}^\star \tag{C4}$$

Appendix 4. Adding general bound constraints on the assets using volatility measure in the single-factor model

In a single risk factor model, we have $\mathbf{R}_t = \boldsymbol{\beta}\mathbf{F}_t + \boldsymbol{\epsilon}_t$ where $\mathbf{A} = \boldsymbol{\beta}$ is the vector of assets beta and $\boldsymbol{\Gamma} = 0$. Using the singular value decomposition of $\mathbf{B}$ and a well-known result (e.g. Proposition 9.2.2, Lange 2010), we derive the expression of $B^+ = \frac{1}{\|\boldsymbol{\beta}\|^2}\boldsymbol{\beta}$. The Lagrangian function of the corresponding optimization problem is given by

$$\begin{aligned}\mathcal{L}(y_1, \lambda_c, \boldsymbol{\lambda}_u, \boldsymbol{\lambda}_d) &= \sigma_F^2 y_1^2 + \lambda_c\left(c - b_1 \ln y_1\right) \\ &\quad + \left(\boldsymbol{\lambda}_u - \boldsymbol{\lambda}_d\right)^\top \mathbf{B}^+ y_1 + \boldsymbol{\lambda}_d^\top \mathbf{x}_d - \boldsymbol{\lambda}_u^\top \mathbf{x}_u. \end{aligned} \tag{D5}$$

The KKT conditions for optimality gives the following equations

$$\frac{\partial \mathcal{L}}{\partial y_1} = 2\sigma_F^2 y_1 - \lambda_c \frac{b_1}{y_1} + \frac{1}{\|\boldsymbol{\beta}\|^2}\boldsymbol{\beta}^\top\left(\boldsymbol{\lambda}_u - \boldsymbol{\lambda}_d\right) = 0 \tag{D6}$$

$$\min\left(c - b_1 \ln y_1, \lambda_c\right) = 0 \tag{D7}$$

$$\forall i \in \{1, \ldots, n\} \min\left(\frac{\beta_i}{\|\boldsymbol{\beta}\|^2} y_1 - x_{u,i}, \lambda_{u,i}\right) = 0 \tag{D8}$$

$$\forall i \in \{1, \ldots, n\} \min\left(-\frac{\beta_i}{\|\boldsymbol{\beta}\|^2} y_1 + x_{d,i}, \lambda_{d,i}\right) = 0 \tag{D9}$$

Using a similar argument to Bruder and Roncalli (2012, cf. appendix), we find that $\lambda_c > 0$ and that the constraint (D7) is always active. There are then two possible cases.

Case 1 No active constraint: $\lambda_{u,i} = \lambda_{d,i} = 0 \quad \forall i = 1, \ldots, n$. In this case, we are in a CAPM-world and the risk factor is the market portfolio. The KKT conditions lead to the following solution: $y_1^\star = \sqrt{\frac{\lambda_c}{\sigma_F^2} b_1}$. It is easy to verify in this case that the risk contribution of the portfolio is then proportional to the budgeted risk b_1.

Case 2 There exist $\mathcal{I}_d, \mathcal{I}_u \subset \{1, \ldots, n\}$ with $\mathcal{I}_d \cap \mathcal{I}_u = \{\emptyset\}$ and

- $\forall i \in \mathcal{I}_u, x_i = \left(\mathbf{B}^+ y_1\right)_i = x_{u,i}$ and $\lambda_{u,i} > 0$ and $\lambda_{d,i} = 0$;
- $\forall i \in \mathcal{I}_d, x_i = \left(\mathbf{B}^+ y_1\right)_i = x_{d,i}$ and $\lambda_{d,i} > 0$ and $\lambda_{u,i} = 0$;
- $\forall i \in \mathcal{I}_{NA} = \{1, \ldots, n\} \setminus \{\mathcal{I}_u \cup \mathcal{I}_d\}, \lambda_{d,i} = \lambda_{u,i} = 0$.

Furthermore, we have $\left(\mathbf{B}^+ y_1\right)_i = \frac{\beta_i}{\|\boldsymbol{\beta}\|^2} y_1$. If we note $y_d = \max_i\left\{\frac{x_{d,i}}{\beta_i}\right\}$ and $y_u = \min_i\left\{\frac{x_{u,i}}{\beta_i}\right\}$, the general bound constraints translate to† $y_d \leq \frac{y_1}{\|\boldsymbol{\beta}\|^2} \leq y_u$. We then distinguish three cases:

Case 2a If $y_d > y_u$, there is no solution.

Case 2b If $y_d < y_u$, then either $\frac{y_1^\star}{\|\boldsymbol{\beta}\|^2} = y_d$ or $\frac{y_1^\star}{\|\boldsymbol{\beta}\|^2} = y_u$ or else we are in the scenario *Case 1*. Hence, either $\mathcal{I}_u = \{\emptyset\}$ or $\mathcal{I}_d = \{\emptyset\}$.

Let us assume $\mathcal{I}_d = \{\emptyset\}$ and $y_1^\star = \|\boldsymbol{\beta}\|^2 y_u = \|\boldsymbol{\beta}\|^2 \min_i\left\{\frac{x_{u,i}}{\beta_i}\right\}$. Then, for all $i \in \mathcal{I}_u$, $x_i = x_{u,i} = \frac{\beta_i}{\|\boldsymbol{\beta}\|^2} y_1^\star$. We find

$$y_1^\star = \sqrt{\frac{\lambda_c b_1 - y_u \sum_{i \in \mathcal{I}_u} \beta_i \lambda_{u,i}}{\sigma_F^2}} \quad \text{and}$$

$$\mathcal{RC}(\mathcal{F}) = \lambda_c b_1 - y_u \sum_{i \in \mathcal{I}_u} \beta_i \lambda_{u,i}. \tag{D10}$$

In particular, when $\sharp\mathcal{I}_u = 1$, then there exists $i_0 \in \mathcal{I}_u$ such that $y_u = \frac{x_{u,i_0}}{\beta_{i_0}}$. The expressions in (D10) simplify to $y_1^\star = \sqrt{\frac{\lambda_c b_1 - \lambda_{u,i_0} x_{u,i_0}}{\sigma_F^2}}$ and $\mathcal{RC}(\mathcal{F}) = \lambda_c b_1 - \lambda_{u,i_0} x_{u,i_0}$.

Similarly, if we assume $\mathcal{I}_u = \{\emptyset\}$ and $y_1^\star = \|\boldsymbol{\beta}\|^2 y_d = \|\boldsymbol{\beta}\|^2 \max_i\left\{\frac{x_{d,i}}{\beta_i}\right\}$. Then, for all $i \in \mathcal{I}_d$, $x_i = x_{d,i} = \frac{\beta_i}{\|\boldsymbol{\beta}\|^2} y_1^\star$. We find

$$y_1^\star = \sqrt{\frac{\lambda_c b_1 + y_d \sum_{i \in \mathcal{I}_d} \beta_i \lambda_{d,i}}{\sigma_F^2}} \quad \text{and}$$

$$\mathcal{RC}(\mathcal{F}) = \lambda_c b_1 + y_d \sum_{i \in \mathcal{I}_d} \beta_i \lambda_{d,i}. \tag{D11}$$

Again, when $\sharp\mathcal{I}_d = 1$, then there exists $i_0 \in \mathcal{I}_d$ such that $y_d = \frac{x_{d,i_0}}{\beta_{i_0}}$. The expressions in (D11) simplify to $y_1^\star = \sqrt{\frac{\lambda_c b_1 + \lambda_{d,i_0} x_{d,i_0}}{\sigma_F^2}}$ and $\mathcal{RC}(\mathcal{F}) = \lambda_c b_1 + \lambda_{d,i_0} x_{d,i_0}$.

Case 2c If $y_d = y_u = \frac{y_1^\star}{\|\boldsymbol{\beta}\|^2}$, then $y_1^\star = \sqrt{\frac{\lambda_c b_1 - y_u\left(\sum_{i \in \mathcal{I}_u} \beta_i \lambda_{u,i} - \sum_{i \in \mathcal{I}_d} \beta_i \lambda_{d,i}\right)}{\sigma_F^2}}$ and $\mathcal{RC}(\mathcal{F}) = \lambda_c b_1 - y_u\left(\sum_{i \in \mathcal{I}_u} \beta_i \lambda_{u,i} - \sum_{i \in \mathcal{I}_d} \beta_i \lambda_{d,i}\right)$.

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†Under the assumption that $\boldsymbol{\beta} \succeq \mathbf{0}$. If a particular asset has a negative sensitivity to the risk factor, $\beta_i < 0$, then the problem can be equivalently be solved by replacing the asset x_i by its opposite $x_i' = -x_i$, and switching the bounds to $x_{u,i}^{\text{new}} = -x_{d,i}$ and $x_{d,i}^{\text{new}} = -x_{u,i}$.