# Rational expectations

School of Economics, University College Dublin

Spring 2018

**Lucas** (1976)

> *Given that the structure of an econometric model consists of optimal decision rules of economic agents, and that optimal decision rules vary systematically with changes in the structure of series relevant to the decision maker, it follows that any change in policy will systematically alter the structure of econometric models.*

$$y_t = a\tilde{y}_{t+1} + bx_t \tag{1}$$
$$|a| < 1; b \neq 0$$

$\tilde{y}_{t+1}$ is expectation of $y_{t+1}$: how to specify?

$$\tilde{y}_{t+1} = y_t \tag{2}$$

$$y_t = ay_t + bx_t \tag{3}$$
$$y_t = \frac{b}{1-a}x_t$$

$$\tilde{y}_{t+1} = \mathbb{E}_t y_{t+1} \tag{4}$$

$$y_t = a\tilde{y}_{t+1} + bx_t \tag{5}$$

$$y_t = a\mathbb{E}_t y_{t+1} + bx_t \tag{6}$$

**DSGE** models have both backward and forward looking elements: Backward looking dynamics come from identities

$$K_t = (1-\gamma)K_{t-1} + I_t \tag{7}$$

Forward looking dynamics from optimising behaviour

- What agents expect to happen tomorrow is very important for what they decide to do today.

Modeling this requires assumptions about how people formulate expectations

- DSGE approach relies on the idea that people have **rational expectations**

**Rational expectations**

> *Agents formulate expectations in such a way that their subjective probability distribution of economic variables - conditional on the available information - coincides with the objective probability distribution of the same variable - according to a measure of the state of nature- in an equilibrium.*

In other words agents

1. Use publicly available information efficiently (i.e. no systematic mistakes)
2. Understand the structure of the economy and base their expectations on this knowledge

Rational expectations (RE) is a **baseline** assumption about people's behaviour.

$$y_t = x_t + a\mathbb{E}_t y_{t+1} \tag{8}$$

Today's value of $y$ is determined by

1. $x$
2. Tomorrow's expected value of $y$

Under RE agents formulate expectations that is consistent with

$$\begin{aligned}\mathbb{E}_t y_{t+1} &= \mathbb{E}_t x_{t+1} + a\mathbb{E}_t\mathbb{E}_{t+1} y_{t+2} \\ \mathbb{E}_t y_{t+1} &= \mathbb{E}_t x_{t+1} + a\mathbb{E}_t y_{t+2}\end{aligned} \tag{9}$$

$$\mathbb{E}_t\mathbb{E}_{t+1} y_{t+2} = \mathbb{E}_t y_{t+2} \tag{10}$$

This is the **Law of Iterated Expectations**

- It is not rational for me to expect to have a different expectation next period for $y_{t+2}$ than the one that I have today (based on the information set available at $t$).

Use repeated substitution to get solution for model

$$y_t = x_t + a\mathbb{E}_t x_{t+1} + a^2\mathbb{E}_t y_{t+2} \tag{11}$$

Include additional periods

$$\begin{aligned} y_t &= x_t + a\mathbb{E}_t x_{t+1} + \ldots + a^{N-1}\mathbb{E}_t x_{t+N-1} + a^N\mathbb{E}_t y_{t+N} \\ &= \sum_{k=0}^{N-1} a^k\mathbb{E}_t x_{t+k} + a^N\mathbb{E}_t y_{t+N} \end{aligned} \tag{12}$$

Assuming

$$\lim_{N\to\infty} a^N\mathbb{E}_t y_{t+N} = 0 \tag{13}$$

We get

$$y_t = \sum_{k=0}^{\infty} a^k\mathbb{E}_t x_{t+k} \tag{14}$$

## Asset pricing

1. Asset with price $p_t$, yielding dividend $d_t$
2. Close alternative with guaranteed rate of return $r$

Risk neutral investor will only hold the asset if the yield is the same as the rate of return

$$\frac{d_t + \mathbb{E}_t p_{t+1}}{p_t} = 1 + r \tag{15}$$

Rearranging

$$\frac{d_t + \mathbb{E}_t p_{t+1}}{p_t} = 1 + r \tag{16}$$

Gives

$$p_t = \frac{d_t}{1+r} + \frac{\mathbb{E}_t p_{t+1}}{1+r} \tag{17}$$

Repeated substitution solution

$$p_t = \sum_{k=0}^{\infty} \left( \frac{1}{1+r} \right)^{k+1} \mathbb{E}_t d_{t+k} \tag{18}$$

**Dividend-discount model**: Asset prices should equal a discounted present-value sum of expected future dividends.

## Backward solution

$$y_t = x_t + a\mathbb{E}_t y_{t+1} \tag{19}$$
$$y_t = x_t + a y_{t+1} + a\epsilon_{t+1}$$

Forecast error $\epsilon_{t+1}$ cannot be predicted at time $t$.
Can move time index back one period

$$\begin{aligned} y_{t-1} &= x_{t-1} + a y_t + a\epsilon_t \\ a y_t &= y_{t-1} - x_{t-1} - a\epsilon_t \\ y_t &= a^{-1} y_{t-1} - a^{-1} x_{t-1} - \epsilon_t \end{aligned} \tag{20}$$

$$y_t = -\sum_{k=0}^{\infty} a^{-k}\epsilon_{t-k} - \sum_{k=1}^{\infty} a^{-k} x_{t-k} \tag{21}$$

**Forward & backward**

$$y_t = \sum_{k=0}^{\infty} a^k \mathbb{E}_t x_{t+k}$$

$$y_t = -\sum_{k=0}^{\infty} a^{-k} \epsilon_{t-k} - \sum_{k=1}^{\infty} a^{-k} x_{t-k}$$

Both are correct; which one we use depends on $a$

$$|a| > 1$$

Weights on forward solution are explosive

- Forward solution will not converge to finite sum
- $y_t$ depends more on values of $x_t$ far in the distant future than on today's values

$\Rightarrow$ Use backward solution; will be indeterminate though

$$\mathbb{E}_{t-1}\epsilon_t = 0 \tag{22}$$

Cannot predict $y_t$ even if we know full path of $x_t$

$$|a| < 1$$

Weights on backward solution are explosive; use forward solution

- Path of $x_t$ will tell path of $y_t$

Most cases assume that $|a| < 1$

$$\lim_{N\to\infty} a^N \mathbb{E}_t y_{t+N} = 0 \tag{23}$$

$y_t$ can't grow too fast (*transversality condition*)

**Rational bubbles**

If transversality condition is not imposed, model can have any other solution

$$y_t^* = \sum_{k=0}^{\infty} a^k \mathbb{E}_t x_{t+k} \tag{24}$$

Consider other solution

$$y_t = y_t^* + b_t \tag{25}$$

With bubble component $b_t$ model must satisfy

$$y_t^* + b_t = x_t + a\mathbb{E}_t y_{t+1}^* + a\mathbb{E}_t b_{t+1} \tag{26}$$

Model is by construction

$$y_t^* = x_t + a\mathbb{E}_t y_{t+1}^* \tag{27}$$

Bubble component will satisfy

$$b_t = a\mathbb{E}_t b_{t+1} \tag{28}$$

Because $|a| < 1$, $|b|$ is always expected to get larger

$$b_t = \mathbb{E}_t \left[ \frac{1}{1+r} b_{t+1} \right] \tag{29}$$

$b_t$ grows in expectations at rate $r$

There may be restrictions in the real economy that stop $b$ growing forever

- e.g. constant growth

Following would do the trick

$$b_{t+1} \begin{cases} (aq)^{-1} b_t + e_{t+1} & with & \Pr(q) \\ e_{t+1} & with & Pr(1-q) \end{cases} \tag{30}$$

With

$$\mathbb{E}_t e_{t+1} = 0 \tag{31}$$

Imposing

$$\lim_{N \to \infty} a^N \mathbb{E}_t y_{t+N} = 0 \tag{32}$$

rules out bubbles of this or any other form.

**Structural to reduced form**

$$y_t = \sum_{k=0}^{\infty} a^k \mathbb{E}_t x_{t+k} \tag{33}$$

Requires assumption on how $x_t$ evolves over time

- Otherwise cannot be used for predictions on $y_t$ dynamics

Assumption that exogenous variables $x_t$ are generated by backward-looking time-series models

- Therefore strong link between DSGE and VAR models

Consider $x_t$ with DGP

$$x_t = \rho x_{t-1} + \epsilon_t \tag{34}$$
$$|\rho| < 1$$

$$\mathbb{E}_t x_{t+k} = \rho^k x_t \tag{35}$$

Model solution given by

$$y_t = \left[ \sum_{k=0}^{\infty} (a\rho)^k \right] x_t \tag{36}$$

Assuming $|a\rho| < 1$: infinite sum converges to

$$\sum_{k=0}^{\infty} (a\rho)^k = \frac{1}{1 - a\rho} \tag{37}$$

Gives reduced-form solution

$$y_t = \frac{1}{1 - a\rho} x_t \tag{38}$$

Can be combined with equation describing $x_t$ and simulated on computer

**Reduced-form** model has VAR-like representation

$$y_t = \frac{1}{1 - a\rho}(\rho x_{t-1} + \epsilon_t) \tag{39}$$

$$= \rho y_{t-1} + \frac{1}{1 - a\rho}\epsilon_t \tag{40}$$

$x_t, y_t$ series have purely backward-looking representations

- Theoretical models tend to predict that data can be described by VAR

## DSGE recipe

1. Obtain structural equations involving expectations of future driving variables

$$\mathbb{E}_t x_{t+k}$$

2. Make assumptions about time-series process for the driving variables

$$x_t$$

3. Solve for reduced-form solution that can be simulated on a computer

$$y_t = \frac{1}{1 - a\rho} x_t$$

**Permanent income hypothesis**

Let consumption depend on present discounted value of after-tax income

$$c_t = \gamma \sum_{k=0}^{\infty} \beta^k \mathbb{E}_t y_{t+k} \tag{41}$$

Income generating process is

$$y_t = (1+g) y_{t-1} + \epsilon_t \tag{42}$$

$$\mathbb{E}_t y_{t+k} = (1+g)^k y_t \tag{43}$$

Reduced-form representation

$$c_t = \gamma \left[ \sum_{k=0}^{\infty} (\beta(1+g))^k \right] y_t \tag{44}$$

Assuming $\beta(1+g)^k < 1$ this becomes

$$c_t = \frac{\gamma}{1-\beta(1+g)} y_t \tag{45}$$

**Lucas critique**

Two important implications

1. Structural equation is always true for the model

$$c_t = \gamma \sum_{k=0}^{\infty} \beta^k \mathbb{E}_t y_{t+k}$$

2. Reduced-form depends on process $y_t$ taking particular form

$$c_t = \frac{\gamma}{1 - \beta(1+g)} y_t$$

Reduced-form parameters change when $\mathbb{E}_t y_{t+k}$ changes

- Reduced-form models using historical data useless for policy analysis

**Temporary tax cut**

Suppose following parameter values

$$g = 0.02; \beta = 0.95$$

If consumers expect $g$, tax cut will increase consumption by

$$\frac{\gamma}{1 - \beta(1+g)} = \frac{\gamma}{1 - 0.95(1 + 0.02)} \approx 32\gamma$$

Under rational expectations increase will be

$$\gamma$$

Off by a factor of 32.

## VAR vs. DSGE

VARs fit data well but cannot be used for policy analysis

- VARs do not allow reduced-form correlations to change over time

DSGE can as it includes policy equations based on rational expectations

i. Relate interest rates to inflation and unemployment
ii. Dependence of fiscal variables on other macro variables
iii. Exchange rate regime

DSGE can explain pattern as result of structural changes in policy rules

**Jump variables**

$$y_t = \sum_{k=0}^{\infty} a^k \mathbb{E}_t x_{t+k}$$

- Depend on what happens today and what's expected to happen tomorrow
- Will jump if expectations about future change
- Past does not restrict their movement

Useful to characterise stock prices

For real economy use **second-order stochastic difference equations**

$$y_t = ay_{t-1} + b\mathbb{E}_t y_{t+1} + x_t \tag{46}$$

How to solve these? Suppose there is value for $\lambda$ such that

$$v_t = y_t - \lambda y_{t-1} \tag{47}$$

follows SDE of form

$$v_t = \alpha \mathbb{E}_t v_{t+1} + \beta x_t \tag{48}$$

Can solve for $v_t$ and back out the values for $y_t$

Given

$$y_t = v_t + \lambda y_{t-1} \tag{49}$$

Rewrite as

$$\begin{aligned} v_t + \lambda y_{t-1} &= a y_{t-1} + b(\mathbb{E}_t v_{t+1} + \lambda y_t) + x_t \\ &= a y_{t-1} + b\mathbb{E}_t v_{t+1} + b\lambda(v_t + \lambda y_{t-1}) + x_t \end{aligned} \tag{50}$$

Rearrange to

$$(1 - b\lambda)v_t = b\mathbb{E}_t v_{t+1} + x_t + (b\lambda^2 - \lambda + a)y_{t-1} \tag{51}$$

By definition $\lambda$ is such that the $v_t$ it defined followed a first-order SDE: $\lambda$ satisfies

$$b\lambda^2 - \lambda + a = 0 \tag{52}$$

This is a quadratic equation so there are two values for $\lambda$ that satisfy it.

For either value can characterise $v_t$ by

$$v_t = \frac{b}{1-b\lambda}\mathbb{E}_t v_{t+1} + \frac{1}{1-b\lambda}x_t \tag{53}$$

$$= \frac{1}{1-b\lambda}\sum_{k=0}^{\infty}\left(\frac{b}{1-b\lambda}\right)^k \mathbb{E}_t x_{t+k} \tag{54}$$

$y_t$ obeys

$$y_t = \lambda y_{t-1} + \frac{1}{1-b\lambda}\sum_{k=0}^{\infty}\left(\frac{b}{1-b\lambda}\right)^k \mathbb{E}_t x_{t+k} \tag{55}$$

Usually have one potential value for which $|\lambda| < 1$: gives unique stable solution

**Systems of RE equations:**

1. Generalise solution for single equation to vector

$$Z_t = \begin{pmatrix} z_{1t} \\ z_{2t} \\ . \\ z_{nt} \end{pmatrix} \tag{56}$$

2. Specify as macroeconomic model

$$Z_t = B\mathbb{E}_t Z_{t+1} + X_t \tag{57}$$

   $B$ is an $n$ $x$ $n$ matrix

3. Get solution using repeated substitution

$$Z_t = \sum_{k=0}^{\infty} B^k \mathbb{E}_t X_{t+k} \tag{58}$$

**NB**- Will give stable non-explosive solution under certain conditions

## Eigenvalues

$\lambda_i$ is an eigenvalue of the matrix $B$ if there exists a vector $e_i$ such that

$$Be_i = \lambda_i e_i \tag{59}$$

Many $n$ x $n$ matrices have $n$ eigenvalues

- Denote $P$ which has columns $n$ eigenvectors corresponding to these eigenvalues

$$BP = P\Omega \tag{60}$$

$$\Omega = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & . & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix} \tag{61}$$

$\Omega$ is a diagonal matrix of eigenvalues.

## Stability condition

$$BP = P\Omega \tag{62}$$

Implies

$$B = P\Omega P^{-1} \tag{63}$$

Provides information on relation between eigenvalues and higher powers of $B$

$$B^n = P\Omega^n P^{-1} = P \begin{pmatrix} \lambda_1^n & 0 & 0 & 0 \\ 0 & \lambda_2^n & 0 & 0 \\ 0 & 0 & . & 0 \\ 0 & 0 & 0 & \lambda_n^n \end{pmatrix} P^{-1} \tag{64}$$

Higher powers of $B$ depend on eigenvalues taken to power $n$

$B^n$ will tend towards zero as $n \to \infty$

- If all of the eigenvalues are in the unit circle $|\lambda| < 1$

Therefore we can get unique stable forward-looking solution for

$$Z_t = B\mathbb{E}_t Z_{t+1} + X_t \tag{65}$$

if the eigenvalues of $B$ are all inside the unit circle.

## Calculating eigenvalues

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{66}$$

Suppose two eigenvalues for $A$: $\lambda_1, \lambda_2$

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{67}$$

There are eigenvectors when multiplied by $A - \lambda I$ equal a vector of zeroes: determinant of matrix

$$A - \lambda I = \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda_2 \end{pmatrix} \tag{68}$$

will equal zero: solving quadratic formula will give the two eigenvalues of A

$$(a_{11} - \lambda_1 a_{12})(a_{22} - \lambda_2) - a_{12} a_{21} = 0 \tag{69}$$

## Binder-Pesaran method

$$Z_t = AZ_{t-1} + B\mathbb{E}_t Z_{t+1} + HX_t \tag{70}$$

One-lag/one-lead form is only for illustration

- Equation summarises all possible linear RE models

Binder & Pesaran (1996) solved this model in manner exactly analogous to second-order SDE, for

$$W_t = Z_t - CZ_{t-1} \tag{71}$$

find matrix $C$ such that (70) obeys a first-order matrix

$$W_t = F\mathbb{E}_t W_{t+1} + GX_t \tag{72}$$

i.e. turn second-order into simpler first-order

To determine $C$ can use fact that

$$Z_t = W_t + CZ_{t-1} \tag{73}$$

Can be rewritten as

$$\begin{aligned} W_t + CZ_{t-1} &= AZ_{t-1} + B(\mathbb{E}_t W_{t+1} + CZ_t) + HX_t \\ &= AZ_{t-1} + B(\mathbb{E}_t W_{t+1} + C(W_t + CZ_{t-1})) + HX_t \end{aligned} \tag{74}$$

Rearranges to

$$(I - BC)W_t = B\mathbb{E}_t W_{t+1} + (BC^2 - C + A)Z_{t-1} + HX_t \tag{75}$$

$C$ is matrix such that $W_t$ follow a first-order forward-looking matrix equation

- No extra $Z_{t-1}$ terms

Follows that

$$BC^2 - C + A = 0 \tag{76}$$

This equation can be solved to give $C$: which is non-trivial

Can use

$$C = BC^2 + A \tag{77}$$

and solve reiteratively:

1. Provide initial guess $C_0 = I$
2. Iterate on $C_n = BC_{n-1}^2 + A$ until all entries in $C_n$ converge

Once we know $C$, we have

$$W_t = F\mathbb{E}_t W_{t+1} + GX_t \tag{78}$$

where

$$F = (I - BC)^{-1}B \tag{79}$$
$$G = (I - BC)^{-1}H \tag{80}$$

Has a stable forward-looking solution

- Assuming all eigenvalues of $F$ are inside the unit circle

$$W_t = \sum_{k=0}^{\infty} F^k \mathbb{E}_t(GX_{t+k}) \tag{81}$$

**Reduced form representation**
Sub. (81) in original equation

$$Z_t = CZ_{t-1} + \sum_{k=0}^{\infty} F^k \mathbb{E}_t(GX_{t+k}) \tag{82}$$

Suppose variables $X_t$ can be represented using VAR

$$X_t = DX_{t-1} + \epsilon_t \tag{83}$$

$D$ has eigenvalues inside unit circle, implies

$$\mathbb{E}_t X_{t+k} = D^k X_t \tag{84}$$

Model solution is given by

$$Z_t = CZ_{t-1} + \left[ \sum_{k=0}^{\infty} F^k G D^k \right] X_t \tag{85}$$

Infinite sum in (85) will converge to matrix $P$: model has reduced-form representation

- Can be simulate along with the VAR process driving $X_t$

$$Z_t = CZ_{t-1} + PX_t \tag{86}$$

Recipe for simulating DSGE model:

1. Specify matrices $A, B, H$

$$Z_t = AZ_{t-1} + B\mathbb{E}_t Z_{t+1} + HX_t$$

2. Solve for $C, F, G$

$$\begin{aligned} W_t &= Z_t - CZ_{t-1} \\ W_t &= F\mathbb{E}_t W_{t+1} + GX_t \end{aligned}$$

3. Specify a VAR process for the driving variables

$$X_t = DX_{t-1} + \epsilon_t$$

4. Obtain the reduced-form representations

$$Z_t = CZ_{t-1} + PX_t$$

Finally, there are often multiple values for different variables at $t$: Can estimate

$$KZ_t = AZ_{t-1} + B\mathbb{E}_t Z_{t+1} + HX_t \tag{87}$$

Computer will multiply both sides by $K^{-1}$ and solve using Binder-Pesaran model.

$$Z_t = K^{-1}AZ_{t-1} + K^{-1}B\mathbb{E}_t Z_{t+1} + K^{-1}HX_t \tag{88}$$

Need to figure out what model implies for $K, A, B, H$; computer will provide following representation

$$Z_t = CZ_{t-1} + PX_t \tag{89}$$
$$X_t = DX_{t-1} + \epsilon_t \tag{90}$$

These can be used for further calculations