The Real Business Cycle Model

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An RBC model

The basic RBC model assumes perfectly functioning competitive markets (along with rational expectations). This means that the outcomes generated by decentralized decisions by firms and households can be replicated as the solution to a social planner problem. The social planner wants to maximise

C_t is consumption, N_t hours worked, and β is the household's rate of time preference

$$E_t \left[\sum_{i=0}^{\infty} \beta^i (U(C_{t+i}) - V(N_{t+i})) \right]$$

Where

$$U(C_t) - V(N_t) = \frac{C_t^{1-\eta}}{1-\eta} - \nu N_t$$

The economy faces constraints by

$$Y_t = C_t + I_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha}$$
$$K_t = I_t + (1-\gamma) K_{t-1}$$

Where the process for the technology term A_t is usually a log-linear AR(1) process

For simplicity here we assume that A_t does not trend over time, so the model economy has an average growth rate of zero. A^* indicates the steady-state for technology.

$$lnA_t = (1-\rho) lnA^* + \rho lnA_{t-1} + \epsilon_t$$

Solving the RBC model

Solving the RBC model involves a number of steps

1. Formulating the Lagrangean
2. Finding the first order conditions (FOCs)
3. Log-linearisation of the FOCs
4. Finding the steady-state

Now the social planner faces two constraints,

$$Y_t = C_t + I_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha}$$
$$K_t = I_t + (1-\gamma) K_{t-1}$$

which we can simplify by combining them into one equation

$$A_t K_{t-1}^{\alpha} N_t^{1-\alpha} = C_t + K_t - (1-\gamma) K_{t-1}$$

This problem can be formulated as a Langrangian problem which involves picking a series of values for consumption and labour, subject to satisfying a series of constraints of the form just described:

$$L = E_t \sum_{i=0}^{\infty} \beta^i [U(C_{t+i}) - V(N_{t+i})] +$$
$$E_t \sum_{i=0}^{\infty} \beta^i \lambda_{t+i} [A_t K_{t+i-1}^{\alpha} N_t^{1-\alpha} + (1-\gamma)K_{t+i-1} - C_{t+i} - K_{t+i}]$$

Now note that this equation sums to infinity, so there is an infinite number of first-order conditions for current and expected values of C_t, K_t, N_t. However, things can be simplified by looking at when exactly the time t and $t+n$ variables appear.

Note that the derivative for capital is $\frac{\delta L}{\delta K_t}$, so we just need to find out when K_t appears.

$$U(C_t) - V(N_t) + \lambda_t (A_t K_{t-1}^{\alpha} N_t^{1-\alpha} - C_t - K_t + (1-\gamma)K_{t-1})$$
$$+\beta E_t [\lambda_{t+1} (A_{t+1} K_t^{\alpha} N_{t+1}^{1-\alpha} + (1-\gamma)K_t)]$$

- t variables only appear once: their FOCs consist of differentiating the model end setting the derivatives equal to zero
- $t+n$ appear exactly as the t variables, only in expectation form and multiplied by discount β^n: their FOCs are identical to the t variables.

Differentiating we get the following FOCs

Recall that

$$Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha}$$

$$\frac{\delta L}{\delta C_t} : U'(C_t) - \lambda_t = 0$$
$$\frac{\delta L}{\delta K_t} : -\lambda_t + \beta E_t \left[\lambda_{t+1} \left(\alpha \frac{Y_{t+1}}{K_t} + 1 - \gamma \right) \right] = 0$$
$$\frac{\delta L}{\delta N_t} : -V'(N_t) + (1-\alpha)\lambda_t \frac{Y_t}{N_t} = 0$$
$$\frac{\delta L}{\delta \lambda_t} : A_t K_{t-1}^{\alpha} N_t^{1-\alpha} - C_t - K_t + (1-\gamma)K_{t-1} = 0$$

The Keynes-Ramsey condition

Now in order to make the system a bit easier to understand, it helps to define the marginal value of an additional unit of capital next year as

$$R_{t+1} = \alpha \frac{Y_{t+1}}{K_t} + 1 - \gamma$$
$$FOC : \lambda_t = \beta E_t (\lambda_{t+1} R_{t+1})$$

This can then be combined with the FOC for consumption to give

$$U'(C_t) = \beta E_t [U'(C_{t+1}) R_{t+1}]$$

This means that

$$\frac{\Delta L}{\Delta C_t} = U'(C_t) - \lambda_t = 0$$
$$\lambda_t = U'(C_t)$$

- The marginal utility of consumption must equal the marginal utility of capital
- And the marginal utility of capital must equal the expected value of capital at $t+1$ times the return of capital times a discount factor

  The interpretation of the Keynes-Ramsey condition is that

- A Δ decrease in consumption today will lead to a loss of $U'(C_t)\Delta$ in utility
- Invest to get $R_{t+1}\Delta$ tomorrow
- Which is worth $\beta E_t[U'(C_{t+1})R_{t+1}]$ in terms of today's utility.
- Along an optimal path, the household must be indifferent

CCRA Consumption and Separable Consumption-Leisure

The model uses the utility function

$$U(C_t) - V(N_t) = \frac{C_t^{1-\eta}}{1-\eta} - \nu N_t$$

This formulation of the Constant Relative Risk Aversion (CRRA) utility from consumption and separate disutility from labour turns out to be necessary for the model to have a stable growth path solution.

The Keynes-Ramsey condition becomes

$$C_t^{-\eta} = \beta E_t(C_{t+1}^{-\eta} R_{t+1})$$

And the condition for optimal worked hours becomes

$$-\nu + (1-\alpha)C_t^{-\eta}\frac{Y_t}{N_t} = 0$$

$$\frac{Y_t}{N_t} = \frac{v}{1-\alpha}C_t^{\eta}$$

Full set of model equations

The RBC model can be defined by six equations

1. three identities describing resource constraints
2. one definition
3. and two FOCs describing optimal behaviour

Process for the technology variable is

$$lnA_t = (1-\rho)lnA^* + \rho lnA_{t-1} + \epsilon_t$$

Note that the model is a set of linear and nonlinear equations.

$$Y_t = C_t + I_t$$
$$Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha}$$
$$K_t = I_t + (1-\gamma)K_{t-1}$$
$$R_t = \alpha\frac{Y_t}{K_{t-1}} + 1 - \gamma$$
$$C_t^{-\eta} = \beta E_t(C_{t+1}^{-\eta} R_{t+1})$$
$$\frac{Y_t}{N_t} = \frac{v}{1-\alpha}C_t^{\eta}$$

Log-linearisation

Nonlinear systems can generally not be solved analytically.

The solution can be approximated however using a corresponding set of linear equations.

The idea is to use Taylor series approximation: any nonlinear function $F(x_t, y_t)$ can be approximated around any point (x_t^*, y_t^*) using the formula

$$F(x_t, y_t) = F(x_t^*, y_t^*) + F_x(x_t^*, y_t^*)(x_t - x_t^*) + F_y(x_t^*, y_t^*)(y_t - y_t^*) + \\ F_{xx}(x_t^*, y_t^*)(x_t - x_t^*)^2 + F_{xy}(x_t^*, y_t^*)(x_t x_t^*)(y_t - y_t^*) + \\ F_{yy}(x_t^*, y_t^*)(y_t - y_t^*) + ...$$

If the gap between (x_t, y_t) and (x_t^*, y_t^*) is small, then terms in second and higher order powers and cross-terms will all be very small and can be ignored leaving something like

If we linearise around point that is far away from (x_t, y_t), then the approximation will not be accurate.

$$F(x_t, y_t) \approx \alpha + \beta_1 x_t + \beta_2 y_t$$

DSGE MODEL USE A PARTICULAR VERSION OF THIS TECHNIQUE, by taking logs and linearise the logs of the variables around a steady-state path in which all real variables are growing at the same rate.

The steady-state path is relevant because the stochastic economy will, on average, tend to fluctuate around the values given by this path, making the approximation an accurate one. Remember that log-differences are approximately percentage deviations

This gives us a set of linear equations in the deviations of the logs of these variables from their steady-state values.

$$lnX - lnY \approx \frac{X - Y}{Y}$$

This approach gives us

- A system that expresses variables in terms of their percentage deviations from the steady-state paths
- A system of variables that can be thought of representing the business-cycle component of the model
- Coefficients are elasticities and IRFs are easy to interpret
- It is also easy to implement

How log-linearisation works

THE KEY TO THE LOG-LINEARIZATION METHOD is that every variable can be written as

A log-deviation of a variable from its steady-state value is noted as $x_t = lnX_t - lnX^*$.

$$X_t = X^* \frac{X_t}{X^*} = X^* e^{x_t}$$

A first-order Taylor approximation for e^{x_t} can be given by

$$e^{x_t} \approx 1 + x_t$$

Meaning that the variables can be written as

$$X_t \approx X^*(1 + x_t)$$

Additionally we can set terms like $x_t y_t = 0$ when multiplying variables since we are looking at small deviations from the steady state; multiplying these small deviations together one will get a term close to zero.

$$X_t Y_t \approx X^* Y^* (1 + x_t)(1 + y_t) \approx X^* Y^* (1 + x_t + y_t)$$

Examples log-linearisation

Start with

$$Y_t = C_t + I_t$$

Re-write as

$$Y^* e^{y_t} = C^* e^{c_t} + I^* e^{i_t}$$

Using first-order approximation this becomes

$$Y^*(1 + y_t) = C^*(1 + c_t) + I^*(1 + i_t)$$

Since the steady-state terms must obey identities so

$$Y^* = C^* + I^*$$

Cancelling these terms on both sides we get

$$Y^* y_t = C^* c_t + I^* i_t$$

Which we will write as

$$y_t = \frac{C^*}{Y^*} c_t + \frac{I^*}{Y^*} i_t$$

Now consider

$$Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha}$$

This can be re-written in terms of steady-states and log-deviations as

$$Y^* e^{y_t} = (A^* e^{a_t})(K^*)^{\alpha} e^{\alpha k_{t-1}} (N^*)^{1-\alpha} e^{(1-\alpha)n_t}$$

Again, the steady-state values obey identities so that

$$Y^* = A^*(K^*)^{\alpha}(N^*)^{1-\alpha}$$

Cancelling will give

$$e^{y_t} = e^{a_t} e^{\alpha k_{t-1}} e^{(1-\alpha)n_t}$$

Using first-order Taylor approximation this becomes

$$(1 + y_t) = (1 + \alpha_t)(1 + \alpha k_{t-1})(1 + (1-\alpha)n_t)$$

Ignoring the cross-products of the log-deviations this simplifies to

$$y_t = a_t + \alpha k_{t-1} + (1-\alpha)n_t$$

The full log-linearised model

ONCE ALL THE EQUATIONS HAVE BEEN LOG-LINEARISED , we have a system of seven equations of the form

The model used assumes that technology, the source of all long-term growth in the economy, is given by $a_t = \rho a_{t-1} + \epsilon_t$. This means that there is no trend growth in the economy, and as a result all the steady-state variables are constants.

$$\begin{aligned}
y_t &= \frac{C^*}{Y^*} c_t + \frac{I^*}{Y^*} i_t \\
y_t &= a_t + \alpha k_{t-1} + (1-\alpha)n_t \\
k_t &= \frac{I^*}{K^*} i_t + (1-\gamma)k_{t-1} \\
n_t &= y_t - \eta c_t \\
c_t &= E_t c_{t+1} - \frac{1}{\eta} E_t r_{t+1} \\
r_t &= \left(\frac{\alpha}{R^*}\frac{Y^*}{K^*}\right)(y_t - k_{t-1}) \\
a_t &= \rho a_{t-1} + \epsilon_t
\end{aligned}$$

Calculating the steady state

THE LOG-LINEARISED SYSTEM contains three variables related to the steady-state path which needs to be calculated. We do this by taking the original non-linearized RBC system and figuring out what things look like along a zero growth path. We start with the steady-state interest rate which is linked to consumption behaviour via the so called Euler equation (or Keynes-Ramsey condition)

These are $\frac{C^*}{Y^*}, \frac{I^*}{Y^*}, \frac{I^*}{K^*}, \frac{\alpha}{R^*}\frac{Y^*}{K^*}$

In the vicinity of the steady state we have that $y_t = y_{t+1} = y^*$ so $\frac{y_t}{y_{t+1}} = 1$

$$
\begin{aligned}
C_t^{-\eta} &= \beta E_t(C_{t+1}^{-\eta} R_{t+1}) \\
1 &= \beta E_t\left(\left(\frac{C_t}{C_{t+1}}\right)^{\eta} R_{t+1}\right)
\end{aligned}
$$

Because we have no trend growth in technology in our model, the steady-state features consumption, investment, and output will all take constant values with no uncertainty. In steady-state we have

$$
\begin{aligned}
C_t^* &= C_{t+1}^* = C^* \\
R^* &= \beta^{-1}
\end{aligned}
$$

In a no-growth economy, the rate of return on capital is determined by the rate of time preference.

Let's look at the rate of return on capital

$$R_t = \alpha \frac{Y_t}{K_{t-1}} + 1 - \gamma$$

In steady-state we have

$$R^* = \beta^{-1} = \alpha \frac{Y^*}{K^*} + 1 - \gamma$$

So we get

$$\frac{Y^*}{K^*} = \frac{\beta^{-1} + \gamma - 1}{\alpha}$$

Together with the steady-state interest equation this tells us that

$$
\begin{aligned}
\frac{\alpha}{R^*}\frac{Y^*}{K^*} &= \alpha\beta\left(\frac{\beta^{-1} + \gamma - 1}{\alpha}\right) \\
&= 1 - \beta(1 - \gamma)
\end{aligned}
$$

Now we only have to find the ratios for

- investment-capital
- investment-output

Here we can use the identity

$$K_t = I_t + (1-\gamma)K_{t-1}$$

This identity is in steady-state and combined with the fact that $K_t^* = K_{t-1}^* = K^*$ we get

$$\frac{I^*}{K^*} = \gamma$$

$$\begin{aligned} K^* &= I^* + (1-\gamma)K^* \\ K^* &= I^* + K^* - \gamma K^* \\ I^* &= \gamma K^* \\ \frac{I^*}{K^*} &= \gamma \end{aligned}$$

This can be combined with the previous steady-state ratio to give

$\frac{Y^*}{K^*} = \frac{\beta^{-1}+\gamma-1}{\alpha}$

$$\frac{I^*}{Y^*} = \frac{\frac{I^*}{K^*}}{\frac{Y^*}{K^*}} = \frac{\alpha\gamma}{\beta^{-1}+\gamma-1}$$

From this it follows that the consumption-output ratio must be

$$\frac{C^*}{Y^*} = 1 - \frac{\alpha\gamma}{\beta^{-1}+\gamma-1}$$

The final system

USING THE STEADY-STATE IDENTITIES, the system becomes

$$\begin{aligned} y_t &= \left(1 - \frac{\alpha\gamma}{\beta^{-1}+\gamma-1}\right) c_t + \left(\frac{\alpha\gamma}{\beta^{-1}+\gamma-1}\right) i_t \\ y_t &= a_t + \alpha k_{t-1} + (1-\alpha)n_t \\ k_t &= \gamma i_t + (1-\gamma)k_{t-1} \\ n_t &= y_t - \eta c_t \\ c_t &= E_t c_{t+1} - \frac{1}{\eta} E_t r_{t+1} \\ r_t &= (1-\beta(1-\gamma))(y_t - k_{t-1}) \\ a_t &= \rho a_{t-1} + \epsilon_t \end{aligned}$$

Once we make assumptions about the underlying parameter values a solution algorithm such as the Binder-Pesaran program can be used to obtain the reduced-form solution and simulate the model.

Criticism of the RBC approach

THE RBC MODEL IS OFTEN CRITICISED on a number of points

1. Perfect markets and rational expectations
   - Markets are not always competitive and people are not always rational (in their economic decisions)

- RBC model should be seen as a benchmark against which more complicated models can be assessed.
- Separate modeling of the decisions of firms and households to account for imperfect competition can be done

2. Monetary and fiscal policy
   - RBC models exhibit complete monetary neutrality, so there is no role at all for monetary policy, something which many people think is crucial to understanding the macroeconomy
   - Most models build on the RBC approach introducing mechanisms that are allowed to have Keynesian effects, such as sticky prices and wages
3. Skepticism about technology shocks
   - RBC models give primacy to technology shocks as the source of economic fluctuations (all variables apart from A_t are deterministic). But what are these shocks?
   - Link between long-term growth and TFP

Simulating the model

Can check the parameterizing of the the model and simulate and check the impluse response functions. The following graphs are based on a model with parameter values intended for the analysis of quarterly time series

Graphs were produced by Karl Whelan.

$$\alpha = \frac{1}{3}$$
$$\beta = 0.99$$
$$\gamma = 0.015$$
$$\rho = 0.95$$
$$\eta = 1$$

Figure 1 shows a 200-period simulation of the model and illustrates the main feature of the RBC model, namely that it can generate business cycles that don't look too far-fetched. We can notice two things here

1. The model roughly matches the observed fluctuations in output
2. The model reflects the fact that investment cycles are more volatile than consumption

Part of the early hype surrounding RBC models stemmed from the idea that the model contained important propagation mechanisms

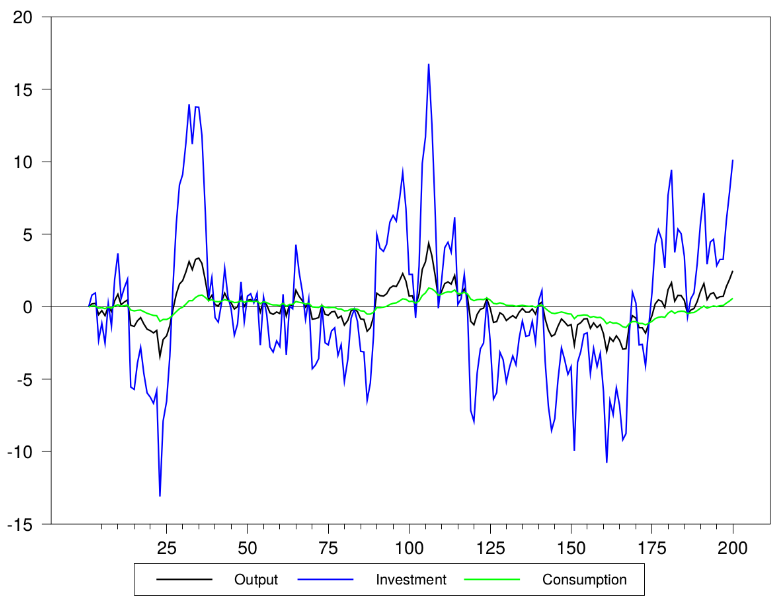

Figure 1: 200-period simulation of a RBC model

turning technology shocks into business cycles. The idea behind this is that increases in technology would lead to extra output through higher capital accumulation and by inducing people to work more. This entails, as suggested in early research, that in a world with identical technology level one would expect the RBC model still to generate business cycles. However, these propagation mechanisms are quite weak as shown in figure 2 which illustrates that fluctuations in output follow fluctuations in technology quite closely. Another issue with the RBC model is the distribution of output growth over time. Have a look at figure 3 which shows output growth in the simulated model. Notice that the growth rates are fairly random. However, as Cogley & Nason (1995) showed, in real life output growth is positively autocorrelated.

The autocorrelation coefficient is 0.34, which is relatively low. Note that the RBC can only generate positively autocorrelated output growth if the technology process is positively autocorrelated.

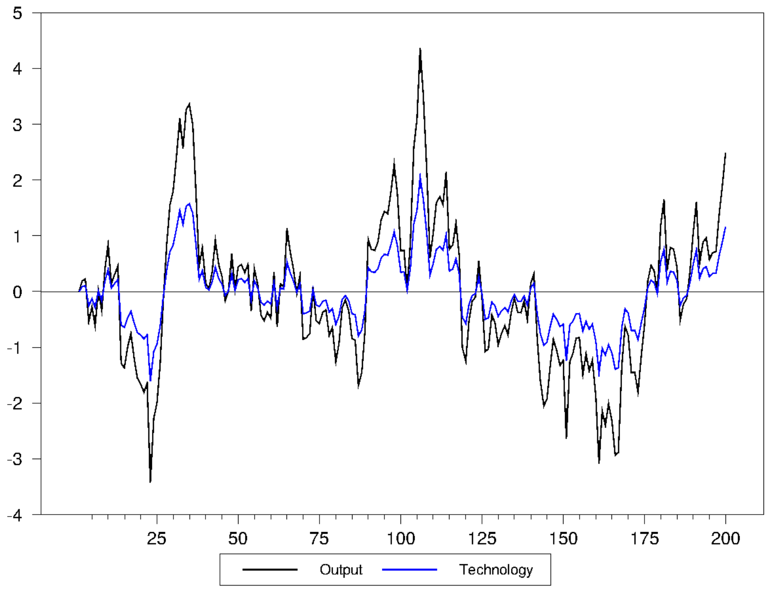

Figure 2: Link between output and technology fluctuations in RBC model

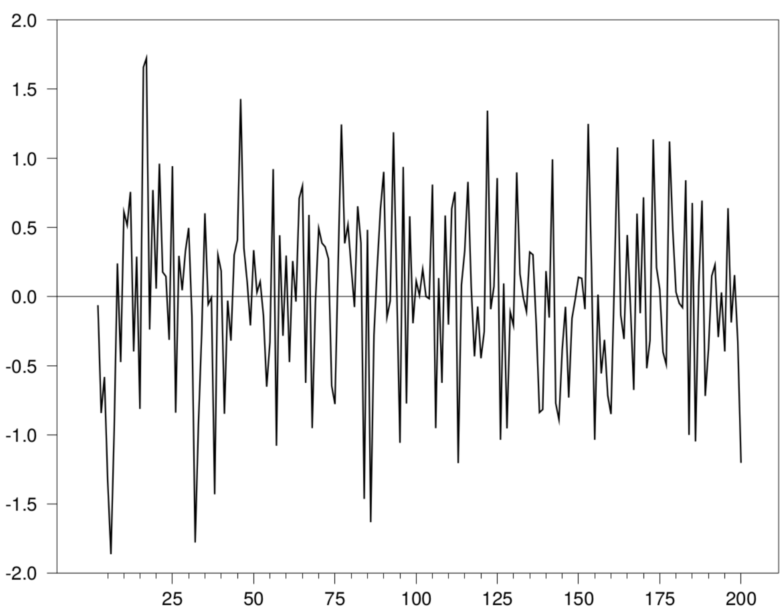

Figure 3: Output growth in a RBC model

Cogley & Nason relate this to the shape of the Impulse Response Functions, which are pretty flat and not hump-shaped (figure 4 and 5). If a model matches that fact that output growth is autocorrelated, than the IRF to a shock should be hump-shaped: a growth rate increase needs to be followed by another growth rate increase. This is not the case for instance for the technology shock, figure 5 shows that the response of output to a technology shocks is basically the same as the response of technology itself.

Note also that the RBC model does not have any other shocks in the model, such as government-spending. Cogley & Nason show that the model does not generate hump-shaped IRF to these shocks either.

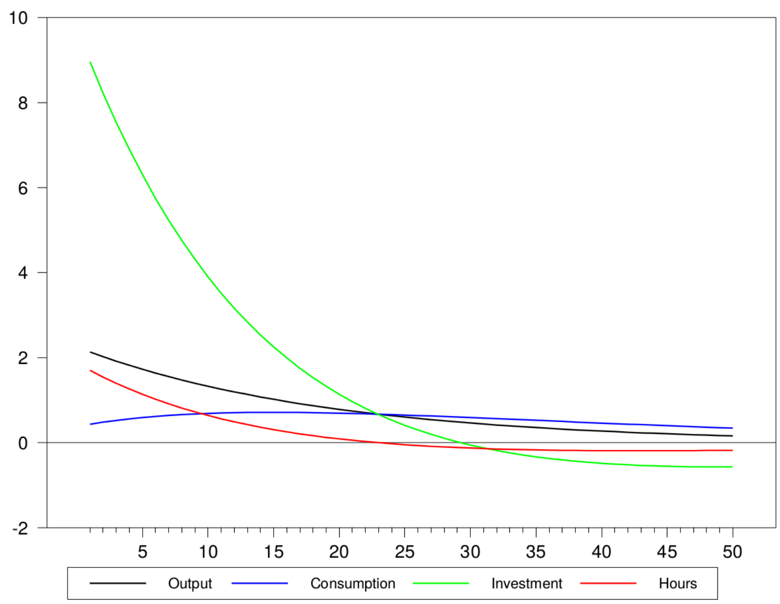

Figure 4: Response of output, consumption, investment, and hours worked to a unit shock to ϵ_t.

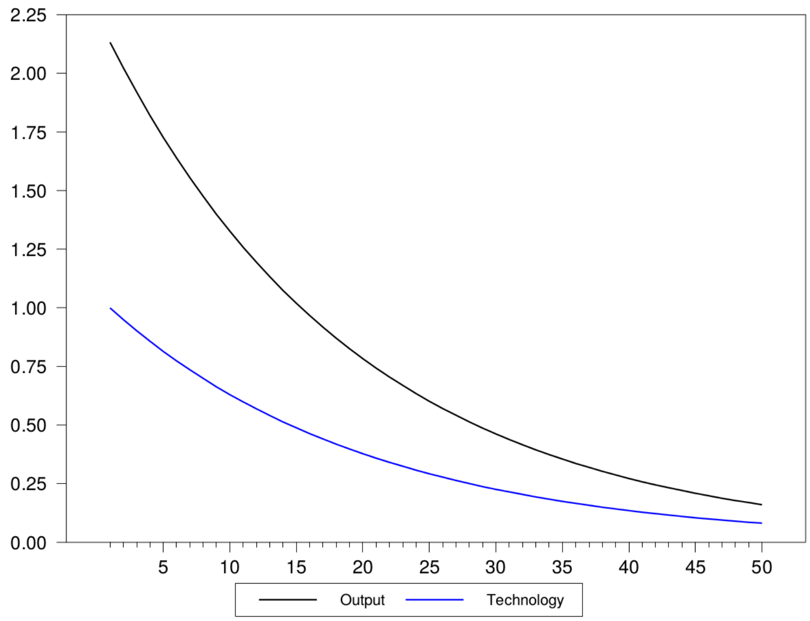

Figure 5: Response of output to technology shock.