# Kalman filter

School of Economics, University College Dublin

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**Latent variables:** Variables that are not directly observed but inferred from other variables

- Unobserved, but variable can still play important role in theoretical model

Example: Potential output

- Keynesian model: inflationary pressure determined by deviation of output from potential output
- Consider GDP increase in last quarter but no sign of inflationary pressure: potential output increased

**Q:** Can we assume that there has been a change in potential output?

**A:** Potential output probably stable from quarter to quarter while there is likely random noise fluctuations in inflation

- Signal vs. noise: Need to have a method that extracts useful signal from data that also contains lot of noise

One way to extract a signal is the **Kalman filter**

- Recursive method to estimate state of process; minimises MSE

**Conditional expectations:**
We want an estimate of the value of variable $X$

- Problem: we don't observe $X$

Do observe $Z$ which is correlated with $X$: Assume that $X, Z$ are jointly normally distributed

$$\begin{pmatrix} X \\ Z \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_X \\ \mu_Z \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \sigma_{XZ} \\ \sigma_{XZ} & \sigma_Z^2 \end{pmatrix}\right) \tag{1}$$

We get

$$\mathbb{E}(X|Z) = \mu_X + \frac{\sigma_{XZ}}{\sigma_Z^2}(Z - \mu_Z) \tag{2}$$

Alternatively, define $\rho$ as correlation between $X$ and $Z$

$$\rho = \frac{\sigma_{XZ}}{\sigma_X \sigma_Z} \tag{3}$$

Inserting in (2) we get

$$\mathbb{E}(X|Z) = \mu_X + \rho \frac{\sigma_X}{\sigma_Z}(Z - \mu_Z) \tag{4}$$

Weight put on information from $Z$ depends on

1. Correlation between $X$ and $Z$ ($\rho$)
2. The relative standard deviation ($\frac{\sigma_X}{\sigma_Z}$)

If $Z$ has high standard deviation, it is a poor signal.

## Multivariate conditional expectations

Can generalise from 2 to $n$ variables

- Let $X$ be 1 $x$ $n$ vector of variables and $Z$ 1 $x$ $m$

$$\begin{pmatrix} X \\ Z \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_X \\ \mu_Z \end{pmatrix}, \begin{pmatrix} \sum_{XX} & \sum_{XZ} \\ \sum'_{XZ} & \sum_{ZZ} \end{pmatrix}\right) \tag{5}$$

Expected value of $X$ conditional on $Z$ is

$$\mathbb{E}(X|Z) = \mu_X + \sum_{xz} \sum_{zz}^{-1} (Z - \mu_Z) \tag{6}$$

**State-space models:** Linear time-series models that mix observable and unobservable variables.

$$S_t = FS_{t-1} + u_t \tag{7}$$

**State equation** - or transition equation - describes how unobservables $S_t$ evolve over time

$$Z_t = HS_t + v_t \tag{8}$$

**Measurement equation**, relates set of observable variables $Z_t$ to unobservable variables $S_t$

**Errors:** Both $u_t$ and $v_t$ can include either

1. Normally distributed errors
2. Zeros, if the described equation is an identity

$$u_t \sim N(0, \sum^u) \tag{9}$$
$$v_t \sim N(0, \sum^v) \tag{10}$$

$\sum$ might not have full matrix rank

- Or rank deficient
- i.e. not enough information in data to estimate equation

**Estimation:** Observed data described by

$$Z_t = HS_t + v_t \tag{11}$$

Cannot observe $S_t$ but can replace it with unbiased guess based on information available at time $t$

$$S_{t|t-1} \tag{12}$$

Assume that errors are normally distributed with known covariance matrix

$$S_t - S_{t|t-1} \sim N(0, \textstyle\sum_{t|t-1}^{S}) \tag{13}$$

Can express observed variables as

$$Z_t = HS_{t|t-1} + v_t + H(S_t - S_{t|t-1}) \tag{14}$$

- $S_{t|t-1}$ is observable
- $v_t, S_t - S_{t|t-1}$ are unobservable but normally distributed

Can estimate model using ML ; with observed data

$$Z_t = HS_{t|t-1} + v_t + H(S_t - S_{t|t-1}) \tag{15}$$

Variance of error , after conditioning on $t-1$ state-variable estimate, is given by

$$v_t + H(S_t - S_{t|t-1}) \sim N(0, \Omega_t) \tag{16}$$

$$\Omega_t = \sum^{v} + H \sum_{t|t-1}^{S} H' \tag{17}$$

Parameters of model are given by

$$\theta = (F, H, \sum^{u}, \sum^{v}) \tag{18}$$

Log-likelihood function for $Z_t$ given observables at $t-1$ is

$$\begin{aligned} log\ f(Z_t|Z_{t-1}, \theta) = & -log\ 2\pi - log\ |\Omega_t| - \\ & \frac{1}{2}(Z_t - HS_{t|t-1})'\Omega_t^{-1}(Z_t - HS_{t|t-1}) \end{aligned} \tag{19}$$

Combined likelihood is given by

- Based on initial estimate of first period unobservable state $S_{1|0}$

$$f(Z_1, ..., Z_T | S_{1|0}, \theta) = f(Z_1 | S_{1|0}, \theta) \prod_{i=2}^{i=T} f(Z_i | Z_{i-1}, \theta) \qquad (20)$$

$$\begin{aligned} log\ f(Z_1, ..., Z_T | S_{1|0}, \theta) = & -T\ log\ 2\pi - \sum_{i=2}^{T} log\ |\Omega| \qquad (21) \\ & - \frac{1}{2} \sum_{i=1}^{T} (Z_i - HS_{i|i-1})' \Omega_i^{-1} (Z_i - HS_{i|i-1}) \end{aligned}$$

ML parameter estimates will be set of matrices
$\theta = (F, H, \sum^v, \sum^u)$ that provides largest value for this function

MLE will estimate model's parameters; only need an unbiased guess based on information available at $t-1$ ($S_{t|t-1}$): Use Kalman filter

- Iterative method
- Provides estimates of state variables for $t$, uses observable data for $t+1$ to update estimates

**Estimating state variables:** Formulate estimate of state variable at time $t$ given information at $t-1$

$$S_t = FS_{t-1} + u_t \Rightarrow S_{t|t-1} = FS_{t-1|t-1} \tag{22}$$

At $t-1$, expected value for the observables at $t$ are

$$Z_{t|t-1} = HS_{t|t-1} = HFS_{t-1|t-1} \tag{23}$$

At $t$ we observe $Z_t$; need to update estimate of state variable given information

$$Z_t - HFS_{t-1|t-1} \tag{24}$$

Model assumptions imply

$$\begin{pmatrix} S_t \\ Z_t \end{pmatrix} \sim N\left(\begin{pmatrix} FS_{t-1|t-1} \\ HFS_{t-1|t-1} \end{pmatrix}, \begin{pmatrix} \sum_{t|t-1}^{S} & \left(H\sum_{t|t-1}^{S}\right)' \\ H\sum_{t|t-1}^{S} & \sum^{V} + H\sum_{t|t-1}^{S} H' \end{pmatrix}\right) \tag{25}$$

Use conditional expectations to state that minimum variance unbiased estimate of $S_t|Z_t$ is

$$\mathbb{E}(S_t|Z_t) = S_{t|t} = FS_{t-1|t-1} + K_t(Z_t - HFS_{t-1|t-1}) \tag{26}$$

$K_t$ is the **Kalman gain** matrix

$$K_t = \left(H \sum_{t|t-1}^{S}\right)' \left(\sum^{V} + H \sum_{t|t-1}^{S} H'\right)^{-1} \tag{27}$$

Covariance matrices required to compute $K_t$ are updated by

$$\sum_{t|t-1}^{S} = F \sum_{t-1|t-1}^{S} F' + \sum^{U} \tag{28}$$

$$\sum_{t|t}^{S} = (I - K_t H) \sum_{t|t-1}^{S} \tag{29}$$

Initialising the Kalman filter we still need

1. Initial estimate $S_{1|0}$
2. Covariance matrix

In many macroeconomic models $S$ can be assumed to have zero mean. For the covariance matrix we use

$$\sum_{t|t-1}^{S} = F \sum_{t-1|t-1}^{S} F' + \sum^{u} \tag{30}$$

Values of this covariance matrix will converge; for unconditional covariance matrix can use value for $\sum$ that solves

$$\sum = F \sum F' + \sum^{u} \tag{31}$$

**Kalman smoother:** Kalman filter is a one-sided filter

- $\hat{S}_t$ based on data available at $t$

Good for real-time, but we often have access to full historical dataset

- Time-varying models can be estimated using Kalman smoother
- Two-sided filter using all available data to compute $\hat{S}_t$

## Hodrick-Prescott filter

$$\sum_{t=1}^{N}[(y_t - y_t^*)^2 + \lambda(\Delta y_t^* - \Delta y_{t-1}^*)] \tag{32}$$

Consider state-space model

$$y_t = y_t^* + C_t \tag{33}$$

$$\Delta y_t^* = \Delta y_{t-1}^* + \epsilon_t^g \tag{34}$$

$$C_t = \epsilon_t^c \tag{35}$$

Here

$$Var(\epsilon_t^g) = \sigma_g^2; Var(\epsilon_t^c) = \sigma_c^2 \tag{36}$$

HP filter = Kalman filter when

$$\lambda = \frac{\sigma_c^2}{\sigma_g^2} \tag{37}$$

It is assumed that $\sigma_c^2$ is 5 percentage points and $\sigma_g^2$ one-eight percentage point: $\lambda = 1600$

**Laubach & Williams** (2001): Estimate model with two unobservable time-varying series

1. Potential output
2. Natural rate of interest

$$
\begin{aligned}
\tilde{y}_t &= y_t - y_t^* \\
\tilde{y}_t &= A_y(L)\tilde{y}_{t-1} + A_r(L)(r_{t-1} - r_{t-1}^*) + \epsilon_{1t} \\
\pi_t &= B_\pi(L)\pi_{t-1} + B_y(L)\tilde{y}_{t-1} + B_x(L)x_t + \epsilon_{2t} \\
r_t^* &= cg_t + z_t \\
z_t &= D_z(L)z_{t-1} + \epsilon_{3t} \\
y_t^* &= y_{t-1}^* + g_{t-1} + \epsilon_{4t} \\
g_t &= g_{t-1} + \epsilon_{5t}
\end{aligned}
$$

Figure 1: Smoothed Estimates of Unobserved States

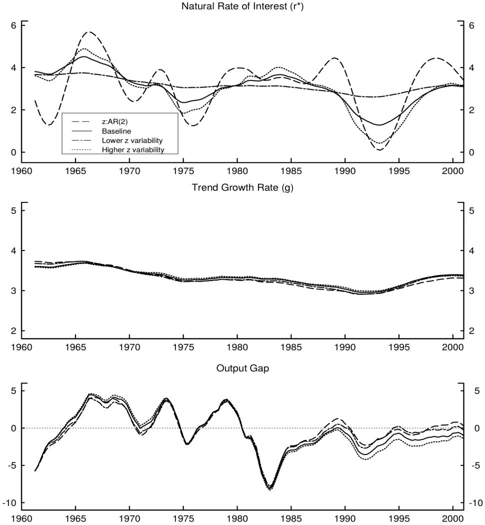

Figure 4: Real-time Estimates of the Natural Rate of Interest