The Real Business Cycle Model

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An RBC model

THE BASICS RBC MODEL ASSUMES PERFECTLY FUNCTIONING COMPETITIVE MARKETS, so the outcomes generated by decentralized decisions by firms and households can be replicated as the solution to a social planner problem. The social planner wants to maximise

C_t is consumption, N_t hours worked, and β is the household's rate of time preference

$$E_t \left[\sum_{i=0}^{\infty} \beta^i (U(C_{t+i}) - V(N_{t+i})) \right]$$

The economy faces constraints by

The process for technology term A_t is usually a log-linear AR(1) process

$$lnA_t = (1-\rho)lnA^* + \rho logA_{t-1} + \epsilon_t$$

$$Y_t = C_t + I_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha}$$
$$K_t = I_t + (1-\gamma)K_{t-1}$$

Criticism of the RBC approach

THE RBC MODEL IS OFTEN CRITICISED on a number of points

1. Perfect markets and rational expectations
   - Markets are not always competitive and surely people are not always rational (in their economic decisions)
   - RBC model should be seen as a benchmark against which more complicated models can be assessed.
   - Separate modeling of the decisions of firms and households to account for imperfect competition can be done
2. Monetary and fiscal policy
   - RBC models exhibit complete monetary neutrality, so there is no role at all for monetary policy, something which many people think is crucial to understanding the macroeconomy.
   - Most models build on the RBC approach introducing mechanisms that are allowed to have Keynesian effects, such as sticky prices and wages.
3. Skepticism about technology shocks

- RBC models give primacy to technology shocks as the source of economic fluctuations (all variables apart from A_t are deterministic). But what are these shocks?
- Link between long-term growth and TFP

Solving the RBC model

The social planner faces two constraints

$$
\begin{aligned}
Y_t &= C_t + I_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha} \\
K_t &= I_t + (1-\gamma) K_{t-1}
\end{aligned}
$$

We can simplify this by combining them into one equation

$$A_t K_{t-1}^{\alpha} N_t^{1-\alpha} = C_t + K_t - (1-\gamma) K_{t-1}$$

The problem can now be formulated as a Langrangian problem which involves picking a series of values for consumption and labour, subject to satisfying a series of constraints of the form just described:

$$L = E_t \sum_{i=0}^{\infty} \beta^i [U(C_{t+i}) - V(N_{t+i})] + E_t \sum_{i=0}^{\infty} \beta^i \lambda_{t+i} [A_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\gamma) K_{t+i-1} - C_{t+i} - K_{t+i}]$$

Given that this equation sums to infinity, there is an infinite number of first-order conditions for current and expected values of C_t, K_t, N_t. We can simplify things a bit though since the time t variables appear as follows

$$
\begin{aligned}
U(C_t) - V(N_t) + \lambda_t (A_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\gamma) K_{t-1} \\
+\beta E_t [\lambda_{t+1} (A_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\gamma) K_t)]
\end{aligned}
$$

- The t-variables only appear once. So their FOCs consist simply of differentiating the model end setting the derivatives equal to zero
- The time $t+n$ appear exactly as the t variables, only in expectation form and multiplied by discount β^n. Their FOCs are identical to the t variables.

Differentiating we get the following FOCs

$$
\begin{aligned}
&\frac{\delta L}{\delta C_t} : U'(C_t) - \lambda_t = 0 \\
&\frac{\delta L}{\delta K_t} : -\lambda_t + \beta E_t \left[\lambda_{t+1} \left(\alpha \frac{Y_{t+1}}{K_t} + 1 - \gamma \right) \right] = 0 \\
&\frac{\delta L}{\delta N_t} : -V'(N_t) + (1-\alpha) \lambda_t \frac{Y_t}{N_t} = 0 \\
&\frac{\delta L}{\delta \lambda_t} : A_t K_{t-1}^{\alpha} N_t^{1-\alpha} - C_t - K_t + (1-\gamma) K_{t-1} = 0
\end{aligned}
$$

The Keynes-Ramsey condition

DEFINE THE MARGINAL VALUE of an additional unit of capital next year as

$$R_{t+1} = \alpha \frac{Y_{t+1}}{K_t} + 1 - \gamma$$
$$FOC : \lambda_t = \beta E_t(\lambda_{t+1} R_{t+1})$$

This can be combined with the FOC for consumption to give

$$U'(C_t) = \beta E_t[U'(C_{t+1})R_{t+1}]$$

The interpretation of this rule is that

- A Δ decrease in consumption today will lead to a loss of $U'(C_t)\Delta$ in utility
- Invest to get $R_{t+1}\Delta$ tomorrow
- Which is worth $\beta E_t[U'(C_{t+1})R_{t+1}]$ in terms of today's utility.
- Along an optimal path, the household must be indifferent

CCRA Consumption and Separable Consumption-Leisure

In the model we will work with the following utility function

This formulation of the Constant Relative Risk Aversion (CRRA) utility from consumption and separate disutility from labour turns out to be necessary for the model to have a stable growth path solution.

$$U(C_t) - V(N_t) = \frac{C^{1-\eta}}{1-\eta} - aN_t$$

The Keynes-Ramsey condition becomes

$$C_t^{-\eta} = \beta E_t(C_{t+1}^{-\eta} R_{t+1})$$

And the condition for optimal worked hours becomes

$$-a + (1-\alpha)C_t^{-\eta} \frac{Y_t}{N_t} = 0$$

Full set of model equations

The RBC model can be defined by six equations

1. three identities describing resource constraints
2. one definition
3. and two FOCs describing optimal behaviour

$$
\begin{aligned}
Y_t &= C_t + I_t \\
Y_t &= A_t K_{t-1}^{\alpha} N_t^{1-\alpha} \\
K_t &= I_t + (1-\gamma)K_{t-1} \\
R_t &= \alpha \frac{Y_t}{K_{t-1}} + 1 - \gamma \\
C_t^{-\eta} &= \beta E_t (C_{t+1}^{-\eta} R_{t+1}) \\
\frac{Y_t}{N_t} &= \frac{a}{1-\alpha} C_t^{\eta}
\end{aligned}
$$

Process for the technology variable is

$$lnA_t = (1-\rho)lnA^* + \rho lnA_{t-1} + \epsilon_t$$

Note that the model is a set of linear and nonlinear equations.

Log-linearisation

NONLINEAR SYSTEMS CAN GENERALLY NOT BE SOLVED ANALYTICALLY.

The solution can be approximated however using a corresponding set of linear equations.

The idea is to use Taylor series approximation: any nonlinear function $F(x_t, y_t)$ can be approximated around any point (x_t^*, y_t^*) using the formula

$$
\begin{aligned}
F(x_t, y_t) = F(x_t^*, y_t^*) + F_x(x_t^*, y_t^*)(x_t - x_t^*) + F_y(x_t^*, y_t^*)(y_t - y_t^*) + \\
F_{xx}(x_t^*, y_t^*)(x_t - x_t^*)^2 + F_{xy}(x_t^*, y_t^*)(x_t x_t^*)(y_t - y_t^*) + \\
F_{yy}(x_t^*, y_t^*)(y_t - y_t^*) + ...
\end{aligned}
$$

If the gap between (x_t, y_t) and (x_t^*, y_t^*) is small, then terms in second and higher order powers and cross-terms will all be very small and can be ignored leaving something like

$$F(x_t, y_t) \approx \alpha + \beta_1 x_t + \beta_2 y_t$$

If we linearise around point that is far away from (x_t, y_t), then the approximation will not be accurate.

DSGE MODEL USE A PARTICULAR VERSION OF THIS TECHNIQUE, by taking logs and linearise the logs of the variables around a steady-state path in which all real variables are growing at the same rate.

The steady-state path is relevant because the stochastic economy will, on average, tend to fluctuate around the values given by this path, making the approximation an accurate one. Remember that log-differences are approximately percentage deviations

$$lnX - lnY \approx \frac{X - Y}{Y}$$

This gives us a set of linear equations in the deviations of the logs of these variables from their steady-state values.

This approach gives us

- A system that expresses variables in terms of their percentage deviations from the steady-state paths

- A system of variables that can be thought of representing the business-cycle component of the model
- Coefficients are elasticities and IRFs are easy to interpret
- It is also easy to implement

How log-linearisation works

THE KEY TO THE LOG-LINEARIZATION METHOD is that every variable can be written as

A log-deviation of a variable from its steady-state value is noted as $x_t = lnX_t - lnX^*$.

$$X_t = X * \frac{X_t}{X^*} = X^* e^{x_t}$$

A first-order Taylor approximation for e^{x_t} can be given by

$$e^{x_t} \approx 1 + x_t$$

Meaning that the variables can be written as

$$X_t \approx X^*(1 + x_t)$$

Additionally we can set terms like $x_t y_t = 0$ when multiplying variables since we are looking at small deviations from the steady state; multiplying these small deviations together one will get a term close to zero.

$$X_t Y_t \approx X^* Y^* (1 + x_t)(1 + y_t) \approx X^* Y^* (1 + x_t + y_t)$$

Examples log-linearisation

Start with

$$Y_t = C_t + I_t$$

Re-write as

$$Y^* e^{y_t} = C^* e^{c_t} + I^* e^{i_t}$$

Using first-order approximation this becomes

$$Y^*(1 + y_t) = C^*(1 + c_t) + I^*(1 + i_t)$$

Since the steady-state terms must obey identities so

$$Y^* = C^* + I^*$$

Cancelling these terms on both sides we get

$$Y^* y_t = C^* c_t + I^* i_t$$

Which we will write as

$$y_t = \frac{C^*}{Y^*}c_t + \frac{I^*}{Y^*}i_t$$

Now consider

$$Y_t = A_t K_{t-1}^{\alpha} N_t^{t-\alpha}$$

This can be re-written in terms of steady-states and log-deviations as

$$Y^* e^{y_t} = (A^* e^{a_t})(K^*)^{\alpha} e^{\alpha k_{t-1}} (N^*)^{1-\alpha} e^{(1-\alpha)n_t}$$

Again, the steady-state values obey identities so that

$$Y^* = A^*(K^*)^{\alpha}(N^*)^{1-\alpha}$$

Cancelling will give

$$e^{y_t} = e^{a_t} e^{\alpha k_{t-1}} e^{(1-\alpha)n_t}$$

Using first-order Taylor approximation this becomes

$$(1 + y_t) = (1 + \alpha_t)(1 + \alpha k_{t-1})(1 + (1-\alpha)n_t)$$

Ignoring the cross-products of the log-deviations this simplifies to

$$y_t = a_t + \alpha k_{t-1} + (1-\alpha)n_t$$

The full log-linearised model

ONCE ALL THE EQUATIONS HAVE BEEN LOG-LINEARISED , we have a system of seven equations of the form

$$\begin{aligned}
y_t &= \frac{C^*}{Y^*}c_t + \frac{I^*}{Y^*}i_t \\
y_t &= a_t + \alpha k_{t-1} + (1-\alpha)n_t \\
k_t &= \frac{I^*}{K^*}i_t + (1-\gamma)k_{t-1} \\
n_t &= y_t - \eta c_t \\
c_t &= E_t c_{t+1} - \frac{1}{\eta}E_t r_{t+1} \\
r_t &= \left(\frac{\alpha}{R^*}\frac{Y^*}{K^*}\right)(y_t - K_{t-1}) \\
a_t &= \rho a_{t-1} + \epsilon_t
\end{aligned}$$

The model used assumes that technology, the source of all long-term growth in the economy, is given by $a_t = \rho a_{t-1} + \epsilon_t$. This means that there is no trend growth in the economy, and as a result all the steady-state variables are constants.

Calculating the steady state

THE LOG-LINEARISED SYSTEM contains three variables related to the steady-state path which needs to be calculated. We do this by taking the original non-linearized RBC system and figuring out what things look like along a zero growth path. We start with the steady-state interest rate which is linked to consumption behaviour via the so called Euler equation (or Keynes-Ramsey condition)

These are $\frac{C^*}{Y^*}$, $\frac{I^*}{K^*}$, $\frac{\alpha}{R^*}\frac{Y^*}{K^*}$

$$1 = \beta E_t \left(\left(\frac{C_t}{C_{t+1}} \right)^{\eta} R_{t+1} \right)$$

Because we have no trend growth in technology in our model, the steady-state features consumption, investment, and output all taking on constant values with no uncertainty. In steady-state we have

$$C_t^* = C_{t+1}^* = C^*$$
$$R^* = \beta^{-1}$$

In a no-growth economy, the rate of return on capital is determined by the rate of time preference.

Let's look at the rate of return on capital

$$R_t = \alpha \frac{Y_t}{K_{t-1}} + 1 - \gamma$$

In steady-state we have

$$R^* = \beta^{-1} = \alpha \frac{Y^*}{K^*} + 1 - \gamma$$

So we get

$$\frac{Y^*}{K^*} = \frac{\beta^{-1} + \gamma - 1}{\alpha}$$

Together with the steady-state interest equation this tells us that

$$\begin{aligned} \frac{\alpha}{R^*}\frac{Y^*}{K^*} &= \alpha\beta \left(\frac{\beta^{-1} + \gamma - 1}{\alpha} \right) \\ &= 1 - \beta(1-\gamma) \end{aligned}$$

Next find the investment-capital and investment-output ratios. We use the identity

$$K_t = I_t + (1-\gamma)K_{t-1}$$

This identity is in steady-state and combined with the fact that $K_t^* = K_{t-1}^* = K^*$ we get

$$\frac{I^*}{K^*} = \gamma$$

This can be combined with the pervious steady-state ratio to give

$$\frac{I^*}{Y^*} = \frac{\frac{I^*}{K^*}}{\frac{Y^*}{K^*}} = \frac{\alpha\gamma}{\beta^{-1} + \gamma - 1}$$

From this it easily follows that

$$\frac{C^*}{Y^*} = 1 - \frac{\alpha\gamma}{\beta^{-1} + \gamma - 1}$$

The final system

USING THE STEADY-STATE IDENTITIES, the system becomes

$$\begin{aligned}
y_t &= \left(1 - \frac{\alpha\gamma}{\beta^{-1} + \gamma - 1}\right) c_t + \left(\frac{\alpha\gamma}{\beta^{-1} + \gamma - 1}\right) i_t \\
y_t &= a_t + \alpha k_{t-1} + (1-\alpha) n_t \\
k_t &= \gamma i_t + (1-\gamma) k_{t-1} \\
n_t &= y_t - \eta c_t \\
c_t &= E_t c_{t+1} - \frac{1}{\eta} E_t r_{t+1} \\
r_t &= (1 - \beta(1-\gamma))(y_t - k_{t-1}) \\
a_t &= \rho a_{t-1} + \epsilon_t
\end{aligned}$$

Once we make assumptions about the underlying parameter values a solution algorithm such as the Binder-Pesaran program can be used to obtain the reduced-form solution and simulate the model.