Delineating Efficient Portfolios

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Abstract

The purpose of this vignette is to demonstrate the efficient frontier of portfolios as outlined in Chapter 3 of Foundations of Risk Management.

Contents

1 Portfolio of Two Risky Assets

First, we consider a portfolio of two risky assets. The expected return for the portfolio of two assets is given by:

$$\bar{R}_P = X_A\bar{R}_A + X_B\bar{R}_B \tag{1}$$

where:

X_A is the fraction of the portfolio held in asset A

X_B is the fraction of the portfolio held in asset B

R_P is the expected return of the portfolio

R_A is the expected return of asset A

R_B is the expected return of asset B

We can impose the full investment constraint such that the fraction of the portfolio invested in asset A and the the fraction of the portfolio invested in asset B sum to 1, $X_A + X_B = 1$.

With the full investment constraint, equation (1) can be written as

$$\bar{R}_P = X_A \bar{R}_A + (1 - X_A)\bar{R_B} \tag{2}$$

Imposing the full investment constraint, the standard deviation of a portfolio with two assets is given by

$$\sigma_P = (X_A^2 \sigma_A^2 + (1 - X_A)^2 \sigma_B^2 + 2X_A(1 - X_A)\sigma_{AB})^{1/2} \tag{3}$$

where:

σ_P is the standard deviation of the portfolio

σ_A^2 is the variance of the returns of asset A

σ_B^2 is the variance of the returns of asset A

σ_{AB} is the covariance between the returns on asset A and asset B

Recall that $\sigma_{AB} = \rho_{AB}\sigma_A\sigma_B$. Substituting the equation for the covariance into equation (3) results in

$$\sigma_P = (X_A^2 \sigma_A^2 + (1 - X_A)^2 \sigma_B^2 + 2X_A(1 - X_A)\rho_{AB}\sigma_A\sigma_B)^{1/2} \tag{4}$$

This equation is useful in that it allows one to explore the effect of correlation on the portfolio returns and standard deviation.

For the following examples, unless noted otherwise, consider the following two stocks with characteristics given in the following table.

	Expected Return	Standard Deviation
Colonel Motors (C)	14%	6%
Separated Edison (S)	8%	3%

1.1 Case 1 - Perfect Positive Correlation ($\rho = +1$)

Here we calculate the portfolio expected return and standard deviation of the portfolio assuming the two assets are perfectly positively correlated.

```r
suppressPackageStartupMessages(library(GARPFRM))

# Colonel Motors expected return
R_C <- 0.14

# Colonel Motors standard deviation
sigma_C <- 0.06

# Separated Edison expected return
R_S <- 0.08
```

```r
# Separated Edison standard deviation
sigma_S <- 0.03

# Correlation coefficient
rho <- 1

# Create a vector of portfolio weights
X_C <- seq(from=0, to=1, by=0.2)

# Calculate the portfolio expected return
R_P <- portReturnTwoAsset(R_C, R_S, X_C)

## Error:  could not find function "portReturnTwoAsset"

# Calculate the portfolio standard deviation
sigma_P <- portSDTwoAsset(R_C, R_S, X_C, sigma_C, sigma_S, rho)

## Error:  could not find function "portSDTwoAsset"

# Combine the portfolio returns and standard deviations in a data.frame object.
# Note that this is the transpose t() of the data.frame object to match the
# layout of the tables in the FRM book.
df <- t(data.frame(R_P=R_P, sigma_P=sigma_P))

## Error:  object 'R_P' not found

colnames(df) <- X_C

## Error:  attempt to set 'colnames' on an object with less than two dimensions

df

## function (x, df1, df2, ncp, log = FALSE)
## {
##     if (missing(ncp))
##         .External(C_df, x, df1, df2, log)
##     else .External(C_dnf, x, df1, df2, ncp, log)
## }
## <bytecode: 0x103a71140>
## <environment: namespace:stats>
```

Now we can plot the portfolio return and standard deviation to understand the risk and return of the portfolio. Figure 1 shows that risk increases linearly with return and there is no gain by diversifying compared to purchasing the individual assets.

```r
# Create and plot the efficient frontier object
ef <- efficientFrontierTwoAsset(R_C, R_S, sigma_C, sigma_S, rho)

## Error:  could not find function "efficientFrontierTwoAsset"

plot(ef)

## Error:  object 'ef' not found
```

1.2 Case 2 - Perfect Negative Correlation ($\rho = -1$)

Here we calculate the portfolio expected return and standard deviation of the portfolio assuming the two assets are perfectly negatively correlated. Note that we just need to change the vaue of the correlation coefficient and recalculate the portfolio expected return and standard deviation.

```
# Correlation coefficient
rho <- -1

# Calculate the portfolio expected return
R_P <- portReturnTwoAsset(R_C, R_S, X_C)

## Error:  could not find function "portReturnTwoAsset"

# Calculate the portfolio standard deviation
sigma_P <- portSDTwoAsset(R_C, R_S, X_C, sigma_C, sigma_S, rho)

## Error:  could not find function "portSDTwoAsset"

# Combine the portfolio returns and standard deviations in a data.frame object.
# Note that this is the transpose t() of the data.frame object to match the
# layout of the tables in the FRM book.
df <- t(data.frame(R_P=R_P, sigma_P=sigma_P))

## Error:  object 'R_P' not found

colnames(df) <- X_C

## Error:  attempt to set 'colnames' on an object with less than two dimensions

df

## function (x, df1, df2, ncp, log = FALSE)
## {
##     if (missing(ncp))
##         .External(C_df, x, df1, df2, log)
##     else .External(C_dnf, x, df1, df2, ncp, log)
## }
## <bytecode: 0x103a71140>
## <environment: namespace:stats>
```

Now we plot the portfolio return and standard deviation to understand the risk and return of the portfolio. Figure 2 shows an interesting result, ther is a combination of assets that results in a portfolio with zero risk. This demonstrates how diversification can reduce risk when assets do not have perfect positive correlation.

```
# Create and plot the efficient frontier object
ef <- efficientFrontierTwoAsset(R_C, R_S, sigma_C, sigma_S, rho)

## Error:  could not find function "efficientFrontierTwoAsset"

plot(ef)
```

```
## Error:  object 'ef' not found
```

1.3 Case 3 - No Relationship between Returns on the Assets ($\rho = 0$)

Here we calculate the portfolio expected return and standard deviation of the portfolio assuming the two assets are uncorrelated. Note that we just need to change the vaue of the correlation coefficient and recalculate the portfolio expected return and standard deviation.

```
# Correlation coefficient
rho <- 0

# Calculate the portfolio expected return
R_P <- portReturnTwoAsset(R_C, R_S, X_C)

## Error:  could not find function "portReturnTwoAsset"

# Calculate the portfolio standard deviation
sigma_P <- portSDTwoAsset(R_C, R_S, X_C, sigma_C, sigma_S, rho)

## Error:  could not find function "portSDTwoAsset"

# Combine the portfolio returns and standard deviations in a data.frame object.
# Note that this is the transpose t() of the data.frame object to match the
# layout of the tables in the FRM book.
df <- t(data.frame(R_P=R_P, sigma_P=sigma_P))

## Error:  object 'R_P' not found

colnames(df) <- X_C

## Error:  attempt to set 'colnames' on an object with less than two dimensions

df

## function (x, df1, df2, ncp, log = FALSE)
## {
##     if (missing(ncp))
##         .External(C_df, x, df1, df2, log)
##     else .External(C_dnf, x, df1, df2, ncp, log)
## }
## <bytecode: 0x103a71140>
## <environment: namespace:stats>
```

Now we plot the portfolio return and standard deviation to understand the risk and return of the portfolio.

```
# Create and plot the efficient frontier object
ef <- efficientFrontierTwoAsset(R_C, R_S, sigma_C, sigma_S, rho)

## Error:  could not find function "efficientFrontierTwoAsset"
```

```r
plot(ef)

## Error:  object 'ef' not found
```

Figure 4 also shows that there is a point along the curve that is a portfolio with minimum risk. The minimum risk portfolio can be solved for using the equation for portfolio standard deviation. By taking the first derivative of the equation with respect to X_C, we solve for the value of X_C that minimizes portfolio risk.

$$\sigma_P = (X_C^2\sigma_C^2 + (1 - X_C)^2\sigma_S^2 + 2X_C(1 - X_C)\rho_{CS}\sigma_C\sigma_S)^{1/2} \tag{5}$$

The derivative of this equation is

$$\frac{\partial \sigma_P}{\partial X_c} = \frac{1}{2}\frac{2X_c\sigma_c^2 - 2\sigma_s^2 + 2X_c\sigma_s^2 + 2\sigma_c\sigma_s\rho_{cs} - 4X_c\sigma_c\sigma_s\rho_{cs}}{(X_c^2\sigma_c^2 + (1 - X_c)^2\sigma_s^2 + 2X_c(1 - X_c)\sigma_c\sigma_s\rho_{cs})^{1/2}} \tag{6}$$

Set the derivative equal to zero and solve for X_C.

$$X_c = \frac{\sigma_s^2 - \sigma_c\sigma_s\rho_{cs}}{\sigma_c^2 + \sigma_s^2 - 2\sigma_c\sigma_s\rho_{cs}} \tag{7}$$

We can easily write an R function to solve for the weights of the minimum risk portfolio.

```r
minRisk <- function(R_A, R_B, sigma_A, sigma_B, rho){
  top_term <- sigma_B^2 - sigma_A * sigma_B * rho
  bottom_term <- sigma_A^2 + sigma_B^2 - 2 * sigma_A * sigma_B * rho
  # x is the fraction of asset A
  x <- top_term / bottom_term
  # calculate the weights for the minimum risk portfolio
  weights <- c(x, 1-x)
  names(weights) <- c("Asset_A", "Asset_B")
  # calculate the portfolio return and standard deviation of the minimum
  # risk portfolio
  port_ret <- portReturnTwoAsset(R_A, R_B, x)
  port_sd <- portSDTwoAsset(R_A, R_B, x, sigma_A, sigma_B, rho)
  return(list(weights=weights,
              portfolio_return=port_ret,
              portfolio_sd=port_sd))
}
```

Now we can solve for the fraction of the portfolio to invest in each risky asset that minimizes risk.

```r
minRisk(R_C, R_S, sigma_C, sigma_S, rho)

## Error:  could not find function "portReturnTwoAsset"
```

1.4 Case 4 - Intermediate Risk ($\rho = 0.5$)

Here we calculate the portfolio expected return and standard deviation of the portfolio assuming the two assets have a correlation coefficient of 0.5. Note that we just need to change the vaue of the correlation coefficient and recalculate the portfolio expected return and standard deviation.

```
# Correlation coefficient
rho <- 0.5

# Calculate the portfolio expected return
R_P <- portReturnTwoAsset(R_C, R_S, X_C)

## Error:  could not find function "portReturnTwoAsset"

# Calculate the portfolio standard deviation
sigma_P <- portSDTwoAsset(R_C, R_S, X_C, sigma_C, sigma_S, rho)

## Error:  could not find function "portSDTwoAsset"

# Combine the portfolio returns and standard deviations in a data.frame object.
# Note that this is the transpose t() of the data.frame object to match the
# layout of the tables in the FRM book.
df <- t(data.frame(R_P=R_P, sigma_P=sigma_P))

## Error:  object 'R_P' not found

colnames(df) <- X_C

## Error:  attempt to set 'colnames' on an object with less than two dimensions

df

## function (x, df1, df2, ncp, log = FALSE)
## {
##     if (missing(ncp))
##         .External(C_df, x, df1, df2, log)
##     else .External(C_dnf, x, df1, df2, ncp, log)
## }
## <bytecode: 0x103a71140>
## <environment: namespace:stats>
```

Now we plot the portfolio return and standard deviation to understand the risk and return of the portfolio. The plot shows an interesting result, ther is a combination of assets that results in a portfolio with zero risk. This demonstrates how diversification can reduce risk when assets do not have perfect positive correlation.

```
# Create and plot the efficient frontier object
ef <- efficientFrontierTwoAsset(R_C, R_S, sigma_C, sigma_S, rho)

## Error:  could not find function "efficientFrontierTwoAsset"

plot(ef)
```

```
## Error:  object 'ef' not found
```

Now we can solve for the fraction of the portfolio to invest in each risky asset that minimizes risk.

```
minRisk(R_C, R_S, sigma_C, sigma_S, rho)

## Error:  could not find function "portReturnTwoAsset"
```

Figure 5 shows the relationship between portfolio expected return and standard deviation for the four cases of correlation coefficients. This plot offers insight into the shape of the efficient frontier along which all combinations of portfolio weights in the two risky assets must lie as the correlation coefficient is varied.

```
## Error:  could not find function "portReturnTwoAsset"
## Error:  could not find function "portSDTwoAsset"
## Error:  could not find function "portSDTwoAsset"
## Error:  could not find function "portSDTwoAsset"
## Error:  could not find function "portSDTwoAsset"
## Error:  plot.new has not been called yet
```

2 The Efficient Frontier with Short Sales Allowed

Here we consider the case where short selling is allowed.

```
# correlation coefficient
rho <- 0.5

# Fraction of the portfolio invested in Colonel Motors
X_C <- seq(from=-1, to=2, by=0.2)

# Calculate the portfolio expected return
R_P <- portReturnTwoAsset(R_C, R_S, X_C)

## Error:  could not find function "portReturnTwoAsset"

# Calculate the portfolio standard deviation
sigma_P <- portSDTwoAsset(R_C, R_S, X_C, sigma_C, sigma_S, rho)

## Error:  could not find function "portSDTwoAsset"

# Combine the portfolio returns and standard deviations in a data.frame object.
# Note that this is the transpose t() of the data.frame object to match the
# layout of the tables in the FRM book.
df <- t(data.frame(R_P=R_P, sigma_P=sigma_P))

## Error:  object 'R_P' not found

colnames(df) <- X_C
```

```
## Error:  attempt to set 'colnames' on an object with less than two dimensions

print(df, digits=4)

## function (x, df1, df2, ncp, log = FALSE)
## {
##     if (missing(ncp))
##         .External(C_df, x, df1, df2, log)
##     else .External(C_dnf, x, df1, df2, ncp, log)
## }
## <bytecode: 0x103a71140>
## <environment: namespace:stats>
```

Now we calculate and plot the efficient frontier with short sales allowed. For comparison, we also calculate and plot the efficient frontier for the long only case where short selling is not allowed

```
# Create and plot the efficient frontier object
ef_short <- efficientFrontierTwoAsset(R_C, R_S, sigma_C, sigma_S, rho,
                                       allowShorting=TRUE)

## Error:  could not find function "efficientFrontierTwoAsset"

ef_long <- efficientFrontierTwoAsset(R_C, R_S, sigma_C, sigma_S, rho)

## Error:  could not find function "efficientFrontierTwoAsset"

plot(ef_short)

## Error:  object 'ef_short' not found

lines(ef_long$efficient_frontier[,2], ef_long$efficient_frontier[,1],
      lty=2, lwd=2, col="blue")

## Error:  object 'ef_long' not found
```

3 Examples

This first example considers an allocation between equity and bonds.

```
# Estimated inputs for equity
R_SP <- 0.125
sigma_SP <- 0.149

# Estimated inputs for bonds
R_B <- 0.06
sigma_B <- 0.048

# Estimated correlation between equity and bonds
rho <- 0.45
```

Here we calculate the allocation to equities and bonds to achieve the portfolio which minimizes risk where risk is defined as the standard deviation of the portfolio. We see that we should short sell approximately 5.1% of equity.

```
# Calculate the allocation and values for the minimum variance portfolio
minRisk(R_SP, R_B, sigma_SP, sigma_B, rho)

## Error:  could not find function "portReturnTwoAsset"
```

Now we create and plot the efficient frontier so we can better understand the portfolio expected return and standard deviation as the allocation as we vary the allocation between equity and bonds with short selling allowed. We also consider the riskless borrowing and lending rate as 5%.

```
ef <- efficientFrontierTwoAsset(R_SP, R_B, sigma_SP, sigma_B, rho,
                                rf=0.05, allowShorting=TRUE)

## Error:  could not find function "efficientFrontierTwoAsset"

plot(ef)

## Error:  object 'ef' not found
```

For the second example, we consider an allocation between a domestic portfolio represented by the S&P index and an international portfolio represented by an average international fund. We proceed with the same analysis as the first example. We first solve for the portfolio that minimizes risk and the create and plot the efficient frontier.

```
# Estimated inputs for domestic portfolio
R_SP <- 0.125
sigma_SP <- 0.149

# Estimated inputs for international portfolio
R_int <- 0.105
sigma_int <- 0.14

# Estimated correlation between domestic and international portfolio
rho <- 0.33
```

Here we calculate the allocation to domestic stocks and international stocks to achieve the portfolio which minimizes risk where risk is defined as the standard deviation of the portfolio. We see that we should have an allocation of approximately 45% of the S&P index and 55% of the international fund for the minimum risk portfolio.

```
# Calculate the allocation and values for the minimum variance portfolio
minRisk(R_SP, R_int, sigma_SP, sigma_int, rho)

## Error:  could not find function "portReturnTwoAsset"
```

Now we create and plot the efficient frontier so we can better understand the portfolio expected return and standard deviation as the allocation as we vary the allocation between the domestic and international portfolios. We also consider the riskless borrowing and lending rate as 5%.

```
ef <- efficientFrontierTwoAsset(R_SP, R_int, sigma_SP, sigma_int, rho, rf=0.05)

## Error:  could not find function "efficientFrontierTwoAsset"

plot(ef)

## Error:  object 'ef' not found
```

4 Extended Examples for Portfolios with Multiple Assets

The previous sections all considered the case of a portfolios with two assets. Those assets, in general, could be portfolios or individual securities. The code used in this section depends on the `PortfolioAnalytics` package (Peterson et al., 2014). `PortfolioAnalytics` is an R package designed to provide numerical solutions and visualizations for portfolio problems with complex constraints and objectives.

Load weekly returns data for a set of large cap, mid cap, and small cap stocks.

```
data(crsp_weekly)
R_large <- largecap_weekly[,1:5]

## Error:  object 'largecap_weekly' not found

R_mid <- midcap_weekly[,1:5]

## Error:  object 'midcap_weekly' not found

R_small <- smallcap_weekly[,1:5]

## Error:  object 'smallcap_weekly' not found
```

Here we create and plot an efficient frontier of a selection of large cap stocks.

```
# Create and plot an efficient frontier of large cap stocks
ef_large <- efficientFrontier(R_large)

## Error:  could not find function "efficientFrontier"

plot(ef_large, main="Large Cap Efficient Frontier", cexAssets=0.6)

## Error:  object 'ef_large' not found
```

Here we add box constraints to the large cap portfolio such that each asset must have a weight greater than 15% and less than 55%.

```
# Create an efficient frontier object with box constraints
# Add box constraints such that each asset must have a weight greater than 15%
# and less than 55%
ef_large_box <- efficientFrontier(R_large, minBox=0.15, maxBox=0.55)
```

```
## Error:  could not find function "efficientFrontier"

plot(ef_large_box, main="Large Cap Efficient Frontier\nwith Box Constraints",
     cexAssets=0.6)

## Error:  object 'ef_large_box' not found
```

Here we combine the asset returns for large cap, mid cap, and small cap stocks. Next we specify group constraints for three groups such that each market cap is in its own group. We then specify weight limits on the groups. Finally, we create an efficient frontier and plot it.

```
# Combine the large cap, mid cap, and small cap stocks
R <- cbind(R_large, R_mid, R_small)

## Error:  object 'R_large' not found

# Specify the list of groups such that each market cap is in its own group
groups <- list(c(1:5), c(6:10), c(11:15))

# Specify the sum of weights for each group:
# largecap: no more than 60% and no less than 30%
# midcap: no more than 50% and no less than 20%
# smallcap: no more than 40% and no less than 15%
groupMin <- c(0.3, 0.2, 0.15)
groupMax <- c(0.6, 0.5, 0.4)

# Create the efficient frontier with the group constraints
efGroup <- efficientFrontier(R, groupList=groups,
                             groupMin=groupMin,
                             groupMax=groupMax)

## Error:  could not find function "efficientFrontier"

plot(efGroup, labelAssets=FALSE)

## Error:  object 'efGroup' not found
```

In this vignette, we have demonstrated the behavior of efficient frontiers of two assets as the correlation varies under the full investment constraint with and without short selling allowed. We have also presented the more general case for generating portfolios along an efficient frontier for more than 2 assets. We also demonstrated functionality to generate portfolios along the efficient frontier with box constraints as well as group constraints.

References

B. G. Peterson, P. Carl, R. Bennett, and K. Boudt. *PortfolioAnalytics: Portfolio Analysis, including Numerical Methods for Optimization of Portfolios*, 2014. R package version 0.9.0.