

Partial Ordering

Equivalence relations are very strong & completely break up sets, giving us info on which elements are the same etc. There are other properties Relations can have to give info too.

Def: A relation R on S is called a partial ordering if it is reflexive, anti-symmetric & transitive. (S, R) is then called a poset (partially ordered set).

Ex: $S = \mathbb{Z}$ $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a \leq b\}$

Reflexive: $a \leq a$ ✓

anti-sym: If $a \leq b$ & $b \leq a \Rightarrow a = b$ ✓

transitive: $a \leq b$ & $b \leq c \Rightarrow a \leq b \leq c \Rightarrow a \leq c$ ✓

Thus (\mathbb{Z}, \geq) is a poset.

Ex: Let $S =$ Set of all people $x R y$ iff x is older than y .

Not a Partial order why? Not reflexive.

However it is anti symmetric. \leftarrow if statement if $a R b$ & $b R a \Rightarrow a = b$
we find the hypothesis if x older than y then y not older than x !

A partial ordering says we can follow the chain in one direction only.

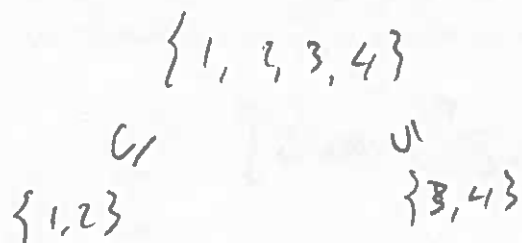
However not every thing need be related.

For partial orderings the Symbols \leq is often used.

Def: The elements a, b in the poset (S, \leq) are comparable if $a \leq b$ or $b \leq a$. Otherwise they are incomparable.

In our first example any two numbers are comparable. In the second two people are comparable if they are born at the same time.

Ex: (S, \subseteq) is a poset for any collection of sets, but many are incomparable. $S = \{ \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\} \}$



but $\{1, 2\} \nsubseteq \{3, 4\}$ are incomparable

Def: If any two elements are comparable (S, \leq) is the poset it is called a total ordering or totally ordered

Def: (S, \leq) is well ordered if it is totally ordered and there is a least (first) element.

Theorem Induction can be done over any well-ordered set.