The growth of functions

Def: Let f & g be function from $\overset{\mathbb{R}}{\mathbb{Z}} \to \mathbb{R}$. We say $f \in O(g)$ if $\exists\, C, k$ constants s.t.

$$|f(x)| \leq C \cdot |g(x)|$$

$\forall x > k$.

This is read as f is big-oh of g.

Intuitively this is saying f grows slower than g

$\exists k$ - means there is some k that after that f grows slower than g.

$\exists C$ - to get around possible differences in size.

Ex: $f(x) = x$
$g(x) = x^2 + 7$

then $\forall x > 0$ $\quad |f(x)| \leq |g(x)|$ $\quad$ so $\cancel{R} \in O(x^2+7)$

Remember this is about growth!

Ex: $f(x) = 100x + 1000$
$g(x) = x^2$.

well x^2 grows faster but is smaller than f for a while

to capture this we can say $k = 1$ $\quad C = 2000$

$\forall x > 1$ $\quad |100x + 1000| \leq 2000|x^2|$. $\quad f \in O(x^2)$.

Ex: $x^2 \notin O(x)$.

Pf by Contradiction Suppose $\exists C, k$ s.t. $x^2 \le C \cdot x \quad \forall x > k$.

AWLOG $k > 0 \Rightarrow x > 0$ so $x \le C$ which means x is bounded.

This contradicts that our inequality holds for all $x > k$.

Thus $x^2 \notin O(x)$.

$\square$.

Theorem: Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 \quad a_i \in \mathbb{R}$.

Then $f \in O(x^n)$.

Pf: $|f(x)| = |a_n x^n + \dots + a_0|$ Assume $x > \overset{k}{1}$

$\le |a_n| x^n + \dots + |a_1| x + |a_0|$ (triangle inequality).

$\le x^n \underbrace{\left(|a_n| + \dots + |a_0|\right)}_{C}$

$\square$.