CSCI 2824 - Discrete Structures
Homework 2

You MUST show your work. If you only present answers you will receive minimal credit. This homework is worth 50pts.

Due: Friday June 10

1. (2 points) Prove that for all rational numbers x and y, $x + y$ is rational.

> **Solution:** This is a direct proof. Since x is rational we can write $x = \frac{p}{q}$ for some integers p, q. Similarly we can write $y = \frac{a}{b}$ for some integers a, b. Then:
>
> $$\begin{aligned} x + y &= \frac{p}{q} + \frac{a}{b} \\ &= \frac{pb}{qb} + \frac{aq}{qb} \\ &= \frac{pb + aq}{qb} \end{aligned}$$
>
> Since a, b, p, q are all integers, $pb + aq$ and qb are integers. Thus $x + y$ is a rational number.
>
> $\square$

2. (4 points) Prove that for all real numbers x and y, if $xy \leq 2$, then either $x \leq \sqrt{2}$ or $y \leq \sqrt{2}$.

> **Solution:** Proof by contrapositive. That is we prove that if $x > \sqrt{2}$ **and** $y > \sqrt{2}$ then $xy > 2$. Simply multiply the numbers: Since $x, y > \sqrt{2}$ we have that $xy > \sqrt{2} \cdot \sqrt{2}$. Or more simply, $xy > 2$. This proves the original claim that $xy \leq 2$ implies that $x \leq \sqrt{2}$ or $y \leq \sqrt{2}$
>
> This can also be proved by contradiction, but you'll probably just contradict the assumption that $xy \leq 2$, meaning you're really doing a contrapositive argument.
>
> $\square$

3. (6 points) Prove or disprove the following:

   (a) There exist rational numbers a and b such that a^b is rational.

   > **Solution:** This is true, choose $a = b = 1$ then $a^b = 1^1 = 1$ which is rational.
   >
   > $\square$

   (b) There exist rational numbers a and b such that a^b is irrational.

   > **Solution:** This is also true. Choose $a = 2$ and $b = \frac{1}{2}$. Then $a^b = 2^{\frac{1}{2}} = \sqrt{2}$. We showed in class that $\sqrt{2}$ is irrational.
   >
   > $\square$

4. (7 points) Prove that for all *positive* integers m, n: $2m + 5n^2 = 20$ has no solution.

Solution: This solutions is more long than difficult. In order to use positive integers to satisfy the equation we must have that $1 \leq m \leq 10$. Note that 0 is NOT positive.

Similarly to satisfy the equation we must have that $1 \leq n \leq 2$.

However note that when $n = 2$ then $5n^2 = 20$ so $m = 0$, this is not allowed so the ONLY possible option for n is 1. Thus we must have $2m = 15$ (after reducing our equation). This is not possible for integral m. So no solution exists, with positive integers.

$\square$

5. (5 points) Prove that that the difference between an irrational number x and a rational number y is irrational.

Solution: Proof by contradiction. That is suppose the difference between an irrational number x and a rational number y is rational. Then we have the following circumstance:

Let $y = \frac{p}{q}$ for integers p, q, we can't write x like this but by our assumption we can write $x - y = \frac{a}{b}$ for some integers a, b, or :

$$\begin{aligned} x - y &= \frac{a}{b} \\ x - \frac{p}{q} &= \frac{a}{b} \\ x &= \frac{a}{b} + \frac{p}{q} \end{aligned}$$

And by the proof of Question 1, the sum of two rationals is rational, so that x is rational. This is a contradiction to our assumption, so it must be the case that the difference $x - y$ is NOT rational, i.e., irrational.

$\square$

6. (4 points) Prove that for all integers n if $n^3 + 5$ is odd then n is even.

Solution: We prove this by contrapostive. That is we prove that if n is odd then $n^3 + 5$ is even. If n is odd then we can write $n = 2k + 1$ for some integer k. Then we compute:

$$\begin{aligned} n^3 + 5 &= (2k+1)^3 + 5 \\ &= 8k^3 + 12k^2 + 6k + 1 + 5 \\ &= 8k^3 + 12k^2 + 6k + 6 \\ &= 2(4k^3 + 6k^2 + 3k + 3) \end{aligned}$$

Thus $n^3 + 5$ is even. This proves the original claim that if $n^3 + 5$ is odd then n is even.

$\square$

7. (8 points) Verify the following equation:

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n+1}n^2 = \frac{(-1)^{n+1}n(n+1)}{2}$$

Solution: We prove this by Induction: **Base case:** $n = 1$, $1^2 = \frac{1(2)}{2} = 1$ ✓.

IH: If $1^2 - 2^2 + \cdots + (-1)^{n+1}n^2 = \frac{(-1)^{n+1}n(n+1)}{2}$ then we show $1^2 - 2^2 + \cdots + (-1)^{n+2}(n+1)^2 = \frac{(-1)^{n+2}(n+1)(n+2)}{2}$.

For simplicity: We're trying to show that

$$\sum_{j=1}^{n}(-1)^{j+1}j^2 = \frac{(-1)^{n+1}n(n+1)}{2} \implies \sum_{j=1}^{n+1}(-1)^{j+1}j^2 = \frac{(-1)^{n+2}(n+1)(n+2)}{2}$$

Lets begin:

$$\begin{aligned}
\sum_{j=1}^{n+1}(-1)^{j+1}j^2 &= \sum_{j=1}^{n}(-1)^{j+1}j^2 + (-1)^{n+2}(n+1)^2 \\
\text{(by } \textbf{IH:}\text{)} \qquad &= \frac{(-1)^{n+1}n(n+1)}{2} + (-1)^{n+2}(n+1)^2 \\
&= \frac{(-1)^{n+1}n(n+1)}{2} + \frac{2(-1)^{n+2}(n+1)^2}{2} \\
&= \frac{(-1)^{n+1}(n^2+n) + (n^2+2n+1)(2)(-1)^{n+2}}{2}
\end{aligned}$$

Re-grouping terms:

$$= \frac{(-1)^{n+1}n^2 + (-1)^{n+1}(2)n^2 + (-1)^{n+1}n + (-1)^{n+2}(2)(2n) + (-1)^{n+2}(2)(1)}{2}$$

The next step is kind of involved. We're using that $(-1)^n + (-1)^{n+1} = 0$ for all n, since one is -1 and the other is 1. However in our case one of each term (the one with $(-1)^{n+2}$) has a factor of 2 in front, so it takes over and replaces the zero.

$$\begin{aligned}
&= \frac{(-1)^{n+2}n^2 + (-1)^{n+2}3n + (-1)^{n+2}2}{2} \\
&= \frac{(-1)^{n+2}(n+1)(n+2)}{2}
\end{aligned}$$

□

8. (6 points) Prove that $7^n - 1$ is divisible by 6 for all integers $n \geq 1$.

Solution: We prove this by induction. **Base case:** $n = 1$ gives $7 - 1 = 6$ clearly $6 \mid 6$.

IH: If $7^n - 1$ is divisible by 6 we show that $7^{n+1} - 1$ is divisible by 6.

Thus:

$$\begin{aligned}7^{n+1} - 1 &= 7^n \cdot 7 - 1\\&= 7^n \cdot (1+6) - 1\\&= 7^n - 1 + 6 \cdot 7^n\end{aligned}$$

By **IH** we have that $7^n - 1 = 6k$ for some integer k.

$$\begin{aligned}&= 6k + 6 \cdot 7^n\\&= 6(k + 7^n)\end{aligned}$$

we see that $7^{n+1} - 1 = 6p$ for an integer p. This proves our inductions hypothesis, and closes our induction.

□

9. (8 points) Prove the following by cases:

(a)

$$\max\{x, y\} = \frac{x + y + |x - y|}{2}$$

for all real numbers x and y.

Solution: We prove this by cases.
Case 1: $x \geq y$. Then $\max\{x, y\} = x$, and $x - y > 0 \implies |x - y| = x - y$. So:

$$\begin{aligned}\frac{x + y + |x - y|}{2} &= \frac{x + y + x - y}{2}\\&= \frac{2x}{2}\\&= x\end{aligned}$$

Case 2: $x \leq y$. Then $\max\{x, y\} = y$ and $x - y < 0 \implies |x - y| = -(x - y)$. So:

$$\begin{aligned}\frac{x + y + |x - y|}{2} &= \frac{x + y - (x - y)}{2}\\&= \frac{x + y - x + y}{2}\\&= \frac{2y}{2}\\&= y\end{aligned}$$

In either case we get that

$$\max\{x, y\} = \frac{x + y + |x - y|}{2}$$

proving our claim.

□

(b)

$$\min\{x, y\} = \frac{x + y - |x - y|}{2}$$

for all real numbers x and y.

Solution: We prove this by cases.
Case 1: $x \geq y$. Then $\min\{x, y\} = y$, and $x - y > 0 \implies |x - y| = x - y$. So:

$$\begin{aligned}\frac{x + y - |x - y|}{2} &= \frac{x + y - (x - y)}{2} \\ &= \frac{x + y - x + y}{2} \\ &= \frac{2y}{2} \\ &= y\end{aligned}$$

Case 2: $x \leq y$. Then $\min\{x, y\} = x$ and $x - y < 0 \implies |x - y| = -(x - y)$. So:

$$\begin{aligned}\frac{x + y - |x - y|}{2} &= \frac{x + y + (x - y)}{2} \\ &= \frac{x + y + x - y}{2} \\ &= \frac{2x}{2} \\ &= x\end{aligned}$$

In either case we get that

$$\min\{x, y\} = \frac{x + y - |x - y|}{2}$$

proving our claim.

$\square$