CSCI 2824 - Discrete Structures
Homework 3

You MUST show your work. If you only present answers you will receive minimal credit. This homework is worth 100pts.

Due: Wednesday June 28

1. (4 points) For each of the following determine the number of elements in the given set.
   (a) $\{\}$
   (b) $\{\{\}, \{\{\}\}\}$.
   (c) $\{a, b, \{\}, \{\{\{\}\}\}\}$
   (d) $\{a, b, \{a, b\}, \{a, c\}, \{a\}\}$
2. (5 points) For the following pairs of sets, determine which operator goes between the sets to make a true statement: $\in$, $\ni$, $\subseteq$, $\supseteq$, or none.
   (a) $\{1, 2\}$, $\{1, 2, \{1, 2\}\}$
   (b) $\{1, 2\}$, $\mathbb{N}$
   (c) $\{\mathbb{N}, \mathbb{R}\}$, $\{\mathbb{R}\}$
   (d) $\{\mathbb{R}\}$, $\{1, 3, 4\}$
   (e) $\mathbb{R}$, $\{1, \pi, \sqrt{2}, \sqrt{-1}\}$
3. (5 points) For each of the following determine whether or not it is a function, if not explain why not.
   (a) $f : A \to B$ where $A = \{1, 2, 3, 4, 5\}$ and $B = \{b, x, t, m, z, y, a\}$ given by the following set $\{(1, a), (4, b), (2, b)(5, t), (2, a)\}$.
   (b) $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \tan(x)$.
   (c) $h : \mathbb{N} \to \mathbb{Z}^{>0}$ given by $h(x) = x - 1$
   (d) $k : A \to B$ where $A = \{18, 38, 485, 382385, 25\}$ and $B = \{1, 2, 3, 4, 5\}$ given by the following set $\{(18, 1), (38, 1), (285, 1), (382385, 1), (25, 1)\}$.
   (e) $l : \mathbb{R} \to \mathbb{R}$ given by $l(x) = \log(|x|)$.
4. (5 points) Prove that if $X \subseteq Y$ then $X \cap Z \subseteq Y \cap Z$ for all sets X, Y, Z.
5. (5 points) Prove that $\mathcal{P}(X) \cup \mathcal{P}(Y) \subseteq \mathcal{P}(X \cup Y)$. Are they equal? If not give a counterexample.
6. (20 points) For the following statements either give a proof or a counterexample. The sets X, Y, Z are subsets of a universal set U. Counter examples must also include the definition for U.
   (a) For all sets X and Y, either $X \subseteq Y$ or $Y \subseteq X$.
   (b) $\overline{Y \backslash X} = X \cup \overline{Y}$
   (c) $X \cup (Y \backslash Z) = (X \cup Y) \setminus (X \cup Z)$
   (d) $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$
7. (3 points) For the following problems a function definition is given. You must describe a domain and co-domain that ensures it is actually a function. The co-domain you provide need not be precisely the range.
   (a) $m(x) = \log(x)$.
   (b) $n(x) = 12$
   (c) $o(x)$ defined by the set $\{(1, 2), (\pi, 3), (12, 3)\}$

8. (12 points) For the following functions determine whether they are one-to-one or onto or both or neither.
   (a) $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = n + 1$
   (b) $g : \mathbb{Z} \to \mathbb{Z}$, $g(n) = \lceil \frac{n}{2} \rceil$.
   (c) $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, $h(m, n) = m - n$.
   (d) $j : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, $j(m, n) = m^2 + n^2 + 2$.

9. (5 points) Give examples of functions from $\mathbb{N}$ to $\mathbb{N}$ that are:
   (a) one-to-one, but not surjective
   (b) surjective but not injective
   (c) injective and surjective (but not the identity function)
   (d) neither injective nor surjective.

10. (11 points) Prove that the function $f : \mathbb{Z}^{>0} \times \mathbb{Z}^{>0} \to \mathbb{Z}^{>0}$ defined by $f(m, n) = 2^m \cdot 3^n$ is injective but not surjective.

11. (10 points) Solve the following recurrence relations (provide a closed form solution):
    (a) $a_n = -3a_{n-1}$, $a_0 = 4$
    (b) $a_n = a_{n-1} + 1$, $a_0 = 12$

12. (10 points) Solve the following recurrence relations (provide a closed form solution):
    (a) $a_n = 6a_{n-1} - 8a_{n-2}$, $a_0 = 1$, $a_1 = 0$.
    (b) $a_n = 2a_{n-1} + 8a_{n-2}$, $a_0 = 4$, $a_1 = 10$.

13. (5 points) Give an example of of two uncountable sets A and B such that $A \cap B$ is:
    (a) finite
    (b) countably infinite
    (c) uncountably infinite