$3n + 1$ Problem

A survey into the $3n + 1$ problem

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Goals

1. Look at a "simple" problem
2. Discover why this is a difficult problem
3. Learn about recent advances in the problem

Collatz function

Definition

The function $C : \mathbb{N} \to \mathbb{N}$ is given by

$$C(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases}$$

Behavior of function

We can look at how the function behaves at certain values:

$$C(5) = 16, C(12) = 6, C(49) = 148$$

or more interestingly we can look at the sequence of iterates of the function: $C^{(k)}(n)$:

$$(12, 6, 3, 10, 5, 16, 8, 4, 2, 1, 2, 1, \ldots)$$
$$(3, 10, 5, 16, 8, 4, 2, 1, 2, 1, \ldots)$$
$$(17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, \ldots)$$
$$(256, 128, 64, 32, 16, 8, 4, 2, 1, \ldots)$$

Behavior of sequences

Three cases for sequences:

1. The sequence eventually hits 1, and then enters the cycle $\ldots, 4, 2, 1, 4, 2, 1, \ldots$. We will call this cycle the **trivial cycle**, and in some sense consider the sequence as having converged, or stopped.
2. The sequence continues to grow forever, or diverges.
3. The sequence eventually enters a different cycle.

$3n + 1$ problem

First statement of problem:

$3n + 1$ problem

Prove (or find a counter example) that every positive integer, n, eventually reaches the number 1, under iterations of the function $C(n)$.

Brief History

This problem is generally credited to Lother Collatz (hence the common name **Collatz Problem**). Having distributed the problem during the International Congress of Mathematicians in Cambridge, in the 1930's.
Helmut Hasse is often attributed with this problem as well, and sometimes the iteration is referred to by the name **Hasse's algorithm**.
Another common name is the **Syracuse problem**.
Other names associated with the problem are S. Ulam, H. S. M. Coxeter, John Conway, and Sir Bryan Thwaites.

Observable behavior of the Collatz function

Proposition

Under the given Collatz function an even integer always follows an odd integer.

Proof.

$$\begin{aligned} C(2k+1) &= 3(2k+1)+1 \\ &= 6k+3+1 \\ &= 6k+4 \\ &= 2(3k+2) \end{aligned}$$

□

Simplifying the iteration

Definition

The function $T : \mathbb{N} \to \mathbb{N}$ is given by

$$T(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ \dfrac{3n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Really $T(n)$ is defined in terms of $C(n)$, we just iterate $C(n)$ twice when n is odd.

Definitions

Definition

The least positive k for which $T^{(k)}(n) < n$ is called the **stopping time**, $\sigma(n)$ of n. Or $\sigma(n) = \infty$ if no k occurs with $T^{(k)}(n) < n$.

Example

Examine the trajectory of 15:

$$(15, 23, 35, 53, 80, 40, 20, 10, 5, 8, 4, 2, 1, \ldots)$$

Thus $\sigma(15) = 7$, since $T^{(7)}(15) = 10$ is the first time our iteration is below 15

Definitions (cont.)

Definition

The least positive k for which $T^{(k)}(n) = 1$ is called the **total stopping time**, $\sigma_\infty(n)$ of n, or $\sigma_\infty(n) = \infty$ if no k occurs with $T^{(k)}(n) = 1$.

Example

With the same iteration:

$$(15, 23, 35, 53, 80, 40, 20, 10, 5, 8, 4, 2, 1, \dots)$$

$\sigma_\infty(n) = 12$ since $T^{(12)}(15) = 1$ is the first time our iteration reaches 1.

Problem restatement

$3n+1$ problem

Every integer $n \geq 2$ has a finite total stopping time, under the function $T(n)$.

In fact it is enough to show that every integer has a finite stopping time.

A failed attempt at a proof

(Failed) Proof

Base case: $n = 2$, $2 \to 1$

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<u>IH</u>: for $n \leq k - 1$ the Collatz function converges to 1.

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(Failed) Proof

Base case: $n = 2$, $2 \to 1$

IH: for $n \leq k - 1$ the Collatz function converges to 1.

Induction step: When $n = k$ if n is even $T(n) < n$ and we're done. But if n is odd $T(n) = \frac{3n+1}{2}$ but who knows if this is even or odd...

Difficulty of the $3n + 1$ problem

- Pseudorandomness:
  - Next iterate even or odd?
  - $T(n)$ increases odds by factor of $\frac{3}{2}$ but shrinks evens by factor of $\frac{1}{2}$.
  - Iterations could quickly plummet or skyrocket, or behave like hailstones

Examples: $n = 2^k$, and $n = 2^k - 1$

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Examples: $n = 2^k$, and $n = 2^k - 1$
Notice:

$$2^k \to 2^{k-1} \to 2^{k-2} \to \cdots \to 4 \to 2 \to 1$$

but

$$\begin{aligned} T(2^k - 1) &= \frac{3 \cdot 2^k - 3 + 1}{2} \\ &= \frac{3 \cdot 2^k - 2}{2} \\ &= 3 \cdot 2^{k-1} - 1 \end{aligned}$$

Difficulty of the $3n + 1$ problem (cont.)

- Factorization of integers into primes
  - If we have the factorization of $a \in \mathbb{Z}$, we do not immediately have the factorization of $a + 1$.
  - If n is even $T(n) = \frac{n}{2}$ changes the factorization very little.
  - If n is odd $3n$ changes the factorization very little, but $3n + 1$ can drastically change the factorization, besides knowing we have at least one power of 2. Thus $T(n)$ leaves us completely in the dark.

Why are we so stubborn?

Silva, Tomás Oliverira e Silva (2011)

All integers up to $5 \cdot 2^{60} \approx 5.754 \times 10^{18}$ eventually reach one under The Collatz iteration.

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This is NOT a proof, note: Pólya's Conjecture had its first found counterexample at 1.845×10^{361}.

Heuristic Algorithm

Pick an arbitrary odd integer n_0. Now we iterate T until we get to the next odd number, n_1.

$$\underbrace{\tfrac{3n_0+1}{2} \text{ is even}}_{Pr = \frac{1}{2}} \qquad \text{or} \qquad \underbrace{\tfrac{3n_0+1}{2} \text{ is odd}}_{Pr = \frac{1}{2}}$$

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$$\downarrow$$

$$\underbrace{\frac{3n_0+1}{4} \text{ is even}}_{Pr = \frac{1}{4}} \qquad \text{or} \qquad \underbrace{\frac{3n_0+1}{4} \text{ is odd}}_{Pr = \frac{1}{4}}$$

Heuristic Algorithm

Pick an arbitrary odd integer n_0. Now we iterate T until we get to the next odd number, n_1.

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$$\underbrace{\frac{3n_0+1}{8} \text{ is even}}_{Pr = \frac{1}{8}} \quad \text{or} \quad \underbrace{\frac{3n_0+1}{8} \text{ is odd}}_{Pr = \frac{1}{8}}$$

Heuristic Algorithm (cont.)

The expected growth between successive odd terms is:

$$\left(\frac{3}{2}\right)^{\frac{1}{2}} \cdot \left(\frac{3}{4}\right)^{\frac{1}{4}} \cdot \left(\frac{3}{8}\right)^{\frac{1}{8}} \cdots = \frac{3}{4} < 1$$

Heuristic Algorithm (cont.)

This infinite product represents the growth between consecutive odd terms in iterations of the Collatz function. Since our growth is less than one, this means that consecutive odd terms *should* be shrinking.

We can use this to approximate the total stopping time of an odd integer, i.e., if $\sigma_\infty(n_0) = k$ we have $T^{(k)}(n_0) = 1$. Roughly:

$$\begin{aligned} \left(\frac{3}{4}\right)^k \cdot n_0 &\approx 1 \\ k \log\left(\frac{3}{4}\right) &\approx \log\left(\frac{1}{n_0}\right) \\ k &\approx -\frac{\log n_0}{\log\left(\frac{3}{4}\right)} \end{aligned}$$

A minor correction

Our model was a bit too simplistic, we actually need to take into account the even terms that will show up. We assumed our function, $T(n)$, "mixed" evens and odds equally, so we should have as many evens as odds. So our corrected model should actually be:

$$k \approx -\frac{2 \log n}{\log\left(\frac{3}{4}\right)}$$

Some graphs of our model

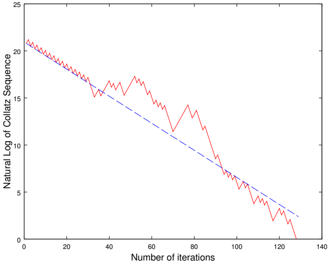

Figure: $n = 2^{30} + 1$

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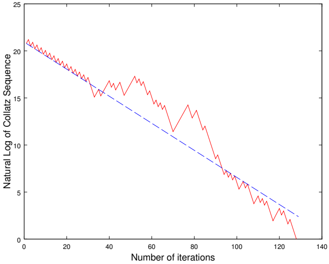

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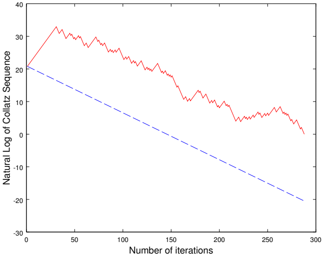

Figure: $n = 2^{30} - 1$

More graphs

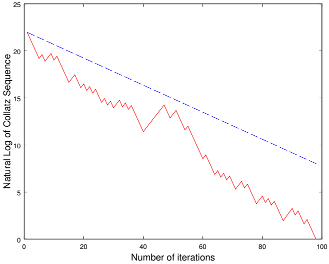

Figure: $n = 3^{20} - 1$

More graphs

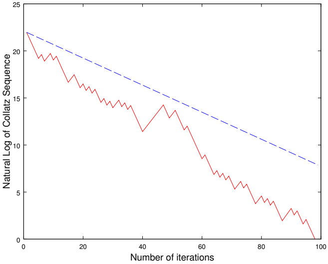

Figure: $n = 3^{20} - 1$

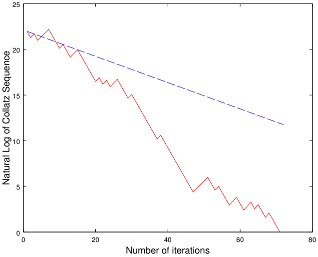

Figure: $n = 3^{20} + 1$

Analysis of our model

We saw some graphs that fit very nicely with our model, but we didn't need to change the size of our number much to find a number that did not fit well.
This is an example of how the Collatz function behaves "randomly". We can't use common tricks of saying if we take a large enough number, this model will be a good fit.
This can be summarized: The trajectory of a number n has no influence on the trajectory of its neighbors.

What else can we do?

Since we can't directly look at the function $T(n)$ to deduce a pattern we look instead at the stopping time.

Definition

A subset $S \subseteq \mathbb{N}$ has **natural density** α where $0 \leq \alpha \leq 1$ if the proportion of elements of S among $[n]$ is asymptotic to α.

More explicitly, define $S_n = S \cap [n]$. Then S has natural density α if and only if

$$\lim_{n \to \infty} \frac{|S_n|}{n} = \alpha$$

Example 1

Let S be the set of even positive numbers, i.e., $S = \{2, 4, 6, 8, \ldots\}$. A standard result in Mathematics is that $|S| = |\mathbb{N}|$.
To get a more intuitive result we can use natural density. We can first compute $S_n = S \cap [n]$ for a given n this is just all the even integers up to n. Thus $|S_n| = \frac{n}{2}$. So our limit becomes:

$$\begin{aligned} \lim_{n\to\infty} \frac{|S_n|}{n} &= \lim_{n\to\infty} \frac{\frac{n}{2}}{n} \\ &= \lim_{n\to\infty} \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

Example 2

Let S be the set of prime numbers, i.e., $S = \{2, 3, 5, 7, \ldots\}$. A result in mathematics is that: $|S| = |\mathbb{N}|$.
To get a more intuitive result we can use natural density. We can first compute $S_n = S \cap [n]$ for a given n this is just all the prime integers up to n. This is not as intuitive but there is a result which gives $|S_n| = \log n$. So our limit becomes:

$$\begin{aligned}\lim_{n\to\infty} \frac{|S_n|}{n} &= \lim_{n\to\infty} \frac{\log n}{n} \\ &= 0\end{aligned}$$

Results

Terras (1976,1979), Everett (1977)

The set of natural numbers with finite stopping time under the Collatz function, i.e., set of numbers that reach one, have natural density **one**.

This shows that our iteration will converge to one for almost every natural number!

Other cycles

If a number does not converge to one, it could diverge to ∞ (though our model from before suggests that is unlikely) or it could be part of a cycle. We know of the trivial cycle

$$(4, 2, 1, 4, 2, 1, \dots)$$

But we could have other cycles of the form:

$$(n_0, n_1, n_2, \dots, n_0)$$

Size of a cycle

Theorem

Let Ω be a non-trivial cycle of T. Provided that $\min \Omega > 1.08 \times 2^{60}$ *we have*

$$|\Omega| = 630\,138\,877a + 10\,439\,860\,591b + 103\,768\,467\,013c$$

where a, b, c are non-negative integers with $b > 0$ and $ac = 0$. Specifically the smallest possible values for $|\Omega|$ are $10,439,860,591$*;* $11,069,999,488$*;* $11,700,138,385$*, etc.*

Why study the $3n + 1$ problem?

The $3n + 1$ problem has some nice properties:

- Very easy to state
- Can be examined from many different areas in mathematics
    - Number Theory
    - Dynamical Systems
    - Mathematical Logic & Theory of Computation
    - Probability & Stochastic Processes
    - Computer Science

Final Thoughts

"Hopeless, absolutely hopeless"
"Mathematics is not yet ready for such problems"
–Paul Erdős
"Don't try to solve these problems"
–Richard Guy

Thank you!

References

- "The Ultimate Challenge: The $3x + 1$ Problem" Jeffrey C. Lagarias.
- "The $3x + 1$ problem and its generalizations" Jeffrey C. Lagarias (1985).
- "A $3x + 1$ survey: Number theory and dynamical systems" Marc Chamberland.