Quantum Stochastic Processes: A Case Study

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Abstract

We present a detailed study of a simple quantum stochastic process, the quantum phase space Brownian motion, which we obtain as the Markovian limit of a simple model of open quantum system. We show that this physical description of the process allows us to specify and to construct the dilation of the quantum dynamical maps, including conditional quantum expectations. The quantum phase space Brownian motion possesses many properties similar to that of the classical Brownian motion, notably its increments are independent and identically distributed. Possible applications to dissipative phenomena in the quantum Hall effect are suggested.

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1 Introduction and motivations

Quantum Brownian motion [1] is usually defined by coupling quantum mechanically a massive particle to a bath of harmonic oscillators, see e.g. refs.[2, 3, 4] and references therein. This is not the quantum analogue of what probabilists call Brownian motion (which is the continuum limit of random walks) but that of a Newtonian particle submitted to friction and random force (which is also called Brownian motion in the physics literature), whose dynamics is described by $m\ddot{x} + \alpha \dot{x} = f$ with α the friction coefficient and f the random forcing. These two notions are related: the former is the over-damped limit, or the massless limit, of the latter. The mass plays a role of short distance cut-off, and the velocity $\dot{x}$ scales as $1/\sqrt{m}$. It diverges in the probabilistic Brownian motion which is known to be nowhere differentiable. This causes troubles when defining the quantum analogue of the probabilistic Brownian motion as the massless limit of the quantum Brownian motion because the momentum $p = m\dot{x}$ scales as $p \sim \sqrt{m}$, whose massless limit is incompatible with the canonical commutation relation $[x, p] = i\hbar$. The first aim of the present paper is to understand how to consistently formulate the over-damped quantum Brownian motion.

A possible way to overcome these difficulties may consist in going one dimension higher and consider the two dimensional Brownian motion. An advantage is that one may then couple the Newtonian particle to an out-of-plane external magnetic field B, so that the classical equation of motion becomes

$$m\ddot{z} + (\alpha + ieB)\dot{z} = f$$

in complex coordinate $z = x + iy$, with e the electric charge. If the force f is white-noice in time, i.e. $\langle f(t)\bar{f}(s)\rangle \propto \delta(t - s)$, the massless limit still describes the probabilistic 2D Brownian motion, and its quantification will provide a candidat for the 2D (over-damped) quantum Brownian motion. The magnetic field only rotates and dilates the Brownian trajectories. Since these trajectories are known to be statistically conformally invariant [6], the effect of the magnetic field is classically quite innocent. Quantum mechanically the situation is different. In absence of random forcing, the energy spectrum is that of the Landau levels. The magnetic length $\ell_B = \sqrt{\hbar/eB}$ provides a short-distance cut-off which would allow us to take the massless limit. Since the cyclotron frequency $\omega_B = eB/m$ diverges as m vanishes (or as B increases), taking the massless (or the over-damped) limit amounts to project on the first Landau level. Thus a model candidate for the over-damped 2D quantum Brownian motion is a particle confined to the first

Landau level and coupled to a harmonic oscillator bath [4]. Recall that, once projected onto the first Landau level, the two coordinate operators x and y are not commuting [9] but form a pair of canonical operators, $[x, y] = i\ell_B^2$, so that the motion actually takes place in the quantum phase plane. The model obtained by following this projection strategy leads to a coupling between the oscillator bath and the particle which (although quadratic) is yet too complicated [5]. Hoping, or claiming, for universality property, we consider an alternative model with a much simpler coupling between the quantum phase plane and a bath of harmonic oscillators, see eq.(1).

In this way, in a weak-coupling/long-time/continuous-density limit, we obtain a Markovian quantum process, which we call "quantum phase Brownian motion" and which bares many similarities with classical Brownian motion. A similar, but not identical, model has been considered by E. Davies in ref.[5] but without analysing the quantum process.

The two dimensional Brownian motion possesses very peculiar geometrical properties linked to conformal invariance, see e.g. ref.[7]. It would be very interesting to know how these properties are deformed after quantisation. To understand the quantum geometry of the quantum 2D Brownian motion is another long term motivation of the present work [8].

The second aim of the paper is to provide a description – as complete as possible – of the quantum stochastic process defined by this simple model. We aim at presenting the algebraic structure underlying the process which goes beyond the dynamical maps and the associated master equation, which code for the evolution of the reduced density matrix [10, 11, 12]. This is in particular needed if one is willing to evaluate, or even define, multi-point expectations. Recall that a classical stochastic process is defined over a probability space equipped with a probability measure and a filtration (i.e. an increasing family of σ-algebras) to which the process is adapted. The process is specified by a consistent set of finite-dimensional distributions (encoded in transition kernels for Markov processes) and the filtration allows to define conditional expectations. The filtration encodes the increase of knowledge as time and the process go on. In Appendix 6 we recall the algebraic structure induced by these data. Quantum mechanically, transition kernels are replaced by completely positive dynamical maps acting on the algebra of observables of the quantum system (but not of the bath). By duality, this specifies the evolution of the system density matrix and it is

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[4]One may wonder whether this model may have relevance to the description of dissipative effects in the integer quantum Hall effect, although it does not deal with the edge currents.

[5]We nevertheless hope to report on this model in the near future [8].

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enough for evaluating one-point functions (but not multi-point). Most, but not all, of the physics literature only deals with density matrices. This is however not enough to fully specify the process and in particular to define the quantum analogue of conditional expectations. This requires extending the algebra of observables and the dynamical maps to a set of embedded algebras (quantum analogues of the filtration, which roughly keeps track of the process up to time t) with flows intertwining them and projectors (quantum analogues of conditional expectations) with compatibility properties. This extended algebraic structure, which mimics that of the classical processes (see Appendix 6), is called a dilation of the dynamical maps [10, 11, 12, 13]. Given completely positive maps, there is generically not a unique dilation but a set of compatible dilations. We present this structure in the simple (but quite generic) model we describe and show that the physical description of the process given by coupling the quantum system to an oscillator bath specifies the dilation [6].

Exemple of dilations of dynamical maps have of course been given in the literature, see e.g. refs.[10, 11, 12] and references therein. Elements of the construction we present here also bare some similarities with the quantum stochastic calculus of ref.[16]. However, as far as we know, our construction of the dilation from the microscopic physical model has not been described earlier. Our model provides a simple framework in which concepts of quantum stochastic processes may be explained and exemplified.

The paper is organised as follows: In the section 2, we define the model and describe the long-time/weak-coupling limit in which it becomes Markovian. In section 3 we describe the quantum process with a detailed presentation of the quantum filtration, of the dynamical maps and the associated Lindbladian, of the flows and the conditional expectations, etc... The quantum process we have defined enjoys many properties analogous to those of Brownian motion. In particular we show that its increments are independent and identically distributed and we show that the conditional expectations are neutral with respect to both left and right mesurable multiplications[7]. We also make contact with processes introduced earlier on the mathematical literature. In Appendix 6 we recall the algebraic structure underlying classical stochastic processes and in Appendix 7 we present an alternative approach to the weak-coupling/long-time limit.

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[6]We show this property in the case of this simple quadratic model, but we hope to also proof it in a future publication [8] in the case of an arbitrary finite dimensional quantum system linearly coupled to a bath.

[7]This is explained precisely below.

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2 A simple heat bath model

2.1 The model

Out-of-equilibrium quantum systems may often be described by coupling them to reservoirs which, quantum mechanically, may be modelled as collections of harmonic oscillators. Here we consider one of the simplest model: a harmonic oscillator (which we call the system oscillator) linearly coupled to a reservoir set of independent harmonic oscillators (which we call the bath). We denote by ϵ the energy of the system oscillator and by ω_α the pulsations of the reservoir oscillators. The system creation-annihilation operators will be denoted $w^\dagger$ and w and those of the reservoir by $a_\alpha^\dagger$ and a_α. They satisfy canonical commutation relations:

$$[a_\alpha, a_{\alpha'}^\dagger] = \delta_{\alpha;\alpha'}, \quad [w, w^\dagger] = 1,$$

The other commutators vanish: $[a_\alpha, a_{\alpha'}] = 0$ and $[w^\dagger, a_\alpha] = [w, a_\alpha] = 0$.

The hamiltonian of the system oscillator plus the reservoir is the sum of three pieces: $H = H_B + \lambda H_{SB} + H_S$ where H_B is the bath hamiltonian, H_S the system oscillator hamiltonian and H_{SB} a coupling hamiltonian. Explicitely

$$H = \sum_\alpha \omega_\alpha a_\alpha^\dagger a_\alpha + \lambda \big(w^\dagger \sum_\alpha \varphi_\alpha a_\alpha + w \sum_\alpha \bar{\varphi}_\alpha a_\alpha^\dagger \big) + \epsilon w^\dagger w \tag{1}$$

We set $\hbar = 1$ and λ is a dimensionless controlling parameter and φ_α are coupling constants. The Hilbert space is the tensor product of the Fock spaces associated to each of the pairs of creation-annihilation operators: $\mathcal{H} = \Gamma_S \otimes \Gamma_B$ with $\Gamma_B = \bigotimes_\alpha \Gamma_\alpha$.

At initial time, the system is supposed to be prepared with a density matrix $\rho_0 \otimes \rho_B$ where the system density matrix ρ_0 is yet unspecified. Density matrices are normalised, $\mathrm{Tr}\rho = 1$. The bath density matrix ρ_B is choosen to be gaussian, $\rho_B = \bigotimes_\alpha \rho_\alpha$ with $\rho_\alpha = Z_\alpha^{-1} e^{-\beta_\alpha a_\alpha^\dagger a_\alpha}$ and $Z_\alpha^{-1} = 1 - e^{-\beta_\alpha}$. We shall parametrize the bath density matrix via the mean occupation numbers $n_\alpha \equiv \mathrm{Tr}_{\Gamma_\alpha}(\rho_\alpha a_\alpha^\dagger a_\alpha)$ with $n_\alpha = 1/(e^{\beta_\alpha} - 1)$.

Expectations of product of observables (which may be evaluated at different times) are by definition given by their traces weighted by the density matrix: $\langle \mathcal{O} \rangle \equiv \mathrm{Tr}_{\mathcal{H}}(\rho_0 \otimes \rho_B \, \mathcal{O})$. As usual with composite systems, we first trace over the reservoir Hilbert space as we are not aiming at observing it. We denote by $\mathbb{E}$ the trace over the reservoir:

$$\mathbb{E}[\, \mathcal{O} \,] \equiv \mathrm{Tr}_{\Gamma_B}(\rho_B \, \mathcal{O}).$$

By abuse of language we shall call $\mathbb{E}$ the expectation, it maps general operators into operators acting on the system Hilbert space Γ_S. The (complete) expectations are thus given by $\langle \mathcal{O} \rangle = \text{Tr}_{\Gamma_S}(\rho_0 \, \mathbb{E}[\mathcal{O}])$.

2.2   Weak-coupling/long-time behaviour

The aim of this section is to describe the model in the weak-coupling/long-time/continous-density limit, corresponding to $\lambda \to 0$ with $\lambda^2 t$ fixed, in which the dynamics is known to become Markovian [14, 15]. The continuous-density limit ensures dissipation because the bath relaxation time scales as the inverse of the bath energy level spacing. Computations presented in this Section are a bit technical but they are needed to get an explicit description of this limiting behaviour. The output formulas are summarised in eqs.(3,4). They are going to be the starting point of the construction of the quantum stochastic process which we call quantum phase space Brownian motion. Computations may be skipped (in a first reading) if the reader is only interested in the output.

In the Heisenberg picture, the equation of motion for an operator $\mathcal{O}$ is:

$$\dot{\mathcal{O}} = i[H, \mathcal{O}].$$

In our model, this leads to linear equations for the oscillators:

$$
\begin{aligned}
\dot{w} + i\epsilon w &= -i\lambda \sum_\alpha \varphi_\alpha a_\alpha \\
\dot{a}_\alpha + i\omega_\alpha a_\alpha &= -i\lambda \, \bar{\varphi}_\alpha w
\end{aligned}
$$

(together with their adjoints). Our aim is to understand these equations in the weak coupling limit for an idealized bath involving a continuum of frequencies. This can be done routinely using Laplace transform.

Define the Laplace transform $\tilde{f}$ of a function f by $\tilde{f} \equiv \int_0^{+\infty} f(t) e^{-pt}$. Then the equations of motion become:

$$
\begin{aligned}
p\tilde{w}(p) - w(0) + i\epsilon w(p) &= -i\lambda \sum_\alpha \varphi_\alpha \tilde{a}_\alpha(p) \\
p\tilde{a}_\alpha(p) - a_\alpha(0) + i\omega_\alpha \tilde{a}_\alpha(p) &= -i\lambda \, \bar{\varphi}_\alpha \tilde{w}(p).
\end{aligned}
$$

We solve the second equation for $\tilde{a}_\alpha$ and reinject in the first to get

$$\left(p + i\epsilon + \lambda^2 \sum_\alpha \frac{|\varphi_\alpha|^2}{p + i\omega_\alpha} \right) \tilde{w}(p) = w(0) - i\lambda \sum_\alpha \frac{\varphi_\alpha}{p + i\omega_\alpha} a_\alpha(0).$$

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As we shall discuss in details in Appendix 7, the function $p + i\epsilon + \lambda^2 \sum_\alpha \frac{|\varphi_\alpha|^2}{p + i\omega_\alpha}$ can only vanish for purely imaginary values of p. So Laplace inversion tells that

$$w(t) = \int_{c-i\infty}^{c-i\infty} \frac{dp}{2i\pi} \, e^{pt} \, \frac{w(0) - i\lambda \sum_\alpha \frac{\varphi_\alpha}{p + i\omega_\alpha} a_\alpha(0)}{p + i\epsilon + \lambda^2 \sum_\alpha \frac{|\varphi_\alpha|^2}{p + i\omega_\alpha}}$$

where c is an arbitrary positive constant.

In the weak coupling limit the exchanges between the bath and the system take a long time, and the short time evolution of the system is dominated by the energy scale ϵ. Once this is factored out, the relevant times are longer. As we shall see below, if the bath is idealized by a continuum of oscillators, the relevant long time scale will be of order $1/\lambda^2$ and there is a non-trivial zero-coupling limit for

$$z(t) \equiv w(t/\lambda^2) \, e^{i\epsilon t/\lambda^2}.$$

Let us define

$$\bar{\Gamma}(p) \equiv \sum_\alpha \frac{|\varphi_\alpha|^2}{\lambda^2 p - i\epsilon + i\omega_\alpha}.$$

After translating and rescaling p in the above formula for $w(t)$, we get a formula for $z(t)$ which is a sum of a term proportional to the initial value $z(0)$ and terms proportional to $a_\alpha(0)$, that is:

$$z(t) = \bar{f}(t)z(0) + \xi(t), \quad \text{with } \xi(t) \equiv \sum_\alpha \bar{f}_\alpha(t)a_\alpha(0),$$

where

$$\bar{f}(t) \equiv \int_{c-i\infty}^{c-i\infty} \frac{dp}{2i\pi} \, e^{pt} \, \frac{1}{p + \bar{\Gamma}(p)},$$

$$\bar{f}_\alpha(t) \equiv -i\lambda \int_{c-i\infty}^{c-i\infty} \frac{dp}{2i\pi} \, e^{pt} \, \frac{\varphi_\alpha}{[\lambda^2 p - i\epsilon + i\omega_\alpha]} \frac{1}{[p + \bar{\Gamma}(p)]}$$

Taking complex conjugates, we note that

$$f_\alpha(s) = i\lambda \int_{c-i\infty}^{c-i\infty} \frac{dq}{2i\pi} \, e^{qs} \, \frac{\bar{\varphi}_\alpha}{[\lambda^2 q + i\epsilon - i\omega_\alpha]} \frac{1}{[q + \Gamma(\bar{q})]}.$$

Because the bath is Gaussian and $\xi(t)$ is linear in bath oscillators, its properties are fully specified by the value of the commutator $[\xi(t), \xi^\dagger(s)]$

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and the two point function $\mathbb{E}[\xi^\dagger(s)\xi(t)]$. From the basic oscillator commutation relations, we have $[\xi(t), \xi^\dagger(s)] = \sum_\alpha \bar{f}_\alpha(t) f_\alpha(s)$. Using the identity $\frac{1}{p+\omega} \frac{1}{q-\omega} = \frac{1}{p+q} \left(\frac{1}{p+\omega} + \frac{1}{q-\omega} \right)$ we get

$$[\xi(t), \xi^\dagger(s)] = \int_{c-i\infty}^{c-i\infty} \frac{dp}{2i\pi} \frac{dq}{2i\pi} e^{pt} e^{qs} \frac{\bar{\Gamma}(p) + \Gamma(\bar{q})}{[p + \bar{\Gamma}(p)][q + \Gamma(\bar{q})][p + q]}.$$

In the same vein, we find $\mathbb{E}[\xi^\dagger(s)\xi(t)] = \sum_\alpha n_\alpha \bar{f}_\alpha(t) f_\alpha(s)$, which can be written as:

$$\mathbb{E}[\xi^\dagger(s)\xi(t)] = \int_{c-i\infty}^{c-i\infty} \frac{dp}{2i\pi} \frac{dq}{2i\pi} e^{pt} e^{qs} \frac{\bar{\Upsilon}(p) + \Upsilon(\bar{q})}{[p + \bar{\Gamma}(p)][q + \Gamma(\bar{q})][p + q]},$$

where

$$\bar{\Upsilon}(p) \equiv \sum_\alpha \frac{n_\alpha |\varphi_\alpha|^2}{\lambda^2 p - i\epsilon + i\omega_\alpha}.$$

Turning now to the situation when there is a density of oscillators, we replace $\sum_\alpha |\varphi_\alpha|^2 \delta(\omega - \omega_\alpha)$ by $r(\omega)$ and, interpreting n_α as $n(\omega_\alpha)$, we find:

$$\bar{\Gamma}(p) = \int_0^{+\infty} d\omega \frac{r(\omega)}{\lambda^2 p - i\epsilon + i\omega}, \quad \bar{\Upsilon}(p) = \int_0^{+\infty} d\omega \frac{n(\omega) r(\omega)}{\lambda^2 p - i\epsilon + i\omega}.$$

Taking naively the limit $\lambda^2 \to 0^+$ and remembering that the integration contours stay in the half-plane $\Re e\, p > 0$, we find that $\bar{\Gamma}(p) \to \bar{\gamma}$ with

$$\bar{\gamma} \equiv \bar{\Gamma}(0^+) = \pi r(\epsilon) + i \fint_0^{+\infty} d\omega \frac{r(\omega)}{\epsilon - \omega}. \tag{2}$$

Analogously, $\bar{\Upsilon}(p) + \Upsilon(\bar{q}) \to 2\pi n(\epsilon) r(\epsilon)$.

Thus, in the weak coupling limit we get:

$$\bar{f}(t) = \int_{c-i\infty}^{c-i\infty} \frac{dp}{2i\pi} e^{pt} \frac{1}{p + \bar{\gamma}},$$

$$\left[\xi(t), \xi^\dagger(s) \right] = 2\pi r(\epsilon) \int_{c-i\infty}^{c-i\infty} \frac{dp}{2i\pi} \frac{dq}{2i\pi} e^{pt} e^{qs} \frac{1}{[p + \bar{\gamma}][q + \gamma][p + q]},$$

$$\mathbb{E}[\xi^\dagger(s)\xi(t)] = 2\pi n(\epsilon) r(\epsilon) \int_{c-i\infty}^{c-i\infty} \frac{dp}{2i\pi} \frac{dq}{2i\pi} e^{pt} e^{qs} \frac{1}{[p + \bar{\gamma}][q + \gamma][p + q]}.$$

Note that $\mathbb{E}[\xi^\dagger(s)\xi(t)] = n(\epsilon) \left[\xi(t), \xi^\dagger(s) \right]$.

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The remaining integrals are obtained by application of the residue theorem. For instance, $\bar{f}(t) = e^{-\bar{\gamma}t}$. The commutator requires a slight discussion. If $t \geq s \geq 0$,

$$\int_{c-i\infty}^{c-i\infty} \frac{dq}{2i\pi} e^{qs} \frac{1}{q+\gamma} \frac{1}{p+q} = \frac{e^{-ps} - e^{-\gamma s}}{\gamma - p},$$

because the integration contour can be pushed to the left. This has no pole at $p = \gamma$. Then,

$$\int_{c-i\infty}^{c-i\infty} \frac{dp}{2i\pi} e^{pt} \frac{1}{p+\bar{\gamma}} \frac{e^{-ps} - e^{-\gamma s}}{\gamma - p} = \frac{e^{-\bar{\gamma}(t-s)} - e^{-\gamma s - \bar{\gamma}t}}{\gamma + \bar{\gamma}},$$

because the integration contour can be pushed to the left again. The case $s \geq t \geq 0$ is treated analogously.

Hence, defining

$$G(t,s) \equiv e^{i(t-s)\Im m\, \gamma} \left(e^{-|t-s|\Re e\,\gamma} - e^{-(t+s)\Re e\,\gamma} \right),$$

we end up with $f(t) = e^{-\bar{\gamma}t}$ and

$$z(t) = e^{-\bar{\gamma}t} z(0) + \xi(t),$$
$$\left[\xi(t), \xi^\dagger(s) \right] = G(t,s)$$
$$\mathbb{E}[\xi^\dagger(s)\xi(t)] = n(\epsilon)G(t,s).$$

The above derivation of the long-time/weak-coupling/continuous-density limit relies heavily on the magic of contour deformation. Though the final result, damping when the bath is infinite, is physically satisfactory, it may seem disturbing that taking the limit of a function with many poles but all purely imaginary, one finds functions with single poles having a negative real part. The model we consider is simple enough that one can make a more down-to-earth derivation which shows more clearly that damping arises from destructive interferences. It also leads to an explicit spectral representation (21) of the process directly inherited from the discrete harmonic bath oscillator decomposition. This is presented in Appendix 7.

3   Quantum phase space Brownian motion

Let us reformulate (and summarise) the output of the weak-coupling/long-time limit of previous section. The time evolutions of the system canonical operators z and $z^\dagger$ are:

$$z(t) = e^{-\bar{\gamma}t}z + \xi(t) \ , \quad z^\dagger(t) = e^{-\gamma t}z^\dagger + \xi^\dagger(t) \tag{3}$$

with γ a complex parameter, $\Re e\,\gamma \geq 0$. We shall set $\gamma = \kappa + i\nu$ and $\bar{\gamma} = \kappa - i\nu$. The fields $\xi(t)$ and $\xi^\dagger(t)$, which represent "the quantum noice", are linear combinations of the reservoir creation-annihilation operators. Commutation relations are:

$$[z, z^\dagger] = 1$$

and

$$[\xi(t), \xi^\dagger(s)] = G(t, s) \equiv \begin{cases} (e^{\gamma t} - e^{-\bar{\gamma}t})e^{-\gamma s} \; ; & \text{for } t \leq s \\ e^{-\bar{\gamma}t}(e^{\bar{\gamma}s} - e^{-\gamma s}) \; ; & \text{for } t \geq s \end{cases} \tag{4}$$

The system operators commute with the reservoir operators so that $[z, \xi(t)] = [z^\dagger, \xi(t)] = 0$.

The measure $\mathbb{E}$, induced by tracing over the reservoir degrees of freedom with the bath density matrix, is such that $\xi^\dagger(t)$ and $\xi(s)$ are gaussian with two-point function:

$$\mathbb{E}[\xi^\dagger(t)\,\xi(s)] = \mathfrak{n}_0\,G(s, t)$$

with $\mathfrak{n}_0$ real positive. In previous Section 2.2, this parameter was denoted $n(\epsilon)$. Since physical phenomena concentrate at the energy scale ϵ in the long-time/weak-coupling limit, all details of the reservoir parameters have been erased and summarised in the very few parameters γ, $\mathfrak{n}_0$ and in the form of the commutation relations $G(t, s)$ and in the two-point functions $\mathfrak{n}_0\,G(s, t)$.

The aim of this section is to extract (and illustrate) the structure of quantum processes in this particular example which possesses all generic properties plus some peculiar ones. It has most properties for being a quantum analogue of a (two-dimensional) Brownian motion. In particular its increments are independent and identically distributed.

3.1   Flow and expectations

• *Algebras:* We first have to identify where the flow takes place. Let A_0 be the algebra generated by the system canonical operators z and $z^\dagger$, and let $B_{[0,t]}$ be the algebra generated by all the $\xi(s)$ and $\xi^\dagger(s)$ for $s \in [0, t]$. We set

$$A_t \equiv A_0 \otimes B_{[0,t]} \; , \quad A_\infty \equiv A_0 \otimes B_{[0,\infty)}. \tag{5}$$

Although understandable by "common sense" and useful in practice, the definition of $B_{[0,t]}$ as "the algebra generated by the $\xi(s)$'s with $s \leq t$" is not precise. Thus, let $\mathbb{L}_\kappa^2(\mathbb{R}_+)$ be the Hilbert space of functions $s \to f(s)$, such that $e^{\kappa s}f(s)$ is square integrable on the positive real line, equipped

with the scalar product $(g|f)_\kappa = \int_0^\infty ds\,\overline{g(s)}\,2\kappa e^{2\kappa s}\,f(s)$. Let $\Gamma(\mathbb{L}_\kappa^2(\mathbb{R}_+))$ be the Fock space over $\mathbb{L}_\kappa^2(\mathbb{R}_+)$, and let $\mathfrak{V}_f$ be the canonical operators on $\Gamma(\mathbb{L}_\kappa^2(\mathbb{R}_+))$ depending linearly on functions f with commutation relations $[\mathfrak{V}_f, \mathfrak{V}_g^\dagger] = (g|f)_\kappa$. Then

$$\xi(t) = e^{-\bar\gamma t}\,\mathfrak{V}_{\mathbf{1}_{[0,t]}}.$$

More explicitely, the bath operators $\xi(t)$ may be represented as (recall that $\gamma = \kappa + i\nu$):

$$\xi(t) = e^{-\bar\gamma t} \int_0^t du\,(2\kappa e^{2\kappa u})^{\frac{1}{2}}\,a(u),$$

where $a(s)$ and $a^\dagger(s)$ are bare canonical operators with commutation relations:

$$[a(s), a(s')] = 0, \quad [a(s), a^\dagger(s')] = \delta(s - s').$$

One may use this representation of $\xi(t)$ to write a "quantum stochastic differential equation" for $z(t)$. Indeed differentiating eq.(3), we get:

$$dz(t) = -\bar\gamma\,z(t)\,dt + \sqrt{2\kappa}\,e^{i\nu t}\,d\mathfrak{A}(t), \tag{6}$$

where we set $d\mathfrak{A}(t) \equiv \int_t^{t+dt} a(u)du$. A similar representation is of course at the basis of the quantum stochastic calculus due to Hudson and Parthasarathy [16].

Let $B_\infty \equiv B(\Gamma(\mathbb{L}_\kappa^2(\mathbb{R}_+))$ be the algebra of bounded operators on $\Gamma(\mathbb{L}_\kappa^2(\mathbb{R}_+)$, which we identify with $B_{[0,\infty)}$. Recall that $\Gamma(\mathbb{L}_\kappa^2(\mathbb{R}_+) = \Gamma(\mathbb{L}_\kappa^2([0,t]) \otimes \Gamma(\mathbb{L}_\kappa^2(]t,\infty))$. We may then define $B_{[0,t]}$ as the B_∞-subalgebra (this is more an embedding than an inclusion):

$$B_{[0,t]} \equiv B\big(\Gamma(\mathbb{L}_\kappa^2([0,t]))\big) \subset B_\infty.$$

Notice that $B_{[0,s]} \subset B_{[0,t]}$ for $s < t$ and $B_\infty = B_{[0,t]} \otimes B_{]t,\infty)}$. There is some freedom in the choice of the algebras $B_{[0,t]}$, the above choice is a minimal one, maybe not adapted to all situations one may encounter. However, most of the following computations are algebraic and valid for a larger class of operators on $\Gamma(\mathbb{L}_\kappa^2(\mathbb{R}_+))$ than those in $B_{[0,t]}$.

• *Flow:* The time evolution of the system operator defines a flow J_t from A_0 to A_t by:

$$J_t : A_0 \to A_t, \quad J_t(z) \equiv z(t) = e^{-\bar\gamma t} z + \xi(t). \tag{7}$$

Similarly, $J_t(z^\dagger) \equiv z^\dagger(t) = e^{-\gamma t}z^\dagger + \xi^\dagger(t)$. It is a $*$-homomorphism since $[z(t), z^\dagger(t)] = 1$, because $G(t,t) = 1 - e^{-(\gamma+\bar\gamma)t}$, so that

$$J_t(e^{\mu z^\dagger}e^{\bar\mu z}) = e^{\mu z^\dagger(t)}e^{\bar\mu z(t)} = e^{\mu(t)z^\dagger}e^{\bar\mu(t)z}\, e^{\mu\xi^\dagger(t)}e^{\bar\mu\xi(t)}\,.$$

with

$$\mu(t) = \mu\, e^{-\gamma t}, \quad \bar\mu(t) = \bar\mu\, e^{-\bar\gamma t}.$$

Eq.(7) is the quantum analogue of a stochastic classical flow with $\xi(t)$ playing the role of quantum noise.

• *Measure:* The quantum noises $\xi(t)$ are elements of B_∞. The measure over it is $\mathbb{E}$. It is gaussian with two-point function $\mathbb{E}[\xi^\dagger(t)\xi(s)] = \mathfrak{n}_0 G(s,t)$ so that

$$\mathbb{E}[\, e^{\mu_1\xi^\dagger(t_1)}e^{\bar\mu_1\xi(t_1)}\cdots e^{\mu_N\xi^\dagger(t_N)}e^{\bar\mu_N\xi(t_N)}\,] = \exp[\sum_{i,j}\bar\mu_i\mu_j G(t_i,t_j)(\mathfrak{n}_0 + \mathbf{1}_{\{i<j\}})].$$

Here μ_j and $\bar\mu_j$ are formal parameters not necessary complex conjugate. If one chooses $\bar\mu_j = -\mu_j^*$, then the operators involved in the above equation are in B_∞.

The measure $\mathbb{E}$ is neutral with respect to elements of the system oscillator algebra, that is $\mathbb{E}[a_0\, b] = a_0\mathbb{E}[b]$ for any $a_0 \in A_0$, so that it extends to a map from A_∞ to A_0:

$$\mathbb{E}: A_\infty \to A_0.$$

On the bare canonical operators $a(s)$ the measure is $\mathbb{E}[a^\dagger(s)a(s')] = \mathfrak{n}_0\delta(s-s')$, and:

$$\mathbb{E}[d\mathfrak{A}(t)\, d\mathfrak{A}(t)] = 0, \quad \mathbb{E}[d\mathfrak{A}^\dagger(t)\, d\mathfrak{A}(t)] = \mathfrak{n}_0\, dt.$$

To complete the setting, one has to provide the measure $\mathrm{Tr}_{\Gamma_S}(\rho_0 \cdots)$ on A_0 specified by the system density matrix. Classically, $\mathbb{E}$ would be the measure on the noise and $\mathrm{Tr}_{\Gamma_S}(\rho_0 \cdots)$ the measure on the initial position of the stochastic process.

3.2   Dynamical maps and density matrices

• *Dynamical map:* The map Φ_t on A_0 is obtained by evaluating one-point functions: $\Phi_t(a) = \mathbb{E}[J_t(a)]$ for $a \in A_0$. In the present case, it yields:

$$\Phi_t(e^{\mu z^\dagger}e^{\bar\mu z}) = e^{\mu(t)z^\dagger}e^{\bar\mu(t)z}\, e^{\mathfrak{n}_0[\mu\bar\mu - \mu(t)\bar\mu(t)]}. \tag{8}$$

Of course $\Phi_t(1) = 1$. As can be checked by a direct computation, it defines a semi-group on A_0:

$$\Phi_t \circ \Phi_s = \Phi_{t+s}.$$

The form of the generator of the dynamical maps that we are going to find, in eq.(9) below, implies that Φ_t are completely positive maps, as required by the theory of quantum processes [10, 11, 12]. It is the quantum analogue of transition probability kernels for classical stochastic processes.

• *Lindbladian:* The Lindbladian is the generator of the dynamical semi-group. Indeed, the semi-group law (with extra continuous properties) implies that

$$\Phi_t = \exp(\,tL\,)$$

where $L = \frac{d}{dt}\Phi_t|_{t=0}$ is a so-called Lindbladian, with a structure imposed by the general theory of completely positive dynamical maps [17]:

$$L(a) = i[h_s, a] + \sum_j g_j \,(2D_j a D_j^\dagger - D_j D_j^\dagger a - a D_j D_j^\dagger), \quad a \in A_0$$

with some effective hamiltonian $h_s = h_s^\dagger$ and some operators D_j and $g_j > 0$. In the present case, L is quadratic in z and $z^\dagger$ and, for all $a \in A_0$,

$$L(a) = -i\nu\,[z^\dagger z, a] + g_1\,(2zaz^\dagger - zz^\dagger a - azz^\dagger) + g_2\,(2z^\dagger az - z^\dagger za - az^\dagger z), \quad (9)$$

where we set $\gamma = \kappa + i\nu$ and

$$g_1 = \kappa \mathfrak{n}_0, \quad g_2 = \kappa(\mathfrak{n}_0 + 1).$$

Of course $L(1) = 0$ and $L(a^\dagger) = L(a)^\dagger$. Set $L(a) = -i\nu[z^\dagger z, a] + D(a)$ where $D(a)$ is the so-called dissipative part of the Linbladian. It may alternatively be written as

$$D(a) = -\kappa \mathfrak{n}_0 \left([z^\dagger, [z, a]] + [z, [z^\dagger, a]]\right) + \kappa\,(2z^\dagger az - z^\dagger za - az^\dagger z).$$

The double commutator term coincides with the (quantum analogue of the) two dimensional Laplacian. As expected, the imaginary part of γ only enters into the hamiltonian part of the Linbladian, but note that the frequency ν is not the bare frequency ϵ. The dissipative part of L only depends on $\Re\,\gamma$ and $\mathfrak{n}_0$. This Lindbladian has been considered in ref.[3] to describe damped harmonic oscillator.

• *Master equation:* This refers to the evolution equation for the system reduced density matrix. It is obtained from the observable time evolution by duality since $\mathrm{Tr}_{\Gamma_S}(\rho_0\,\Phi_t(a)) = \mathrm{Tr}_{\Gamma_S}(\rho_t\,a)$ for $a \in A_0$. Hence $\rho_t = e^{tL^*}\cdot\rho_0$. So that

$$\frac{d}{dt}\rho_t = L^* \cdot \rho_t \tag{10}$$

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with

$$L^* \cdot \rho = i\nu[z^\dagger z, \rho] + g_2 \left(2z\rho z^\dagger - \rho z^\dagger z - z^\dagger z\rho\right) + g_1 \left(2z^\dagger \rho z - \rho zz^\dagger - zz^\dagger \rho\right).$$

Eq.(10) is the quantum analogue of Fokker-Planck equations for Markov processes. It preserves the normalisation of ρ, i.e. $\mathrm{Tr}_{\Gamma_S}(L^* \cdot \rho) = 0$.

The system admits an invariant measure which is a density matrix ρ_{inv} such that $L^* \cdot \rho_{\mathrm{inv}} = 0$. It is gaussian:

$$\rho_{\mathrm{inv}} = \frac{1}{Z}\, e^{-\sigma\, z^\dagger z}, \quad \text{with } e^\sigma = \frac{\mathfrak{n}_0 + 1}{\mathfrak{n}_0},$$

and $Z = 1/(1 - e^{-\sigma})$. Alternatively, $\mathfrak{n}_0 = 1/(e^\sigma - 1)$. An effective temperature may be defined either by $\sigma = \nu/T$ or $\sigma = \epsilon/T$. The invariant measure depends on $\mathfrak{n}_0$ but not on κ.

Since κ represents damping effects, the approach to equilibrium is κ-dependent. It is exponentially fast as $e^{-2\kappa t}$ with relaxation time $1/\kappa$. It is governed by the smallest eigenvalue of L^* which is -2κ:

$$L^* \cdot \rho_1 = -2\kappa\, \rho_1,$$

with eigenvector $\rho_1 = e^{-\sigma\, z^\dagger z}[z^\dagger z - 2\mathfrak{n}_0]$. Other eigenvalues of L^* (in the zero charge sector) are $-2p\kappa$ with p integer and eigenvector $\rho_p = e^{-\sigma\, z^\dagger z}P_p(z^\dagger z)$ with P_p polynomial of degree p.

The invariant measure is stable since all eigenvalues of L^* are non positive. This is a consequence of the complete positivity of the dynamical maps which implies that the dissipative part of $L^* \cdot \rho$ is the sum of terms of the form $g_j(2D_j\rho D_j^\dagger - \rho D_j^\dagger D_j - D_j^\dagger D_j\rho)$ which are all non positive operators for $g_j > 0$. Indeed $\mathrm{Tr}_{\Gamma_S}(\rho\,(L^* \cdot \rho))$ is the sum of $2g_j(\mathrm{Tr}_{\Gamma_S}(\rho D_j\rho D_j^\dagger) - \mathrm{Tr}_{\Gamma_S}(\rho D_j D_j^\dagger \rho)) \leq 0$ by the Cauchy-Schwarz inequality.

3.3 Quantum filtration and conditional expectations

• *Filtration:* The set of embedded algebras $A_t = A_0 \otimes B_{[0,t]}$ defines a filtration [8]:

$$A_0 \subset A_s \subset A_t \subset A_\infty, \quad \text{for } 0 < s < t < \infty.$$

The algebra A_s has to be understood as the past of the process up to time s. The image $J_s(A_0)$ of the system algebra by the flow is identified as the present of the process at time s: $J_s(A_0) \subset A_s$. With these definitions the

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[8]We do not enter here in the subtleties associated to left/right time limits as the construction is clearly time continuous.

past includes the present. The algebra $A_0 \otimes B_{]t,\infty)}$ may be identified with the futur of the process.

• *Conditional expectations:* These are projectors $\mathbb{E}_s$ from A_∞ into A_s such that

$$\mathbb{E}_s : A_\infty \to A_s, \quad \mathbb{E}_{s_1} \circ \mathbb{E}_{s_2} = \mathbb{E}_{\min(s_1,s_2)}, \quad \text{for } s_1, s_2 \geq 0, \tag{11}$$

and compatible with the flow and the stochastic map in the sense that, for all $a \in A_0$:

$$\mathbb{E}_s[\,J_t(a)\,] = J_s[\,\Phi_{t-s}(a)\,]\,, \quad \text{for } t > s \geq 0. \tag{12}$$

$\mathbb{E}_s$ has to be understood as the expectation conditioned on the past of the process up to time s. Eq.(12) expresses the Markov property of the process. In the present case, compatibility with the flow imposes that, for $t > s$,

$$\mathbb{E}_s[\,e^{\mu\xi^\dagger(t)}e^{\bar\mu\xi(t)}\,] = e^{\mu(t-s)\xi^\dagger(s)}e^{\bar\mu(t-s)\xi(s)}\,e^{\mathfrak{n}_0[\mu\bar\mu - \mu(t-s)\bar\mu(t-s)]}.$$

This defines $\mathbb{E}_s$ on single-time operator but we have to define it on product of multi-time operators. Since we know the commutation relations in A_∞ it is enough to define it on product of time ordered operators. That is, we have to define

$$\mathbb{E}_s[\,e^{\mu_1\xi^\dagger(t_1)}e^{\bar\mu_1\xi(t_1)} \cdots e^{\mu_N\xi^\dagger(t_N)}e^{\bar\mu_N\xi(t_N)}\,]$$

with $t_1 < \cdots < t_N$. A constructive way to do it consists in (recursively) imposing that $\mathbb{E}_s$ is neutral with respect to left multiplication by elements of A_s, i.e. $\mathbb{E}_s[ab] = a\mathbb{E}_s[b]$ for $a \in A_s$. After a few computations summarised in Appendix 8, we get:

$$\mathbb{E}_s[e^{\mu_1\xi^\dagger(t_1)}e^{\bar\mu_1\xi(t_1)} \cdots e^{\mu_N\xi^\dagger(t_N)}e^{\bar\mu_N\xi(t_N)}\,] \tag{13}$$
$$= \quad e^{\mu_1(t_{1;s})\xi^\dagger(s)}e^{\bar\mu_1(t_{1;s})\xi(s)} \cdots e^{\mu_N(t_{N;s})\xi^\dagger(s)}e^{\bar\mu_N(t_{N;s})\xi(s)}\,e^{X_s^{(N)}}\,e^{\mathfrak{n}_0 Y_s^{(N)}}$$

for $s < t_1 < \cdots < t_N$, with

$$X_s^{(N)} = \sum_{i<j} \bar\mu_i\mu_j(t_{j;i}) - \sum_{i<j} \bar\mu_i(t_{i;s})\mu_j(t_{j;s})$$

$$Y_s^{(N)} = \sum_i \bar\mu_i\mu_i + \sum_{i<j}[\bar\mu_i\mu_j(t_{j;i}) + \mu_i\bar\mu_j(t_{j;i})] - \sum_{i,j} \bar\mu_i(t_{i;s})\mu_j(t_{j;s})$$

where $t_{j;i} = t_j - t_i$ and $t_{j;s} = t_j - s$. Notice that the r.h.s. of eq.(13) only involves $\xi(s)$ and $\xi^\dagger(s)$. One checks that $\mathbb{E}_{s=0} = \mathbb{E}$ as it should be.

• *Markov and other properties:*. The previous definition is designed to ensure that relations (11) and (12) hold, but this can be checked directly. See Appendix 8.

Since it only involves $\xi(s)$ and $\xi^\dagger(s)$, it also satisfies a (weak) Markov property because the image by $\mathbb{E}_s$ of the future algebra $B_{]s,\infty)}$ maps into the present algebra $J_s(A_0)$:

$$\mathbb{E}_s : B_{]s,\infty)} \to J_s(A_0) \tag{14}$$

By construction, $\mathbb{E}_s$ is neutral with respect to left multiplication by elements in A_s, but remarkably it is also neutral with respect to right multiplication. Namely,

$$\mathbb{E}_s[ab] = a\,\mathbb{E}_s[b] \ , \quad \mathbb{E}_s[ba] = \mathbb{E}_s[b]\,a \ , \quad \text{for } a \in A_s \tag{15}$$

for all $b \in A_\infty$. The first relation is true by construction and the second one follows from the relation

$$\mu\,[\xi(t_0), \xi^\dagger(t)] = \mu(t - s)\,[\xi(t_0), \xi^\dagger(s)],$$

for $t_0 \leq s \leq t$. See Appendix 8. This property is remarkable. In particular it means that we would have got the same conditional expectations if we would have constructed it using neutrality under right multiplication instead of neutrality under left multiplication as we did. In this sense, the conditional expectation is unique.

By construction (also), the conditional expectation satisfies the nested formula, similar to that valid in the classical theory, but for time ordered products $s < t_1 < \cdots < t_N$:

$$\begin{aligned} &\mathbb{E}_s[\,J_{t_1}(a_1) \cdots J_{t_{N-1}}(a_{N-1}) J_{t_N}(a_N)\,] \tag{16} \\ =\ & J_s\big[\Phi_{t_1-s}(a_1 \Phi_{t_2-t_1}(\cdots a_{N-1}\Phi_{t_N-t_{N-1}}(a_N)))\big] \end{aligned}$$

for $a_j \in A_0$. Note that the nesting is ordered from right to left. A similar nested formula applies for an opposite ordering of the time but with an opposite nesting. Both formula follow from eqs.(11) and from the left and right neutralities.

The nested formula eq.(16) is of course an echo that the bath dynamics has been frozen by the infinite-volume/continuous-density limit.

• *Martingales:* As for classical processes, we may define martingales – which, for instance, are instrumental for computing probabilities of events related to stopping times. A family of elements M_t in A_t is called martingale if

$$\mathbb{E}_s[\,M_t\,] = M_s, \quad \text{for } s < t.$$

Examples are for instance given by:

$$M_t = e^{\mu e^{\gamma t}\xi^\dagger(t)}\, e^{\bar\mu e^{\bar\gamma t}\xi(t)}\, e^{-\mathfrak{n}_0\mu\bar\mu e^{(\gamma+\bar\gamma)t}}.$$

To first order in $\bar\mu$, this gives $e^{\bar\gamma t}\xi(t)$ as a martingale.

3.4   Time evolution in A_∞

• *Flow:* By eq.(7) we have defined the flow of system observables. We may extend this flow to elements of A_∞. That is, we may define a semi-group of A_∞-endomorphisms, σ_s with $\sigma_0 = \mathrm{Id}$, such that:

$$\sigma_s: \ A_\infty \to A_\infty, \quad \sigma_s \circ \sigma_t = \sigma_{s+t}, \quad \sigma_s \circ J_t = J_{t+s}, \tag{17}$$

for $s, t \geq 0$. The second relation, namely $\sigma_s \circ J_t = J_{t+s}$, means that σ_s extends the flow J_s. In particular σ_s acts like J_s on elements of A_0. It implies that:

$$\sigma_s(z) \equiv e^{-\bar\gamma s}z + \xi(s), \quad \sigma_s(\xi(t)) \equiv \xi(t+s) - e^{-\bar\gamma t}\xi(s). \tag{18}$$

Similarly $\sigma_s(z^\dagger) = J_s(z^\dagger)$ and $\sigma_s(\xi^\dagger(t)) = \xi^\dagger(t+s) - e^{-\gamma t}\xi^\dagger(s)$. One may check by a direct computation that σ_s are $*$-endomorphisms and

$$[\,\sigma_s(\xi(t)), \sigma_s(\xi^\dagger(t'))\,] = [\,\xi(t), \xi^\dagger(t')\,] = G(t, t'),$$

for all $t, t' \geq 0$. Of course, $\sigma_s \circ \sigma_t = \sigma_{s+t}$. Since both σ_s and J_t are morphisms, the compatibility relation $\sigma_s \circ J_t = J_{t+s}$ is valid on A_0 because it is true for the generators of A_0.

• *Covariance:* The flow σ_t is compatible with the conditional expectations in the sense that

$$\sigma_t \circ \mathbb{E}_s = \mathbb{E}_{t+s} \circ \sigma_t. \tag{19}$$

This is a direct consequence of the previous construction. Indeed, neutrality of $\mathbb{E}_s$ for elements of A_s implies that this is true on A_∞ if it is true when applied to elements of the future algebra $B_{]s,\infty)}$. To check it on $B_{]s,\infty)}$, it is enough to check that it holds true for product elements of the form $J_{t_1}(a_1) \cdots J_{t_N}(a_N)$ with $a_j \in A_0$ and times ordered $s < t_1 < \cdots < t_N$. On such elements, eq.(19) is a direct consequence of the nested formula eq.(16) for conditional expectations and of the compatibility relation $\sigma_t(J_s(a)) = J_{t+s}(a)$ on A_0.

3.5 Independence of increments

We call $\sigma_s(\xi(t)) = \xi(t+s) - e^{-\bar{\gamma}t}\xi(s)$ the increment from s to $t+s$.

Since σ_s is an endomorphism and since the two-point functions are proportional to the commutators $G(t, t')$, the increments are identically distributed:

$$\mathbb{E}[\,\sigma_s(\xi^\dagger(t))\,\sigma_s(\xi(t'))\,] \;\; = \;\; \mathbb{E}[\,\xi^\dagger(t)\,\xi(t')\,] = \mathfrak{n}_0\, G(t', t)$$

for all $s > 0$.

Furthermore, the relation $[\xi(t_0), \xi^\dagger(t+s)] = \mu(t)\,[\xi(t_0), \xi^\dagger(s)]$ for $t_0 \leq s$ and $t \geq 0$, which was instrumental in proving the left/right neutrality of the conditional expectations, implies that:

$$\begin{aligned}
[\,\sigma_s(\xi(t))\,,\; \xi^\dagger(t_0)\,] &= 0, \quad \text{for } t_0 \leq s & (20)\\
\mathbb{E}[\,\sigma_s(\xi(t))\,\xi^\dagger(t_0)\,] &= 0, \quad \text{for } t_0 \leq s
\end{aligned}$$

These two relations then imply that $[\,\sigma_s(\xi(t)), a\,] = 0$ and $\mathbb{E}[\,\sigma_s(\xi(t))\,a\,] = 0$ for any $a \in A_s$. It means that the increments commute with the past algebra A_s and are independent of the past. Another remarkable property.

4 Limiting cases and decoherence

4.1 The quantum t-Brownian limit

The process simplifies in the limit $\gamma \to 0$. Set $\gamma = \kappa + i\nu$ and let $\chi(t) = \frac{1}{\sqrt{\kappa}}\xi(t)e^{-i\nu t}$. In the limit $\kappa \to 0$, these can be written as $\chi(t) = \int_0^t ds\, a(s)$, in terms of the bare canonical operators $a(s)$, with $[a(s), a^\dagger(s')] = \delta(s - s')$. Then

$$\begin{aligned}
[\chi(t), \chi^\dagger(s)] &= 2\min(t, s)\\
\mathbb{E}[\chi^\dagger(t)\chi(s)] &= 2\mathfrak{n}_0\min(t, s)
\end{aligned}$$

in the limit $\kappa \to 0$. The real and imaginary parts of $\chi(t)$ are two Brownian motions but which do not commute except at time $t = 0$.

This coincides with the quantum process (also called a quantum Brownian motion) defined in ref.[18] using Hopf algebra techniques. Actually, our initial algebra A_0 has to be slightly modified in order to take care of the limit $\kappa \to 0$. Following ref.[18], we define the initial algebra as the Heisenberg algebra $\tilde{A}_0$ generated by elements $\tilde{z}$, $\tilde{z}^\dagger$ and τ with commutation relations

$$[\tilde{z}, \tilde{z}^\dagger] = \tau, \quad [\tau, \tilde{z}^\dagger] = 0 = [\tau, \tilde{z}].$$

The flow is defined by $J_t(\tilde{z}) = \chi(t)$ and $J_t(\tau) = t$. It is a $*$-homomorphism. The dynamical maps is the $\gamma \to 0$ limit of eq.(8), namely:

$$\Phi_t(e^{\eta \tilde{z}^\dagger} e^{\bar{\eta} \tilde{z}}) = e^{\eta \tilde{z}^\dagger} e^{\bar{\eta} \tilde{z}} \, e^{2\eta\bar{\eta}\,\mathfrak{n}_0 t}.$$

In this limit, the conditional expectations are

$$\mathbb{E}_s[e^{\eta \chi^\dagger(t)} e^{\bar{\eta}\chi(t)}] = e^{\eta\chi^\dagger(s)} e^{\bar{\eta}\chi(s)} \, e^{2\eta\bar{\eta}\,\mathfrak{n}_0(t-s)}.$$

It indicates that the increments $\chi(t+s) - \chi(s)$ are independent and identically distributed.

Note that the above limit is not the same as the limit $\kappa \to 0$ but with $D \equiv \kappa \mathfrak{n}_0$ fixed which would gives

$$[\xi(t), \xi^\dagger(s)] = 0, \quad \mathbb{E}[\xi^\dagger(t)\xi(s)] = 2D\, e^{i\nu(t-s)} \min(t, s)$$

and conditional expectations

$$\mathbb{E}_s[e^{\mu\xi^\dagger(t)} e^{\bar{\mu}\xi(t)}] = e^{\mu e^{i\nu(t-s)}\xi^\dagger(s)} e^{\bar{\mu}e^{-i\nu(t-s)}\xi(s)} \, e^{2\mu\bar{\mu}\,D(t-s)},$$

for $t > s$. It describes two commuting Brownian motions. The system algebra is still that of the harmonic oscillator, $[z, z^\dagger] = 1$. The flow simply corresponds to add the random $\mathbb{C}$-number $\xi(t)$ to the canonical operators z, an operation which obviously preserves the commutation relations. The Linbladian also simplifies in this limit, since only the double commutator remains.

4.2 Semi-classical limit

To take the semi-classical limit we put back the $\hbar$ factors and set $\epsilon = \hbar\omega_0$ and replace φ_α by $\hbar\varphi_\alpha$ so that φ_α has the dimension of a frequency. The hamiltonian is then:

$$H = \sum_\alpha \hbar\omega_\alpha a_\alpha^\dagger a_\alpha + \lambda\hbar\big(w^\dagger \sum_\alpha \varphi_\alpha a_\alpha + w \sum_\alpha \bar{\varphi}_\alpha a_\alpha^\dagger\big) + \hbar\omega_0\, w^\dagger w.$$

The large time limit is taken in the same way. The factors $\hbar$ compensate in evaluating $z(t)$ so that we again have $z(t) = e^{-\bar{\gamma}t}z + \xi(t)$ with $[\xi(t), \xi^\dagger(s)] = G(t, s)$ where

$$G(t, s) = e^{-\bar{\gamma}t - \gamma s}\left(e^{(\gamma+\bar{\gamma})\min(t,s)} - 1\right)$$

The two point function is still given by $\mathbb{E}[\xi^\dagger(t)\xi(s)] = \mathfrak{n}_\circ G(s, t)$. The occupation number $\mathfrak{n}_0$ may be estimated as if it is given by thermal equilibrium at some temperature T, that is $\mathfrak{n}_0 = 1/(e^{\hbar\omega_0/T} - 1)$ and $\mathfrak{n}_0 \simeq T/\hbar\omega_0$ as $\hbar$ goes to 0.

The semi-classical limit is thus $\hbar \to 0$ with $\mathfrak{n}_0 \hbar \equiv T/\omega_0$ fixed.

Set $\gamma = \kappa + i\nu$ and let $X(t) = \sqrt{\hbar} \, \Re \, z(t) e^{-i\nu t}$ and $Y(t) = \sqrt{\hbar} \, \Im \, z(t) e^{-i\nu t}$. Then, in the semi-classical limit, $X(t)$ and $Y(t)$ are two commuting gaussian processes with [9]

$$\mathbb{E}[\, X(t)X(s)\,] = \mathbb{E}[\, Y(t)Y(s)\,] = \frac{T}{2\omega_0} \, e^{-\kappa(t+s)} \big(e^{2\kappa \min(t,s)} - 1\big).$$

They are two commuting Ornstein-Uhlenbeck processes. As $\kappa \to 0$, $X(t)/\sqrt{\kappa}$ and $Y(t)/\sqrt{\kappa}$ goes over two (un-normalized) commuting Brownian motions.

4.3   Decoherence and thermalisation

Let us first consider the evolution of a density matrix which initially is simply the pure state $\rho_0 = |\alpha\rangle\langle\alpha|$ with $|\alpha\rangle$ the coherent state $|\alpha\rangle \equiv e^{-\alpha\bar{\alpha}/2} \, e^{\alpha z^\dagger} |0\rangle$. At time t, the density matrix is determined by duality via $\mathrm{Tr}_{\Gamma_S}(\rho_t \, a) = \mathrm{Tr}_{\Gamma_S}(\rho_0 \, \Phi_t(a))$ with Φ_t given in eq.(8). Using $\mathrm{Tr}_{\Gamma_S}(\rho_0 \, e^{\mu z^\dagger} e^{\bar{\mu} z}) = e^{\alpha\bar{\mu} + \bar{\alpha}\mu}$, this gives

$$\mathrm{Tr}_{\Gamma_S}(\rho_t \, e^{\mu z^\dagger} e^{\bar{\mu} z}) = e^{\alpha\bar{\mu}(t) + \bar{\alpha}\mu(t)} \, e^{\mathfrak{n}_0(\mu\bar{\mu} - \mu(t)\bar{\mu}(t))}.$$

This is solved in terms of superposition of coherent states by:

$$\rho_t = \int \frac{da d\bar{a}}{\pi\sigma_t} \, |a\rangle \, e^{-\frac{1}{\sigma_t}(a - \alpha(t))(\bar{a} - \bar{\alpha}(t))} \, \langle a|.$$

with $\sigma_t = \mathfrak{n}_0(1 - e^{-2\kappa t})$ and $\alpha(t) = e^{-\bar{\gamma}t}\alpha$, an evolution equation similar to that of z but without the quantum noice. The interpretation is clear: the density matrix is an (incoherent) sum of coherent states centred around the un-noisy solution $\alpha(t)$ with dispersion σ_t as in the classical case. As it should be, it converges at large time towards the thermal invariant measure, since:

$$\rho_\infty = \int \frac{da d\bar{a}}{\pi\mathfrak{n}_0} \, |a\rangle \, e^{-\frac{a\bar{a}}{\mathfrak{n}_0}} \, \langle a| = \frac{1}{\mathfrak{n}_0 + 1} \, e^{-\sigma \, z^\dagger z},$$

with $e^\sigma = (\mathfrak{n}_0 + 1)/\mathfrak{n}_0$. The thermalisation time is

$$\tau_{\mathrm{therm}} = 1/\kappa.$$

Let us now consider an initial density matrix which is still that of a pure state but for a state linear combination of coherent states, a kind of Schrodinger cat:

$$\rho_0 = \big(u|\alpha\rangle + v|\beta\rangle\big)\big(\bar{u}\langle\alpha| + \bar{v}\langle\beta|\big),$$

---

[9]With our convention, X has dimension of $\sqrt{\hbar}$ since $X \simeq \sqrt{\hbar} \, \Re \, z$. To go to the physical picture with X having dimension of a length, one would have to introduce the mass scale m of the harmonic oscillator and set $X \simeq \sqrt{\hbar/m\omega_0} \, \Re \, z$.

with $|\alpha - \beta|$ large. We explicitely write $\rho_0 = u\bar{u}\,\rho_0^{\alpha\alpha} + u\bar{v}\,\rho_0^{\alpha\beta} + v\bar{u}\,\rho_0^{\beta\alpha} + v\bar{v}\,\rho_0^{\beta\beta}$. By linearity of the evolution equation, the density matrix at time t will be of form

$$\rho_t = u\bar{u}\,\rho_t^{\alpha\alpha} + u\bar{v}\,\rho_t^{\alpha\beta} + v\bar{u}\,\rho_t^{\beta\alpha} + v\bar{v}\,\rho_t^{\beta\beta}.$$

The evolution of each of the terms is determined as above. For instance:

$$\rho_t^{\beta\alpha} = \langle \alpha|\beta\rangle \int \frac{da\,d\bar{a}}{\pi\sigma_t}\,|a\rangle\,e^{-\frac{1}{\sigma_t}(a-\beta(t))(\bar{a}-\bar{\alpha}(t))}\,\langle a|,$$

or equivalently,

$$\langle b|\rho_t^{\beta\alpha}|c\rangle = \langle \alpha|\beta\rangle\langle b|c\rangle\,e^{-\frac{1}{\sigma_t+1}(c-\beta(t))(\bar{b}-\bar{\alpha}(t))},$$

with $|b\rangle$ and $|c\rangle$ coherent states. In order to code for the decoherence, let us consider the ratio of the matrix elements of the 'mixed' density matrices, $\rho_t^{\beta\alpha}$ or $\rho_t^{\alpha\beta}$, over the pure ones, $\rho_t^{\alpha\alpha}$ or $\rho_t^{\beta\beta}$, defined by:

$$\mathfrak{R}_t = \frac{\langle b|\rho_t^{\beta\alpha}|c\rangle\,\langle b|\rho_t^{\alpha\beta}|c\rangle}{\langle b|\rho_t^{\alpha\alpha}|c\rangle\,\langle b|\rho_t^{\beta\beta}|c\rangle},$$

with $\mathfrak{R}_0 = 1$. An explicit computation shows that this ratio is independent of b and c and given by

$$\mathfrak{R}_t = |\langle\alpha|\beta\rangle|^2\,|\langle\alpha(t)|\beta(t)\rangle|^{-2/(\sigma_t+1)}.$$

Recall that $|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}$. At short time, $\sigma_t \simeq 2\mathfrak{n}_0\kappa\,t$, so that

$$\log\mathfrak{R}_t \simeq -2(\mathfrak{n}_0+1)\kappa\,t\,|\alpha-\beta|^2,$$

at short time. Hence, the mixed density matrices decrease exponentially faster than the pure ones.

Thus there are two regimes: first a short decoherence time $\tau_{\text{coher}} = \frac{1}{\kappa(\mathfrak{n}_0+1)|\alpha-\beta|^2}$, whose semi-classical limit is (recall that $\mathfrak{n}_0 \simeq T/\hbar\omega_0$ and $\delta X \simeq \sqrt{\hbar}\,\delta\alpha$)

$$\tau_{\text{coher}} \simeq \frac{\omega_0}{\kappa T}\frac{\hbar^2}{|\delta X|^2},$$

and which decreases as the inverse power of the square distance between the states, and then a longer thermalisation time $\tau_{\text{therm}} \simeq 1/\kappa$.

5   Conclusion

Motivated by an initial problem related to the quantum Hall effect, these notes have investigated a simple open quantum system. Starting from a traditional coupled system-reservoir model, we were able to study most of its salient features in the weak-coupling/long-time limit, from the more physical to the more mathematical.

The model is simple enough to be solved by several methods. One of these concentrates on the limiting behavior of expectations and commutators, without paying attention to the limit of operators themselves. The output is an abstract (almost axiomatic) representation result for the limiting process in Fock spaces. Another approach, which leads to the same result, pays more attention to the relationship between the bath oscillators of the initial model and their weak-coupling/long-time cousins. As expected, dissipation arises via destructive interferences.

Along the way, we retrieve a number of well-studied objects that we connect together, like the damped harmonic oscillator on the physical side, or the quantum stochastic calculus and quantum geometry on the mathematical side. We pay a particular attention to the explicit construction, in this simple case, of a quantum filtration and the associated quantum conditional expectations. This leads to the specification of the (mathematical) dilation of the quantum dynamical maps. It gives access to multi-time correlation functions that cannot be computed via the knowledge of the system time dependent density matrix only. The quantum phase space Brownian motion we obtain possesses peculiar properties analogous to those of the classical one. We observe the remarkable fact that despite non-trivial non-commutative effects, the conditional expectations have the property to be transparent to multiplication by measurable observables (on the left or on the right). We also do a sample computation of decoherence.

One of the missing probabilistic concepts is that of stopping time – which is instrumental for computing exit probabilities or harmonic measures. The concept is defined in the mathematical literature, but the examples are few and a bit artificial. In the present case, and if the system starts in the ground state, one can define the time at which the system leaves the ground state. This can be checked to be a stopping time, i.e. deciding if it occurs before time t can be done by looking only at conditional expectations up to time t. Alas, due to continuous time, this time vanishes almost surely. However, our discussion of filtration and conditional expectations can be extended straightforwardly to a discrete version of eq.(6), namely $z_0 = z$ and $z_n = \lambda z_{n-1} + (1 - |\lambda|^2)^{1/2} a_n$ for $n \geq 1$ (where $|\lambda| \in [0, 1[$ and the a_n's

are standard independent annihilation operators). Then one checks without difficulty that, starting the system in the ground state at $n = 0$, the time at which it leaves the ground state is a stopping time, which follows a geometric distribution. Other examples of meaningful stopping times, in the discrete and especially in the continuous time case, would surely be desirable.

We hope to come back in a near future to all these themes in other simple open quantum systems – including generalisations to finite dimensional quantum systems and the associated quantum stochastic processes, aspects of quantum geometry of quantum phase space Brownian motion – and to the more puzzling case of dissipative effects in the quantum Hall effect and their relations to the behaviour of the first Landau level coupled to a reservoir.

6 Appendix: Algebraisation of stochastic processes

The aim of this appendix is to recap the algebraic structure underlying classical stochastic processes for comparison with the quantum theory. Although we did not find it gathered in one reference, we have no doubt that the material presented here is already somewhere (dispersed) in the literature.

Suppose we are given an adapted stochastic process, defined on a probability space Ω, with measure $\mathbb{E}$ and filtration $\mathcal{F}_t$, and valued in some measured space V with measure dy. For $\omega \in \Omega$, let $t \to X_t(\omega)$, or simply X_t, be the realisation of the process. By construction this defines a path on V (for simplicity we may assume that it is continuous). The measure $\mathbb{E}$ then induces a measure on paths on V.

To simplify the writing, we shall restrict ourselves to time-homogeneous Markov processes but most of the algebraic structure we shall describe remains valid for non Markovian processes (up to simple reformulations). Let $P_t(dy, x)$ be the transition kernel, i.e. the probability density for the process started at x at time $t = 0$ to be in the neighbourhood of y at time t, so that for any (good enough) function f from V to, say, $\mathbb{C}$ we have:

$$\mathbb{E}_x[\, f(X_t)] = \int f(y)\, P_t(dy, x).$$

Let $\mathcal{A}_0$ be the algebra of measurable functions on V with values, say, in $\mathbb{C}$ (with the Borel σ-algebra), $\mathcal{A}_t$ be the algebra of $\mathcal{F}_t$-measurable functions and $\mathcal{A}_\infty$ some inductive limit of these algebras adapted to the filtration: $\mathcal{A}_0 \equiv \mathrm{Func}(V \to \mathbb{C})$ and

$$\mathcal{A}_t \equiv \{f \in \mathrm{Func}(\Omega \to \mathbb{C}),\ \text{s.t.}\ f\ \text{is}\ \mathcal{F}_t\text{-measurable}\}.$$

Since to any $\omega \in \Omega$ we associate a path $t \to X_t(\omega)$ on V, function of $\mathcal{A}_t$ are functions on parameterized paths $[0, t] \to V$ of "length" t drawn on V. The algebra $\mathcal{A}_\infty$ is the algebra of $\mathbb{E}$-measurable functions. We identify $\mathcal{A}_0$ with $\mathcal{A}_{t=0}$. Of course, since $\mathcal{F}_t$ is a filtration, we have the embeddings:

$$\mathcal{A}_0 \subset \mathcal{A}_s \subset \mathcal{A}_t \subset \mathcal{A}_\infty, \quad \text{for } s < t.$$

Naively, $\mathcal{F}_s$-measurable functions are functions only of the portion of the path $t \to X_t(\omega)$ for $t \in [0, s]$, i.e. they code about the knowledge of the process from time 0 to time s.

The data of the stochastic flow X_t (tautologically) induces a map J_t from $\mathcal{A}_0$ to $\mathcal{A}_t$ which to any test function f on V associate an $\mathcal{F}_t$-measurable function by:

$$J_t:\ \mathcal{A}_0 \to \mathcal{A}_t, \quad J_t(f)(\omega) = f(X_t(\omega)).$$

Given P_t, we may define the stochastic map Φ_t on $\mathcal{A}_0$ by:

$$\Phi_t(f)(x) \equiv \mathbb{E}_x[\,f(X_t)\,], \quad f \in \mathcal{A}_0.$$

Alternatively, $\Phi_t = \mathbb{E} \circ J_t$, with a slight abuse of notation. As P_t, they define a semi-group $\Phi_t \circ \Phi_s = \Phi_{t+s}$. Note that Φ_t is defined using only one-point functions (involving only one time).

Conditional expectations are maps from $\mathcal{A}_t$ onto $\mathcal{A}_s$ for $s < t$:

$$\mathbb{E}[\,F_t\,|\mathcal{F}_s] \in \mathcal{A}_s, \quad \text{for } F_t \in \mathcal{A}_t.$$

They are neutral with respect to multiplication by $\mathcal{F}_s$-measurable functions:

$$\mathbb{E}[\,F_s\,F\,|\mathcal{F}_s] = F_s\,\mathbb{E}[\,F\,|\mathcal{F}_s].$$

As a consequence, they are nested projectors since $\mathbb{E}[F_s|\mathcal{F}_s] = F_s$ for $F_s \in \mathcal{A}_s$ and

$$\mathbb{E}[\,\mathbb{E}[\,F\,|\mathcal{F}_s]\,|\mathcal{F}_t] = \mathbb{E}[\,F\,|\mathcal{F}_{\min(s;t)}],$$

for any $\mathbb{E}$-measurable function F.

The data of $P_t(dy, x)$ specifies the process. In particular, by conditioning recursively on the position of the path at intermediate times, multipoint correlation functions may be evaluated as

$$\begin{aligned}
\mathbb{E}_x[\,f_1(X_{t_1}) &\cdots f_n(X_{t_n})\,] \\
&= \int f_1(y_1) P_{t_1}(dy_1, x) \cdots \int f_n(y_n) P_{t_n - t_{n-1}}(dy_n, y_{n-1}),
\end{aligned}$$

for $t_1 < \cdots < t_n$. By definition conditioning on $X_{t=0} = x$ amounts to put the Dirac point measure on $\mathcal{F}_0$, so that these multipoint expectations may also be written in terms of the maps Φ_t as:

$$\mathbb{E}[\,f_1(X_{t_1}) \cdots f_n(X_{t_n})|\mathcal{F}_0] = \Phi_{t_1}\big(f_1\,\Phi_{t_2 - t_1}\big(\cdots f_{n-1}\,\Phi_{t_n - t_{n-1}}(f_n)\big)\big)$$

More intrinsically (and for non Markovian processes), using the nested property of conditional expectations, for any $\mathcal{F}_{t_j}$-measurable functions F_j, we can recursively write

$$\begin{aligned}
\mathbb{E}[\,F_1 \cdots F_n|\mathcal{F}_0] &= \mathbb{E}[\,F_1 \cdots F_{n-1}\mathbb{E}[F_n|\mathcal{F}_{t_{n-1}}]|\mathcal{F}_0\,] \\
&= \mathbb{E}[\,F_1\mathbb{E}[F_2 \cdots \mathbb{E}[F_n|\mathcal{F}_{t_{n-1}}] \cdots |\mathcal{F}_{t_2}]|\mathcal{F}_0\,]
\end{aligned}$$

We see that this formula can be written in terms of maps $\Phi_{t;s}$, which map $\mathcal{F}_t$-measurable functions onto $\mathcal{F}_s$-measurable functions, defined for $s < t$ by:

$$\Phi_{t;s}(F) \equiv \mathbb{E}[\,F\,|\mathcal{F}_s] \in \mathcal{A}_s, \quad \text{for } F \in \mathcal{A}_t.$$

Indeed,

$$\mathbb{E}[\,F_1 \cdots F_n | \mathcal{F}_0\,] = \Phi_{t_1,0}\big(F_1 \, \Phi_{t_2;t_1}\big(\cdots F_{n-1} \, \Phi_{t_n;t_{n-1}}(F_n)\big)\big)$$

For $F_j = f_j(X_{t_j})$ and time homogeneous Markov processes, this last formula coincides with the previous one since then $\Phi_{t;s}(f(X_s)) = \Phi_{t-s}(f)$ for $s < t$.

7 Appendix: Another approach to the long time limit

The aim of this Appendix is to re-derive the long-time/weak-coupling/continuous-density limit of Section 2.2 differently in way which makes more transparent the fact that damping arises from of an infinite number of destructive interferences.

Without loss of generality, we assume that the φ_α's are non vanishing. We also assume that the ω_α's are all distinct.

Define a rational function

$$S(\Omega) \equiv \Omega - \epsilon - \lambda^2 \sum_\alpha \frac{|\varphi_\alpha|^2}{\Omega - \omega_\alpha}.$$

Its degree is equal to the number of oscillators in the bath plus one. Direct substitution shows that Ω is an eigen-frequency of the coupled system (i.e. the coupled equations have a solution for which both w and the a_α's have a time dependence $e^{-i\Omega t}$) if and only if $S(\Omega) = 0$. The variations of S show immediately that the solutions are real and intertwine the ω_α's: as expected, there is one more solution than the number of oscillators in the bath. In particular, there is at most one negative eigenfrequency. Let $\{\Omega_k\}$ be the set of eigenfrequencies of the coupled system.

The Hamiltonian is bounded below if and only no solution is negative, i.e. if

$$\epsilon > \sum_\alpha \frac{|\varphi_\alpha|^2}{\omega_\alpha}.$$

We assume that this condition is satisfied in what follows.

Writing

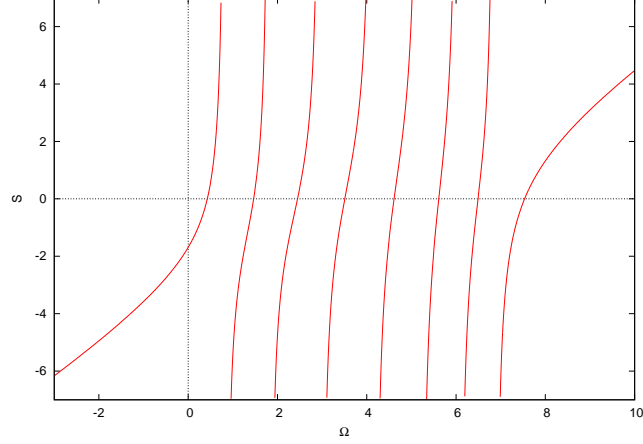$$\frac{1}{S(\Omega)} = \sum_k \frac{R_k}{\Omega - \Omega_k},$$

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Figure 1: *The typical shape of the function $S(\Omega)$.*

we claim that the general solution of the coupled system is

$$
\begin{aligned}
w(t) &= \sum_k R_k e^{-i\Omega_k t}\left(w(0) + \sum_\alpha a_\alpha(0)\frac{\lambda\varphi_\alpha}{\Omega_k - \omega_\alpha}\right) \\
a_\beta(t) &= \sum_k R_k e^{-i\Omega_k t}\left(w(0) + \sum_\alpha a_\alpha(0)\frac{\lambda\varphi_\alpha}{\Omega_k - \omega_\alpha}\right)\frac{\lambda\bar\varphi_\beta}{\Omega_k - \omega_\beta}.
\end{aligned}
$$

These formulæ could be obtained mechanically via residue calculus using the inverse Laplace transform solution presented in Section 2.2. But a direct argument is easy too. For fixed k, the coefficient for $a_\beta(0)$ is just $\frac{\lambda\bar\varphi_\beta}{\Omega_k - \omega_\beta}$ times the coefficient for w, so this is indeed a family of solutions of the coupled equations. What remains to be checked is that the initial conditions are satisfied. This is the case because

$$
\begin{aligned}
\sum_k R_k &= 1, \\
\sum_k \frac{R_k}{\Omega_k - \omega_\alpha} &= 0 \text{ for any } \alpha, \\
\sum_k \frac{R_k}{(\Omega_k - \omega_\alpha)(\Omega_k - \omega_\beta)} &= \delta_{\alpha;\beta}\,\frac{1}{\lambda^2\bar\varphi_\beta\varphi_\alpha} \text{ for any } \alpha, \beta.
\end{aligned}
$$

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The first equation expresses that $S(\Omega)$ behaves like Ω for large Ω, so that $1/S(\Omega)$ behaves like $1/\Omega$. The second equation expresses that $S(\Omega)$ has a pole at each ω_α so that $1/S(\omega_\alpha) = 0$. For the third identity, we consider two cases. If $\alpha \neq \beta$, writing $\frac{1}{(\Omega_k - \omega_\alpha)(\Omega_k - \omega_\beta)} = \frac{1}{\omega_\alpha - \omega_\beta} \left(\frac{1}{\Omega_k - \omega_\alpha} - \frac{1}{\Omega_k - \omega_\beta} \right)$, the third identity is a consequence of the second. For $\alpha = \beta$, the left-hand side is nothing but $-\frac{d}{d\Omega} \frac{1}{S(\Omega)}$ evaluated at $\Omega = \omega_\alpha$. As $S(\Omega) \sim \frac{-\lambda^2 |\varphi_\alpha|^2}{\Omega - \omega_\alpha}$ for Ω close to ω_α the result follows.

From now on, our aim is to compute the behavior of the Ω_k's and R_k's when the ω_α's and φ_α's are adjusted to mimic a smooth density. We choose a smooth positive density function $r(\omega)$ and consider Riemman sums with a mesh η, a small parameter with the dimension of a frequency, leading to

$$S(\Omega) = \Omega - \epsilon - \eta \sum_{l \in [1,L]} \frac{\lambda^2 r(\eta l)}{\Omega - \eta l}.$$

Later, we can let $\eta \to 0^+$ and $L \to +\infty$. We assume that $\epsilon > \sum_{l \in [1,L]} \frac{r(\eta l)}{l}$, which is ensured for η small enough if $\epsilon > \int_0^{+\infty} r(\omega) \frac{d\omega}{\omega}$, an integral that we assume to be convergent. The zeroes of S can be ordered as $0 < \Omega_0 < \eta < \Omega_1 < 2\eta < \cdots < \Omega_{L-1} < L\eta < \Omega_L$. Though not stricly evenly spaced, the Ω_k's are good enough to define Riemman sums.

The phases of the φ_α's can be reabsorbed in the bath oscillators, so we may assume that the φ_α's are real and positive. Then, we can write the coupled time evolution as

$$w(t) = \sum_{k \in [0,L]} R_k e^{-i\Omega_k t} \left(w(0) + \sum_{l \in [1,L]} a_l(0) \frac{\lambda \sqrt{\eta r(\eta l)}}{\Omega_k - \eta l} \right)$$

$$a_{l'}(t) = \sum_{k \in [0,L]} R_k e^{-i\Omega_k t} \left(w(0) + \sum_{l \in [1,L]} a_l(0) \frac{\lambda \sqrt{\eta r(\eta l)}}{\Omega_k - \eta l} \right) \frac{\lambda \sqrt{\eta r(\eta l')}}{\Omega_k - \eta l'}.$$

Fix $\Omega < \eta L$ and take k such that $k\eta \leq \Omega < (k+1)\eta$. We want to find an approximate formula for $S(\Omega)$ for fixed Ω when η is small and L large. We shall see that the condition $\eta \log L$ small is important.

More precisely, we let $\eta \to 0$ but write $\Omega = \eta(k+\delta)$, $\delta \in\,]0,1[$, so that the interval between two adjacent eigenvalues is blown up to size 1. We need to compute $\eta(k+\delta) - \epsilon - \lambda^2 \sum_{l \in [1,L]} \frac{r(\eta l)}{k - l + \delta} = 0$. We split the sum as

$$\sum_{l \in [1,L]} \frac{r(\eta l)}{k - l + \delta} = \frac{r(\eta k)}{\delta} + \sum_{l \in [1,L], l \neq k} \left(\frac{r(\eta l)}{k - l + \delta} - \frac{r(\eta l)}{k - l} \right) + \sum_{l \in [1,L], l \neq k} \frac{r(\eta l)}{k - l}.$$

The last sum is well approximated by

$$\hat{r}(\Omega) \equiv \fint_0^{+\infty} d\omega \frac{r(\omega)}{\Omega - \omega}$$

up to $O(\eta)$. In the remaining sum, writing $r(\eta l) = r(\eta k) + (r(\eta l) - r(\eta k))$, the second term is $O(\eta \log L)$. When these errors can be neglected, a good approximation for $S(\Omega)$ is

$$S(\Omega) = \Omega - \epsilon - \lambda^2 \left(r(\Omega) \frac{\pi}{\tan \pi \delta} + \fint_0^{+\infty} d\omega \frac{r(\omega)}{\Omega - \omega} \right).$$

This equation gives an approximation for the Ω_k's close to Ω by choosing $\delta = \delta_k$ such that the right-hand side vanishes. This is not directly needed, but it is the crucial ingredient to control R_k in this region. Indeed, to get $S'(\Omega)$ we can take the derivative with respect to δ and divide by η. This leads to $\eta S'(\Omega) = \lambda^2 \pi^2 r(\Omega) \left(1 + \frac{1}{\tan^2 \pi \delta}\right)$. But $S'(\Omega_k) = 1/R_k$, leading to

$$R_k = \frac{\eta r(\Omega)}{\lambda^2 \left(\pi^2 r(\Omega)^2 + \left(\frac{\Omega - \epsilon}{\lambda^2} - \fint_0^{+\infty} d\omega \frac{r(\omega)}{\Omega - \omega} \right)^2 \right)} \equiv \eta R(\Omega)$$

for $k\eta$ close to Ω. So indeed, in this approximation R_k is obtained by discretizing a smooth function. For instance,

$$\sum_{k \in [0,L]} R_k e^{-i\Omega_k t} \sim \int_0^{+\infty} d\Omega \, e^{-i\Omega t} R(\Omega).$$

This is enough to treat the term proportional to $w(0)$ in $w(t)$. To deal with the oscillator contribution in $w(t)$, we need to control the sum

$$\sum_{k \in [0,L]} R_k e^{-i\Omega_k t} \frac{1}{\Omega_k - \eta l}$$

By contruction, at $t = 0$ this sum vanishes, and we may subtract $0 = e^{-i\eta l t} \sum_{k \in [0,L]} R_k \frac{1}{\Omega_k - \eta l}$ to get

$$\sum_{k \in [0,L]} R_k e^{-i\Omega_k t} \frac{1}{\Omega_k - \eta l} \sim \int_0^{+\infty} d\Omega \frac{e^{-i\Omega t} - e^{-i\eta l t}}{\Omega - \eta l} R(\Omega).$$

If we define $a_l = \sqrt{\eta} a(\eta l)$ then in the continuum limit we get $[a(\omega), a^\dagger(\omega')] = \delta(\omega - \omega')$, and we find

$$w(t) = w \int_0^{+\infty} d\Omega \, e^{-i\Omega t} R(\Omega) + \lambda \int_0^{+\infty} d\Omega d\omega \frac{e^{-i\Omega t} - e^{-i\omega t}}{\Omega - \omega} a(\omega) \sqrt{r(\omega)} R(\Omega).$$

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The limit $\lambda \to 0^+$ in $z(t) = w(t/\lambda^2)e^{i\epsilon t/\lambda^2}$ is now straigthforward. We set $\Omega \to \epsilon + \lambda^2\Omega$, $\omega \to \epsilon + \lambda^2\omega$. Observe that $\lambda a(\epsilon + \lambda^2\omega)$ is still a normalized oscillator, which we denote by a_ω. Noting that $R(\epsilon + \lambda^2\Omega) \sim \frac{r(\epsilon)}{\lambda^2(\pi^2 r(\epsilon)^2 + (\Omega - \hat{r}(\epsilon))^2)}$, which we denote by R_Ω, we find that

$$
\begin{aligned}
z(t) &= z \int_{-\infty}^{+\infty} d\Omega\, e^{-i\Omega t} R_\Omega + \int_{-\infty}^{+\infty} d\Omega d\omega\, \frac{e^{-i\Omega t} - e^{-i\omega t}}{\Omega - \omega} a_\omega \sqrt{r(\epsilon)} R_\Omega \\
&= z \int_{-\infty}^{+\infty} d\Omega e^{-i\Omega t} \frac{r(\epsilon)}{\pi^2 r(\epsilon)^2 + (\Omega - \hat{r}(\epsilon))^2} \\
&\quad + \int_{-\infty}^{+\infty} d\Omega d\omega\, \frac{e^{-i\Omega t} - e^{-i\omega t}}{\Omega - \omega} a_\omega \frac{r(\epsilon)^{3/2}}{\pi^2 r(\epsilon)^2 + (\Omega - \hat{r}(\epsilon))^2}.
\end{aligned}
$$

Integration over Ω is responsible for destructive interferences. Performing this integral and using that $\bar{\gamma} \equiv \pi r(\epsilon) + i\hat{r}(\epsilon)$ leads to

$$
z(t) = z(0)e^{-\bar{\gamma}t} + \sqrt{\frac{\Re e\,\bar{\gamma}}{\pi}} \int_{-\infty}^{+\infty} d\omega\, a_\omega \frac{e^{-i\omega t} - e^{-\bar{\gamma}t}}{\omega + i\bar{\gamma}}. \tag{21}
$$

This gives the spectral representation of $z(t) = ze^{-\bar{\gamma}t} + \xi(t)$. One checks that $[\xi(t), \xi^\dagger(s)] = G(t, s)$ as should be.

The spectral representation also allows to check easily that the coupled bath-reservoir system is still hamiltonian, with Hamiltonian

$$
H = \int d\omega\, \omega a_\omega^\dagger a_\omega + \sqrt{\frac{\Re e\,\bar{\gamma}}{\pi}} \int d\omega\, \left(z^\dagger a_\omega + z a_\omega^\dagger\right) + \Im m\,\bar{\gamma} z^\dagger z
$$

Checking that this leads to the spectral representation is done routinely via the Laplace transform, interpreting at some point $\int d\omega\, \frac{1}{p+i\omega}$ as $\oint d\omega\, \frac{1}{p+i\omega} = \pi$ for $\Re e\, p > 0$. Note that the hamiltonian is not bounded below due to the appearance of negative frequencies. This is simply due to the fact that taking the long time limit leads to shift the origin of energies by ϵ.

The relation between the coupling constant $\sqrt{\frac{\Re e\,\bar{\gamma}}{\pi}}$ and the friction coefficient $\Re e\,\bar{\gamma}$ is no surprise (though it is usually interpreted as the Einstein relation only after introduction of a temperature).

To diagonalize the dynamics, define (recall that $R_\Omega = r(\epsilon)/|\Omega + i\bar{\gamma}|^2$)

$$
b_\Omega \equiv R_\Omega^{1/2} z + \left(\frac{r(\epsilon)}{R_\Omega}\right)^{1/2} \left(\frac{a_\Omega}{\Omega + i\bar{\gamma}} - R_\Omega \int_{-\infty}^{+\infty} d\omega\, \frac{a_\omega}{\omega + i\bar{\gamma}}\right).
$$

One checks that the b's are normalized oscillators

$$
[b_\Omega, b_{\Omega'}^\dagger] = \delta(\Omega - \Omega'),
$$

such that $[H, b_\Omega] = -\Omega\, b_\Omega$, and that

$$z(t) = \int_{-\infty}^{+\infty} d\Omega\, e^{-i\Omega t}\, R_\Omega^{1/2}\, b_\Omega.$$

Just as $z = \int_{-\infty}^{+\infty} d\Omega\, R_\Omega^{1/2} b_\Omega$ commutes with the operators a_ω and $a_\omega^\dagger$, one checks that $\int_{-\infty}^{+\infty} d\omega\, \frac{r(\epsilon)^{1/2}}{\omega + i\bar{\gamma}} a_\omega$ is a normalized anihilation operator that commutes with the operators b_Ω and $b_\Omega^\dagger$.

8 Appendix: Proofs

Here we present the computation which leads to the formula eq.(13) for the conditional expectations. Eq.(12) defines $\mathbb{E}_s$ on single-time operator. We have to define it on product of time ordered operators, that is:

$$\mathbb{E}_s\big[e^{\mu_1 \xi^\dagger(t_1)} e^{\bar{\mu}_1 \xi(t_1)} \cdots e^{\mu_N \xi^\dagger(t_N)} e^{\bar{\mu}_N \xi(t_N)}\big]$$

with $t_1 < \cdots < t_N$. We impose that $\mathbb{E}_s$ is neutral with respect to left multiplication by elements of A_s, i.e. $\mathbb{E}_s[ab] = a\mathbb{E}_s[b]$ for $a \in A_s$. As a consequence, it is enough to consider times bigger than s, that is $s < t_1 < \cdots < t_N$. We may then recursively compute the conditional expectation by imposing the required condition $\mathbb{E}_{s_1} \circ \mathbb{E}_{s_2} = \mathbb{E}_{\min(s_1,s_2)}$. Indeed, inserting first $\mathbb{E}_s = \mathbb{E}_s \circ \mathbb{E}_{t_{N-1}}$ for $s < t_{N-1}$ in the above conditional expectation and using that all operators $e^{\mu_j \xi^\dagger(t_j)} e^{\bar{\mu}_j \xi(t_j)}$ with $j < N-1$ are left neutral with respect $\mathbb{E}_{t_{N-1}}$ (by construction), we obtain

$$\mathbb{E}_s[e^{\mu_1 \xi^\dagger(t_1)} e^{\bar{\mu}_1 \xi(t_1)} \cdots e^{\mu_N \xi^\dagger(t_N)} e^{\bar{\mu}_N \xi(t_N)}]$$
$$= \mathbb{E}_s\big[e^{\mu_1 \xi^\dagger(t_1)} e^{\bar{\mu}_1 \xi(t_1)} \cdots e^{\mu_{N-1} \xi^\dagger(t_{N-1})} e^{\bar{\mu}_{N-1} \xi(t_{N-1})}\, \mathbb{E}_{t_{N-1}}\big[e^{\mu_N \xi^\dagger(t_N)} e^{\bar{\mu}_N \xi(t_N)}\big]\big]$$
$$= \mathbb{E}_s\big[e^{\mu_1 \xi^\dagger(t_1)} e^{\bar{\mu}_1 \xi(t_1)} \cdots e^{\mu_{N-1} \xi^\dagger(t_{N-1})} e^{\bar{\mu}_{N-1} \xi(t_{N-1})} \times$$
$$\times\, e^{\mu_N(t_{N;N-1}) \xi^\dagger(t_{N-1})} e^{\bar{\mu}_N(t_{N;N-1}) \xi(t_{N-1})}\big] e^{\mathsf{n}_0[\mu_N \bar{\mu}_N - \mu_N(t_{N;N-1}) \bar{\mu}_N(t_{N;N-1})]}$$

with $t_{N;N-1} = t_N - t_{N-1}$ and where in the last equality we used the formula for conditional expectations of single-time operators. The key point is that now the conditional expectation $\mathbb{E}_s$ only involves the $N-1$ times $t_1 < \cdots < t_{N-1}$, so that we can recursively apply this procedure to fully compute the conditional expectation. After appropriate reordering of all terms, we get the formula (13).

One now verifies that the conditional expectations (13) satisfy

$$\mathbb{E}_t \circ \mathbb{E}_s = \mathbb{E}_{\min(s,t)}.$$

This is true by construction for $s \leq t$. The proof for $t < s$ requires a tiny computation. One first gets the factors $X_s^{(N)}$ and $Y_s^{(N)}$ by applying $\mathbb{E}_s$. Applying next $\mathbb{E}_t$ we get an extra contribution to X coming from the difference between reordering the operators before and after the projection by $\mathbb{E}_t$ which is

$$\hat{X}_{t;s} \equiv \sum_{i<j} [\bar{\mu}_i(t_{i;s})\mu_j(t_{j;s}) - \bar{\mu}_i(t_{i;t})\mu_j(t_{j;t})].$$

One checks that $X_s^{(N)} + \hat{X}_{t;s} = X_t^{(N)}$. Similarly, acting next with $\mathbb{E}_t$ we get an extra contribution to Y which is

$$\hat{Y}_{t;s} = \sum_{i,j} [\bar{\mu}_i(t_{i;s})\mu_j(t_{j;s}) - \bar{\mu}_i(t_{i;t})\mu_j(t_{j;t})].$$

One also checks that $Y_s^{(N)} + \hat{Y}_{t;s} = Y_t^{(N)}$.

Lastly we prove that the conditional expectations are right neutral by measurable multiplication, i.e. $\mathbb{E}_s[ba] = \mathbb{E}_s[b]a$ for $a \in A_s$. It is enough to check it for $a_0 = e^{\mu_0 \xi^\dagger(t_0)} e^{\bar{\mu}_0 \xi(t_0)}$ and b a product of operators as above. If a_0 multiplies this product of operators on the right, we first have to move it to the left by using the commutation relations, then to use the left neutrality and apply $\mathbb{E}_s$ and, finally, to move it back to the right using again the commutation relations. The difference between the commutators before and after having applied $\mathbb{E}_s$ is:

$$\sum_j \left(\bar{\mu}_j \mu_0 [\xi(t_j), \xi^\dagger(t_0)] + \bar{\mu}_0 \mu_j [\xi^\dagger(t_j), \xi(t_0)] \right)$$
$$+ \sum_j \left(\bar{\mu}_0 \mu_j(t_{j;s}) [\xi(t_0), \xi^\dagger(t_s)] + \mu_0 \bar{\mu}_j(t_{j;s}) [\xi^\dagger(t_0), \xi(t_s)] \right).$$

This vanishes thanks to the relations $\mu_j[\xi(t_0), \xi^\dagger(t_j)] = \mu_j(t_{j;s})[\xi(t_0), \xi^\dagger(t_s)]$, and their complex conjugate, valid for $t_0 < s < t_j$.

References

[1] A.O. Caldeira and A.J. Leggett, "Influence of dissipation on quantum tunneling in macroscopic systems", Phys. Rev. Lett., 46, (1981) 211.

[2] C.W. Gardiner and P. Zoller, *"Quantum Noise"*, 2nd edition, Springer, 2000.

[3] H.P. Breuer and F. Petruccione, *"The theory of open quantum systems"*, Oxford University Press, 2006.

[4] A. Leggett, S. Chakravarty, A. Dorsey, M. Fisher, A. Garg, and W. Zwerger, *"Dynamics of the dissipative two-state system"*, Rev. Mod. Phys., 59, (1987) 1.

[5] E.B. Davies, "The harmonic oscillator in a heat bath", Commun. Math. Phys. 33 (1973) 171-186.

[6] P. Levy, *"Processus stochastiques et mouvement Brownien"*, Gauthier-Villars, Paris, 1965.

[7] G. Lawler, *"Conformally invariant process in the plane"*, American Mathematical Society, 2005

[8] M. Bauer and D. Bernard, to appear.

[9] R.E. Prange and S.M. Girving eds. *"The quantum Hall effect"*, 2nd edition, Springer-Verlag N.Y., 1990.

[10] S. Attal, A. Joyce and C.A. Pillet, eds. *"Open quantum systems, vol. I, II and III"*, Lectures notes in mathematics, vol.1880-1882, Springer, 2006.

[11] Ph. Biane, "Introduction to random walks on non-commuative spaces", in *"Quantum potential theory"*, eds. Ph. Biane et al, Springer 2008;

[12] S. Attal, *"Quantum Noise"*, book in preparation, preprint 2010.

[13] J.L. Sauvageot, "Markov quantum semigroups admit covariant markov C*-dilations", Commun. Math. Phys. 106 (1986) 91-103.

[14] E.B. Davies, "Markovian master equations, Commun. Math. Phys 39 (1974) 91-110;
E.B. Davies, "The classical limit for quantum dynamical semigroups", Commun. Math. Phys. 49 (1976) 113-129.

[15] J. Lebowitz and H. Spohn, "Irreversible themodynamics for quantum systems weakly coupled to thermal reservoirs", Adv. Chem. Phys. 39 (1978) 109-142.

[16] R.L. Hudson and K.R. Parthasarathy, "Quantum Ito formula and stochastic evolutions", Commun. Math. Phys. 93 (1984) 301-323.

K.R. Parthasarathy, *"An introduction to quantum stochastic calculus"*, Monographs in Mathematcis 85, Birkhauser Verlag, 1992.

[17] G. Lindblad, "On the generators of quantum dynamical semigroups", Commun. Math. Phys. 48 (1976) 119-130.

[18] Ph. Biane, "Ito stochastic calculus and Heisenberg commutation relations", Sto. Proc. Appl. 120 (2010) 698-720.