Convex Optimization:

CVX 101

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Week 2: Convex Sets

Affine sets

- Defn: Contains the line through any two distinct points in a set
- Proof: Affine set contains line through any two points

  Given:

  1) $\{x|Ax = b\}$ (Set constructor)

  2) x = θx_1 + $(1-\theta)x_2$ with $\theta \in \boldsymbol{R}$ (Definition of affine)

  3) Ax_1 = b and Ax_2 = b (Two points chosen arbitrarily)

$$\begin{aligned} Ax &= A(\theta x_1 + (1-\theta)x_2) && \text{(Fact 2)} \\ Ax &= \theta A x_1 + A(1-\theta)x_2 \\ Ax &= \theta b + (1-\theta)b && \text{(Fact 3)} \\ &= b \end{aligned}$$

- Conversely, any affine set can be expressed as solution set of system of linear equations

Convex sets

- Line segment: between x_1 and x_2: all points

$$x = \theta x_1 + (1-\theta)x_2, \text{ with } 0 \leq \theta \leq 1$$

- Convex sets: contains line segment between any two points in the set

$$x_1, x_2 \in \mathcal{C}, 0 \leq \theta \leq 1 \implies \theta x_1 + (1-\theta)x_2 \in \mathcal{C}$$

Convex combination and convex hulls

- Convex combination of $x_1, ..., x_k$: any point x of the form

$$x = \theta_1 x_1 + ... + \theta_k x_k \text{ with}$$

$$1)\theta_1 + ... + \theta_k = 1$$

$$2)\theta_i \geq 0$$

- Convex hull conv S: set of all convex combinations of points in S

  E.g: Consider a square with dots on all 4 sides. The area within the set is the convex hull of the set S (which contains all 4 dots), as we can represent any point *WITHIN* the area as some convex combination of the 4 points (or less, if a θ is 0)

Convex cone

- Conic (nonnegative) combination of x_1, x_2: any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 \text{ with}$$

$$\theta_1, \theta_2 \geq 0$$

- Convex cone: set that contains all conic combinations of points in the set
- visually: a pie slice. Lies in a plane even in high dimensions, since we have 3 points

Hyperplanes and halfspaces

- hyperplane: set of the form $\{x|a^T x = b\}$ with a $\neq 0$
- halfspace: set of the form $\{x|a^T x \leq b\}$ with a $\neq 0$ such that a is the normal vector
- hyperplanes are affine and convex, whereas halfspaces are convex
- **Proof: Verifying why the halfspace is convex and NOT affine**

  1) $\{x|a^T x \leq b\}$ (Set constructor)

  2) x = θx_1 + $(1-\theta)x_2$ with $\theta \in \boldsymbol{R}$(Definition of affine)

  3) x = θx_1 + $(1-\theta)x_2$ with $0 \leq \theta \leq 1$ (Definition of convex)

  4) $a^T x_1 \leq$ b and $a^T x_2 \leq$ b (Two points chosen arbitrarily)

$$\begin{aligned}
\text{WTS: } a^T x \leq b \\
a^T &= a^T(\theta x_1 + (1-\theta)x_2) && \text{(By definition in 2/3)} \\
&= \theta a^T x_1 + (1-\theta)a^T x_2 \\
&\leq \theta b + (1-\theta)b && \text{(Explanation 1)} \\
&\leq b
\end{aligned}$$

  1) We can put in the inequality, because both are non-negative. At this point, we cannot use 2 anymore, because if we chose any number larger than 1, we would end up with a negative now. Thus, it cannot be affine but is instead convex

Euclidean Ball and Ellipsoids

- Euclidean ball: center x_c and radius r:

$$B(x_c, r) = \{x| \, ||x - x_c||_2 \leq r \tag{1}$$
$$= \{x_c + ru| \, ||u||_2 \leq 1\} \tag{2}$$

1) any point that is within r from the center is contained

2) any point that is some multiple of the radius, where that multiple is between 0 and 1 is contained

- Ellipsoid: set of the form

$$E(x_c, r) = \{x|(x - x_c)^T P^{-1}(x - x_c) \leq 1\} \tag{1}$$
$$= \{x_c + Au| \, ||u||_2 \leq 1\} \tag{2}$$

1) P $\in \boldsymbol{S}^n_{++}$ i.e P symmetric positive definite

2) A is square and non-singular, and can be thought of as a transformation on the unit ball, u

The ellipsoid is a generalization of the Euclidean ball. If P = r^2 $\boldsymbol{I}$, we have that they are equal

Polyhedra or Polytope

- Solution set of finitely many linear inequalities and equalities

  Ax $\prec$ b, Cx = d

  Where $A \in \boldsymbol{R}^{mxn}, C \in \boldsymbol{R}^{pxn},$ and $\prec$ is componentwise inequality

- Is the intersection of finite number of halfspaces and hyperplanes

Positive Semidefinite Cone

- $\boldsymbol{S}^n$ is set of symmetric n x n matrices
- $\boldsymbol{S}^n_+ = \{$X $\in \boldsymbol{S}^n | X \succeq 0\}$: positive semidefinite n x n matrices

  $$X \in \boldsymbol{S}^n_+ \iff z^T X z \geq 0 \forall z$$

  Note, that a linear combination of

  $\boldsymbol{S}^n_+$ is a convex cone

- $\boldsymbol{S}^n_{++} = \{$X $\in \boldsymbol{S}^n | X \succ 0\}$: positive definite n x n matrices

Operations that preserve Convexity

- Can always work:

$$\text{arbitrary } x_1, x_2 \in \mathcal{C}, 0 \leq \theta \leq 1 \implies \theta x_1 \text{ and } (1-\theta)x_2 \text{ is in the set}$$

- Show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, ..) by operations that preserve convexity
  - intersection
  - affine functions
  - perspective function
  - linear-fractional functions

Preserving Convexity: Intersection

- Intersection of any number of convex functions is convex

$$\text{eg: } S = \{x \in \boldsymbol{R}^m | abs(p(t)) \leq 1 \text{ for } abs(t) \leq \frac{\pi}{3}\}$$

  where p(t) = x_1 cos t + x_2 cos 2t + ... + x_m cos mt for m = 2

- Intuition behind example

  Repheasing S:

$$S_t = \{x \in \boldsymbol{R}^m || p(t)| \leq 1\}$$

  - S_t is a single point. It is the intersection of two parallel halfspaces (which are linear, as they are a linear combination of cos bla) with opposite orientations (hyperslab)
  - As we modify t, we see that the slab produced by the two parallel lines rotates, and changes width.
  - Thus, S is the intersection of those slabs.

Preserving Convexity: Affine Function

- Suppose f: $\boldsymbol{R}^n \to \boldsymbol{R}^m$ is affine (f(x) = Ax + b with A $\in \boldsymbol{R}^{mxn}, b \in \boldsymbol{R}^m$)
  - The image of a convex set under f is convex

$$S \subseteq \boldsymbol{R^n} \text{ convex } \implies f(S) = \{f(x) | x \in S\} \text{ convex}$$

  - The inverse image f^{-1}(C) of a convex set under f is convex

$$C \subseteq \boldsymbol{R^m} \text{ convex } \implies f^{-1}(C) = \{x \in \boldsymbol{R^n} | f(x) \in C\} \text{ convex}$$

- Examples
  - scaling, translation, projection
  - solution set of linear matrix inequality $\{x | x_1 A_1 + ... + x_m A_m \succeq B\}$ with A_i, B $\in \boldsymbol{S}^P$
  - hyperbolic cone $\{x | x^T P x \leq (c^T x)^2, c^T x \geq 0\}$ (with P $\in \boldsymbol{S}^n_+$)