

Convex Optimization:

CVX 101

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Week 2: Convex Sets

Affine sets

- Defn: Contains the line through any two distinct points in a set

- Proof: Affine set contains line through any two points

Given:

- 1) $\{x | Ax = b\}$ (Set constructor)
- 2) $x = \theta x_1 + (1 - \theta)x_2$ with $\theta \in \mathbf{R}$ (Definition of affine)
- 3) $Ax_1 = b$ and $Ax_2 = b$ (Two points chosen arbitrarily)

$$Ax = A(\theta x_1 + (1 - \theta)x_2) \quad (\text{Fact 2})$$

$$Ax = \theta Ax_1 + A(1 - \theta)x_2$$

$$Ax = \theta b + (1 - \theta)b \quad (\text{Fact 3})$$

$$= b$$

- Conversely, any affine set can be expressed as solution set of system of linear equations

Convex sets

- Line segment: between x_1 and x_2 : all points

$$x = \theta x_1 + (1 - \theta)x_2, \text{ with } 0 \leq \theta \leq 1$$

- Convex sets: contains line segment between any two points in the set

$$x_1, x_2 \in \mathcal{C}, 0 \leq \theta \leq 1 \implies \theta x_1 + (1 - \theta)x_2 \in \mathcal{C}$$

Convex combination and convex hulls

- Convex combination of x_1, \dots, x_k : any point x of the form

$$x = \theta_1 x_1 + \dots + \theta_k x_k \text{ with}$$

$$1) \theta_1 + \dots + \theta_k = 1$$

$$2) \theta_i \geq 0$$

- Convex hull $\text{conv } S$: set of all convex combinations of points in S

E.g: Consider a square with dots on all 4 sides. The area within the set is the convex hull of the set S (which contains all 4 dots), as we can represent any point *WITHIN* the area as some convex combination of the 4 points (or less, if a θ is 0)

Convex cone

- Conic (nonnegative) combination of x_1, x_2 : any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 \text{ with}$$

$$\theta_1, \theta_2 \geq 0$$

- Convex cone: set that contains all conic combinations of points in the set
- visually: a pie slice. Lies in a plane even in high dimensions, since we have 3 points

Hyperplanes and halfspaces

- hyperplane: set of the form $\{x | a^T x = b\}$ with $a \neq 0$
- halfspace: set of the form $\{x | a^T x \leq b\}$ with $a \neq 0$ such that a is the normal vector
- hyperplanes are affine and convex, whereas halfspaces are convex

- **Proof: Verifying why the halfspace is convex and NOT affine**

- 1) $\{x | a^T x \leq b\}$ (Set constructor)
- 2) $x = \theta x_1 + (1 - \theta)x_2$ with $\theta \in \mathbf{R}$ (Definition of affine)
- 3) $x = \theta x_1 + (1 - \theta)x_2$ with $0 \leq \theta \leq 1$ (Definition of convex)
- 4) $a^T x_1 \leq b$ and $a^T x_2 \leq b$ (Two points chosen arbitrarily)

$$\text{WTS: } a^T x \leq b$$

$$a^T = a^T(\theta x_1 + (1 - \theta)x_2) \quad (\text{By definition in 2/3})$$

$$= \theta a^T x_1 + (1 - \theta)a^T x_2$$

$$\leq \theta b + (1 - \theta)b \quad (\text{Explanation 1})$$

$$\leq b$$

- 1) We can put in the inequality, because both are non-negative. At this point, we cannot use 2 anymore, because if we chose any number larger than 1, we would end up with a negative now. Thus, it cannot be affine but is instead convex

Euclidean Ball and Ellipsoids

- Euclidean ball: center x_c and radius r :

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\} \quad (1)$$

$$= \{x_c + ru \mid \|u\|_2 \leq 1\} \quad (2)$$

- 1) any point that is within r from the center is contained
- 2) any point that is some multiple of the radius, where that multiple is between 0 and 1 is contained

- Ellipsoid: set of the form

$$E(x_c, r) = \{x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1\} \quad (1)$$

$$= \{x_c + Au \mid \|u\|_2 \leq 1\} \quad (2)$$

- 1) $P \in \mathbf{S}_{++}^n$ i.e P symmetric positive definite
 - 2) A is square and non-singular, and can be thought of as a transformation on the unit ball, u
- The ellipsoid is a generalization of the Euclidean ball. If $P = r^2 \mathbf{I}$, we have that they are equal

Polyhedra or Polytope

- Solution set of finitely many linear inequalities and equalities

$$Ax \preceq b, \quad Cx = d$$

Where $A \in \mathbf{R}^{m \times n}, C \in \mathbf{R}^{p \times n}$, and \preceq is componentwise inequality

- Is the intersection of finite number of halfspaces and hyperplanes

Positive Semidefinite Cone

- \mathbf{S}^n is set of symmetric $n \times n$ matrices
- $\mathbf{S}_+^n = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: positive semidefinite $n \times n$ matrices

$$X \in \mathbf{S}_+^n \iff z^T X z \geq 0 \forall z$$

Note, that a linear combination of

\mathbf{S}_+^n is a convex cone

- $\mathbf{S}_{++}^n = \{X \in \mathbf{S}^n \mid X \succ 0\}$: positive definite $n \times n$ matrices

Operations that preserve Convexity

- Can always work:

arbitrary $x_1, x_2 \in \mathcal{C}, 0 \leq \theta \leq 1 \implies \theta x_1$ and $(1 - \theta)x_2$ is in the set

- Show that \mathcal{C} is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, ..) by operations that preserve convexity
 - intersection
 - affine functions
 - perspective function
 - linear-fractional functions

Preserving Convexity: Intersection

- Intersection of any number of convex functions is convex

$$\text{eg: } S = \{x \in \mathbf{R}^m \mid \text{abs}(p(t)) \leq 1 \text{ for } \text{abs}(t) \leq \frac{\pi}{3}\}$$

where $p(t) = x_1 \cos t + x_2 \cos 2t + \dots + x_m \cos mt$ for $m = 2$

- Intuition behind example

Rephrasing S :

$$S_t = \{x \in \mathbf{R}^m \mid |p(t)| \leq 1\}$$

- S_t is a single point. It is the intersection of two parallel halfspaces (which are linear, as they are a linear combination of \cos bla) with opposite orientations (hyperslab)
- As we modify t , we see that the slab produced by the two parallel lines rotates, and changes width.
- Thus, S is the intersection of those slabs.

Preserving Convexity: Affine Function

- Suppose $f: \mathbf{R}^n \rightarrow \mathbf{R}^m$ is affine ($f(x) = Ax + b$ with $A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^m$)

- The image of a convex set under f is convex

$$S \subseteq \mathbf{R}^n \text{ convex} \implies f(S) = \{f(x) \mid x \in S\} \text{ convex}$$

- The inverse image $f^{-1}(C)$ of a convex set under f is convex

$$C \subseteq \mathbf{R}^m \text{ convex} \implies f^{-1}(C) = \{x \in \mathbf{R}^n \mid f(x) \in C\} \text{ convex}$$

- Examples

- scaling, translation, projection
- solution set of linear matrix inequality $\{x \mid x_1 A_1 + \dots + x_m A_m \succeq B\}$ with $A_i, B \in \mathbf{S}^P$
- hyperbolic cone $\{x \mid x^T P x \leq (c^T x)^2, c^T x \geq 0\}$ (with $P \in \mathbf{S}_+^n$)