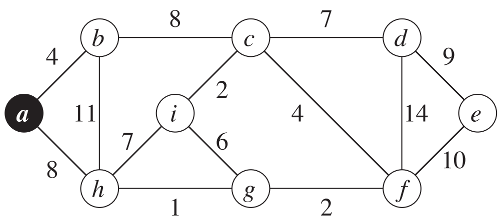
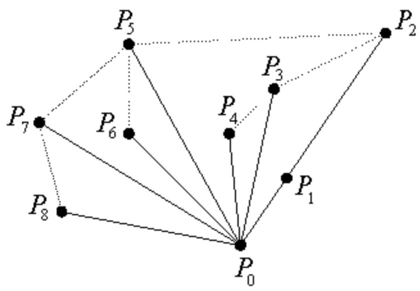
Graph Algorithm
Final Exam
December 27, 2023, 05:00 – 08:00 pm

1. (20%) Minimum Spanning Trees:
   (a) Find a minimum spanning tree in G by Prim's algorithm, starting from vertex a. Analyze the time complexity of each algorithm and show the state of each phase.
   (b) Let $G = (V, E)$ be an undirected graph. Suppose that each edge e has a cost $c(e)$, with $c(e) \in \{1, 2, 3\}$. Describe an $O(n+m)$ time algorithm to compute a minimum spanning tree for G. HINT: Observe that if all the edge costs were exactly the same, then any spanning tree would be a minimum spanning tree of the graph.



2. (20%) Find the convex hull of the set Q of points in the following figure by using Jarvis's and Graham's algorithms, respectively. Analyze the time complexity of each algorithm and show the state of each phase.



3. (20%) Friends Dal and Sean want to take a car trip across the country from Yew Nork to Fan Sancrisco by driving between cities during the day, and staying at a hotel in some city each night. There are n cities across the country. For each city c_i, Dal and Sean have compiled:

   - the positive integer expense $h(c_i)$ of staying at a hotel in city c_i for one night; and
   - a list L_i of the at most 10 other cities they could drive to in a single day starting from city c_i, along with the positive integer expense $g(c_i, c_j)$ required to drive directly from c_i to c_j for each $c_j \in L_i$.

   Describe an $O(nd)$-time algorithm to determine whether it is possible for Dal and Sean to drive from Yew Nork to Fan Sancrisco in at most d days, spending at most b on expenses along the way.

4. (20%) All-Pairs Shortest-Paths:

   (a) Describe and analysis the time complexity of the Floyd-Warshall algorithm.

   (b) Given matrices $D^{(0)}, D^{(3)}, \Pi^{(0)}$, and $\Pi^{(3)}$, determine the matrices $d_{25}^{(4)}$, $d_{35}^{(4)}$, $\pi_{25}^{(4)}$ and $\pi_{35}^{(4)}$.

$$D^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad \Pi^{(0)} = \begin{pmatrix} \text{NIL} & 1 & 1 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\ 4 & \text{NIL} & 4 & \text{NIL} & \text{NIL} \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

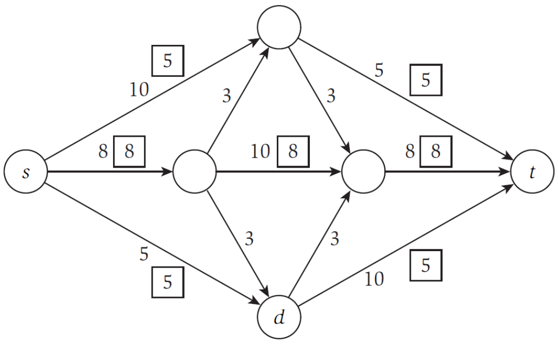
$$D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad \Pi^{(3)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

5. (20%) The following figure shows a flow network on which an s-t flow has been computed. The capacity of each edge appears as a label next to the edge, and the numbers in boxes give the amount of flow sent on each edge. (Edges without boxed numbers have no flow being sent on them.)

   (a) What is the value of this flow?

   (b) Is this a maximum (s, t) flow in this graph?

   (c) Find a minimum s-t cut in the flow network pictured in the graph, and also say its capacity.



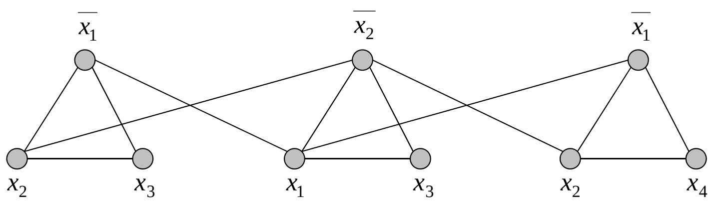
6. (20%) Polynomial-Time Reductions:

   (a) Describe the 3-SAT Problem.

   (b) Describe the Independent Set Problem.

   (c) Suppose that 3-SAT Problem is NP-complete, then prove Independent Set is also NP-complete.
   Hint:

G



k = 3

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$