Graph Algorithm

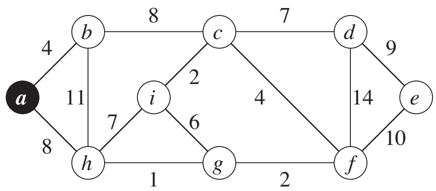
Final Exam

December 29, 2022, 9:20 – 12:05 am

1. (20%) Minimum Spanning Trees:

   (a) Find a minimum spanning tree in G by Kruskal's algorithm. Analysis the time complexity of each algorithm and show the state of each phase.

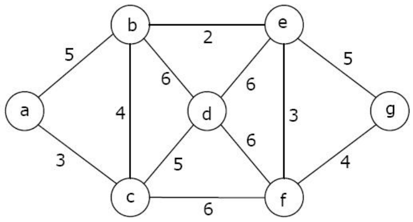
   (b) Find a minimum spanning tree in G by Prim's algorithm, starting from vertex f. Analysis the time complexity of each algorithm and show the state of each phase.



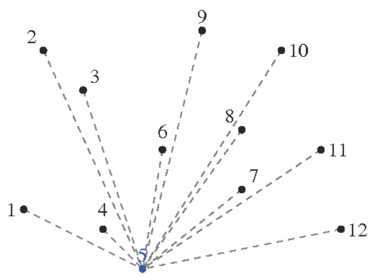
2. (20%) Single-Source Shortest Paths:

   (a) Find the lengths of the shortest paths starting from vertex f, going through all other vertices in G by Dijkstra's algorithm. Show the state of each phase.

   (b) In a weighted graph G where all edges have weight 1, how can we use Dijkstra's algorithm to find a minimum spanning tree?



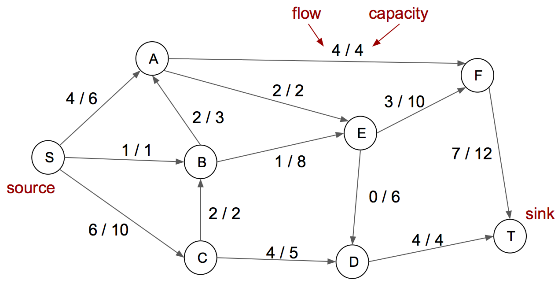
3. (20%) Convex Hull:
   Find the convex hull of the set Q of points in the following figure by using Jarvis and Graham's algorithms, respectively. Analysis the time complexity of each algorithm and show the state of each phase.



4. (20%) Network Flow:
   The following graph shows a flow network on which an s-t flow has been computed.

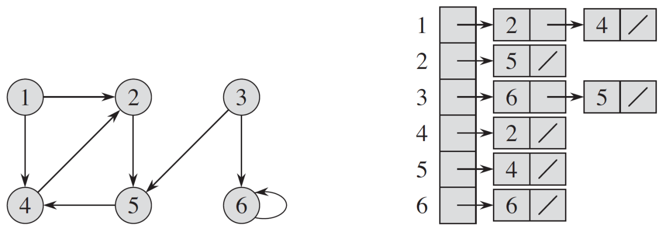
   (a) What is the value of this flow? Is this a maximum (s, t) flow in this graph?

   (b) Find a minimum s-t cut in the flow network pictured in the graph, and also say what its capacity is.



5. (15%) Depth-First Search:

   Assume that the DFS procedure considers the vertices in numerical order. Show how depth-first search works on the graph of the following figure, including the discovery and finishing times for each vertex, and show the classification of each edge.

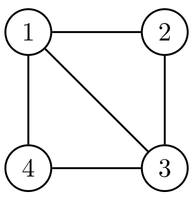


(a) (b)

6. (15%) Polynomial-Time Reductions:

   In the DOMINATING SET problem, we are given an undirected graph $G = (V, E)$ with n vertices and a number k ($1 \leq k \leq n$). A vertex u dominates itself and all of its neighbors. That is, vertex u dominates vertex v if $v = u$ or v is adjacent to u. A set S of the vertices is called a *dominating set* if every vertex $v \in V$ is dominated by at least one vertex $u \in S$. DOMINATING SET problem asks you to check whether there is a dominating set of size k in graph G. It is well-known that DOMINATING SET is an NP-complete problem.

   In this problem, we consider a variant called DOUBLE DOMINATING SET. The input is an undirected graph $G' = (V', E')$ with n' vertices, and a number k' ($1 \leq k' \leq n'$). A set $S' \subset V'$ is called a *double dominating set*, if every vertex $v \in V'$ is dominated by at least two vertices in S'.



Example: For the graph on the left, vertex $\{1\}$ is a dominating set of size 1; vertices $\{2, 4\}$ form a dominating set of size 2. However, neither $\{1\}$ nor $\{2, 4\}$ is a double dominating set. The set $\{1, 3\}$ is a double dominating set of size 2.

   (a) To show DOUBLE DOMINATING SET is NP-hard based on the fact that DOMINATING SET is NP-complete, what is the correct direction of reduction?
   (Please answer in the form A to B)

   (b) Prove that DOUBLE DOMINATING SET is in NP.

   (c) Do a reduction (related to the DOMINATING SET problem) to show DOUBLE DOMINATING SET is NP-hard.
   Hint: The intended solution only creates 2 extra vertices in the new instance.