Population Stability and Sex Ratios: A Mathematical Perspective

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Abstract

The purpose of this project is to explore the population and sex ratio dynamics of sea lampreys. Sea Lampreys are a species which are largely dependent on their living environment; resource abundance and availability is a main factor that influences lamprey population growth and their sex ratio. By using a system of differential equations, we model how resource availability influences population growth and sex ratio. Our assumptions include a closed environment, where no emigration or immigration occurs, and resource dependency, which affects growth rates, carrying capacity, and changes in the sex ratio. Birth rates are directly impacted by the female population, while mortality rates are proportional rate to the overall population size. Results from our model suggest that abundant resources favor female population growth, leading to overall increased rates population growth. Furthermore, we are able to extrapolate equilibrium points which the model approaches as time passes for given initial conditions.

1 Introduction

The goal of this project is to understand the population dynamics of sea lampreys, primarily through the lens of how the influence of resource availability and sex ratio impact their population growth. By understanding the factors that lead to population changes and sex ratio dynamics, we can provide useful strategies that support effective ecological management strategies for sea lamprey populations and overall aquatic species development. However, while the MCM problem itself discusses the impact of sea lamprey population on the environment as a whole, we want to primarily focus on the sea lamprey population itself and how sea lampreys interact with the resources in their environment and how the population settles into some kind of stable equilibrium. Our analysis will aim to establish how the population, resources, and sex ratio will change and interact over time. We ultimately discover as discussed in section 4 the importance of the resource regeneration and consumption parameter and how it relates to all three areas we model.

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2 Background

Our model is based on a system of ordinary differential equations (ODEs) to describe the dynamics of sea lamprey populations. The logistic growth equation is used to represent population changes constrained by carrying capacity. Additional terms account for resource dependency, which influences both population growth and the sex ratio. The male sex ratio (S) is modeled linearly to reflect shifts based on resource availability, with adjustments toward upper or lower bounds under resource scarcity or abundance. By incorporating these mathematical principles, the model simulates how resources drive changes in population size (N) and composition. Our approach offers a mathematical framework to analyze these ecological dynamics.

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3 Our Model

Below is our differential equation model. $\boldsymbol{N}, \boldsymbol{R}, \boldsymbol{S}$ represents the total population of Lampreys, the amount of resources available/usable by the Lampreys, and the male sex ratio respectively.

3.1 System of Equations

$$\frac{dN}{dt} = rN\,(1-S)\left(1-\frac{N}{k}\right)\left(\frac{R}{R_{\max}}\right) - mN \tag{1}$$

$$\frac{dR}{dt} = \beta(R_{\max}-R) - \delta NR \tag{2}$$

$$S(R) = s_{\max} - (s_{\max}-s_{\min})\left(\frac{R}{R_{\max}}\right) \tag{3}$$

3.2 Total Population Model

$\dot{N}$ is derived from the prototypical logistic growth model. rN and $\left(1-\frac{N}{k}\right)$ are both elements of this logistic growth model. The rate of change of the population is dependent on the population itself–adjusted by some constant r–and limited by some carrying capacity k. In addition, the growth rate is affected by both the resources of the population compared to their maximum resource level and the ratio of females. Lastly, because this is dependent on the female population, we have a separate death rate that is simply dependent on N, represented by a $-mN$. m will be indicative of a small natural death rate as the birth rate is constrained by a variety of other factors already.

3.3 Resource Model

$\dot{R}$ is derived similarly for a logistical growth model with some slight modifications. We made the assumption that the resources would replenish at a faster rate if the current resource level R was further from some threshold. Thus, we have some factor of β and $(R_{\max}-R)$ to represent this growth. We opted to not have the growth dependent on a factor of R itself, assuming that the pool of resources is replenished by factors other than merely the resources themselves. More importantly, we want to assume that $R=0$ is not a fixed point and the resources could always rebound. Additionally, resources are depleted by some factor δNR, dependent on the total population of Lampreys, total resources, and some constant δ.

3.4 Sex Ratio Model

$S(R)$ is simply a linear model that obtains its minimum value given by the MDM problem of $s_{\min}=0.56$ when $\frac{R}{R_{\max}}$ obtains its maximum possible value of 1 and $S(R)$ obtains its maximum value given by the MDM problem $s_{\max}=0.78$ when $\frac{R}{R_{\max}}$ obtains its minimum value of 0. We considered a non-linear model which likely could be slightly more precise; however, this would massively complicate our analysis of the model and also is not necessarily clearly biologically justified, i.e., there is not a clear alternative that would be more effective.

3.5 Fixed Points of N and R

We essentially have a 2D system of ODEs. However, these equations are quite complicated and do not have easily solvable closed-form fixed points where one component of (N^*,R^*) is not zero. In general, the fixed points are determined by setting $\frac{dN}{dt}=0$ and $\frac{dR}{dt}=0$.
For $\frac{dR}{dt}=0$, the singular fixed point of R can occur when :

$$R = \frac{\beta R_{\max}}{\beta+\delta N}.$$

This is by design. We do not want to have $R=0$ to be a fixed point because we want resources to quickly replenish. From this, we find one fixed point which, for analysis purposes, is trivial: $(N^*,R^*)=(0,R_{\max})$. It is worth noting that $(0,0)$ is not a fixed point in this system as R will grow until it reaches its maximum.

For $\frac{dN}{dt}=0$, the non-trivial fixed point of N is:

$$N = k\left(1-\frac{mR_{\max}}{r(1-S)R}\right),$$

where S is a function of R:

$$S(R) = s_{\max} - (s_{\max}-s_{\min})\frac{R}{R_{\max}}.$$

The system does not have non-trivial independent equilibrium points for N or R that are reasonable to calculate in a closed form way by hand. However, it makes sense that there should exist some equilibrium state. We do not have any randomness and this model should eventually settle into some equilibrium after enough time elapses. By our conditions, we prevent the model from having a population about our carrying capacity, prevent it from having resources above a maximum threshold, and prevent resources from ever being driven out entirely. Thus, there should exist some solution. In our cases below, we will offer a phase portrait along and intersect our two nullclines to find fixed points for our given system. It is also worth observing that our system approaches those fixed points in both of the trajectories drawn below starting with arbitrary resource levels and population numbers within our parameters.

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4 Results

We tested our model under different parameters to simulate various conditions. The main parameters that control the model behaviors are resource-related parameters, including the resource regeneration rate (β), maximum resource the environment can hold ($R_{\max}$), and the resource consumption rate (δ). All other parameters, such as population capacity($K = 1000$), growth rate($r = 0.1$), and death rate($m = 0.001$), are kept constant since our main objective is to test the effect of resources on population and gender dynamics. Overall, we found our model to behave as expected: the male proportion reaches an upper bound as resources become scarce, and vice versa.

Hyperlink to code repository:
`https://colab.research.google.com/drive/1ruphOrpjNkDYn-uYRXMamqG9t30621ZO?usp=sharing`

4.1 Scarce Resource

- $\beta = 1$: each unit of missing resource regenerates one unit of new resource.
- $\delta = 0.01$: each member of the population depletes 0.01 unit of resource.
- $R_{\mathbf{max}} = 200$: maximum resource is 200 units, relatively low compared to the carrying capacity of 1000.

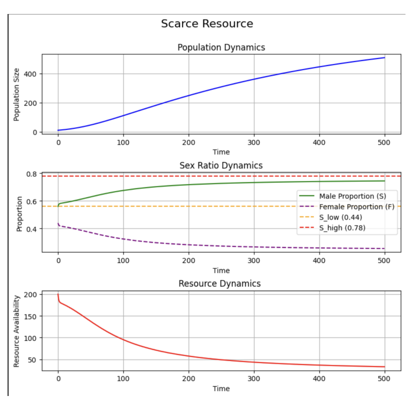

Figure 1: N, R, S behavior when resource is scarce

The figure above shows that the population grows slowly as expected, and it does not reach the capacity. While resource approaches depletion, we see the male sex ratio approach the upper bound, behaving as desired.

With these set parameters, we can also consider a phase portrait. If we graph the the nullclines, we can determine the equilibrium conditions from their intersection. We have the following

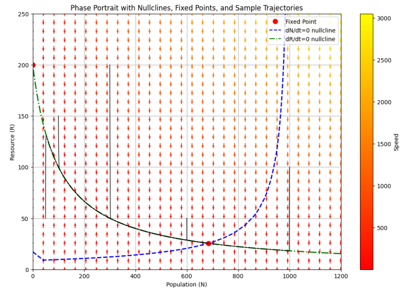

Figure 2: Phase Portrait

We can see that our system is approaching these equilibrium conditions as asymptotes. We computationally found the equilibrium values to be $(N^*, R^*) = (683.9648, 25.5113)$. Using the same code and model as above, if we substitute in initial conditions $(N_0, R_0) = (684, 25)$, this becomes obvious.

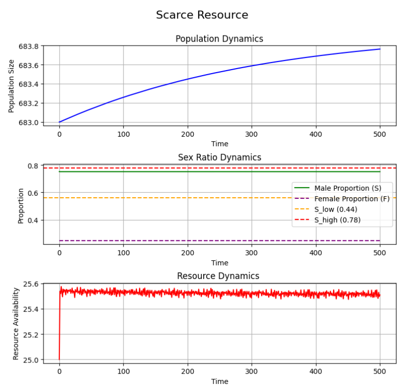

Figure 3: N, R, S behavior when resource is scarce and conditions are approximately the fixed points

4.2 Abundant Resource

- $\beta = 4$: each unit of missing resource regenerates four unit of new resource.
- $\delta = 0.001$: each member of the population depletes 0.001 unit of resource.
- $R_{\mathbf{max}} = 800$: maximum resource is 800 units, relatively high compared to the carrying capacity of 1000.

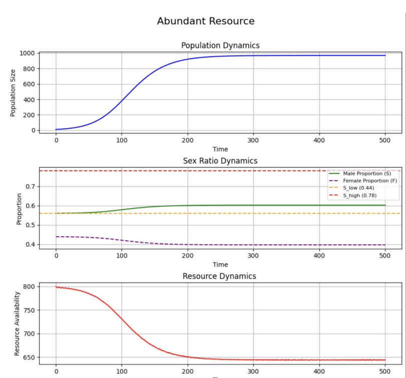

Figure 4: N, R, S behavior when resource is abundant

The figure above shows that the population grows quickly as expected, and it also reaches the capacity quickly. Though, resource maintains at a steady level even after population capacity is reached due to a large resource cap and regeneration rate. As a result, we see the male sex ratio approach the lower bound, behaving as desired.

In terms of a fixed point, we can do the exact same analysis as above. With our new constants, it produces a much higher fixed point as a result of our faster resource regeneration rate (higher β) and lower resource consumption rate (lower δ) in addition to a higher resource maximum. For the sake of space, we will not include the graphics for this scenario but it produces $(N^*, R^*) = (968.7193, 644.0291)$.

In general, considering both of our examples, it is worth noting that in each of these cases, the male sex ratio will clearly stabilize proportionally to whatever the resource level is compared to the maximum resource level. Furthermore, it provides interesting context to compare how the equilibrium points relate the carrying capacity and maximum resource level, respectively. For example, in the scenario of scarcity of resources, the population hovered around 68% of its carrying capacity and approximately 13% of its maximum resource level. In contrast, in the abundant resource scenario, the population stabilized around 97% of its carrying capacity and approximately 80% of its maximum resource level. This provides an interesting insight into how the population is affected by its parameters. The ability of resources to regenerate rapidly favors a much higher population in addition to a larger female sex ratio. To the extent resource levels can be controlled for sea lampreys, a high resource level in a small environment (i.e., low carrying capacity) logically would favor this second scenario. In contrast, sea lampreys with minimal resources–compared to their carrying capacity–will reach lower population equilibrium.

4.3 Stochastic

To further simulate the real-life scenario, where random fluctuations due to environmental factors exist, we added a stochastic term to the population growth model to capture the variability more realistically. To ensure fairness, we graph the modified model 10 times for each noise intensity level when resources are scarce and abundant.

4.3.1 Deterministic Part

$$\dot{N}_{\text{det}} = r\,(1 - S)\,N\left(1 - \frac{N}{K_0}\right)\left(\frac{R}{R_{\max}}\right) - mN$$

This part of the equation describes the deterministic logistic growth of the population, or initial differential equation model for N we proposed earlier.

4.3.2 Stochastic Part

$$\dot{N}_{\text{stoch}} = \text{noise\_intensity} \cdot \text{np.random.normal}(0, 1)$$

This part introduces a normally distributed random variable scaled by `noise_intensity`. The `np.random.normal(0, 1)` function generates a random number from a standard normal distribution (mean 0, standard deviation 1). Multiplying this random number by `noise_intensity` controls the magnitude of the stochastic fluctuations.

4.3.3 Combined Equation

$$\dot{N} = \dot{N}_{\text{det}} + \dot{N}_{\text{stoch}}$$

The combined equation includes both the deterministic logistic growth and the stochastic term, resulting in a population model that accounts for both predictable growth patterns and random fluctuations.

4.3.4 Scarce Resource

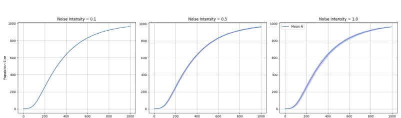

Figure 5: stochastic process for N when resource is scarce

The figure above shows that despite different noise intensity levels, the stochastic term doesn't change the overall shape for N while resource is scarce.

4.3.5 Abundant Resource

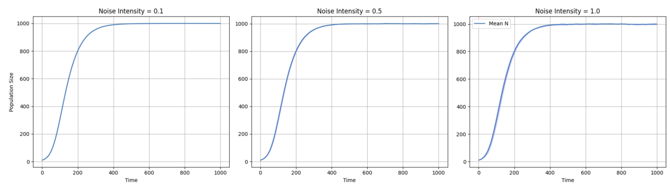

Figure 6: stochastic process for N when resource is abundant

The figure above shows that despite different noise intensity levels, the stochastic term doesn't change the overall shape for N while resource is abundant.

4.3.6 Model Robustness

The observation that the stochastic graphs follow the same shape for different noise intensity levels for two scenarios (Scarce and Abundant) suggests that the model is robust to these perturbations. This implies that the deterministic components of the model are strong enough to maintain the overall dynamics, even in the presence of random fluctuations.

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5 Conclusion and Discussion

In conclusion, our model demonstrates the dynamics between lamprey sex ratio and resource scarcity given by the MCM prompt. As resource becomes more abundant, a higher proportion of the population converts to female, prompting a higher birth rate, and vice versa. Though, we did not use any mathematical methods to establish a formal relationship between our parameters and our model: we simply tuned the relevant resource parameters to a magnitude large or small enough to overshadow or be overshadowed by the effects of other factors in the model. In the future, we could specifically explore the effects of these parameters to create more specific and delicate conditions to test our model on, thus shedding more nuanced insights beyond just those relevant to resource dependency on our model's behaviors. However, while we did not use powerful numerical methods to formalize these relationships between our parameters, we were able to find some general trends as outlined in section 4. Looking into equilibrium trends allowed us to compare the results with different parameters and see dramatic differences in results. In particular, the stable point for resource level was dramatically different with different parameters, as well as the stable ratio of $\frac{R}{R_{\max}}$.

6 Acknowledgments

We collaborated primarily among the four of us. Theo gave some brief advice on our equations after discussion one week which was helpful in making sure we were on the right track. We also utilized Open AIs Chat GPT 4o for help with debugging, formatting, and adjusting some LaTeXand Python code.

7 References

[1] Redelmeier, R. (2022, September 1). In the Great Lakes, the pandemic disrupted sea lamprey control. *Undark Magazine*. `https://undark.org/2022/09/05/in-the-great-lakes-the-pandemic-disrupted-sea-lamprey-control/`

[2] Johnson, N. S., Swink, W. D., & Brenden, T. O. (2017). Field study suggests that sex determination in sea lamprey is directly influenced by larval growth rate. *Proceedings of the Royal Society B: Biological Sciences*, 284(1851), 20170262. `https://doi.org/10.1098/rspb.2017.0262`