# Theory – Boussinesq Approximation

To develop the theory of free convection within the toaster and visualize the fluid flow due to temperature, an approximation is necessary. This approximation relates the temperature to the conservation of momentum equations and is included within the gravitational body force and density. This approximation is known as the Boussinesq approximation for buoyant flow. This approximation first assumes that all fluid properties are constant except for density, which varies with temperature. This temperature variant density is represented by the following equation:

The density term is the density defined by the type of fluid and known physical properties, and it is a constant. The temperature variation term involves a temperature gradient in the density definition with the wall temperature subtracting the temperature of the flow at any given position in two dimensions. This temperature gradient inclusion is important for flows with significant overall temperature gradients such as a house fire or heat generated by a laser in many experimental environments.

For the case of flow out of a toaster, the temperatures present in the gradient are typically of the same order, so the temperature of the flow can potentially be treated as constant for the assumption present in the conservation of momentum. This is not necessary but can serve to reduce the amount of dependent variables in the momentum conservation equation and simplify analysis.

The second aspect of this approximation involves the gravitational body force. Density is present in the momentum conservation equation along with gravity, and the temperature gradient is kept when simplifying the conservation equations.

# Theory – Assumptions

To simplify the governing equations, some assumptions must be made. By assuming a steady state process, the time dependence of the governing equations is neglected. A 2D problem is assumed because of the minimal expected influence of the thin side walls within the toaster channel. To use the Boussinesq approximation the problem is assumed to be incompressible in all parameters but temperature. The boundary conditions for this problem are the no slip condition at all surfaces within the toaster channel, and an insulated bottom surface. The heating walls of the channel have a constant temperature defined as, present in the temperature gradient term.

The four simplified equations used are the continuity equation, conservation of momentum in and, and conservation of energy. With the pressure gradient and inertia terms maintained, these dimensional equations are represented as follows:

**Continuity**

**X – Momentum Conservation**

**Y – Momentum Conservation**

**Conservation of Energy**

# Theory – Scaling of Governing Equations

To simplify the analysis and better understand the contribution of each term, a scaling approach is used. Both the and velocities and respectively are treated as both scaling an unknown velocity scaling term. The dimension scales with the channel width, and the dimension scales the channel width multiplied by some factor, resulting in. The temperature scales the wall temperature. The pressure within the flow is scaled with the dynamic pressure for this problem due to it containing the density and scaled velocity term. Since this scaling process is dealing in order of magnitude, the present in the dynamic pressure is left out of the analysis. These scaling terms are summarized as follows:

With the factor present in the scaling of, the factor arises throughout the dimensionless governing equations. While implementing a factor to the geometry scales is reasonable for problems where the geometries of and are significantly different from each other, toaster channels typically have a factor on the range of 6 or 7 multiplied by the channel width. Because this is relatively small for the case of this problem, the two geometries and are assumed to both scale the channel width to simplify the governing equations further. Another simplification is the removal of the pressure gradients. When using the Boussinesq approximation, the flow is typically treated as temperature driven, resulting in neglecting the pressure gradient terms due to their assumed insignificant influence. These scaled governing equations with dimensionless dependent and independent variables are as follows:

**Continuity, Dimensionless**

**X – Momentum, Dimensionless**

**Y – Momentum, Dimensionless**

**Conservation of Energy, Dimensionless**

# Dimensionless Equations - Analysis

From scaling, the Reynolds number arises with respect to the channel width. With the inverse of the Reynolds number tied to the viscous terms in the dimensionless Navier Stokes equations, at high Reynolds numbers the viscous terms overall influence on the flow will diminish. For this case of flow within the toaster channel, the flow is assumed well into the laminar region and therefore the viscous terms remain. The parameter serves to compare the influence of the temperature gradient on the presence of a forcing function within the conservation of momentum equation in. As the temperature gradient approaches zero with the dimensionless temperature approaching unity, the y-momentum equation becomes homogenous and no longer has a forcing function or temperature dependence. The loss of the temperature dependence implies that the flow is no longer temperature driven and thus the Boussinesq approximation no longer must be applied. The dimensionless parameter implies a loss of the second order terms as the flow increases in size and speed, which also corresponds with an increase in Reynolds number. As the flow gains size and speed, the conservation of energy changes from a second order differential equation to a first order one, but introduces complications through the presence of turbulence once the Reynolds number increases enough.

# Solution

The governing equations (with the exception of continuity) are each second order differential equations with non-linear velocity coefficients and two or more dependent variables present. Typically a problem with these properties can be solved through approximation by infinite series expansion, but this method requires the selection of a term that defines the order of the naïve expansion terms. The momentum and conservation of energy equations mean that no consistent parameter can be chosen to implement such an approximation without a meaningful relation between and. Seems to be a reasonable choice for this parameter, but without its inclusion into the conservation of energy equation a reliable expansion cannot be implemented. Applying the stream functions to the governing equations only serves to increase the equations to 3rd order and does not remove the non-linearity. Searching for a similarity solution proves difficult due to the three dependent variables in both the y momentum and conservation of energy equations, even when implementing a power law similarity solution. The presence of the temperature in the momentum equation essentially means that either a numerical solution or simulation is required to solve the governing equations. Regarding a numerical solution, the existence of non-linear coefficients means that many available numerical solvers cannot approach the problem without running into significant issues.

As will be discussed in the simulation portion of this report, an extended computational domain is required to properly define the boundary conditions for the toaster channel exhaust. Instead of defining the velocity and temperature boundary conditions at the channel exit, the conditions are defined for the far field air past the channel exhaust. The computational domain now includes the channel flow and the exhaust flow past the channel opening.