On Sampling

1 Continuous Sampling

Consider sampling in a 1D real line L. Let $S_c = \{q_x | \forall x \in L\}$, where q_x is a sphere centered at x. Let $C(S_c)$ be the space covered by S_c.

Theorem 1.1 *Continuously sample spheres from left to right in the real line L, let the set of sphere be S_n, such that no sphere in S_n is centered within any sphere of S_n. Then $C(S_n) = C(S_c)$.*

Proof:

$\forall x \in L, \exists q_x \in S_c$, meaning every point in L is covered by one sphere centered at x in S_c, $C(S_c) = L$. If sphere q_p is not centered inside any spheres in S_n, then $q_p \in S_n$ and $p \in C(S_n)$. Assume point p is inside some existing spheres centered to the left of p, then $p \in C(S_n)$, but we don't sample spheres at p.

To sum up, if point p is sampled, then $p \in C(S_c)$ and $p \in C(S_n)$, if point p is not sampled, still $p \in C(S_c)$ and $p \in C(S_n)$. Therefore, $C(S_n) = C(S_c)$.

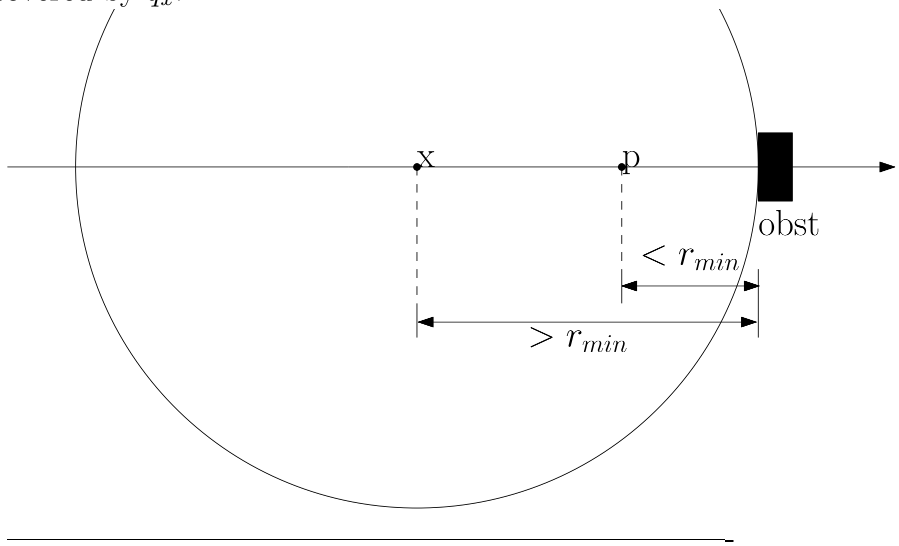
This holds true also for 2D and 3D cases. If a point p is sampled, then $p \in C(S_n)$, if p is not sampled, that is because it is already inside some existing spheres, so $p \in C(S_n)$.

Continuously sample points that are not inside any sphere is equivalent to continuously sample on the boundary of existing spheres.

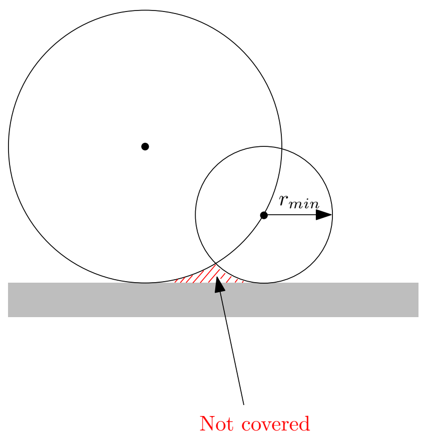
Theorem 1.2 *Continuously sample spheres from left to right in the real line L, such that no sphere has radius less than r_{min} and no sphere in S_m is centered within any sphere of S_n. $C(S_m) = C(S_c)$.*

Proof:

For point p that is within r_{min} distance away from obstacles, there exist a point x that has clearance larger than r_{min}, such that $|p - x| > 0$, then p is covered by q_x.



For 2D and 3D this is different:



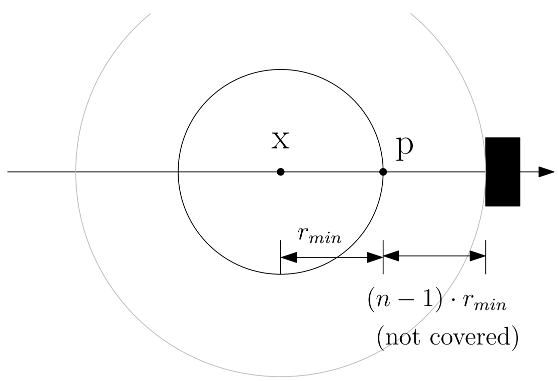
A possible way to reduce the uncovered area is to more densely sample areas close to obstacles. For example, allow spheres to center inside existing spheres in this area.

Theorem 1.3 *Let S_{in} be a subset of S_m by continuously sampling spheres from left to right in the real line L with inaccurate metric. Assume the metric returns 1/n of the accurate metric, then any point p within $(n-1) \cdot r_{min}$ distance away from obstacles, $p \notin C(S_(in))$.*

Proof:

Assume p is $(n-1) \cdot r_{min}$ distance away from obstacles, if there exists point x that is $n \cdot r_{min}$ distance away from obstacles, then sphere q_x has radius r_{min}, $p \in q_x$.

If p is $(n-1) \cdot r_{min} - \epsilon$ distance away from obstacles, where $\epsilon >= 0$, we need a point x that is within $n \cdot r_{min} - \frac{n \cdot \epsilon}{n-1}$ distance from obstacle. The sphere sampled at x has radius less than r_{min} thus will not cover point p.

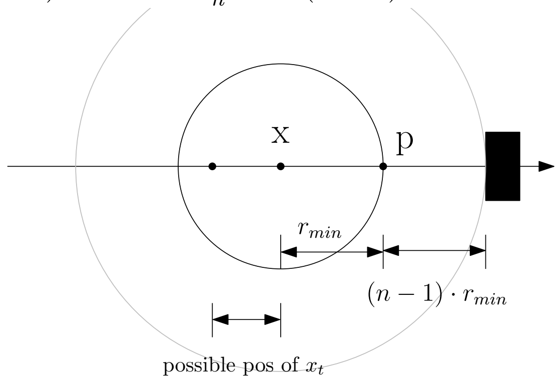


This is also true for 2D and 3D cases. Introduce a line from the closest point in obstacle in normal direction, then the proof is the same. However, uncovered area as shown in Theorem 1.2 stays still.

Theorem 1.4 *Let S_d be a subset of S_{in} by discrete sampling in the real line. Assume every two neighbor samples are d distance away from each other. If $d <= \frac{r_{min}}{k}, k >= 1$, in the worst case, any point p that has clearance less than $\frac{r_{min} \cdot (n-1)}{k \cdot n} + (n-1) \cdot r_{min}$, $p \notin C(S_d)$.*

As is shown in last theorem, the smallest clearance a point x should have in order to be sampled is $n \cdot r_{min}$. $\exists \epsilon > 0$, point x_t with clearance $n \cdot r_{min} + d - \epsilon$. The point the sphere q_{x_t} can cover has clearance more than

$(n-1) \cdot r_{min} + \frac{d \cdot (n-1)}{n} - \frac{(n-1) \cdot \epsilon}{n}$. The worst case is $\epsilon = 0$, so the clearance is $(n-1) \cdot r_{min} + \frac{d \cdot (n-1)}{n} = (n-1) \cdot r_{min} + \frac{r_{min} \cdot (n-1)}{k \cdot n}$



In 2D the possible position for x_t is a square, it will be a cube in 3D, the result is very similar except for that the uncovered area introduced by Theorem 1.2 is still can't be removed.