Small Time Optimality

1 Model

A Reed Shepp car model can be described by two variables $\boldsymbol{v}$ and $\boldsymbol{\omega}$ which are the speed and angular velocity. The turning radius is $r = \frac{\boldsymbol{v}}{\boldsymbol{\omega}}$. To simplify the model, $|\boldsymbol{v}| = 1$, $|\boldsymbol{\omega}| \in [\omega_{min}, \omega_{max}]$.

We use (x, y, θ) to describe the state of a R.S car, (x, y) being the position of the car, while θ being the orientation.

2 Optimal Path Upper Bound

Theorem 2.1 *There exist a minimium lenth l for a R.S car to turn angle $\Delta\theta$.*

Proof: To turning a car with angle $\Delta\theta$ with the least amount of time, The control has to be $l^+l^-l^+l^-l^+$... or $r^+r^-r^+r^-r^+$... (keep turning counter-clockwisely or clockwisely), with the max angular velocity. Therefore the turning radius is $r_{min} = \frac{|\boldsymbol{v}|}{\omega_{max}}$.

Assume the i-th operation turns $\Delta\theta_i$ angles, the number of operations is n ($\forall\Delta\theta_i \geq 0$ or $\forall\Delta\theta_i \leq 0$, and $\sum_{i=1}^{n} \Delta\theta_i = \Delta\theta$). The total path length of these operation is:

$l = \sum_{i=1}^{n} r_{min}\Delta\theta_i = r_{min}\sum_{i=1}^{n} \Delta\theta_i = r_{min}\Delta\theta$.

which has nothing to do with the number of operations, neither with the angle of each turn.

Theorem 2.2 *For a start configuration $s = (x_s, y_s, \theta_s)$, a goal configuration $g = (x_g, y_g, \theta_g)$ and a path σ with length l. σ cannot be the optimal path if $l > r_{min}\pi + |SG|$, where S and G are (x_s, y_s) and (x_g, y_g).*

Proof: Consider steering a R.S car with small-time controlability using $l^+l^-l^+l^-l^+$... or $r^+r^-r^+r^-r^+$... operations. when the number of operation keeps increasing, the position offset decreases(limit to 0). A naive way of moving the car is to 1. steer the car from start orientation to $\vec{SG}$ direction, 2. move the car from S to G, 3. steer the car from $\vec{SG}$ direction to goal orientation.

It can be proved that 1 and 3 steps can steer the car with a maxinum angle of π. (list all kinds of possible orientations then it can be shown. I omit it here.)

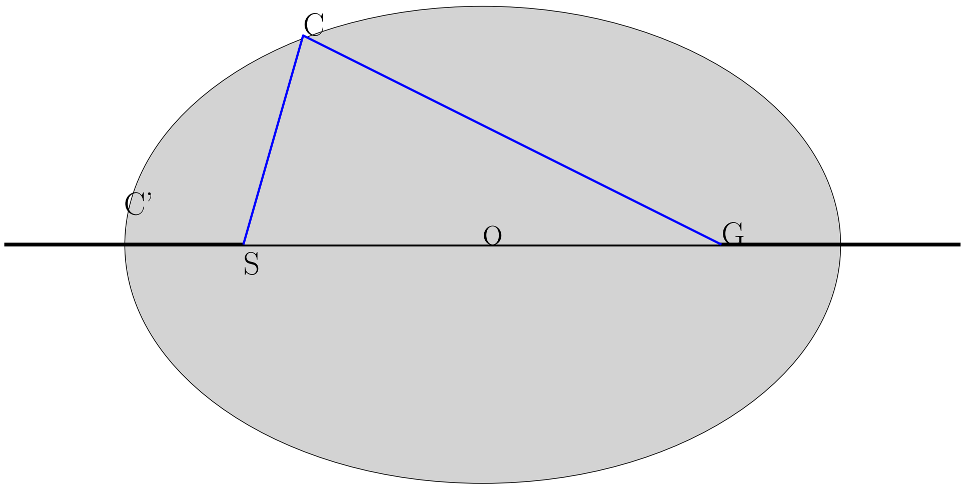
Thus the path length of this strategy is $L = \frac{\pi \boldsymbol{v}}{\boldsymbol{\omega}} + |SG| \geq r_{min}\pi + |SG|$

If a path planning algorithm gives a path σ with length $l > r_{min}\pi + |SG|$, then the naive strategy is even better. σ is clearly not the optimal path.

Theorem 2.3 *If there is clearance of size $R > \frac{r_{min}\pi}{2}$ at any point p in workspace, there exist a R_{in}-ball centered at p, $R_{in} \in (0, R)$, such that between any points in the R_{in}-ball with any orientaion, there exists a feasible optimal trajectory.*

Proof: Assume $S = (x_s, y_s)$ and $G = (x_g, y_g)$ with orientations θ_s and θ_g are start and goal positions in workspace. We denode $|SG| = 2\delta$.

With Theorem 2.2, we know that the optimal path σ must satisfy the property of lenth $l <= r_{min}\pi + 2\delta$. Assume C being a point in the workspae and $|SC| + |GC| \leq r_{min}\pi + 2\delta$. C is in an ellipse with foci's S and G.



Since any point P out of the ellipse will have $|SP| + |PG| > r_{min}\pi + 2\delta$, the optimal path from S to G with any orientation must be within the ellipse.

C' is a intersect point of line SG and the ellipse. O is the mid point of S and G.

$\implies |SO| = |OG| = \delta.\ |C'S| + |C'G| = r_{min}\pi + 2\delta = 2|C'S| + |SG|$
$\implies |C'S| = \frac{r_{min}\pi}{2}$

By rotating S and G about their mid point O, the trajectory of C' is the edge if another ball with radius $\delta + \frac{r_{min}\pi}{2}$. And between any points in the inner ball, there must exist an optimal trajectory without leaving the outter ball.

This two balls have radius difference of $\frac{r_{min}\pi}{2}$.

3 A Planning Algorithm

A possible non-optimal planning algorithm is as below:

1. Sample a random sphere in workspace.
2. Get the inner sphere of the previously sampled sphere.

3. Sample spheres along the boundary of existing inner spheres.
4. Repeat 1 - 3 until converege.
5. Find a path using boundary points of these inner spheres.

Suppose an algorithm gives a series of points $P_1, P_2, P_3, ..., P_n$ which are points that the car should go through. Then we can have the upper bound of total length of optimal path:
$L \leq (n-1)r_{min}\pi + \sum_{i=1}^{n-1} l_i$
l_i being the distance between P_i and P_{i+1}.

3.1 Problems to solve:

1. How to minimize the number of points a car should go through?

2. How to choose the orientation at each point such that the total upper bound of optimal path that goes through these points is minimized?

3. How to find the lower bound of optimal path between S and G?

4. How to minimize the upper bound of path length found by algorithm, such that the upper bound is close to the lower bound?