Small Time Optimality

1 Model

A Reed Shepp car model can be described by two variables $\boldsymbol{v}$ and $\boldsymbol{\omega}$ which are the speed and angular velocity. The turning radius is $r = \frac{\boldsymbol{v}}{\boldsymbol{\omega}}$. To simplify the model, $|\boldsymbol{v}| = 1$, $|\boldsymbol{\omega}| \in [\omega_{min}, \omega_{max}]$.

We use (x, y, θ) to describe the state of a R.S car, (x, y) being the position of the car, while θ being the orientation.

2 Optimal Path Upper Bound

Theorem 2.1 *There exist a minimium lenth l for a R.S car to turn angle $\Delta\theta$.*

Proof: To turning a car with angle $\Delta\theta$ with the least amount of time, The control has to be $l^+l^-l^+l^-l^+$... or $r^+r^-r^+r^-r^+$... (keep turning counter-clockwisely or clockwisely), with the max angular velocity. Therefore the turning radius is $r_{min} = \frac{|\boldsymbol{v}|}{\omega_{max}}$.

Assume the i-th operation turns $\Delta\theta_i$ angles, the number of operations is n ($\forall\Delta\theta_i \geq 0$ or $\forall\Delta\theta_i \leq 0$, and $\sum_{i=1}^{n} \Delta\theta_i = \Delta\theta$). The total path length of these operation is:

$l = \sum_{i=1}^{n} r_{min}\Delta\theta_i = r_{min} \sum_{i=1}^{n} \Delta\theta_i = r_{min}\Delta\theta.$

which has nothing to do with the number of operations, neither with the angle of each turn.

Theorem 2.2 *For a start configuration $s = (x_s, y_s, \theta_s)$, a goal configuration $g = (x_g, y_g, \theta_g)$ and a path σ with length l. σ cannot be the optimal path if $l > r_{min}\pi + |SG|$, where S and G are (x_s, y_s) and (x_g, y_g).*

Proof: Consider steering a R.S car with small-time controlability using $l^+l^-l^+l^-l^+$... or $r^+r^-r^+r^-r^+$... operations. when the number of operation keeps increasing, the position offset decreases(limit to 0). A naive way of moving the car is to 1. steer the car from start orientation to $\vec{SG}$ direction, 2. move the car from S to G, 3. steer the car from $\vec{SG}$ direction to goal orientation.

It can be proved that 1 and 3 steps can steer the car with a maxinum angle of π. (list all kinds of possible orientations then it can be shown. I omit it here.)

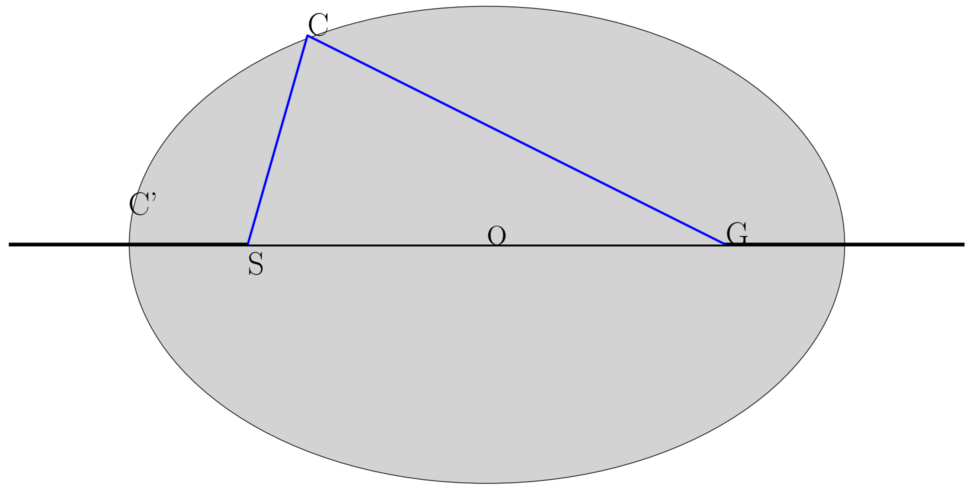
Thus the path length of this strategy is $L = \frac{\pi \boldsymbol{v}}{\boldsymbol{\omega}} + |SG| \geq r_{min}\pi + |SG|$

If a path planning algorithm gives a path σ with length $l > r_{min}\pi + |SG|$, then the naive strategy is even better. σ is clearly not the optimal path.

Theorem 2.3 *If there is clearance of size $R > \frac{r_{min}\pi}{2}$ at any point p in workspace, there exist a R_{in}-ball centered at p, $R_{in} \in (0, R)$, such between any points in the R_{in}-ball with any orientaion, there exists a feasible optimal trajectory.*

Proof: Assume $S = (x_s, y_s)$ and $G = (x_g, y_g)$ with orientations θ_s and θ_g are start and goal positions in workspace. We denode $|SG| = 2\delta$.

With Theorem 2.2, we know that the optimal path σ must satisfy the property of lenth $l <= r_{min}\pi + 2\delta$. Assume C being a point in the workspae and $|SC| + |GC| \leq r_{min}\pi + 2\delta$. C is in an ellipse with foci's S and G.



Since any point P out side the ellipse will have $|SP| + |PG| > r_{min}\pi + 2\delta$, the optimal path from S to G with any orientation must be within the ellipse.

C' is a intersect point of line SG and the ellipse. O is the mid point of S and G.

$\Longrightarrow |SO| = |OG| = \delta.\ |C'S| + |C'G| = r_{min}\pi + 2\delta = 2|C'S| + |SG|$
$\Longrightarrow |C'S| = \frac{r_{min}\pi}{2}$

By rotating S and G about their mid point O, the trajectory of C' is the edge if another ball with radius $\delta + \frac{r_{min}\pi}{2}$. And between any points in the inner ball, there must exist an optimal trajectory without leaving the outter ball.

This two balls have radius difference of $\frac{r_{min}\pi}{2}$.