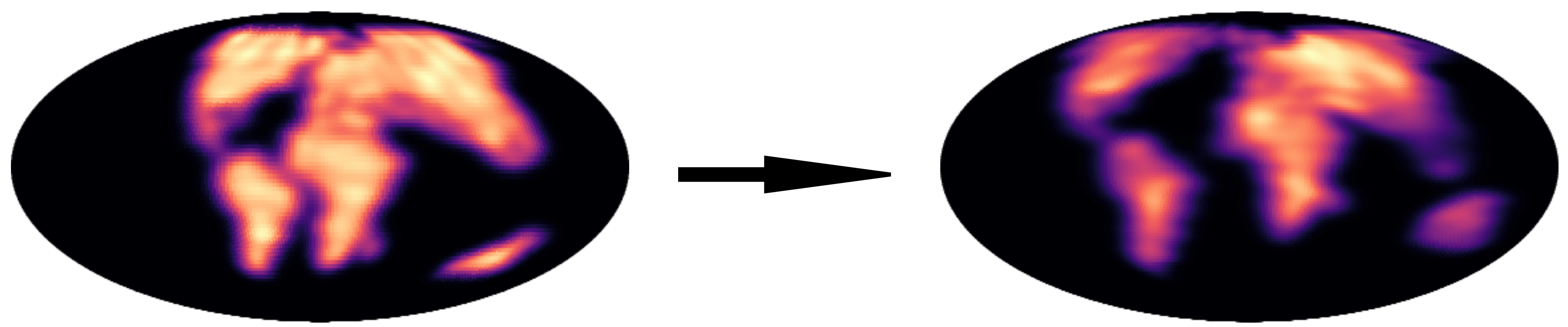
Riemannian Convex Potential Maps

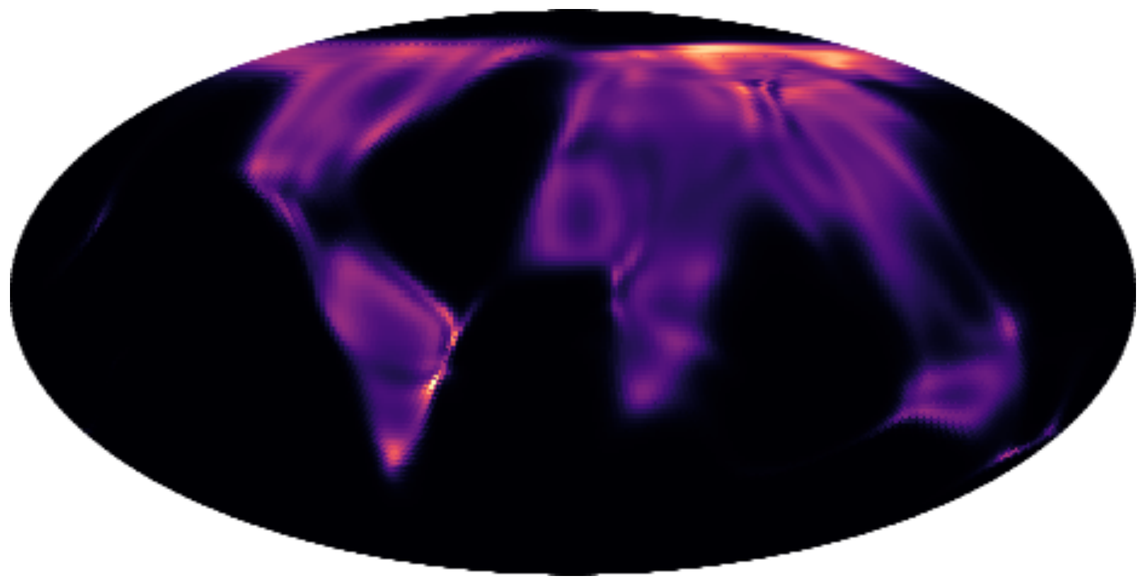
Samuel Cohen*, Brandon Amos*, Yaron Lipman

Motivation

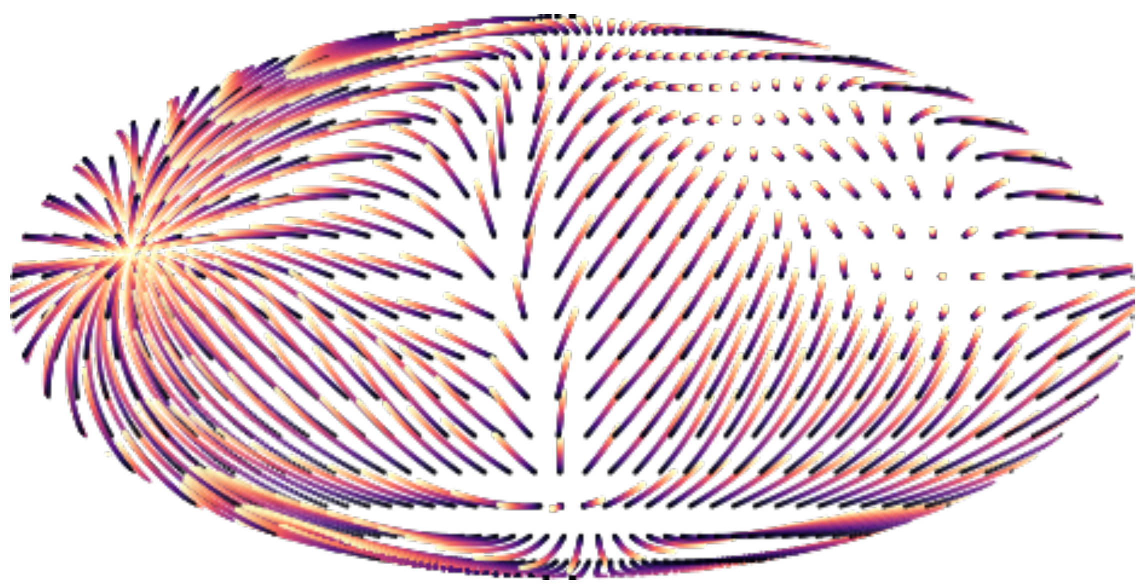
Real-world data on Riemannian manifolds



- Density Transportation



- Density Estimation

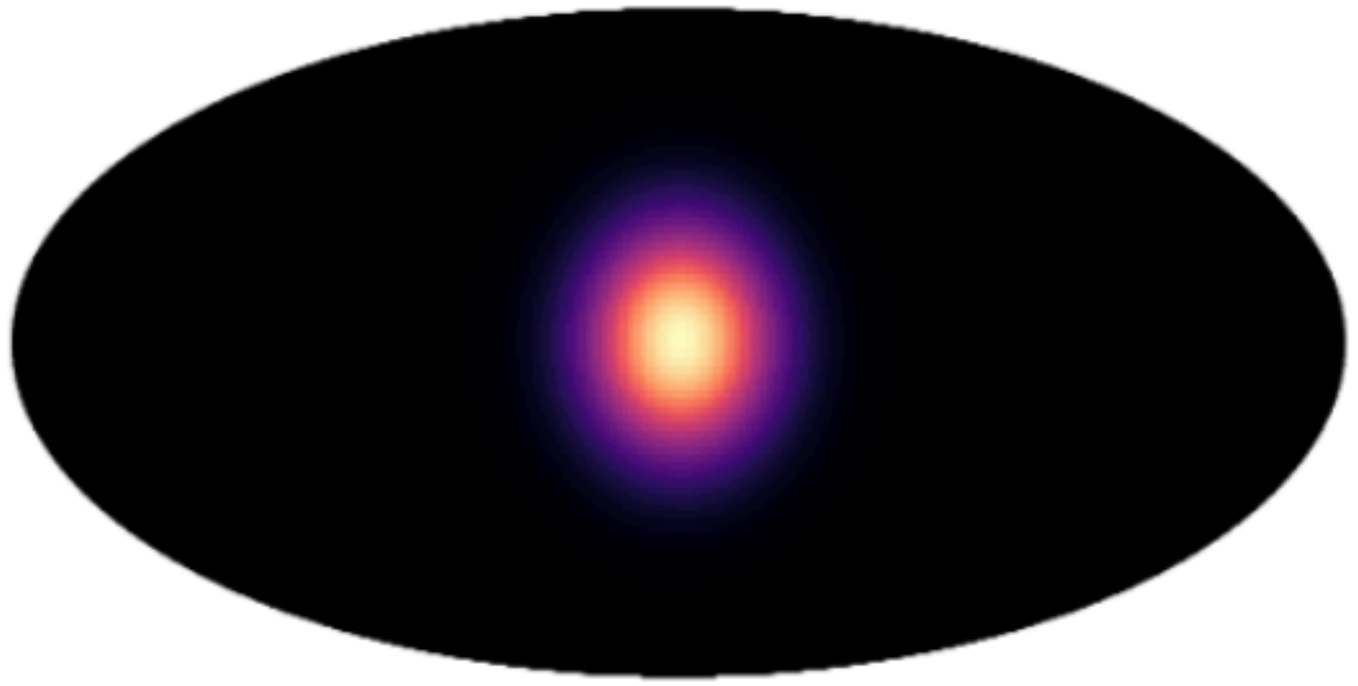


- Geodesics Estimation

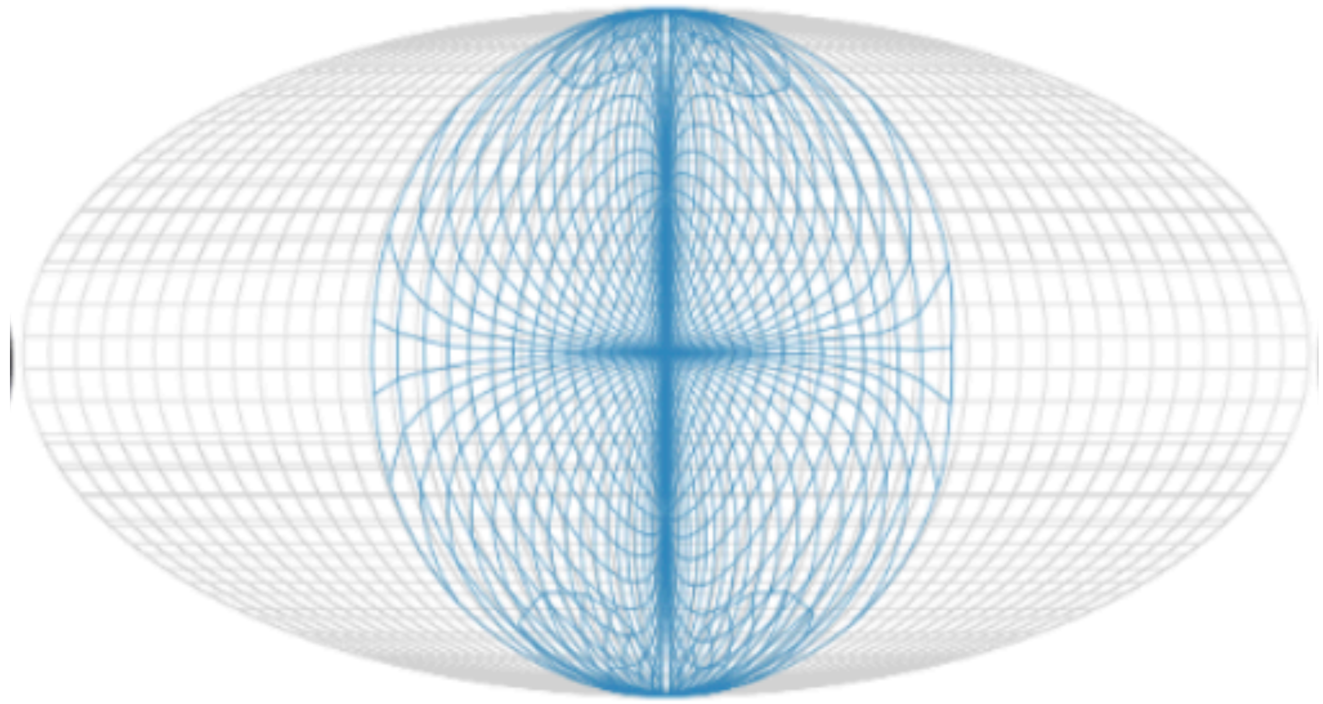
Riemannian Optimal Transport

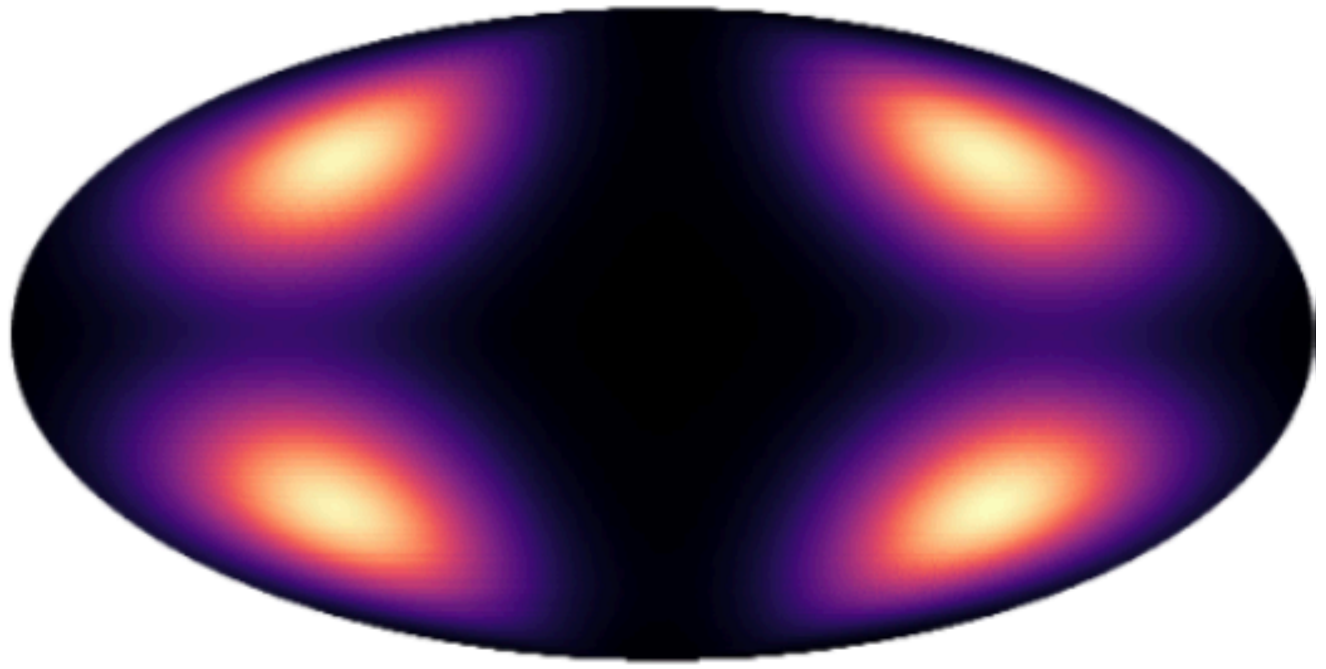
- Given source μ and target ν distributions on manifolds find an (OT) map pushing source to target.

$$\operatorname*{argmin}_{t:t_{\#}\mu=\nu} \int_{\mathcal{M}} c[x, t(x)]\, d\mu$$



μ





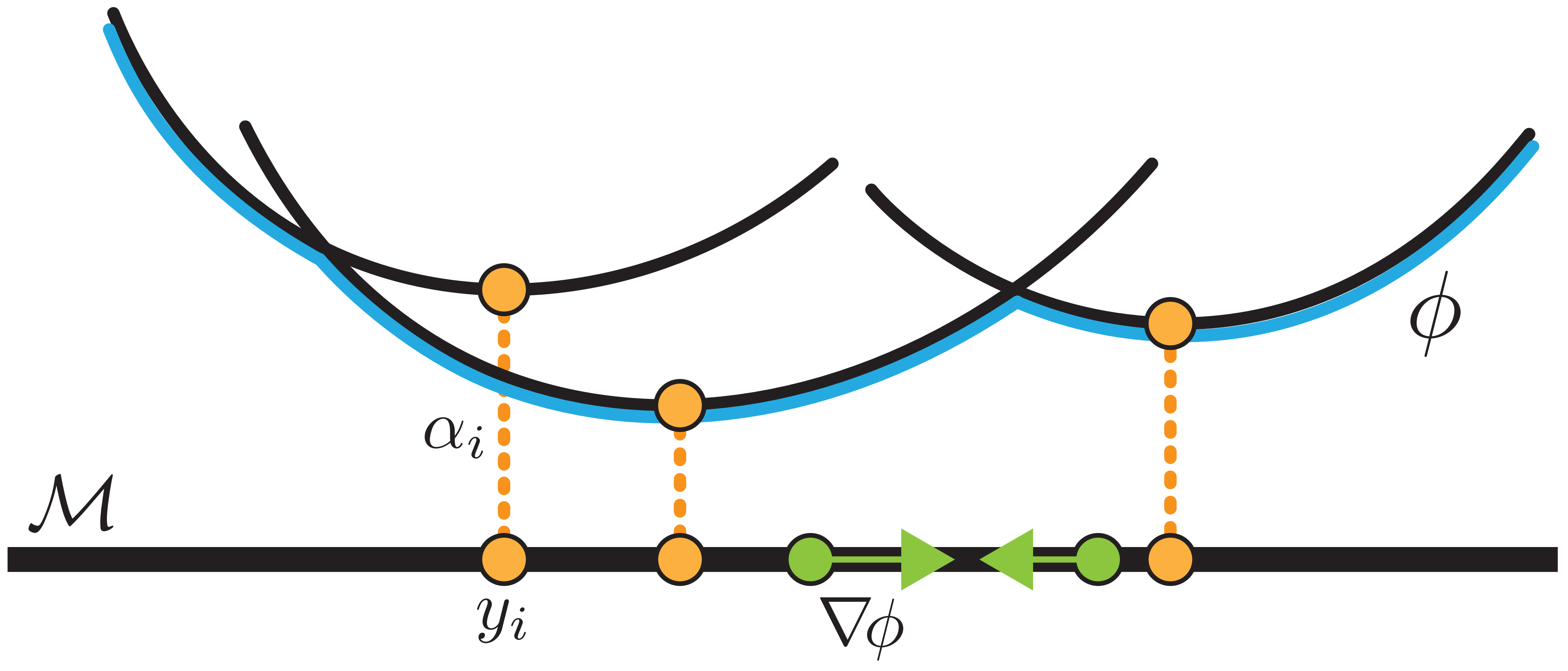
ν

Approach: Semidiscrete OT

- **McCann's Theorem**: OT is of the form $t(x) = \exp(-\nabla\phi)$, ϕ is c-concave
- **Our semidiscrete OT on manifolds:**
  Use discrete c-concave potentials of the form

$$\phi(x) = \min_{i\in[n]} c(x, y_i) + \alpha_i$$

Learnable parameters



Theory: Universality

Theorem 1: For compact, boundaryless, smooth manifolds, $\left\{f \mid f(x) = \min_{i \in [n]} c(x, y_i) + \alpha_i\right\}$ is dense in $\{f \mid f$ is c-concave$\}$.

Theorem 2: If μ, ν are regular, there exists a sequence of discrete c-concave potentials ϕ_ϵ such that $\exp[-\nabla \phi_\epsilon] \xrightarrow{p} t$

where t is the OT map.

Implementation Details

- **Map architecture**: stack of multiple blocks of the form

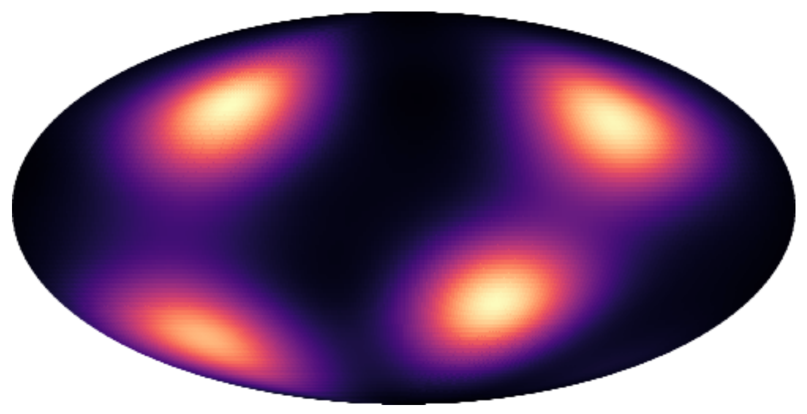
$$s_j(x) = \exp[-\nabla_x \phi_j(x)], \qquad j = 1, \ldots, T$$

- **Smoothing**: applied to discrete c-concave layers

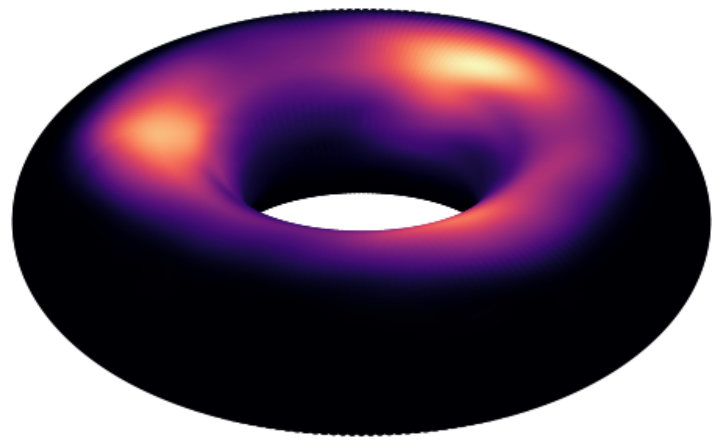
$$\min_\gamma(a_1, \ldots, a_n) = -\gamma \log \sum_{i=1}^{n} \exp - \frac{a_i}{\gamma}$$

- **Loss**: standard density estimation losses (NLL, KL)

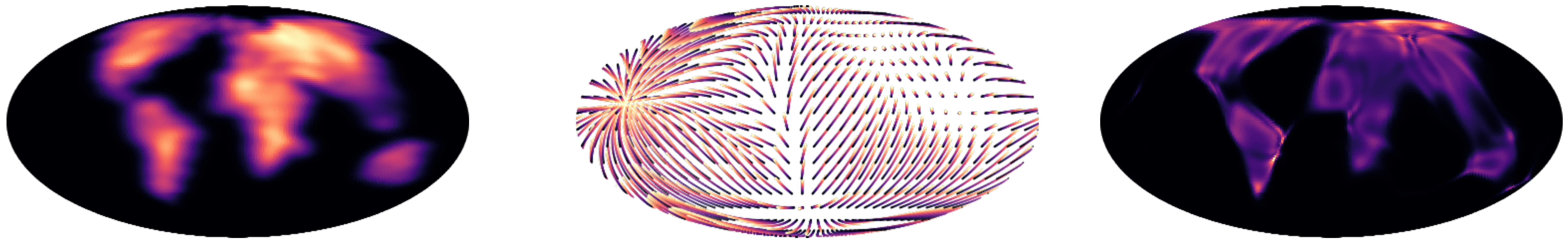
Results



• Synthetic, sphere



• Synthetic, torus



• Earth case study, sphere

For more details, have a look at our paper!