Regression

Expert Testimony

Predicting a numerical outcome variable

- What salary should I expect, *given my relevant characteristics*?
- What demand for transformers should I expect, *given a price I set*?
- What price should I expect for a hotel, *given its location and rating*?

Recipe

Given X, expect Y.

Predictions are never exact

- Cannot make perfectly accurate prediction $\rightarrow$ learn to live with *error*.
- Capture the *mean* of the outcome.
- Minimize prediction error.

Our first prediction

I expect the mean salary, mean demand, mean price.

$$\hat{Y}_{\text{mine}} = E(Y_{\text{all}})$$

"regression to the mean" $\rightarrow$ **regression**

This does not hold relevant characteristics fixed.

Our second prediction

Take the mean of Y for the group of cases with exactly the same X as mine.

$$\hat{Y}_{\text{mine}} = E(Y) \text{ for cases where } X = X_{\text{mine}}$$

This is called **conditional mean**, or mean of Y **conditional on** X:

$$\hat{Y}_{\text{mine}} = E(Y|X = X_{\text{mine}})$$

Can hold multiple things fixed

$$\widehat{\text{wage}}_{\text{mine}} = E(\text{wage}|\text{occupation} = \text{economist}, \text{eye color} = \text{blue})$$

How to compute this?

If all Xs are categorical:

1. select comparison group with exact same Xs
2. compute sample mean within this group

(Pivot table, AVG() ... GROUP BY ...)

What price for a 3-star hotel in Favoriten?

Limits of pivot tables

Often no or few exact matches with $X = X_{\text{mine}} \rightarrow$ noisy prediction.

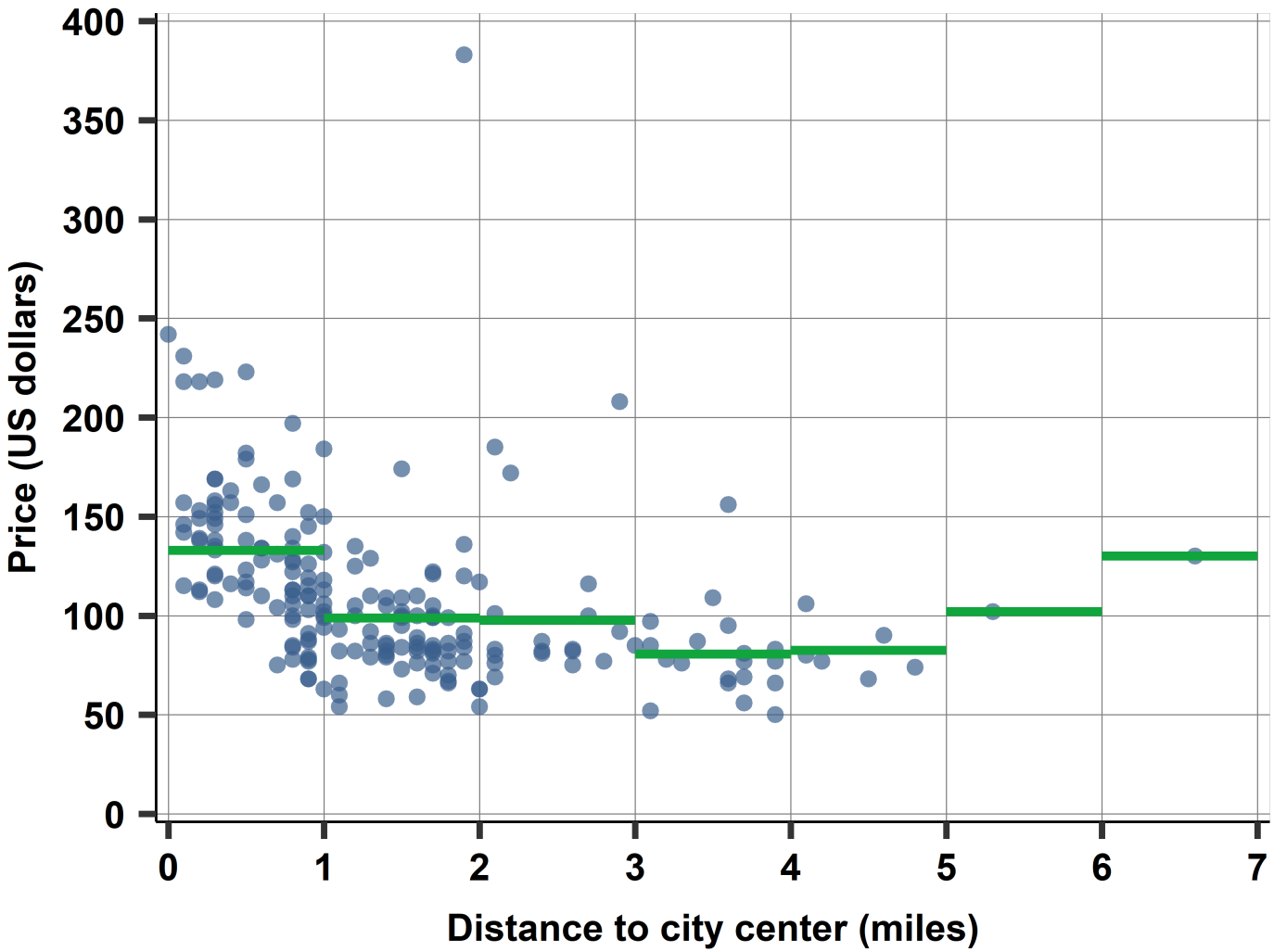
Particularly if X is a numerical variable.

What price for a 3-star hotel in Favoriten with a rating of 3.7?

Two solutions

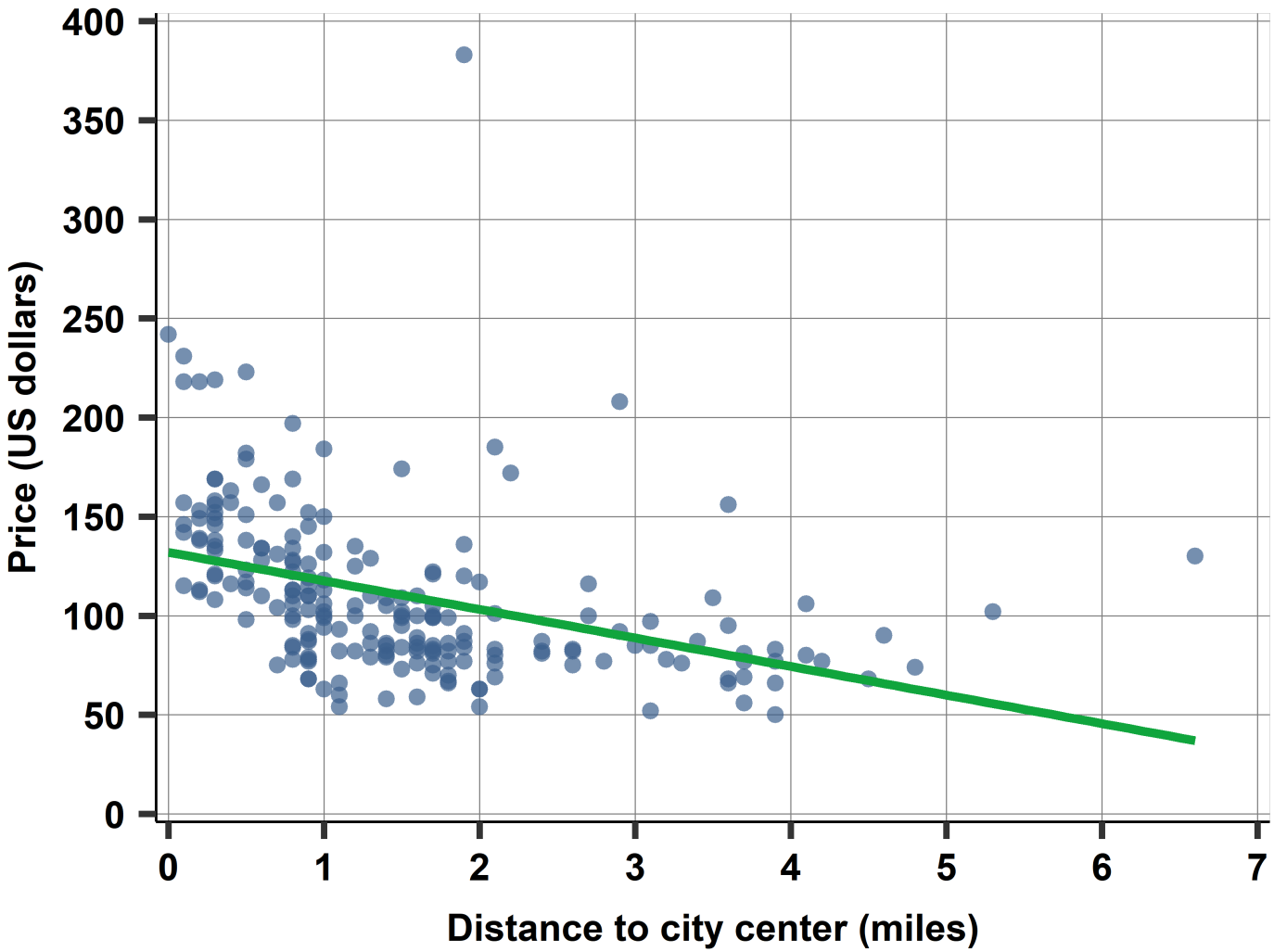
1. Merge some groups
   - Which ones? Machine Learning: regression tree, random forest
2. Interpolate between data points
   - Assume a relationship with a simple mathematical functional form

Split distance into bins



Békés and Kézdi (2021, Chapter 07)

Fit a line on the scatter plot



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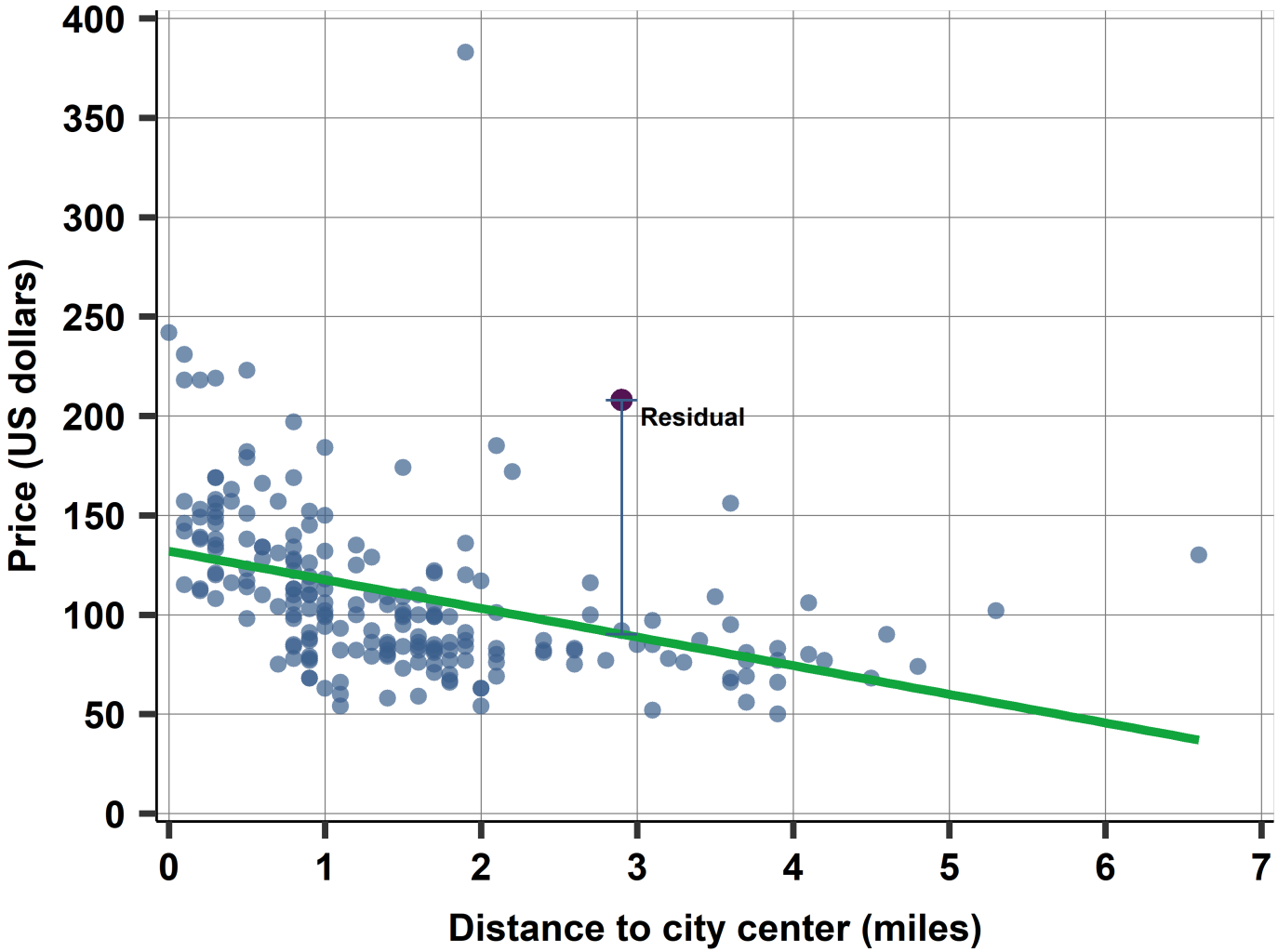
Try to minimize residuals



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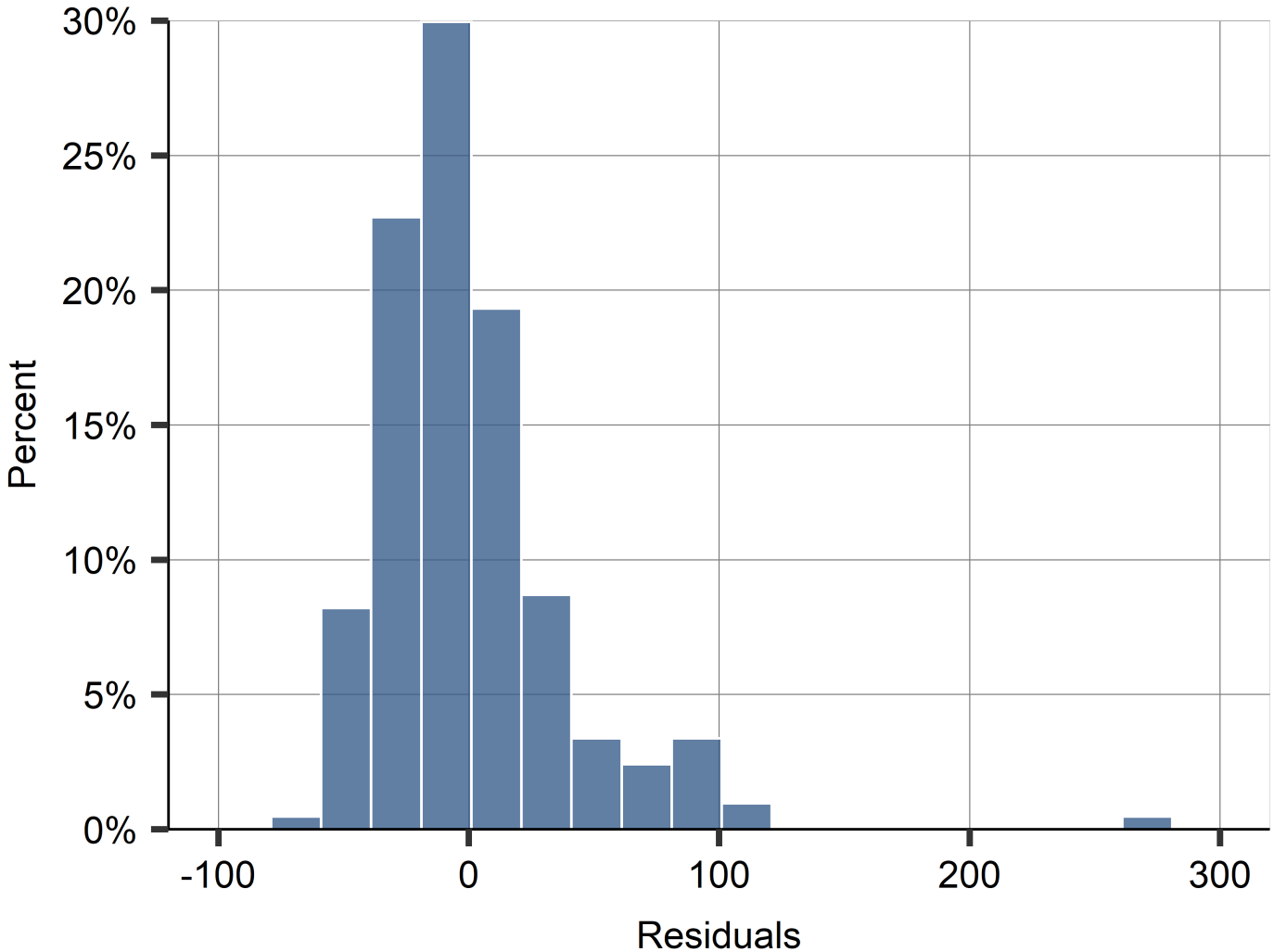
Unbiased but dispersed residuals



Békés and Kézdi (2021, Chapter 07)

Linear regression

Assume linear relationship:

$$E(Y|X) = a + bX$$

Need to estimate a and b.

But can use even for X for which we have no data (yet), like rating=3.7.

Interpret coefficients

$$E(Y|X) = a + bX$$

a mean value of Y when $X = 0$:

$$a + b \cdot 0 = a$$

b difference in mean value of Y when X *increases by 1 unit*:

$$[a + b(x + 1)] - [a + bx] = b(x + 1) - bx = b$$

Interpret regression of hotel price on distance

```
. regress price distance
```

Source	SS	df	MS		
				Number of obs	= 428
				F(1, 426)	= 19.51
Model	156858.055	1	156858.055	Prob > F	= 0.0000
Residual	3424389.35	426	8038.47266	R-squared	= 0.0438
				Adj R-squared	= 0.0416
Total	3581247.41	427	8386.99627	Root MSE	= 89.658

price	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
distance	-12.01145	2.719123	-4.42	0.000	-17.35602	-6.666883
_cons	151.2924	6.255225	24.19	0.000	138.9974	163.5873

What is the expected price 2.2 miles from the center?

$$E(\text{price}|\text{distance}) = 151.29 - 12.01 \cdot \text{distance}$$

Can hold multiple things fixed

Multiple linear regression

$$E(Y|X,Z) = a + bX + cZ$$

Interpret coefficients

$$E(Y|X,Z) = a + bX + cZ$$

a mean value of Y when **both** $X = 0$ and $Z = 0$:

$$a + b \cdot 0 + c \cdot 0 = a$$

b difference in mean value of Y when X increases by 1 unit, **holding** Z **fixed**:

$$[a + b(x+1) + cz] - [a + bx + cz] = b(x+1) - bx = b$$

c difference in mean value of Y when Z increases by 1 unit, **holding** X **fixed**:

$$[a + bx + c(z+1)] - [a + bx + cz] = c(z+1) - cz = c$$

Hotel prices depend both on distance and rating

```
. regress price distance rating
```

Source	SS	df	MS		
				Number of obs	= 393
				F(2, 390)	= 13.97
Model	211966.86	2	105983.43	Prob > F	= 0.0000
Residual	2959529.45	390	7588.53705	R-squared	= 0.0668
				Adj R-squared	= 0.0620
Total	3171496.31	392	8090.55181	Root MSE	= 87.112

price	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
distance	-9.027104	2.764378	-3.27	0.001	-14.46205	-3.592156
rating	26.42263	7.786677	3.39	0.001	11.11351	41.73174
_cons	38.14768	32.52452	1.17	0.242	-25.79765	102.093

What is the expected price 2.2 miles from the center with a 3.7 rating?

$$E(\text{price}|\text{distance}) = 38.14 - 9.03 \cdot \text{distance} + 26.42 \cdot \text{rating}$$

Multiplicative models

Multiplicative models

So far we studied

$$E(Y|X) = a + bX,$$

but often we want

$$E(Y|X) = aX^b.$$

Why? Because we may be interested in percentage or proportional changes, not changes by one unit.

$$\frac{E(Y|X = 2x)}{E(Y|X = x)} = \frac{a(2x)^b}{ax^b} = 2^b$$

Interpreting coefficients

Suppose

$$E(\text{wage}|\text{hours}) = 100 \times \text{hours}^{0.5}.$$

What is the effect of doubling working hours?

$$\frac{100 \times (2h)^{0.5}}{100 \times h^{0.5}} = 2^{0.5} \approx 1.41,$$

wages increase by 41 percent.

Categorical variables

Categorical variables

What X is *categorical*? Say, taking the values `male` and `female`.

One-hot encoding ("dummy variables") to the rescue.

$$\text{FEMALE}_i = \begin{cases} 1 & \text{if } X_i = \text{female} \\ 0 & \text{if } X_i = \text{male} \end{cases}$$

Similarly,

$$\text{MALE}_i = \begin{cases} 0 & \text{if } X_i = \text{female} \\ 1 & \text{if } X_i = \text{male} \end{cases}$$

One-hot encoded variables in a regression

Suppose

$$E(\text{WAGE}_i|X_i) = \begin{cases} 80 & \text{if } X_i = \text{female} \\ 100 & \text{if } X_i = \text{male} \end{cases}$$

This can be written as

$$E(\text{WAGE}_i|X_i) = 100 - 20 \times \text{FEMALE}_i,$$

which is a *linear regression*!

Huh?!

$$E(\text{WAGE}_i|X_i) = 100 - 20 \times \text{FEMALE}_i$$

$$= \begin{cases} 100 - 20 \times 1 & \text{if } X_i = \text{female} \\ 100 - 20 \times 0 & \text{if } X_i = \text{male} \end{cases}$$

$$= \begin{cases} 80 & \text{if } X_i = \text{female} \\ 100 & \text{if } X_i = \text{male} \end{cases}$$

Interpreting the effect of gender

$$E(\text{WAGE}_i|X_i) = 100 - 20 \times \text{FEMALE}_i,$$

is equal to

$$E(\text{WAGE}_i|X_i) = 80 + 20 \times \text{MALE}_i$$

Females earn 20 dollars *less than males* = Males earn 20 dollars *more than females*.

Question: why not enter both MALE and FEMALE?

Multiple categories

Suppose X captures the area of law.

$$E(\text{WAGE}_i|X_i) = \begin{cases} 80 & \text{if } X_i = \text{tax} \\ 100 & \text{if } X_i = \text{M\&A} \\ 95 & \text{if } X_i = \text{IP} \end{cases}$$

How to one-hot encode this? We can encode two of them, say `tax` and `IP`.

$$E(\text{WAGE}_i|X_i) = 100 - 20 \times \text{tax}_i - 5 \times \text{IP}_i$$

Questions: Why not include all three? Will expected wage take the value of 75?