GARCH-type Model Based Volatility Forecast For Implied Volatility

Fan Gao

1 Introduction

The purpose of this research is to investigate various approaches for forecasting implied volatility, especially GARCH family. My study intends to find whether the two streams of volatility, which are estimation by GARCH-type models and implied volatility by derived from Black-Scholes Model, can be bridged together.

Data, which I download from Bloomberg terminal, are price series and implied volatility series of at-the-money call option with 30 days maturity. The underlying assets are AAPL, BAC, BA and WMT. After introducing GARCH-type models formulations and the implied volatility, I will do an empirical experiment. I will use GARCH-type models to recursively fit the return series. To simulate the implied volatility, I average the 30-days-ahead, that is also 22 trading days, forecasts. This averaged series is my volatility prediction from GARCH-type models. I will compare the performance of the forecasts on the volatility of stock of APPLE Inc.(AAPL) obtained with various GARCH-type models. Here, the measurement of model's capability of forecast 30-day-ahead IV is based on mean absolute error(MAE) which is given by,

$$\theta_t = IV_t - \sigma_t^i$$

$$MAE(i) = \sum_{t=1}^{T} |\sigma_t^i| \Big/ T$$

Where i represents some GARCH-type model.

In order to capture the relation of GARCH estimates and implied volatility, hoping to improve the prediction performance, I will run a regression, using log of implied volatility as dependent variable and log of GARCH-type model estimated volatility as explanatory variable. Then, I use this regression to predict out-of-sample IV estimates. Again, using MAE as criteria, I hope this regression's predictions can reduce MAE value, thus improving the performance. Concluding remarks will follow.

2 GARCH-type Models

In this section, I want to briefly summarize the main features of some model specifications in the GARCH family. As noted by Engle and Patton (2007), GARCH-type models can capture some stylized facts about volatility: pronounced persistence and mean-reversion, asymmetry such that the sign of an innovation also affects volatility and the possibility of exogenous or pre-determined variables influencing volatility.

2.1 GARCH

A GARCH(1,1) model introduced by Bollerslev (1986) for the residual process, y_t, can be expressed as

$$y_t = \varepsilon_t \sqrt{h_t}$$

$$h_t = \omega + \beta h_{t-1} + \alpha y_{t-1}^2$$

Where $\varepsilon_t \sim N(0,1), h_t = E_{t-1}(y_t^2)$.

The log likelihood function of the GARCH(1,1) model is

$$L(\Theta: y_1, y_2, \cdots, y_t) = -\frac{Tln(2\pi)}{2} - \frac{1}{2\sum_{t=1}^{T}\left[\ln(h_t) + \frac{y_t^2}{h_t}\right]}$$

Where $\Theta' = (\omega, \alpha, \beta)$. The likelihood function is maximized using the Berndt, Hall, Hall and Hausman (1974) algorithm. Further, Bollerslev and Wooldridge (1992) show that even if the assumption that ε_t is i.i.d. N(0,1) is not valid, the quasi-maximum likelihood estimates are still consistent and asymptotically normally distributed.

Here, I only give the example of GARCH(1,1). The f-step-ahead forecast of implied variance from the GARCH(1,1) model can be represented as

$$\begin{aligned} E_t(y_{t+f}^2) &= \omega \sum_{i=0}^{f-1} (\alpha+\beta)^i + (\alpha+\beta)^{f-1}\beta h_t + (\alpha+\beta)^{f-1}\alpha y_t^2 \qquad f > 1 \\ &= \omega \sum_{i=0}^{f-2} (\alpha+\beta)^i + (\alpha+\beta)^{f-1} h_{t+1} \qquad f > 2 \end{aligned}$$

When $\alpha + \beta < 1$, for large f, the conditional expectation is given by

$$\begin{aligned} E_t(y_{t+f}^2) &= \omega \sum_{i=0}^{f-2} (\alpha+\beta)^i + (\alpha+\beta)^{f-1} h_{t+1} \\ &= \omega/(1-\alpha-\beta) \end{aligned}$$

When $0 < \alpha + \beta < 1$, $1/(1-\alpha-\beta)$ is always positive. From this formula, we can see that, for large forecasting horizons, the forecasts will converge to a constant at an exponential rate.

2.2 EGARCH

The exponential GARCH model assumes a specific parametric form for this conditional heteroskedasticity. An EGARCH(1,1) can generally be specified as

$$\varepsilon_t = \sigma_t z_t$$

$$\ln(\sigma_t^2) = \omega + \alpha(|z_{t-1}| - E[|z_{t-1}|]) + \gamma z_{t-1} + \beta \ln(\sigma_{t-1}^2)$$

The impact is asymmetric if $\gamma \neq 0$, and leverage is present if $\gamma < 0$. This model differs from the GARCH variance structure because of the log of variance.

2.3 GJR-GARCH

The Glosten-Jagannathan-Runkle GARCH model models positive and negative shocks on the conditional variance asymmetrically via the use of the indicator function,

$$\varepsilon_t = \sigma_t z_t$$

$$\sigma_t^2 = \omega + (\alpha + \gamma I_{t-1})\varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2$$

where

$$I_{t-1} = \begin{cases} 0 \ \ if \ r_{t-1} \geq \mu \\ 1 \ \ if \ r_{t-1} < \mu \end{cases}$$

The impact of good news is α and bad news $\alpha + \gamma$. Thus, $\gamma \neq 0$ implies asymmetry. Leverage exists if $\gamma > 0$.

2.4 TGARCH

The threshold GARCH is similar to the GJR-GARCH. The only difference is using standard deviation instead of variance. In the specification, TGARCH(1,1) can be written as

$$\varepsilon_t = \sigma_t z_t$$

$$\sigma_t^2 = \omega + \alpha(|\varepsilon_{t-1}| - \eta\varepsilon_{t-1}) + \beta\sigma_{t-1}$$

3 An In-sample Estimation Comparison

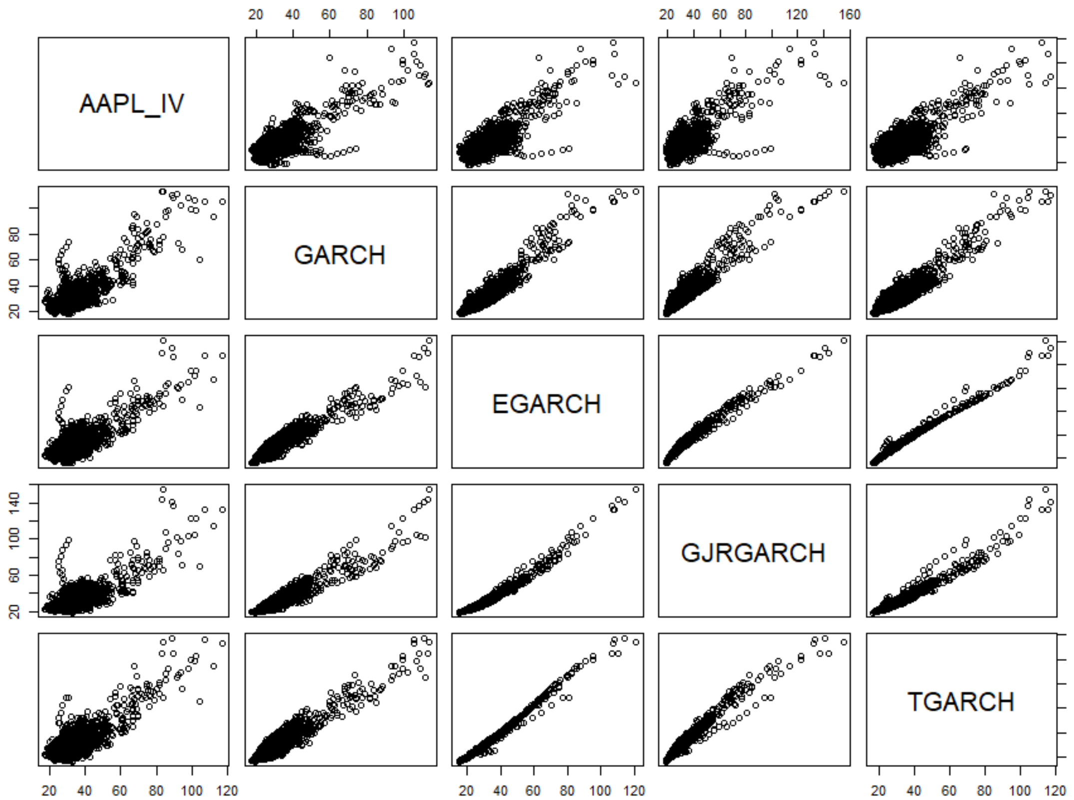
In this section, I will use GARCH, EGARCH, GJR-GARCH and TGARCH to do in-sample estimation. Price data ranging from 2008-04-29 to 2013-04-26. Taken autocorrelation into consideration, I use annualized log return. The model fit summary is shown in Table 1.

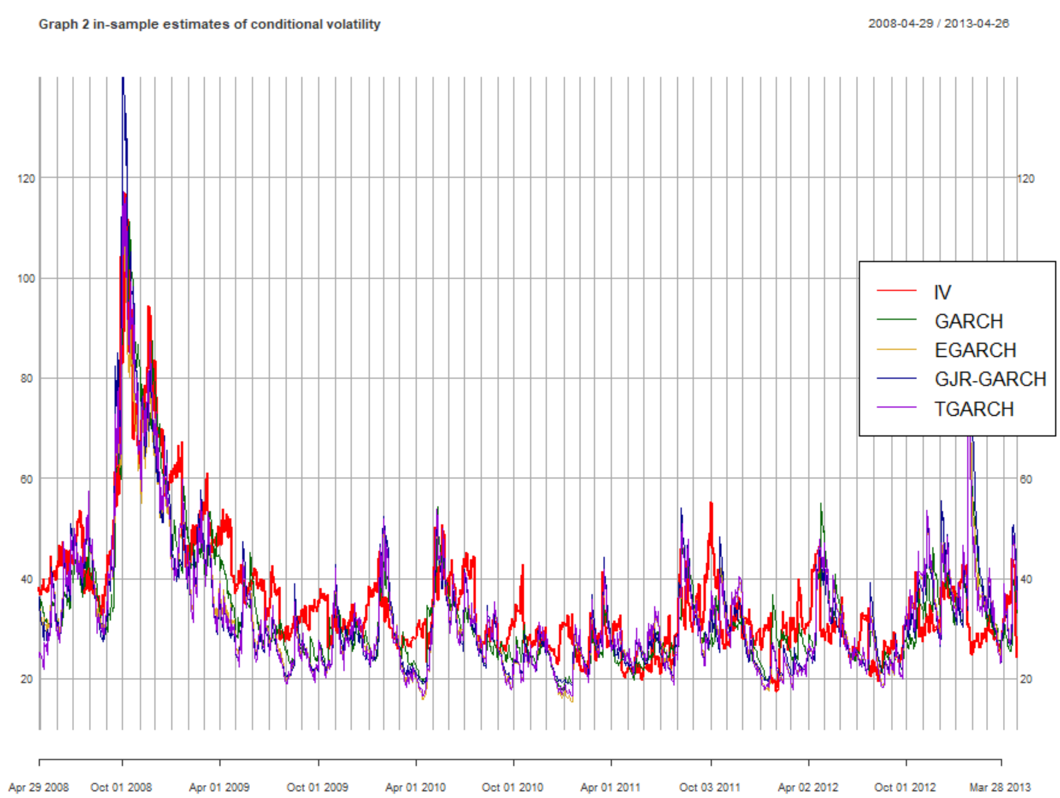
Table 1 In-Sample GARCH-type model Summary

Parameters	GARCH(1,1)	EGARCH(1,1)	GJR-GARCH(1,1)	TGARCH(1,1)
μ	0.194017 (0.000139)	0.139486 (0.006577)	0.133843 (0.007646)	0.131753 (0.021017)
ω	0.115385 (0.002052)	0.062299 (0.000098)	0.186202 (0.000008)	0.098429 (0.000000)
α_1	0.095214 (0.000002)	-0.113191 (0.000098)	0.013317 (0.360797)	0.102946 (0.000000)
β_1	0.882466 (0.000000)	0.956626 (0.000000)	0.854032 (0.000000)	0.871187 (0.000000)
γ_1		0.185993 (0.000000)	0.183743 (0.000000)	
η_1				0.701662 (0.000000)
$Log - likelihood$	-2645.814	-2623.095	-2624.578	-2616.019
AIC	4.2127	4.1782	4.1806	4.1670
BIC	4.2291	4.1986	4.2010	4.1874

From the summary, the TGARCH(1,1) has the highest log-likelihood and the lowest AIC/BIC value, thus making it the most suitable model. To have further insights on the characteristic of the estimated conditional volatility, I compare the estimated series with one another within sample and I draw the estimated in-sample conditional volatility scatterplot matrix, Graph 1, and in-sample conditional volatility graph, Graph 2. We can see from the Graph 1 and Graph 2 that the estimation of EGARCH and TGARCH are quite similar. And GARCH model's predictions are quite different from that from others, which is not surprising in that the GARCH model does not include the asymmetric term of return.

Graph 1 Estimated In-Sample Conditional Volatility Scatterplot Matrix

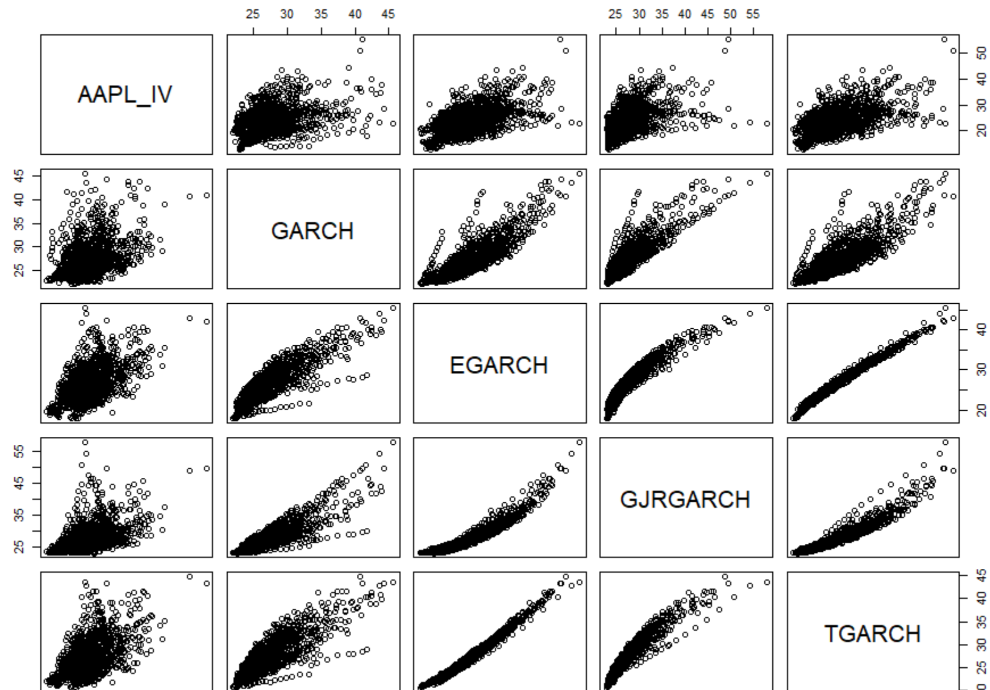




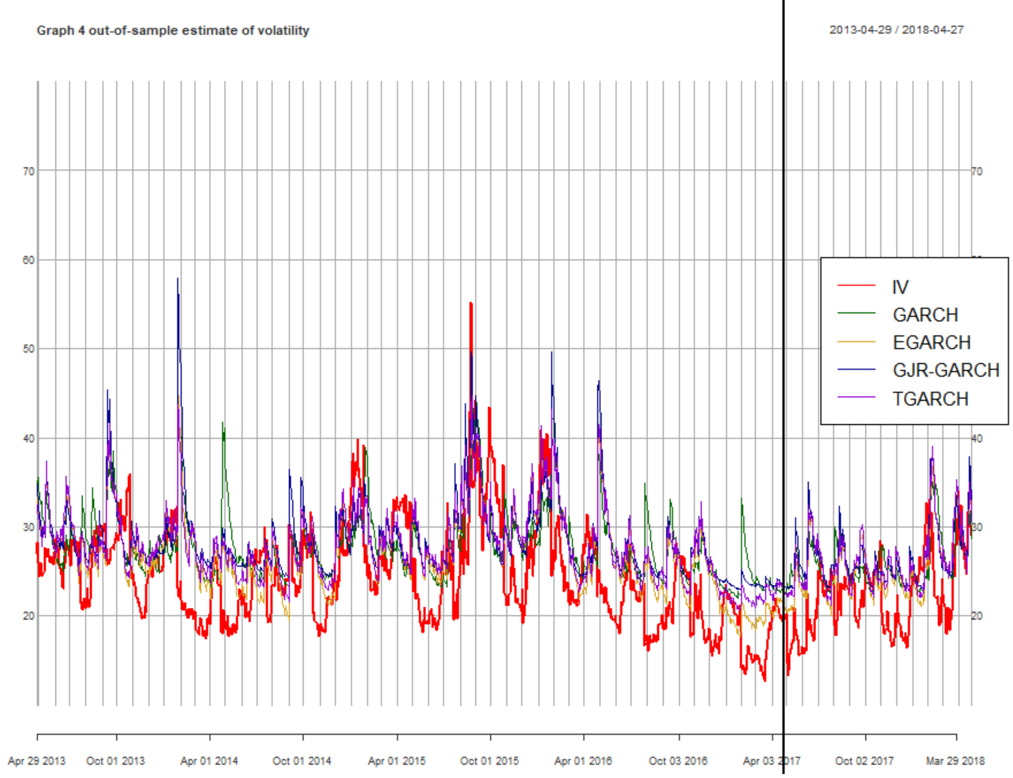
4 Out-of-Sample Recursive forecasts

In this section, I use four GARCH-type models to forecast 30-day-ahead conditional volatility. To estimate the parameters, I use a recursive window, re-estimating the model's parameters each time. By

Graph 3 Out-Of-Sample Volatility Forecast Scatterplot Matrix



Graph 4 out-of-sample estimate of volatility



averaging the 30-day-ahead volatility, I generate a series of volatility forecasts ranging from 2013-04-29 to 2018-04-27, which has a length of 1260.

As can be seen from Graph 3, 4, EGARCH and TGARCH have a better capacity of predicting implied volatility. The forecasts from EGARCH and TGARCH are almost the same pattern. This is not very surprising because they have similar model structure, the former use natural logarithm and the latter close to binary logarithm.

Here, I calculate the MAE of these different forecasts. The MAE, Table 2, confirms that TGARCH and EGARCH have a better predicting ability among these four GARCH-type models. But, I still think the error is large for annualized volatility.

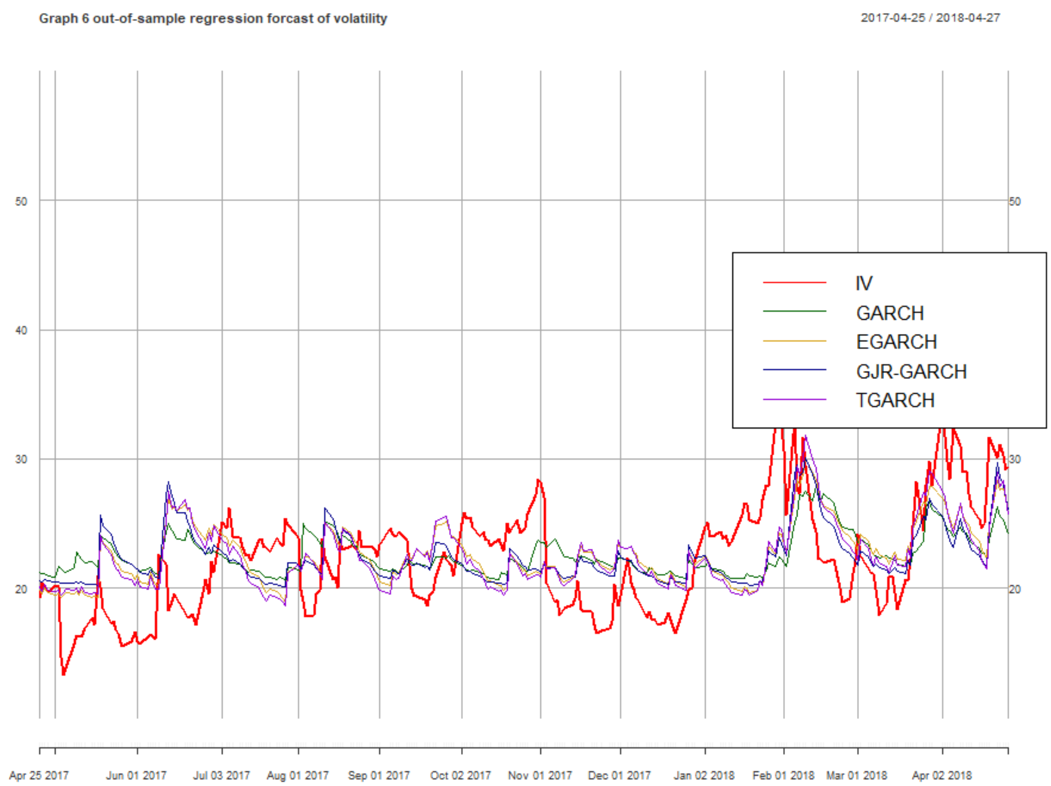
Table 2 MAE of GARCH-type Model Forecast

MODEL	MAE(1260 data)
GARCH	5.08399
EGARCH	4.535667
GJR-GARCH	5.293241
TGARCH	4.874501

Since there is still a gap between implied volatility and GARCH-type model's out-of-sample forecast, I want to use the forecast to re-estimate implied volatility. Here, I use OLS regression.

5 Regression and Recursive Forecast

Note that there is a straight line in Graph 4, this line, pointing to the date "2017-04-27", separates the dataset into a train set and a test set. Here, I begin forecasting after 2017-04-27 using a recursive fit. Specifically, I use log implied volatility as dependent variable and log GARCH-type model out-of-sample forecast volatility as explanatory variable. At each time T, I use data before time T to fit a regression model and forecast a-day-ahead implied volatility at time T+1. I use regression to forecast 255 day's implied volatility ranging from 2017-04-28 to 2018-04-27. I want to figure out whether the regression parameters are statistically significant and whether this approach can improve implied volatility forecast.



From Table 3 below, we can see that the regression can reduce MAE using volatility forecast from all these four GARCH-type models. I can conclude that EGARCH and TGARCH have similarly better prediction capacities, which is consistent with the result from the in-sample estimation and out-of-sample forecast of conditional volatility. The coefficients of the constant and external regressor term are all statistical significant. What is not surprising is that the coefficients of external regressor are all positive. This is not surprising because the regressor itself is an estimate of implied volatility.

Table 3 MAE Comparison Using Recursive fitting

MODEL	MAE(255 data, BEFORE)	MAE(255 data, AFTER REG.)
GARCH	4.494272	3.507913
EGARCH	4.309127	3.376324
GJR-GARCH	4.755725	3.318599
TGARCH	4.670321	3.430297

Then, I need to test the residual assumption of OLS regression. Unfortunately, the residuals are not normally distributed. We can see from Table 4 below that we can only not reject the null hypothesis of the test of residuals using simple GARCH forecast. We should reject the null hypothesis of tests using EGARCH or TGARCH.

Table 4 Regression Residual Diagnostics					
	GARCH			EGARCH	
Test	Statistic	pvalue	Test	Statistic	pvalue
Shapiro-Wilk	0.9978	0.0916	Shapiro-Wilk	0.9871	0.0000
Kolmogorov-Smirnov	0.024	0.4648	Kolmogorov-Smirnov	0.0555	9e-04
Cramer-von Mises	269.8424	0.0575	Cramer-von Mises	279.3907	0.0605
Anderson-Darling	0.9191	0.0194	Anderson-Darling	5.036	0.0000
	GJR-GARCH			TGARCH	
Test	Statistic	pvalue	Test	Statistic	pvalue
Shapiro-Wilk	0.9963	0.0044	Shapiro-Wilk	0.9888	0.0000
Kolmogorov-Smirnov	0.0345	0.1000	Kolmogorov-Smirnov	0.0569	6e-04
Cramer-von Mises	275.9931	0.0594	Cramer-von Mises	279.3766	0.0605
Anderson-Darling	1.3694	0.0015	Anderson-Darling	4.4979	0.0000

The residual diagnostics result is not pleasant. I am thinking how this non-normality will bring disaster to the regression forecast. Instead of recursively fitting the regression and forecasting one day implied volatility each time, this time, I use the whole data before 2017-04-27 to fit one and only one regression model and predict the following 255 day's implied volatility using GARCH-type model's volatility forecast. If the non-normality does cause big problems, I think the prediction will be quite different from that above because this time I forecast a much longer out-of-sample period instead of just one day.

From Table 5, we can see that the forecast capacities of these two out-of-sample forecast approaches are almost the same. It seems that non-normality does not cause big trouble to the forecast. I also run a regression with GARCH error term. The forecast performance is worse. The result is in the Appendix.

Table 5 MAE Comparison (one day each time vs. a 255-day period)		
MODEL	MAE(255 data, one day each time)	MAE(255 data, a 255-day period)
GARCH	3.507913	3.498469
EGARCH	3.376324	3.321263

GJR-GARCH	3.318599	3.379504
TGARCH	3.430297	3.435477

6 Concluding Remarks

In this research, I fit the four types of GARCH family models, GARCH, EGARCH, GJR-GARCH and TGARCH, and investigate whether they can provide out-of-sample forecasts of implied volatility which are related to the market valuations of volatility. By calculating 30-day-ahead forecasts (22 trading days) of volatility, I get GARCH-type model-based volatility using mean absolute error as a measurement of forecast performance. In order to improve the performance, I think of fitting a regression, using implied volatility as dependent variable and GARCH-type model-based volatility as explanatory variable. The out-of-sample recursively forecasts show that the regression does reduce the MAE, thus, improving the forecast performance. When I do the regression residual normality diagnostics, it shows that the residual from the regression using EGARCH or TGARCH are not normally distributed. To investigate whether this non-normality will cause trouble, I do a longer out-of-sample forecasts. The MAE table shows that the forecast results are almost the same as that using recursively fitting and forecast. It seems that the non-normality does not cause a big problem to the forecast. Actually, I also run a regression using GARCH(1,1) error term. But the forecast gets worse than using OLS regression. Due to the time limit, I do not show the result.

Reference

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