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8

Tools for Stat Theory HW3

1

Determine whether each following set is linearly independent or linearly dependent.

(a) $S_1 = \{(1,1,0)', (0,1,1)', (1,1,1)'\}$

A set is linearly independent if (1) A_j for $2 \leq j \leq k$ is not expressible as a linear combination of of $A_1, ...A_{j-1}$.

$$S_{12} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \neq x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

The above implies that $0 = x_1$ and $1 = x_1$ at the same time. Therefore, S_{12} is not expressible by S_{11}.

$$S_{13} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \neq x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$1 = x_1$

$1 = x_1 + x_2$

$1 = x_2$

There is no possible solution for x_1 or x_2, therefore S_{13} is not expressible by S_{12}, S_{11}.

S_1 is linearly independent.

(b) $S_2 = \{(1,1,1)', (1,2,3)'\}$

$$S_{22} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \neq x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The above implies that $1 = x_1$, $2 = x_1$, $3 = x_1$ at the same time.

S_2 is linearly independent.

(c) $S_3 = \{(6,2,-3)', (-2,-4,1)', (4,-7,-2)'\}$

$$S_{32} = \begin{bmatrix} -2 \\ -4 \\ 1 \end{bmatrix} = x_1 \begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix}$$

There is no solution for the above.

$$S_{33} = \begin{bmatrix} 4 \\ -7 \\ -2 \end{bmatrix} = x_1 \begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -4 \\ 1 \end{bmatrix}$$

There indeed is a solution of $x_1 = 3/2$ and $x_2 = 5/2$ that can express S_{33}, so S_3 is linearly dependent.

2

For what values of the scalar k are the three vectors $(k, 1, 0)'$, $(1, k, 1)'$, and $(0, 1, k)'$ linearly dependent, and for what values are they linearly independent?

$$S_{k2} = \begin{bmatrix} 0 \\ 1 \\ k \end{bmatrix} \neq x_1 \begin{bmatrix} k \\ 1 \\ 0 \end{bmatrix}$$

$$S_{k3} = \begin{bmatrix} 0 \\ 1 \\ k \end{bmatrix} \neq x_1 \begin{bmatrix} k \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ k \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ k \end{bmatrix} = \begin{bmatrix} x_1 k + x_2 \\ x_1 + k x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 k + k \\ x_1 + k^2 \\ k \end{bmatrix} = \begin{bmatrix} k(x_1 + 1) \\ x_1 + k^2 \\ k \end{bmatrix}$$

Linear Dependence

To find values of k that make the 3 vectors linear dependent, we find a k that allows the solution to work. These are the only 2 possible solutions for the above.

Solution 1: $k = 0$, $x_1 = 1$, $x_2 = k = 0$

Solution 2: $k = \sqrt{2}$, $x_1 = -1$, $x_2 = k = \sqrt{2}$

Solution3: k=-sqrt(2)

Linear Independence

For linear independence, we have to find k such that each vector is not expressible by a linear combination of its previous vectors. As long as $k \neq 0 \neq \sqrt{2}$ then we should have a linearly independent set of 3 vectors.

For example, $k = 2$

$$\begin{bmatrix} 1 \\ k \\ 1 \end{bmatrix} = x_1 \begin{bmatrix} k \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

There is no solution for x_1.

$$\begin{bmatrix}0\\1\\k\end{bmatrix} = x_1 \begin{bmatrix}k\\1\\0\end{bmatrix} + x_2 \begin{bmatrix}1\\k\\1\end{bmatrix}$$

$$\begin{bmatrix}0\\1\\2\end{bmatrix} = x_1 \begin{bmatrix}2\\1\\0\end{bmatrix} + x_2 \begin{bmatrix}1\\2\\1\end{bmatrix}$$

There is no solution of x_1 or x_2 to satisfy the above. Since the vectors cannot be expressed as linear combinations of its previous vectors, it is linearly independent.

3

Show that if a set $S = \{u_1, .., u_n\}$ is linearly independent, then any non-empty subset of S is also linearly independent.

We use Proof By Contradiction. Instead of trying to prove that the non-empty subset is linearly independent, we assume that it is NOT linearly independent, then show that it's impossible.

Given $R \subseteq S$, $R = \{S_i, .., S_j\}$, $1 \leq i \leq j; i \leq j \leq n$, assume R is linearly dependent. If R is linearly dependent, then there must be some S_k in $\{S_i, .., S_j\}$ which can be represented as a linear combination: $S_k = x_i S_i + ... + S_{k-1}x_{k-1} + S_{k+1}x_{k+1} + ... + S_j x_j$

By definition S_k must be in S but if S is linearly independent then S_k cannot be represented as a linear combination of other vectors/matrices in S and therefore it presents a contradiction so S_k cannot exist. Since S_k cannot exist then all vectors/matrices in R are expressible as a linear combination of others so R is linearly independent.

4

Consider $V = R^2$. Show that $W_1 = \{u = (x, y)' : ax + by = 0, a, b \neq 0\}$ is a subspace of V while $W_2 = \{u = (x, y)' : ax + by + c = 0, a, b, c \neq 0\}$ is not.

To show that W_1 is a subspace of V we need to show that W_1 is a subset of V and W_1 is a linear space. To show that W_2 is not a subspace of V we need to show that W_2 is not a subset of V or W_2 is not a linear space.

By definition R^2 includes all 2-dimensional vectors so $W_1 \subseteq R^2$ and W_1 is a linear space if for every $A, B \in W_1$, $A + B \in W_1$.

$$A = \begin{bmatrix}x_1\\y_1\end{bmatrix}$$

You also need to check the scalar multiplication.

$$B = \begin{bmatrix}x_2\\y_2\end{bmatrix}$$

$$A + B = \begin{bmatrix}x_1 + x_2\\y_1 + y_2\end{bmatrix}$$

From the above, $A + B$ is also in V so W_1 is a subspace of V.

To show that W_2 is not in V, we only need 1 example.

$$C = \begin{bmatrix}1\\0\end{bmatrix}$$

$$D = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$C + D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Both C and D are in W_2 because they satisfy the corresponding equation. However, their sum $C + D$ is not in the subspace of W_2 because $a(0) + b(0) + c = 0$ would mean that $c = 0$ which violates the definition of the set and therefore it is not a linear space and thus it cannot be a subspace of V.

5

Show that a set of vectors $(x_1, x_2, x_3, x_4)'$ that satisfy the following equations is a subspace of $\mathbb{R}^4$:

$3x_1 - 2x_2 - x_3 - 4x_4 = 0$ Linear space has two assumptions, scalar multiplication and sum.

$x_1 + x_2 - 2x_3 - 3x_4 = 0$

By definition $\mathbb{R}^4$ includes all 4-dimensional real column vectors. Hence, $(x_1, x_2, x_3, x_4)' \subseteq \mathbb{R}^4$ and each of its linear combinations are also in $\mathbb{R}^4$. Assume A is a set of vectors that satisfies equation 1 in the above and B is a set of vectors that satisfy the 2nd equation. Then, $A \cap B$ is the set of vectors that satisfy both equations. By definition $A \subseteq \mathbb{R}^4$ and $B \subseteq \mathbb{R}^4$. Since $(A \cap B) \subseteq A$ and $A \subseteq \mathbb{R}^4$, then $A \cap B \subseteq A \subseteq \mathbb{R}^4$. The set of vectors that satisfy both equations are also a subset and therefore a subspace of $\mathbb{R}^4$.

6

Let W_1 and W_2 be subspaces of V. Show by example that $W_1 \cup W_2$ may not be a subspace of V.

$$W_1 = \left\{ \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}, x_1 \in \mathbb{R} \right\}$$

$$W_2 = \left\{ \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, x_1 \in \mathbb{R} \right\}$$

Assume $A \subseteq W_1, B \subseteq W_2$.

$$A = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$A + B \notin W_1$, $A + B \notin W_2$ and therefore $A + B \notin W_1 \cup W_2$. By definition, $W_1 \cup W_2$ is a subspace of V if for any matrix $A, B \in V$, $A + B \in V$. Since A and B are each in the subspace of W_1 and W_2, they each must also be in the subspace of V and in the subspace of $W_1 \cup W_2$. However, their sum is not in $W_1 \cup W_2$ and therefore we cannot say that $W_1 \cup W_2$ is a subspace of V because vector addition is not closed.