Midterm Practice

1

Show that if the set $\{u;\ v;\ w\}$ is linearly independent then so is the set $\{2u+v+3w; u+v+2w; u+v+w\}$.

$x(2u+v+3w)+y(u+v+2w)+z(u+v+w)=0=(2x+y+z)u+(x+y+z)v+(2x+2y+z)w$

u, v, w are linear independent then

$$\begin{bmatrix} 2x+y+z \\ x+y+z \\ 3x+2y+z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We convert into the form $Ax = B$, as

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A is the coefficient matrix.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

We can solve for X, but we can also find the determinant of A. If the determinant of A $\neq 0$ then it is linearly independent.

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

I used row operations to simplify the matrix so as to simplify determinant calculation. $|A| = 1*(0-1) = -1 \neq 0$ so set is linearly independent.

If I did not use the simplification, it would be as follows: $|A| = 2*(1-2) - 1*(1-3) + 1*(2-3) = -2+2+1 = 1 \neq 0$

2

Let A represent a $q \times p$ matrix, B a $p \times n$ matrix, and C an $m \times q$ matrix. Show that if $\text{rank}(CAB) = \text{rank}(B)$, then $\text{rank}(AB) = \text{rank}(B)$.

If $\text{rank}(CAB) = \text{rank}(B)$, $R(CAB) = R(B)$ by Corollary 4.4.7.

$R(B) \subset R(CAB)$ since $R(CAB) = R(B)$ (by definition).

$R(B) \subset R(CAB)$ if and only if there is F such that $B = FCAB$ (by Lemma 4.2.2).

$\text{rank}(B) = \text{rank}(FCAB) \leq \text{rank}(AB) \leq \text{rank}(B)$

$\text{rank}(B) = \text{rank}(AB)$

3

Let A represent an $n \times n$ nonnull symmetric matrix, and let B represent an $n \times r$ matrix of full column rank r and T an $r \times n$ matrix of full row rank r such that $A = BT$. Show that the $r \times r$ matrix TB is nonsingular. (Hint: $A'A = A^2 = BTBT$.)

$\text{rank}(A) = \text{rank}(A'A)$ by Corollary 7.4.5.

According to Lemma 8.3.2, If A has full column rank, then $\text{rank}(AB) = \text{rank}(B)$. If B has full row rank, $\text{rank}(AB) = \text{rank}(A)$.

$\text{rank}(A) = \text{rank}(BT)$. Since B has full column rank, we can drop B so that $\text{rank}(BT) = \text{rank}(T) = r$.

We know that $\text{rank}(A'A) = \text{rank}(BTBT) = \text{rank}(TBT)$ because B has full column rank.

$\text{rank}(TBT) = \text{rank}(TB)$ because T has full row rank.

Thus, $r = \text{rank}(A) = \text{rank}(A'A) = \text{rank }(BTBT) = \text{rank }(TBT) = \text{rank }(TB)$

4

Calculate the general inverse matrix of

$$\begin{pmatrix} 4 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

The generalized inverse is given by Corollary 9.6.3:

$$\begin{bmatrix} A_{11}^- & 0 \\ 0 & A_{22}^- \end{bmatrix}$$

Neither A_{11}^- or A_{22}^- are invertible because their determinant = 0. But that does not mean we cannot find its generalized inverse. For A_{11}^-:

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4x + 2z & 4y + 2w \\ 2x + z & 2y + w \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4(4x + 2z) + 2(4y + 2w) & 2(4x + 2z) + 1(4y + 2w) \\ 4(2x + z) + 2(2y + w) & 2(2x + z) + 1(2y + w) \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 16x + 8z + 8y + 4w & 8x + 4z + 4y + 2w \\ 8x + 4z + 4y + 2w & 4x + 2z + 2y + w \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

One solution is

$$A_{11}^{-} = \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$x = 1, z = -1, y = -1, w = 1$.

For A_{22}^{-}:

$$\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0x + 0z & 0y + 0w \\ 1x + 2z & 1y + 2w \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1x + 2z & 1y + 2w \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 + 0 & 0 + 0 \\ 0 + 1(1y + 2w) & 0 + 2(1y + 2w) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1y + 2w & 2(1y + 2w) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$$

Since $1y + 2w = 1$, one solution could be $w = 0$, $y = 1$. Another solution could be $y = 2$, w$= -1/2$.

One solution is

$$A_{22}^{-} = \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus, the generalized inverse

$$\begin{bmatrix} A_{11}^{-} & 0 \\ 0 & A_{22}^{-} \end{bmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

5

Let A represent an $n \times n$ matrix such that $A'A = AA' = AA$.

(a) Show that $\text{tr}[(A - A')'(A - A')] = 0$.

$(A - A')'(A - A') = ((-A')' + A')(A - A') = (-A + A')(A - A') = -AA + A'A + AA' - A'A'$

$\text{tr}(-AA+A'A+AA'-A'A') = \text{tr}(-AA)+\text{tr}(A'A)+\text{tr}(AA')+\text{tr}(-A'A') = -\text{tr}(AA)+\text{tr}(A'A)+\text{tr}(AA')-\text{tr}(A'A')$

By (2.8) we know that $\text{tr}(AB) = \text{tr}(B'A') = \text{tr}(BA) = \text{tr}(A'B')$. Thus we know that $\text{tr}(AA) = \text{tr}(A'A')$ and that $\text{tr}(A'A) = \text{tr}(AA')$. Hence, $-\text{tr}(AA)+\text{tr}(A'A)+\text{tr}(AA')-\text{tr}(A'A') = 2\text{tr}(A'A) - 2\text{tr}(AA)$

By definition, $\text{tr}(AA) = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}a_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}^2$ and by (2.5) $\text{tr}(A'A) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2$. Therefore, $\text{tr}(AA) =\text{tr}(A'A)$ and thus $2\text{tr}(A'A) - 2\text{tr}(AA) = 0$

(b) Show that A is symmetric.

Lemma 5.3.1 states: For any $m{\times}n$ matrix $A = \{a_{ij}\}$, $A = 0$ if and only if $\text{tr}(A'A) = 0$.

Set $F = A - A'$ so that $\text{tr}(F'F) =\text{tr}((A - A')'(A - A'))$

We have shown in 5a that $\text{tr}((A - A')'(A - A')) = 0$ so we show that $F = 0$. If $F = 0$ it means that $A - A' = 0$ and since the definition of symmetric is for $A' = A$ or $A' - A = 0$ or or $A - A' = 0$, we have shown that A is symmetric.