Tools for Stat Theory HW8

1

Let $X = \{x_{ij}\}$ represent an $m \times n$ matrix of mn ”independent” variables, and suppose that X is free to range over all of $R^{m\text{Ö}n}$. Show that, for any $p \times m$ and $n \times p$ matrices of constants A and B,

$$\frac{\partial tr(AXB)}{\partial X} = A'B'$$

By (6.5), $\frac{\partial tr(AX)}{\partial X} = A'$.

$$\frac{\partial tr(AXB)}{\partial X} = \frac{\partial tr(BAX)}{\partial X} = \frac{\partial tr((BA)X)}{\partial X} = (BA)' = A'B'$$

2

Let $X = \{x_{st}\}$ represent an $m \times n$ matrix of mn ”independent” variables, and suppose that X is free to range over all of $R^{m\text{Ö}n}$. Show that, for any $m \times m$ matrix of constants A,

$$\frac{\partial tr(X'AX)}{\partial X} = (A + A')X$$

By (6.5), $\frac{\partial tr(AX)}{\partial X} = A' = \frac{\partial tr(XA)}{\partial X}$. See $X'A$ as B_2 and AX as B_1.

$$\frac{\partial tr(X'AX)}{\partial X} = B_1 + B_2' = AX + (X'A)' = AX + A'X = (A + A')X$$

3

Let $X = \{x_{ij}\}$ represent an $m \times m$ matrix of m^2 ”independent” variables (where $m \geq 2$), and suppose that X is free to range over all of $R^{m\text{Ö}m}$. Show that (for any positive integer k) the function f defined (on $R^{m\text{Ö}m}$) by $f(X) = |X|^k$ is continuously differentiable at every X and that

$$\frac{\partial |X|^k}{\partial X} = k|X|^{k-1}[adj(X)]'$$

We can define $y = |X|$.

$$\frac{\partial |X|^k}{\partial X} = \frac{\partial y^k}{\partial y}\frac{\partial y}{\partial X} = ky^{k-1}[adj(X)]' = k|X|^{k-1}[adj(X)]'$$

The determinant is a polynomial and the kth power of the determinant is still a polynomial. Thus, they are all continuously differentiable.

4

Let $F =$

$$\begin{pmatrix} sinx & sin2x \\ cosx & cos3x \end{pmatrix}$$

Calculate

a) $\frac{dF^{-1}}{dx}$

$$F^{-1} = \frac{1}{|F|}\begin{bmatrix} cos3x & -sin2x \\ -cosx & sinx \end{bmatrix} = \frac{1}{cos(3x)sin(x) - cos(x)sin(2x)}\begin{bmatrix} cos3x & -sin2x \\ -cosx & sinx \end{bmatrix}$$

$$\frac{\partial F}{\partial x} = \begin{bmatrix} cosx & 2cos2x \\ -sinx & -3sin3x \end{bmatrix}$$

By (8.15),

$$\frac{\partial F^{-1}}{\partial x} = -F^{-1}\frac{\partial F}{\partial x}F^{-1} =$$

$$-\frac{1}{cos(3x)sin(x) - cos(x)sin(2x)}\begin{bmatrix} cos3x & -sin2x \\ -cosx & sinx \end{bmatrix}\frac{\partial F}{\partial x}\frac{1}{cos(3x)sin(x) - cos(x)sin(2x)}\begin{bmatrix} cos3x & -sin2x \\ -cosx & sinx \end{bmatrix} =$$

$$-\frac{1}{cos(3x)sin(x) - cos(x)sin(2x)}\frac{1}{cos(3x)sin(x) - cos(x)sin(2x)}\begin{bmatrix} cos3x & -sin2x \\ -cosx & sinx \end{bmatrix}\frac{\partial F}{\partial x}\begin{bmatrix} cos3x & -sin2x \\ -cosx & sinx \end{bmatrix} =$$

$$-\frac{1}{[cos(3x)sin(x) - cos(x)sin(2x)]^2}\begin{bmatrix} cos3x & -sin2x \\ -cosx & sinx \end{bmatrix}\frac{\partial F}{\partial x}\begin{bmatrix} cos3x & -sin2x \\ -cosx & sinx \end{bmatrix} =$$

$$-\frac{1}{[cos(3x)sin(x) - cos(x)sin(2x)]^2}\begin{bmatrix} cos3x & -sin2x \\ -cosx & sinx \end{bmatrix}\begin{bmatrix} cosx & 2cos2x \\ -sinx & -3sin3x \end{bmatrix}\begin{bmatrix} cos3x & -sin2x \\ -cosx & sinx \end{bmatrix} =$$

b) $\frac{\partial det(F)}{\partial x}$

By (8.5), $\frac{\partial det(F)}{\partial x} = |F|tr(F^{-1}\frac{\partial F}{\partial x})$

$$\frac{\partial det(F)}{\partial x} = [cos(3x)sin(x) - cos(x)sin(2x)]tr\left(\frac{1}{cos(3x)sin(x) - cos(x)sin(2x)}\begin{bmatrix} cos3x & -sin2x \\ -cosx & sinx \end{bmatrix}\begin{bmatrix} cosx & 2cos2x \\ -sinx & -3sin3x \end{bmatrix}\right) =$$

$$tr\left(\begin{bmatrix} cos3x \cdot cosx + -sin2x \cdot -sinx & cos3x \cdot 2cos2x + -sin2x \cdot -3sin3x \\ -cosx \cdot cosx + sinx \cdot -sinx & -cosx \cdot 2cos2x + sinx \cdot -3sin3x \end{bmatrix}\right) =$$

$$cos3x \cdot cosx + -sin2x \cdot -sinx + -cosx \cdot 2cos2x + sinx \cdot -3sin3x$$