Tools for Stat Theory HW6

1

Let A represent an m $\times$ n matrix. Show that if $A'A$ is idempotent, then AA' is idempotent.

If $A'A$ is idempotent, then $A'AA'A = A'A = A'AI$.Based on Corollary 5.5.3, we can conclude that $AA'A = AI = A$. Therefore $|AA'|^2 = AA'AA' = AA'$, implying that AA' is idempotent.

2

Let A and B represent n $\times$ n symmetric idempotent matrices. Show that if $C(A) = C(B)$, then $A = B$.

If $C(A) = C(B)$, suppose there exists matrix F_1, F_2 such that $A = BF_1$ and $A = AF_2$. Also,

$B = B' = (AF_2)' = F_2'A' = F_2'A$.

$A = BBF_1 = BA = F_2'AA = F_2'A = B$

3

Let A represent an r$\times$m matrix and B an m$\times$n matrix. (a) Show that B^-A^- is a generalized inverse of AB if and only if A^-ABB^- is idempotent.

(i) If A^-ABB^- is idempotent show that B^-A^- is a generalized inverse of AB.

If B^-A^- is the generalized inverse of AB, then $ABB^-A^-AB = AB$.

$A^-ABB^-A^-ABB^- = A^-ABB^-$. Thus A^-ABB^- is idempotent.

(ii) If B^-A^- is a generalized inverse of AB show that A^-ABB^- is idempotent.

If A^-ABB^- is idempotent, then $A^-ABB^-A^-ABB^- = A^-ABB^-$

$AA^-ABB^-A^-ABB^-B = AA^-ABB^-B$

$ABB^-A^-AB = AB$

Hence, B^-A^- is the generalized inverse of AB.

4

Show that, for any matrix A, $C(A) = \mathcal{N}(I - AA^-)$.

Corollary 9.3.6. Let A represent an m $\times$ n matrix. Then, for any m-dimensional column vector x, $x \in C(A)$ if and only if $x = AA^-x$, and, for any n-dimensional row vector y', $y' \in R(A)$ if and only if $y' = y'A^-A$.

Let x be the column vector whose dimension equals the number of rows in A. By Corollary 9.3.6, we know $x \in C(A)$ if and only if $x = AA^-x$. Equivalently, $(I - AA^-)x = 0$ and hence if and only if $x \in \mathcal{N}(I - AA^-)$. Therefore, we can conclude that $C(A) = \mathcal{N}(I - AA^-)$.