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Tools for Stat Theory HW4

1 Lemma 8.1.1 Prove Column rank part.

We can use apply transpose to the row rank proof to achieve the same results. If A has full column rank, then it has $\text{rank}(A) = n$.

Theorem 4.4.1 "The row rank of any matrix A equals the column rank of A" and $\text{rank}(A') = \text{rank}(A)$ as stated in the textbook.

Therefore, $\text{rank}(A') = \text{rank}(A) = n =$ row rank of A'. If $\text{rank}(A') = n$, Theorem 7.2.2 ("If the coefficient matrix A of a linear system$AX = B$ (in X) has full row rank, then $AX = B$ is consistent.") shows that matrix R must exist where $A'R = I_n$ (we set $B = I_n$ and $X = R$). In other words, A' must have a right inverse. If there is matrix R such that $A'R = I_n$ then $\text{rank}(A') \geq \text{rank}(A'R) = \text{rank}(I_n) = n$ implying (Lemma 4.4.3: For any m × n matrix A, rank(A) ≤ m and rank(A) ≤ n.) that rank(A') = n. If $A'R = I_n$ then $(A'R)' = I_n'$ and $R'(A')' = I_n$ and hence $R'A = I_n$. We have shown that given full column rank, we must have a left inverse matrix $L = R'$.

2 Corollary 8.1.2 Prove.

Corollary 8.1.2: A matrix A has both a right inverse and a left inverse if and only if A is a (square) nonsingular matrix. According to Lemma 8.1.1, A has right inverse if and only if there is full row rank (rank(A)= m) and A has left inverse if and only if there is full column rank (rank(A)=n). Thus, if A is to have both a right and left inverse, it needs to have full column and full row rank. In other words, it needs to have rank(A) = m = n. To have m = n, A must be a square matrix. To have full row rank and full column rank at the same time, A must be nonsingular (nonsingular means a full rank square matrix).

if and only if, you only proved one way -0.5

3 Let A represent an $m \times n$ matrix, C an $n \times q$ matrix, and B a $q \times p$ matrix. Show that if rank(AC)=rank(C), then $R(ACB) = R(CB)$ and $\text{rank}(ACB) = \text{rank}(CB)$.

Lemma 4.2.2. For any m × n matrix A and m × p matrix B, C(B) ⊂ C(A) if and only if there exists an n × p matrix F such that B = AF. Similarly, for any m × n matrix A and q × n matrix C, R(C) ⊂ R(A) if and only if there exists a q × m matrix L such that C = LA.

Corollary 4.2.3. For any $m \times n$ matrix A and $n \times p$ matrix F, $C(AF) \subset C(A)$. Similarly, for any $m \times n$ matrix A and $q \times m$ matrix L, $R(LA) \subset R(A)$.

Corollary 4.4.5. For any $m \times n$ matrix A and $n \times p$ matrix F, rank(AF) ≤ rank(A) and rank(AF) ≤ rank(F).

Corollary 4.4.7. Let A represent an $m \times n$ matrix and F an $n \times p$ matrix. If rank(AF) = rank(A), then $C(AF) = C(A)$. Similarly, if rank(AF) = rank(F), then $R(AF) = R(F)$.

Corollary 7.4.4: For any $m \times n$ matrix A and $n \times s$ matrix T, rank($T'A'A$) = rank($T'A'$) and rank($A'AT$) = rank(AT).

means C=LAC for some L, so R(CB)=R(LACB)⊂R(ACB)

By Corollary 4.4.7, $R(AC) = R(C)$ because $\text{rank}(AC) = \text{rank}(C)$. Since $R(AC) = R(C)$ so $R(AC) \subset R(C)$ and $R(C) \subset R(AC)$. According to Corollary 4.2.3 $R(ACB) \subset R(CB)$. According to Lemma 4.2.2, if $R(ACB) \subset R(CB)$ then there must be a matrix L such that ACB = LCB. Since $\text{rank}(AC) = \text{rank}(C)$, and $\text{rank}(ACB) = \text{rank}(LCB)$, L must equal I and ACB.

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According to Lemma 4.2.2, if $R(AC) \subset R(C)$ then there must exist a matrix L such that AC = LC and if $R(C) \subset R(AC)$ then there must exist a matrix L such that C = LAC.

By Corollary 4.4.5, rank(ACB) ≤ rank(B), rank(ACB) ≤ rank(A), rank(ACB) ≤ rank(CB), and rank(ACB) ≤ rank(AC) = rank(C).

According to Corollary 7.4.4, rank($ACBB'$) = rank(ACB) and rank($A'ACB$) = rank (ACB). $ACBB'$ results in a m x q matrix. rank($ACBB'$) ≤ m and rank($ACBB'$) ≤ q by Lemma 4.4.3. Hence rank($ACBB'$) = rank (ACB) ≤ q and rank($ACBB'$) = rank (ACB) ≤ m

By Corollary 4.4.7, if rank(AC)=rank(C), then $R(AC) = R(C)$.

If rank(AC)=rank(C), then rank(AC) = rank(C) ≤ n and rank(AC) = rank(C) ≤ q because of Lemma 4.4.3 as aforementioned. Since AC is an mxq matrix, then rank(AC) ≤ m as well. Rank (CB) <= Rank (C) Rank (CB) <= n, Rank (CB) <= p, Rank (CB) <= q, Rank (CB) <= m rank (ACB) <= rank(CB) <= rank(C) = rank(AC) rank(ACB) <= rank(CB) <= rank(AC) = rank(C)

4 Let A and B represent m×n matrices. Show that if C is an $r \times q$ matrix and D a $q \times m$ matrix such that $\text{rank}(CD) = \text{rank}(D)$, then $CDA = CDB$ implies $DA = DB$, thereby extending the result of Part (1) of Corollary 5.3.3. {Hint. To show that $DA = DB$, it suffices to show that $\text{rank}[D(A - B)] = 0$.}

Since $\text{rank}(CD) = \text{rank}(D)$, by Corollary 4.4.7, we know that $R(CD) = R(D)$. Like #3, we know that $R(CDE) = R(DE)$ where E is a m x n matrix. We can set $E = A - B$ so that $R(CD(A - B)) = R(D(A - B))$ => $R(CDA - CDB) = R(DA - DB)$. If $CDA = CDB$ then $CDA - CDB = 0$ which means that rank($CDA - CDB$) = rank(0) = 0. By Corollary 4.4.7, we know that if rank($CDA - CDB$) = 0 then $R(CDA - CDB) = 0$ and that $0 = R(CDA - CDB) = R(DA - DB) = 0$, thereby implying that $DA - DB = 0$.

5 Let V be the vector space spanned by the vectors $u_1 = (1, 2, 3)'$ and $u_2 = (1, 1, -1)'$. Compute the projection of $v = (1, 1, 1)'$ on

u_1 and u_2, respectively. Then compute the projection of v onto V.

$\text{proj}_L(v) = \frac{v \cdot y}{y \cdot y} y$

$||u_1|| = \sqrt{1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3} = \sqrt{1 + 4 + 9} = \sqrt{13}$ 14

$||u_2|| = \sqrt{1 \cdot 1 + 1 \cdot 1 + (-1) \cdot (-1)} = \sqrt{1 + 1 + 1} = \sqrt{3}$

$v \cdot u_1 = 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 = 1 + 2 + 3 = 6$

$v \cdot u_2 = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot -1 = 1 + 1 + -1 = 1$

Projection of v on u_1 $= \frac{6}{13}(1, 2, 3)' = (\frac{6}{13}, \frac{12}{13}, \frac{18}{13})'$

Projection of v on u_2 $= \frac{1}{3}(1, 1, -1)' = (\frac{1}{3}, \frac{1}{3}, \frac{-1}{3})'$

Projection of v on V $= (\frac{6}{13} + \frac{1}{3}, \frac{12}{13} + \frac{1}{3}, \frac{18}{13} + \frac{-1}{3})' = (0.7948718, 1.25641, 1.051282)'$