Tools for Stat Theory HW5

1 Lemma 9.2.8 Prove part (2).

Lemma 9.2.8: Let A represent a matrix of full column rank and B a matrix of full row rank. Then, (1) a matrix G is a generalized inverse of A if and only if G is a left inverse of A. And, (2) a matrix G is a generalized inverse of B if and only if G is a right inverse of B.

First we show that if B has right inverse it must be a generalized inverse. By Lemma 8.1.1, B has a right inverse R such that $BR = I_m$. Then $BRB = IB = B$ and thus R is a generalized inverse of B. We have shown that if R is a right inverse of B, it must be a generalized inverse of B. Therefore R = G.

Now we need to show that if G is a generalized inverse of B, it is a right inverse of B (G=R). Suppose R is a right inverse and G is a generalized inverse of B, $BG = BGI = BGBR = (BGB)R = BR$. Thus we show that $BG = BR$ and that G, the generalized inverse of B, is also a right inverse of B.

2. Prove Lemma 9.2.9.

Lemma 9.2.9: If a matrix A has full column rank, then the matrix $(A'A)^{-1}A'$ is a left inverse of A. Similarly, if A has full row rank, then $A'(AA')^{-1}$ is a right inverse of A.

By Corollary 7.4.6, if a matrix A has full column rank, then $A'A$ is nonsingular and by Corollary 8.1.2, $A'A$ has both left and right inverse and by Theorem 8.1.4 it has a unique inverse $(A'A)^{-1}$ and no other left or right inverse.Thus, we have $(A'A)^{-1}A'A = (A'A)^{-1}(A'A) = I$

If A has full row rank, then by Lemma 8.1.1, we know that A has left inverse and that it is a generalized inverse (Lemma 9.2.8). If A has full row rank, we know that A' has full column rank and therefore by Corollary 7.4.6 AA' is nonsingular and has a unique inverse $(AA')^{-1}$by Corollary 8.1.2 and Theorem 8.1.4. Thus, $AA'(AA')^{-1} = (AA')(AA')^{-1} = I$ thus we show that $A'(AA')^{-1}$ is a right inverse of A.

3. Lemma 9.3.5 Prove row space part

Lemma 9.3.5. Let A represent an m × n matrix. Then, for any m × p matrix B, $C(B) \subset C(A)$ if and only if $B = AA^-B$, or, equivalently, if and only if $(I - AA^-)B = 0$. And, for any q×n matrix C, $R(C) \subset R(A)$ if and only if $C = CA^-A$, or, equivalently, if and only if $C(I - A^-A) = 0$.

Show if $C = CA^-A$ then $R(C) \subset R(A)$: By Corollary 4.2.3, $R(C) = R(CA^-A) \subset R(A)$.

Show if $R(C) \subset R(A)$ then $C = CA^-A$: By Lemma 4.2.2, $R(C) \subset R(A)$ if and only if there exists a matrix L such that $C = LA$. By Corollary 9.1.4, $A = AA^-A$. As such, $C = LA = L(AA^-A) = (LA)A^-A = CA^-A$.

We thus show that $R(C) \subset R(A)$ if and only if $C = CA^-A$.