9.5

Tools for Stat Theory HW7

1

Prove Lemma 12.1.1: Let Y represent a matrix in a linear space V, let U and W represent subspaces of V, and take $\{X_1, ..., X_s\}$ to be a set of matrices that spans U and $\{Z_1, ..., Z_t\}$ to be a set that spans W. Then, $Y \perp U$ if and only if $Y \cdot X = 0$ for $i = 1, ..., s$; that is, Y is orthogonal to U if and only if Y is orthogonal to each of the matrices $X_1, ..., X_s$. And, similarly, $U \perp W$ if and only if $X \cdot Z = 0$ for $i = 1, ..., s$ and $j = 1, ..., t$; that is, U is orthogonal to W if and only if each of the matrices $X_1, ..., X_s$ is orthogonal to each of the matrices $Z_1, ..., Z_t$.

Since $\{X_1, ..., X_s\}$ span U, then U can be represented as a linear combination of $\{X_1, ..., X_s\}$: $U = a_1X_1 + ... + a_sX_s$.

1) Prove Y is orthogonal to U if Y is orthogonal to each of the matrices $X_1, ..., X_s$

If Y is orthogonal to each of the matrices $X_1, ..., X_s$, then

$$Y \cdot a_1X_1 + ... + Y \cdot a_sX_s = 0$$

. By the property of matrix multiplication, we can convert it to the following form

$$Y \cdot (a_1X_1 + ... + a_sX_s) = 0$$

Since U can be represented as a linear combination $U = a_1X_1 + ... + a_sX_s$, then if Y is orthogonal to all linear combinations of $\{X_1, ..., X_s\}$, then Y is orthogonal to U.

2) Prove Y is orthogonal to each of the matrices $X_1, ..., X_s$ if Y is orthogonal to U **-0.5**

If Y orthogonal to U, then Y orthogonal to $\{X_1, ..., X_s\}$ by definition because $\{X_1, ..., X_s\} \subset U$.

2

For the example of Section 12.2c,recompute the projection of y on U by taking X to be the 3×2 matrix

$$\begin{pmatrix} 0 & -2 \\ 3 & 2 \\ 6 & 4 \end{pmatrix}$$

. . . .

and carrying out the two steps:

(1) compute the solution to the normal solutions $X'Xb = X'y$;
(2) postmultiply X by the solution you computed in Step (1).

1) Compute solution to normal solutions

$$X'y = \begin{bmatrix} 6 & 3 & 0 \\ 4 & 2 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ -\frac{38}{5} \\ \frac{74}{5} \end{bmatrix}$$

$$b = X^{-1}(X')^{-1}X'y$$

```
X <- matrix(c(0,3,6, -2,2,4), nrow=3)
X_prime <- t(X)
y <- matrix(c(3, -38/5, 74/5), ncol =1)

XtX <- t(X) %*% X
Xty <- t(X) %*% y

b <- solve(XtX, Xty)

print("SOLUTION:")
```

```
## [1] "SOLUTION:"
```

```
print(b)
```

```
##           [,1]
## [1,]  2.466667
## [2,] -1.500000
```

```
print("XtXb:")
```

```
## [1] "XtXb:"
```

```
print(XtX %*% b)
```

```
##      [,1]
## [1,]   66
## [2,]   38
```

```
print("Xty")
```

```
## [1] "Xty"
```

```
print(Xty)
```

```
##      [,1]
## [1,]   66
## [2,]   38
```

2) Postmultiply X by b

```r
print(X %*% b)
```

```
##      [,1]
## [1,]  3.0
## [2,]  4.4
## [3,]  8.8
```

3

Prove Theorem 13.2.6: If an $n \times n$ matrix $B = \{b_{ij}\}$ is formed from an $n \times n$ matrix $A = \{a_{ij}\}$ by interchanging two rows or two columns of A, then $|B| = -|A|$.

We will first try to prove the theorem using a special case of swapping 2 rows of A to obtain B, as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & ... & b_{1n} \\ b_{21} & b_{22} & ... & b_{2n} \\ ... & ... & ... & ... \\ b_{n1} & b_{n2} & ... & b_{nn} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & ... & a_{2n} \\ a_{11} & a_{12} & ... & a_{1n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix}$$

We can expand $|A|$ along its first row to obtain its value: $|A| = \sum_{j=1}^{n} a_{1j}(-1)^{1+j}|A_{1j}|$. The property of the cofactor expansion is such that whichever row or column we choose to expand by, the results are the same.

To obtain $|B|$ we can expand along the 2nd row as well, $|B| = \sum_{j=1}^{n} b_{2j}(-1)^{2+j}|B_{2j}| = \sum_{j=1}^{n} a_{1j}(-1)^{2+j}|A_{1j}|$

In this case, we show that $|B_{2j}| = |A_{1j}|$ because it's along the same row. Since the cofactor expansions along the other row are the same, $|B| = -|A|$.

Now that we show that $|B| = -|A|$ in the case that B is just A with the 1st and 2nd row swapped. To find B in the general case. We can assume that there is row f and row h. Between row f and row h there are k rows. We can swap the hth row with the rows above it one by one until it reaches the original positions of row f. Now f will be at k+2th row after k swaps. To swap h & f back we need k + 1 more swaps so $(-1)^{2k+1}$.

4

Numerically test: determinant keeps invariant when we expend by every row or every column (with order $n \geq 6$, randomly generate integers as matrix elements), i.e., $\det(A) = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}\det(A_{ij})$ where A_{ij} is the submatrix of A by deleting the ith row and jth column of A, $(-1)^{i+j}\det(A_{ij})$ is the **cofactor of matrix A** or **algebraic cofactor of matrix A**, apply R or Python function to calculate $\det(A_{ij})$ is available.

```python
import numpy as np

def calc_det_row(A, i):
  n = A.shape[0]
  det = 0
  for j in range(0, n):
    A_ij = np.delete(A, i, axis=0)
```

```
    A_ij = np.delete(A_ij, j, axis=1)
    det_A_ij = np.linalg.det(A_ij)
    det += A[i,j] * (-1) ** (i + j + 2) * det_A_ij

  return det

def calc_det_col(A, j):
  n = A.shape[0]
  det = 0
  for i in range(0, n):
    A_ij = np.delete(A, i, axis=0)
    A_ij = np.delete(A_ij, j, axis=1)
    det_A_ij = np.linalg.det(A_ij)
    det += A[i,j] * (-1) ** (i + j + 2) * det_A_ij

  return det

for n in range(6, 9):

  A = np.random.randint(-30, 30, size=(n, n))
  det_A = np.linalg.det(A).round(0)  # compute the determinant of A
  print(det_A)

  dets = []
  for i in range(0, n):
    det = calc_det_row(A, i)
    det = det.round(0)
    assert(det_A - det <= 1)
    dets.append(det)
    det = calc_det_col(A, i)
    det = det.round(0)
    assert(det_A - det <= 1)
    dets.append(det)
  print(dets)
```

```
## 66615165.0
## [66615165.0, 66615165.0, 66615165.0, 66615165.0, 66615165.0, 66615165.0, 66615165.0, 66615165.0, 666
## -12423391964.0
## [-12423391964.0, -12423391964.0, -12423391964.0, -12423391964.0, -12423391964.0, -12423391964.0, -124
## 2630165909022.0
## [2630165909022.0, 2630165909022.0, 2630165909022.0, 2630165909022.0, 2630165909022.0, 2630165909022.0
```

```
print("Determinant is invariant")
```

```
## Determinant is invariant
```