A note on Weyl groups, characters and cocharacters of linear algebraic groups

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Abstract

In this note we show that the rational characters of a linear algebraic group over an algebraically closed field coincide with the Weyl-invariant rational characters of a maximal torus. In the reductive, not necessarily connected, case, we prove an analogue result for integral cocharacters.

1 Introduction

Let k be an algebraically closed field, let G be a smooth linear algebraic group over k, and let T be a maximal torus of G, with Weyl group $W = W(G,T) = N_G(T)/C_G(T)$. The Weyl group embeds in the automorphism group scheme $\underline{\mathrm{Aut}}(T)$ of the torus T, and it is thus a finite constant group.

We denote $\Gamma_{\mathbb{Z}}(G), \Gamma^{\mathbb{Z}}(G), \Gamma_{\mathbb{Q}}(G), \Gamma^{\mathbb{Q}}(G)$ the set of characters, cocharacters, rational characters and rational cocharacters of G. Let U be the unipotent radical of G and let $R = G/U$. The group G is said to be *reductive* if $U = 1$, and R is always reductive. We denote T' the image of T in R, which is a maximal torus of R isomorphic to T, and let $W' = W(R,T')$ be the associated Weyl group.

The first result of this note is the fact that the unipotent radical does not play a role in the formation of the Weyl group:

Proposition 1. *If G has a Levi subgroup, then there is a canonical isommorphism $W \cong W'$.*

By a Levi subgroup of G we mean a section of $G \to R$, which allows us to write $G = U \rtimes R$. If the characteristic of k is 0, then a Levi subgroup always exists.

Secondly, we describe the rational characters of G in terms of those of T and the Weyl group.

Proposition 2. *If G has a Levi subgroup, then the natural map $\Gamma_{\mathbb{Q}}(G) \to \Gamma_{\mathbb{Q}}(T)^W$ is an isomorphism. Moreover, if G is reductive, then $\Gamma_{\mathbb{Q}}(G) \cong \Gamma_{\mathbb{Q}}(Z(G))$.*

Above, $\Gamma_{\mathbb{Q}}(T)^W$ denotes those rational characters of T that are fixed under the action of W.

Finally, in the reductive case, the integral cocharacters of the centre $Z(G)$ of G coincide with the Weyl-invariant cocharacters of T.

Proposition 3. *If G is reductive, then the natural map $\Gamma^{\mathbb{Z}}(Z(G)) \to \Gamma^{\mathbb{Z}}(T)^W$ is an isomorphism.*

While these results might be known to experts, the authors could not find them in the literature. The purpose of this note is to provide a reference beynd the well-studied connected reductive case.

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2 Proof of Proposition 1

We have a diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_U(T) & \longrightarrow & N_U(T) & \longrightarrow & N_U(T)/C_U(T) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_G(T) & \longrightarrow & N_G(T) & \longrightarrow & W & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_R(T') & \longrightarrow & N_R(T') & \longrightarrow & W' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where all rows and columns are exact. Surjectivity of $N_G(T) \to N_R(T')$ and of $C_G(T) \to C_R(T')$ follows from the existence of a Levi subgroup.

Let A be a commutative k-algebra and let $g \in N_U(T)(A)$. For every A-algebra B and every $t \in T(B)$, we have that $tg_{|B}^{-1}t^{-1} \in U(B)$, because U is normal, and $g_{|B}tg_{|B}^{-1} \in T(B)$, because g normalises T. Therefore $g_{|B}tg_{|B}^{-1}t^{-1} \in T(B) \cap U(B)$. Since $T \cap U$ is trivial, we must have $g \in C_U(T)(A)$. Thus $C_U(T) = N_U(T)$ and we conclude that the map $W \to W'$ is an isomorphism.

3 Proof of Proposition 3

We are assuming that G is reductive. The identity component $Z(G)^\circ$ of the centre $Z(G)$ of G is a torus contained in T, so we have an injection

$$\Gamma^{\mathbb{Z}}(Z(G)) = \Gamma^{\mathbb{Z}}(Z(G)^\circ) \hookrightarrow \Gamma^{\mathbb{Z}}(T)^W.$$

Let $\lambda \in \Gamma^{\mathbb{Z}}(T)^W$. We want to prove that $\lambda\colon \mathbb{G}_{m,k} \to T$ factors through $Z(G)$ or, equivalently, that the centraliser $L_G(\lambda)$ of λ in G equals the whole of G.

We denote $P_G(\lambda)$ and $U_G(\lambda)$ the subgroups of G defined functorially by

$$P_G(\lambda)(A) = \{g \in G(A) \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} \text{ exists in } G\},$$

$$U_G(\lambda)(A) = \{g \in P_G(\lambda)(A) \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} = 1\},$$

for any k-algebra A. By existence of the limit we mean that the corresponding morphism $\mathbb{G}_{m,A} \to G$ extends to a morphism $\mathbb{A}^1_A \to G$. The subgroups $P_G(\lambda)$ and $U_G(\lambda)$

are smooth because G is, $U_G(\lambda)$ is connected (even if G is not) because it retracts to a point, and it is also unipotent.

Let $g \in G(k)$ and denote $\lambda^g = g\lambda g^{-1}$. Since the intersection of two parabolic subgroups of G°, $P_{G^\circ}(\lambda)$ and $P_{G^\circ}(\lambda^g)$ in this case, contains a maximal torus of G°, there exist $h_1 \in P_{G^\circ}(\lambda^g)(k)$ and $h_2 \in P_{G^\circ}(\lambda)(k)$ such that $\lambda^{h_1 g}$ and λ^{h_2} commute. Since $P_{G^\circ}(\lambda^g) = gP_{G^\circ}(\lambda)g^{-1}$, there are $h, h' \in P_{G^\circ}(\lambda)(k)$ such that $\lambda^{hgh'}$ and λ commute (just take $h = h_2^{-1}$ and $h' = g^{-1}h_1 g$).

Let T_1 be a maximal torus containig both $\lambda^{hgh'}$ and λ. Both T_1 and T are maximal tori of $L_G(\lambda)$, so there is $u \in L_G(\lambda)(k)$ such that $uT_1u^{-1} = T$. Let $v = uhgh'$, then $\lambda^v \in \Gamma^{\mathbb{Z}}(T)$.

Now $v^{-1}Tv$ is a maximal torus containing λ, so there is $r \in L_G(\lambda)(k)$ such that $rTr^{-1} = v^{-1}Tv$. Then $vr \in N_G(T)(k)$ so, since λ is fixed by W, we have $\lambda = \lambda^{vr} = (\lambda^r)^v = \lambda^v$, and hence $v = uhgh' \in L_G(\lambda)(k)$. Since $u, h, h' \in P_G(\lambda)(k)$, this implies that $g \in P_G(\lambda)(k)$ as well.

Since $P_G(\lambda)$ and G are smooth and g was arbitrary, we have $G = P_G(\lambda)$. Now, $P_G(\lambda) = U_G(\lambda) \rtimes L_G(\lambda)$, and $U_G(\lambda)$ is smooth, connected and unipotent. Since G is reductive, $U_G(\lambda) = 1$, and thus $G = L_G(\lambda)$, as desired.

Remark 4. If one is willing to accept Proposition 3 in the connected case, one can also argue as follows in the general case.

Since G° is connected reductive, the automorphism group can be written as $\mathrm{Aut}(G^\circ) = G^\circ/Z(G^\circ) \rtimes \mathrm{Out}(G)$, where $\mathrm{Out}(G)$ preserves pinning (in particular, the torus T). Thus if $g \in G(k)$, then conjugation by g in G° is the same as conjugation by ah, with $a \in G^\circ(k)$ and $h \in N_G(T)$. If $\lambda \in \Gamma^{\mathbb{Z}}(T)^W$, then $\lambda \in \Gamma^{\mathbb{Z}}(T)^{W(G^\circ,T)} = \Gamma^{\mathbb{Z}}(Z(G^\circ))$ by the connected case. Then $\lambda^g = (\lambda^h)^a = \lambda^a = \lambda$. Since this is true for all $g \in G(k)$, we have $\lambda \in \Gamma^{\mathbb{Z}}(Z(G))$, as desired.

4 Proof of Proposition 2

If the result is true in the reductive case, then we have

$$\Gamma_{\mathbb{Q}}(G) = \Gamma_{\mathbb{Q}}(R) \cong \Gamma_{\mathbb{Q}}(T')^{W'} = \Gamma_{\mathbb{Q}}(T)^W,$$

because every character $G \to \mathbb{G}_{m,k}$ factors through R, because the result is true for R, and by Proposition 1. Therefore we may assume that G is reductive.

We have a chain of inclusions

$$\Gamma^{\mathbb{Q}}(Z(G)) \subset \Gamma^{\mathbb{Q}}(Z(G^\circ))^{G(k)} \subset \Gamma^{\mathbb{Q}}(Z(G^\circ))^W \subset \Gamma^{\mathbb{Q}}(T)^W,$$

and $\Gamma^{\mathbb{Q}}(Z(G)) = \Gamma^{\mathbb{Q}}(T)^W$ by Proposition 3. Thus $\Gamma^{\mathbb{Q}}(T)^W = \Gamma^{\mathbb{Q}}(Z(G^\circ))^{G(k)} = \Gamma^{\mathbb{Q}}(Z(G^\circ))^F$, where $F = G/G^\circ$, which is a finite constant group.

Recall that if a finite group S acts on a finite dimensional $\mathbb{Q}$-vector space V, then there is a canonical bijection $(V^S)^\vee = (V^\vee)^S$. Hence

$$\Gamma_{\mathbb{Q}}(T)^W = \left(\Gamma^{\mathbb{Q}}(T)^W\right)^\vee = \left(\Gamma^{\mathbb{Q}}(Z(G^\circ))^F\right)^\vee = \Gamma_{\mathbb{Q}}(Z(G^\circ))^F.$$

On the other hand, since G° is connected and reductive, we have the central isogeny $Z(G^\circ) \to G^\circ/\mathcal{D}(G^\circ)$, where $\mathcal{D}(G^\circ)$ is the derived subgroup. This gives a F-equivariant isomorphism $\Gamma_{\mathbb{Q}}(Z(G^\circ)) = \Gamma_{\mathbb{Q}}(G^\circ/\mathcal{D}(G^\circ)) = \Gamma_{\mathbb{Q}}(G^\circ)$. Therefore $\Gamma_{\mathbb{Q}}(T)^W = \Gamma_{\mathbb{Q}}(G^\circ)^F$.

To conclude, it is enough to show that $\Gamma_{\mathbb{Q}}(G) = \Gamma_{\mathbb{Q}}(G^\circ)^F$. Denote $\varphi\colon \Gamma_{\mathbb{Q}}(G) \to \Gamma_{\mathbb{Q}}(G^\circ)^F$ the obvious map. First we show injectivity of φ. Let $\chi \in \Gamma_{\mathbb{Q}}(G)$ such that $\varphi(\chi) = 0$. After scaling, we may assume that χ is represented by a character $\chi\colon G \to \mathbb{G}_{m,k}$ such that $\chi|_{G^\circ}$ is trivial. For all $g \in G(k)$, $\chi^{\#F}(g) = \chi(g^{\#F}) = 1$, because $g^{\#F}$ is in G°. In additive notation, $(\#F)\chi = 0$. This shows injectivity.

For surjectivity of φ, we take a finite constant subgroup $N \leq G$ such that $N \to G \to F$ is surjective [1]. Let $\chi \in \Gamma_{\mathbb{Q}}(G^\circ)^F$. After scaling, we may assume that χ is represented by a character $\chi\colon G^\circ \to \mathbb{G}_{m,k}$ that is F-invariant. Let $l = \#N$. We define a map $\alpha\colon N \ltimes G^\circ \to \mathbb{G}_{m,k}$ by $\alpha(h, n) = \chi^l(h)$. By F-invariance,

$$\alpha((n,h)(n',h')) = \alpha((nn', (n')^{-1}hn'h')) = \chi^l((n')^{-1}hn')\chi^l(h') = \alpha((n,h))\alpha((n',h')),$$

so α is a group homomorphism. If $nh = n'h'$ in G, then $h'h^{-1}$ is in N and thus $\chi^l(h') = \chi^l(h)$. Therefore α descends to a character $\overline{\alpha}$ of G and by construction $\varphi(\frac{1}{l}\overline{\alpha}) = \chi$. This shows surjectivity.

The last statement in the proposition follows because

$$\Gamma_{\mathbb{Q}}(Z(G)) = \Gamma^{\mathbb{Q}}(Z(G))^\vee = (\Gamma^{\mathbb{Q}}(T)^W)^\vee = \Gamma_{\mathbb{Q}}(T)^W$$

by Proposition 3.

References

[1] M. Brion. On extensions of algebraic groups with finite quotient. *Pacific Journal of Mathematics*, 279(1-2):135–153, 2015.