Good moduli space for the stack of graded points

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In this note we provide a proof that in reasonable situations, if an algebraic stack $\mathcal{X}$ has a good moduli space, then $\mathrm{Grad}(\mathcal{X})$ has a good moduli space. More precisely:

Proposition 1. *Let $\mathcal{X}$ be an algebraic stack over an algebraic space B satisfying Assumption 2 below and suppose that $\mathcal{X}$ has affine diagonal. If $\mathcal{X}$ has a good moduli space, then* $\mathrm{Grad}(\mathcal{X})$ *has a good moduli space too.*

Assumption 2 (For $(\mathcal{X}, B)$). The symbol B denotes a quasi-separated excellent algebraic space; and $\mathcal{X}$ is a quasi-separated and locally finitely presented algebraic stack over B, with affine stabilisers and separated inertia.

Under Assumption 2, it follows from [5, Theorem 14.9] that $\mathrm{Filt}(\mathcal{X})$ and $\mathrm{Grad}(\mathcal{X})$ are algebraic. See also [8, Theorem 5.1.1] for a related algebraicity result. Also, the "evaluation at 1" map $\mathrm{Filt}(\mathcal{X}) \to \mathrm{Grad}(\mathcal{X})$ is representable [7, Proposition 1.1.13].

Assumption 2 is stable under taking Grad.

Proposition 3. *If $\mathcal{X}$ satisfies Assumption 2, then so does* $\mathrm{Grad}(\mathcal{X})$.

Proof. The stack $\mathrm{Grad}(\mathcal{X})$ is locally of finite presentation and quasi-separated over B by [7, Proposition 1.1.2]. We need to prove that $\mathrm{Grad}(\mathcal{X})$ has affine stabilisers and that the inertia $\mathcal{I}_{\mathrm{Grad}(\mathcal{X})} \to \mathrm{Grad}(\mathcal{X})$ is separated.

Let $x\colon T \to \mathrm{Grad}(\mathcal{X})$. We consider $\mathrm{Aut}(x)$, which is the base change of the inertia morphism along x. Using the description of [7, Lemma 1.1.5], the T-point x corresponds to a cocharacter $\lambda\colon \mathbb{G}_{m,T} \to \mathrm{Aut}(u(x))$, where $u\colon \mathrm{Grad}(\mathcal{X}) \to \mathcal{X}$ is the forgetful map. Describing x as descent data on the smooth groupoid of the trivial action of $\mathbb{G}_{m,T}$ on T, we see that an automorphism of x over a T-scheme S is an S-point g of $\mathrm{Aut}(u(x))$ commuting with $\lambda_S\colon \mathbb{G}_{m,S} \to \mathrm{Aut}(u(x)) \times_T S$. Thus $\mathrm{Aut}(x) = L(\lambda)$, the centraliser of λ in $\mathrm{Aut}(u(x))$, which is the fixed-point locus of the action of $\mathbb{G}_{m,T}$ on $\mathrm{Aut}(u(x))$ by conjugation via λ. Since $\mathrm{Aut}(u(x)) \to T$ is separated and locally of finite type, an application of [7, Proposition 1.4.1] gives that $L(\lambda) \to \mathrm{Aut}(u(x))$ is a closed immersion. Thus $\mathrm{Aut}(x) \to T$ is separated. This proves that $\mathrm{Grad}(\mathcal{X})$ has separated inertia. If T is the spectrum of a field, then $\mathrm{Aut}(u(x))$ is affine, so $\mathrm{Aut}(x)$ is affine as well. Thus $\mathrm{Grad}(\mathcal{X})$ has affine automorphism groups. □

The proof of Proposition 1 will require several lemmas and the introduction of the key concept of Θ-surjectivity, important for glueing good moduli spaces. Recall from [6, Definition 3.3] that a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks satisfying Assumption 2 is said to be Θ-*surjective* if the induced map $\mathrm{Filt}(\mathcal{X}) \to \mathcal{X} \times_{\mathcal{Y},\mathrm{ev}_1} \mathrm{Filt}(\mathcal{Y})$

is surjective. This is equivalent to asking that for every algebraically closed field k and any commutative diagram

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[d] & \Theta_k \arrow[d] \arrow[dl, dashed] \\ \mathcal{X} \arrow[r, "f"'] & \mathcal{Y} \end{tikzcd}$$

of solid arrows, there exists a dotted doted lift (in the 2-categorical sense).

We need to introduce another convenient concept.

Definition 4. A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is said to be *inertia-preserving* if the induced map $\mathcal{I}_{\mathcal{X}} \to f^*\mathcal{I}_{\mathcal{Y}}$ of inertia stacks is an isomorphism.

Inertia-preserving morphisms are stable under arbitrary base change.

Lemma 5. *Let $\mathcal{X}$ and $\mathcal{X}_1$ be locally noetherian algebraic stacks and let $f\colon \mathcal{X}_1 \to \mathcal{X}$ be étale, affine, inertia-preserving, surjective and Θ-surjective. If $\mathcal{X}_1$ has a good moduli space, then so does $\mathcal{X}$.*

Proof. This result is contained in the proof of [6, Theorem 4.1], which is itself based on [3, Proposition 3.1]. We sketch the argument for the convenience of the reader.

Let $\mathcal{X}_2 = \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1$ and consider the two projections $p_1, p_2\colon \mathcal{X}_2 \to \mathcal{X}_1$. Since p_1 (and p_2) is affine and $\mathcal{X}_1$ has a good moduli space, $\mathcal{X}_2$ has a good moduli space too [1, Lemma 4.14]. The same is true for $\mathcal{X}_3 = \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1$. The étale groupoid $\mathcal{X}_\bullet$ induces, after taking good moduli spaces, a groupoid $X_\bullet$ on algebraic spaces. All projections between the $\mathcal{X}_i$ are also étale, affine, Θ-surjective and inertia-preserving. Thus, by [6, Proposition 4.2], the diagram

$$\begin{tikzcd} \mathcal{X}_3 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & \mathcal{X}_2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathcal{X}_1 \arrow[d] \\ X_3 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X_2 \arrow[r, shift left] \arrow[r, shift right] & X_1 \end{tikzcd}$$

is cartesian, and all projections between the X_i are étale. Therefore $\mathcal{X}_\bullet$ is an étale groupoid. It can be seen that $X_2 \to X_1 \times X_1$ is a monomorphism (see the proof of [3, Proposition 3.1]), thus $\mathcal{X}_\bullet$ is actually an étale equivalence relation, and its quotient is an algebraic space X. There is an induced map $\pi\colon \mathcal{X} \to X$. The square

$$\begin{tikzcd} \mathcal{X}_1 \arrow[r] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\ulcorner"] & \mathcal{X} \arrow[d, "\pi"] \\ X_1 \arrow[r] & X \end{tikzcd}$$

is cartesian (see argument at the end of the proof of [3, Proposition 3.1]). Therefore, by descent, π is a good moduli space. □

Lemma 6. *Let $\mathcal{X}$ be a locally noetherian algebraic stack with affine diagonal. Let $f_i\colon \mathcal{Y}_i \to \mathcal{X}$ be a family of jointly surjective morphisms that are affine, étale, Θ-surjective and inertia-preserving. If each $\mathcal{Y}_i$ is quasi-compact and has a good moduli space, then $\mathcal{X}$ has a good moduli space.*

Proof. The image $f_i(|\mathcal{Y}_i|)$ of f_i in $|\mathcal{X}|$ is an open subset that defines an open substack $\mathcal{U}_i$ of $\mathcal{X}$. The restriction $\mathcal{Y}_i \to \mathcal{U}_i$ satisfies the hypothesis of 5, so each $\mathcal{U}_i$ has a good moduli space $\mathcal{U}_i \to U_i$, which is of finite type by [4, Theorem A.1]. Moreover, each open immersion $\mathcal{U}_i \to \mathcal{X}$ is Θ-surjective, so the glueing lemma [6, Lemma 4.4] implies that $\mathcal{X}$ has a good moduli space too. □

Lemma 7. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a representable morphism between algebraic stacks over a base algebraic space and assume that $\mathcal{X}$ and $\mathcal{Y}$ satisfy Assumption 2. Suppose f has one of the following properties:*

1. *étale,*
2. *affine,*
3. *surjective and étale,*
4. *Θ-surjective,*
5. *inertia-preserving.*

Then so does $\operatorname{Grad}(f)$.

Proof. For étaleness, this is [7, Proposition 1.3.1]. For affineness, note the fact that, if $T \to \operatorname{Grad}(\mathcal{Y})$ is a scheme-valued point corresponding to $B\mathbb{G}_{m,T} \to \mathcal{Y}$, and

(1)
$$\begin{tikzcd} Z/\mathbb{G}_{m,T} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B\mathbb{G}_{m,T} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

is cartesian, then

$$\begin{tikzcd} Z^{\mathbb{G}_{m,T}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & T \arrow[d] \\ \operatorname{Grad}(\mathcal{X}) \arrow[r] & \operatorname{Grad}(\mathcal{Y}) \end{tikzcd}$$

is cartesian. Indeed, the 1-category of representable algebraic stacks over $B\mathbb{G}_{m,T}$ is equivalent to the category of algebraic spaces over T endowed with a $\mathbb{G}_{m,T}$-action, and the equivalence is given by pullback along $T \to B\mathbb{G}_{m,T}$. Therefore $\mathcal{X} \times_{\mathcal{Y}} B\mathbb{G}_{m,T} = Z/\mathbb{G}_{m,T}$ for a T-algebraic space Z acted on by $\mathbb{G}_{m,T}$. Now, given a T-scheme S, a map $S \to \operatorname{Grad}(\mathcal{X}) \times_{\operatorname{Grad}(\mathcal{Y})} T$ over T is a section of $Z/\mathbb{G}_{m,T} \to B\mathbb{G}_{m,T}$ over $B\mathbb{G}_{m,S} \to B\mathbb{G}_{m,T}$, which is in turn a $\mathbb{G}_{m,T}$-equivariant map $S \to Z$. Therefore $\operatorname{Grad}(\mathcal{X}) \times_{\operatorname{Grad}(\mathcal{Y})} T = Z^{\mathbb{G}_{m,T}}$, the fixed points of Z, as a stack over T. Since $Z \to T$ is affine and the inclusion $Z^{\mathbb{G}_{m,T}} \to Z$ of the fixed-point locus is a closed immersion [7, Proposition 1.4.1], it follows that $Z^{\mathbb{G}_{m,T}} \to T$ is affine.

Now suppose that f is surjective and étale, take $T \to \operatorname{Grad}(\mathcal{Y})$ as before, and assume that T is the spectrum of an algebraically closed field k. Then Z in 1 is a nonempty disjoint union of copies of $\operatorname{Spec}(k)$. Thus the $\mathbb{G}_{m,k}$-action on Z is trivial and $Z^{\mathbb{G}_{m,k}} = Z$ is, in particular, nonempty. Thus $\operatorname{Grad}(f)$ is surjective.

Now assume that f is Θ-surjective. Let k be an algebraically closed field and consider a solid commutative diagram

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "1"] \arrow[d, "y"] & \Theta_k \arrow[d, "x"] \arrow[dl, dashed] \\ \operatorname{Grad}(\mathcal{X}) \arrow[r, "\operatorname{Grad}(f)"'] & \operatorname{Grad}(\mathcal{Y}) \end{tikzcd} \tag{2}$$

for which we want to find the dashed lift. Write $\Theta_k \times B\mathbb{G}_{m,k} = \mathbb{A}^1_k/(\mathbb{G}_{m,k} \times \mathbb{G}_{m,k})$, where $\mathbb{G}_{m,k} \times 1$ acts with weight 1 and $1 \times \mathbb{G}_{m,k}$ acts trivially. The morphism x corresponds to a map $\underline{x} \colon \mathbb{A}^1_k/\mathbb{G}^2_{m,k} \to \mathcal{X}$. Form the fibre product

$$\begin{tikzcd} Y/\mathbb{G}^2_{m,k} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{A}^1_k/\mathbb{G}^2_{m,k} \arrow[d, "\underline{x}"] \\ \mathcal{Y} \arrow[r, "f"] & \mathcal{X}. \end{tikzcd}$$

Then there is a cartesian square

$$\begin{tikzcd} Y^{1\times\mathbb{G}_{m,k}}/\mathbb{G}_{m,k} \times 1 \arrow[r, "v"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{A}^1_k/\mathbb{G}_{m,k} \times 1 = \Theta_k \arrow[d, "x"] \\ \operatorname{Grad}(\mathcal{X}) \arrow[r, "\operatorname{Grad}(f)"] & \operatorname{Grad}(\mathcal{Y}) \end{tikzcd}$$

Since f is Θ-surjective, the map $Y/\mathbb{G}_{m,k} \times 1 \to \mathbb{A}^1_k/\mathbb{G}_{m,k} \times 1$ is Θ-surjective too, so $u \colon Y \to \mathbb{A}^1_k$ has a $\mathbb{G}_{m,k} \times 1$-equivariant section $s \colon \mathbb{A}^1_k \to Y$ such that $s(1)$ gives the point y of $\operatorname{Grad}(\mathcal{X})$. Since the restriction $s \mid \mathbb{A}^1_k \setminus 0 \to Y$ factors through $Y^{1\times\mathbb{G}_{m,k}} \to Y$, which is a closed immersion, and since $\mathbb{A}^1_k \setminus 0$ is schematically dense in $\mathbb{A}^1_k$, the map s itself factors through $Y^{1\times\mathbb{G}_{m,k}} \to Y$, thus giving a section of v that defines the dotted lift of 2.

Suppose now that f is inertia-preserving. Call $g = \operatorname{Grad}(f)$. We have to prove that for any scheme-valued point $x \colon T \to \operatorname{Grad}(\mathcal{X})$, the induced group homomorphism $\operatorname{Aut}(x) \to \operatorname{Aut}(g(x))$ is an isomorphism. Denote $u(x)$ the underlying T-point of $\mathcal{X}$. The map x is determined by $u(x)$ and a cocharacter $\lambda \colon \mathbb{G}_{m,T} \to \operatorname{Aut}(u(x))$ [7, Lemma 1.1.5], and there is a natural isomorphism $\operatorname{Aut}(x) = L(\lambda)$ (see the proof of Proposition 3). Similarly $\operatorname{Aut}(g(x)) = L(h \circ \lambda)$, where $h \colon \operatorname{Aut}(u(x)) \to \operatorname{Aut}(f(u(x)))$ is the homomorphism induced by f and $u(x)$. since f is an isomorphism by hypothesis, we get by the above description that $\operatorname{Aut}(x) \to \operatorname{Aut}(g(x))$ is an isomorphism too. □

Lemma 8. *Let A be a commutative ring, and consider an action of GL_N on $X = \operatorname{Spec} A$ (over $\mathbb{Z}$) such that X/GL_N has a good moduli space. Let $\lambda \colon \mathbb{G}_m \to GL_N$ be a cocharacter. Then $X^{\lambda,0}/L(\lambda)$ has a good moduli space, where $X^{\lambda,0}$ is the fixed point locus of the $\mathbb{G}_m$-action on X induced by λ and $L(\lambda)$ is the centraliser of λ.*

Proof. It is enough [1, Lemma 4.14] to prove that the natural map $X^{\lambda,0}/L(\lambda) \to X/\operatorname{GL}_N$ is affine. There is a cartesian square

$$\begin{tikzcd} \operatorname{GL}_N \times^{L(\lambda)} X^{\lambda,0} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\ X^{\lambda,0}/L(\lambda) \arrow[r] & X/\operatorname{GL}_N \end{tikzcd}$$

where $\mathrm{GL}_N \times^{L(\lambda)} X^{\lambda,0}$ is the stack quotient of $\mathrm{GL}_N \times X^{\lambda,0}$ by the diagonal action of $L(\lambda)$. Since the action is free, $\mathrm{GL}_N \times^{L(\lambda)} X^{\lambda,0}$ is an algebraic space. Now, $L(\lambda)$ is isomorphic to a product of GL_{N_i}'s and it is thus geometrically reductive [2, Definition 9.1.1]. Since $\mathrm{GL}_N \times X^{\lambda,0} = \operatorname{Spec} B$ is affine, the $L(\lambda)$-invariants give an adequate moduli space $\mathrm{GL}_N \times^{L(\lambda)} X^{\lambda,0} \to \operatorname{Spec}\left(B^{L(\lambda)}\right)$ [2, Theorem 9.1.4]. By universality for adequate moduli spaces [5, Theorem 3.12], we get an isomorphism $\mathrm{GL}_N \times^{L(\lambda)} X^{\lambda,0} = \operatorname{Spec} B^{L(\lambda)}$. Therefore $\mathrm{GL}_N \times^{L(\lambda)} X^{\lambda,0}$ is affine and we are done by descent. □

Remark 9. Since $L(\lambda)$ is geometrically reductive and $X^{\lambda,0}$ is affine, taking $L(\lambda)$-invariants gives an adequate moduli space for $X^{\lambda,0}/L(\lambda)$. However, unless A is of characteristic 0, an extra argument is needed to show that the adequate moduli space is indeed a good moduli space.

Proof of Proposition 1. By [5, Theorem 1.1] and by [6, Proposition 4.3], there is a jointly surjective family of morphisms $f_i \colon \mathcal{Y}_i = \operatorname{Spec}(A_i)/\mathrm{GL}_{N_i} \to \mathcal{X}$ that are affine, étale, Θ-surjective and inertia-preserving, and where each $\mathcal{Y}_i$ has a good moduli space. Thus $\operatorname{Grad}(f_i) \colon \operatorname{Grad}(\mathcal{Y}_i) \to \operatorname{Grad}(\mathcal{X})$ is a jointly surjective family of morphisms with the same properties, by 7. Moreover, each $\operatorname{Grad}(\mathcal{Y}_i)$ is a disjoint union of quasi-compact quotient stacks $\mathcal{Y}_{i,\lambda}$ of the form $\operatorname{Spec}(A_i)^{\lambda,0}/L(\lambda)$, where $\lambda \colon \mathbb{G}_m \to \mathrm{GL}_{N_i}$ is a cocharacter, by [7, Theorem 1.4.7]. Therefore, by Lemma 8, each $\mathcal{Y}_{i,\lambda}$ has a good moduli space. Thus, the family of morphisms $\mathcal{Y}_{i,\lambda} \to \operatorname{Grad}(\mathcal{X})$ satisfies the hypothesis of Lemma 6. Hence $\operatorname{Grad}(\mathcal{X})$ has a good moduli space. □

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