Blow-ups of stacks, instability and modular lattices

Draft

Andrés Ibáñez Núñez

November 2023

Abstract

We show that, for moduli stacks of objects in an abelian category, the iterated balanced filtration, defined using blow-ups and semistability, agrees with the HKKP filtration, which is defined in terms of a normed artinian lattice.

Contents

1 Introduction

In [7], Haiden-Katzarkov-Kontsevich-Pandit defined a canonical filtration $0 < a_0 < \cdots < a_n = 1$ of a normed artinian lattice L, together with labels $c_0 > \cdots > c_n$ where each $c_i = (c_i^{(0)}, c_i^{(1)}, \dots) \in \mathbb{Q}^\infty$ is an eventually zero sequence of rational numbers. We call this filtration the *iterated HKKP filtration* of the lattice L. If Q is a quiver with central charge Z and E is a semistable representation of Q over the field $\mathbb{C}$ of complex numbers, then the lattice L of semistable subrepresentations of Q is artinian and naturally normed, so one may consider its iterated HKKP filtration.

Remarkably, this filtration describes the asymptotics of a certain natural gradient flow associated to the moduli problem. Let S be the Riemannian manifold of hermitian metrics on the representation E. The so called *Kempf potential* is a map $p\colon S \to \mathbb{R}$ that depends on the central charge Z. By the celebrated Kempf-Ness theorem [12], the algebraic notion of *semistability* of E can be detected from the potential p: the representation E is semistable if and only if p is bounded below. Similarly, E is polystable if and only if p attains a minimum m, in which case the negative gradient flow of p starting at any point of S converges to m. The question arises of whether something more precise can be said about this flow in the case where E is strictly semistable. This is answered by the iterated HKKP filtration: in the piece of the filtration labelled by the sequence $(s_0, s_2, \dots)$ of rational numbers, asymptotically the gradient flow equals

$$t^{s_0} \log(t)^{s_1} \log(\log(t))^{s_2} \cdots$$

up to bounded terms.

The Kempf potential is not specific to moduli spaces of quiver representations. Every point in a Geometric Invariant Theory situation has an associated Kempf potential, and many other moduli problems in practice (moduli of K-semistable varieties, vector bundles or principal bundles on a curve) have associated potentials. One would expect a similar result describing the asymptotic behaviour of the associated gradient flow. The first problem with this is that most moduli problems do not have associated lattices, so there is no hope that the general machinery of the iterated HKKP filtration works in general.

In the author's previous work [10], an analogue of the HKKP filtration, the iterated balanced filtration, was introduce in the vast setting of algebraic stacks admitting a good moduli space and endowed with a norm on graded points. Examples of this situation are all GIT quotient stacks, stacks of Bridgeland semistable objects in the derived category $D^{\mathrm{b}}_{\mathrm{coh}}(X)$ of coherent sheaves on a projective scheme X, stacks of semistable G-bundles on a curve for a reductive group G, stacks of semistable G-Higgs bundles and stacks of K-semistable Fano varieties, so the applicability of this methods is wide.

Thus we now have two constructions that live a priori in completely different worlds: the iterated HKKP filtration for artinian lattices on one hand, and the iterated balanced filtrations for points on stacks on the other. Our aim is to develop a framework to be able to relate these two different worlds and we prove that the two perspectives agree whenever a "lattice of subobjects" exists.

To make this precise, we introduce the notion of *linearly lit good moduli stack*. These are stacks $\mathcal{X}$ together with a piece of extra structure, what we call the *lamp*, that remembers what filtrations have "only nonnegative weights". In the definition, we ask that $\mathcal{X}$ has a good moduli space $\pi \colon \mathcal{X} \to X$ and that each fibre $\mathcal{F}$ of $\mathcal{X}$ embeds into a moduli stack $\mathscr{Rep}(Q, d)$ of quiver representations for some quiver Q and dimension vector d. While this may seem like a restrictive condition at first, we prove (Theorem 5.0.12) that any open substack $\mathcal{X}$ of the moduli stack of objects $\mathcal{M}_{\mathcal{A}}$ of an abelian category $\mathcal{A}$ is linearly lit provided it admits a good moduli space. The moral is that any stack parametrising objects of linear type, like complexes of sheaves, and admitting a good moduli space will automatically be linearly lit.

We show that if $\mathcal{X}$ is linearly lit, then for any point x of $\mathcal{X}$, there is a canonical associated artinian lattice L_x and a bijection $\mathbb{Q}^\infty\text{-}\operatorname{Filt}(\mathcal{X}, x) = \mathbb{Q}^\infty\text{-}\operatorname{Filt}(L_x)$ between the set of $\mathbb{Q}^\infty$-filtrations of x (in the stacky sense) and the set of $\mathbb{Q}^\infty$-filtrations of the lattice L_x (in the lattice-theoretic sense). Our main result (Corollary 6.0.11) establishes that, under this bijection, the iterated balanced filtration of x equals the iterated HKKP filtration of L_x.

In order to establish this comparison, we use a new characterisation of the HKKP filtration for lattices as a minimiser of certain norm function [9]. Then we introduce the concept of a chain of lattices, in analogy with a chain of stacks, and see the iterated HKKP filtration as a shadow of the HKKP chain. If one has a chain of pointed linearly lit stacks, one can apply the "associated lattice" functor to it, to get a chain of lattices. We show that, when applied to the cone-balanced chain, the chain of lattices obtained is nothing but the HKKP chain (Theorem 6.0.9). From this our main comparison result (Corollary 6.0.11) follows.

We think of our results as strong evidence that the iterated balanced filtration should govern the asymptotics of natural gradient flows in moduli theory.

2 Formal fans and the degeneration fan

We recall Halpern-Leistner's notion degeneration fan [8, Definition 3.2.2], with slightly different conventions.

Let A be a subring of $\mathbb{R}$, endowed with the inherited order. We let Cone_A be the category that has

1. objects: $[n]$, where $n \in \mathbb{N}$,
2. morphisms: a map $[n] \to [m]$ is an A-linear homomorphism $\varphi\colon A^n \to A^m$ such that $\varphi(A^n_{\geq 0}) \subset A^m_{\geq 0}$.

We denote $\mathrm{Cone} \coloneqq \mathrm{Cone}_{\mathbb{Z}}$. It is naturally a subcategory of $\mathrm{Cone}_{\mathbb{Q}}$ consisting of maps $\varphi\colon \mathbb{Q}^n \to \mathbb{Q}^m$ such that $\varphi(\mathbb{Z}^n) \subset \mathbb{Z}^m$.

Definition 2.0.1. An *A-linear formal fan* is a functor $\mathrm{Cone}_A^{\mathrm{op}} \to \mathbf{Set}$.

A $\mathbb{Z}$-linear formal fan will simply be called a *formal fan*, while by *rational formal fan* we mean a $\mathbb{Q}$-linear formal fan. A-linear formal fans form categories, the corresponding functor categories.

If $F_\bullet$ is a formal fan, then the multiplicative monoid $\mathbb{Z}_{>0}$ acts on each F_n: if $l \in \mathbb{Z}_{>0}$, multiplication by l gives a map $[n] \to [n]$, which in turns gives a map $F_k \to F_k$. We can localise this action by setting $F^{\mathbb{Q}}_\bullet \coloneqq \varinjlim_{l \in \mathbb{Z}_{>0}} F_n$, which is the set of symbols a/l with $a \in F_n$ and $l \in \mathbb{Z}_{>0}$, and were we identify $a/l = a'/l'$ if there is $m \in \mathbb{Z}_{>0}$ such that $ml'a = mla'$. This gives a rational formal fan $F^{\mathbb{Q}}_\bullet$, that we will refer to as the *rational formal fan associated to F*.

If $\mathcal{X}$ is an algebraic stack satisfying Assumption **??** and $x \in \operatorname{Spec} k \to \mathcal{X}$ is a geometric point, then the degeneration fan $\mathbf{DF}(\mathcal{X}, x)_\bullet$ is the formal fan given by $\mathbf{DF}(\mathcal{X}, x)_n = \operatorname{Hom}((\Theta^n_k, 1), (\mathcal{X}, x))$, the set of pointed morphisms from $(\Theta^n_k, 1)$ to $(\mathcal{X}, x)$, see [8, Definition 3.2.2]. In principle, $\operatorname{Hom}((\Theta^n_k, 1), (\mathcal{X}, x))$ is a groupoid, but because $\mathcal{X}$ has separated inertia, it is equivalent to a set. Note that we consider all morphisms $\lambda\colon (\Theta^n_k, 1) \to (\mathcal{X}, x)$ and not just nondegenerate ones, i.e. those for which the induced homomorphism $\mathbb{G}^n_{m,k} \to \operatorname{Aut}(\lambda(0))$ has finite kernel. The associated rational formal fan $\mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_\bullet$ will be called the *rational degeneration fan* of x. The definitions are such that $\mathbf{DF}(\mathcal{X}, x)_1 = \mathbb{Z}\text{-}\mathrm{Filt}(x)$ and $\mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1 = \mathbb{Q}\text{-}\mathrm{Filt}(x)$.

For $\gamma \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_n$, we denote $v_1\gamma, \dots, v_n\gamma \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1$ the pullbacks of γ along the maps $[1] \to [n]$ given by the standard basis of $\mathbb{Z}^n$. We can think of the $v_i\gamma$ as being the vertices of γ.

Proposition 2.0.2. *Let k be a field, and let $\mathcal{X} = X/G$, where X is a quasi-separated scheme over k and G is a smooth affine algebraic group over k. Let $x \in \mathcal{X}(k)$ and let $x^* \in X(k)$ be a point representing x. For a homomorphism $\gamma\colon \mathbb{G}^n_{m,k} \to G$, which is given by commuting cocharacters $\gamma_1, \dots, \gamma_n$, we say that* $\lim \gamma x^*$ exists *if the map $\mathbb{G}^n_{m,k} \to X\colon t \mapsto \gamma(t)x^*$ extends to $\mathbb{A}^n_k$. Then*

$$\mathbf{DF}(\mathcal{X}, x)_n = \{\gamma\colon \mathbb{G}^n_{m,k} \to G \mid \lim \gamma x^* \textit{ exists}\}/\sim,$$

where $\gamma \sim \gamma^g$ if $g \in P(\gamma)$. Here $P(\gamma)$ is the attractor [6, Definition 1.3.2] for the conjugation action of $\mathbb{G}^n_{m,k}$ on G induced by γ.

Proof. This follows directly from [8, Theorem 1.4.8]. □

Proposition 2.0.3. *Let $\mathcal{X}$ be an algebraic stack satisfying Assumption* **??**, *and assume that $\mathcal{X}$ has a good moduli space $\pi\colon \mathcal{X} \to X$. Let $x\colon \operatorname{Spec}(k) \to \mathcal{X}$ be a geometric point. Then the map $\mathbf{DF}(\mathcal{X}, x)_n \to \mathbf{DF}(\mathcal{X}, x)_1^n \colon \gamma \mapsto (v_1\gamma, \dots, v_n\gamma)$ is injective for all $n \in \mathbb{Z}_{>0}$ and a bijection for $n = 2$. The same holds for $\mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_n \to (\mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1)^n$.*

Proof. It is enough to prove the result for $\mathbf{DF}(\mathcal{X}, x)_\bullet$. Since $\mathbf{DF}(\mathcal{X}, x)_\bullet$ only depends on the fibre of the good moduli space of $\mathcal{X}$ containing x, we may assume by **??** that $\mathcal{X} = X/\operatorname{GL}_{N,k}$, where $X = \operatorname{Spec} A$ is affine. Let $x^* \in X(k)$ be a point representing x. By Proposition 2.0.2 we have

$$\mathbf{DF}(\mathcal{X}, x)_n = \{\gamma\colon \mathbb{G}_{m,k}^n \to \operatorname{GL}_{N,k} \mid \lim \gamma x^* \text{ exists}\}/\sim .$$

Since X is affine, $\lim \gamma x^*$ exists if and only if $\lim \gamma_i x^*$ exists for all i. To see this, we look at the $\mathbb{Z}^n$-grading $\oplus_{l \in \mathbb{Z}^n} A_l = A$ that γ induces on A. Let $\varphi\colon A \to k$ be the homomorphism defining x. We have that $\lim \gamma x^*$ exists if and only if $\varphi(A_l) = 0$ unless $l \in \mathbb{Z}^n_{\geq 0}$, and that $\lim \gamma_i x^*$ exists if and only if $\varphi(A_l) = 0$ unless $l_i \geq 0$. From this the statement follows. In particular we have $P(\gamma) = P(\gamma_1) \cap \dots \cap P(\gamma_n)$.

Now suppose that $\gamma'\colon \mathbb{G}_{m,k}^n \to \operatorname{GL}_{N,k}$ is such that γ_i and γ_i' define the same element of $\mathbf{DF}(\mathcal{X}, x)_1$ for each i, that is, there are $g_i \in P(\gamma_i)$ such that $\gamma_i' = \gamma_i^{g_i}$. The γ_i' are contained in some maximal torus T' of $P(\gamma)$. If T is another maximal torus of $P(\gamma)$ containing the γ_i and $g \in P(\gamma)$ is such that $gTg^{-1} = T'$, then, for each i, γ_i^g and γ_i' are commuting representatives of the same element of $\mathbf{DF}(\mathcal{X}, x)_\bullet$, so $\gamma_i^g = \gamma_i'$. Therefore $\gamma^g = \gamma'$. This proves injectivity of $\mathbf{DF}(\mathcal{X}, x)_n \to (\mathbf{DF}(\mathcal{X}, x)_1)^n$.

Now, if $\alpha_1, \alpha_2 \in \mathbf{DF}(\mathcal{X}, x)_1$, then there is a maximal torus of $\operatorname{GL}_{N,k}$ contained in $P(\alpha_1) \cap P(\alpha_2)$. Thus there are two commuting representatives of α_1 and α_2, which gives $\gamma \in \mathbf{DF}(\mathcal{X}, x)_2$ such that $v_1\gamma = \alpha_1$ and $v_2\gamma = \alpha_2$. $\square$

For $a, b \in \mathbb{N}$, we denote $\binom{a}{b}$ the homomorphism $\mathbb{Z} \to \mathbb{Z}^2 \colon l \mapsto (la, lb)$, which gives a map $[1] \to [2]$ in Cone.

Definition 2.0.4 (Sum of two filtrations)**.** Let $\mathcal{X}$ be an algebraic stack satisfying Assumption **??** and assume that $\mathcal{X}$ has a good moduli space. Let $x\colon \operatorname{Spec}(k) \to \mathcal{X}$ be a geometric point. For $\lambda_1, \lambda_2 \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1$, let γ be the unique element of $\mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_2$ such that $v_1\gamma = \lambda_1$ and $v_2\gamma = \lambda_2$. We define the *sum* $\lambda_1 + \lambda_2 := \binom{1}{1}^*\gamma$, which is again an element of $\mathbf{DF}(\mathcal{X}, x)_1^{\mathbb{Q}}$.

Remark 2.0.5. The sum in $\mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1$ satisfies the following properties:

1. $\forall \lambda, \mu \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1, \quad \lambda + \mu = \mu + \lambda$,
2. $\forall \lambda \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1, \quad \lambda + 0 = \lambda$,
3. $\forall a, b \in \mathbb{Q}_{\geq 0}, \forall \lambda \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1, \quad (a + b)\lambda = a\lambda + b\lambda$.
4. $\forall \lambda, \mu \in \mathbf{DF}(\mathcal{X}, x)_1,\ \lambda + \mu \in \mathbf{DF}(\mathcal{X}, x)_1$.

Thus we can see $(\mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1, +, 0)$ as a commutative magma with zero on which $\mathbb{Q}_{\geq 0}$ acts in a compatible way. Note that the addition $+$ on $\mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1$ need not be associative.

We say that $\mu \in \mathbf{DF}(\mathcal{X}, x)_1$ is an opposite of $\lambda \in \mathbf{DF}(\mathcal{X}, x)_1$ if $\lambda + \mu = 0$. An element $\lambda \in \mathbf{DF}(\mathcal{X}, x)_1$ may have several opposites.

The degeneration fan encodes geometric information about the stack around a point. The following proposition is an example of this.

Proposition 2.0.6. *Let $\mathcal{X}$ be an algebraic stack satisfying Assumption* **??** *and assume that $\mathcal{X}$ has a good moduli space $\pi\colon \mathcal{X} \to X$. Let $x\colon \operatorname{Spec}(k) \to \mathcal{X}$ be a geometric point. Then x is closed in the fibre $\pi^{-1}\pi(x)$ if and only if every element of $\mathbf{DF}(\mathcal{X}, x)_1$ has an opposite.*

Proof. We can write $\pi^{-1}\pi(x) = \operatorname{Spec} A/G$ by **??**, where G is reductive and x is given by a closed point $x^* \in (\operatorname{Spec} A)(k)$ whose stabiliser is G.

If x is closed, then $\mathbf{DF}(\mathcal{X}, x)_\bullet$ is isomorphic to the degeneration fan of a point of BG. Every element λ of $\mathbf{DF}(\mathcal{X}, x)_1$ has thus an opposite, which is given by the inverse of a cocharacater of G defining λ.

If x is not closed, then there is $\lambda \in \mathbf{DF}(\mathcal{X}, x)_1$ such that $\lambda(0)$ is closed [3, Lemma 3.24]. Choose a representative $\lambda^* \in \Gamma^{\mathbb{Z}}(G)$ of λ such that $(\lambda^*)^{-1}$ represents an opposite of λ. This gives a λ^*-equivariant map $u\colon \mathbb{A}^1_k \to \operatorname{Spec} A$. The $(\lambda^*)^{-1}$-equivariant map $v\colon \mathbb{A}^1_k \to \operatorname{Spec} A$ corresponding to the choice of opposite for λ glues with u to give $w\colon \mathbb{P}^1_k \to \operatorname{Spec} A$. The image of w is a single point, since the target is affine. Thus $\lambda(0) = \lambda(1)$, a contradiction. $\square$

3 The case of nilpotent quiver representations

In this section, k is an algebraically closed field.

We start by recalling certain basic definitions and fixing notations. A *quiver* Q is a finite directed graph. It consists of a finite set Q_0 of *vertices*, a finite set Q_1 of *arrows* and *source* and *target* maps $s, t\colon Q_1 \to Q_0$. If $\alpha \in Q_1$, we write $\alpha\colon i \to j$ if $i = s(\alpha)$ and $j = t(\alpha)$. A *representation* E of Q over the field k consists of a vector space E_i for every vertex $i \in Q_0$ and a linear map $x_\alpha\colon E_i \to E_j$ for every arrow $\alpha\colon i \to j$ of Q. We will sometimes denote $E = ((E_i)_{i\in Q_0}, (x_\alpha)_{\alpha\in Q_1})$. We say that E is *finite dimensional* if every E_i is. A morphism of representations $E \to E'$, where $E' = ((E'_i)_i, (x'_\alpha)_\alpha)$, is a family $(f_i)_{i\in Q_0}$ of maps $f_i\colon E_i \to E'_i$ such that $x'_\alpha \circ f_i = f_j \circ x_\alpha$ for every arrow $\alpha\colon i \to j$.

A *dimension vector* d for Q is the data of a number $d_i \in \mathbb{N}$ for every vertex i of Q. The *representation space* for the dimension vector d is

$$\operatorname{Rep}(Q, d) = \bigoplus_{\alpha\colon i\to j} \operatorname{Hom}(k^{d_i}, k^{d_j}).$$

The group $G(d) = \prod_{i\in Q_0} \mathrm{GL}_{d_i,k}$ acts on $\operatorname{Rep}(Q, d)$ by

$$(g_i)\cdot(x_\alpha)_\alpha = (g_j x_\alpha g_i^{-1})_{\alpha\colon i\to j}.$$

The quotient stack $\mathscr{R}ep(Q, d) := \operatorname{Rep}(Q, d)/G(d)$ is referred to as the *moduli stack of representations of Q of dimension vector d.*

We now fix a quiver Q, a dimension vector d and we denote $G = G(d)$, $V = \operatorname{Rep}(Q, d)$ and $\mathcal{X} = \mathscr{R}ep(Q, d)$. If k is of positive characteristic, then we assume that G is a torus, that is $d_i = 0$ or 1 for every vertex i. In any case, G is linearly reductive and $\mathscr{R}ep(Q, d)$ has a good moduli space. We fix a k-point x of V, and, by slight abuse of notation, we denote the image of x under $V \to \mathcal{X}$ also by x. The point x gives a representation $E = ((k^{d_i})_i, (x_\alpha)_\alpha)$, and we will denote $L = L_E$ the lattice of subrepresentations of E, which is an artinian lattice **??**.

Proposition 3.0.1. *There is a canonical isomorphism*

$$\mathbf{DF}(\mathcal{X}, x)_\bullet \cong \mathbf{DF}(L)_\bullet$$

of formal fans.

Proof. By Proposition 2.0.2, there is a injection $\mathbf{DF}(\mathcal{X}, x)_\bullet \to \mathbf{DF}(BG, \mathrm{pt})_\bullet$ that identifies $\mathbf{DF}(\mathcal{X}, x)_\bullet$ with a subfan of $\mathbf{DF}(BG, \mathrm{pt})_\bullet$ by

$$\mathbf{DF}(\mathcal{X}, x)_n = \{\gamma \in \mathbf{DF}(BG, \mathrm{pt})_n \mid \lim \gamma x \text{ exists}\},$$

and where

$$\mathbf{DF}(BG, \mathrm{pt})_n = \mathrm{Hom}(\mathbb{G}_{m,k}^n, G)/\sim,$$

with $\gamma \sim \gamma^g$ if $g \in P(\gamma)(k)$.

Let Q° be the quiver with $Q_0^\circ = Q_0$ and $Q_1^\circ = \varnothing$. Then E defines a representation E° of Q° which is just the vector space $\bigoplus_{i \in Q_0} E_i$ endowed with its grading by the set Q_0. The lattice L_{E° of subrepresentations of E° is the product

$$L_{E^\circ} = \prod_{i \in Q_0} L_{E_i}$$

of the lattices of vector subspaces of each E_i. Therefore Proposition **??** gives a canonical isomorphism

$$\mathbf{DF}(BG, \mathrm{pt})_\bullet = \prod_{i \in Q_0} \mathbf{DF}(B\,\mathrm{GL}_{d_i,k}, \mathrm{pt})_\bullet \cong \prod_{i \in Q_0} \mathbf{DF}(L_{E_i})_\bullet \cong \mathbf{DF}(L^\circ)_\bullet \colon \gamma \mapsto F^\gamma.$$

The injection of lattices $L \to L^\circ$ gives an injection of formal fans $\mathbf{DF}(L)_\bullet \to \mathbf{DF}(L^\circ)_\bullet$. Now if $\gamma \in \mathbf{DF}(BG, \mathrm{pt})_n$ is represented by a group homomorphism $\mathbb{G}_{m,k}^n \to G$ (that we also denote γ), then the associated filtration $F^\gamma \in \mathbf{DF}(L^\circ)_n$ is given by

$$(F^\gamma_{\geq c})_i = \bigoplus_{\substack{\chi \in \mathbb{Z}^n \\ \chi \geq c}} (E_i)^\gamma_\chi, \quad c \in \mathbb{Z}^n,$$

where $(E_i)^\gamma_\chi$ is the eigenspace of E_i where $\mathbb{G}_{m,k}^n$ acts, via γ, through the character $\chi \in \Gamma_{\mathbb{Z}}(\mathbb{G}_{m,k}^n) \cong \mathbb{Z}^n$. The direct sum decomposition of the E_i allows us to write x in coordinates $x = (x_{\alpha,\chi,\chi'})_{\alpha \in Q_1, \chi, \chi' \in \mathbb{Z}^n}$ where, if $\alpha \colon i \to j$, then $x_{\alpha,\chi,\chi'} \colon (E_i)^\gamma_{\chi'} \to (E_j)^\gamma_\chi$ is the corresponding component of x_α. For F^γ to be a filtration of E we need that

$$x_\alpha((E_i)^\gamma_{\chi'}) \subset \bigoplus_{\substack{\chi \in \mathbb{Z}^n \\ \chi \geq \chi'}} (E_j)^\gamma_\chi$$

for all $\alpha \colon i \to j$ and all $\chi' \in \mathbb{Z}^n$. This is equivalent to $x_{\alpha,\chi,\chi'}$ being 0 whenever $\chi < \chi'$, for all $\alpha \in Q_1$. On the other hand,

$$\gamma(t)x = ((\chi - \chi')(t) x_{\alpha,\chi,\chi'})_{\alpha,\chi,\chi'},$$

so $\lim \gamma x$ exists if and only if $x_{\alpha,\chi,\chi'} = 0$ whenever $\chi < \chi'$. Therefore $\mathbf{DF}(\mathcal{X}, x)_\bullet$ and $\mathbf{DF}(L)_\bullet$ are the same subfan under the isomorphism $\mathbf{DF}(BG, \mathrm{pt})_\bullet \cong \mathbf{DF}(L^\circ)_\bullet$, so we get an isomorphism between $\mathbf{DF}(\mathcal{X}, x)_\bullet$ and $\mathbf{DF}(L)_\bullet$, as desired. □

We denote $\mathbf{DF}(\mathcal{X}, x)_n \to \mathbf{DF}(L)_n \colon \gamma \mapsto F^\gamma$ the isomorphism above.

Definition 3.0.2. We say that a representation $M = ((M_i)_{i \in Q_0}, (y_\alpha)_{\alpha \in Q_1})$ of Q is null if $y_\alpha = 0$ for all $\alpha \in Q_1$.

Proposition and Definition 3.0.3. Let M be a finite dimensional representation of Q of dimension vector d, corresponding to the k-point y of $\mathscr{Rep}(Q, d)$. The following are equivalent:

1. There is a filtration of M whose associated graded is null.
2. We have $0 \in \overline{\{y\}}$ inside $\mathscr{Rep}(Q, d)$.

If these conditions hold we say that M is a *nilpotent* representation of Q.

Proof. Since $\mathscr{R}ep(Q,d)$ has a good moduli space, $0 \in \overline{\{y\}}$ if and only if there is $\lambda \in \mathbf{DF}(\mathscr{R}ep(Q,d),y)_1$ with $\mathrm{ev}_0\,\lambda = 0$ by [3, Lemma 3.24]. This gives a filtration F^λ of M whose associated graded representation is null, and conversely any such filtration gives a suitable λ. □

Proposition 3.0.4. *Let M be a nilpotent representation of Q of dimension vector d. Then:*

1. *Any subquotient of M is also nilpotent.*
2. *The representation M is semisimple if and only if M is null.*

Proof. A Jordan-Hölder filtration of a subquotient N of M can be extended to a Jordan-Hölder filtration of M, whose associated graded must be null by hypothesis. Thus the associated graded of a Jordan-Hölder filtration of N is also null.

The representation M is semisimple if it is isomorphic to the associated graded representation of a Jordan-Hölder filtration, which is the null representation by hypothesis. □

Proposition 3.0.5. *Suppose that E is nilpotent and let $\lambda \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X},x)_1$. Then*

$$\langle \lambda, \mathcal{X}^{\max} \rangle = \langle F^\lambda, \mathfrak{l} \rangle,$$

where $\langle \lambda, \mathcal{X}^{\max} \rangle$ denotes Kempf's intersection number (Definition **??***) and $\langle F^\lambda, \mathfrak{l}\rangle$ is the complementedness of F^λ as a filtration of the lattice L (Definition* **??***).*

Proof. It is enough to assume that $\lambda \in \mathbf{DF}(\mathcal{X},x)_1$. We choose a cocharacter $\mathbb{G}_{m,k} \to G$ representing λ that we still denote by λ. Then λ induces a direct sum decomposition $\mathrm{Rep}(Q,d) = V = \bigoplus_{l\in\mathbb{Z}} V_l$. Let $p_l \colon V \to V_l$ be the induced projections and let $\Xi_{x,\lambda} = \{l \in \mathbb{Z} \mid\ p_l(x) \neq 0\}$. Note that $\Xi_{x,\lambda} \subset \mathbb{N}$ since $\lim \lambda x$ exists.

Claim 3.0.6. $\langle \lambda, \mathcal{X}^{\max} \rangle = \inf \Xi_{x,\lambda}$.

Proof. Note that $\mathcal{X}^{\max} = V^G/G$. Since G is linearly reductive, there is a unique splitting $V = V^G \oplus V'$, with V' a G-subrepresentation of V. Choose a basis $v_1,\dots,v_m$ of V of eigenvectors for λ and $0 \le r \le r' \le m$ such that $v_1,\dots,v_r$ is a basis of V^G, $v_1,\dots,v_{r'}$ is a basis of V_0 and the v_i with $i > r$ are in V'. Let $n_i \in \mathbb{Z}$ such that $v_i \in V_{n_i}$. Taking Spec of the cartesian square

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & V^G \arrow[d] \\ \mathbb{A}^1_k \arrow[r, "t\mapsto\lambda(t)x"] & V \end{tikzcd}$$

we get

$$\begin{tikzcd} k[t]/I \arrow[dr, phantom, "\ulcorner"] & k[v_1^\vee,\dots,v_r^\vee] \arrow[l] \\ k[t] \arrow[u] & k[v_1^\vee,\dots,v_m^\vee], \arrow[l, "x_it^{n_i}\leftarrow\!\shortmid v_i^\vee"'] \arrow[u] \end{tikzcd}$$

where I is the ideal generated by the t^{n_i} with $x_i \neq 0$ and $i > r$. Since $0 \in \overline{Gx}$, we have $x_i = 0$ for $i \le r$. Therefore $I = (t^a \mid\ a \in \Xi_{x,\lambda}) = \left(t^{\inf \Xi_{x,\lambda}}\right)$ and the claim follows. □

Claim 3.0.7. For any $a \in \mathbb{N}$, we have that $F^\lambda_{\ge c}/F^\lambda_{\ge c+a}$ is semisimple for all $c \in \mathbb{Z}$ if and only if $\inf \Xi_{x,\lambda} \ge a$.

Proof. This induces a direct sum decomposition $E_i = \bigoplus_{l\in\mathbb{Z}}(E_i)_l$ of every E_i, $i \in Q_0$. The subquotient $F^\lambda_{\geq c}/F^\lambda_{\geq c+a}$ can be written in coordinates as $(x_{\alpha,nm})_{\alpha,c\leq n,m<c+a}$ where, if $\alpha\colon i \to j$, then $x_{\alpha,nm}\colon (E_i)_m \to (E_j)_n$ is the corresponding component of $x_\alpha\colon E_i \to E_j$. Since it is nilpotent, the representation $F^\lambda_{\geq c}/F^\lambda_{\geq c+a}$ is semisimple if and only if $x_{\alpha,nm} = 0$ for all $\alpha \in Q_1$ and $c \leq n, m < c + a$. This holds for all $c \in \mathbb{Z}$ if and only if $x_{\alpha,nm} = 0$ whenever $|n - m| < a$. Note that $\lambda(t)x = (t^{n-m}x_{\alpha,nm})_{\alpha\in Q_1,\, n,m\in\mathbb{Z}}$. Therefore $\Xi_{x,\lambda} = \{n - m \mid \exists\alpha \in Q_1,\ x_{\alpha,nm} \neq 0\}$. The claim follows. □

The complementedness of F^λ is the supremum of the set of those $a \in \mathbb{N}$ such that $F_{\geq c}/F_{\geq c+a}$ is semisimple for all $c \in \mathbb{Z}$. Thus $\inf \Xi_{x,\lambda} = \langle F^\lambda, \mathfrak{l}\rangle$ by the second claim, and the result follows from the first claim. □

4 Lamps and linearly lit good moduli stacks

In this section we define a family of algebraic stacks, linearly lit good moduli stacks stacks, for which there is an artinian lattice L_x associated to every geometric point x, and a canonical isomorphism between the set of $\mathbb{Q}$-filtrations of L_x and the set $\mathbb{Q}\text{-}\mathrm{Filt}(x)$ of $\mathbb{Q}$-filtrations of x, the latter defined in the stacky sense. For the lattice L_x to be canonical, we need to introduce a piece of extra structure on stacks, what we call a lamp. Morally, a lamp contains the information of what graded points have nonnegative weights only. We will see that stacks parameterising objects in an abelian category and admitting a good moduli space are linear for the natural lamp.

Definition 4.0.1. Let $\mathcal{X}$ be an algebraic stack satisfying Assumption **??**. A *lamp* on $\mathcal{X}$ is a closed and open substack $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})_{\geq 0}$ of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ satisfying the following convexity property: If k is a field and $f \in \mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X})(k)$ is a k-point such that for the maps $e_i\colon \mathbb{Q} \to \mathbb{Q}^n$ corresponding to the standard basis of $\mathbb{Q}^n$ we have that e_i^*f is in $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})_{\geq 0}$, then for every order-preserving map $h\colon \mathbb{Q} \to \mathbb{Q}^n$ we have that h^*f is in $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})_{\geq 0}$. A *lit stack* $\mathcal{X}$ over an algebraic space B is a stack $\mathcal{X}$ over B satisfying Assumption **??** endowed with a lamp.

If $f\colon \mathcal{X} \to \mathcal{Y}$ is a morphism between stacks satisfying Assumption **??** and $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y})_{\geq 0}$ is a lamp on $\mathcal{Y}$, then $f^*(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y})_{\geq 0}) := \mathrm{Grad}(f)^{-1}(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y})_{\geq 0}) \subset \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})_{\geq 0}$ is a lamp on $\mathcal{X}$, called the *pullback lamp.* If $\mathcal{X}$ is lit with lamp $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})_{\geq 0}$, then we say that the morphism f is *lit* if $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})_{\geq 0} = f^*(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y})_{\geq 0})$.

Example 4.0.2. Let G be a linear algebraic group over a field k. One source of lamps on BG are partial compactifications $\overline{G}$ of G. By this we mean an affine algebraic monoid $(\overline{G}, e)$ over k such that the unit group $\overline{G}^\times$, which is an open subscheme of $\overline{G}$, is dense in $\overline{G}$, together with an isomorphism $G \cong \overline{G}^\times$ of algebraic groups (compare with [11, Definition 2.1]).

Given such a partial compactification, we can define the associated lamp $\mathrm{Grad}_{\mathbb{Q}}(BG)_{\geq 0}$ to be the closed and open substack of $\mathrm{Grad}_{\mathbb{Q}}(BG)$ whose components form the image of the map

$$\mathbb{Q}\text{-}\mathrm{Filt}(\overline{G}/G, e) \to \pi_0(\mathrm{Grad}_{\mathbb{Q}}(\overline{G}/G)) \to \pi_0(\mathrm{Grad}_{\mathbb{Q}}(BG))$$

induced by taking associated graded points. The convexity property in the definition of lamp holds because $\overline{G}$ is affine.

Example 4.0.3. Let k be a field. A particularly important example of the above is the partial compactification $\mathrm{GL}_{n,k} \subset \mathrm{Mat}_{n\times n,k}$ of $\mathrm{GL}_{n,k}$ given by the algebraic monoid of $n \times n$ matrices. More generally, we consider the partial compactification $\mathrm{GL}(d) =$

$\prod_{i\in Q_0} \mathrm{GL}_{d_i,k} \subset \mathrm{Mat}(d) \coloneqq \prod_{i\in Q_0} \mathrm{Mat}_{d_i\times d_i,k}$ of $\mathrm{GL}(d)$, where d is a dimension vector of some quiver Q. This gives a lamp $\mathrm{Grad}_{\mathbb{Q}}(B\,\mathrm{GL}(d))_{\geq 0}$ that we refer to as the *standard lamp* on $B\,\mathrm{GL}(d)$.

A point p of $\mathrm{Grad}_{\mathbb{Q}}(B\,\mathrm{GL}_{n,k})$ represented by a rational cocharacter

$$\lambda(t) = \mathrm{diag}(t^{a_1}, \ldots, t^{a_n})$$

of $\mathrm{GL}_{n,k}$ is in $\mathrm{Grad}_{\mathbb{Q}}(B\,\mathrm{GL}_{n,k})_{\geq 0}$ precisely when $a_i \geq 0$ for all i.

Example 4.0.4. We can use the example above to define a canonical lamp on stacks of quiver representations over some field k. Let Q be a quiver and let d be a dimension vector for Q. We have a canonical map

$$\mathscr{Rep}(Q,d) \to B\,\mathrm{GL}(d)$$

and we define the *standard lamp* on $\mathscr{Rep}(Q,d)$ to be the pullback of the standard lamp on $B\,\mathrm{GL}(d)$ along this map.

We will consider the following assumption on algebraic stacks $\mathcal{X}$ defined over an algebraically closed field k.

Assumption 4.0.5. The stack $\mathcal{X}$ is of finite presentation over k, it has affine diagonal and a good moduli space $\pi\colon \mathcal{X} \to X$.

Definition 4.0.6. Let $\mathcal{X}$ be an algebraic stack over an algebraically closed field k satisfying Assumption 4.0.5, endowed with a lamp $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})_{\geq 0}$. Let us denote $\pi\colon \mathcal{X} \to X$ the good moduli space of $\mathcal{X}$. We say that $\mathcal{X}$ is *linearly lit* if for every closed k-point y of $\mathcal{X}$, there is a quiver Q, a dimension vector d of Q and a lit pointed closed immersion

$$i\colon (\mathcal{F}, y) \to (\mathscr{Rep}(Q,d), 0)\,,$$

where $\mathcal{F} = \pi^{-1}\pi(y)$ is the fibre of π containing y.

Remark 4.0.7. Since every point x of $\mathcal{F}$ above specialises to y, $i(x)$ specialises to 0, that is, it corresponds to a nilpotent representation of Q.

Remark 4.0.8. Note that if k is of positive characteristic, the assumption that $\mathcal{X}$ has a good moduli space forces $G(d)$ to be linearly reductive, so a torus.

Definition 4.0.9. Let $\mathcal{X}$ be a linearly lit good moduli stack over k, and let $x \in \mathcal{X}(k)$. We define a relation $\leq$ on $\mathbf{DF}(\mathcal{X},x)_1$ as follows. For $\lambda, \mu \in \mathbf{DF}(\mathcal{X},x)_1$, let $\gamma \in \mathbf{DF}(\mathcal{X},x)_2$ be the unique element such that $v_1\gamma = \lambda$ and $v_2\gamma = \mu$. Consider the graded point g given by the composition

$$B\mathbb{G}_{m,k} \xrightarrow{B\binom{-1}{1}} B\mathbb{G}_{m,k}^2 \xrightarrow{0} \Theta_k^2 \xrightarrow{\gamma} \mathcal{X}.$$

Then we say $\lambda \leq \mu$ if g lies in $\mathrm{Grad}_{\geq 0}(\mathcal{X})$. Here we denote $\binom{-1}{1}\colon \mathbb{G}_{m,k} \to \mathbb{G}_{m,k}^2\colon t \mapsto (t^{-1}, t)$.

This gives $(\mathbf{DF}(\mathcal{X},x)_\bullet, \leq)$ the structure of a formal fan with relation (Definition **??**). We denote $L_x = L_{DF(x)}$, endowed with the relation $\leq$.

Lemma 4.0.10. *Let Q be a quiver and d a dimension vector for Q. Let x be a point of $\mathscr{Rep}(Q,d)$ corresponding to a representation E with lattice of subrepresentations L. Then the isomorphism $\mathbf{DF}(\mathscr{Rep}(Q,d),x)_\bullet \xrightarrow{\sim} \mathbf{DF}(L)_\bullet$ of Proposition 3.0.1 is indeed an isomorphism of formal fans with relation, where $\mathscr{Rep}(Q,d)$ is equipped with the standard lamp.*

Proof. Omitted. ◻

Proposition 4.0.11. *Let $\mathcal{X}$ be a linearly lit good moduli stack over k, and let $x \in \mathcal{X}(k)$. Then L_x is an artinian lattice and there is a canonical isomorphism $(\mathbf{DF}(L_x)_\bullet, \leq) \to (\mathbf{DF}(\mathcal{X}, x)_\bullet, \leq)$ of formal fans with relation.*

Proof. By Definition 4.0.6, there is a quiver Q, a dimension vector d and a point $z \in \mathscr{Rep}(Q, d)(k)$ corresponding to a nilpotent representation E of Q such that $(\mathbf{DF}(\mathcal{X}, x)_\bullet, \leq)$ is isomorphic to $(\mathbf{DF}(\mathscr{Rep}(Q, d), z)_\bullet, \leq)$ as formal fans with relation. By Proposition 3.0.1 and Lemma 4.0.10, $(\mathbf{DF}(\mathscr{Rep}(Q, d), z)_\bullet, \leq)$ is isomorphic to $(\mathbf{DF}(M)_\bullet, \leq)$, where M is the lattice of subrepresentations of E, which is an artinian lattice. Therefore, by Proposition **??**, $(L_x, \leq)$ is isomorphic to $(M, \leq)$ and there is a canonical isomorphism $(\mathbf{DF}(L_x)_\bullet, \leq) \cong (\mathbf{DF}(\mathcal{X}, x)_\bullet, \leq)$ of formal fans with relation, independent of the choice of Q, d, E or of the isomorphism $(\mathbf{DF}(\mathcal{X}, x)_\bullet, \leq) \cong (\mathbf{DF}(\mathscr{Rep}(Q, d), z)_\bullet, \leq)$. ◻

If $\lambda \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1$, we will use the notation F^λ to denote the $\mathbb{Q}$-filtration of L_x under the isomorphism above.

If $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})_{\geq 0} \subset \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ is a lamp on the algebraic stack $\mathcal{X}$, we endow $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ with the pullback lamp along the canonical map $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \to \mathcal{X}$.

Proposition 4.0.12. *Let $\mathcal{X}$ be a linearly lit good moduli stack over k. Then every quasi-compact component of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ is linearly lit. Moreover, if $x \in \mathcal{X}(k)$ and $\lambda \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1$, then there is a canonical isomorphism (defined in the proof) of lattices $L_{\mathrm{gr}\,\lambda} \cong \mathrm{Grad}_{F^\lambda}(L_x)$.*

Remark 4.0.13. If $\mathcal{X}$ has a norm on graded points, then every component of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ is quasi-compact by [8, Proposition 3.8.2].

Proof. A quasi-compact component $\mathcal{Z}$ of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ satisfies Assumption 4.0.5. Let $\pi\colon \mathcal{X} \to X$ be the good moduli space of $\mathcal{X}$, and let $\rho\colon \mathcal{Z} \to Z$ be that of $\mathcal{Z}$. A point $g \in \mathcal{Z}(k)$ lies over a k-point $x = u(g)$ of $\mathcal{X}$. Let $\mathcal{F} = \pi^{-1}\pi(x)$. Then $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{F}) \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ is a closed immersion (for example by the same argument as in Proposition **??**). Let $\mathcal{Y}$ be the component of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{F})$ containing g. The good moduli space of $\mathcal{F}$ is $\mathrm{Spec}(k) = \mathrm{pt}$. Let $Y = \mathrm{pt} \times_{\pi(x), X} Z$ and form a cube

$$\begin{tikzcd}
& Y \arrow[rr] \arrow[dd] \arrow[drr, phantom, "\ulcorner", very near start] & & Z \arrow[dd] \\
\mathcal{Y} \arrow[ur] \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\ulcorner", very near start] & & \mathcal{Z} \arrow[ur, "\rho"] \arrow[dd] & \\
& \mathrm{pt} \arrow[rr] & & X \\
\mathcal{F} \arrow[ur] \arrow[rr] \arrow[urr, phantom, "<", very near start] & & \mathcal{X} \arrow[ur, "\pi"'] &
\end{tikzcd}$$

where the marked faces are cartesian. Thus the face $Y, Z, \mathcal{Y}, \mathcal{Z}$ is also cartesian and therefore $\mathcal{Y} \to Y$ is a good moduli space. It is thus enough to prove the statement for the linearly lit stack $\mathcal{F}$. By embedding $\mathcal{F} \hookrightarrow \mathscr{Rep}(Q, d)$, we may assume $\mathcal{X} = \mathscr{Rep}(Q, d)$, which is linearly lit by Theorem 5.0.6.

The component of $\mathrm{Grad}_{\mathbb{Q}}(\mathscr{Rep}(Q, d))$ containing g is isomorphic to $\mathrm{Rep}(Q, d)^{\lambda, 0}/L(\lambda)$ for some rational one-parameter subgroup λ of $G(d)$ [8, Theorem 1.4.8]. Now λ induces a direct sum decomposition $k^{d_i} = \bigoplus_{c \in \mathbb{Q}} k_c^{d_i}$ for each vertex $i \in Q_0$. Define a quiver Q' with set of vertices

$$Q'_0 = \{(i, c) \in Q_0 \times \mathbb{Q} \mid \; k_c^{d_i} \neq 0\}.$$

An arrow $\alpha\colon (i,c) \to (j,c)$ in Q' is just an arrow $\alpha\colon i \to j$ in Q, and there are no arrows $(i,c) \to (j,c')$ for $c \neq c'$.

We have a dimension vector d' on Q' given by $d'_{i,c} = \dim k_c^{d_i}$. The fixed points of λ in $\mathrm{Hom}(k^{d_i}, k^{d_j})$ are $\bigoplus_{c\in\mathbb{Q}} \mathrm{Hom}(k_c^{d_i}, k_c^{d_j})$, and the centraliser $L(\lambda)$ of λ in $G(d)$ is identified with $G(d')$. Therefore we have an isomorphism $\mathrm{Rep}(Q,d)^{\lambda,0}/L(\lambda) \cong \mathscr{Rep}(Q',d')$.

If $\beta\colon \mathbb{G}_{m,k} \to L(\lambda)$ is a cocharacter, β defines a grading $k_c^{d_i} = \bigoplus_{c'\in\mathbb{Z}} (k_c^{d_i})_{c'}^{\beta}$. If $\beta'\colon \mathbb{G}_{m,k} \to L(\lambda) \to G(d)$ denotes the composition of β and the inclusion $L(\lambda) \to G(d)$, then β' defines a grading $k^{d_i} = \bigoplus_{c'\in\mathbb{Z}} (k^{d_i})_{c'}^{\beta'}$ and we have $(k^{d_i})_{c'}^{\beta'} = \bigoplus_{c\in\mathbb{Q}} (k_c^{d_i})_{c'}^{\beta}$. We have $\beta \geq 0$ if and only if $(k_c^{d_i})_{c'}^{\beta} = 0$ whenever $c' < 0$, while $\beta' \geq 0$ if and only if $(k^{d_i})_{c'}^{\beta'} = 0$ whenever $c' < 0$. Thus $\beta \geq 0$ if and only if $\beta' \geq 0$ and hence the isomorphism $\mathrm{Rep}(Q,d)^{\lambda,0}/L(\lambda) \cong \mathscr{Rep}(Q',d')$ is compatible with lamps. This proves the first part of the proposition.

Now let $\lambda \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X},x)_1$. We define a map $o\colon \mathbf{DF}^{\mathbb{Q}}(\mathcal{X},x)_1 \to \mathbf{DF}^{\mathbb{Q}}(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}), \mathrm{gr}\,\lambda)_1$ as follows. If $\mu \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X},x)_1$, let $\gamma \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X},x)_2$ be the unique element such that $v_1\gamma = \mu$ and $v_2\gamma = \lambda$ (Proposition 2.0.3). Noting that γ can be seen as an element of $\mathbf{DF}^{\mathbb{Q}}(\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}), \lambda)_1$, we define $o(\mu) \in \mathbf{DF}^{\mathbb{Q}}(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}), \mathrm{gr}\,\lambda)_1$ to be the image of γ under the associated graded map $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$. The definition of o does not depend on any choices.

If $\mathcal{X} = \mathscr{Rep}(Q,d)$, then the point x corresponds to a representation E of Q and $L_x \cong L_E$. One then sees easily by hand that there is an identification $v\colon L_{\mathrm{gr}\,\lambda} \xrightarrow{\sim} \mathrm{Grad}_{F^\lambda}(L_x)$ which a priori depends on the choices made. The map o is then identified with

$$o'\colon L_x \to \mathrm{Grad}_{F^\lambda}(L_x)\colon a \mapsto ((a \wedge F^\lambda_{\geq c}) \vee F^\lambda_{>c})_{c\in\mathbb{Q}}.$$

This proves that o is injective, that $o(L_x) \subset L_{\mathrm{gr}\,\lambda}$ and that the induced map $o\colon L_x \to L_{\mathrm{gr}\,\lambda}$ is a map of lattices.

In general, a choice of embedding $\mathcal{F} \hookrightarrow \mathscr{Rep}(Q,d)$ gives a commuting triangle

$$\begin{tikzcd} & L_{\mathrm{gr}\,\lambda} \arrow[d, "v", "\wr"'] \\ L_x \arrow[ur, "o"] \arrow[r, "o'"'] & \mathrm{Grad}_{F^\lambda}(L_x). \end{tikzcd}$$

For each $c \in \mathbb{Q}$, $o(F^\lambda_{>c})$ has a unique complement $o(F^\lambda_{>c})^\circ$, since this is true for o'. We define a map

$$\begin{aligned} u\colon\ & \mathrm{Grad}_{F^\lambda}(L_x) \to L_{\mathrm{gr}\,\lambda} \\ & (a_c)_{c\in\mathbb{Q}} \mapsto \bigvee_{c\in\mathbb{Q}} o(a_c) \wedge o(F^\lambda_{>c})^\circ. \end{aligned}$$

This map is an isomorphism, since the analogous map for o' is. Now u does not depend on choices, and it is the desired canonical isomorphism. In fact, $u = v^{-1}$. □

Corollary 4.0.14. *Let $\mathcal{X}$ be a linearly lit good moduli stack over k and let $x \in \mathcal{X}(k)$. Then there is a canonical isomorphism $\mathbb{Q}^\infty$-$\mathrm{Filt}(\mathcal{X},x) \cong \mathbb{Q}^\infty$-$\mathrm{Filt}(L_x)$ between the set of polynomial filtrations of x and that of polynomial filtrations of the lattice L_x.*

5 Examples of linearly lit stacks

We show here that many algebraic stacks parametrising objects in an abelian category and admitting a good moduli space are naturally linearly lit. Out main tool for that purpose is Theorem 5.0.4, that we hope can be applied to essentially all good moduli stacks of linear origin.

Let k be an algebraically closed field and let $\mathcal{X}$ be a quasi-separated algebraic stack locally of finite type over k. If $x \in \mathcal{X}(k)$ is a closed points, then there is a unique reduced closed substack $\mathcal{Z}$ of $\mathcal{X}$ whose topological space is just $\{x\}$. In fact, $\mathcal{Z} \cong BG_x$ is the residual gerbe at x. Let $\mathcal{I}$ be the ideal sheaf defining $\mathcal{Z}$ and let $\iota\colon BG_x \to \mathcal{X}$ denote the closed immersion. Then $\iota^*\mathcal{I} = \mathcal{I}/\mathcal{I}^2$ is a coherent sheaf on BG_x, that is, a finite dimension representation of G_x. Its dual $N_x = (\iota^*\mathcal{I})^\vee$ is called the *normal space* of $\mathcal{X}$ at x. If G_x is smooth, then N_x is also the tangent space of $\mathcal{X}$ at x. We may also consider the quotient stack $\mathcal{N}_x \coloneqq N_x/G_x$, which equals the relative spectrum $\mathcal{N}_x = \operatorname{Spec}_{BG_x} \operatorname{Sym} \iota^*\mathcal{I}$, and call it the *normal stack* of $\mathcal{X}$ at x. We have representable morphisms $\mathcal{N}_x \to BG_x$ and $BG_x \to \mathcal{X}$, so if $\mathcal{X}$ is lit, then BG_x and $\mathcal{N}_x$ inherit lamps from $\mathcal{X}$ by pullback along these maps.

Proposition 5.0.1. *Let k be an algebraically closed field, and let $\mathcal{X}$ be a quasi-separated algebraic stack locally of finite type over k. Suppose $\mathcal{X}$ has a good moduli space $\pi\colon \mathcal{X} \to X$ with affine diagonal. Let $x \in \mathcal{X}(k)$ be a closed point and denote $\mathcal{F} = \pi^{-1}\pi(x)$ the fibre of π containing x. Let $\mathcal{N}_x$ be the normal stack of $\mathcal{X}$ at x.*

Then there is a pointed closed immersion $\iota\colon (\mathcal{F}, x) \to (\mathcal{N}_x, 0)$ and a commuting triangle

(1)
$$\begin{tikzcd} & BG_x \arrow[dl] \arrow[dr] & \\ \mathcal{F} \arrow[rr, "\iota"] & & \mathcal{N}_x \end{tikzcd}$$

where the arrows $BG_x \to \mathcal{F}$ and $BG_x \to \mathcal{N}_x$ are the natural identifications of BG_x with the residual gerbes of $\mathcal{F}$ and $\mathcal{N}_x$ at x and 0.

Moreover, if $\mathcal{X}$ is lit, then ι is a lit closed immersion, where $\mathcal{F}$ and $\mathcal{N}_x$ are endowed with the induced lamps.

Proof. Since x is closed and $\mathcal{X}$ has a good moduli space, the stabiliser G_x is linearly reductive and thus $\mathcal{N}_x$ has a good moduli space $\rho\colon \mathcal{N}_x \to W$. Call $\mathcal{R} = \rho^{-1}\rho(0)$ the fibre. By [1, Theorem 4.16, (3)], there are cartesian diagrams

$$\begin{tikzcd} \widehat{\mathcal{X}}_x \arrow[r] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\ulcorner"] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} \widehat{\mathcal{O}}_{X,\pi(x)} \arrow[r] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \widehat{\mathcal{N}}_x \arrow[r] \arrow[d, "\rho'"'] \arrow[dr, phantom, "\ulcorner"] & \mathcal{N}_x \arrow[d, "\rho"] \\ \operatorname{Spec} \widehat{\mathcal{O}}_{W,\rho(0)} \arrow[r] & W \end{tikzcd}$$

where $\widehat{\mathcal{X}}_x$ is the completion of $\mathcal{X}$ at x (that is, the completion of the adic sequence of thickenings of x, see [2, Definition 1.9]) and $\widehat{\mathcal{N}}_x$ is the completion of $\mathcal{N}_x$ at 0. Thus $\mathcal{F}$ and $\mathcal{R}$ are the special fibres of π' and ρ'.

In [1, Proof of Theorem 1.1, p.688-689] it is shown that there is a closed immersion $\widehat{\mathcal{X}}_x \to \widehat{\mathcal{N}}_x$. We now summarise the argument for the convenience of the reader.

For $n \in \mathbb{N}$, let us denote $\mathcal{X}^{[n]}$ the nth infinitesimal thickening of $\mathcal{X}$ at x, that is, if $\mathcal{I}$ is the ideal sheaf of the closed substack BG_x of $\mathcal{X}$, then $\mathcal{X}^{[n]} = \operatorname{Spec}_{\mathcal{X}} \mathcal{O}_{\mathcal{X}}/\mathcal{I}^{n+1}$. Similarly, let $\mathcal{N}^{[n]}$ be the nth infinitesimal thickening of $\mathcal{N}_x$ at 0. We have natural identifications

$\mathcal{X}^{[0]} \cong BG_x$ and $\mathcal{N}^{[0]} \cong BG_x$. We denote $g_{ij}\colon \mathcal{X}^{[j]} \to \mathcal{X}^{[i]}$ the relevant closed immersions for $j \leq i$. We have a diagram of solid arrows

$$\begin{tikzcd} & \mathcal{X}^{[0]} \arrow[dl, "g_{10}"'] \arrow[dr, "\sim"] & \\ \mathcal{X}^{[1]} \arrow[rr, dashed, "f"] & & BG_x \end{tikzcd}$$

Since g_{10} is a square zero extension of ideal sheaf $\mathcal{I}/\mathcal{I}^2$, by [14, Theorem 8.5] the obstruction to the existence of the dotted arrow f lives in $\mathrm{Ext}^1_{BG_x}(\mathbb{L}_{BG_x/k}, \mathcal{I}/\mathcal{I}^2)$, which is 0 because BG_x is smooth, and hence the cotangent complex $\mathbb{L}_{BG_x/k}$ is concentrated in degrees $[0,1]$, and BG_x is linearly reductive, so $\mathbb{L}_{BG_x/k} = H^0(\mathbb{L}_{BG_x/k}) \oplus H^1(\mathbb{L}_{BG_x/k})[-1]$. Since $\mathcal{X}^{[0]} \to BG_x$ is affine and $\mathcal{X}^{[1]}$ is a square zero extension of $\mathcal{X}^{[0]}$, by dévissage and Serre's criterion for algebraic spaces, we have that f is affine too. We have a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}/\mathcal{I}^2 \arrow[r] & \mathcal{O}_{\mathcal{X}^{[1]}} \arrow[r] & \mathcal{O}_{\mathcal{X}^{[0]}} \arrow[r] \arrow[l, dashed, bend right] & 0 \end{tikzcd}$$

of which f provides the dotted splitting. Thus $\mathcal{O}_{\mathcal{X}^{[1]}} \cong \mathcal{O}_{BG_x} \oplus \varepsilon(\mathcal{I}/\mathcal{I}^2)$, where $\varepsilon^2 = 0$, and hence $\mathcal{X}^{[1]} \cong \mathcal{N}^{[1]}$.

We now produce compatible morphisms $g_n\colon \mathcal{X}^{[n]} \to \mathcal{N}_x$. For $i = 0, 1$, we let g_i be the composition $\mathcal{X}^{[i]} \cong \mathcal{N}^{[i]} \to \mathcal{N}_x$. If g_n has been constructed, the obstruction for g_n to lift to a map $\mathcal{X}^{[n+1]} \to \mathcal{N}_x$ is an element of $\mathrm{Ext}^1_{\mathcal{X}^{[n]}}(g_n^*\mathbb{L}_{\mathcal{N}_x/k}, (g_{n0})_*\mathcal{I}^{n+1}/\mathcal{I}^{n+2})$. By adjunction, we have

$$\mathrm{Ext}^1_{\mathcal{X}^{[n]}}(g_n^*\mathbb{L}_{\mathcal{N}_x/k}, (g_{n0})_*\mathcal{I}^{n+1}/\mathcal{I}^{n+2}) = \mathrm{Ext}^1_{BG_x}(g_0^*\mathbb{L}_{\mathcal{N}_x/k}, \mathcal{I}^{n+1}/\mathcal{I}^{n+2}).$$

This group is 0 because $\mathcal{N}_x$ is smooth, and so $\mathbb{L}_{\mathcal{N}_x/k}$ has Tor-amplitude $[0,1]$, and BG_x is linearly reductive. Hence the g_n have been constructed for all n. By [1, Proposition A.8, (1)] applied to each g_n, we see that g_n is a closed immersion for all n. Each of the g_n factors through the closed immersion $\mathcal{N}^{[n]} \to \mathcal{N}_x$, and after passing to the completion the closed immersions $\mathcal{X}^{[n]} \to \mathcal{N}^{[n]}$ induce, by Tannaka duality, a closed immersion $\widehat{\mathcal{X}}_x \to \widehat{\mathcal{N}}_x$.

The map $\widehat{\mathcal{X}}_x \to \widehat{\mathcal{N}}_x$ yields a closed immersion $\mathcal{F} \to \mathcal{R}$, and thus the desired result, since $\mathcal{R}$ is a closed substack of $\mathcal{N}$. By construction, there is a triangle

$$\begin{tikzcd} & BG_x \arrow[dl] \arrow[dr] & \\ \widehat{\mathcal{X}}_x \arrow[rr] & & \widehat{\mathcal{N}}_x \end{tikzcd}$$

identifying BG_x with the 0th infinitesimal thickenings $\mathcal{X}^{[0]}$ and $\mathcal{N}^{[0]}$. After taking special fibres of the good moduli spaces, we get the triangle 1.

The map $BG_x \to \mathcal{F}$ induces a bijection $\pi_0(\mathrm{Grad}_{\mathbb{Q}}(BG_x)) \to \pi_0(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{F}))$. This can be seen by expressing $\mathcal{F} = \operatorname{Spec} A/G_x$ where $\operatorname{Spec} A$ has a unique closed G_x-orbit consisting of a single point. The same property thus holds for any maximal torus T of G_x and we can then use the description of [8, Theorem 1.4.8] of the stack of graded points. The same holds for the maps $BG_x \to \mathcal{N}_x$ and $\mathcal{N}_x \to BG_x$. These facts imply, together with the commutativity of 1, that $\iota\colon \mathcal{F} \to \mathcal{N}_x$ is lit. □

Now let R be a finite-dimensional associative k-algebra. We can give the groups of units $R^\times$ a structure of algebraic group $\underline{R}^\times$ over k by setting $\underline{R}^\times(A) = (R \otimes_k A)^\times$ for a commutative k-algebra A. Actually, we can enhance R to an algebraic monoid $\underline{R}$ by setting $\underline{R}(A) = R \otimes_k A$ with the multiplication. Note that $\underline{R}$ is isomorphic to $\mathbb{A}_k^{\dim R}$.

There is a map $l\colon \underline{R} \to \underline{\mathrm{End}}_k(R)$, sending an element a of $R \otimes_k A$ to the endomorphism of $R \otimes_k A$ given by left multiplication by a. Then $\underline{R}^\times = l^{-1}(\mathrm{GL}(R))$. Therefore $\underline{R}^\times$ is an affine open subscheme of $\underline{R}$, and it is therefore an affine smooth connected algebraic group over k.

Lemma 5.0.2. *Let R be a finite dimensional associative algebra over k. Then R is semisimple if and only if $\underline{R}^\times$ is reductive.*

Proof. By definition, R is semisimple if its Jacobson radical $\mathrm{Jac}(R)$ is trivial. The Jacobson radical is a nilpotent two-sided ideal, so $1 + \underline{\mathrm{Jac}(R)}$ is a smooth connected unipotent normal subgroup of $\underline{R}^\times$. Therefore if $\underline{R}^\times$ is reductive, $\mathrm{Jac}(R)$ has to be trivial and hence R is semisimple.

Conversely, if R is semisimple, by Artin-Wedderburn Theorem there is a k-algebra isomorphism $R \cong \prod_i \mathrm{Mat}_{n_i \times n_i, k}$, and thus $\underline{R}^\times \cong \prod_i \mathrm{GL}_{n_i,k}$, which is reductive. □

Lemma 5.0.3. *Let R be a semisimple finite dimensional associative algebra over k and let M be a finite-dimensional R-R-bimodule. Then the quotient stack $\underline{M}/\underline{R}^\times$, where $\underline{R}^\times$ acts on $\underline{M}$ by $g \cdot m = gmg^{-1}$, is isomorphic as a lit stack to $\mathscr{Rep}(Q, d)$ for some quiver Q and some dimension vector d.*

Note that above, since $\underline{R}^\times$ is the group of units of an algebraic monoid $\underline{R}$, there is a natural lamp on $B\underline{R}^\times$ from Example 4.0.2 and thus a lamp on $\underline{M}/\underline{R}^\times$ by pulling back along $\underline{M}/\underline{R}^\times \to B\underline{R}^\times$.

Proof. By Artin-Wedderburn theorem, there are finite-dimensional vector spaces $V_1, \ldots, V_n$ and a k-algebra isomorphism $R \cong \mathrm{End}(V_1) \times \cdots \times \mathrm{End}(V_n)$. Let us call $e_i = (0, \cdots, \mathrm{id}_{V_i}, \cdots, 0)$. The e_i are orthogonal central idempotents and $1 = \sum_{i=1}^n e_i$. Therefore each $e_i M e_j$ is an R-R-bimodule and there is a splitting $M = \bigoplus_{i,j} e_i M e_j$ as bimodules. The R-R-bimodule structure on $e_i M e_j$ is induced by restriction of scalars from the obvious $\mathrm{End}(V_i)$-$\mathrm{End}(V_j)$-bimodule structure, which is the same as a left $\mathrm{End}(V_i \otimes V_j^\vee)$-module structure, since $\mathrm{End}(V_i) \otimes_k \mathrm{End}(V_j)^{\mathrm{op}} = \mathrm{End}(V_i) \otimes_k \mathrm{End}(V_j^\vee) = \mathrm{End}(V_i \otimes V_j^\vee)$. The $\mathrm{End}(V_i \otimes V_j^\vee)$-$k$-bimodule $V_i \otimes V_j^\vee$ gives a Morita equivalence between k and $\mathrm{End}(V_i \otimes V_j^\vee)$, so there is $n_{ij} \in \mathbb{N}$ and an isomorphism $e_i M e_j \cong (V_i \otimes V_j^\vee)^{\oplus n_{ij}}$ of left $\mathrm{End}(V_i \otimes V_j^\vee)$-modules, or equivalently of $\mathrm{End}(V_i)$-$\mathrm{End}(V_j)$-bimodules. Thus M is isomorphic to $\bigoplus_{i,j} \mathrm{Hom}(V_j, V_i)^{n_{ij}}$ as an R-R-bimodule, and thus also as $\underline{R}^\times = \prod_i \mathrm{GL}(V_i)$-representations, where the action is as in the statement of the lemma. Let Q be the quiver with set of vertices $\{1, \ldots, n\}$ and with n_{ij} arrows from j to i, and take the dimension vector d with $d_i = \dim V_i$. Then

$$\underline{M}/\underline{R}^\times \cong \bigoplus_{i,j} \mathrm{Hom}(V_j, V_i)^{\oplus n_{ij}} / \prod_i \mathrm{GL}(V_i) \cong \mathscr{Rep}(Q, d),$$

as desired. The isomorphism is lit because the standard lamp on $B \prod_i \mathrm{GL}(V_i)$ comes from the partial compactification $\prod_i \mathrm{End}(V_i)$ of $\prod_i \mathrm{GL}(V_i)$, exactly as the lamp in $B\underline{R}^\times$. □

The following theorem is our main tool for proving that a lit stack is linearly lit.

Theorem 5.0.4. *Let $\mathcal{X}$ be an algebraic stack over k satisfying Assumption 4.0.5, endowed with a lamp and having a good moduli space $\pi\colon \mathcal{X} \to X$. Suppose that for every closed k-point x of $\mathcal{X}$ there exists a finite-dimensional associative k-algebra R and an isomorphism $G_x \cong \underline{R}^\times$ such that the lamp on BG_x inherited from $\mathcal{X}$ agrees with the one coming from the partial compactification $\underline{R}$ of G_x as in Example 4.0.2. Suppose further that the G_x-action on the normal space N_x comes from an R-R-bimodule structure on N_x. Then $\mathcal{X}$ is linearly lit.*

Proof. This follows readily from Lemma 5.0.2, Lemma 5.0.3, Proposition 5.0.1 and the fact that every k-point of $\mathcal{X}$ specialises to a closed k-point. $\square$

Remark 5.0.5. In positive characteristic, for the condition in Theorem 5.0.4 to hold, the stabiliser G_x of $\mathcal{X}$ at x should be linearly reductive, and hence a torus. Thus the only dimension vectors that can appear consist of 0s and 1s.

Moduli of semistable quiver representation

We fix an algebraically closed field k. Let Q be a quiver and let d be a dimension vector for Q.

Theorem 5.0.6. *Let $\mathcal{X}$ be an open substack of $\mathscr{Rep}(Q,d)$ that admits a good moduli space. Then $\mathcal{X}$ is linearly lit with the lamp inherited from $\mathscr{Rep}(Q,d)$.*

Proof. It is clear that $\mathcal{X}$ satisfy Assumption 4.0.5 because $\mathscr{Rep}(Q,d)$ does. Let $x \in \mathcal{X}(k)$ be a point with linearly reductive stabiliser. The point x corresponds to a representation E of Q. The automorphism group G_x is identified with the group of units of $\mathrm{End}(E)$, so it is in particular smooth and thus the normal space at x coincides with the tangent space. The normal stack is then $\mathcal{N}_x = \mathrm{Ext}^1(E,E)/G_x$ by deformation theory, where the action on G_x on $\mathrm{Ext}^1(E,E)$ comes from the natural $\mathrm{End}(E)$-$\mathrm{End}(E)$-bimodule structure of $\mathrm{Ext}^1(E,E)$. For details about this last statement, we refer to the proof of Theorem 5.0.12. We conclude by Theorem 5.0.4. $\square$

Remark 5.0.7. The prototypical example of open substack $\mathcal{X}$ as in the Theorem is the semistable locus for a line bundle on $\mathscr{Rep}(Q,d)$.

The moduli stack of Artin and Zhang

We fix an algebraically closed field k and consider a k-linear locally noetherian Grothendieck abelian category $\mathcal{A}$. This means that $\mathcal{A}$ is a k-linear abelian category that is cocomplete, where filtered colimits are exact, and that has a set of noetherian generators (we recall the notion of noetherian object below). An object E in $\mathcal{A}$ is said to be

1. of *finite type*, if for every filtered diagram $(F_i)_{i\in I}$ in $\mathcal{A}$ where all maps $F_i \to F_j$ in the diagram are monomorphisms, we have a canonical isomorphism
$$\mathrm{Hom}(E, \varinjlim_{i\in I} F_i) \cong \varinjlim_{i\in I} \mathrm{Hom}(E, F_i);$$
2. of *finite presentation*, if for every filtered diagram $(F_i)_{i\in I}$ in $\mathcal{A}$, we have a canonical isomorphism
$$\mathrm{Hom}(E, \varinjlim_{i\in I} F_i) \cong \varinjlim_{i\in I} \mathrm{Hom}(E, F_i);$$
3. *noetherian*, if every subobject of E is of finite type.

For every commutative k-algebra R, there is a notion of base change category $\mathcal{A}_R$ which is an R-linear abelian category. An object of $\mathcal{A}_R$ is a pair (E,ρ) where E is an object of $\mathcal{A}$ and $\rho\colon R \to \mathrm{End}(E)$ is a k-algebra homomorphism. A map from (E,ρ) to (E',ρ') is a map $f\colon E \to E'$ in $\mathcal{A}$ such that for all $r \in R$ we have $f \circ \rho(r) = \rho'(r) \circ f$. It turns out that $\mathcal{A}_R$ is an R-linear Grothendieck abelian category [4, Proposition B2.2] and that it is noetherian if R is essentially of finite type [4, Corollary B6.3].

If R is a commutative k-algebra, N is an R-module, and M is an object of $\mathcal{A}_R$, there is a notion of tensor product $N \otimes_R M$. One can describe it by taking a free presentation

$$R^{\oplus J} \longrightarrow R^{\oplus I} \longrightarrow N \longrightarrow 0$$

of N. Then $N \otimes_R M$ can be defined by exactness of the sequence

$$M^{\oplus J} \longrightarrow M^{\oplus I} \longrightarrow N \otimes_R M \longrightarrow 0.$$

In fact, $(-)\otimes_R M \colon R\text{-Mod} \to \mathcal{A}_R$ is characterised by being the left adjoint to $\mathrm{Hom}(M, -) \colon \mathcal{A}_R \to R\text{-Mod}$.

The tensor product can be used to give an alternative description of the base change categories $\mathcal{A}_R$. An object of $\mathcal{A}_R$ is given by an object E of $\mathcal{A}$ and a map $R \otimes_k E \to E$ such that the usual diagrams commute.

If R' is a commutative R-algebra, we have the base-change functor $R' \otimes_R (-) \colon \mathcal{A}_R \to \mathcal{A}_{R'}$, left adjoint to the restriction of scalars functor $\mathcal{A}_{R'} \to \mathcal{A}_R$.

Definition 5.0.8. An object $M \in \mathcal{A}_R$ is said to be *flat* (over R), if the functor $(-) \otimes_R M \colon R\text{-Mod} \to \mathcal{A}_R$ is exact.

The properties of being flat or finitely presented are preserved under base-change. The following is [3, Definition 7.8].

Definition 5.0.9. The *moduli stack* $\mathcal{M}_\mathcal{A}$ *of objects of* $\mathcal{A}$ is the prestack over k defined by setting, for a commutative k-algebra R, the groupoid $\mathcal{M}_\mathcal{A}(R)$ to be that of finitely presented flat objects in $\mathcal{A}_R$.

The prestack $\mathcal{M}_\mathcal{A}$ is actually a stack for the fppf topology [4, Theorem C8.6].

Suppose now that $\mathcal{M}_\mathcal{A}$ is algebraic and locally of finite type over k. By [3, Lemma 7.20], $\mathcal{M}_\mathcal{A}$ has affine diagonal, and thus it satisfies Assumption **??**. We can define a lamp $\mathrm{Grad}_\mathbb{Q}(\mathcal{M}_\mathcal{A})_{\geq 0} \subset \mathrm{Grad}(\mathcal{M}_\mathcal{A})$ on $\mathcal{M}_\mathcal{A}$ as follows. A graded point $\mathrm{Spec}\, R \to \mathrm{Grad}(\mathcal{M}_\mathcal{A})$ is a $\mathbb{Z}$-graded object $\bigoplus_{n\in\mathbb{Z}} E_n$ in $\mathcal{A}_R$ with $E_n = 0$ for all but finitely many n and such that each E_n is flat and finitely presented [3, Proposition 7.12]. Suppose that R is a finite type k-algebra. Then the condition that $E_n = 0$ for $n < 0$ is open on $\mathrm{Spec}\, R$ by Nakayama's lemma [4, Theorem C4.3] and thus it defines an open substack $\mathrm{Grad}(\mathcal{M}_\mathcal{A})_{\geq 0}$ of $\mathrm{Grad}(\mathcal{M}_\mathcal{A})$.

Denoting $E = \bigoplus_{n\in\mathbb{Z}} E_n$ the underlying object of the grading, we may define $\underline{\mathrm{End}}(E)$ as a functor from R-algebras to sets by $\underline{\mathrm{End}}(E)(R') = \mathrm{End}_{R'}(R' \otimes_R E)$. It is proven in [3, Proof of Lemma 7.19] that $\underline{\mathrm{End}}(E)$ is represented by an affine scheme of finite type over R. In fact, $\underline{\mathrm{End}}(E)$ is an algebraic monoid with unit group $\mathrm{Aut}(E)$ The grading $E = \bigoplus_{n\in\mathbb{Z}} E_n$ corresponds to a one-parameter subgroup $\lambda \colon \mathbb{G}_{m,R} \to \mathrm{Aut}(E)$. The multiplicative group acts on E diagonally with respect to the direct sum decomposition, and it acts in E_n with weight n. Therefore, the condition that $E_n = 0$ for $n < 0$ is equivalent to the existence of $\lim_{t\to 0} \lambda(t)$ in $\underline{\mathrm{End}}(E)$. This is in turn equivalent to the identity section $\mathrm{id}_E \in \underline{\mathrm{End}}(E)(R)$ factoring through the attractor $\underline{\mathrm{End}}(E)^+$ (where the $\mathbb{G}_m$-action is given by λ). Since $\underline{\mathrm{End}}(E)$ is affine, $\underline{\mathrm{End}}(E)^+$ is a closed subscheme of $\underline{\mathrm{End}}(E)$. Therefore the open substack $\mathrm{Grad}(\mathcal{M}_\mathcal{A})_{\geq 0}$ is also closed. Since $\mathrm{Grad}(\mathcal{M}_\mathcal{A})_{\geq 0}$ is equivariant for the $\mathbb{N}_{>0}$-action on $\mathrm{Grad}(\mathcal{X})$, it defines a closed and open substack $\mathrm{Grad}_\mathbb{Q}(\mathcal{M}_\mathcal{A})_{\geq 0}$ of $\mathrm{Grad}_\mathbb{Q}(\mathcal{M}_\mathcal{A})$.

Definition 5.0.10. Suppose $\mathcal{M}_\mathcal{A}$ is an algebraic stack locally of finite type over k. The *standard lamp* on $\mathcal{M}_\mathcal{A}$ is the lamp $\mathrm{Grad}_\mathbb{Q}(\mathcal{M}_\mathcal{A})_{\geq 0}$ defined above.

Remark 5.0.11. By the description in terms of $\underline{\mathrm{End}}(E)$, we can regard the lamp on $\mathcal{M}$ as a shadow of the enhancement of $\mathcal{M}_\mathcal{A}$ to a *stack in categories*, where we remember all morphisms between objects and not just isomorphisms. See [5, Section 4] for a precise definition of this notion.

Theorem 5.0.12. *Suppose that $\mathcal{M}_{\mathcal{A}}$ is an algebraic stack, locally of finite type over k, and let $\mathcal{X}$ be a quasi-compact open substack of $\mathcal{M}_{\mathcal{A}}$, endowed with the lamp inherited from the standard lamp on $\mathcal{M}_{\mathcal{A}}$. Suppose that $\mathcal{X}$ has a good moduli space $\pi\colon \mathcal{X} \to X$. Then $\mathcal{X}$ has affine diagonal and is linearly lit.*

Proof. By [3, Lemma 7.20], the algebraic stack $\mathcal{M}_{\mathcal{A}}$ has affine diagonal over k, and thus the same holds for $\mathcal{X}$. Therefore $\mathcal{X}$ satisfies Assumption 4.0.5.

Let $x \in \mathcal{X}(k)$ be a point with linearly reductive stabiliser, corresponding to an object $E \in \mathcal{A}$. By [4, Proposition B3.21], we have $\underline{\mathrm{End}}(E)(R) = \mathrm{End}(E) \otimes_k R = \underline{\mathrm{End}(E)}(R)$ for any k-algebra R, and thus the stabiliser group G_x is the group of units of the associative k-algebra $\mathrm{End}(E)$, which has to be finite-dimensional because $\underline{\mathrm{End}}(E)$ is an affine scheme of finite type over k. In particular, G_x is smooth, and thus the normal space N_x coincides with the tangent space T_x of $\mathcal{M}_{\mathcal{A}}$ at x. By [4, Proposition E1.1], the tangent space T_x is canonically identified with $\mathrm{Ext}^1(E, E)$, which is thus finite dimensional, since $\mathcal{M}_{\mathcal{A}}$ is locally of finite type over the field k.

An element of T_x is a pair (E', α) where $E' \in \mathcal{M}_{\mathcal{A}}(k[\varepsilon])$, with $\varepsilon^2 = 0$, and $\alpha\colon k \otimes_{k[\varepsilon]} E' \to E$ is an isomorphism. An element $g \in G_x(k)$ acts on T_x by $g(E', \alpha) = (E', g\alpha)$. One gets an element u of $\mathrm{Ext}^1(E, E)$ by tensoring the short exact sequence

$$0 \longrightarrow k \longrightarrow k[\varepsilon] \longrightarrow k \longrightarrow 0$$

of $k[\varepsilon]$-modules with E', obtaining a self-extension

$$0 \longrightarrow E \longrightarrow E' \longrightarrow E \longrightarrow 0.$$

The $\mathrm{End}(E)$-$\mathrm{End}(E)$-bimodule structure on $\mathrm{Ext}^1(E, E)$ can be described as follows. If $a \in \mathrm{End}(E)$, then au corresponds to the pulled back short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & E'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & E \arrow[r] \arrow[d, "a"] & 0 \\
0 \arrow[r] & E \arrow[r] & E' \arrow[r] & E \arrow[r] & 0,
\end{tikzcd}$$

while ua corresponds to the pushed forward sequence

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d, "a"'] \arrow[dr, phantom, "\lrcorner"] & E' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E \arrow[r] & E'' \arrow[r] & E \arrow[r] & 0.
\end{tikzcd}$$

From these descriptions we see that the G_x-action on T_x is the one coming from the $\mathrm{End}(E)$-$\mathrm{End}(E)$-bimodule structure on $\mathrm{Ext}^1(E, E)$. Note that it is indeed enough to check the previous statement on k-points of G_x, since it is smooth.

We conclude by Theorem 5.0.4. $\square$

6 Main comparison result

Linear norms

Let Q be a quiver, d a dimension vector for Q, and fix a family $(m_i)_{i\in Q_0}$ of positive rational numbers $m_i \in \mathbb{Q}_{>0}$. We assume as usual that $G(d)$ is a torus k is of positive characteristic. Let x be a k-point of $\mathscr{Rep}(Q,d)$, corresponding to a quiver representation E, and let λ be a $\mathbb{Q}$-grading of x (i.e. a rational one-parameter subgroup of $\mathrm{Aut}(x)$), corresponding to direct sum decompositions

$$E_i = \bigoplus_{c\in\mathbb{Q}} E_{i,c}$$

for each $i \in Q_0$. We set

$$\|\lambda\|^2 = \sum_{i\in Q_0}\sum_{c\in\mathbb{Q}} m_i c^2 \dim(E_{i,c}).$$

It is not hard to see that this formula defines a norm on graded points of $\mathscr{Rep}(Q,d)$.

Definition 6.0.1. A norm on graded points of $\mathscr{Rep}(Q,d)$ is said to be *standard* if it is induced by a family $(m_i)_{i\in Q_0} \in (\mathbb{Q}_{>0})^{Q_0}$ as above.

Now let $\mathcal{X}$ be a linearly lit good moduli stack over k endowed with a norm on graded points q. For any k-point of $\mathcal{X}$, recall that we have a canonical isomorphism $\mathbf{DF}(\mathcal{X},x)_\bullet \cong \mathbf{DF}(L_x)_\bullet$ of formal fans and an inclusion $L_x \hookrightarrow \mathbf{DF}(\mathcal{X},x)_1$, so we see elements a of the lattice L_x as filtrations of x and in particular we can consider their norm $\|a\|^2 = q(a)$.

Definition 6.0.2. We say that the norm q on $\mathcal{X}$ is linear if the following holds: for every k-point x of $\mathcal{X}$ and for every $\mathbb{Q}$-grading λ of x, corresponding to a $\mathbb{Q}$-grading $(a_c)_{c\in\mathbb{Q}}$ of L_x (Definition **??**) we have

$$\|\lambda\|^2 = \sum_{c\in\mathbb{Q}} c^2\|a_c\|^2. \tag{2}$$

Proposition 6.0.3. *A norm q on graded points of $\mathscr{Rep}(Q,d)$ is linear if and only if it is standard.*

Proof. If q is standard, then it clearly is linear. To prove the converse, we can assume that Q has no arrows, and thus that we are working with $BG(d)$. Indeed, since $\mathrm{Grad}^n(\mathscr{Rep}(Q,d))$ and $\mathrm{Grad}^n(BG(d))$ have the same connected components for all n, norms on graded points of $\mathscr{Rep}(Q,r)$ are the same as norms on $BG(d)$. We can also assume that $d_i \geq 1$ for all $i \in Q_0$.

Fix a k-point x corresponding to a representation E. Let $V(i)$ be the skyscraper representation at vertex i. For any embedding $\alpha\colon V(i) \hookrightarrow E$, consider the corresponding filtration with weights $1,0$ and set $m_i = \|F_i\|^2$. This number does not depend on the choice of α, since any two different α's give conjugate F_i's.

Let E_1, E_2, E_3 be subrepresentations of E with $E = E_1 \oplus E_2 \oplus E_3$. Recall we are regarding subrepresentations of E as filtrations, and thus we can consider their norm. Let F_1 (resp. F_2) be the grading E_1, E_2, E_3 with weights $1,0,0$ (resp. $0,1,0$). We have $q(F_1) = q(E_1)$, $q(F_2) = q(E_2)$, $q(F_1+F_2) = q(E_1\oplus E_2)$ and $q(F_1-F_2) = q(E_1)+q(E_2)$ by linearity of q. From the equality $q(F_1+F_2)+q(F_1-F_2) = 2q(F_1)+2q(F_2)$, it follows that $q(E_1\oplus E_2) = q(E_1)+q(E_2)$. By applying this formula repeatedly, we get that

$$q(E') = \sum_{i\in Q_0} m_i \dim(E'_i)$$

for any subrepresentation E' of E. Combining this with (2), it follows that q is standard. $\square$

Proposition 6.0.4. *Let $\pi\colon \mathcal{X} \to X$ be the good moduli space of $\mathcal{X}$. The norm q on $\mathcal{X}$ is linear if and only for all closed k-points y of $\mathcal{X}$ the lit embedding*

$$\iota\colon (\pi^{-1}\pi(y), y) \hookrightarrow (\mathscr{Rep}(Q,d), 0)$$

in Definition 4.0.6 can be chosen to preserve norms for some (uniquely determined) standard norm on $\mathscr{Rep}(Q,d)$. Equivalently, all such embeddings have this property.

Proof. Call $\mathcal{F} = \pi^{-1}\pi(x)$. Given a lit embedding ι as above, $\mathrm{Grad}^n(\mathcal{F})$ and $\mathrm{Grad}^n(\mathscr{Rep}(Q,d))$ have the same components for all n. Thus giving a norm on $\mathcal{F}$ is equivalent to giving a norm on $\mathscr{Rep}(Q,d)$. If the norm on $\mathcal{X}$ is linear, then so will be the induced norm on $\mathscr{Rep}(Q,d)$, and thus it will be standard by Proposition 6.0.3. $\square$

Definition 6.0.5. A *linearly lit normed good moduli stack* is a linearly lit good moduli stack endowed with a linear norm on graded points.

Definition 6.0.6. Let $\mathcal{X}$ be a linearly lit normed good moduli stack and let x be a k-point of $\mathcal{X}$. For $a \le b$ in L_x, we let $X_x([a,b]) = \|b\|^2 - \|a\|^2$. Then X_x is a norm on the lattice L_x, and we will regard L_x as a normed lattice endowed with X_x.

That X_x defines a norm on L_x is immediate in the case $\mathcal{X} = \mathscr{Rep}(Q,d)$, and the general case follows from this by the embedding in Definition 4.0.6.

Lemma 6.0.7. *Let $(\mathcal{X}, x)$ be a pointed linearly lit normed good moduli stack with norm on graded points q, and let $\lambda \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1$. Let ℓ_q be the canonical linear form on graded points of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ (Definition ??), and let $(F^\lambda)^\vee$ be the linear form on $\mathrm{Grad}_{F^\lambda}(L_x)$ induced by the norm on L_x (see discussion after Definition ??). Then for all $\mu \in \mathbf{DF}(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}), \mathrm{gr}\,\lambda)_1$, we have the equality*

$$\langle \mu, \ell_q \rangle = \langle F^\mu, (F^\lambda)^\vee \rangle.$$

Proof. We may assume $\mathcal{X} = \mathscr{Rep}(Q,d)$ with standard norm given by $(m_i)_{i\in Q_0}$. The point x corresponds to a representation $E = ((E_i)_{i\in Q_0}, (f_\alpha)_{\alpha\in Q_1})$, and the filtration λ of x is represented by a rational one-parameter subgroup β of $G(d)$ that induces a grading $E_i = \bigoplus_{c\in\mathbb{Q}} E_{i,c}$ for each $i \in Q_0$. With respect to this decomposition, each f_α is written in coordinates as $(f_{\alpha,c,c'})_{c,c'\in\mathbb{Q}}$.

The component of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ containing $\mathrm{gr}\,\lambda$ is of the form $\mathscr{Rep}(Q', d')$, with Q', d' as in the proof of Proposition 4.0.12. The point $\mathrm{gr}\,\lambda$ has coordinates $(f_{\alpha,c,c})_{c\in\mathbb{Q}}$. If a rational cocharacter μ of $L(\beta)$ defines a filtration of $\mathrm{gr}\,\lambda$, then $\langle \mu, \ell_q\rangle = \langle \mu, \beta^\vee\rangle = (\mu, \beta)$. A grading $E_{i,c} = \bigoplus_{c\in\mathbb{Q}} E_{i,c,c'}$ for each $E_{i,c}$ is determined by μ. From the definitions, we have the formula

$$(\mu, \beta) = \sum_{i,c,c'} cc' \dim(E_{i,c,c'}) m_i.$$

Also from the definitions, we have

$$\begin{aligned}\langle F^\mu, (F^\lambda)^\vee \rangle &= \sum_{c'\in\mathbb{Q}} c' (F^\lambda)^\vee([F^\mu_{>c'}, F^\mu_{\ge c'}]) = \sum_{c'\in\mathbb{Q}} c' \sum_{i\in Q_0} (F^\lambda)^\vee (E_{i,\bullet,c'}) \\ &= \sum_{c'\in\mathbb{Q}} c' \sum_{i\in Q_0} \sum_{c\in\mathbb{Q}} c m_i \dim(E_{i,c,c'}),\end{aligned}$$

as desired. $\square$

A lemma about semistability

Let k be a field. Let G be a geometrically reductive algebraic group over k, endowed with a norm q on cocharacters, and let V be a finite dimensional representation of G. Let $\lambda \in \Gamma^{\mathbb{Q}}(G)$ be a rational cocharacter of G, inducing a direct sum decomposition $V = \bigoplus_{s\in\mathbb{Q}} V_s$. The commutator subgroup $L(\lambda)$ acts on V_1. We endow $V_1/L(\lambda)$ with the linear form $\ell_1 = -\lambda^\vee$ on graded points, where $\langle \gamma, \lambda^\vee\rangle = (\gamma, \lambda)$ for any $\gamma \in \Gamma^{\mathbb{Q}}(L(\lambda))$, and $(-,-)$ is the inner product that q induces in some split subtorus of $L(\lambda)$ containing γ and λ. Let ℓ be the linear form on $\mathbb{P}(V_1)/L(\lambda)$ determined by $\mathcal{O}(1)$, and we consider semistability on $\mathbb{P}(V_1)/L(\lambda)$ with respect to the linear form $\ell_2 = \ell - \lambda^\vee/\|\lambda\|^2$.

Lemma 6.0.8. *In the situation above, we have a cartesian square*

$$\begin{tikzcd}
(V_1/L(\lambda))^{ss(\ell_1)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & (V_1 \setminus \{0\})/L(\lambda) \arrow[d] \\
(\mathbb{P}(V_1)/L(\lambda))^{ss(\ell_2)} \arrow[r] & \mathbb{P}(V_1)/L(\lambda).
\end{tikzcd}$$

Proof. Let $V_1 \setminus \{0\} \to \mathbb{P}(V_1)\colon x \mapsto [x]$ be the projection and let $v\colon \mathrm{Spec}(k) \to V_1 \setminus \{0\}$ be a geometric point. We wish to show that v is semistable for ℓ_1 if and only if $[v]$ is semistable for ℓ_2. To this end we may assume $k = k$ is algebraically closed after base change.

If T is a maximal torus of $L(\lambda)$, then v (resp. $[v]$) is semistable for $L(\lambda)$ if and only if gv (resp. $g[v]$) is semistable for T for all $g \in L(\lambda)(k)$. It is thus enough to assume that $L(\lambda) = T$ is a torus.

We write $\Xi \subset \Gamma_{\mathbb{Q}}(T)$ for the *state* of v. That is, if $V_1 = \bigoplus_{\chi\in\Gamma_{\mathbb{Q}}(T)} (V_1)_\chi$ is the grading induced by the T-action on V_1 and $p_\chi\colon V_1 \to (V_1)_\chi$ are the projections, then $\Xi = \{\chi \in \Gamma_{\mathbb{Q}}(T)\colon p_\chi(v) \neq 0\}$. Now we have

1. $v \in V_1^{\mathrm{ss}(\ell_1)}(k)$ if and only if $\lambda^\vee \in \mathrm{cone}(\Xi)$, and
2. $[v] \in \mathbb{P}(V_1)^{\mathrm{ss}(\ell_2)}(k)$ if and only if $\lambda^\vee/\|\lambda\|^2 \in \mathrm{conv}(\Xi)$.

Here, $\mathrm{cone}(\Xi)$ is the convex cone in $\Gamma_{\mathbb{Q}}(T)$ generated by Ξ, while $\mathrm{conv}(\Xi)$ is the convex hull of Ξ.

Let $P = \{\alpha \in \Gamma_{\mathbb{Q}}(T) \mid (\alpha, \lambda^\vee) = 1\}$. Note that, since $\Xi \subset P$, we have $\mathrm{conv}(\Xi) = \mathrm{cone}(\Xi) \cap P$. Moreover, since $\lambda^\vee/\|\lambda\|^2 \in P$, we have that $\lambda^\vee \in \mathrm{cone}(\Xi)$ if and only if $\lambda^\vee/\|\lambda\|^2 \in \mathrm{conv}(\Xi)$, as desired. □

The cone-balanced chain and the HKKP chain

Theorem 6.0.9. *Let $(\mathcal{X}, x)$ be a pointed linearly lit normed good moduli stack over k, and let $(\mathcal{Y}_n, y_n, \eta_n, v_n)$ be its cone-balanced chain. Then:*

1. *Each $\mathcal{Y}_n$ is a linearly lit normed good moduli stack.*
2. *Under the canonical identification $\mathbf{DF}^{\mathbb{Q}}(\mathcal{Y}_n, y_n)_1 \cong \mathbf{DF}^{\mathbb{Q}}(L_{y_n})_1\colon \lambda \mapsto F^\lambda$, the balanced filtration η_n of $(\mathcal{Y}_n, y_n)$ (Definition* **??***) coincides with the HKKP filtration of the normed lattice L_{y_n} (Definition* **??***).*
3. *Each map $v_n\colon (\mathcal{Y}_{n+1}, y_{n+1}) \to (\mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y}_n), \mathrm{gr}\,\eta_n)$ induces an injection of lattices $c_n\colon L_{y_{n+1}} \hookrightarrow \mathrm{Grad}_{F^{\eta_n}}(L_{y_n})$, thus giving a chain of lattices $(L_{y_n}, F^{\eta_n}, c_n)_{n\in\mathbb{N}}$.*
4. *The chain $(L_{y_n}, F^{\eta_n}, c_n)_{n\in\mathbb{N}}$ is the HKKP chain of the normed artinian lattice L_x.*

Proof. We may assume that the good moduli space of $\mathcal{X}$ is a point after replacing $\mathcal{X}$ with the fibre of its good moduli space containing x, since this does not change the $(\mathcal{Y}_n, y_n)$ for $n \geq 1$. By embedding $\mathcal{X}$ in quiver moduli, we can assume by Proposition **??** that

$\mathcal{X} = \mathscr{Rep}(Q, d)$ for some quiver Q and dimension vector d, endowed with the standard lamp and a standard norm on graded points given by positive rational numbers $(m_i)_{i \in Q_0}$, and that x corresponds to a nilpotent representation E of Q.

If x is a closed point, then 1 holds trivially. In fact, x is a closed point if and only if L_x is complemented, by propositions 2.0.6, **??** and 4.0.11. In this case both the balanced filtration and the HKKP filtration are zero, so 2 holds, and 3 and 4 hold trivially. We now assume that x is not closed.

We now prove 2 for $n = 0$. We identify $L_x \cong L_E$, the lattice of subrepresentations of E. We have $\|\lambda\|^2 = \|F^\lambda\|^2$, where $\|\lambda\|^2$ is the norm-squared of λ with respect to the norm on graded points on $\mathcal{X}$, and $\|F^\lambda\|^2$ is the norm-squared of F^λ with respect to the induced norm on the lattice L_x.

On the other hand, Kempf's intersection number of λ and $\mathcal{X}^{\max}$ agrees with the complementedness of F^λ, $\langle \lambda, \mathcal{X}^{\max} \rangle = \langle F^\lambda, \mathfrak{l} \rangle$, by Proposition 3.0.5. The balanced filtration of $(\mathcal{X}, x)$ is the element λ of $\mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1$ with $\langle \lambda, \mathcal{X}^{\max} \rangle \geq 1$ and smallest norm. The HKKP filtration of L_x is the element F of $\mathbf{DF}^{\mathbb{Q}}(L_x)_1$ with $\langle F, \mathfrak{l} \rangle \geq 1$ and smallest norm. Thus λ is the balanced filtration of $(\mathcal{X}, x)$ if and only if F^λ is the HKKP filtration of L_x. This proves 2 for $n = 0$.

We write $\mathrm{Rep}(Q, d) = V$, $G = G(d)$ and $V = W \oplus V^G$, where W is the Reynolds operator of G applied to V, that is, the sum of all simple nontrivial subrepresentations of V.

Let $\pi \colon \mathcal{X} = V/G \to W /\!\!/ G \times V^G$ be the good moduli space. Since E is nilpotent, x lies in W, and the fibre $\mathcal{F} = \pi^{-1}\pi(x) = \pi^{-1}(0)$ is also the fibre of $W/G \to W /\!\!/ G$ at 0, and so a closed substack of W/G containing 0. We may replace $\mathcal{F}$ by W/G since this will only change $\mathcal{Y}_1$ by a stack of which it is a closed substack. To construct $\mathcal{Y}_1$, we blow-up W/G at $(W/G)^{\max} = \{0\}/G$ and consider the centre $\mathcal{Z}$ of the weak Θ-stratum of $\mathrm{Bl}_0\, W/G$ containing x. By Lemma **??**, $\mathcal{Z}$ is an open substack of $\mathrm{Grad}_{\mathbb{Q}}(\mathbb{P}(W)/G)$, since $\mathbb{P}(W)/G$ is the exceptional divisor. In fact, $\mathcal{Z}$ is the centre of the weak Θ-stratum of $\mathbb{P}(W)/G$ containing $u(\mathrm{gr}\, \eta_0)$. Here, the linear form ℓ on graded points considered is given by the line bundle $\mathcal{O}_{\mathbb{P}(W)/G}(1)$. The balanced filtration η_0 is given by a rational cocharacter β of G which induces a direct sum decomposition $W = \bigoplus_{c \in \mathbb{Q}} W_c$. Since $\langle \eta_0, \ell \rangle = 1$, the component of $\mathrm{Grad}_{\mathbb{Q}}(\mathbb{P}(W)/G)$ containing $u(\mathrm{gr}\, \eta_0)$ is $\mathbb{P}(W_1)/L(\beta)$. We also denote ℓ the linear form on $\mathbb{P}(W_1)/L(\beta)$ given by $\mathcal{O}_{\mathbb{P}(W_1)/G}(1)$, and we consider also the linear form $\beta^\vee$ given by $\langle \lambda, \beta^\vee \rangle = (\lambda, \beta)$, where $(-,-)$ is the inner product induced by the norm on graded points.

By [13, Remark 12.21], we can write $\mathcal{Z} = \mathbb{P}(W_1)^{\mathrm{ss}(\ell - \beta^\vee/\|\beta\|^2)}/L(\beta)$, and $\mathcal{Y}_1$ is the $\mathbb{G}_{m,k}$-torsor over $\mathcal{Z}$ determined by $\mathcal{O}(1)$. Thus $\mathcal{Y}_1 = W_1^{\mathrm{ss}(-\beta^\vee)}/L(\beta)$ by Lemma 6.0.8.

Claim 6.0.10. There is a quiver Q', a dimension vector d' and an isomorphism $W_1/L(\beta) \cong \mathscr{Rep}(Q', d')$ as lit stacks such that the induced norm on $\mathscr{Rep}(Q', d')$ is linear.

Proof of Claim. Let $V = \bigoplus_{c \in \mathbb{Q}} V_c$ be the grading that β induces on V. Note that $W_1 = V_1$.

The rational cocharacter β also induces a direct sum decomposition $E_i = \bigoplus_{c \in \mathbb{Q}} E_{i,c}$ for each $i \in Q_0$. We define a quiver Q' as follows. Its set of vertices is

$$Q'_0 = \{(i, c) \in Q_0 \times \mathbb{Q} \mid \; E_{i,c} \neq 0\}.$$

The set of arrows from (i, c) to (j, c') is empty if $c' \neq c + 1$ and otherwise it is the set of arrows from i to j in Q. We set a dimension vector d' of Q' by $d'_{(i,c)} = \dim E_{i,c}$.

In this way we have identifications $V_1 = \mathrm{Rep}(Q', d')$ and $L(\beta) = G(d')$, so we get an isomorphism $V_1/L(\beta) \cong \mathscr{Rep}(Q', d')$. The induced lamp on $V_1/L(\beta)$ is the standard one on $\mathscr{Rep}(Q', d')$: if $\lambda\colon \mathbb{G}_{m,k} \to L(\beta)$ is a cocharacter, then the induced grading on $E_{i,c}$ has only nonnegative weights for all i and c if and only if the grading on E_i has only nonnegative weights for all i. The induced norm is standard with $m_{(i,c)} = m_i$. $\square$

By the claim, $\mathcal{Y}_1$ is an open substack of $\mathscr{Rep}(Q', d')$ admitting a good moduli space, so it is linearly lit by Theorem 5.0.6. This proves 1 for $n = 1$ and, by induction, for all n. Since we knew 2 for $n = 0$, now we know it for all n.

It is left to show that the map $(\mathcal{Y}_1, y_1) \to (\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}), \mathrm{gr}\,\eta_0)$ induces a map $L_{y_1} \hookrightarrow \mathrm{Grad}_{F^{\eta_0}}(L_x)$ of lattices that identifies L_{y_1} with $\Lambda(F^{\eta_0})^{\mathrm{ss}(-(F^{\eta_0})^\vee)}$ (definitions **??** and **??**). By induction, this is enough to prove 3 and 4 for all n.

Recall that $V_1 = W_1$, so $\mathcal{Y}_1 = V_1^{\mathrm{ss}}/L(\beta)$, where semistability is with respect to $\beta^\vee$. Let $\mathcal{U} = V_1/L(\beta) = \mathscr{Rep}(Q', d')$, and let $u \in \mathcal{U}(k)$ be the point y_1 seen as a point of $\mathcal{U}$. The map $(\mathcal{Y}_1, y_1) \to (\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}), \mathrm{gr}\,\eta_0)$ factors as

$$(\mathcal{Y}_1, y_1) \to (\mathcal{U}, u) \to (BL(\beta), 0) \to (V_0/L(\beta), 0) \to (\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}), \mathrm{gr}\,\eta_0).$$

By Proposition 4.0.12, the lattice $L_{(V_0/L(\beta),0)} = L_{(BL(\beta),0)}$ is canonically identified with $\mathrm{Grad}_{F^{\eta_0}}(L_x)$. First we show that the map $(\mathcal{U}, u) \to (BL(\beta), 0)$ induces an injection of lattices $L_u \hookrightarrow \mathrm{Grad}_{F^{\eta_0}}(L_x)$ that identifies L_u with $\Lambda(F^{\eta_0})$.

The representation E of Q, corresponding to the point x, can be written in coordinates as $E = ((E_i)_{i\in Q_0}, (f_\alpha)_{\alpha\in Q_1})$, where $f_\alpha \in \mathrm{Hom}(E_{s(\alpha)}, E_{t(\alpha)})$. Since β induces a direct sum decomposition $E_i = \bigoplus_{c\in\mathbb{Q}} E_{i,c}$ for each $i \in Q_0$, each f_α can be written in coordinates as $f_\alpha = (f_{\alpha,c',c})_{c,c'\in\mathbb{Q}}$, with $f_{\alpha,c',c} \in \mathrm{Hom}(E_{s(\alpha),c}, E_{t(\alpha),c'})$.

The point y_1 is then $(f_{\alpha,c+1,c})_{\alpha\in Q_1, c\in\mathbb{Q}}$ in coordinates, corresponding to a representation E' of Q'. From these descriptions it is clear that $\mathbf{DF}(\mathcal{U}, u)_1 \to \mathbf{DF}(BL(\beta), 0)_1$ induces an injection of lattices $L_u = L_{E'} \hookrightarrow \mathrm{Grad}_{F^{\eta_0}}(L_x)$.

An element of $\mathrm{Grad}_{F^{\eta_0}}(L_x)$ is the data of a subspace $M_{i,c}$ of each $E_{i,c}$. The $M_{i,c}$ define a subrepresentation M of E' if and only if $f_{\alpha,c+1,c}(M_{s(\alpha),c}) \subset M_{t(\alpha),c+1}$ for all $\alpha \in Q_1$ and $c \in \mathbb{Q}$.

Let $M_{i,\geq c} = \left(\bigoplus_{c'>c} E_{i,c'}\right) \oplus M_{i,c}$ and $E_{i,\geq c} = \bigoplus_{c'\geq c} E_{i,c'}$. The $E_{i,\geq c}$ define a subrepresentation $E_{\geq c}$ of E just because β defines a filtration of x. Moreover, $E_{\geq c}/E_{>c}$ is semisimple, and actually null (Definition 3.0.2) because E is nilpotent. Therefore the $M_{i,\geq c}$ define a subrepresentation $M_{\geq c}$ of E.

The condition for the $M_{i,c}$ to define an element of $\Lambda(F^{\eta_0})$ is that $M_{\geq c}/M_{\geq c+1}$ is semisimple (or equivalently, since $M_{\geq c}$ is nilpotent, null) for all $c \in \mathbb{Q}$. Choose a splitting $E_{i,c} = M_{i,c} \oplus M'_{i,c}$ as vector spaces. Then we can write

$$M_{i,\geq c}/M_{i,\geq c+1} = M'_{i,c+1} \oplus \left(\bigoplus_{c<c'<c+1} E_{i,c'}\right) \oplus M_{i,c}.$$

Because F^{η_0} is paracomplemented, we have that $f_{\alpha,c_2,c_1} = 0$ for $c_1 \leq c_2 < c_1 + 1$. Therefore $M_{\geq c}/M_{\geq c+1}$ has coordinates $\left(f_{i,c+1,c}|_{M_i}^{M'_{i,c+1}}\right)$. These are all 0 if and only if $f_{i,c+1,c}(M_{i,c}) \subset M_{i,c+1}$ for all c and i. Therefore $L_u = L_{E'} = \Lambda(F^{\eta_0})$, as desired.

Now by [8, Lemma 5.5.11 and Lemma 5.5.12], we have

$$\mathbf{DF}(\mathcal{Y}_1, y_1)_1 = \{\lambda \in \mathbf{DF}(\mathcal{U}, u)_1 \mid \beta^\vee(\lambda) = 0\}.$$

Therefore

$$\begin{aligned} L_{y_1} = \{\lambda \in \mathbf{DF}(\mathcal{U}, u)_1 \mid \ &\lambda \geq 0,\ \beta^\vee(\lambda) = 0,\ \text{and if } \mu \in \mathbf{DF}(\mathcal{U}, u)_1 \\ &\text{satisfies } 0 \leq 2\mu \leq \lambda \text{ and } \beta^\vee(\mu) = 0,\ \text{then } \mu = 0\}. \end{aligned}$$

Note that if $\mu \in \mathbf{DF}(\mathcal{U}, u)_1$ and $0 \leq 2\mu \leq \lambda$, then $0 \leq 2\beta^\vee(\mu) \leq \beta^\vee(\lambda)$, so $\beta^\vee(\lambda) = 0$ implies $\beta^\vee(\mu) = 0$. Therefore the map $\mathbf{DF}(\mathcal{Y}_1, y_1)_1 \to \mathbf{DF}(\mathcal{U}, u)_1$ sends L_{y_1} injectively into L_u, and $L_{y_1} = \{\lambda \in L_u \mid \beta^\vee(\lambda) = 0\}$. Note that $\beta^\vee(\lambda) = (F^{\eta_0})^\vee(\lambda)$ by Lemma 6.0.7, so we get $L_{y_1} = \Lambda(F^{\eta_0})^{\mathrm{ss}}$, as desired. □

Corollary 6.0.11. *In the setting of Theorem 6.0.9 we have that, under the canonical bijection $\mathbb{Q}^\infty$-Filt$(\mathcal{X}, x) = \mathbb{Q}^\infty$-Filt$(L_x)$, the iterated balanced filtration of x equals the iterated HKKP filtration of L_x.*

References

[1] J. Alper, J. Hall, and D. Rydh. A Luna étale slice theorem for algebraic stacks. *Annals of Mathematics*, 191(3):675, 2020.

[2] J. Alper, J. Hall, and D. Rydh. The étale local structure of algebraic stacks. Preprint, arXiv:1912.06162v3, 2021.

[3] J. Alper, D. Halpern-Leistner, and J. Heinloth. Existence of moduli spaces for algebraic stacks. *Inventiones Mathematicae*, aug 2023.

[4] M. Artin and J. J. Zhang. Abstract Hilbert schemes. *Algebras and representation theory*, 4(4):305–394, 2001.

[5] C. Bu. Enumerative invariants in self-dual categories. i. motivic invariants, 2023.

[6] V. Drinfeld. On algebraic spaces with an action of $\mathbb{G}_m$, 2015.

[7] F. Haiden, L. Katzarkov, M. Kontsevich, and P. Pandit. Semistability, modular lattices, and iterated logarithms. *Journal of Differential Geometry*, 123(1):21 – 66, 2023.

[8] D. Halpern-Leistner. On the structure of instability in moduli theory. `http://pi.math.cornell.edu/~danielhl/theta_stability_2019_10_13.pdf`, 2021.

[9] A. Ibáñez Núñez. On canonical filtrations of modular lattices. In preparation.

[10] A. Ibáñez Núñez. Stratifications and canonical sequential filtrations for normed good moduli stacks. In preparation.

[11] Ł. Jelisiejew, Joachim; Sienkiewicz. Białnicki-Birula decomposition for reductive groups. *Journal de Mathématiques Pures et Appliquées*, 131:290–325, 2019.

[12] G. Kempf and L. Ness. The length of vectors in representation spaces. In K. Lønsted, editor, *Algebraic Geometry*, pages 233–243, Berlin, Heidelberg, 1979. Springer Berlin Heidelberg.

[13] F. C. Kirwan. *Cohomology of quotients in symplectic and algebraic geometry*. Princeton University Press. Mathematical notes ; 31. Princeton University Press, Princeton, 1984.

[14] J. P. Pridham. Presenting higher stacks as simplicial schemes. *Advances in Mathematics*, 238:184–245, 2013.