Stratifications and canonical sequential filtrations for normed good moduli stacks

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Abstract

We define a canonical stratification on any noetherian stack $\mathcal{X}$ with affine diagonal that admits a good moduli space and is endowed with a norm on graded points. This generalises a construction of Kirwan in Geometric Invariant Theory. The strata live in the *stack of sequential filtrations* of $\mathcal{X}$, which we define. Therefore the stratification gives a canonical sequential filtration, the *iterated balanced filtration*, for each point of $\mathcal{X}$. We conjecture that this filtration describes the asymptotics of the negative gradient flow of the Kempf-Ness potential in Geometric Invariant Theory, in analogy with work of Haiden-Katzarkov-Kontsevich-Pandit, and we expect it to describe the asymptotics of natural flows in other moduli problems as well. In the case of quotient stacks by diagonalisable algebraic groups, we give an explicit description of the iterated balanced filtration in terms of convex geometry.

Contents

1 Introduction

There is a fascinating relation in moduli theory between three *a priori* quite different phenomena: stratifications, filtrations and flows. To illustrate this, consider the moduli stack $\mathrm{Bun}(C)$ of vector bundles on a smooth projective curve C over the complex numbers. Every vector bundle $E \in \mathrm{Bun}(C)(\mathbb{C})$ has a canonical filtration, its Harder-Narasimhan filtration $\lambda_{\mathrm{HN}}(E)$, and the assignment of $\lambda_{\mathrm{HN}}(E)$ for every E defines a stratification of $\mathrm{Bun}(C)$ by Harder-Narasimhan type. This stratification was used by Atiyah and Bott to compute Betti numbers of moduli spaces of semistable vector bundles [9]. On the other hand, one can consider the Yang-Mills flow on the space M_E of hermitian metrics on the vector bundle E, and it turns out that the slope at infinity of the flow from any initial metric is the Harder-Narasimhan filtration of E.

However, the Harder-Narasimhan filtration is insufficient to detect more subtle asymptotic behaviour of the Yang-Mills flow. In recent work, Haiden-Katzarkov-Kontsevich-Pandit gave a complete description of the asymptotics of the Yang-Mills flow for E in terms of a refinement of the Harder-Narasimhan filtration, which we will call in this work the HKKP filtration. Following the parallelism between flows, filtrations and stratifications, the assignment of the HKKP filtration for every E should define a stratification of $\mathrm{Bun}(C)$ by type of HKKP filtration, refining the stratification by Harder-Narasimhan type. Refinements of this kind had previously been considered by Kirwan for smooth GIT quotients [39] and for $\mathrm{Bun}(C)$ [38].

In this work, we propose a simultaneous generalisation of the HKKP filtration and Kirwan's refined stratifications. We create a framework where they can be compared, as well as extending both theories to a general setup that covers a wide range of new examples. Our construction produces new stratifications and canonical filtrations in GIT over general bases, for moduli stacks of principal bundles on a curve, for moduli of objects at the heart of a Bridgeland stability condition, and for moduli of K-semistable Fano varieties. In all these examples there is an analogue of the Yang-Mills flow, and we expect our filtration to describe its asymptotic behaviour. We provide a conjectural statement in the case of GIT for affine spaces.

1.1 Normed good moduli stacks

The stratification of $\mathrm{Bun}(C)$ by Harder-Narasimhan type is a Θ-stratification [26], so every stratum $\mathcal{S}$ has an $\mathbb{A}^1$-retraction $\mathcal{S} \to \mathcal{Z}$ onto what is called its centre $\mathcal{Z}$. In the case of $\mathrm{Bun}(C)$, as well as in many other examples, $\mathcal{Z}$ enjoys the property of admitting a good moduli space $\mathcal{Z} \to Z$, that is, a map to an algebraic space Z that best approximates the stack $\mathcal{Z}$ (we recall the precise definition from [1] in 2.1.1). Having a good moduli space is a strong condition on the stack that implies many desirable properties.

We may hope that good moduli stacks have canonical stratifications, and that when pulled back along the retraction $\mathcal{S} \to \mathcal{Z}$ from each centre $\mathcal{Z}$ would produce the sought-after stratification by HKKP type. This is close to being true. What is missing is stability type data on $\mathcal{Z}$ on which this stratification depends. The correct notion for our purposes turns out to be a *norm on graded points* of $\mathcal{Z}$, a concept from the Beyond GIT programme [26].

A *graded point* of a stack $\mathcal{X}$ is a map $g\colon B\mathbb{G}_{m,k} \to \mathcal{X}$, where $B\mathbb{G}_{m,k}$ is the classifying stack of the multiplicative group over a field k, and they can also be seen as ordinary points on the mapping stack $\mathrm{Grad}(\mathcal{X}) = \underline{\mathrm{Hom}}(B\mathbb{G}_m, \mathcal{X})$. A norm on graded points of $\mathcal{X}$ is roughly speaking the data of a positive real number $\|g\|$ for every graded point $g\colon B\mathbb{G}_{m,k} \to \mathcal{Z}$, and this data is required to satisfy some local constancy and nondegeneracy conditions (see Definition 2.3.3). For $\mathrm{Bun}(C)$, a natural norm is given by the rank of vector bundles. In this case, a graded point $g\colon B\mathbb{G}_{m,\mathbb{C}} \to \mathrm{Bun}(C)$ corresponds to a vector bundle E together with a direct sum decomposition $E = \bigoplus_{c\in\mathbb{Z}} E_c$, and its norm is then defined by the formula $\|g\|^2 = \sum_{c\in\mathbb{Z}} c^2\,\mathrm{rk}(E_c)$. This restricts to a norm on graded points on each of the centres $\mathcal{Z}$ of the Harder-Narasimhan stratification.

Our aim thus becomes to stratify and produce canonical filtrations for algebraic stacks $\mathcal{X}$ admitting a good moduli space and endowed with a norm on graded points, what we call *normed good moduli stacks* for simplicity.

1.2 Sequential filtrations and stratifications

The first obstacle we encounter is the very meaning of filtration in this generality. The HKKP filtration of a vector bundle E consists of both a chain

$$0 \neq E_0 \subset E_1 \subset \cdots \subset E_n = E \tag{1}$$

of subbundles and a chain

$$c_0 > c_1 > \cdots > c_n \tag{2}$$

of labels $c_i \in \mathbb{Q}^\infty$. Here, $\mathbb{Q}^\infty$ is the set of eventually zero sequences of rational numbers, ordered lexicographically. In this sense the HKKP filtration is a *sequential filtration*, or $\mathbb{Q}^\infty$*-filtration*, of E.

We first look at the well-studied case of $\mathbb{Z}$-filtrations, that is, when the labels c_i are integers. These are closely related to the quotient stack $\mathbb{A}^1/\mathbb{G}_m$ of the affine line $\mathbb{A}^1$ by the scaling action of the multiplicative group, also denoted $\Theta = \mathbb{A}^1/\mathbb{G}_m$. Indeed, the mapping stack $\mathrm{Filt}(\mathrm{Bun}(C)) = \underline{\mathrm{Hom}}(\mathbb{A}^1/\mathbb{G}_m, \mathrm{Bun}(C))$, parametrises vector bundles endowed with a $\mathbb{Z}$-filtration [26, 30]. Therefore it makes sense to define a $\mathbb{Z}$-filtration of a k-point x in a general stack $\mathcal{X}$ to be a map $f\colon \Theta_k \to \mathcal{X}$ together with an isomorphism $x \sim f(1)$, and to define the stack of filtrations on $\mathcal{X}$ to be the mapping stack $\mathrm{Filt}(\mathcal{X}) = \underline{\mathrm{Hom}}(\Theta, \mathcal{X})$. There is an *associated graded map* $\mathrm{gr}\colon \mathrm{Filt}(\mathcal{X}) \to \mathrm{Grad}(\mathcal{X})$ and a forgetful map $\mathrm{ev}_1\colon \mathrm{Filt}(\mathcal{X}) \to \mathcal{X}$.For our purposes, it is better to consider $\mathbb{Q}$-filtrations, and we work instead with the stacks of $\mathbb{Q}$-filtrations $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ and of $\mathbb{Q}$-gradings $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$, that are constructed by formally localising with respect to the natural action of the monoid $(\mathbb{Z}_{>0}, \cdot, 1)$ on $\mathrm{Filt}(\mathcal{X})$ and $\mathrm{Grad}(\mathcal{X})$ (Definition 2.2.6).

Our first goal is to construct an algebraic stack $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ that, when $\mathcal{X} = \mathrm{Bun}(C)$, parametrises vector bundles E endowed with a sequential filtration. To this aim, we observe that giving a $\mathbb{Q}^\infty$-filtration on a vector bundle E on C is equivalent to first giving a $\mathbb{Q}$-filtration $F_\bullet$ of E, then a $\mathbb{Q}$-filtration of the associated graded object $\mathrm{gr}\, F_\bullet$, and so on, until the process finishes in finitely many steps. For general stacks, we formalise this idea by defining the stack $\mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X})$ of $\mathbb{Q}^n_{\mathrm{lex}}$-filtrations (filtrations labelled by $\mathbb{Q}^n$ with lexicographic order) inductively as a fibre product $\mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) = \mathrm{Filt}_{\mathbb{Q}^{n-1}_{\mathrm{lex}}}(\mathcal{X}) \times_{\mathrm{gr},\mathrm{Grad}_{\mathbb{Q}^{n-1}}(\mathcal{X}),\mathrm{ev}_1} \mathrm{Filt}_{\mathbb{Q}}\left(\mathrm{Grad}_{\mathbb{Q}^{n-1}}(\mathcal{X})\right)$. Then we define the stack $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ of *sequential filtrations* on $\mathcal{X}$ as a colimit of the stacks $\mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X})$ when n tends to ∞ (Definition 3.2.2). This provides a meaningful notion of sequential filtration for general stacks.

The relation between stratifications of $\mathcal{X}$ and sequential filtrations is encapsulated in the definition of *sequential stratification*.

Definition 1.2.1 (Definition 3.3.1)**.** A *sequential stratification* of an algebraic stack $\mathcal{X}$ is a family $(\mathcal{S}_\alpha)_{\alpha\in\Gamma}$ of locally closed substacks of $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ indexed by a partially ordered set Γ such that

1. each composition $\mathcal{S}_c \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathcal{X}$ is a locally closed immersion,
2. the topological spaces $|\mathcal{S}_c|$ are pairwise disjoint and cover $|\mathcal{X}|$, and
3. for every $c \in \Gamma$, the union $\bigcup_{c'\leq c} |\mathcal{S}_{c'}|$ is open in $\mathcal{X}$.

Thus the strata $\mathcal{S}_c$ are locally closed substacks of $\mathcal{X}$ together with a choice of lift to $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$. Therefore a sequential stratification provides each point x of $\mathcal{X}$ with a choice of sequential filtration of x.

Our main construction (Theorem 3.5.2) produces a canonical sequential stratification for every noetherian normed good moduli stack $\mathcal{X}$ with affine diagonal. We call it the *balancing stratification*, since the adjective *balanced* was used both by Kirwan and Haiden-Katzarkov-Kontsevich-Pandit to describe the filtrations they studied. The balancing stratification produces a canonical sequential stratification, the *iterated balanced filtration*, for every point of the stack $\mathcal{X}$.

The definition of sequential stratification is inspired in Halpern-Leistner's definition of Θ-stratification of a stack $\mathcal{X}$ [26, Definition 2.1.2] (recalled in Definition 2.5.1), which roughly speaking is a partition of $\mathcal{X}$ into locally closed substacks $\mathcal{S}_c$ of $\mathcal{X}$ that are also open substacks of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$. Very importantly, each stratum $\mathcal{S}_c$ retracts onto what is called its *centre* $\mathcal{Z}_c$, which is an open substack of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$. Note that the concept of sequential stratification is weaker in the sense that we do not demand the existence of centres for the strata. Establishing existence and nice properties of certain Θ-stratifications is a crucial ingredient of our construction of the balancing stratification.

If $\mathcal{X}$ is a normed good moduli stack, $f\colon \mathcal{Y} \to \mathcal{X}$ is representable projective, and $\mathcal{L}$ is an f-ample line bundle on $\mathcal{Y}$, it is a result of Halpern-Leistner that the norm on $\mathcal{X}$ and line bundle $\mathcal{L}$ define a (weak) Θ-stratification $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$ of $\mathcal{Y}$. This generalises Kirwan's construction of the instability stratification in GIT [40]. For our purposes, it is important that the centres $\mathcal{Z}_c$ of the strata $\mathcal{S}_c$ have good moduli spaces, what constitutes Theorem 2.6.3. For the proof, we need to consider the more general case when f is proper and the line bundle $\mathcal{L}$ is replaced by the weaker notion of *linear form on graded points*, and then use the concepts of Θ-monotonicity and S-monotonicity developed in [26] to check Θ-reductivity and S-completeness of $\mathcal{Z}_c$, which implies the existence of a good moduli space [7].

1.3 The construction

In this paper, we construct a totally ordered set $\mathbf{\Gamma}$ and, for every normed good moduli stack $\mathcal{X}$, a canonical sequential stratification $(\mathcal{S}^{\mathcal{X}}_\alpha)_{\alpha\in\mathbf{\Gamma}}$ of $\mathcal{X}$ indexed by $\mathbf{\Gamma}$, that we call the *balancing stratification*. The poset $\mathbf{\Gamma}$ is defined very explicitly. It consists of sequences $((d_0, c_0), \ldots, (d_n, c_n))$ with $d_0, d_1, \cdots, d_n$ in $\mathbb{N}$, $c_0, \ldots, c_{n-1} \in \mathbb{Q}_{>0}$, $c_n = \infty$, and satisfying some other conditions (see Definition 3.5.1), and the poset structure is given by lexicographic order. The balancing stratification is constructed using three main ingredients: the substack $\mathcal{X}^{\max}$ of point with maximal stabiliser dimension, existence and nice properties of Θ-stratifications as above, and the concept of *central rank* of a stack, that is useful for inductive arguments. The balancing stratification is characterised as the unique assignment that gives a sequential stratification for every normed good moduli stack and satisfies certain defining properties (Theorem 3.5.2). We sketch the construction here.

The *central rank* of $\mathcal{X}$, denoted $z(\mathcal{X})$, is the biggest natural number n such that $B\mathbb{G}_m^n$ acts on $\mathcal{X}$ in a nondegenerate way (Definition 3.4.1). The condition can be thought of as every stabiliser of $\mathcal{X}$ containing a copy of $\mathbb{G}_m^n$ in its centre. The maximal dimension of a stabiliser of $\mathcal{X}$ is denoted $d(\mathcal{X})$. Our construction is by induction on $N(\mathcal{X}) := d(\mathcal{X}) - z(\mathcal{X})$.

The main observation that will allow us to perform induction is that, whenever we have a blow-up $f\colon \mathcal{Y} = \mathrm{Bl}_{\mathcal{R}}\,\mathcal{X} \to \mathcal{X}$ of $\mathcal{X}$ along some closed substack $\mathcal{R}$, if the Θ-stratification of $\mathcal{Y}$ is $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$, then for every unstable stratum $\mathcal{S}_c$, its centre $\mathcal{Z}_c$ has bigger central rank than $\mathcal{X}$: $z(\mathcal{Z}_c) > z(\mathcal{X})$ (Lemma 3.4.6). We also have $d(\mathcal{Z}_c) \leq d(\mathcal{X})$ by representability of f, so that $N(\mathcal{Z}_c) < N(\mathcal{X})$.

For our construction we will make a canonical choice of $\mathcal{R}$, namely the locus of points of $\mathcal{X}$ with maximal stabiliser dimension, which can be given a natural closed substack structure $\mathcal{X}^{\max}$ [19, Appendix C]. The most unstable stratum in the balancing stratification for $\mathcal{X}$ is $\mathcal{S}^{\mathcal{X}}_{(d(\mathcal{X}),\infty)} := \mathcal{X}^{\max}$, embedded in $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ via the trivial filtration map $\mathcal{X} \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$. If $N(\mathcal{X}) = 0$, then $|\mathcal{X}^{\max}| = |\mathcal{X}|$ and this is the only stratum. In the general case, for the other strata we consider the blow-up $\mathcal{Y} = \mathrm{Bl}_{\mathcal{X}^{\max}}\,\mathcal{X} \to \mathcal{X}$, which has a Θ-stratification $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$ where the centre $\mathcal{Z}_c$ of each unstable stratum $\mathcal{S}_c$ is again a normed good moduli stack and $N(\mathcal{Z}_c) < N(\mathcal{X})$. By induction, the balancing stratification $(\mathcal{S}^{\mathcal{Z}_c}_\alpha)_{\alpha\in\mathbf{\Gamma}}$ is defined for $\mathcal{Z}_c$. We can pullback each $\mathcal{S}^{\mathcal{Z}_c}_\alpha$ along the associated graded map $\mathcal{S}_c \to \mathcal{Z}_c$ to get a stack $\mathcal{V}_{c,\alpha}$. The stack $\mathcal{Z}_c$ lives as an open substack in $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y})$, and so $\mathcal{S}^{\mathcal{Z}_c}_\alpha$ lives in $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y}))$. The iterative definition of $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{Y})$ gives a cartesian square

$$\begin{tikzcd} \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{Y}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y})) \arrow[d] \\ \mathrm{Filt}_{\mathbb{Q}}(\mathcal{Y}) \arrow[r, "\mathrm{gr}"] & \mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y}) \end{tikzcd}$$

and hence $\mathcal{V}_{c,\alpha}$ lives in $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{Y})$. After subtracting the exceptional divisor $\mathcal{E}$, we get the $\mathbb{Q}^\infty$-stratum $\mathcal{S}^{\mathcal{X}}_{\alpha'} := \mathcal{V}_{c,\alpha} \setminus \mathcal{E}$ of $\mathcal{X}$, where α' is the element of $\mathbf{\Gamma}$ obtained by concatenating $(d(\mathcal{X}), c)$ and α.

In this way we get strata for $\mathcal{X}$ that cover $\mathcal{F} = \pi^{-1}\pi(\mathcal{X}^{\max})$, where $\pi\colon \mathcal{X} \to X$ is the good moduli space. Since $N(\mathcal{X} \setminus \mathcal{F}) < N(\mathcal{X})$, we get the rest of the strata for $\mathcal{X}$ from the balancing stratification of $\mathcal{X} \setminus \mathcal{F}$ by induction.

We show that the balancing stratification has some nice functorial properties. If $f\colon \mathcal{X} \to \mathcal{X}'$ is either a closed immersion or a base change from a flat map $X \to X'$ between the good moduli spaces, then for all $\alpha \in \mathbf{\Gamma}$, the stratum $\mathcal{S}^{\mathcal{X}}_\alpha$ equals the pullback $\mathcal{S}^{\mathcal{X}'}_\alpha \times_{\mathcal{X}',f} \mathcal{X}$ of $\mathcal{S}^{\mathcal{X}'}_\alpha$ along f (Proposition 3.5.5).

1.4 Relation to convex geometry and artinian lattices

It turns out that, despite its seemingly convoluted definition involving several blow-ups and Θ-stratifications, the iterated balanced filtration has a particularly simple description for a point x in quotient stacks of the form $\mathrm{Spec}\, A/G$, where G is a diagonalisable algebraic group over a field k (for example a split torus $\mathbb{G}^n_{m,k}$) and A is a finite type k-algebra, and where the norm on graded points comes from a norm on cocharacters of G. We denote $N = \Gamma^{\mathbb{Q}}(G)$ the set of rational cocharacters of G. The norm on graded points corresponds to a rational inner product $(-,-)$ on $N = N_0$, and we identify N and its dual via this inner product. For simplicity, in this introduction we will assume that $\operatorname{Spec} A$ is the total space of a vector space $V = k^l$, that G acts on V via the characters $\chi_1, \dots, \chi_l \in \Gamma_{\mathbb{Z}}(G) = \operatorname{Hom}(G, \mathbb{G}_{m,k})$ and that we are interested in computing the iterated balanced filtration of the point $x = (1, \dots, 1)$ in the quotient stack V/G.

The iterated balanced filtration of x is determined by a finite sequence $\lambda_0, \dots, \lambda_n \in N$ of one-parameter subgroups of G (Remark 3.2.16). We now describe how to compute the λ_i in terms of the *state* $\Xi_0 = \{\chi_1, \dots, \chi_l\}$ of $x \in V$.

Let F_0 be the smallest face of the cone $\operatorname{cone}(\Xi_0)$ generated by Ξ_0 inside N that contains 0. Then λ_0 is the unique element of the orthogonal complement $N_1 \coloneqq F_0^{\perp}$ such that $(\lambda_0, \alpha) \geq 1$ for all $\alpha \in \Xi_0 \setminus F_0$ and $\|\lambda_0\|$ is minimal. This is a convex optimisation problem.

To compute $\lambda_1, \cdots, \lambda_n$, we proceed as follows. We let $\Xi_1 = \{p_1(\alpha) \in \Xi_0 \mid (\lambda_0, \alpha) = 1\} \subset N_1$, where $p_1 \colon N \to N_1$ is the orthogonal projection. It will always be the case that $\lambda_0 \in \operatorname{cone}(\Xi_1) \subset N_1$ (Theorem 5.1.19), and thus there is a smallest face F_1 of $\operatorname{cone}(\Xi_1)$ containing λ_0. Let N_2 be the orthogonal complement of F_1 inside N_1. Then λ_1 is the unique element in N_2 such that $(\lambda_1, \alpha) \geq 1$ for all $\alpha \in \Xi_1 \setminus F_1$ and $\|\lambda_1\|$ is minimal. We define $\Xi_2 = \{p_2(\alpha) \in \Xi_1 \mid (\lambda_1, \alpha) = 1\} \subset N_2$, where $p_2 \colon N_1 \to N_2$ is the orthogonal projection. Repeating this process, we get $\lambda_2, \dots, \lambda_n$. The algorithm terminates when we get to $\Xi_n \setminus F_n = \varnothing$.

Our results showing that this procedure computes the iterated balanced filtration are Theorem 5.2.16 and Corollary 5.2.18. For the proof, we use the machinery of *chains of stacks*. A chain of stacks is the data of a sequence of pointed k-stacks $(\mathcal{X}_n, x_n)$, together with a $\mathbb{Q}$-filtration λ_n of each x_n and linking maps $(\mathcal{X}_{n+1}, x_{n+1}) \to (\operatorname{Grad}(\mathcal{X}_n), \operatorname{gr} \lambda_n)$ (Definition 4.1.1). Associated to such chain there is a $\mathbb{Q}^\infty$-filtration of the point x_0 in $\mathcal{X}_0$. For a pointed normed good moduli stack $(\mathcal{X}, x)$, we give two different constructions of chains with $(\mathcal{X}_0, x_0) = (\mathcal{X}, x)$, the *balancing chain* (Construction 4.2.1) and the *torsor chain* (Construction 4.3.1). The former is closely related to the balancing stratification, while the former tends to be closer to combinatorial structures, like states or lattices. We show that both chains compute the iterated balanced filtration of $(\mathcal{X}, x)$ (Proposition 4.2.6 and Theorem 4.3.4).

In order to relate the convex-geometrical pictures of states to chains of stacks, we define a category of normed semistable polarised states (combinatorial data), a notion of chain in this category, and a canonical *balancing chain* for every object. Then we define a functor from normed semistable polarised states to pointed normed good moduli stacks, and show that the functor sends the balancing chain to the torsor chain.

The theory of chains of stacks will be used to establish a correspondence between the iterated balanced filtration as defined in this article and the HKKP filtration for normed artinian lattices in a sequel to this paper [33].

Theorem 1.4.1. *Let k be an algebraically closed field and let $\mathcal{A}$ be a locally noetherian k-linear Grothendieck abelian category. Suppose that the moduli stack of objects $\mathcal{M}_{\mathcal{A}}$ is an algebraic stack locally of finite type over k, and let $\mathcal{X}$ be a quasi-compact open substack of $\mathcal{M}_{\mathcal{A}}$ admitting a good moduli space $\pi \colon \mathcal{X} \to X$, and endowed with a linear norm on graded points. For any k-point $x \in \mathcal{X}(k)$, there is a canonically defined normed artinian lattice L_x and a canonical bijection $\mathbb{Q}^\infty\text{-}\operatorname{Filt}(L_x) \cong \mathbb{Q}^\infty\text{-}\operatorname{Filt}(\mathcal{X}, x)$ under which the HKKP filtration of L_x and the iterated balanced filtration of $(\mathcal{X}, x)$ agree.*

The norm being linear means it is compatible with the underlying abelian category in a precise sense. Examples of this setup include moduli spaces of Bridgeland semistable objects and moduli spaces of semistable vector bundles on a curve C (see Section 3.6.5 and Section 3.6.4). For the proof of this theorem, we will need a new characterisation of the HKKP filtration for artinian lattices established in [34].

1.5 Asymptotics of flows

In view of Theorem 1.4.1 and the results of Haiden-Katzarkov-Kontsevich-Pandit on asymptotics of gradient flows in the case of quiver representations and vector bundles on a smooth projective complex algebraic curve [23, 24], we expect the iterated balanced filtration to play a role in describing asymptotics of natural gradient flows in moduli theory. We now make this expectation precise in the case of Geometric Invariant Theory in the affine setting.

Let us begin by recalling the Kempf-Ness potential in a more general framework. Let G be a connected reductive algebraic group over $\mathbb{C}$, endowed with a norm on cocharacters (Definition 2.3.7), and let K be a maximal compact subgroup of G. We denote $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{k}$ the Lie algebra of K. We consider a smooth projective-over-affine scheme X over $\mathbb{C}$, endowed with an action of G. An ample line bundle $\mathcal{L}$ on X/G (that is, an ample line bundle L on X with a G-equivariant structure) defines an open semistable locus $(X/G)^{\mathrm{ss}} = X^{\mathrm{ss}}/G$, which is the open stratum of a Θ-stratification of X/G (Theorem 2.6.3). The line bundle $\mathcal{L}$ restricts to a line bundle $\overline{\mathcal{L}}$ on the differentiable stack "of metrics" X/K, and we endowed $\overline{\mathcal{L}}$ with a hermitian norm $\|-\|$ (that is, we endow L with a K-equivariant hermitian norm). For a point $x \in X(\mathbb{C})$, after a choice of nonzero lift x^* of x to the total space of $\overline{\mathcal{L}}$, the *Kempf-Ness potential* is defined to be

$$p_x\colon G/K \to \mathbb{R}\colon Kg \mapsto \log\|x^*\| - \log\|gx^*\|.$$

Here, G/K denotes the quotient by the action of K on G by right translations. Note that the Kempf-Ness potential is independent of the choice of lift x^* of x. Kempf-Ness type theorems state that, under some conditions, the following equivalences hold:

1. x is semistable if and only if p_x is bounded below.
2. x is polystable if and only if p_x attains a minimum.

This is the case, for example, if either X is projective [21, 35, 40] or X is affine and some additional conditions are satisfied [31, 37, 42]. In these cases, if x is polystable, then the negative gradient flow of p_x converges to a minimum from any starting point. In the strictly semistable case, we are interested in understanding the asymptotic behaviour of this flow.

In order to define the gradient flow, we need a Riemannian metric on G/K. This comes from the norm on cocharacters of G, which gives an invariant euclidean inner product on $\mathfrak{k}$ that is then be used to define a Riemannian metric on G/K (see [21, Appendix C] for details). This allows us to define the gradient vector field ∇p_x on G/K.

The semistable locus X^{ss}/G is a good moduli stack, endowed with a norm on graded points coming from the norm on cocharacters of G. If $x \in X(\mathbb{C})$ is a semistable point, then the iterated balanced filtration $\lambda_{\mathrm{ib}}(x)$ of x is defined, and by Remark 3.2.16 it is identified with an equivalence class of sequences of commuting rational one-parameter subgroups $\lambda_1, \dots, \lambda_n$. We may choose representatives λ_i that are compatible with K, in the sense that for some power $\lambda_i^l\colon \mathbb{C}^\times \to G$ that is integral (and hence for any such power), the inclusion $\lambda_i^l(S^1) \subset K$ holds. The λ_i induce well-defined linear maps on Lie algebras $\mathrm{Lie}(\lambda_i)\colon \mathbb{C} \to \mathfrak{g}$ defined over $\mathbb{R}$, giving elements $\nu_i = \mathrm{Lie}(\lambda_i)(1) \in \mathfrak{k}$.

We now shift our attention to a particular case of the above setting. We will take $X = V$ to be a finite dimensional G-representation. Let $\alpha\colon G \to \mathbb{G}_{m,\mathbb{C}}$ be a character, and take $\mathcal{L} = \mathcal{O}_{X/G}(\alpha) = (X/G \to B\mathbb{G}_{m,\mathbb{C}})^*\alpha$, where we regard $\alpha \in \operatorname{Pic}(BG)$. Choose a K-invariant hermitian metric $\|-\|$ on V. The total space of $\mathcal{L}$ is $(V \times \mathbb{C})/G$, where the action is $g(x, c) = (gx, \alpha(g)c)$. The norm $\|-\|$ defines a hermitian norm $\|-\|_{\overline{\mathcal{L}}}$ on $\overline{\mathcal{L}}$ by the formula

$$\|(v, c)\|_{\overline{\mathcal{L}}} = e^{-\|v\|^2}\,|c|\,.$$

The associated Kempf-Ness potential for a point $x \in V$ is

$$p_x(Kg) = \|gx\|^2 - \log|\alpha(g)| - \|x\|^2.$$

Suppose that $x \in V$ is semistable and let $\nu_1, \ldots, \nu_n \in \mathfrak{k}$ represent the iterated balanced filtration of x as above. By Hadamard's theorem, the exponential map $\exp\colon \mathfrak{g} \to G$ composed with the projection $G \to G/K$ restricts to a diffeomorphism $\mathfrak{k} \cong i\mathfrak{k} \to G/K$ (see [21, Appendix A]), whose inverse we denote $\log\colon G/K \to \mathfrak{k}$. With this setup, we conjecture:

Conjecture 1.5.1. *Let $h\colon (0, \infty) \to G/K$ be a flow line for $-\nabla p_x$. Then the expression*

$$\log h(t) + \log(t)\nu_1 + \log\log(t)\nu_2 + \cdots + \log^{\circ n}(t)\nu_n$$

in $\mathfrak{k}$ is bounded for $t >> 0$.

In the case where $V = \bigoplus_{a\in Q_1} \operatorname{Hom}(\mathbb{C}^{d_{s(a)}}, \mathbb{C}^{d_{t(a)}})$ is the representation space of a quiver Q with dimension vector d and $G = \prod_{i\in Q_0} \mathrm{GL}_{d_i,\mathbb{C}}$ with the standard action, the conjecture is true for specific choices of hermitian norm on V and norm on cocharacters of G by Theorem 1.4.1 and [24, Theorem 5.11]. See Example 5.2.19 for a simple example where the conjecture is checked beyond the quiver case.

Our expectation is that the iterated balanced filtration describes also the asymptotics of natural gradient flows in other moduli problems. Examples include the Calabi flow for a K-semistable Fano variety, the Yang-Mills flow for semistable G-bundles on a smooth projective curve and the gradient flow for the Kempf-Ness potential in more general GIT situations. In all these examples, there is an underlying normed good moduli stack of semistable objects, so the iterated balanced filtration is defined.

1.6 Notation and conventions.

The set of nonnegative integers, or natural numbers, is denoted $\mathbb{N} = \mathbb{Z}_{\geq 0}$.

We follow the definitions and conventions of [50] regarding algebraic stacks. For an algebraic stack $\mathcal{X}$, we denote $|\mathcal{X}|$ its topological space. If $x\colon T \to \mathcal{X}$ a T-point, with T a scheme, we denote $\operatorname{Aut}(x)$ the automorphism group of x. If T is the spectrum of a field and $\mathcal{X}$ is quasi-separated, then $\operatorname{Aut}(x)$ is a scheme [50, Tag 0DTS]. The multiplicative group over $\mathbb{Z}$ is denoted $\mathbb{G}_m = \operatorname{Spec}\mathbb{Z}[t, t^{-1}]$ and, for a scheme T, we use the notation $\mathbb{G}_{m,T} = \mathbb{G}_m \times T$.

For a group scheme G acting on an algebraic space X, we use the notation X/G for the quotient stack, omitting the customary brackets.

For a field k and an algebraic group G over k, we denote $\Gamma_{\mathbb{Z}}(G), \Gamma^{\mathbb{Z}}(G), \Gamma_{\mathbb{Q}}(G)$ and $\Gamma^{\mathbb{Q}}(G)$ the sets of characters, cocharacters, rational characters and rational cocharacters of G, respectively. If $\lambda \in \Gamma^{\mathbb{Z}}(G)$ is a cocharacter, and G acts on a scheme X over k, then we denote $X^{\lambda,0}$ the fixed point locus of the induced $\mathbb{G}_m$-action on X and $X^{\lambda,+}$ the *attractor*, defined functorially on k-schemes T by the formula $\operatorname{Hom}(T, X^{\lambda,+}) = \operatorname{Hom}^{\mathbb{G}_{m,k}}(\mathbb{A}^1_T, X)$, where $\operatorname{Hom}^{\mathbb{G}_{m,k}}$ denotes $\mathbb{G}_{m,k}$-equivariant maps and $\mathbb{A}^1_k$ is endowed with the usual scaling action [18]. For the particular case of the conjugation action of G on itself, we denote

$L(\lambda) = G^{\lambda,0}$ and $P(\lambda) = G^{\lambda,+}$. If G is reductive, then $P(\lambda)$ is a parabolic subgroup with Levi factor $L(\lambda)$. If $g \in G(k)$, we denote $\lambda^g = g\lambda g^{-1}$.

If $\mathcal{F}$ is a vector bundle on an algebraic stack $\mathcal{X}$, the *total space* of $\mathcal{F}$ is $\mathbb{A}(\mathcal{F}) := \operatorname{Spec}_{\mathcal{X}} \operatorname{Sym}_{\mathcal{O}_{\mathcal{X}}} \mathcal{F}^\vee$ and the associated projective bundle is $\mathbb{P}(\mathcal{F}) := \operatorname{Proj}_{\mathcal{X}} \operatorname{Sym}_{\mathcal{O}_{\mathcal{X}}} \mathcal{F}^\vee$.

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2 Preliminaries

In this section we recall, mainly following [26], the kind of stability structures on algebraic stacks that we will use in the rest of the paper and how they give rise to stratifications. The two main concepts are that of *norm on graded points* (Definition 2.3.3) and of *linear form on graded points* (Definition 2.4.1), and these give rise to Θ-stratifications (Definition 2.5.1).

While in [26] the stacks $\operatorname{Grad}(\mathcal{X})$ of graded points and $\operatorname{Filt}(\mathcal{X})$ of filtrations of an algebraic stack $\mathcal{X}$ are used, in this work we will need a generalisation of these, what we call the stack of *rational graded points* $\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X})$ and of the stack of *rational filtrations* $\operatorname{Filt}_{\mathbb{Q}}(\mathcal{X})$, that we define in Section 2.2.

For a representable projective morphism $f\colon \mathcal{Y} \to \mathcal{X}$ into a stack $\mathcal{X}$ with a good moduli space and a norm on graded points, Halpern-Leistner proved [26, Theorem 5.5.10] that $\mathcal{Y}$ carries a natural Θ-stratification $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$. The main result is section is Theorem 2.6.3, where we show that the centre $\mathcal{Z}_c$ of each stratum $\mathcal{S}_c$ admits a good moduli space, as well as observing that the weak Θ-stratification $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$ is actually a Θ-stratification.

2.1 Good moduli spaces and local structure theorems

We start by recalling the definition of good moduli space from [1, Definition 4.1], with the slightly modified conventions of [6, 1.7.3, 1.7.4].

Definition 2.1.1 (Good moduli space)**.** A morphism $\pi\colon \mathcal{X} \to X$ from an algebraic stack $\mathcal{X}$ to an algebraic space X is said to be an *good moduli space* if

1. the map π is quasi-compact and quasi-separated;
2. the map $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism; and
3. the pushforward functor π_* on quasi-coherent sheaves is exact, and the same is true after any base change $X' \to X$, where X' is an algebraic space.

If X is quasi-separated, then the third condition can be simplified to π_* being exact, the statement for any base change being then automatic by [1, Proposition 3.10, (vii)]. We will often use the term *good moduli stack* meaning an algebraic stack $\mathcal{X}$ that admits a good moduli space $\pi\colon \mathcal{X} \to X$.

A good moduli space $\pi\colon \mathcal{X} \to X$ enjoys many special properties, for example:

1. Any map $\mathcal{X} \to Y$ with Y an algebraic space factors uniquely through π [6, Theorem 3.12]. In particular, the good moduli space π is uniquely determined by $\mathcal{X}$.
2. Any base change of π along a morphism $X' \to X$ with X' an algebraic space is a good moduli space [1, Proposition 4.7, (i)].
3. If $h\colon \mathcal{X}' \to \mathcal{X}$ is an affine morphism, then $\mathcal{X}'$ has a good moduli space $\mathcal{X}' \to X'$ and the induced map $X' \to X$ is affine with $X' = \operatorname{Spec}_X \pi_* h_* \mathcal{O}_{\mathcal{X}'}$ [1, Lemma 4.14].
4. For every point $p \in |X|$, the fibre $\pi^{-1}(p)$ has a unique closed point q, and the dimension of the stabiliser of q is bigger than that of any other point of $\pi^{-1}(p)$ [1, Proposition 9.1]. Moreover, the stabiliser of q is linearly reductive [1, Proposition 12.14].

Stacks with good moduli spaces are étale locally quotient stacks. More precisely:

Theorem 2.1.2 (Local structure [6])**.** *Let $\mathcal{X}$ be an algebraic stack and $\pi\colon \mathcal{X} \to X$ a good moduli space. Assume that $\mathcal{X}$ is of finite presentation over a quasi-compact and quasi-separated algebraic space B and that $\mathcal{X}$ has affine diagonal.*

Then there is a natural number n, an affine scheme $\operatorname{Spec} A$ *endowed with an action of GL_n, and a cartesian square*

$$\begin{tikzcd}
(\operatorname{Spec} A)/\, GL_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X} \arrow[d, "\pi"] \\
\operatorname{Spec}(A^{GL_n}) \arrow[r, "h"] & X
\end{tikzcd}$$

with h an affine Nisnevich cover (in particular, étale). Here, A^{GL_n} denotes the ring of invariants. Moreover, $X \to B$ is of finite presentation and X has affine diagonal.

The theorem is [6, Theorem 6.1], together with the argument at the end of the proof of [6, Theorem 5.3] to guarantee that h can be taken to be affine. To see that X has affine diagonal, just take good moduli spaces for the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$, which is affine, to obtain the diagonal of X.

From Theorem 2.1.2, it follows that stacks whose good moduli space is a point are necessarily quotient stacks.

Corollary 2.1.3. *Let $\mathcal{X}$ be an algebraic stack of finite presentation over a field k and assume that $\pi\colon \mathcal{X} \to \operatorname{Spec} k$ is a good moduli space. Then $\mathcal{X} \cong (\operatorname{Spec} A)/\, GL_n$, where A is a k-algebra of finite type and* $\operatorname{Spec} A$ *is endowed with a GL_n-action.*

Over an algebraically closed field, there is a stronger result.

Corollary 2.1.4 (of [5, Theorem 4.12])**.** *Let $\mathcal{X}$ be an algebraic stack of finite presentation over an algebraically closed field k and suppose that $\pi\colon \mathcal{X} \to \operatorname{Spec} k$ is a good moduli space. Let $x \in \mathcal{X}(k)$ be the unique closed k-point of $\mathcal{X}$ and let G be the stabiliser of x. Then $\mathcal{X} \cong (\operatorname{Spec} A)/G$, where A is a finite type k-algebra and* $\operatorname{Spec} A$ *is endowed with an action of G.*

2.2 Stacks of rational filtrations and graded points

In [26], Halpern-Leistner defines the stacks $\operatorname{Grad}(\mathcal{X})$ of graded points and $\operatorname{Filt}(\mathcal{X})$ of filtrations of an algebraic stack $\mathcal{X}$ as mapping stacks. If $\mathcal{X}$ parametrises objects in an abelian category, then $\operatorname{Grad}(\mathcal{X})$ parametrises objects endowed with a $\mathbb{Z}$-grading and $\operatorname{Filt}(\mathcal{X})$ parametrises objects endowed with a $\mathbb{Z}$-filtration [7, 7.11, 7.12], but $\operatorname{Grad}(\mathcal{X})$

and $\mathrm{Filt}(\mathcal{X})$ can be defined for very general $\mathcal{X}$. In this section we revisit the construction of $\mathrm{Grad}(\mathcal{X})$ and $\mathrm{Filt}(\mathcal{X})$, and extend it to consider rational filtrations and gradings.

Following [26], we define the stack Θ^n over $\mathrm{Spec}(\mathbb{Z})$ to be the quotient stack $\Theta^n = \mathbb{A}^n_{\mathbb{Z}}/\mathbb{G}_{m,\mathbb{Z}}$, the action of $\mathbb{G}_{m,\mathbb{Z}}$ on $\mathbb{A}^n_{\mathbb{Z}}$ being the usual scaling action where each coordinate has weight 1, and we denote $\Theta = \Theta^1$. For an algebraic space S we denote $\Theta_S = \Theta \times S$.

Recall that for $\mathcal{Y}$ and $\mathcal{Z}$ two stacks over a base space S, an object of the *mapping stack* $\underline{\mathrm{Hom}}_S(\mathcal{Y}, \mathcal{Z})$ over a scheme T is a map $T \to S$ together with a morphism $\mathcal{Y} \times_S T \to \mathcal{X}$ over S. In the case $S = \mathrm{Spec}(\mathbb{Z})$, we omit the subindex from the notation. The following definition is in [26, Section 1.1], except that we do not work relative to a base algebraic stack.

Definition 2.2.1 (Stacks of filtrations and graded points)**.** Let $\mathcal{X}$ be an algebraic stack and n, a positive integer. We define the stack $\mathrm{Grad}^n(\mathcal{X})$ of $\mathbb{Z}^n$*-graded points* of $\mathcal{X}$ to be the mapping stack $\mathrm{Grad}^n(\mathcal{X}) \coloneqq \underline{\mathrm{Hom}}(B\mathbb{G}^n_{m,\mathbb{Z}}, \mathcal{X})$. Similarly, we define the stack $\mathrm{Filt}^n(\mathcal{X})$ of $\mathbb{Z}^n$*-filtrations* of $\mathcal{X}$ to be $\mathrm{Filt}^n(\mathcal{X}) \coloneqq \underline{\mathrm{Hom}}(\Theta^n, \mathcal{X})$.

We will simply denote $\mathrm{Filt}(\mathcal{X}) = \mathrm{Filt}^1(\mathcal{X})$ and $\mathrm{Grad}(\mathcal{X}) = \mathrm{Grad}^1(\mathcal{X})$.

Lemma 2.2.2 (Independence of base for mapping stacks)**.** *Let $\mathcal{Y}$ be an algebraic stack such that $\mathcal{Y} \to \mathrm{Spec}\,\mathbb{Z}$ is a good moduli space, and let $\mathcal{X}$ be an algebraic stack defined over an algebraic space B. Then there is a canonical isomorphism*

$$\underline{\mathrm{Hom}}(\mathcal{Y}, \mathcal{X}) \cong \underline{\mathrm{Hom}}_B(\mathcal{Y}_B, \mathcal{X})$$

of mapping stacks. In particular, there is a canonical map $\underline{\mathrm{Hom}}(\mathcal{Y}, \mathcal{X}) \to B$.

Proof. For a scheme T, an object of the groupoid $\underline{\mathrm{Hom}}_B(\mathcal{Y}_B, \mathcal{X})(T)$ is a pair (a, b) with $b \colon T \to B$ and $a \colon T \times_B \mathcal{Y}_B = T \times \mathcal{Y} \to \mathcal{X}$ a morphism over B. Since $T \times \mathcal{Y} \to T$ is a good moduli space, for any given $a \colon T \times \mathcal{Y} \to \mathcal{X}$, an object of $\underline{\mathrm{Hom}}(\mathcal{Y}, \mathcal{X})(T)$, the composition $T \times \mathcal{Y} \to \mathcal{X} \to B$ factors uniquely through $T \times \mathcal{Y} \to T$, giving a unique $b \colon T \to B$ such that (a, b) is an object of $\underline{\mathrm{Hom}}_B(\mathcal{Y}_B, \mathcal{X})(T)$. □

Applying the Lemma for $\mathcal{Y} = \Theta^n$ or $\mathcal{Y} = B\mathbb{G}^n_m$, we see that $\mathrm{Filt}^n(\mathcal{X})$ and $\mathrm{Grad}^n(\mathcal{X})$ are independent of the base algebraic space considered.

To guarantee that $\mathrm{Filt}^n(\mathcal{X})$ and $\mathrm{Grad}^n(\mathcal{X})$ are well-behaved, we consider the following assumption on an algebraic stack $\mathcal{X}$ defined over an algebraic space B.

Assumption 2.2.3. The algebraic space B is quasi-separated and locally noetherian, and the map $\mathcal{X} \to B$ is locally finitely presented and has affine diagonal.

Example 2.2.4. Suppose that $\mathcal{X}$ is a noetherian algebraic stack with affine diagonal and $\pi \colon \mathcal{X} \to X$ is a good moduli space. Then X is also noetherian by [1, Theorem 4.16], and π is of finite type by [5, Theorem A.1]. The diagonal of X is affine since it is obtained from the diagonal of $\mathcal{X}$ by taking good moduli spaces. In particular, $\mathcal{X}$ satisfies Assumption 2.2.3 with $B = X$.

Under Assumption 2.2.3, the stacks $\mathrm{Filt}^n(\mathcal{X})$ and $\mathrm{Grad}^n(\mathcal{X})$ are algebraic and also satisfy Assumption 2.2.3 [6, Theorem 6.22]. Note that by [6, Remark 6.16] it is not necessary to assume that B is excellent, since the stacks $B\mathbb{G}^n_{m,B}$ and Θ^n_B satisfy condition (N) in [6]. See also [29, Theorem 5.1.1] for a related algebraicity result.

There are several maps relating $\mathrm{Grad}(\mathcal{X})$, $\mathrm{Filt}(\mathcal{X})$ and $\mathcal{X}$:

1. The "evaluation at 1" map $\mathrm{ev}_1 \colon \mathrm{Filt}(\mathcal{X}) \to \mathcal{X}$, defined by precomposition along $\{1\} \to \Theta$. It is representable and separated [26, Proposition 1.1.13].

2. The “associated graded” map $\mathrm{gr}\colon \mathrm{Filt}(\mathcal{X}) \to \mathrm{Grad}(\mathcal{X})$, defined by precomposition along $B\mathbb{G}_m = \{0\}/\mathbb{G}_m \to \Theta$.
3. The “forgetful” map $u\colon \mathrm{Grad}(\mathcal{X}) \to \mathcal{X}$, defined by precomposition along $\mathrm{Spec}(\mathbb{Z}) \to B\mathbb{G}_m$.
4. The “evaluation at 0” map $\mathrm{ev}_0\colon \mathrm{Filt}(\mathcal{X}) \to \mathcal{X}$, which is the composition $\mathrm{ev}_0 = u \circ \mathrm{gr}$.
5. The “split filtration” map $\sigma\colon \mathrm{Grad}(\mathcal{X}) \to \mathrm{Filt}(\mathcal{X})$, defined by precomposition along the canonical representable morphism $\Theta \to B\mathbb{G}_m$.
6. The “trivial grading” map $\mathcal{X} \to \mathrm{Grad}(\mathcal{X})$, given by precomposition along $B\mathbb{G}_m \to \mathrm{Spec}\,\mathbb{Z}$. It is an open and closed immersion [26, Proposition 1.3.9].
7. The “trivial filtration” map $\mathcal{X} \to \mathrm{Filt}(\mathcal{X})$, defined by precomposing along $\Theta \to \mathrm{Spec}\,\mathbb{Z}$. It is an open and closed immersion [26, Proposition 1.3.9].

Remark 2.2.5. Some times, the assumption that $\mathcal{X} \to B$ has affine diagonal can be relaxed to $\mathcal{X}$ having affine stabilisers and being quasi-separated over B. This is the case for the construction and algebraicity of $\mathrm{Filt}^n_{\mathbb{Q}}(\mathcal{X})$, $\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X})$, $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ and $\mathrm{Grad}_{\mathbb{Q}^\infty}(\mathcal{X})$ (Definition 2.2.6 and Definition 3.2.2). Representability of $\mathrm{ev}_1\colon \mathrm{Filt}(\mathcal{X}) \to \mathcal{X}$ follows under the additional hypothesis that $\mathcal{X}$ has separated inertia.

The assumption that B is locally noetherian guarantees that the topological spaces of the algebraic stacks considered are locally connected, and hence their connected components are open.

The monoid $(\mathbb{N}, \cdot, 1)$ acts[1] on the stacks $\mathrm{Filt}(\mathcal{X})$ and $\mathrm{Grad}(\mathcal{X})$. A natural number n acts on $\mathrm{Filt}(\mathcal{X})$ by the map $\mathrm{Filt}(\mathcal{X}) \xrightarrow{(\bullet)^n} \mathrm{Filt}(\mathcal{X})$ given by precomposition along the nth power map $\Theta_T \to \Theta_T$, and similarly in the case of $\mathrm{Grad}(\mathcal{X})$. Now denote $\mathcal{Y}$ one of the stacks $\mathrm{Grad}(\mathcal{X})$ or $\mathrm{Filt}(\mathcal{X})$ with its $(\mathbb{N}, \cdot, 1)$-action. We have a diagram $D_{\mathcal{Y}}\colon (\mathbb{N}^*, |) \to \mathrm{St}_{\mathrm{fppf}}$ (i.e. a pseudofunctor), indexed by the filtered poset $(\mathbb{N}^*, |)$ of positive integers with the divisibility order, defined by $D_{\mathcal{Y}}(n) = \mathcal{Y}$ for all n and $D_{\mathcal{Y}}(n|m)$ being the “rising to the $\frac{m}{n}$th power” map $\mathcal{Y} \xrightarrow{(\bullet)^{m/n}} \mathcal{Y}$ defined above.

Definition 2.2.6. The stacks $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ of *rational filtrations* and $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ of *rational graded points* are the colimits

$$\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \coloneqq \varinjlim D_{\mathrm{Filt}(\mathcal{X})} \quad \text{and} \quad \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \coloneqq \varinjlim D_{\mathrm{Grad}(\mathcal{X})}$$

in the 2-category $\mathrm{St}_{\mathrm{fppf}}$ of stacks on the fppf site of schemes.

Remark 2.2.7. There are also maps $\mathrm{ev}_1\colon \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \to \mathcal{X}$, $\mathrm{gr}\colon \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$, etcetera, relating the stacks $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$, $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ and $\mathcal{X}$, just because the version of these maps for Filt and Grad are compatible with the colimits defining $\mathrm{Filt}_{\mathbb{Q}}$ and $\mathrm{Grad}_{\mathbb{Q}}$.

Proposition 2.2.8. *Let $\mathcal{X}$ be an algebraic stack over an algebraic space B satisfying Assumption 2.2.3. Then $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ and $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ are algebraic and satisfy Assumption 2.2.3.*

Proof. By [26, 1.3.11], the “rising to the nth power” maps $\mathrm{Filt}(\mathcal{X}) \to \mathrm{Filt}(\mathcal{X})$ and $\mathrm{Grad}(\mathcal{X}) \to \mathrm{Grad}(\mathcal{X})$ are open (and closed) immersions. Thus the algebraicity result follows from Lemma 2.2.9. Since $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ and $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ are increasing unions of stacks isomorphic to $\mathrm{Filt}(\mathcal{X})$ and $\mathrm{Grad}(\mathcal{X})$ respectively, they also satisfy Assumption 2.2.3. □

[1] Formally, an action of $(\mathbb{N}, \cdot, 1)$ on a stack $\mathcal{Y}$ is a pseudofunctor $B(\mathbb{N}, \cdot, 1) \to \mathrm{St}_{\mathrm{fppf}}$ sending the unique object of $B(\mathbb{N}, \cdot, 1)$ to $\mathcal{Y}$. Here, we are denoting $B(\mathbb{N}, \cdot, 1)$ the category with one object and endomorphism monoid equal to $(\mathbb{N}, \cdot, 1)$, and $\mathrm{St}_{\mathrm{fppf}}$ is the 2-category of stacks on the category of schemes with the fppf topology.

Lemma 2.2.9. *Let I be a filtered poset, seen as a category, and let $D\colon I \to \mathrm{St}_{fppf}$ be a pseudofunctor such that for all arrows $s \to t$ in I the induced $D(s) \to D(t)$ is an open immersion. Let $\mathcal{Y} = \varinjlim D$ in the 2-category St_{fppf}. If $D(s)$ is algebraic for all objects s of D, then so is $\mathcal{Y}$.*

Proof. In this proof, we consider the site of *affine* schemes with the fppf topology. This does not change the 2-category $\mathrm{St}_{\mathrm{fppf}}$ of stacks, but it will be useful to consider only quasi-compact test schemes.

The stack $\mathcal{Y}$ is the stackification of the colimit $\mathcal{Y}^{\mathrm{pre}} = \varinjlim D$ in the 2-category of prestacks (meaning presheaves of groupoids). A morphism $T \to \mathcal{Y}^{\mathrm{pre}}$ is a pair (s, f), with s an object of I and $f\colon T \to D(s)$ a morphism. A 2-map $(s, f) \to (s', f')$ is a pair (t, r) with $t \in I$ such that there are arrows $t/s\colon s \to t$ and $t/s'\colon s' \to t$, and $r\colon D(t/s) \circ f \to D(t/s') \circ f'$ a 2-morphism. From this description, and using the facts that (1) I is filtered and (2) every object in the site considered, i.e. every affine scheme, is quasi-compact, it follows that $\mathcal{Y}^{\mathrm{pre}}$ is already a stack, so $\mathcal{Y} = \mathcal{Y}^{\mathrm{pre}}$. Moreover, each of the maps $D(i) \to \mathcal{Y}$ is an open immersion. Indeed, if $(s, f)\colon T \to \mathcal{Y}$ is a map, with T affine, then $D(i) \times_{\mathcal{Y}} T = D(i) \times_{D(s')} T$ if $i, s \leq s'$, which is open in T. Thus $\bigsqcup_{s\in I} D(s) \to \mathcal{Y}$ is a smooth representable surjection, so $\mathcal{Y}$ is algebraic. □

Remark 2.2.10 (Functor of points of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ and $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$)*.* From the proof of 2.2.9 we get a simple description of points in $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ and $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$. Namely, if T is a quasi-compact scheme, then a T-point of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ will be denoted as $\lambda^{1/n}$, where λ is a T-point of $\mathrm{Filt}(\mathcal{X})$ and n is a positive integer. An isomorphism between T-points $\lambda^{1/n}$ and $(\lambda')^{1/n'}$ of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ is an isomorphism between $\lambda^{n'}$ and $(\lambda')^{n}$ in $\mathrm{Filt}(\mathcal{X})$. A similar description applies to $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$.

Remark 2.2.11*.* Since $(\mathbb{N}, \cdot, 1)$ also acts on $\mathrm{Filt}^n(\mathcal{X})$ and $\mathrm{Grad}^n(\mathcal{X})$ for all positive integers n, we can form the stacks $\mathrm{Filt}^n_{\mathbb{Q}}(\mathcal{X})$ and $\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X})$ in a similar fashion. For the same reasons, they are algebraic and satisfy Assumption 2.2.3.

The formation of $\mathrm{Filt}^n_{\mathbb{Q}}(\mathcal{X})$ and $\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X})$ is well-behaved with respect to base change from a target algebraic space.

Proposition 2.2.12. *Let $\mathcal{X} \to B$ and $\mathcal{X}' \to B'$ satisfy Assumption 2.2.3 and let*

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X} \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

be a cartesian square with X and X' algebraic spaces. Then

$$\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}') \cong \mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}) \times_{\mathcal{X}} \mathcal{X}' \cong \mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}) \times_X X'$$

and

$$\mathrm{Filt}^n_{\mathbb{Q}}(\mathcal{X}') \cong \mathrm{Filt}^n_{\mathbb{Q}}(\mathcal{X}) \times_{\mathrm{ev}_1, \mathcal{X}} \mathcal{X}' \cong \mathrm{Filt}^n_{\mathbb{Q}}(\mathcal{X}) \times_X X'$$

for all n. The same holds for Filt^n and Grad^n.

Proof. The case of $\mathrm{Filt}^n(\mathcal{X}')$ and $\mathrm{Grad}^n(\mathcal{X}')$ is [26, Corollary 1.3.17]. The result follows for $\mathrm{Filt}^n_{\mathbb{Q}}(\mathcal{X}')$ and $\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}')$ after covering these by copies of $\mathrm{Filt}^n(\mathcal{X}')$ and $\mathrm{Grad}^n(\mathcal{X}')$. □

Proposition 2.2.13. *Let $\mathcal{X}$ be an algebraic stack defined over an algebraic space B, satisfying Assumption 2.2.3, and let $\mathcal{X}' \to \mathcal{X}$ be a closed immersion. Then*

$$\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}') \cong \mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}) \times_{\mathcal{X}} \mathcal{X}'$$

and

$$\mathrm{Filt}^n_{\mathbb{Q}}(\mathcal{X}') \cong \mathrm{Filt}^n_{\mathbb{Q}}(\mathcal{X}) \times_{\mathrm{ev}_1, \mathcal{X}} \mathcal{X}'$$

for all n. The same holds for Filt^n *and* Grad^n*.*

Proof. As above, it is enough to see the fact for Filt^n and Grad^n, which is [26, Proposition 1.3.1]. □

It will be useful for the sequel the fact that Grad preserves properness.

Proposition 2.2.14. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a representable proper finitely presented morphism of algebraic stacks over a base algebraic space B. Then*

$$\mathrm{Grad}(f)\colon\ \mathrm{Grad}(\mathcal{X}) \to \mathrm{Grad}(\mathcal{Y})$$

and

$$\mathrm{Grad}_{\mathbb{Q}}(f)\colon\ \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y})$$

are representable and proper.

Proof. It is enough to prove the statement for $\mathrm{Grad}(f)$. Let T be a scheme and $T \to \mathrm{Grad}(\mathcal{Y})$ a map, corresponding to $B\mathbb{G}_{m,T} \to \mathcal{Y}$. Form a cartesian square

$$\begin{tikzcd} \mathcal{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & B\mathbb{G}_{m,T} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y}. \end{tikzcd}$$

The 1-category of representable algebraic stacks over $B\mathbb{G}_{m,T}$ is equivalent to the category of algebraic spaces over T endowed with a $\mathbb{G}_{m,T}$-action, and the equivalence is given by pullback along $T \to B\mathbb{G}_{m,T}$. Therefore $\mathcal{Z} = Z/\mathbb{G}_{m,T}$ for a T-algebraic space Z acted on by $\mathbb{G}_{m,T}$. Forming now the fibre product

$$\begin{tikzcd} \mathcal{U} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & T \arrow[d] \\ \mathrm{Grad}(\mathcal{X}) \arrow[r] & \mathrm{Grad}(\mathcal{Y}) \end{tikzcd}$$

and given a T-scheme S, a map $S \to \mathcal{U}$ over T is a section of $\mathcal{Z} \to B\mathbb{G}_{m,T}$ over $B\mathbb{G}_{m,S} \to B\mathbb{G}_{m,T}$, which is in turn a $\mathbb{G}_{m,T}$-equivariant map $S \to Z$. Therefore $\mathcal{U} = Z^{\mathbb{G}_{m,T}}$, the fixed points of Z, as a stack over T. Since $Z \to T$ is finitely presented, we have by [26, Proposition 1.4.1] that the map $Z^{\mathbb{G}_{m,T}} \to Z$ is a closed immersion and also, by hypothesis, that $Z \to T$ is proper. Thus $\mathcal{U} \to T$ is proper. □

We recall the definition of $\mathbb{Z}$-*flag spaces* from [26, Definition 1.1.15] and introduce the natural counterpart of $\mathbb{Q}$-*flag spaces*.

Definition 2.2.15 (Flag spaces)**.** Let $\mathcal{X}$ be an algebraic stack over an algebraic space B satisfying Assumption 2.2.3, and let $x\colon T \to \mathcal{X}$ be a scheme-valued point. We define the $\mathbb{Z}$-flag space $\mathrm{Flag}(x)$ to be the fibre product

$$\begin{tikzcd} \mathrm{Flag}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "x"] \\ \mathrm{Filt}(\mathcal{X}) \arrow[r, "\mathrm{ev}_1"] & \mathcal{X} \end{tikzcd}$$

and the $\mathbb{Q}$-flag space $\mathrm{Flag}_{\mathbb{Q}}(x)$ as the fibre product

$$\begin{tikzcd} \mathrm{Flag}_{\mathbb{Q}}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "x"] \\ \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \arrow[r, "\mathrm{ev}_1"] & \mathcal{X}. \end{tikzcd}$$

By representability of $\mathrm{ev}_1\colon \mathrm{Filt}(\mathcal{X}) \to \mathcal{X}$ and $\mathrm{ev}_1\, \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \to \mathcal{X}$, the flag spaces $\mathrm{Flag}(x)$ and $\mathrm{Flag}_{\mathbb{Q}}(x)$ are algebraic spaces over T. Since $\mathrm{Filt}(\mathcal{X}) \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ is an open and closed immersion, we have a natural open and closed immersion $\mathrm{Flag}(x) \to \mathrm{Flag}_{\mathbb{Q}}(x)$ of flag spaces.

In the case of a field-valued point, we can talk about a *set of filtrations*.

Definition 2.2.16 (Set of filtrations of a point)**.** Let $\mathcal{X}$ be an algebraic stack over an algebraic space B satisfying Assumption 2.2.3. Let k be a filed and let $x\colon \mathrm{Spec}(k) \to \mathcal{X}$ be a k-point. The *set of $\mathbb{Z}$-filtrations (or integral filtrations) of x* is defined to be

$$\mathbb{Z}\text{-}\mathrm{Filt}(\mathcal{X}, x) := \mathrm{Flag}(x)(k),$$

the set of k-points of the $\mathbb{Z}$-flag space of x. Similarly, the *set of $\mathbb{Q}$-filtrations (or rational filtrations) of x* is

$$\mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x) := \mathrm{Flag}_{\mathbb{Q}}(x)(k).$$

The filtration of x given by the composition of $\mathbb{A}^1_k/\mathbb{G}_{m,k} \to \mathrm{Spec}\, k$ and $x\colon \mathrm{Spec}\, k \to \mathcal{X}$ is denoted 0 and referred to as the *trivial filtration*.

Remark 2.2.17 (Filtrations of a quotient stack)*.* Let k be a field and let X be a separated scheme over k, endowed with an action $a\colon G \times X \to X$ by a linear algebraic group G. Form the quotient stack $\mathcal{X} = X/G$ and let $x \in X(k)$ be a k-point. We abusively also denote by x the composition $\mathrm{Spec}\, k \xrightarrow{x} X \to \mathcal{X}$. If $\lambda\colon \mathbb{G}_{m,k} \to G$ is a cocharacter, we say that $\lim_{t\to 0} \lambda(t)x$ *exists* in X if the map

$$\mathbb{G}_{m,k} \xrightarrow{\lambda} G \cong G \times \mathrm{Spec}\, k \xrightarrow{\mathrm{id}_G \times x} G \times X \xrightarrow{a} X$$

extends to a map $\overline{\lambda}_x\colon \mathbb{A}^1_k \to X$ (in which case it does so uniquely), where we regard $\mathbb{G}_{m,k} = \mathbb{A}^1_k \setminus \{0\}$. If $\lim_{t\to 0} \lambda(t)x$ exists, we let $\lim_{t\to 0} \lambda(t)x$ denote the k-point $\overline{\lambda}_x(0)$ of X. For $n \in \mathbb{Z}_{>0}$, we have that $\lim_{t\to 0} \lambda(t)x$ exists if and only if $\lim_{t\to 0} \lambda^n(t)x$ exists, so it makes sense to define this notion for a rational one-parameter subgroup $\lambda \in \Gamma^{\mathbb{Q}}(G)$. The k-points of X such that $\lim_{t\to 0} \lambda(t)x$ exists are in bijections with the k-points of the attractor $X^{\lambda,0}$. Thus it follows readily from [26, Theorem 1.4.8 and Remark 1.4.9] that we have an identification:

$$\mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x) = \{\lambda \in \Gamma^{\mathbb{Q}}(G) \mid \lim_{t\to 0} \lambda(t)x \text{ exists}\}/\sim, \tag{3}$$

where $\lambda \sim \lambda'$ if there is $g \in P(\lambda)(k)$ such that $\lambda^g = \lambda'$.

We conclude this section with a couple of facts about maps induced on sets of filtrations.

Proposition 2.2.18. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a schematic proper morphism of algebraic stacks over an algebraic space B satisfying Assumption 2.2.3. Let k be a field, let $x \in \mathcal{X}(k)$ and $y \in \mathcal{Y}(k)$ be k-points, and let $f(x) \to y$ be an isomorphism. Then the induced maps $\mathbb{Z}\text{-Filt}(\mathcal{X}, x) \to \mathbb{Z}\text{-Filt}(\mathcal{Y}, y)$ and $\mathbb{Q}\text{-Filt}(\mathcal{X}, x) \to \mathbb{Q}\text{-Filt}(\mathcal{Y}, y)$ of sets of filtrations are bijective.*

Proof. It is enough to deal with the case of integral filtrations. An element of $\mathbb{Z}\text{-Filt}(\mathcal{Y}, y)$ is a pair (λ, α) fitting in a commutative diagram

$$\begin{tikzcd}
\mathbb{A}^1_k/\mathbb{G}_{m,k} \arrow[r, "\lambda"] & \mathcal{Y} \\
\mathrm{Spec}(k) \arrow[u, "1"] \arrow[ur, "y"'] \arrow[ur, phantom, "\Leftarrow\alpha", near end, shift left=3]
\end{tikzcd}$$

and similarly for $\mathbb{Z}\text{-Filt}(\mathcal{X}, x)$. Fix such a $(\lambda, \alpha) \in \mathbb{Z}\text{-Filt}(\mathcal{Y}, y)$. Now form the fibre product

$$\begin{tikzcd}
X/\mathbb{G}_{m,k} \arrow[r] \arrow[d, "r"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{A}^1_k/\mathbb{G}_{m,k} \arrow[d, "\lambda"] \\
\mathcal{X} \arrow[r, "f"] & \mathcal{Y}.
\end{tikzcd}$$

The base change is indeed of the form $X/\mathbb{G}_{m,k}$ for a scheme X because f is schematic, and $X/\mathbb{G}_{m,k} \to \mathbb{A}^1_k/\mathbb{G}_{m,k}$ is given by a $\mathbb{G}_{m,k}$-equivariant map $X \to \mathbb{A}^1_k$. There is a commutative solid square

$$\begin{tikzcd}
(\mathbb{A}^1_k \setminus 0)/\mathbb{G}_{m,k} \arrow[r] \arrow[d, "u"'] & \mathbb{A}^1_k/\mathbb{G}_{m,k} \arrow[d, "1_{\mathbb{A}^1/\mathbb{G}_m}"] \arrow[dl, dotted, "h"'] \\
X/\mathbb{G}_{m,k} \arrow[r] & \mathbb{A}^1_k/\mathbb{G}_{m,k}
\end{tikzcd} \tag{4}$$

and an isomorphism $r \circ u \to x$. An element of $\mathbb{Z}\text{-Filt}(\mathcal{X}, x)$ mapping to (λ, u) is specified by a lift h of the square 4 in the 2-categorical sense. Thus we want to prove that there is a unique such lift. There is a unique lift g of

$$\begin{tikzcd}
\mathbb{A}^1_k \setminus 0 \arrow[r] \arrow[d] & \mathbb{A}^1_k \arrow[d, "1_{\mathbb{A}^1}"] \arrow[dl, dotted, "g"'] \\
X \arrow[r] & \mathbb{A}^1_k
\end{tikzcd}$$

by properness, see [50, Tag 0BX7]. We just need to prove that g is $\mathbb{G}_{m,k}$-equivariant. Equivariance amounts to the commutativity of

$$\begin{tikzcd}
\mathbb{G}_{m,k} \times \mathbb{A}^1_k \arrow[r] \arrow[d, "1_{\mathbb{G}_m} \times g"'] & \mathbb{A}^1_k \arrow[d, "g"] \\
\mathbb{G}_{m,k} \times X \arrow[r] & X.
\end{tikzcd}$$

Both compositions agree when restricted to $\mathbb{G}_{m,k} \times (\mathbb{A}^1_k \setminus 0)$, which is schematically dense in $\mathbb{G}_{m,k} \times \mathbb{A}^1_k$. Thus, by separatedness of $X \to \mathbb{A}^1_k$, the square commutes. □

Proposition 2.2.19. *Suppose $f\colon \mathcal{X} \to \mathcal{Y}$ is a representable and separated morphism of algebraic stacks over an algebraic space B satisfying Assumption 2.2.3. Let k be a field, let $x \in \mathcal{X}(k)$ and $y \in \mathcal{Y}(k)$ be k-points, and let $f(x) \to y$ be an isomorphism. Then the induced maps $\mathbb{Z}\text{-Filt}(\mathcal{X}, x) \to \mathbb{Z}\text{-Filt}(\mathcal{Y}, y)$ and $\mathbb{Q}\text{-Filt}(\mathcal{X}, x) \to \mathbb{Q}\text{-Filt}(\mathcal{Y}, y)$ of sets of filtrations are injective*

Proof. It is enough to prove the claim for integral filtrations. If λ_1, λ_2 are two filtrations of x that give the same filtration of y, we can form a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "x"] \arrow[d, "1"'] & \mathcal{X} \arrow[d, "\Delta_f"] \\ \mathbb{A}^1_k/\mathbb{G}_{m,k} \arrow[r, "(\lambda_1,\lambda_2)"] \arrow[ur, dashed] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}. \end{tikzcd}$$

Since Δ_f is a closed immersion and $1\colon \operatorname{Spec} k \to \mathbb{A}^1_k/\mathbb{G}_{m,k}$ is a schematically dense open immersion, there is a unique dashed arrow filling the diagram, which gives the isomorphism between λ_1 and λ_2. $\square$

2.3 Normed stacks

We now recall from [26] the notion of norm on graded points of a stack. If G is an algebraic group, we show that norms on graded points of BG are in bijection with norms on cocharacters of G.

Definition 2.3.1 (Nondegenerate graded point)**.** Let $\mathcal{X}$ be a quasi-separated algebraic stack, let k be a field and let $x\colon B\mathbb{G}^n_{m,k} \to \mathcal{X}$ be a $\mathbb{Z}^n$-graded point. We say that the graded point x is *nondegenerate* if $\ker(\mathbb{G}^n_{m,k} \to \operatorname{Aut}(x|_{\operatorname{Spec} k}))$ is finite.

Suppose that $\mathcal{X}$ is defined over some algebraic space B and it satisfies Assumption 2.2.3. We say that a connected component $\mathcal{Z}$ of $\operatorname{Grad}^n(\mathcal{X})$ is *nondegenerate* if there is a field k and a point $x \in \mathcal{Z}(k)$ with x nondegenerate.

Remark 2.3.2. If $\mathcal{Z}$ is a nondegenerate component, then by [26, Proposition 1.3.9] we have that for *all* fields k and points $x \in \mathcal{Z}(k)$, the point x is nondegenerate.

We recall the notion of norm on graded points of a stack from [26, Definition 4.1.12].

Definition 2.3.3. Let $\mathcal{X}$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3. A *(rational quadratic) norm* q on graded points of $\mathcal{X}$ (or simply a *norm* on $\mathcal{X}$) is a locally constant function

$$q\colon\ |\operatorname{Grad}(\mathcal{X})| \to \mathbb{Q}_{\geq 0}$$

such that for every field k and every nondegenerate $\mathbb{Z}^n$-graded point $x\colon B\mathbb{G}^n_{m,k} \to \mathcal{X}$, the induced map $q_x\colon \Gamma^{\mathbb{Z}}(\mathbb{G}^n_{m,k}) \to \mathbb{Q}$ is the quadratic form of a rational inner product on the finite free $\mathbb{Z}$-module $\Gamma^{\mathbb{Z}}(\mathbb{G}^n_{m,k})$ of cocharacters of $\mathbb{G}^n_{m,k}$.

A *normed algebraic stack* is an algebraic stack endowed with a norm on graded points.

Let us clarify what the map q_x is. If $\lambda\colon \mathbb{G}_{m,k} \to \mathbb{G}^n_{m,k}$ is a cocharacter, the composition $B\mathbb{G}_{m,k} \xrightarrow{B\lambda} B\mathbb{G}^n_{m,k} \xrightarrow{x} \mathcal{X}$ defines a point $p \in |\operatorname{Grad}(\mathcal{X})|$, and we let $q_x(\lambda) = q(p)$.

Remark 2.3.4. A norm on graded points $q\colon\ |\operatorname{Grad}(\mathcal{X})| \to \mathbb{Q}$ extends canonically to a map

$$q\colon\ |\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X})| \to \mathbb{Q}$$

by setting $q(\frac{1}{n}\lambda) = \frac{1}{n^2}q(\lambda)$, for a rational graded point $\frac{1}{n}\lambda$.

Remark 2.3.5 (Notation for norms). If $\mathcal{X}$ is endowed with a norm on graded points q and $\lambda \in \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})(k)$ is a filtration, with k a field, then we will often denote $\|\lambda\| := \sqrt{q(\lambda)}$.

In some circumstances we can pull back a norm under a morphism.

Proposition and Definition 2.3.6 (Pulling back norms)**.** Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks over an algebraic space B, satisfying Assumption 2.2.3. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism such that the relative inertia $\mathcal{I}_f \to \mathcal{X}$ has proper fibres (for example if f is representable or separated). Let q be a norm on $\mathcal{Y}$ and denote f^*q the composition $|\mathrm{Grad}(\mathcal{X})| \longrightarrow |\mathrm{Grad}(\mathcal{Y})| \xrightarrow{q} \mathbb{Q}$. Then f^*q is a norm on $\mathcal{X}$, called the *pulled back norm*.

If $\mathcal{X}$ is endowed with a norm q', we say that the morphism f is *norm-preserving* if $f^*q = q'$.

Proof. We need to see that if $u\colon B\mathbb{G}^n_{m,k} \to \mathcal{X}$ is nondegenerate, then so is $f \circ u$. Let $x = u|_{\mathrm{Spec}\,k} \in \mathcal{X}(k)$ be the point that u grades. We have induced algebraic group homomorphisms

$$\begin{tikzcd} \mathbb{G}^n_{m,k} \arrow[d, "s"'] \arrow[dr, "l"] & \\ \mathrm{Aut}(x) \arrow[r, "r"'] & \mathrm{Aut}(f(x)). \end{tikzcd}$$

The kernel of r is proper over $\mathrm{Spec}(k)$, since it is the fibre over x of the relative inertia morphism, and $\ker s$ is finite by hypothesis. We have a sequence

$$\ker l \xrightarrow{a} \ker l/\ker s \xrightarrow{b} \ker r$$

where a is finite and b is a closed immersion. Thus $\ker l$ is proper over k, and therefore finite, because $\mathbb{G}^n_{m,k}$ is affine. This proves that $f \circ u$ is nondegenerate. $\square$

The notion of norm on graded points of an algebraic stack is a generalisation of the classical notion of norm on cocharacters of a group.

Definition 2.3.7 (Norm on cocharacters of a group)**.** Let G be a smooth linear algebraic group over a field k that has a maximal k-split torus. A *(rational quadratic) norm on cocharacters of G* is a map $q\colon \Gamma^{\mathbb{Z}}(G) \to \mathbb{Q}$ that is invariant under the action of $G(k)$ on $\Gamma^{\mathbb{Z}}(G)$ by conjugation and such that for every k-split torus T of G, the restriction of q to $\Gamma^{\mathbb{Z}}(T)$ is the quadratic form of a rational inner product on $\Gamma^{\mathbb{Z}}(T)$.

Now fix a smooth connected linear algebraic group G over a field k. Any two maximal k-split tori are conjugate by an element of $G(k)$ [15, Theorem C.2.3], and there exists one such torus for dimension reasons. The Weyl group $W(G,T) = N_G(T)/Z_G(T)$, for a k-split torus T, is a finite constant group scheme [15, Proposition C.2.10], so we identify it with a finite abstract group. By conjugacy of maximal k-split tori and [44, Lemma 2.8], it follows that $\Gamma^{\mathbb{Z}}(T)/W(G,T) = \Gamma^{\mathbb{Z}}(G)/G(k)$ if T is a maximal k-split torus of G. Thus we deduce:

Proposition 2.3.8. *If T is a maximal k-split torus of G, then the data of a norm on cocharacters of G is equivalent to a rational quadratic inner product on $\Gamma^{\mathbb{Z}}(T)$, invariant under the action of $W(G,T)$.*

The link with the concept of a norm on graded points is given in the following proposition.

Proposition 2.3.9. *Suppose G has a k-split maximal torus T. Then norms on BG are in natural bijection with norms on cocharacters of G.*

Proof. Let $W = W(G,T)$ be the Weyl group. Then, by [26, Theorem 1.4.8], we can explicitly describe the stack of graded points as

$$\mathrm{Grad}(BG) = \bigsqcup_{\lambda \in \Gamma^{\mathbb{Z}}(T)/W} BL(\lambda),$$

where $L(\lambda)$ is the centraliser of a choice of representative of λ. Thus $|\mathrm{Grad}(BG)| = \Gamma^{\mathbb{Z}}(T)/W$, and the identification is compatible in the sense that if $\lambda\colon \mathbb{G}_m \to G$ is a cocharacter, the point in $|\mathrm{Grad}(BG)|$ defined by $B\lambda\colon B\mathbb{G}_m \to BG$ is the class of λ in $\Gamma^{\mathbb{Z}}(T)/W = \Gamma^{\mathbb{Z}}(G)/G(k)$. To conclude, just note that if $T \to T'$ is a map of k-split tori with finite kernel, then $\Gamma^{\mathbb{Z}}(T) \to \Gamma^{\mathbb{Z}}(T')$ is injective, so $\Gamma^{\mathbb{Z}}(T)$ inherits an inner product if $\Gamma^{\mathbb{Z}}(T')$ has one. ☐

Norms on cocharacters are a source of norms on quotient stacks.

Proposition 2.3.10 (Norms on quotient stacks)**.** *Suppose G has a k-split maximal torus and acts on an quasi-separated algebraic space X of finite type over k. If G is endowed with a norm on cocharacters q, then X/G is naturally endowed with a norm on graded points.*

Proof. Since $p\colon X/G \to BG$ is representable, the pullback p^*q is a norm on X/G by Proposition and Definition 2.3.6. ☐

Definition 2.3.11 (Norm on $\mathrm{Grad}_{\mathbb{Q}}(\mathfrak{X})$)**.** Let $\mathfrak{X}$ be an algebraic stack over an algebraic space B satisfying Assumption 2.2.3 and endowed with a norm on graded points q. We denote $\mathrm{Grad}(q)$ (resp. $\mathrm{Grad}_{\mathbb{Q}}(q)$) the norm on $\mathrm{Grad}(\mathfrak{X})$ (resp. $\mathrm{Grad}_{\mathbb{Q}}(\mathfrak{X})$) given as the pullback of q along the forgetful morphism $u\colon \mathrm{Grad}_{\mathbb{Q}}(\mathfrak{X}) \to \mathfrak{X}$ (resp. $\mathrm{Grad}_{\mathbb{Q}}(\mathfrak{X}) \to \mathfrak{X}$), which is representable.

It follows from Proposition 2.3.6 that $\mathrm{Grad}(q)$ is a norm on $\mathrm{Grad}(\mathfrak{X})$. If $\mathfrak{X}$ is a normed stack, we will always regard $\mathrm{Grad}(\mathfrak{X})$ (resp. $\mathrm{Grad}_{\mathbb{Q}}(\mathfrak{X})$) as normed stack, endowed with the norm $\mathrm{Grad}(q)$ (resp. $\mathrm{Grad}_{\mathbb{Q}}(\mathfrak{X})$).

2.4 Linear forms on stacks

We now recall the notion of linear form on graded points of a stack from [26] and show how to get linear forms from line bundles.

Definition 2.4.1 (Linear form)**.** Let $\mathfrak{X}$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3. A *rational linear form ℓ on graded points of* $\mathfrak{X}$ (or simply a *linear form on* $\mathfrak{X}$) is a locally constant function

$$\ell\colon\ |\mathrm{Grad}(\mathfrak{X})| \to \mathbb{Q}$$

such that, for every field k and every $\mathbb{Z}^n$-graded point $B\mathbb{G}_{m,k}^n \to \mathfrak{X}$, the induced map $\Gamma^{\mathbb{Z}}(\mathbb{G}_{m,k}^n) \to \mathbb{Q}$ on cocharacters of the torus is $\mathbb{Z}$-linear.

If $\lambda\colon B\mathbb{G}_{m,k} \to \mathfrak{X}$ is a graded point, we denote by $\langle \lambda, \ell\rangle$ the value of ℓ at the point of $\mathrm{Grad}(\mathfrak{X})$ defined by λ.

Remark 2.4.2. A linear form $\ell\colon\ |\mathrm{Grad}(\mathfrak{X})| \to \mathbb{Q}$ extends canonically to a map $\ell\colon\ |\mathrm{Grad}_{\mathbb{Q}}(\mathfrak{X})| \to \mathbb{Q}$ by setting $\langle \frac{1}{n}\lambda, \ell\rangle = \frac{1}{n}\langle \lambda, \mathcal{L}\rangle$, for a rational graded point $\frac{1}{n}\lambda$.

Line bundles are an important source of linear forms. The following definition essentially comes from [30] and [26].

Definition 2.4.3. Let $\mathcal{X}$ be an algebraic stack such that $\mathrm{Grad}(\mathcal{X})$ is algebraic, and let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. Define a map

$$\langle -, \mathcal{L}\rangle \colon |\mathrm{Grad}(\mathcal{X})| \to \mathbb{Q}$$

as follows. If $g \in |\mathrm{Grad}(\mathcal{X})|$ is represented by a field k and a map $\lambda\colon B\mathbb{G}_{m,k} \to \mathcal{X}$, let $\langle g, \mathcal{L}\rangle := \langle \lambda, \mathcal{L}\rangle := -\operatorname{wt}(\lambda^*\mathcal{L})$, the opposite of the weight of the one-dimensional representation $\lambda^*\mathcal{L}$ of $\mathbb{G}_{m,k}$.

Remark 2.4.4. Our sign convention for weights is as follows. Let k be a field. We denote by $\mathcal{O}_{B\mathbb{G}_{m,k}}(n)$, for $n \in \mathbb{Z}$, the representation of $\mathbb{G}_{m,k}$ with underlying vector space k and such that $t * 1 = t^n$ for $t \in k^\times$. By definition, $\operatorname{wt}\mathcal{O}_{B\mathbb{G}_{m,k}}(n) = n$.

The total space of $\mathcal{O}_{B\mathbb{G}_{m,k}}(n)$ is $\mathbb{A}(\mathcal{O}_{B\mathbb{G}_{m,k}}(n)) = \operatorname{Spec}_{B\mathbb{G}_{m,k}}(\operatorname{Sym}(\mathcal{O}_{B\mathbb{G}_{m,k}})^\vee)$. If we let $\mathbb{G}_{m,k}$ act on $\mathbb{A}^1_k$ by the formula $t * s = t^n s$, for any k-algebra R, $t \in \mathbb{G}_{m,k}(R) = R^\times$ and $s \in \mathbb{A}^1_k(R) = R$, then $\mathbb{A}(\mathcal{O}_{B\mathbb{G}_{m,k}}(n)) = \mathbb{A}^1_k/\mathbb{G}_{m,k}$ for this action. If we write $\mathbb{A}^1_k = \operatorname{Spec} k[x]$, the standard coordinate x has weight $-n$.

Let G be a linear algebraic group over k acting linearly on a finite dimensional vector space V, and let $p\colon \mathbb{P}(V)/G \to B\mathbb{G}_m$ be the map $\mathbb{P}(V)/G = (V \setminus \{0\})/G \times \mathbb{G}_m \to B(G \times \mathbb{G}_m) \to B\mathbb{G}_m$, where $\mathbb{G}_m$ acts by scaling on V. Then $p^*(\mathcal{O}_{B\mathbb{G}_{m,k}}(1)) = \mathcal{O}_{\mathbb{P}(V)/G}(1)$ is the standard ample line bundle on $\mathbb{P}(V)/G$. Let $\lambda\colon \mathbb{G}_{m,k} \to G$ be a one-parameter subgroup, and let $x \in \mathbb{P}(V)(k)$ be a point fixed by λ, defining a graded point $\lambda_x\colon B\mathbb{G}_{m,k} \to \mathbb{P}(V)/G$. The point x corresponds to a one-dimensional subspace $L \subset V$ invariant by λ. Thus there is $n \in \mathbb{Z}$ such that $\lambda(t)v = t^n v$ for all k-algebras R, $v \in R \otimes_k L$ and $t \in R^\times$. In other words, L is regarded as a $\mathbb{G}_{m,k}$-representation and we let $n = \operatorname{wt} L$. We can identify $L = \lambda_x^*\left(\mathcal{O}_{\mathbb{P}(V)/G}(-1)\right)$. Thus our sign conventions are such that $\langle \lambda_x, \mathcal{O}_{\mathbb{P}(V)/G}(1)\rangle = -\operatorname{wt}(L^\vee) = n$.

Proposition 2.4.5. *Let $\mathcal{X}$ and $\mathcal{L}$ be as in Definition 2.4.3. Then $\langle -, \mathcal{L}\rangle$ is well-defined and a linear form on $\mathcal{X}$.*

Proof. If $k'|k$ is a field extension, inducing a map $g\colon B\mathbb{G}_{m,k'} \to B\mathbb{G}_{m,k}$, and U is a line bundle on $B\mathbb{G}_{m,k'}$, then $\operatorname{wt} U = \operatorname{wt}(g^*U)$, so $\langle -, \mathcal{L}\rangle$ is well-defined.

To see that $\langle -, \mathcal{L}\rangle$ is locally constant on $|\mathrm{Grad}(\mathcal{X})|$, it is enough to prove that for any map $f\colon \operatorname{Spec} A \to \mathrm{Grad}(\mathcal{X})$ with A any commutative ring, the composition $|\operatorname{Spec} A| \to |\mathrm{Grad}(\mathcal{X})| \to \mathbb{Q}$ of $|f|$ and $\langle -, \mathcal{L}\rangle$ is locally constant. Let $h\colon B\mathbb{G}_{m,A} \to \mathcal{X}$ be the map corresponding to f. We may assume that $h^*\mathcal{L}$ is trivial when restricted to $\operatorname{Spec} A$, so $h^*\mathcal{L}$ is an A-module direct sum decomposition $A = \bigoplus_{n\in\mathbb{Z}} A_n$. Let 1_n be the degree n part of $1 \in A$. Each A_n is generated by 1_n as an A-module. On the nonvanishing locus $D(1_n)$ of 1_n, we have that the restriction $A_n|_{D(1_n)} = A|_{D(1_n)}$ and $A_m|_{D(1_n)} = 0$ if $m \neq n$. Thus $\operatorname{Spec} A = \bigsqcup_{n\in\mathbb{Z}} D(1_n)$, and the composition $|D(1_n)| \to |\mathrm{Grad}(\mathcal{X})| \xrightarrow{\langle -,\mathcal{L}\rangle} \mathbb{Q}$ is constant with value $-n$.

Now let $\alpha\colon B\mathbb{G}^n_{m,k} \to \mathcal{X}$ be a $\mathbb{Z}^n$-graded point. The pullback $\alpha^*\mathcal{L}$ corresponds to a character $\chi \in \Gamma_{\mathbb{Z}}(\mathbb{G}^n_{m,k})$, and the map $\Gamma^{\mathbb{Z}}(\mathbb{G}^n_{m,k}) \to \mathbb{Q}$ that $\langle -, \mathcal{L}\rangle$ induces is just the pairing $\lambda \mapsto -\langle \lambda, \chi\rangle$. It is thus linear. □

Now suppose that $\mathcal{X}$ is an algebraic stack over an algebraic space B satisfying Assumption 2.2.3, and endowed with a norm on graded points q.

Definition 2.4.6. The *canonical linear form* ℓ_q *on* $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ induced by q is the linear form on graded points of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ defined as follows. A graded point $\mu \in \mathrm{Grad}(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}))(k)$ lying over a point $\lambda/n \in \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})(k)$, with $\lambda \in \mathrm{Grad}(\mathcal{X})(k)$ and $n \in \mathbb{Z}_{>0}$, gives a map $\alpha = (\mu, \lambda)\colon B\mathbb{G}_{m,k}^2 \to \mathcal{X}$ and q gives an inner product on the set $\Gamma^{\mathbb{Z}}(\mathbb{G}_{m,k}^2)$ of cocharacters of $\mathbb{G}_{m,k}^2$. Let e_1, e_2 be the standard basis of $\Gamma^{\mathbb{Z}}(\mathbb{G}_{m,k}^2)$. We define

$$\langle \lambda, \ell_q \rangle = \frac{1}{n}(e_1, e_2)_q.$$

Remark 2.4.7. Note that ℓ_q determines q, since for a graded point $\lambda\colon B\mathbb{G}_{m,k} \to \mathcal{X}$, if $o\colon B\mathbb{G}_{m,k}^2 \to B\mathbb{G}_{m,k}$ is induced by $\mathbb{G}_{m,k}^2 \to \mathbb{G}_{m,k}\colon (t, t') \to tt'$, then $q(\lambda) = \ell_q(\lambda \circ o)$.

Definition 2.4.8. We say that the norm q on $\mathcal{X}$ is *algebraic* if the canonical linear form ℓ_q on $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ is induced, on each component $\mathcal{Z}$ of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ by a rational line bundle on $\mathcal{Z}$.

Remark 2.4.9. If q is an algebraic norm on $\mathcal{X}$ and $f\colon \mathcal{Y} \to \mathcal{X}$ is a morphism with proper relative automorphism groups, then the pulled back norm f^*q (Proposition and Definition 2.3.6) is algebraic. Indeed, if $\ell_q = \langle -, \mathcal{M}\rangle$ for a rational line bundle $\mathcal{M}$ on $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$, then $\ell_{f^*q} = \langle -, \mathrm{Grad}_{\mathbb{Q}}(f)^*\mathcal{M}\rangle$. We will tacitly use this fact in the sequel.

2.5 Θ-stratifications

We now discuss Θ-stratifications on algebraic stacks, which are a generalisation due to Halpern-Leistner of the Hesselink-Kempf-Kirwan-Ness stratification in GIT [40, Chapter 12], and of the stratification by Harder-Narasimhan type for vector bundles on a curve [9]. We follow [26] with slight modifications, using the stack $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ of rational filtrations instead of $\mathrm{Filt}(\mathcal{X})$. We will focus on Θ-stratifications induced by a pair (ℓ, q), where q is a norm and ℓ is a linear form on graded points of a stack $\mathcal{X}$.

We fix an algebraic stack $\mathcal{X}$ over an algebraic space B satisfying Assumption 2.2.3. The following is a variant of [26, Definition 2.1.2].

Definition 2.5.1 (Θ-stratification)**.** Let Γ be a partially ordered set. A *(weak)* Θ-*stratification* of $\mathcal{X}$ indexed by Γ is a family $(\mathcal{S}_c)_{c\in\Gamma}$ of open substacks of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ satisfying:

1. For every $c \in \Gamma$, the composition $\mathcal{S}_c \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \xrightarrow{\mathrm{ev}_1} \mathcal{X}$ is a locally closed immersion (resp. locally finite radicial[2]).
2. The $\mathrm{ev}_1(|\mathcal{S}_c|)$ are pairwise disjoint and cover $|\mathcal{X}|$.
3. For every $c \in \Gamma$, the stratum $\mathcal{S}_c$ is the preimage along $\mathrm{gr}\colon \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ of an open substack $\mathcal{Z}_c$ of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$, called the *centre* of $\mathcal{S}_c$.
4. For every $c \in \Gamma$, the set $|\mathcal{X}_{\le c}| \coloneqq \bigcup_{c' \le c} \mathrm{ev}_1(|\mathcal{S}_{c'}|)$ is open in $|\mathcal{X}|$, thus defining an open substack $\mathcal{X}_{\le c}$ of $\mathcal{X}$.

Remark 2.5.2. If $\sigma\colon \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ denotes the "split filtration" map, then for all $c \in \Gamma$ the centre of $\mathcal{S}_c$ is $\mathcal{Z}_c = \sigma^{-1}(\mathcal{S}_c)$ and thus it is uniquely determined.

Remark 2.5.3. An open substack $\mathcal{S}$ of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ will be referred to as a *locally closed* Θ-*stratum* if it is the preimage along gr of an open substack $\mathcal{Z}$ of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ (its *centre*) and the composition $\mathcal{S} \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \to \mathcal{X}$ is a locally closed immersion.

[2] We say that a morphism $\mathcal{Y} \to \mathcal{Z}$ is *locally finite radicial* if it factors as $\mathcal{Y} \to \mathcal{U} \to \mathcal{Z}$ with $\mathcal{Y} \to \mathcal{U}$ finite and radicial and $\mathcal{U} \to \mathcal{Z}$ an open immersion.

Proposition 2.5.4. *Let Γ be a totally ordered set. Let $(\mathcal{S}_c)_{c\in\Gamma}$ be a (weak) Θ-stratification of $\mathcal{X}$. If each stratum $\mathcal{S}_c$ is contained in the closed and open substack* $\mathrm{Filt}(\mathcal{X}) \subset \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ *of integral filtrations, then $(\mathcal{S}_c)_{c\in\Gamma}$ and $(\mathcal{X}_{\leq c})_{c\in\Gamma}$ define a (weak) Θ-stratification in the sense of [26, Definition 2.1.2]. Conversely, any (weak) Θ-stratification in the sense of [26, Definition 2.1.2] defines a (weak) Θ-stratification.*

Proof. Let $(\mathcal{S}_c)_{c\in\Gamma}$ be a (weak) Θ-stratification of $\mathcal{X}$. It is enough to show that each $\mathcal{S}_c$ is a (weak) Θ-stratum of $\mathcal{X}_{\leq c}$ [26, Definition 2.1.1], that is

1. Each $\mathcal{S}_c \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ factors through $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_{\leq c})$ and $\mathcal{S}_c \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_{\leq c})$ is an open and closed immersion.
2. The composition $\mathcal{S}_c \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_{\leq c}) \to \mathcal{X}_{\leq c}$ is a closed immersion (resp. finite and radicial).

Then the conditions in [26, Definition 2.1.2] are trivially satisfied by construction. Note that Halpern-Leistner also demands the $\mathcal{S}_c$ to be integral and Γ to be a total order.

We have a diagram

$$\begin{tikzcd} \mathcal{Z}_c \arrow[r] \arrow[d] & \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \arrow[r] \arrow[d, "\sigma"] & \mathcal{X} \arrow[d, equal] \\ \mathcal{S}_c \arrow[r] & \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \arrow[r] & \mathcal{X} \end{tikzcd}$$

and $\mathcal{S}_c \to \mathcal{X}$ factors through $\mathcal{X}_{\leq c}$. Therefore $\mathcal{Z}_c \to \mathcal{X}$ also factors through $\mathcal{X}_{\leq c}$ and thus $\mathcal{Z}_c \subset \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_{\leq c})$ since the formation of $\mathrm{Grad}_{\mathbb{Q}}$ is compatible with immersions. Since the natural square

$$\begin{tikzcd} \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_{\leq c}) \arrow[r] \arrow[d, "\mathrm{gr}"] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \arrow[d, "\mathrm{gr}"] \\ \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_{\leq c}) \arrow[r] & \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \end{tikzcd}$$

is cartesian [26, Proposition 1.3.1 (3)] and $\mathcal{S}_c = \mathrm{gr}^{-1}(\mathcal{Z}_c)$, we have $\mathcal{S}_c \subset \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_{\leq c})$.

Now the composition $\mathcal{S}_c \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_{\leq c}) \to \mathcal{X}_{\leq c}$ is a locally closed immersion (resp. locally finite radicial) and has closed image. It is thus a closed immersion (resp. finite radicial).

Form the pullback

$$\begin{tikzcd} \mathcal{U} \arrow[r, "a"] \arrow[d, "b"] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_{\leq c}) \arrow[d, "\mathrm{ev}_1"] \\ \mathcal{S}_c \arrow[r] \arrow[u, bend left, dashed, "s_c"] & \mathcal{X}_{\leq c}. \end{tikzcd}$$

The inclusion $\mathcal{S}_c \xrightarrow{\iota_c} \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_{\leq c})$ gives a section s_c of b. Since ev_1 is representable and separated [26, Proposition 1.1.13], s_c is a closed immersion. Therefore $\iota = a \circ s_c$ is a closed immersion (resp. finite and radicial).

For the converse, just note that if $\mathcal{S}_c \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_{\leq c})$ is open and closed, then $\mathcal{S}_c = \mathrm{gr}^{-1}(\mathcal{Z}_c)$ for some $\mathcal{Z}_c \subset \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_{\leq c})$ open and closed, because $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_{\leq c})$ and $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_{\leq c})$ have the same components [26, Lemma 1.3.8]. $\square$

Remark 2.5.5. If each $\mathcal{S}_c$ intersects only a finite number of connected components of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$, then it becomes integral after scaling up, that is, after replacing $\mathcal{S}_c$ by its image under the "rising to the nth power" map $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ for big enough n. After suitably subdividing each $\mathcal{S}_c$, the condition is always satisfied.

A Θ-stratification of $\mathcal{X}$ assigns, for each geometric point x of $\mathcal{X}$, a canonical rational filtration of x:

Definition 2.5.6 (HN filtrations, [26, Lemma 2.1.4])**.** Consider a Θ-stratification $(\mathcal{S}_c)_{c\in\Gamma}$ of $\mathcal{X}$, and let $x\colon \operatorname{Spec}(k) \to \mathcal{X}$ be a field-valued point. There is a unique $c \in \Gamma$ such that the image of $\mathcal{S}_c \to \mathcal{X}$ contains the topological point defined by x; and there is a unique, up to unique isomorphism, lift of x to $\mathcal{S}_c$, which gives a filtration $\lambda \in \mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{X}, x)$ called the *Harder-Narasimhan filtration* (or the *HN filtration*) of x.

Remark 2.5.7. In the case of a weak Θ-stratification, the HN filtration of a k-point is defined over a finite purely inseparable extension of k. We will only use the case of Θ-stratifications.

The following is a reformulation of [26, Definition 2.3.1], convenient for our purposes.

Proposition and Definition 2.5.8 (Induced Θ-stratifications)**.** Let $\mathcal{X}'$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3, and let $h\colon \mathcal{X}' \to \mathcal{X}$ be either a closed immersion or a base change of a map between algebraic spaces like in Proposition 2.2.12. Let $(\mathcal{S}_c)_{c\in\Gamma}$ be a (weak) Θ-stratification of $\mathcal{X}$. For each $c \in \Gamma$ let $h^*\mathcal{S}_c$ be the pullback

$$\begin{tikzcd} h^*\mathcal{S}_c \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{S}_c \arrow[d] \\ \operatorname{Filt}_{\mathbb{Q}}(\mathcal{X}') \arrow[r] & \operatorname{Filt}_{\mathbb{Q}}(\mathcal{X}). \end{tikzcd}$$

Then the family $(h^*\mathcal{S}_c)_{c\in\Gamma}$ is a (weak) Θ-stratification of $\mathcal{X}'$ called the Θ*-stratification induced by* h and $(\mathcal{S}_c)_{c\in\Gamma}$.

Proof. This is the content of [26, Lemmas 2.3.2 and 2.3.3] in the case of integral filtration, from which the case of rational filtrations follows easily. □

Now we fix a rational quadratic norm q and a rational linear form ℓ on graded points of $\mathcal{X}$. We regard q and ℓ as maps $|\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X})| \to \mathbb{Q}$ by Remarks 2.3.4 and 2.4.2. Moreover, q and ℓ induce functions on $|\operatorname{Filt}_{\mathbb{Q}}(\mathcal{X})|$, by precomposing along $|\mathrm{gr}| : |\operatorname{Filt}_{\mathbb{Q}}(\mathcal{X})| \to |\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X})|$ that we will still denote q and ℓ.

The norm q and the linear form ℓ give rise to two other interesting functions.

Definition 2.5.9 (Associated numerical invariant [26, 4.1.1 and 4.1.14])**.** We define the *numerical invariant* μ associated to q and ℓ to be the function $\mu\colon |\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X})| \to \mathbb{R}_{\geq 0}$ such that

1. on the open and closed substack $\sigma\colon \mathcal{X} \to \operatorname{Grad}_{\mathbb{Q}}(\mathcal{X})$ defined by the "trivial grading" map, μ takes the value 0; and
2. on $|\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X})| \setminus |\mathcal{X}|$, we set $\mu = \dfrac{\ell}{\sqrt{q}}$.

We extend μ to a function on $|\operatorname{Filt}_{\mathbb{Q}}(\mathcal{X})|$ by taking the composition

$$|\operatorname{Filt}_{\mathbb{Q}}(\mathcal{X})| \xrightarrow{|\mathrm{gr}|} |\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X})| \xrightarrow{\mu} \mathbb{R}_{\geq 0},$$

which we will still denote μ.

Definition 2.5.10 (Stability function [26, 4.1.1])**.** The *stability function* $M^\mu\colon |\mathcal{X}| \to [0, \infty]$ associated to μ is defined by

$$M^\mu(x) := \sup\{\mu(\lambda) \mid \lambda \in |\operatorname{Filt}_{\mathbb{Q}}(\mathcal{X})|,\ \operatorname{ev}_1(\lambda) = x\}.$$

Remark 2.5.11. In [26], the stack $\operatorname{Grad}(\mathcal{X})$ is used instead of $\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X})$. This is not an important difference because the numerical invariant μ is scale-invariant.

Definition 2.5.12. The *semistable locus* $|\mathcal{X}^{\mathrm{ss}}|$ with respect to the linear form ℓ on $\mathcal{X}$ is the subset

$$\{x \in |\mathcal{X}| \mid \ell(\lambda) \leq 0, \text{ for all } \lambda \in |\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})| \text{ with } \mathrm{ev}_1(\lambda) = x\} \subset |\mathcal{X}|.$$

If the semistable locus $|\mathcal{X}^{\mathrm{ss}}|$ is open, then it defines an open substack of $\mathcal{X}$ denoted $\mathcal{X}^{\mathrm{ss}}$.

Definition 2.5.13. We say that the pair (ℓ, q) *defines a Θ-stratification* if the following holds:

1. The semistable locus $|\mathcal{X}^{\mathrm{ss}}| \subset |\mathcal{X}|$ is open. If $s\colon \mathcal{X} \to \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ is the "trivial filtration", which is an open and closed immersion, we denote $\mathcal{S}_0 = s(\mathcal{X}^{\mathrm{ss}})$, an open substack of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ isomorphic to $\mathcal{X}^{\mathrm{ss}}$.
2. For all $c \in \mathbb{Q}_{>0}$, the subset $|\mathcal{S}_c| \coloneqq \{\lambda \in |\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})| \mid \langle \ell, \lambda \rangle = 1 \text{ and } \mu(\lambda) = M^{\mu}(\mathrm{ev}_1(\lambda)) = \sqrt{c}\}$ of $|\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})|$ is open, and thus it defines an open substack $\mathcal{S}_c$ of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$.
3. The family $(\mathcal{S}_c)_{c \in \mathbb{Q}_{\geq 0}}$ is a Θ-stratification of $\mathcal{X}$ indexed by $\mathbb{Q}_{\geq 0}$, referred to as the Θ-stratification *induced by* (ℓ, q).

Remark 2.5.14. If (ℓ, q) define a Θ-stratification, then for all $c \in \mathbb{Q}_{\geq 0}$ we have the equality $|\mathcal{X}_{\leq c}| = \{x \in |\mathcal{X}| \mid M^{\mu}(x) \leq \sqrt{c}\}$ of $|\mathcal{X}|$, using the notation of Definition 2.5.1. In particular $\mathcal{X}^{\mathrm{ss}} = \mathcal{X}_{\leq 0}$, and note as well that the centre $\mathcal{Z}_0$ of the minimal stratum $\mathcal{S}_0 = \mathcal{X}^{\mathrm{ss}}$ is canonically isomorphic to $\mathcal{X}^{\mathrm{ss}}$.

Remark 2.5.15. In Halpern-Leistner's definition, the stack $\mathrm{Filt}(\mathcal{X})$ is used instead of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$. This is not an important difference, as we now explain. In [26, Definition 4.1.3], the Θ-stratification depends, in principle, on the choice of a complete set of representatives for $\pi_0(\mathrm{Filt}(\mathcal{X}))/\mathbb{N}^*$, although two different choices give rise to isomorphic Θ-strata. Since $\pi_0(\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}))/\mathbb{N}^* = \pi_0(\mathrm{Filt}(\mathcal{X}))/\mathbb{N}^*$, and since two connected components a and b of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ and $\mathrm{Filt}(\mathcal{X})$ respectively that represent the same class are isomorphic via the action of $\mathbb{N}^*$, we can instead take a complete set of representatives in $\pi_0(\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}))$. We are using canonical set of representatives for the unstable strata, namely the components of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$ on which ℓ takes the value 1. This observation, together with Proposition 2.5.4, implies that the pair (ℓ, q) defines a Θ-stratification in the sense of [26, Definition 4.1.3] if and only if it does so in the sense above, and that in that case the Θ-strata that we get are isomorphic to Halpern-Leistner's. However, Definition 2.5.13 has the advantage that it does not depend on noncanonical choices.

Proposition 2.5.16 (Compatibility with pullback)**.** *Let $\mathcal{X}'$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3, and let $h\colon \mathcal{X}' \to \mathcal{X}$ be either a closed immersion or a base change of a map between algebraic spaces like in Proposition 2.2.12. Suppose that the pair (ℓ, q) defines a Θ-stratification $(\mathcal{S}_c)_{c \in \mathbb{Q}_{\geq 0}}$. Then $(h^*\ell, h^*q)$ defines a Θ-stratification of $\mathcal{X}'$, equal to the induced stratification $(h^*\mathcal{S}_c)_{c \in \mathbb{Q}_{\geq 0}}$ of Proposition and Definition 2.5.8.*

Proof. The claim follows at once from the observation that, for every field k and point $x \in \mathcal{X}'(k)$, the induced map $\mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}', x) \to \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, h(x))$ is a bijection, compatible with the values of (ℓ, q) and $(h^*\ell, h^*q)$. $\square$

2.6 Θ-stratifications for stacks proper over a normed good moduli stack

We now get to the main result (Theorem 2.6.3) about existence and properties of Θ-stratifications that we will use. It is an extension of [26, Theorem 5.6.1] where we also establish existence of good moduli spaces for the centres of the strata, along with other improvements. We first introduce the notion of *positive linear form on graded points*, a slight variant of [26, Definition 5.3.1].

Definition 2.6.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks over an algebraic space B, satisfying Assumption 2.2.3. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a proper representable morphism and let ℓ be a linear form on graded points of $\mathcal{Y}$. We say that ℓ is *f-positive* provided that for all fields k and all diagrams

$$\begin{tikzcd} \mathbb{P}^1_k/\mathbb{G}_{m,k} \arrow[r, "\varphi"] \arrow[d] & \mathcal{Y} \arrow[d, "f"] \\ B\mathbb{G}_{m,k} \arrow[r] & \mathcal{X} \end{tikzcd}$$

such that the induced map $\mathbb{P}^1_k/\mathbb{G}_{m,k} \to B\mathbb{G}_{m,k} \times_{\mathcal{X}} \mathcal{Y}$ is finite, where the action of $\mathbb{G}_{m,k}$ on $\mathbb{P}^1_k$ is given by $t[a,b] = [ta,b]$ in projective coordinates and we denote $0 = [0,1]$ and $\infty = [1,0]$, we have

$$\ell(\varphi|_{\{0\}/\mathbb{G}_{m,k}}) < \ell(\varphi|_{\{\infty\}/\mathbb{G}_{m,k}}).$$

Example 2.6.2. If $\mathcal{L}$ is a rational line bundle on $\mathcal{Y}$ that is f-ample and $\ell = \langle -, \mathcal{L}\rangle$, then ℓ is f-positive. Indeed, for all commutative diagrams as in Definition 2.6.1, the pullback $\varphi^*\mathcal{L}$ is ample relative to $\mathbb{P}^1_k/\mathbb{G}_{m,k} \to B\mathbb{G}_{m,k}$, and the claim follows after embedding $\mathbb{P}^1_k$ in a bigger projective space and looking at the weights of the corresponding $\mathbb{G}_{m,k}$-representation.

Theorem 2.6.3. *Let $\mathcal{X}$ be a noetherian algebraic stack with affine diagonal and a good moduli space $\pi\colon \mathcal{X} \to X$, endowed with a norm on graded points q. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a proper representable morphism and let ℓ be an f-positive linear form on graded points of $\mathcal{Y}$. Then*

1. *The pair (ℓ, f^*q) defines a Θ-stratification $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$ of $\mathcal{Y}$.*
2. *For every $c \in \mathbb{Q}_{\geq 0}$, the centre $\mathcal{Z}_c$ has a good moduli space $\mathcal{Z}_c \to Z_c$. We denote $\mathcal{Y}^{ss} = \mathcal{Z}_0$ and $Y^{ss} = Z_0$.*
3. *For $c \in \mathbb{Q}_{\geq 0}$, let $\mathcal{X}_c$ be the union of connected components of* $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$ *intersecting the image of $\mathcal{Z}_c \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y}) \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$. Then $\mathcal{X}_c$ is quasi-compact and has a good moduli space $\mathcal{X}_c \to X_c$.*
4. *For every $c \in \mathbb{Q}_{\geq 0}$, the induced map $Z_c \to X_c$ is proper.*
5. *If $\ell = \langle -, \mathcal{L}\rangle$ for an f-ample rational line bundle $\mathcal{L}$ on $\mathcal{Y}$, then $Y^{ss} \to X$ is projective. If in addition the norm on graded points q is algebraic, then for all $c \in \mathbb{Q}_{\geq c}$ the map $Z_c \to X_c$ is projective.*

Example 2.6.4. Let $\mathcal{X}$ be as in the statement of the theorem, and let $f\colon \mathcal{Y} \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ along a closed substack $\mathcal{Z}$. Let $\mathcal{E} = f^{-1}(\mathcal{Z})$ be the exceptional divisor. The ideal sheaf $\mathcal{O}_{\mathcal{Y}}(-\mathcal{E})$ of the exceptional divisor is f-ample. Therefore we may apply the theorem with $\ell = \langle -, \mathcal{O}_{\mathcal{Y}}(-\mathcal{E})\rangle$ to get a Θ-stratification of $\mathcal{Y}$. In this case, the semistable locus $\mathcal{Y}^{\mathrm{ss}}$ is the *saturated blow-up* of $\mathcal{X}$ along $\mathcal{Z}$ [19, Definition 3.2], by [19, Proposition 3.17] and [26, Theorem 5.6.1, (2)].

More generally, if f is projective and $\ell = \langle -, \mathcal{L}\rangle$, with $\mathcal{L}$ an f-ample line bundle. Then $\mathcal{Y}^{\mathrm{ss}}$ is the *saturated* Proj [19, Definition 3.1], $\mathcal{Y}^{\mathrm{ss}} = \mathrm{Proj}^{\pi}_{\mathcal{X}}\left(\bigoplus_{n\in\mathbb{N}} f_*(\mathcal{L}_{\otimes n})\right)$.

Proof. Note that $\mathcal{X}$ satisfies Assumption 2.2.3 with $B = X$ (Example 2.2.4). Let μ denote the numerical invariant defined by (ℓ, f^*q).

Step 1. The numerical invariant μ is strictly Θ-monotone over X [26, Definition 5.2.1] and strictly S-monotone [26, Definition 5.5.7] over X.

We first note that the map $\pi\colon \mathcal{X} \to X$ is Θ-reductive [7, Definition 3.10] and S-complete [7, Definition 3.38] by [7, Theorem 5.4]. We follow the argument in the proof of [26, Proposition 5.3.3]. Let R be a discrete valuation ring over X with uniformiser π

and residue field k. Let $\overline{\mathrm{ST}}_R = \mathrm{Spec}(R[s,t]/(st-\pi))/\mathbb{G}_m$, where s has weight 1 and t has weight -1 [7, Section 3.5.1]. Let $\mathcal{V}$ be the stack Θ_R (resp. the stack $\overline{\mathrm{ST}}_R$), and let $\mathcal{V}' = \mathcal{V} \setminus \{(0,0)\}$. Suppose given a morphism $v\colon \mathcal{V}' \to \mathcal{Y}$. Because $\mathcal{X}$ is Θ-reductive (resp. S-complete), the morphism $f \circ v\colon \mathcal{V}' \to \mathcal{X}$ extends to a map $u\colon \mathcal{V} \to \mathcal{X}$. Let $\mathcal{W}$ be the schematic image of $\mathcal{V}'$ inside the pullback $\mathcal{V} \times_{u,\mathcal{X},f} \mathcal{Y}$. We have a diagram

$$\begin{tikzcd} & \mathcal{W} \arrow[r, "\overline{v}"'] \arrow[d, "p"] & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{V}' \arrow[r, hook] \arrow[ur] \arrow[urr, bend left, "v"] & \mathcal{V} \arrow[r, "u"'] & \mathcal{X} \end{tikzcd}$$

and we want to show that $\mathcal{W}$ and $\overline{v}$ satisfy the conditions of [26, Definitions 5.2.1 and 5.5.7], of which the only nontrivial one is (3). For this, suppose given a commutative square

$$\begin{tikzcd} \mathbb{P}^1_k/\mathbb{G}_{m,k} \arrow[r, "\varphi"] \arrow[d] & \mathcal{W} \arrow[d, "p"] \\ B\mathbb{G}_{m,k} \arrow[r, "g"] & \mathcal{V} \end{tikzcd}$$

where g is a multiple of the canonical graded point $\{(0,0)\}/\mathbb{G}_{m,k} \to \mathcal{V}$ and the induced $\mathbb{P}^1_k/\mathbb{G}_{m,k} \to B\mathbb{G}_{m,k} \times_{\mathcal{V}} \mathcal{W}$ is finite. We want to show that

$$\ell_c(\overline{v} \circ \varphi|_{\{0\}/\mathbb{G}_{m,k}}) < \ell_c(\overline{v} \circ \varphi|_{\{\infty\}/\mathbb{G}_{m,k}}).$$

The map $\mathbb{P}^1_k/\mathbb{G}_{m,k} \to B\mathbb{G}_{m,k} \times_{\mathcal{X}} \mathcal{Y}$ is also finite, so the inequality follows from ℓ being f-positive.

We remark that properness of p is one of the conditions of [26, Definitions 5.2.1 and 5.5.7] and it is guaranteed by properness of f.

Step 2. The pair (ℓ, f^*q) defines a weak Θ-stratification.

Since $\mathcal{Y}$ is quasi-compact, the numerical invariant μ satisfies HN-boundedness [26, Proposition 4.4.2]. Therefore (ℓ, f^*q) defines a weak Θ-stratification if μ is strictly Θ-monotone [26, Theorem 5.2.3]. Thus the claim follows from Step 1.

Step 3. The weak Θ-stratification defined by (ℓ, f^*q) is a Θ-stratification.

Checking that a finite morphism $\mathcal{U} \to \mathcal{V}$ is a closed immersion can be done after base change along any geometric point $\mathrm{Spec}\, k \to \mathcal{V}$. Since the stratification is preserved after base change along any map $X' \to X$ Proposition 2.5.16, we may thus assume that $X = \mathrm{Spec}\, k$ is the spectrum of an algebraically closed field. If $\mathrm{Spec}\, k$ is of characteristic 0, then the weak Θ-stratification is a Θ-stratification by [26, Theorem 5.6.1]. If k is of positive characteristic p, then $\mathcal{X} = \mathrm{Spec}\, A/G$ where G is a linearly reductive group over k. We should have $\mathcal{Y} = Y/G$, where $Y \to \mathrm{Spec}\, A$ is G-equivariant and projective. By Nagata's Theorem [16, Chapter IV, Section 3, Theorem 3.6], the identity component G° is a group of multiplicative type and p does not divide the order of G/G°. By [26, Lemma 2.1.7, (2)] it is enough to prove that for any $c \in \mathbb{Q}_{\geq 0}$ and for any k-point λ of $\mathcal{S}_c$, the induced map $\varphi\colon \mathrm{Lie}(\mathrm{Aut}_{\mathcal{S}_c}(\lambda)) \to \mathrm{Lie}(\mathrm{Aut}_{\mathcal{Y}}(\mathrm{ev}_1(\lambda))$ on Lie algebras is surjective. The filtration λ is contained in some open substack of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{Y})$ of the form $Y^{\gamma,+}/P(\gamma)$ for some one-parameter subgroup γ of G, by [26, Theorem 1.4.8], and it corresponds to a point $z \in Y^{\gamma,+}(k)$. Thus $\mathrm{Aut}_{\mathcal{S}_c}(\lambda) = \mathrm{Stab}_{P(\gamma)}(z)$, while $\mathrm{Aut}_{\mathcal{X}}(\mathrm{ev}_1(\lambda)) = \mathrm{Stab}_G(z)$. Both groups have the same identity component, equal to $\mathrm{Stab}_{G^\circ}(z)^\circ$, since G° is contained in $P(\lambda)$. Thus the map φ on Lie algebras is actually an isomorphism.

Step 4. The semistable locus $\mathcal{Y}^{ss}$ has a good moduli space $\mathcal{Y}^{ss} \to Y^{ss}$ and $Y^{ss} \to X$ is separated.

By [7, Theorem 5.4], $\mathcal{X}$ is Θ-reductive and S-complete. Therefore μ is strictly Θ-monotone and strictly S-monotone over X by Step 1. By [26, Theorem 5.5.8], the semistable locus $\mathcal{Y}^{\mathrm{ss}}$ is Θ-reductive and S-complete over X. Therefore it is enough to show, by [7, Theorem 5.4], that the stabiliser of every closed point of $\mathcal{Y}^{\mathrm{ss}}$ is linearly reductive. Let k be an algebraically closed field and let $x \in \mathcal{Y}^{\mathrm{ss}}(k)$ be a closed point. By [7, Proposition 3.47], the automorphism group $\mathrm{Aut}(x)$ is geometrically reductive. If the characteristic of k is 0, then $\mathrm{Aut}(x)$ is also linearly reductive. If k is of positive characteristic p, then $f(x)$ specialises to a k-point y closed in the fibre of $\pi\colon \mathcal{X} \to X$, whose stabiliser $\mathrm{Aut}(y)$ is thus linearly reductive. By Nagata's Theorem [16, Chapter IV, Section 3, Theorem 3.6], the identity component $\mathrm{Aut}(y)^\circ$ is a group of multiplicative type and p does not divide the order of $\mathrm{Aut}(y)/\mathrm{Aut}(y)^\circ$. Since $\mathrm{Aut}(x)$ is a subgroup of $\mathrm{Aut}(y)$, the same holds for $\mathrm{Aut}(x)$ and it is thus linearly reductive.

Step 5. The map $Y^{ss} \to X$ is proper.

Since $\mathcal{Y} \to \mathcal{X}$ is proper and $\mathcal{X} \to X$ is universally closed [1, Theorem 4.16], we have that $\mathcal{Y} \to X$ is universally closed. Now the Semistable Reduction Theorem [7, Corollary 6.12] implies that $\mathcal{Y}^{\mathrm{ss}} \to X$ is also universally closed. Therefore $Y^{\mathrm{ss}} \to X$ is universally closed. Since it is also separated, by Step 4, and of finite type, because f and π are, it is proper.

Step 6. The map $Y^{ss} \to X$ is projective if $\ell = \langle -, \mathcal{L}\rangle$ for a rational line bundle $\mathcal{L}$.

We may scale $\mathcal{L}$ up to assume that it is a line bundle, since this does not change the semistable locus. Then we have, by [26], that $Y^{\mathrm{ss}} = \mathrm{Proj}_X\left(\pi_* f_*(\oplus_{n\in\mathbb{N}}\mathcal{L}^{\otimes n})\right)$.

Step 7. For all $c \in \mathbb{Q}_{\geq 0}$, the stack $\mathcal{X}_c$ is quasi-compact and has a good moduli space $\mathcal{X}_c \to X_c$.

Since $\mathcal{Z}_c$ is quasi-compact, and because of the existence of a norm on graded points on $\mathcal{X}$, we have by [26, Proposition 3.8.2] that $\mathcal{X}_c$ is quasi-compact. The claim follows from Lemma 2.6.6.

Step 8. The centres $\mathcal{Z}_c$ have good moduli spaces $\mathcal{Z}_c \to Z_c$ for all $c \in \mathbb{Q}_{c\geq 0}$ and $Z_c \to X_c$ is proper.

For $c = 0$, the claim is the content of Step 4 and Step 5. If $c \in \mathbb{Q}_{>0}$, let $\overline{\mathcal{Z}}_c$ be the union of those connected components of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y})$ intersecting $\mathcal{Z}_c$. Again, because of the existence of a norm on graded points on $\mathcal{X}$ and $\mathcal{Y}$, we have by [26, Proposition 3.8.2] that $\overline{\mathcal{Z}}_c$ is quasi-compact. Let $f_c\colon \overline{\mathcal{Z}}_c \to \mathcal{X}_c$ be the restriction of $\mathrm{Grad}_{\mathbb{Q}}(f)$ to $\overline{\mathcal{Z}}_c$ and $\mathcal{X}_c$. By Proposition 2.2.14, the map f_c is representable and proper.

Let us denote $\ell|_{\overline{\mathcal{Z}}_c}$ the pullback of the linear form ℓ on $\mathcal{Y}$ along $\overline{\mathcal{Z}}_c \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y}) \to \mathcal{Y}$. We denote ℓ_{f^*q} the linear form on $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y})$ induced by the norm f^*q on $\mathcal{Y}$ (Definition 2.4.6), and $\ell_{f^*q}|_{\overline{\mathcal{Z}}_c}$ its restriction to $\overline{\mathcal{Z}}_c$. Note that $\ell_{f^*q}|_{\overline{\mathcal{Z}}_c} = (f_c)^*(\ell_q|_{\mathcal{X}_c})$ is the pullback of the linear form $\ell_q|_{\mathcal{X}_c}$ on $\mathcal{X}_c$. We will consider the *shifted linear form*

$$\ell_c := \ell|_{\overline{\mathcal{Z}}_c} - c\ell_{f^*q}|_{\overline{\mathcal{Z}}_c} \tag{5}$$

on $\overline{\mathcal{Z}}_c$. We will use the following.

Theorem 2.6.5 (Linear Recognition Theorem [28])**.** *The centre $\mathcal{Z}_c$ is the semistable locus inside $\overline{\mathcal{Z}}_c$ with respect to the shifted linear form ℓ_c on $\overline{\mathcal{Z}}_c$.*

Therefore, we have a representable proper morphism $f_c\colon \overline{\mathcal{Z}}_c \to \mathcal{X}_c$ and a linear form ℓ_c and a norm on graded points $q|_{\mathcal{X}_c}$ on $\mathcal{X}_c$. By steps 4 and 5 applied to f_c in place of f, it is enough to show that ℓ_c is f_c-positive. For this, suppose given a commutative square

$$\begin{tikzcd} \mathbb{P}^1_k/\mathbb{G}_{m,k} \arrow[r, "\varphi"] \arrow[d] & \overline{\mathcal{Z}}_c \arrow[d] \\ B\mathbb{G}_{m,k} \arrow[r, "g"] & \mathcal{X}_c \end{tikzcd}$$

where the induced $\mathbb{P}^1_k/\mathbb{G}_{m,k} \to B\mathbb{G}_{m,k} \times_{\mathcal{X}_c} \overline{\mathcal{Z}}_c$ is finite. We want to show that

$$\ell_c(\varphi|_{\{0\}/\mathbb{G}_{m,k}}) < \ell_c(\varphi|_{\{\infty\}/\mathbb{G}_{m,k}}).$$

First, note that

$$\left(\ell_{f^*q}|_{\overline{\mathcal{Z}}_c}\right)(\varphi|_{\{0\}/\mathbb{G}_{m,k}}) = \ell_q(g) = \left(\ell_{f^*q}|_{\overline{\mathcal{Z}}_c}\right)(\varphi|_{\{\infty\}/\mathbb{G}_{m,k}}).$$

On the other hand, the map $\overline{\mathcal{Z}}_c \to \mathcal{X}_c \times_{\mathcal{X}} \mathcal{Y}$ is proper, since $\overline{\mathcal{Z}}_c \to \mathcal{X}_c$ and $\mathcal{Y} \to \mathcal{X}$ are proper; and also affine, since $\overline{\mathcal{Z}}_c \to \mathcal{Y}$ and $\mathcal{X}_c \to \mathcal{X}$ are affine. Thus the induced map $\mathbb{P}^1_k/\mathbb{G}_{m,k} \to B\mathbb{G}_{m,k} \times_{\mathcal{X}} \mathcal{Y}$ is finite. Therefore we have

$$\begin{aligned} &\ell_c(\varphi|_{\{\infty\}/\mathbb{G}_{m,k}}) - \ell_c(\varphi|_{\{0\}/\mathbb{G}_{m,k}}) = \\ &\ell|_{\overline{\mathcal{Z}}_c}(\varphi|_{\{\infty\}/\mathbb{G}_{m,k}}) - \ell|_{\overline{\mathcal{Z}}_c}(\varphi|_{\{0\}/\mathbb{G}_{m,k}}) > 0, \end{aligned}$$

because ℓ is f-positive.

Step 9. If $\ell = \langle -, \mathcal{L}\rangle$ for an f-ample line bundle $\mathcal{L}$ and q is algebraic, then for all $c \in \mathbb{Q}_{\geq 0}$ the map $Z_c \to X_c$ is projective.

We begin by showing that the pullback $\mathcal{M} := (\mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y}) \to \mathcal{Y})^*\mathcal{L}$ is relatively ample with respect to $g = \mathrm{Grad}_{\mathbb{Q}}(f)\colon\ \mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y}) \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$. For this, we may assume that $\mathcal{L}$ is a line bundle and we want to show that the canonical map $\mathcal{Y} \to \mathrm{Proj}\left(\bigoplus_{n\in\mathbb{N}} g_*(\mathcal{M}^{\otimes n})\right)$ is everywhere defined and an open immersion (although it is actually an isomorphism by properness of g). This can be checked étale locally on $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$. Chose a surjective, affine and strongly étale morphism $\rho\colon\ \mathrm{Spec}\,A/\,\mathrm{GL}_n \to \mathcal{X}$ (Theorem 2.1.2). Pulling back along ρ we get a cube

$$\begin{tikzcd}[row sep=small] & \bigsqcup_{\lambda\in C} Y^{\lambda,0}/L(\lambda) \arrow[rr, "b"] \arrow[dl, "a"'] \arrow[dd] & & \bigsqcup_{\lambda\in C}(\mathrm{Spec}\,A)^{\lambda,0}/L(\lambda) \arrow[dl, "d"'] \arrow[dd] \\ \mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y}) \arrow[rr, crossing over] \arrow[dd] & & \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) & \\ & Y/\,\mathrm{GL}_n \arrow[rr] \arrow[dl, "c"'] & & \mathrm{Spec}\,A/\,\mathrm{GL}_n \arrow[dl] \\ \mathcal{Y} \arrow[rr] & & \mathcal{X} \arrow[from=uu, crossing over] & \end{tikzcd}$$

where we are using [26, Theorem 1.4.7] for the description of the stack of graded points of a quotient stack. Here, Y is a scheme acted on by GL_n with an equivariant map into $\mathrm{Spec}\,A$. We are denoting $C = \Gamma^{\mathbb{Z}}(T)/W$, where T is the standard maximal torus of GL_n

and W is the Weyl group. For $\lambda \in \Gamma^{\mathbb{Z}}(T)$, $Y^{\lambda,0}$ and $(\operatorname{Spec} A)^{\lambda,0}$ denote the fixed point loci by the cocharacter λ, and $L(\lambda)$ is the centraliser of λ in GL_n.

The arrow d is an étale cover of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$. Thus we want to show that $a^*\mathcal{M}$ is ample relative to b. In fact, since the $(\operatorname{Spec} A)^{\lambda,0}$ are affine, it is enough to show that $a^*\mathcal{M}|_{Y^{\lambda,0}}$ is ample on the scheme $Y^{\lambda,0}$. Now, $a^*\mathcal{M}|_{Y^{\lambda,0}} = (\mathcal{L}|_Y)\,|_{Y^{\lambda,0}}$, the line bundle $\mathcal{L}|_Y$ is ample and $Y^{\lambda,0} \to Y$ is affine, so $a^*\mathcal{M}|_{Y^{\lambda,0}}$ is ample too. This shows that $\mathcal{M}$ is relatively g-ample. In particular, if we let $\mathcal{M}_c = \mathcal{M}|_{\overline{\mathcal{Z}}_c}$, then $\mathcal{M}_c$ is relatively ample with respect to f_c.

Since q is algebraic, the linear form $\ell_{f^*q}|_{\overline{\mathcal{Z}}_c}$ is induced by a rational line bundle of the form $f_c^*\mathcal{N}$, where $\mathcal{N}$ is a rational line bundle on $\mathcal{X}_c$. Thus ℓ_c is induced by the rational line bundle $\mathcal{M}_c - cf_c^*\mathcal{N}$, which is relatively f_c-ample because $\mathcal{M}_c$ is. Therefore the claim follows from Step 6 applied to f_c in place of f. □

In the proof we used the following lemmas.

Lemma 2.6.6. *Let $\mathcal{X}$ be a noetherian algebraic stack with affine diagonal and a good moduli space $\pi\colon \mathcal{X} \to X$. For every quasi-compact open and closed substack $\mathcal{Z}$ of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$, the composition*

$$\mathcal{Z} \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \xrightarrow{u} \mathcal{X}$$

is affine. In particular, $\mathcal{Z}$ has a good moduli space Z which is affine over X.

Proof. Let $\operatorname{Spec} A/\mathrm{GL}_n \to \mathcal{X}$ be affine, surjective and strongly étale (Theorem 2.1.2). We have a cartesian diagram

$$\begin{tikzcd}
\mathrm{Grad}_{\mathbb{Q}}(\operatorname{Spec} A/\mathrm{GL}_n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \arrow[d] \\
\operatorname{Spec} A/\mathrm{GL}_n \arrow[r] & \mathcal{X}
\end{tikzcd}$$

and $\mathrm{Grad}_{\mathbb{Q}}(\operatorname{Spec} A/\mathrm{GL}_n)$ is a disjoint union of schemes of the form $\operatorname{Spec}(A)^{\lambda,0}/L(\lambda)$ with λ a rational cocharacter of GL_n. We conclude by Lemma 2.6.7. □

Lemma 2.6.7. *Let A be a commutative ring, and consider an action of GL_N on $X = \operatorname{Spec} A$ (over $\mathbb{Z}$) such that X/GL_N has a good moduli space. Let $\lambda\colon \mathbb{G}_m \to GL_N$ be a cocharacter. Then the natural map $X^{\lambda,0}/L(\lambda) \to X/GL_N$ is affine, where $X^{\lambda,0}$ is the fixed point locus of the $\mathbb{G}_m$-action on X induced by λ and $L(\lambda)$ is the centraliser of λ.*

Proof. There is a cartesian square

$$\begin{tikzcd}
\mathrm{GL}_N \times^{L(\lambda)} X^{\lambda,0} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d] \\
X^{\lambda,0}/L(\lambda) \arrow[r] & X/\mathrm{GL}_N
\end{tikzcd}$$

where $\mathrm{GL}_N \times^{L(\lambda)} X^{\lambda,0}$ is the stack quotient of $\mathrm{GL}_N \times X^{\lambda,0}$ by the diagonal action of $L(\lambda)$. Since the action is free, $\mathrm{GL}_N \times^{L(\lambda)} X^{\lambda,0}$ is an algebraic space. Now, $L(\lambda)$ is isomorphic to a product of GL_{N_i}'s and it is thus geometrically reductive [2, Definition 9.1.1]. Since $\mathrm{GL}_N \times X^{\lambda,0} = \operatorname{Spec} B$ is affine, the $L(\lambda)$-invariants give an adequate moduli space $\mathrm{GL}_N \times^{L(\lambda)} X^{\lambda,0} \to \operatorname{Spec}\left(B^{L(\lambda)}\right)$ [2, Theorem 9.1.4]. By universality for adequate moduli spaces [6, Theorem 3.12], we get an isomorphism $\mathrm{GL}_N \times^{L(\lambda)} X^{\lambda,0} = \operatorname{Spec} B^{L(\lambda)}$. Therefore $\mathrm{GL}_N \times^{L(\lambda)} X^{\lambda,0}$ is affine and we are done by descent. □

Remark 2.6.8. Since $L(\lambda)$ is geometrically reductive and $X^{\lambda,0}$ is affine, taking $L(\lambda)$-invariants gives an adequate moduli space for $X^{\lambda,0}/L(\lambda)$. However, unless A is of characteristic 0, an extra argument is needed to show that the adequate moduli space is indeed a good moduli space.

3 Sequential stratifications and the iterated balanced filtration

The goal of this section is the construction of the *balanced sequential stratification* for every noetherian normed good moduli stack $\mathcal{X}$ with affine diagonal (Theorem 3.5.2 and Definition 3.5.3) and derive from it the definition of the *iterated balanced filtration* of every point of $\mathcal{X}$ (Definition 3.5.8).

We start by defining a first approximation of the iterated balanced filtration, simply called the *balanced filtration* (Definition 3.1.6). We then construct a stack of sequential filtrations $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ for a stack $\mathcal{X}$, and define a notion of *sequential stratification* (Definition 3.3.1), roughly a weak analogue of Θ-stratification with the stack $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ used instead of $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X})$. For the purpose of using induction in the construction of the balancing stratification, we introduce the concept of *central rank* of a stack (Definition 3.4.1). After our main construction (Theorem 3.5.2), we prove some functorial properties of the balancing stratification (Proposition 3.5.5). We finish the section with a collection of natural examples of normed good moduli stacks in moduli theory, including GIT quotients, moduli of Bridgeland semistable objects and moduli of K-semistable Fano varieties.

3.1 The balanced filtration

Let $\mathcal{X}$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3. Let $\mathcal{Z} \to \mathcal{X}$ be a closed substack, let k be a field, x be a k-point of $\mathcal{X}$ and let $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ be a rational filtration of x. We would like to formalise the idea of velocity at which $\lambda(t)$ tends to $\mathcal{Z}$ when t tends to 0. For that, we first write $\lambda = \gamma/m$ with $\gamma \in \mathbb{Z}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ an integral filtration and $m \in \mathbb{Z}_{>0}$. We then form the pullback

$$\begin{tikzcd} R/\mathbb{G}_{m,k} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{Z} \arrow[d] \\ \mathbb{A}^1_k/\mathbb{G}_{m,k} \arrow[r, "\gamma"] & \mathcal{X} \end{tikzcd}$$

of $\mathcal{Z} \to \mathcal{X}$ along γ, which is given by a $\mathbb{G}_{m,k}$-equivariant closed subscheme R of $\mathbb{A}^1_k$, thus necessarily of the form $R = \mathrm{Spec}(k[x]/(x^n))$ for some $n \in \mathbb{N} \cup \{\infty\}$. The following concept is used by Kempf in [36].

Definition 3.1.1. We define *Kempf's intersection number* to be $\langle \lambda, \mathcal{Z} \rangle := n/m \in \mathbb{Q}_{\geq 0} \cup \{\infty\}$.

The definition does not depend on the presentation $\lambda = \gamma/m$ because if $l \in \mathbb{Z}_{>0}$, then $\langle l\gamma, \mathcal{Z} \rangle = ln$. More generally, the linearity property $\langle c\lambda, \mathcal{Z} \rangle = c\langle \lambda, \mathcal{Z} \rangle$ holds for all $c \in \mathbb{Q}_{>0}$. We have $\langle \lambda, \mathcal{Z} \rangle = \infty$ precisely when x is in $\mathcal{Z}$, and $\langle \lambda, \mathcal{Z} \rangle = 0$ if and only if $\mathrm{ev}_0(\lambda)$ is not in $\mathcal{Z}$. If x is not in $\mathcal{Z}$ but $\mathrm{ev}_0(\lambda)$ is in $\mathcal{Z}$, then $\langle \lambda, \mathcal{Z} \rangle$ is a positive rational number that can be thought of as the velocity at which λ approaches $\mathcal{Z}$.

Proposition 3.1.2. *Suppose x is not in $\mathcal{Z}$. Let $p\colon \mathrm{Bl}_{\mathcal{Z}}\,\mathcal{X} \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ along $\mathcal{Z}$, let $\mathcal{L}$ be the canonical p-ample line bundle on $\mathrm{Bl}_{\mathcal{Z}}\,\mathcal{X}$, and let $x' \in \mathrm{Bl}_{\mathcal{Z}}\,\mathcal{X}(k)$ be a lift of*

x to $\mathrm{Bl}_{\mathcal{Z}}\mathcal{X}$. Let $\lambda' \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathrm{Bl}_{\mathcal{Z}}\mathcal{X}, x')$ be the unique lift of λ to a rational filtration of x', which exists by Proposition 2.2.18. Then $\langle \lambda', \mathcal{L}\rangle = \langle \lambda, \mathcal{Z}\rangle$.

Proof. By linearity of $\langle -, \mathcal{Z}\rangle$, we may assume that λ is integral. By definition, $\mathcal{L}$ is the ideal sheaf of the exceptional divisor $\mathcal{E} = p^{-1}(\mathcal{Z})$. We have a diagram

$$\begin{tikzcd}
R/\mathbb{G}_{m,k} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{Z} \arrow[d] \\
\Theta_k \arrow[r, "\lambda'"] & \mathrm{Bl}_{\mathcal{Z}}\mathcal{X} \arrow[r, "p"] & \mathcal{X},
\end{tikzcd}$$

where $R = \operatorname{Spec} k[x]/(x^n)$ for some $n \in \mathbb{N}$, the variable x having weight -1. There is a natural injection $\mathcal{L} \to \mathcal{O}_{\mathrm{Bl}_{\mathcal{Z}}\mathcal{X}}$. Pulling back along λ' we get a map $(\lambda')^*\mathcal{L} \to \mathcal{O}_{\Theta_k}$ whose image $\mathcal{I} = (x^n)$ is the ideal sheaf of $R/\mathbb{G}_{m,k}$. Since there is a surjective map $(\lambda')^*\mathcal{L} \to \mathcal{I}$ and both surce and target are line bundles, the map should be an isomorphism. Thus $(\lambda')^*\mathcal{L} \cong \mathcal{I} \cong \mathcal{O}_{\Theta_k}(-n)$, where $\mathcal{O}_{\Theta_k}(-n)$ is the pullback of $\mathcal{O}_{B\mathbb{G}_{m,k}}(-n)$ along the structure map $\Theta_k \to B\mathbb{G}_{m,k}$, because x^n has weight $-n$. Therefore $\langle \lambda', \mathcal{L}\rangle = -\operatorname{wt}\left(((\lambda')^*\mathcal{L})\,|_{B\mathbb{G}_{m,k}}\right) = n = \langle \lambda, \mathcal{Z}\rangle$, as desired. □

The following result is a generalisation of Kempf's Theorem [36, Theorem 3.4] to stacks with good moduli space. See also [26, Example 5.3.7].

Theorem 3.1.3 (Kempf)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack with affine diagonal and a good moduli space $\pi\colon \mathcal{X} \to X$. Let k be a field, let $\mathcal{Z} \to \mathcal{X}$ be a closed substack and let x be a k-point of $\mathcal{X} \setminus \mathcal{Z}$. Then:*

1. *The intersection $|\mathcal{Z}| \cap |\pi^{-1}\pi(x)|$ is nonempty if and only if there is a filtration $\lambda \in \mathbb{Z}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ with $\mathrm{ev}_0(\lambda)$ in $\mathcal{Z}$.*
2. *Suppose that $\mathcal{X}$ is endowed with a norm on graded points. If $|\mathcal{Z}| \cap |\pi^{-1}\pi(x)| \neq \varnothing$, then there is a unique $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ with $\langle \lambda, \mathcal{Z}\rangle \geq 1$ and such that for all $\gamma \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ with $\langle \gamma, \mathcal{Z}\rangle \geq 1$ we have $\|\lambda\| \leq \|\gamma\|$.*

Proof. By replacing $\mathcal{X}$ by $\pi^{-1}\pi(x)$ we may assume that $X = \operatorname{Spec} k$. By Corollary 2.1.3, $\mathcal{X} = \operatorname{Spec} A/\operatorname{GL}_{n,k}$, where A is a k-algebra of finite type.

We first show Item 1 assuming k is algebraically closed. Suppose $\mathcal{Z} \neq \varnothing$. Since $\mathcal{X}$ has a unique closed point [1, Proposition 9.1], we also have that $\overline{\{x\}} \cap |\mathcal{Z}|$ is nonempty. By [36, Theorem 1.4], there exists $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ such that $\mathrm{ev}_1(\lambda) = x$ and $\mathrm{ev}_0(\lambda)$ is in $\mathcal{Z}$.

Now we show Item 2 for any k. We are given a norm on graded points on $\mathcal{X}$. Let x' be a lift of x to the blow-up $\mathrm{Bl}_{\mathcal{Z}}\mathcal{X}$. By Theorem 2.6.3 there is a Θ-stratification on $\mathrm{Bl}_{\mathcal{Z}}\mathcal{X}$. By Proposition 2.2.18 we have $\mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x|_{\overline{k}}) = \mathbb{Q}\text{-}\mathrm{Filt}(\mathrm{Bl}_{\mathcal{Z}}\mathcal{X}, x'|_{\overline{k}})$, and by Item 1 in the algebraically closed case and Proposition 3.1.2 we have that x' is semistable if and only if $|\mathcal{Z}| = \varnothing$. If x' is unstable, then its HN filtration is the unique $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ in Item 2, also by Proposition 3.1.2.

By choosing any norm on cocharacters of $\mathrm{GL}_{n,k}$ and pulling it back to $\mathcal{X}$, we see that Item 2 readily implies Item 1 for any k. □

Whe now shift attention to the case where $\mathcal{Z}$ is the locus $\mathcal{X}^{\max}$ of maximal dimension of stabiliser groups of $\mathcal{X}$. For each $d \in \mathbb{N}$, the set $\{x \in |\mathcal{X}| \mid \dim \mathrm{Aut}(x) \geq d\}$ is closed [17, Exposé VIb, Proposition 4.1]. Therefore, by quasi-compactness of $\mathcal{X}$, it makes sense to define:

Definition 3.1.4. Let $\mathcal{X}$ be a noetherian algebraic stack. The *maximal stabiliser dimension* $d(\mathcal{X}) \in \mathbb{N}$ of $\mathcal{X}$ is the maximal dimension of the stabiliser group of a point of $\mathcal{X}$. For the empty stack we set $d(\varnothing) = -\infty$.

Remark 3.1.5. Note that for each $d \in \mathbb{N}$, the set $\{x \in |\mathcal{X}| \mid \dim G_x \geq d\}$ is closed [17, Exposé VIb, Proposition 4.1]. By quasi-compactness of $\mathcal{X}$, the maximal dimension of a stabiliser group of $\mathcal{X}$ is thus well-defined.

As a topological space, $|\mathcal{X}^{\max}| = \{x \in |\mathcal{X}| \mid \dim \operatorname{Aut}(x) = d(\mathcal{X})\}$, and it is a closed subset of $|\mathcal{X}|$. It is a nontrivial result [19, Proposition C.5] that if $\mathcal{X}$ is a noetherian good moduli stack with affine diagonal, then $|\mathcal{X}^{\max}|$ can be given a natural structure of closed substack of $\mathcal{X}$, denoted $\mathcal{X}^{\max}$ and called the *maximal dimension stabiliser locus* of $\mathcal{X}$. It can be characterised étale locally by the property that, if $\mathcal{X} = X/G$, where X is an algebraic space and $G \to \operatorname{Spec} \mathbb{Z}$ is an affine flat group scheme of finite type that is either diagonalisable or a Chevalley group, with fibres of dimension $d(\mathcal{X})$, then $\mathcal{X}^{\max} = X^{G_\circ}/G$, where $G_\circ$ is the reduced identity component of G [19, Section C.2].

One reason why Edidin-Rydh's stack structure on $\mathcal{X}^{\max}$ is better behaved than the reduced structure is that it behaves well with respect to base change. If $\mathcal{Y} \to \mathcal{X}$ is a closed immersion and $d(\mathcal{Y}) = d(\mathcal{X})$, then $\mathcal{Y}^{\max} = \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}^{\max}$. Similarly, if $\mathcal{Y}$ has a good moduli space, $\mathcal{Y} \to \mathcal{X}$ is representable, étale and separated, and $d(\mathcal{Y}) = d(\mathcal{X})$, then $\mathcal{Y}^{\max} = \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}^{\max}$. See [19, Proposition C.5]. We get to the main definition of this section.

Definition 3.1.6 (The balanced filtration)**.** Let $\mathcal{X}$ be a normed noetherian algebraic stack with affine diagonal and a good moduli space $\pi\colon \mathcal{X} \to X$. Let $x\colon \operatorname{Spec} k \to \mathcal{X}$ be a field-valued point and denote $\mathcal{F} = \pi^{-1}\pi(x)$. We define the *balanced filtration* $\lambda_{\mathrm{b}}(x)$ of x to be the unique element λ of $\mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{X}, x)$ satisfying $\langle \lambda, \mathcal{F}^{\max}\rangle \geq 1$ and such that for all filtrations $\gamma \in \mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{X}, x)$ with $\langle \gamma, \mathcal{F}^{\max}\rangle \geq 1$ we have $\|\lambda\| \leq \|\gamma\|$.

Note that we have an identification $\mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{X}, x) = \mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{F}, x)$ and that existence and uniqueness of the balanced filtration is guaranteed by Theorem 3.1.3. The balanced filtration of x is 0 precisely when x is closed in $\mathcal{F}$.

In the case where $d(\mathcal{F}) = d(\mathcal{X})$, from the fact that $\mathcal{X}^{\max} \cap \mathcal{F} = \mathcal{F}^{\max}$ it follows that $\langle \lambda, \mathcal{F}^{\max}\rangle = \langle \lambda, \mathcal{X}^{\max}\rangle$. We have a Θ-stratification $(\mathcal{S}_c)_{c \in \mathbb{Q}_{\geq 0}}$ of $\operatorname{Bl}_{\mathcal{X}^{\max}} \mathcal{X}$, and thus a stratification of $\mathcal{X}$ where the strata are $\mathcal{X}^{\max}$ and the $\mathcal{S}_c \setminus \mathcal{E}$ with $c > 0$, where $\mathcal{E}$ is the exceptional divisor. This can be seen as a stratification of $\mathcal{X}$ by type of balanced filtration. Our goal is to refine this stratification by iterating the blowing-up process from the centres $\mathcal{Z}_c$ of the strata $\mathcal{S}_c$ (Theorem 3.5.2). The iterated strata will naturally live inside a stack of sequential filtrations, defined in the following section.

3.2 Stacks of sequential filtrations

The main goal of this section is to define a stack $\operatorname{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ of $\mathbb{Q}^\infty$*-filtrations* (or *sequential filtrations*) for a suitable algebraic stack $\mathcal{X}$. Here, the symbol $\mathbb{Q}^\infty$ denotes the direct sum $\mathbb{Q}^{\oplus \mathbb{N}}$ of countably many copies of $\mathbb{Q}$ with lexicographic order. We start with a simpler version of the stack $\operatorname{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$.

Definition 3.2.1. Let $\mathcal{X}$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3. We define the stack $\operatorname{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X})$ of $\mathbb{Q}^n$*-filtrations of* $\mathcal{X}$ *with lexicographic* *order* to be the limit of the diagram

$$\begin{tikzcd}[column sep=small]
\operatorname{Filt}_{\mathbb{Q}}(\mathcal{X}) \arrow[dr, "\mathrm{gr}"] & & \operatorname{Filt}_{\mathbb{Q}}\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}) \arrow[dl, "\mathrm{ev}_1"'] \arrow[dr, "\mathrm{gr}"] & & \cdots \arrow[dl] \arrow[dr] & & \operatorname{Filt}_{\mathbb{Q}}\operatorname{Grad}_{\mathbb{Q}}^{n-1}(\mathcal{X}) \arrow[dl, "\mathrm{ev}_1"] \\
& \operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}) & & \operatorname{Grad}_{\mathbb{Q}}^{2}(\mathcal{X}) & & \operatorname{Grad}_{\mathbb{Q}}^{n-1}(\mathcal{X}). &
\end{tikzcd}$$

Thus $\operatorname{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X})$ is just the fibre product

$$\operatorname{Filt}_{\mathbb{Q}}(\mathcal{X}) \underset{\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X})}{\times} \operatorname{Filt}_{\mathbb{Q}}\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}) \underset{\operatorname{Grad}_{\mathbb{Q}}^2(\mathcal{X})}{\times} \cdots \underset{\operatorname{Grad}_{\mathbb{Q}}^{n-1}(\mathcal{X})}{\times} \operatorname{Filt}_{\mathbb{Q}}\operatorname{Grad}_{\mathbb{Q}}^{n-1}(\mathcal{X}).$$

We define, for each $n \in \mathbb{Z}_{>0}$, a map $\operatorname{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) \to \operatorname{Filt}_{\mathbb{Q}^{n+1}_{\mathrm{lex}}}(\mathcal{X})$ by

$$\begin{tikzcd}
\operatorname{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) \arrow[d, equal] & \operatorname{Filt}_{\mathbb{Q}^{n+1}_{\mathrm{lex}}}(\mathcal{X}) \arrow[d, equal] \\
\operatorname{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) \times_{\operatorname{Grad}^n_{\mathbb{Q}}(\mathcal{X})} \operatorname{Grad}^n_{\mathbb{Q}}(\mathcal{X}) \arrow[r, "1\times o"] & \operatorname{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) \times_{\operatorname{Grad}^n_{\mathbb{Q}}(\mathcal{X})} \operatorname{Filt}_{\mathbb{Q}}\operatorname{Grad}^n_{\mathbb{Q}}(\mathcal{X}),
\end{tikzcd}$$

where $o\colon \operatorname{Grad}^n_{\mathbb{Q}}(\mathcal{X}) \to \operatorname{Filt}_{\mathbb{Q}}\operatorname{Grad}^n_{\mathbb{Q}}(\mathcal{X})$ is the "trivial filtration" map. If $\mathcal{X}$ satisfies Assumption 2.2.3, the so does $\operatorname{Grad}^n_{\mathbb{Q}}(\mathcal{X})$, so by [26, Proposition 1.3.9], the morphism o is an open and closed immersion. Thus each of the maps $\operatorname{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) \to \operatorname{Filt}_{\mathbb{Q}^{n+1}_{\mathrm{lex}}}(\mathcal{X})$ is an open immersion. We also have morphisms $\operatorname{Grad}^n_{\mathbb{Q}}(\mathcal{X}) \to \operatorname{Grad}_{\mathbb{Q}}\operatorname{Grad}^n_{\mathbb{Q}}(\mathcal{X}) = \operatorname{Grad}^{n+1}_{\mathbb{Q}}(\mathcal{X})$ given by the "trivial grading" maps, that are also open and closed immersions.

Definition 3.2.2. Let $\mathcal{X}$ be an algebraic stack. We define the stack $\operatorname{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ of $\mathbb{Q}^\infty$-*filtrations* (or *sequential filtrations*) of $\mathcal{X}$ as the colimit

$$\operatorname{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) = \varinjlim_{n\in\mathbb{Z}_{>0}} \operatorname{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}).$$

Similarly, we define the stack $\operatorname{Grad}_{\mathbb{Q}^\infty}(\mathcal{X})$ of $\mathbb{Q}^\infty$-graded points of $\mathcal{X}$ as

$$\operatorname{Grad}_{\mathbb{Q}^\infty}(\mathcal{X}) = \varinjlim_{n\in\mathbb{Z}_{>0}} \operatorname{Grad}^n_{\mathbb{Q}}(\mathcal{X}).$$

As a direct application of Lemma 2.2.9, we get:

Proposition 3.2.3. *Let $\mathcal{X}$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3. Then the stacks* $\operatorname{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ *and* $\operatorname{Grad}_{\mathbb{Q}^\infty}(\mathcal{X})$ *are algebraic, defined over B, and also satisfy Assumption 2.2.3.*

Remark 3.2.4. Here we are regarding $\mathbb{Q}^\infty$ as having lexicographic order. It would be more precise to use the notation $\operatorname{Filt}_{\mathbb{Q}^\infty_{\mathrm{lex}}}(\mathcal{X})$ for what we called $\operatorname{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$, to distinguish it from a stack of $\mathbb{Q}^\infty$-filtrations with product order, which can also be defined. Since we will not use such a stack, our notation will not be problematic.

For each n, we have an associated graded map $\operatorname{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) \to \operatorname{Grad}^n_{\mathbb{Q}}(\mathcal{X})$ and these maps glue to a morphism $\mathrm{gr}\colon \operatorname{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \operatorname{Grad}_{\mathbb{Q}^\infty}(\mathcal{X})$. Similarly, we get an "evaluation at 1" map $\mathrm{ev}_1\colon \operatorname{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathcal{X}$, a "forgetful" map $u\colon \operatorname{Grad}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathcal{X}$, and a "split filtration" map $\sigma\colon \operatorname{Grad}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \operatorname{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ as in [26, Figure 2].

Proposition 3.2.5. *Let $\mathcal{X}$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3. Then the "evaluation at 1" map $\mathrm{ev}_1 \colon \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathcal{X}$ is representable and separated.*

Proof. It is enough to prove that $\mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) \to \mathcal{X}$ is representable and separated for each n. There is a cartesian square

$$\begin{tikzcd} \mathrm{Filt}_{\mathbb{Q}^{n+1}_{\mathrm{lex}}}(\mathcal{X}) \arrow[r, "a_n"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) \arrow[d] \\ \mathrm{Filt}_{\mathbb{Q}} \mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}) \arrow[r, "b_n"] & \mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}) \end{tikzcd}$$

for each n, where b_n is the "evaluation at 1" map. Thus by [26, Proposition 1.1.13], the map a_n is representable and separated, being a base change of b_n. Expressing $\mathrm{ev}_1 \colon \mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) \to \mathcal{X}$ as a composition of the a_n, we get the result. $\square$

Remark 3.2.6. One can define stacks $\mathrm{Filt}_{\mathbb{Z}^\infty}(\mathcal{X})$ and $\mathrm{Grad}_{\mathbb{Z}^\infty}(\mathcal{X})$ in a similar vein. The monoid $(\mathbb{N}, \cdot, 1)$ acts on these stacks, and $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ and $\mathrm{Grad}_{\mathbb{Q}^\infty}(\mathcal{X})$ are obtained from these by localising the action as in Definition 2.2.6.

Remark 3.2.7. The formation of $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ is functorial in $\mathcal{X}$. If $f \colon \mathcal{X} \to \mathcal{Y}$ is a morphism, then there is an obvious induced map $\mathrm{Filt}_{\mathbb{Q}^\infty}(f) \colon \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{Y})$.

Remark 3.2.8 (Stack of polynomial filtrations). Our definition of the stack $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ as a colimit of the stack $\mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X})$ is justified by the fact that the poset $\mathbb{Q}^\infty$ can be written as the colimit $\mathbb{Q}^\infty = \varinjlim_n \mathbb{Q}^n_{\mathrm{lex}}$ where the maps are $\mathbb{Q}^n_{\mathrm{lex}} \to \mathbb{Q}^{n+1}_{\mathrm{lex}} \colon (a_0, \dots, a_{n-1}) \mapsto (a_0, \dots, a_{n-1}, 0)$. Alternatively, we can consider the diagram where the maps are $\mathbb{Q}^n_{\mathrm{lex}} \to \mathbb{Q}^{n+1}_{\mathrm{lex}} \colon (a_0, \dots, a_{n-1}) \mapsto (0, a_0, \dots, a_{n-1})$, and the colimit is now $\mathbb{Q}[t]$, the set of polynomials in one variable with rational coefficients, where $p \le q$ if $p(n) \le q(n)$ for $n >> 0$. We define the stack $\mathrm{Filt}_{\mathbb{Q}[t]}(\mathcal{X})$ of *polynomial filtrations* of $\mathcal{X}$ to be the colimit of the corresponding diagram of open and closed immersions $\mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) \to \mathrm{Filt}_{\mathbb{Q}^{n+1}_{\mathrm{lex}}}(\mathcal{X})$. Everything we have said about $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ applies also to $\mathrm{Filt}_{\mathbb{Q}[t]}(\mathcal{X})$ with similar arguments.

The stack of sequential filtrations behaves well with respect to pullback with from an algebraic space.

Proposition 3.2.9. *Let $\mathcal{X} \to B$ and $\mathcal{X}' \to B'$ satisfy Assumption 2.2.3, and let*

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X} \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

be a cartesian square, with X, X' algebraic spaces. Then

$$\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}') \cong \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \times_{\mathrm{ev}_1, \mathcal{X}} \mathcal{X}' \cong \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \times_X X'.$$

Proof. Since $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}')$ is an incrising union of the stacks $\mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}')$, it is enough to show the analogue claim for these stacks. For $n = 1$, this is Proposition 2.2.12. For $n > 1$, we have

$$\begin{aligned} \mathcal{X}' \times_{\mathcal{X}, \mathrm{ev}_1} \mathrm{Filt}_{\mathbb{Q}^n_{\mathrm{lex}}}(\mathcal{X}) &= \mathcal{X}' \times_{\mathcal{X}, \mathrm{ev}_1} \mathrm{Filt}_{\mathbb{Q}^{n-1}_{\mathrm{lex}}}(\mathcal{X}) \times_{\mathrm{gr}, \mathrm{Grad}^{n-1}_{\mathbb{Q}}(\mathcal{X}), \mathrm{ev}_1} \mathrm{Filt}_{\mathbb{Q}}(\mathrm{Grad}^{n-1}_{\mathbb{Q}}(\mathcal{X})) \\ &= \mathrm{Filt}_{\mathbb{Q}^{n-1}_{\mathrm{lex}}}(\mathcal{X}') \times_{\mathrm{gr}, \mathrm{Grad}^{n-1}_{\mathbb{Q}}(\mathcal{X}')} \mathrm{Grad}^{n-1}_{\mathbb{Q}}(\mathcal{X}') \times_{\mathrm{Grad}^{n-1}_{\mathbb{Q}}(\mathcal{X}), \mathrm{ev}_1} \mathrm{Filt}_{\mathbb{Q}}\left(\mathrm{Grad}^{n-1}_{\mathbb{Q}}(\mathcal{X})\right) \\ &= \mathrm{Filt}_{\mathbb{Q}^{n-1}_{\mathrm{lex}}}(\mathcal{X}') \times_{\mathrm{gr}, \mathrm{Grad}^{n-1}_{\mathbb{Q}}(\mathcal{X}'), \mathrm{ev}_1} \mathrm{Filt}_{\mathbb{Q}}\left(\mathrm{Grad}^{n-1}_{\mathbb{Q}}(\mathcal{X}')\right) = \mathrm{Filt}_{\mathbb{Q}^{n-1}_{\mathrm{lex}}}(\mathcal{X}') \end{aligned}$$

again by Proposition 2.2.12. $\square$

Proposition 3.2.10. *Let $\mathcal{X}$ be an algebraic stack defined over an algebraic space B, satisfying Assumption 2.2.3, and let $\mathcal{X}' \to \mathcal{X}$ be a closed immersion. Then*

$$\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}') \cong \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \times_{\mathrm{ev}_1,\mathcal{X}} \mathcal{X}'.$$

Proof. The statement follows in the same way as Proposition 3.2.9, but using Proposition 2.2.13 instead of Proposition 2.2.12. □

Proposition 3.2.11. *Let $\mathcal{X}$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3. Then there is a cartesian diagram*

$$\begin{tikzcd}
\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Filt}_{\mathbb{Q}^\infty} \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \arrow[d, "\mathrm{ev}_1"] \\
\mathrm{Filt}_{\mathbb{Q}} \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \arrow[r, "\mathrm{ev}_1"] & \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}).
\end{tikzcd}$$

Proof. The claim follows from cartesianity of the diagram

$$\begin{tikzcd}
\mathrm{Filt}_{\mathbb{Q}^{n+1}_{\mathrm{lex}}}(\mathcal{X}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Filt}_{\mathbb{Q}^{n}_{\mathrm{lex}}} \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \arrow[d, "\mathrm{ev}_1"] \\
\mathrm{Filt}_{\mathbb{Q}} \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \arrow[r, "\mathrm{ev}_1"] & \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})
\end{tikzcd}$$

by taking the colimit when n tends to ∞. □

We define $\mathbb{Q}^\infty$-flag spaces in analogy with Definition 2.2.15.

Definition 3.2.12 ($\mathbb{Q}^\infty$-Flag spaces)**.** Let $\mathcal{X}$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3, and let $x\colon T \to \mathcal{X}$ be a scheme-valued point. We define the *space of $\mathbb{Q}^\infty$-flags* $\mathrm{Flag}_{\mathbb{Q}^\infty}(x)$ *of x* as the fibre product

$$\begin{tikzcd}
\mathrm{Flag}_{\mathbb{Q}^\infty}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & T \arrow[d, "x"] \\
\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \arrow[r, "\mathrm{ev}_1"] & \mathcal{X},
\end{tikzcd}$$

which is, by Proposition 3.2.5, an algebraic space over T.

In the case of a field-valued point x, it is reasonable to talk about $\mathbb{Q}^\infty$-filtrations.

Definition 3.2.13 ($\mathbb{Q}^\infty$-filtrations of a point)**.** Let $\mathcal{X}$ be an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3. Let k be a field and let $x\colon \operatorname{Spec} k \to \mathcal{X}$ point. We define the set $\mathbb{Q}^\infty\text{-}\mathrm{Filt}(\mathcal{X}, x)$ of *$\mathbb{Q}^\infty$-filtrations* of x to be $\mathbb{Q}^\infty\text{-}\mathrm{Filt}(\mathcal{X}, x) := \mathrm{Flag}_{\mathbb{Q}^\infty}(x)(k)$, the set of k-points of the space of $\mathbb{Q}^\infty$-flags of x.

Remark 3.2.14. The set $\mathbb{Q}^\infty\text{-}\mathrm{Filt}(\mathcal{X}, x)$ can be described as follows. An element λ of $\mathbb{Q}^\infty\text{-}\mathrm{Filt}(\mathcal{X}, x)$ is uniquely determined by a sequence $(\lambda_n)_{n\in\mathbb{N}}$ where

1. $\lambda_0 \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$,
2. $\lambda_n \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}), \mathrm{gr}\,\lambda_{n-1})$ for $n \geq 1$, and
3. $\lambda_n = 0$ for $n >> 0$.

The trivial $\mathbb{Q}^\infty$-filtration corresponds to the sequence where $\lambda_n = 0$ for all n. In general,

$$N := \min\{n \mid \ \lambda_i = 0, \ \forall i \geq n\}$$

is the minimal natural number such that λ is in $\mathrm{Filt}_{\mathbb{Q}^N_{\mathrm{lex}}}(\mathcal{X}) \subset \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$. The description follows at once from the definition of $\mathrm{Filt}_{\mathbb{Q}^N_{\mathrm{lex}}}(\mathcal{X})$ as a fibre product.

Remark 3.2.15. If $f\colon \mathcal{X}' \to \mathcal{X}$ is either a closed immersion or a base change from a map of algebraic spaces, and if $x \in \mathcal{X}'(k)$ is a field-valued point, then, by Propositions 3.2.9 and 3.2.10, we have a canonical bijection $\mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x) \cong \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}', f(x))$. We will use this fact throughout.

Remark 3.2.16 (Sequential filtrations on quotient stacks). Let k be a field, and consider a quotient stack $\mathcal{X} = X/G$ where X is a separated scheme over k and G is a linear algebraic group over k. Let $x \in X(k)$ be a k-point and call also $x \in \mathcal{X}(k)$ its image in $\mathcal{X}$. From Remarks 2.2.17 and 3.2.14, it follows that we have an identification of the set $\mathbb{Q}^\infty\text{-}\mathrm{Filt}(\mathcal{X}, x)$ of sequential filtrations of x with the set of sequences $(\lambda_n)_{n\in\mathbb{N}}$ where

1. each $\lambda_n \in \Gamma^{\mathbb{Q}}(G)$;
2. for all $n, m \in \mathbb{N}$, λ_n and λ_m commute;
3. $\lambda_n = 0$ for $n >> 0$;
4. for every $n \in \mathbb{Z}_{>0}$, the iterated limit

$$\lim_{t_n \to 0} \lambda_n(t_n) \left(\lim_{t_{n-1} \to 0} \lambda_{n-1}(t_{n-1}) \left(\cdots \lim_{t_0 \to 0} \lambda_0(t_0) x \right) \cdots \right)$$

   exists in X;

subject to the equivalence relation that identifies $(\lambda_n)_{n\in\mathbb{N}} \sim (\lambda'_n)_{n\in\mathbb{N}}$ if there are $g_n \in P_{L(\lambda_0,\cdots,\lambda_n)}(\lambda_n)$ such that $(\lambda_n)^{g_0 g_1 \cdots g_n} = \lambda'_n$ for all $n \in \mathbb{N}$. To see this, note that we can explicitly describe components of $\mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X})$ as $X^{\lambda_0,\cdots,\lambda_n,0}/L(\lambda_0, \cdots, \lambda_n)$, where $\lambda_0, \cdots, \lambda_n$ are commuting rational cocharacters of G, $X^{\lambda_0,\cdots,\lambda_n,0}$ denotes the fixed-point locus by $\lambda_0, \cdots, \lambda_n$ and $L(\lambda_0, \cdots, \lambda_n)$ is the centraliser of $\lambda_0, \cdots, \lambda_n$ inside G.

3.3 Sequential stratifications

In this section, we give the definition of *sequential stratification* and we study some pullback and pushforward operations for sequential strata. All algebraic stacks are defined over an algebraic space B and are assumed to satisfy Assumption 2.2.3.

Definition 3.3.1. Let $\mathcal{X}$ be an algebraic stack let Γ be a partially ordered set. A *sequential stratification* (or $\mathbb{Q}^\infty$*-stratification*) of $\mathcal{X}$ indexed by Γ is a family $(\mathcal{S}_i)_{i\in\Gamma}$ of locally closed substacks of $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ such that:

1. Each composition $r_i\colon \mathcal{S}_i \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathcal{X}$ is a locally closed immersion.
2. The $r_i(|\mathcal{S}_i|)$ are pairwise disjoint and cover $|\mathcal{X}|$.
3. For each $i \in \Gamma$, the union $\bigcup_{j\leq i} r_j(|\mathcal{S}_j|)$ is open in $|\mathcal{X}|$.

Remark 3.3.2 (Sequential and polynomial Θ-stratifications). This definition is not quite the analogue of the notion of Θ-stratification for $\mathbb{Q}^\infty$-filtrations. That would require in addition that each $\mathcal{S}_i$ is open in $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ and is the preimage of an open substack of $\mathrm{Grad}_{\mathbb{Q}^\infty}(\mathcal{X})$, but these conditions do not hold for the balancing stratification. We may call this stronger notion *sequential* Θ*-stratifications*. Similarly, if we use the stack $\mathrm{Filt}_{\mathbb{Q}[t]}(\mathcal{X})$ of polynomial filtrations instead of $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$, we get a notion of *polynomial* Θ*-stratification*. The stratification of the stack of pure coherent sheaves on a polarised projective scheme over a noetherian base by polynomial Harder-Narasimhan filtration [45] should be a polynomial Θ-stratification. Another example should be given by the stratifications for moduli spaces of principal ρ-sheaves considered in [22].

Definition 3.3.3 (Pulling back sequential stratifications)**.** Let $f\colon \mathcal{X}' \to \mathcal{X}$ be a morphism of algebraic stacks such that

$$\begin{tikzcd} \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}') \arrow[r] \arrow[d, "\mathrm{ev}_1"'] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \arrow[d, "\mathrm{ev}_1"] \\ \mathcal{X}' \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

is cartesian (for example a closed immersion or a base change from a map of algebraic spaces, see Propositions 3.2.9 and 3.2.10). Let $(\mathcal{S}_i)_{i\in\Gamma}$ be a sequential stratification of $\mathcal{X}$. For each $i \in \Gamma$, set $f^*\mathcal{S}_i := \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}') \times_{\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})} \mathcal{S}_i$, which is a locally closed substack of $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}')$. Then $(f^*\mathcal{S}_i)_{i\in\Gamma}$ is a sequential stratification of $\mathcal{X}'$, called the *pulled back* sequential stratification.

To see that $(f^*\mathcal{S}_i)_{i\in\Gamma}$ is indeed a sequential stratification, just note that we have $f^*\mathcal{S}_i = \mathcal{X}' \times_{\mathcal{X}} \mathcal{S}_i$ for all $i \in \Gamma$.

Definition 3.3.4. A *sequential stratum* $\mathcal{S}$ of $\mathcal{X}$ is a locally closed substack of $\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ such that the composition

$$\mathcal{S} \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathcal{X}$$

is a locally closed immersion. We refer to the morphism $\mathcal{S} \to \mathcal{X}$ as the *structure map*.

Remark 3.3.5. If $a\colon \mathcal{S} \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X})$ is a morphism such that the composition

$$\mathcal{S} \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathcal{X}$$

is a locally closed immersion, then a is a locally closed immersion as well. Indeed, $\mathrm{ev}_1\colon \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathcal{X}$ is representable and separated (Proposition 3.2.5), so $\mathcal{S}\times_{\mathcal{X}}\mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathcal{S}$ is separated, and a defines a section s of this map.

$$\begin{tikzcd} \mathcal{S} \times_{\mathcal{X}} \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \arrow[d] \\ \mathcal{S} \arrow[u, bend left, dashed, "s"] \arrow[ur, "a"] \arrow[r, hook] & \mathcal{X} \end{tikzcd}$$

Thus s is a closed immersion and a is the composition of s and a locally closed immersion.

It will be useful to have a few ways of constructing sequential strata from given ones.

Definition 3.3.6 (Pushforward along a locally closed immersion)**.** If $\iota\colon \mathcal{X} \to \mathcal{Y}$ is a locally closed immersion and $\mathcal{S}$ is a sequential stratum of $\mathcal{X}$, then we define a sequential stratum $\iota_*\mathcal{S}$ as follows. As a stack, $\iota_*\mathcal{S} = \mathcal{S}$, and the structure map is the composition

$$\mathcal{S} \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{Y}),$$

which is a locally closed immersion because $\mathrm{Filt}_{\mathbb{Q}^\infty}(\iota)$ is. The map $\mathcal{S} \to \mathcal{Y}$ factors as $\mathcal{S} \to \mathcal{X} \to \mathcal{Y}$, so it is a locally closed immersion.

Definition 3.3.7 (Induction of a sequential stratum from centre of a Θ-stratum)**.** Suppose $\mathcal{S}$ is a locally closed Θ-stratum of an algebraic stack $\mathcal{X}$, with centre $\mathcal{Z}$, and let $\mathcal{S}'$ be a sequential stratum of $\mathcal{Z}$. We define a sequential stratum $\mathrm{ind}_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{S}')$ of $\mathcal{X}$, as follows. As a stack, $\mathrm{ind}_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{S}')$ is the pullback

$$\begin{tikzcd} \mathrm{ind}_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{S}') \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{S}' \arrow[d] \\ \mathcal{S} \arrow[r, "\mathrm{gr}"] & \mathcal{Z}. \end{tikzcd}$$

The structure morphism a is obtained by pulling back the square

$$\begin{tikzcd} \mathcal{S}' \arrow[r] \arrow[d] & \mathrm{Filt}_{\mathbb{Q}^\infty} \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \arrow[d] \\ \mathcal{Z} \arrow[r] & \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}) \end{tikzcd}$$

along $\mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \xrightarrow{\mathrm{gr}} \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X})$, obtaining a square of the form

$$\begin{tikzcd} \mathrm{ind}_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{S}') \arrow[r, "a"] \arrow[d] & \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \arrow[d] \\ \mathcal{S} \arrow[r] & \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}) \end{tikzcd}$$

by Proposition 3.2.11. The induced morphism $\mathrm{ind}_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{S}') \to \mathcal{X}$ is the composition of the locally closed immersions $\mathrm{ind}_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{S}') \to \mathcal{S}$ and $\mathcal{S} \to \mathcal{X}$ and it is thus also a locally closed immersion.

Definition 3.3.8 (Pushforward along a blow-up)**.** Let $p\colon \mathcal{Y} \to \mathcal{X}$ be a blow-up with exceptional divisor $\mathcal{E} \subset \mathcal{Y}$. Let $\mathcal{S}$ be a sequential stratum of $\mathcal{Y}$. We define a sequential stratum $p_*(\mathcal{S} \setminus \mathcal{E})$ of $\mathcal{X}$ as follows. If $r\colon \mathcal{S} \to \mathcal{Y}$ is the locally closed immersion, then we set $p_*(\mathcal{S} \setminus \mathcal{E}) = \mathcal{S} \setminus r^{-1}(\mathcal{E})$ as a stack. The structure map is the composition

$$\mathcal{S} \setminus r^{-1}(\mathcal{E}) \to \mathcal{S} \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{Y}) \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}).$$

We have a diagram

$$\begin{tikzcd} \mathcal{S} \setminus r^{-1}(\mathcal{E}) \arrow[r] \arrow[d, "a"'] & \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{Y}) \arrow[r] \arrow[d] & \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}) \arrow[d] \\ \mathcal{Y} \setminus \mathcal{E} \arrow[r, "b"'] & \mathcal{Y} \arrow[r, "p"'] & \mathcal{X}. \end{tikzcd}$$

The map a is a locally closed immersion, and $p \circ b$ is an open immersion. Thus $\mathcal{S} \setminus r^{-1}(\mathcal{E}) \to \mathcal{X}$ is a locally closed immersion. By Remark 3.3.5, the structure map is also a locally closed immersion.

3.4 Central rank and $B\mathbb{G}_m^n$-actions

Let $\mathcal{X}$ be a quasi-compact algebraic stack over an algebraic space B, satisfying Assumption 2.2.3.

Definition 3.4.1. We define the *central rank* $z(\mathcal{X}) \in \mathbb{N}$ of $\mathcal{X}$ to be the biggest $n \in \mathbb{N}$ such that there is a nondegenerate component $\mathcal{Z}$ of $\mathrm{Grad}^n(\mathcal{X})$ (Definition 2.3.1) mapping isomorphically onto $\mathcal{X}$. For the empty stack we define $z(\varnothing) = \infty$.

The stack $B\mathbb{G}_m^n$ is a group stack. It turns out that the data of a $B\mathbb{G}_m^n$ action on a stack $\mathcal{X}$ satisfying Assumption 2.2.3 is equivalent to the data of a section $s\colon \mathcal{X} \to \mathrm{Grad}^n(\mathcal{X})$ of $u\colon \mathrm{Grad}^n(\mathcal{X}) \to \mathcal{X}$ such that s is a closed and open immersion. This is [25, Corollary 1.4.2.1] in the case of the group stack $\mathbb{A}^1/\mathbb{G}_m$, but the same proof works for $B\mathbb{G}_m^n$. We say that the $B\mathbb{G}_m^n$-action is *nondegenerate* if all components of $s(\mathcal{X})$ are nondegenerate. If $m\colon B\mathbb{G}_m^n \times \mathcal{X} \to \mathcal{X}$ is the action map, then the action is nondegenerate if for all $x \in \mathcal{X}(k)$, the homomorphism $\mathbb{G}_{m,k}^n \to \mathrm{Aut}(x)$ induced by m has finite kernel.

Using the stack $\mathrm{Grad}_{\mathbb{Q}}^n(\mathcal{X})$ instead of $\mathrm{Grad}^n(\mathcal{X})$ we can talk about rational $B\mathbb{G}_m^n$-actions.

Definition 3.4.2. A *rational $B\mathbb{G}_m^n$-action* on $\mathcal{X}$ is a section $s\colon \mathcal{X} \to \mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X})$ of

$$u\colon \mathrm{Grad}^n_{\mathbb{Q}}(\mathcal{X}) \to \mathcal{X}$$

that is a closed and open immersion.

Remark 3.4.3. It is tacitly understood that an isomorphism $u \circ s \sim \mathrm{id}_{\mathcal{X}}$ is part of the data of a section s.

Lemma 3.4.4. *Suppose that $\mathcal{X}$ has a good moduli space $\pi\colon \mathcal{X} \to X$, and let $p\colon \mathcal{Y} \to \mathcal{X}$ be a blow-up. Then any (rational) $B\mathbb{G}_m^n$-action on $\mathcal{X}$ lifts canonically to $\mathcal{Y}$.*

Proof. We prove the lemma for integral actions, the proof for rational actions being identical after replacing Grad with $\mathrm{Grad}_{\mathbb{Q}}$. The $B\mathbb{G}_m^n$-action corresponds to a section $s\colon \mathcal{X} \to \mathrm{Grad}^n(\mathcal{X})$ of $\mathrm{Grad}^n(\mathcal{X}) \to \mathcal{X}$ that is an open and closed immersion. With this language, what we want to show is that the preimage $\mathcal{Y}'$ of $s(\mathcal{X})$ along $\mathrm{Grad}^n(\mathcal{Y}) \to \mathrm{Grad}^n(\mathcal{X})$ is a closed and open substack of $\mathrm{Grad}^n(\mathcal{Y})$ such that the composition $\mathcal{Y}' \to \mathrm{Grad}^n(\mathcal{Y}) \to \mathcal{Y}$ is an isomorphism.

If $\mathcal{X}$ is of the form $\mathcal{X} = \operatorname{Spec} A/\mathrm{GL}_l$, then

$$\mathrm{Grad}^n(\mathcal{X}) = \bigsqcup_{\lambda \in \mathrm{Hom}(\mathbb{G}_m^n, T)/W} (\operatorname{Spec} A)^{\lambda,0}/L(\lambda),$$

where T is the standard maximal torus of GL_l and W is the symmetric group of degree l [26, Theorem 1.4.7]. Thus $\mathcal{X}$ is isomorphic to one of the $(\operatorname{Spec} A)^{\lambda,0}/L(\lambda)$, with $\lambda\colon \mathbb{G}_m^n \to \mathrm{GL}_l$ having finite kernel. Any blow-up $p\colon \mathcal{Y} \to \mathcal{X}$ is then of the form $Y/L(\lambda)$ with $\lambda(\mathbb{G}_m^l)$ acting trivially on Y. This proves the lemma in the case $\mathcal{X} = \operatorname{Spec} A/\mathrm{GL}_l$.

For general $\mathcal{X}$, the claim can be checked étale locally on $\mathcal{X}$, since for $\mathcal{X}' \to \mathcal{X}$ representable and étale, the square

$$\begin{tikzcd}
\mathrm{Grad}^n(\mathcal{X}') \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Grad}^n(\mathcal{X}) \arrow[d] \\
\mathcal{X}' \arrow[r] & \mathcal{X}
\end{tikzcd}$$

is cartesian [26, Corollary 1.1.7], and since blow-ups commute with flat base change. We can cover $\mathcal{X}$ by representable étale neighbourhoods of the form $\operatorname{Spec} A/\mathrm{GL}_l$ (Theorem 2.1.2), which proves the lemma for general $\mathcal{X}$. $\square$

Remark 3.4.5. It should not be essential that $\mathcal{X}$ has a good moduli space for Lemma 3.4.4, but this is all we will need.

Lemma 3.4.6. *Let $\mathcal{X}$ be a normed noetherian good moduli stack with affine diagonal. Let $p\colon \mathcal{Y} \to \mathcal{X}$ be a blow-up of $\mathcal{X}$ at some closed substack, and let $\mathcal{E}$ be the exceptional divisor. The stack $\mathcal{Y}$ is endowed with the Θ-stratification induced by the norm on $\mathcal{X}$ and the p-ample line bundle $\mathcal{O}_{\mathcal{Y}}(-\mathcal{E})$ (Theorem 2.6.3). Let $\mathcal{Z}_c$ be the centre of some unstable stratum of $\mathcal{Y}$. Then $z(\mathcal{X}) < z(\mathcal{Z}_c)$.*

Proof. Let $n = z(\mathcal{X})$. There is a nondegenerate $B\mathbb{G}_m^n$ action on $\mathcal{X}$ that, since p is a blow-up, lifts canonically to $\mathcal{Y}$ by Lemma 3.4.4. The $B\mathbb{G}_m^n$ action gives a closed and open immersion $\mathcal{Y} \to \mathrm{Grad}^n(\mathcal{Y})$.

Since $\mathcal{Z}_c$ is the centre of a stratum, it comes equipped with a $B\mathbb{G}_m$-action inherited from a closed and open immersion $\mathcal{Z}_c \to \mathrm{Grad}(\mathcal{Y})$. Applying Grad to the closed and open immersion $\mathcal{Y} \to \mathrm{Grad}^n(\mathcal{Y})$ and composing with $\mathcal{Z}_c \to \mathrm{Grad}(\mathcal{Y})$, we get a closed and open immersion $\mathcal{Z}_c \to \mathrm{Grad}^{n+1}(\mathcal{Y})$, which gives a $B\mathbb{G}_m \times B\mathbb{G}_m^n$-action on $\mathcal{Z}_c$.

Let $x \in \mathcal{Z}_c(k)$ be a k-point for some field k. The $B\mathbb{G}_m \times B\mathbb{G}_m^n$-action provides cocharacters $\lambda, \beta_1, \dots, \beta_n$ of the centre $Z(\mathrm{Aut}(x))$. Since the $\beta_1, \dots, \beta_n$ come from $\mathcal{X}$, we have $\langle \beta_i, \mathcal{O}_{\mathcal{Y}}(-\mathcal{E})\rangle = 0$, where $\mathcal{E}$ is the exceptional divisor, while since $\mathcal{Z}_c$ is the centre of an *unstable* stratum, we have $\langle \lambda, \mathcal{O}_{\mathcal{Y}}(-\mathcal{E})\rangle > 0$. Therefore $\lambda(\mathbb{G}_m)$ is not contained in the image of $(\beta_1, \dots, \beta_n)\colon \mathbb{G}_m^n \to \mathrm{Aut}(x)$, and thus $(\lambda, \beta_1, \dots, \beta_n)\colon \mathbb{G}_m^{n+1} \to \mathrm{Aut}(x)$ has finite kernel. Therefore $z(\mathcal{Z}_c) \geq n+1$, as desired. □

3.5 The balanced sequential stratification and the iterated balanced filtration

We now get to the main construction of the paper, namely the *balancing stratification* for normed good moduli stacks (Theorem 3.5.2), and the canonical sequential filtration it defines for every point (Definition 3.5.8). We also show that the balancing stratification is preserved under certain pullbacks (Proposition 3.5.5). Let us first introduce the indexing poset labelling the stratification.

Definition 3.5.1 (Indexing poset for the balancing stratification)**.** We define a totally ordered set $\mathbf{\Gamma}$ as follows. As a set, $\mathbf{\Gamma}$ consists of the sequences

$$((d_0, c_0), (d_1, c_1), \dots, (d_n, c_n))$$

with

1. $n \in \mathbb{N}$,
2. $d_0 \geq d_1 \geq \cdots \geq d_n$ in $\mathbb{N}$,
3. $d_i \geq i$ for all $0 \leq i \leq n$,
4. $c_0, \dots, c_{n-1} \in \mathbb{Q}_{>0}$,
5. $c_n = \infty$.

As a poset, we write

$$((d_0, c_0), \dots, (d_n, c_n)) < ((d'_0, c'_0), \dots, (d'_m, c'_m))$$

if there is $0 \leq i \leq \min(n, m)$ such that $(d_j, c_j) = (d'_j, c'_j)$ for $j < i$ and either $d_i < d'_i$ or $d_i = d'_i$ and $c_i < c'_i$. This makes $\mathbf{\Gamma}$ a totally ordered set.

If $\alpha \in \mathbf{\Gamma}$, we use the notation

$$\alpha = ((d_0^\alpha, c_0^\alpha), \dots, (d_{n(\alpha)}^\alpha, c_{n(\alpha)}^\alpha)).$$

Theorem 3.5.2 (Existence and characterisation of the balancing stratification)**.** *There is a unique way of assigning, to every normed noetherian good moduli stack $\mathcal{X}$ with affine diagonal, a sequential stratification $(\mathcal{S}_\alpha^{\mathcal{X}})_{\alpha \in \mathbf{\Gamma}}$ indexed by $\mathbf{\Gamma}$ in such a way that the following properties are satisfied for every such $\mathcal{X}$:*

1. *The stratum indexed by $(d(\mathcal{X}), \infty)$ is $\mathcal{S}^{\mathcal{X}}_{(d(\mathcal{X}),\infty)} = \mathcal{X}^{\max}$, with structure map*

$$\mathcal{X}^{\max} \to \mathcal{X} \to \mathrm{Filt}_{\mathbb{Q}^\infty}(\mathcal{X}),$$

where the second arrow is the "trivial filtration" morphism.

2. *Let $\pi\colon \mathcal{X} \to X$ be the good moduli space and let $\mathcal{U} = \mathcal{X} \setminus \pi^{-1}\pi(\mathcal{X}^{\max})$. Denote $j\colon \mathcal{U} \to \mathcal{X}$ the open immersion. Then for all $\alpha \in \mathbf{\Gamma}$ with $d_0^\alpha < d(\mathcal{X})$ we have $\mathcal{S}_\alpha^{\mathcal{X}} = j_*(\mathcal{S}_\alpha^{\mathcal{U}})$.*

3. *Let $p\colon \mathcal{Y} = \mathrm{Bl}_{\mathcal{X}^{\max}} \mathcal{X} \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ along $\mathcal{X}^{\max}$ and let $\mathcal{E}$ be the exceptional divisor. Let $\mathcal{Z}_c$ be the centres of the Θ-stratification on $\mathcal{Y}$ induced by the norm on $\mathcal{X}$ and the line bundle $\mathcal{O}_{\mathcal{Y}}(-\mathcal{E})$. The $\mathcal{Z}_c$ are endowed with the induced norm and are thus also normed noetherian good moduli stacks with affine diagonal by Theorem 2.6.3. Then, for all $c \in \mathbb{Q}_{>0}$ and for all $\alpha \in \mathbf{\Gamma}$ such that the concatenation $((d(\mathcal{X}), c), \alpha)$ of $(d(\mathcal{X}), c)$ and α also belongs to $\mathbf{\Gamma}$, we have the equality*

$$\mathcal{S}^{\mathcal{X}}_{((d(\mathcal{X}),c),\alpha)} = p_* \left(\mathrm{ind}^{\mathcal{Y}}_{\mathcal{Z}_c}(\mathcal{S}^{\mathcal{Z}_c}_{\alpha}) \setminus \mathcal{E} \right).$$

Moreover, for every $\alpha \in \mathbf{\Gamma}$ such that $\mathcal{S}^{\mathcal{X}}_{\alpha}$ is nonempty, we have:

4. *$d^{\alpha}_0 \le d(\mathcal{X})$; and*
5. *$d^{\alpha}_i \ge i + z(\mathcal{X})$, for all $0 \le i \le n(\alpha)$.*

Proof. First we note that if $\mathcal{X}$ is empty, then the only stratification is given by $\mathcal{S}^{\mathcal{X}}_{\alpha} = \varnothing$ for all α, and it satisfies the required properties.

For a stack $\mathcal{X}$ as in the statement of the theorem, define the number $N(\mathcal{X}) = d(\mathcal{X}) - z(\mathcal{X})$. We will use induction on $N(\mathcal{X})$.

Assume $\mathcal{X} \neq \varnothing$ and let $\mathcal{U}$ be as in 2. Then clearly $d(\mathcal{U}) < d(\mathcal{X})$ and $z(\mathcal{U}) \ge z(\mathcal{X})$, so $N(\mathcal{U}) < N(\mathcal{X})$. If $\mathcal{Z}_c$ is as in 3, then $d(\mathcal{Z}_c) \le d(\mathcal{X})$, since $\mathcal{Z}_c \to \mathcal{X}$ is representable, and $z(\mathcal{Z}_c) > z(\mathcal{X})$ by Lemma 3.4.6. Thus $N(\mathcal{Z}_c) < N(\mathcal{X})$. Therefore the statement of the theorem makes sense if we fix an $N \in \mathbb{N}$ and we restrict to the class of normed noetherian good moduli stacks $\mathcal{X}$ with affine diagonal and with $N(\mathcal{X}) \le N$. We prove the theorem for this class of stacks by induction on N.

If $N = 0$, and $N(\mathcal{X}) = 0$, then the identity component of every stabiliser group of a point of $\mathcal{X}$ is a split torus of dimension $d(\mathcal{X})$. Therefore $|\mathcal{X}^{\max}| = |\mathcal{X}|$ and $S^{\mathcal{X}}_{(d(\mathcal{X}),\infty)} = \mathcal{X}^{\max}$ is the only nonempty stratum. This gives the desired sequential stratification.

Fix $N > 0$ and assume the theorem is true for $N-1$. For $\mathcal{X}$ with $N(\mathcal{X}) = N$, define $\mathcal{S}^{\mathcal{X}}_{\alpha}$ as in the statement of the theorem when $d^{\alpha}_0 \le d(\mathcal{X})$, which makes sense because $N(\mathcal{U}) < N$ and $N(\mathcal{Z}_c) < N$. Define $\mathcal{S}^{\mathcal{X}}_{\alpha} = \varnothing$ otherwise. We show that $(\mathcal{S}^{\mathcal{X}}_{\alpha})_{\alpha\in\mathbf{\Gamma}}$ is a sequential stratification of $\mathcal{X}$.

Denote $r_{\alpha}\colon \mathcal{S}^{\mathcal{X}}_{\alpha} \to \mathcal{X}$ the induced locally closed immersions. First we show that the $r_{\alpha}\left(\left|\mathcal{S}^{\mathcal{X}}_{\alpha}\right|\right)$ are pairwise disjoint and cover $\mathcal{X}$. For $p \in |\mathcal{X}|$, there is a unique $p' \in \overline{\{x\}}$ that is closed in $\pi^{-1}\pi(p)$ [1, Proposition 9.1]. One and only one of the following situations takes place:

1. The dimension $\dim \mathrm{Aut}(p') < d(\mathcal{X})$. In this case $p \in |\mathcal{U}|$ and it is contained in exactly one of the $j_*\mathcal{S}^{\mathcal{U}}_{\alpha} = \mathcal{S}^{\mathcal{X}}_{\alpha}$ with $d^{\mathcal{X}}_0 < d(\mathcal{X})$, by induction hypothesis.
2. We have $\dim \mathrm{Aut}(p') = d(\mathcal{X})$ and $p = p'$. Then $p \in |\mathcal{X}^{\max}| = \left|\mathcal{S}^{\mathcal{X}}_{d(\mathcal{X}),\infty)}\right|$ and this is the only stratum containing p.
3. Again, $\dim \mathrm{Aut}(p') = d(\mathcal{X})$, but this time $p \neq p'$. In this case, $p \in |\mathcal{X}| \setminus (|\mathcal{U}| \cup |\mathcal{X}^{\max}|)$ and there is a unique point $q \in |\mathcal{Y}| \setminus |\mathcal{E}|$ mapping to p. The point q lies in a unique stratum $\mathcal{S}_c$ ($c \in \mathbb{Q}_{\ge 0}$) of the Θ-stratification of $\mathcal{Y}$. There exists a map $\lambda\colon \Theta_k \to \mathcal{X}$ with $\lambda(0) = p'$ and $\lambda(1) = p$ for some field k [7, Lemma 3.24], and the Kempf number $\langle \lambda, \mathcal{X}^{\max} \rangle > 0$ is positive because $\lambda(0) \in |\mathcal{X}^{\max}|$ and $\lambda(1) \notin |\mathcal{X}^{\max}|$ (Definition 3.1.1). The filtration λ lifts uniquely to $\lambda\colon \Theta_k \to \mathcal{Y}$ (Proposition 2.2.18), and $\langle \lambda, \mathcal{O}_{\mathcal{Y}}(-\mathcal{E}) \rangle = \langle \lambda, \mathcal{X}^{\max} \rangle > 0$ (Proposition 3.1.2). Therefore q is unstable in $\mathcal{Y}$ and thus $c > 0$. Since the $\mathcal{S}^{\mathcal{Z}_c}_{\alpha}$ stratify $\mathcal{Z}_c$, the $\mathrm{ind}^{\mathcal{Y}}_{\mathcal{Z}_c}(\mathcal{S}^{\mathcal{Z}_c}_{\alpha})$ stratify $\mathcal{S}_c$, so q is contained in $\mathrm{ind}^{\mathcal{Y}}_{\mathcal{Z}_c}(\mathcal{S}^{\mathcal{Z}_c}_{\alpha})$ for a unique $\alpha \in \mathbf{\Gamma}$ and thus p is contained in a unique $p_* \left(\mathrm{ind}^{\mathcal{Y}}_{\mathcal{Z}_c}(\mathcal{S}^{\mathcal{Z}_c}_{\alpha}) \setminus \mathcal{E} \right)$. It is left to check that $((d(\mathcal{X}), c), \alpha) \in \mathbf{\Gamma}$. This follows because $d^{\alpha}_i \ge i + z(\mathcal{Z}_c) \ge i + 1$ by 5 and Lemma 3.4.6.

Now we check that $\left|\mathcal{S}^{\mathcal{X}}_{>\alpha}\right| := \bigcup_{\beta>\alpha} r_\beta\left(\left|\mathcal{S}^{\mathcal{X}}_{\beta}\right|\right)$ is closed for all $\alpha \in \mathbf{\Gamma}$. If $\alpha = (d(\mathcal{X}), \infty)$, then $\left|\mathcal{S}^{\mathcal{X}}_{>\alpha}\right| = \varnothing$, which is closed. If $\alpha = ((d(\mathcal{X}), c), \alpha')$ with $c \in \mathbb{Q}_{>0}$, then

$$\left|\mathcal{S}^{\mathcal{X}}_{>\alpha}\right| = p\left(\mathrm{gr}^{-1}\left(\left|\mathcal{S}^{\mathcal{Z}_c}_{>\alpha'}\right|\right) \cup \bigcup_{c'>c} |\mathcal{S}_{c'}|\right) \cup |\mathcal{X}^{\max}|,$$

which is closed by induction and because p is proper. If $d_0^\alpha < d(\mathcal{X})$, then $\left|\mathcal{S}^{\mathcal{X}}_{>\alpha}\right| = \left|\mathcal{S}^{\mathcal{U}}_{>\alpha}\right| \cup |\pi^{-1}\pi(\mathcal{X}^{\max})|$, which is also closed by induction and because π is universally closed.

It is left to show properties 4 and 5. The former is true by construction. The latter is true for $i = 0$ because $d(\mathcal{X}) \geq z(\mathcal{X})$, and it is true if $d_0^\alpha < d(\mathcal{X})$ by induction, using the result for $\mathcal{U}$. The remaining case is when α is of the form $\alpha = ((d(\mathcal{X}), c), \alpha')$. Then $\mathcal{S}^{\mathcal{Z}_c}_{\alpha'} \neq \varnothing$ and thus $d^\alpha_{i+1} = d^{\alpha'}_i \geq i + z(\mathcal{Z}_c) \geq i + 1 + z(\mathcal{X})$ by induction and by Lemma 3.4.6. □

Definition 3.5.3 (The balancing stratification)**.** Let $\mathcal{X}$ be a normed noetherian good moduli stack with affine diagonal. The *balancing stratification* of $\mathcal{X}$ is the sequential stratification $(\mathcal{S}^{\mathcal{X}}_\alpha)_{\alpha\in\mathbf{\Gamma}}$ of $\mathcal{X}$ from Theorem 3.5.2.

Remark 3.5.4 (Refining Θ-stratifications)*.* Suppose that $\mathcal{X}$ is an algebraic stack over an algebraic space B, satisfying Assumption 2.2.3, and endowed with a linear form ℓ and a norm q that define a Θ-stratification $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$ such that all the centres $\mathcal{Z}_c$ are quasi-compact with affine diagonal and have good moduli spaces. Then each $\mathcal{Z}_c$ has a well-defined balanced sequential stratification, and thus $\mathcal{X}$ is endowed with the sequential stratification $(\mathrm{ind}^{\mathcal{X}}_{\mathcal{Z}_c}(\mathcal{S}^{\mathcal{Z}_c}_\alpha))_{(c,\alpha)\in\mathbb{Q}_{\geq 0}\times\mathbf{\Gamma}}$, indexed by the poset $\mathbb{Q}_{\geq 0} \times \mathbf{\Gamma}$ with lexicographic order. This gives a natural sequential refinement of the Θ-stratification $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$.

The balanced sequential stratification is well-behaved under certain pullbacks.

Proposition 3.5.5 (Compatibility with pullback)**.** *Let $f\colon \mathcal{X}' \to \mathcal{X}$ be a morphism between normed noetherian good moduli stacks with affine diagonal. Let $\pi\colon \mathcal{X} \to X$ and $\pi'\colon \mathcal{X}' \to X'$ be the good moduli spaces. Assume further that f is either*

1. *a closed immersion, or*
2. *it fits in a cartesian diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{X} \arrow[d] \\ X' \arrow[r, "h"] & X \end{tikzcd}$$

with h a flat morphism,

and that f is norm-preserving. Then the balancing stratification on $\mathcal{X}'$ is the pullback along f (Definition 3.3.3) of the balancing stratification on $\mathcal{X}$. That is, for all $\alpha \in \mathbf{\Gamma}$, we have $\mathcal{S}^{\mathcal{X}'}_\alpha = f^*\mathcal{S}^{\mathcal{X}}_\alpha$.

Proof. We have an upper semicontinuous function r on $|X|$ given by

$$\begin{aligned} r\colon\ |X| &\longrightarrow \mathbb{N} \\ p &\longmapsto d(\pi^{-1}(p)), \end{aligned}$$

where $d(-)$ denotes "maximal stabiliser dimension" (Definition 3.1.4). For $d \in \mathbb{N}$, let $X_{\leq d}$ be the open subspace of X with $|X_{\leq d}| = \{p \in |X| \mid r(p) \leq d\}$ and let $\mathcal{X}_{\leq d} = \pi^{-1}(X_{\leq d})$.

In the case where $\mathcal{X}' = \mathcal{X}_{\leq d}$ and f is the inclusion $\mathcal{X}_{\leq d} \to \mathcal{X}$, the result follows from Item 2 in Theorem 3.5.2.

Note that in both cases we have $\mathcal{X}'_{\leq d} = f^{-1}(\mathcal{X}_{\leq d})$ for all $d \in \mathbb{N}$. To prove the statement for given $\alpha \in \mathbf{\Gamma}$, we may assume $d(\mathcal{X}) = d_0^\alpha$ by replacing $\mathcal{X}$ by $\mathcal{X}_{\leq d_0^\alpha}$ and $\mathcal{X}'$ by $\mathcal{X}'_{\leq d_0^\alpha} = f^{-1}(\mathcal{X}_{\leq d_0^\alpha})$. If $d(\mathcal{X}') < d(\mathcal{X})$, then $\mathcal{S}_\alpha^{\mathcal{X}'} = \varnothing = f^*(\mathcal{S}_\alpha^{\mathcal{X}})$. Thus we may assume $d := d(\mathcal{X}) = d(\mathcal{X}')$. We prove that $\mathcal{S}_\alpha^{\mathcal{X}'} = f^*\mathcal{S}_\alpha^{\mathcal{X}}$ in this case by induction on $N(\mathcal{X}) = d(\mathcal{X}) - z(\mathcal{X})$.

First, we have $(\mathcal{X}')^{\max} = f^{-1}(\mathcal{X}^{\max})$ by [19, Proposition C.5], since in both cases f is stabiliser-preserving (that is, the map $\mathcal{I}_{\mathcal{X}'} \to f^*\mathcal{I}_{\mathcal{X}}$ between inertia stacks is an isomorphism). Therefore $\mathcal{S}^{\mathcal{X}'}_{(d,\infty)} = f^*\mathcal{S}^{\mathcal{X}}_{(d,\infty)}$, and we may assume $c := c_0^\alpha \in \mathbb{Q}_{>c}$ and write $\alpha = ((d,c),\beta)$ with $\beta \in \Gamma$. If f is a closed immersion, then the induced map $o\colon \mathcal{Y}' := \mathrm{Bl}_{(\mathcal{X}')^{\max}} \mathcal{X}' \to \mathcal{Y} := \mathrm{Bl}_{\mathcal{X}^{\max}} \mathcal{X}$ is also a closed immersion. In the second case, the square

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "o"] \arrow[d] & \mathcal{Y} \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

is cartesian because blow-ups commute with flat base-change. In both cases, the Θ-stratification $(\mathcal{S}'_c)_{c\in\mathbb{Q}_{\geq 0}}$ on $\mathcal{Y}'$ is the pullback of the Θ-stratification $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$ on $\mathcal{Y}$ by Proposition 2.5.16. Let $c > 0$, let $\mathcal{Z}_c$ be the centre of $\mathcal{S}_c$, let $\mathcal{Z}'_c$ be the centre of $\mathcal{S}'_c$, and let $g\colon \mathcal{Z}'_c \to \mathcal{Z}_c$ be the natural map. By cartesianity of the square

$$\begin{tikzcd} \mathcal{Z}'_c \arrow[r, "g"] \arrow[d] & \mathcal{Z}_c \arrow[d] \\ \mathcal{Y}' \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

we see that g is also either a closed immersion or a base change of a flat map between the good moduli spaces of $\mathcal{Z}'_c$ and $\mathcal{Z}_c$. By induction, since $N(\mathcal{Z}_c) < N(\mathcal{X})$, we have $\mathcal{S}_\beta^{\mathcal{Z}'_c} = g^*(\mathcal{S}_\beta^{\mathcal{Z}_c})$. It follows from Item 3 in Theorem 3.5.2 that $\mathcal{S}_\alpha^{\mathcal{X}'} = f^*(\mathcal{S}_\alpha^{\mathcal{X}})$. □

Remark 3.5.6. A crucial ingredient that makes Proposition 3.5.5 possible is that the closed substack structure of the maximal dimension stabiliser locus is compatible with pullbacks. It would not hold if we had used the reduced substacks structure instead.

Proposition 3.5.7 (Compatibility with fibres)**.** *Let $\mathcal{X}$ be a normed noetherian algebraic stack with affine diagonal and a good moduli space $\pi\colon \mathcal{X} \to X$, let k be a field and let $x\colon \mathrm{Spec}\, k \to X$ be a map. Let $\mathcal{F} = \mathrm{Spec}\, k \times_{x,X,\pi} \mathcal{X}$. Then the balanced sequential stratification of $\mathcal{F}$ is the pullback along $\mathcal{F} \to \mathcal{X}$ of the balanced sequential stratification of $\mathcal{X}$.*

Proof. In view of Proposition 3.5.5, it is enough to show that $x\colon \mathrm{Spec}\, k \to X$ factors as

$$\mathrm{Spec}\, k \xrightarrow{a} Z \xrightarrow{b} X,$$

with a flat and b a closed immersion. Let $p \in |X|$ be the image of x. Take $Z = \overline{\{p\}}$ with reduced structure, $b\colon Z \to X$ the closed immersion, and $a\colon \mathrm{Spec}\, k \to Z$ the induced factorisation.

Since Z is quasi-separated, it has an open dense subspace $U \subset Z$ that is a scheme [50, Tag 06NH]. Moreover, U is necessarily an integral scheme with generic point p. The map

$\operatorname{Spec}\kappa(p) \to U$ is flat, being a localisation, and a is therefore flat as well, as it factors as a composition

$$\operatorname{Spec} k \to \operatorname{Spec}\kappa(p) \to U \to Z$$

of flat maps. $\square$

The balanced sequential stratification defines, for every point x of the stack $\mathcal{X}$, a canonical sequential filtration of x.

Definition 3.5.8 (Iterated balanced filtration)**.** Let $\mathcal{X}$ be a normed noetherian good moduli stack with affine diagonal, let k be a field and let $x \in \mathcal{X}(k)$ be a k-point. The *iterated balanced filtration* $\lambda_{\mathrm{ib}}(x) \in \mathbb{Q}^\infty\text{-}\operatorname{Filt}(\mathcal{X}, x)$ of x is defined as follows. There is a unique $\alpha \in \mathbf{\Gamma}$ such that the point x factors through $\mathcal{S}_\alpha^{\mathcal{X}} \to \mathcal{X}$. There is then a unique (up to unique isomorphism) k-point y of $\mathcal{S}_\alpha^{\mathcal{X}}$ mapping to x. The point y naturally belongs to $\mathbb{Q}^\infty\text{-}\operatorname{Filt}(\mathcal{X}, x)$, and we set $\lambda_{\mathrm{ib}}(x) \coloneqq y$.

3.6 Examples

In moduli theory there are many natural instances of normed good moduli stacks, for which the balancing stratification is defined. We collect here a few examples.

3.6.1 Stacks proper over a good moduli stack

Assume the framework of Theorem 2.6.3, that is, $\mathcal{X}$ is a normed noetherian good moduli stack with affine diagonal, $f\colon \mathcal{Y} \to \mathcal{X}$ is representable and proper and ℓ is an f-positive linear form on $\mathcal{Y}$. Then we have a Θ-stratification $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$ of $\mathcal{Y}$ and every centre $\mathcal{Z}_c$ is again a normed noetherian good moduli stack with affine diagonal. Therefore each $\mathcal{Z}_c$ has canonically a balanced sequential stratification $(\mathcal{S}_\alpha^{\mathcal{Z}_c})_{\alpha\in\mathbf{\Gamma}}$. By Remark 3.5.4, the Θ-stratification $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$ is canonically refined by the sequential stratification

$$\left(\operatorname{ind}_{\mathcal{Z}_c}^{\mathcal{Y}}\left(\mathcal{S}_\alpha^{\mathcal{Z}_c}\right)\right)_{(c,\alpha)\in\mathbb{Q}_{\geq 0}\times\mathbf{\Gamma}}.$$

3.6.2 Geometric Invariant Theory

Let k be a field, let G be a linearly reductive affine algebraic group over k admitting a split maximal torus T, with Weyl group W, and endowed with a norm on cocharacters (Definition 2.3.7). Let A be a finite type k-algebra and consider an action of G on $\operatorname{Spec} A$. The quotient stack $\mathcal{X} = \operatorname{Spec}(A)/G$ has a good moduli space $\mathcal{X} \to \operatorname{Spec}(A^G)$. Given any G-equivariant projective morphism $f\colon Y \to \operatorname{Spec} A$ and an ample linearisation on Y (that is, a line bundle on Y/G ample with respect to $h = f/G\colon Y/G \to \operatorname{Spec}(A)/G$). The previous example applied to h gives a sequential stratification of Y/G indexed by $\mathbb{Q}_{\geq 0}\times\mathbf{\Gamma}$. For every k-point $y \in Y^{\mathrm{ss}}(k)$, the iterated balanced filtration $\lambda_{\mathrm{ib}}(y)$ of y can be seen as a sequence $\lambda_0, \dots, \lambda_n \in \Gamma^{\mathbb{Q}}(G)$ of commuting rational cocharacters of G, considered up to certain equivalence relation, by Remark 3.2.16.

This stratification was first defined by Kirwan [39] in the case where $k = \mathbb{C}$, $A = \mathbb{C}$ and $Y \to \operatorname{Spec}\mathbb{C}$ is smooth, building on the ideas introduced in [41]. The indexing set used in [39] is different from the one used here, and it depends on the quotient presentation of Y/G. The strata obtained in [39] are open and closed substacks of the strata defined here, which does not make a substantial difference. This further partition of the strata arises in two different ways. First, instead of the Θ-stratification $(\mathcal{S}_c)_{c\in\mathbb{Q}_{\geq 0}}$ of Y/G considered here, indexed by $\mathbb{Q}_{\geq 0}$, the stratification considered by Kirwan is indexed by $\Gamma^{\mathbb{Q}}(T)/W$ (see also [40]). The set $\Gamma^{\mathbb{Q}}(T)/W$ can be seen as parametrising certain unions of connected components on $\mathrm{Grad}_{\mathbb{Q}}(Y/G)$ by [26, Theorem 1.4.8], and the strata obtained in this way are closed and open substacks of the $\mathcal{S}_c$. The same kind of difference on indexing sets appears each time a blow-up is performed in the construction. Also, each time a locus of maximal dimension stabilisers is considered, for example $(Y^{\mathrm{ss}}/G)^{\max}$, Kirwan writes it as a disjoint union of loci of the form $G\left((Y^{\mathrm{ss}})^R\right)/G$, where R is a reductive subgroup of G of maximal dimension such that $(Y^{\mathrm{ss}})^R$ is nonempty. This further refines the indexing set and the stratification, but again only breaking down the strata into pieces that are closed and open.

Even in the case considered in [39], the fact that the balancing stratification has the structure of a sequential stratification is a novelty of our approach. The definition of the iterated balanced filtration (Definition 3.5.8) is to our knowledge new also in this case.

3.6.3 Quiver representations

Let Q be a quiver with set of vertices Q_0, set of arrows Q_1 and source and target maps $s, t\colon Q_1 \to Q_0$. Let d be a dimension vector for Q and consider the moduli stack $\mathscr{Rep}(Q,d)$ of representations of Q with dimension vector d over an algebraically closed field k of characteristic 0. A *central charge* Z for Q is defined by a family $(a_i)_{i\in Q_0}$ with $a_i \in \mathbb{Q} \oplus i\mathbb{Q}_{>0} \subset \mathbb{C}$. For a finite dimensional representation E of Q we set $Z(E) = \sum_{i\in\mathbb{Q}} a_i \dim E_i$. This defines a linear form ℓ and a norm q on $\mathscr{Rep}(Q,d)$ as follows. A graded point $g\colon B\mathbb{G}_{m,k} \to \mathscr{Rep}(Q,d)$ corresponds to a representation E with dimension vector d and a direct sum decomposition $E = \bigoplus_{c\in\mathbb{Z}} E_c$. We then set

$$\ell(g) = \sum_{c\in\mathbb{Z}} c\,\mathfrak{Re}\,Z(E_c) \tag{6}$$

$$q(g) = \sum_{c\in\mathbb{Z}} c^2\,\mathfrak{Im}\,Z(E_c). \tag{7}$$

Therefore the central charge Z determines a normed good moduli stack $\mathscr{Rep}(Q,d)^{\mathrm{ss}}$ of semistable quiver representations. In fact, semistability with respect to ℓ is a King stability [37] condition. We can also see ℓ-semistable representations as Bridgeland semistable representations of slope $\pi/2$. The linear form ℓ comes from the line bundle on $\mathscr{Rep}(Q,d)$ given by the rational character $\prod_{i\in Q_0} \det_{\mathrm{GL}_{d_i}}^{-\mathfrak{Re}\,a_i}$ of $\prod_{i\in Q_0} \mathrm{GL}_{d_i}$.

The iterated balanced filtration of a semistable representation E coincides with the iterated weight filtration (or HKKP filtration) of E defined by Haiden-Katzarkov-Kontsevich-Pandit in [24]. This fact is proven in [33]. Hence the balancing stratification of $\mathscr{Rep}(Q,d)^{\mathrm{ss}}$ can be seen as a stratification by type of HKKP filtration.

3.6.4 Vector bundles on a curve

Let C be a smooth projective curve over $\mathbb{C}$ and consider the stack $\mathrm{Bun}(C)_{r,d}$ of vector bundles on C of rank r and degree d. The open substack $\mathrm{Bun}(C)^{\mathrm{ss}}_{r,d} \subset \mathrm{Bun}(C)_{r,d}$ of semistable vector bundles admits a good moduli space. This can be proven either using GIT [32] or by intrinsic methods [3]. The stack $\mathrm{Bun}(C)_{r,d}$ has a norm on graded points q given by the rank as follows. A graded point $g\colon B\mathbb{G}_{m,\mathbb{C}} \to \mathrm{Bun}(C)_{r,d}$ corresponds to a vector bundle E of degree d and rank r and a direct sum decomposition $E = \bigoplus_{c\in\mathbb{Z}} E_c$ as sum of subbundles. We set

$$q(g) = \sum_{c\in\mathbb{Z}} c^2\,\mathrm{rk}(E_c).$$

Hence $\mathrm{Bun}(C)^{\mathrm{ss}}_{r,d}$ is a normed good moduli stack (and it is also noetherian with affine diagonal), so the balancing stratification and the iterated balanced filtration of every semistable vector bundle are defined.

In this setting, a coarser version of the balancing stratification was defined and studied by Kirwan [38]. Each stratum in Kirwan's stratification is a connected component of a locally closed substack of the form $\bigcup_{\alpha\in\mathbf{\Gamma}} \mathcal{S}^{\mathcal{X}}_{((n,c),\alpha)}$, for $n \in \mathbb{N}$ and $c \in \mathbb{Q}_{>0} \cup \{\infty\}$, where $\mathcal{X} = \mathrm{Bun}(C)^{\mathrm{ss}}_{r,d}$. Therefore Kirwan's stratification can be thought of as the stratification by type of balanced filtration. Kirwan calls the filtrations associated to this stratification *balanced δ-filtrations of maximal triviality*.

In [33], we show that the iterated balanced filtration for a semistable vector bundle E coincides with the iterated HKKP filtration of the lattice of semistable subbundles of E of the same slope. Hence the (first step of) the HKKP filtration of E should correspond to Kirwan's balanced δ-filtration of maximal triviality of E.

3.6.5 Bridgeland semistable objects

Let X be a projective scheme over an algebraically closed field k of characteristic 0, and consider a Bridgeland stability condition [12] given by the heart $\mathcal{C} \subset \mathrm{D^b}(X)$ of a t-structure on the derived category of X and a central charge $Z\colon K_0^{\mathrm{num}}(\mathrm{D^b}(X)) \to \mathbb{C}$, where $K_0^{\mathrm{num}}(\mathrm{D^b}(X))$ is the numerical Grothendieck group. Let $\mathcal{A} = \mathrm{Ind}(\mathcal{C})$ be the ind-completion of $\mathcal{C}$. Under certain natural assumptions on $(\mathcal{C}, Z)$, good moduli stacks of Bridgeland semistable objects can be constructed as open substacks of $\mathcal{M}_{\mathcal{A}}$, the moduli stack of objects in $\mathcal{A}$ defined in [7, Section 7] following [8]. We assume that Z is algebraic in the sense that $Z\left(\mathrm{D^b}(X)\right) \subset \mathbb{Q} \oplus i\mathbb{Q}$, that Z factors through a finite free quotient of $K_0^{\mathrm{num}}(\mathrm{D^b}(X))$, that $\mathcal{C}$ satisfies the generic flatness condition and that certain boundedness conditions also hold (see [26, Theorem 6.5.3] and [7, Example 7.29] for details). Under these assumptions, $\mathcal{M}_{\mathcal{A}}$ is an algebraic stack with affine diagonal locally of finite type over k.

For a numerical class $v \in K_0^{\mathrm{num}}(\mathrm{D^b}(X))$, there is an open and closed substack $\mathcal{M}_v \subset \mathcal{M}_{\mathcal{A}}$ of objects in class v and, by our boundedness hypothesis, there is a quasicompact open substack $\mathcal{M}_v^{\mathrm{ss}}$ of semistable objects. From the general results in [7], it follows that $\mathcal{M}_v^{\mathrm{ss}}$ admits a (proper) good moduli space [7, Example 7.29]. The imaginary part of the central charge defines a norm q on graded points of $\mathcal{M}_{\mathcal{A}}$ as in the previous examples. If $g\colon B\mathbb{G}_{m,k} \to \mathcal{M}_{\mathcal{A}}$ corresponds to $E = \bigoplus_{c\in\mathbb{Z}} E_c$ in $\mathcal{C}$, then we define $q(g)$ by the formula

$$q(g) = \sum_{c\in\mathbb{Z}} c^2\, \mathfrak{Im}\, Z(E_c)$$

as above (this is the norm used in [26, Chapter 6] to define the numerical invariant on $\mathcal{M}_v$, by [26, Lemma 6.4.8]). Therefore $\mathcal{M}_v^{\mathrm{ss}}$ is naturally a normed good moduli stack, noetherian and with affine diagonal, and thus the balancing stratification and the iterated balanced filtration of every point are defined. For $E \in \mathcal{M}_v^{\mathrm{ss}}(k)$, it is proven in [33] that the iterated balanced filtration of E coincides with the HKKP filtration of the lattice of semistable subobjects of E of the same slope. This follows from the more general Theorem 1.4.1 mentioned in the introduction.

3.6.6 K-semistable Fano varieties

Let k be a field of characteristic 0. It has recently been established that there is an algebraic stack $\mathcal{X}^{\mathrm{K}}_{n,V}$ of finite type over k with affine diagonal parametrising families of

K-semistable Fano varieties over k of dimension n and volume V, and that $\mathcal{X}^{\mathrm{K}}_{n,V}$ admits a proper good moduli space [4, 10, 11] (see [51] for a book account of these results). The L^2-norm of a test configuration [10, Section 2.3] defines a norm on graded points of $\mathcal{X}^{\mathrm{K}}_{n,V}$ [51, Lemmas 2.36 and 8.45], and thus the balancing stratification of $\mathcal{X}^{\mathrm{K}}_{n,V}$ is defined.

It is plausible that the iterated balanced filtration of a smooth K-semistable Fano variety X over $\mathbb{C}$ is related to the asymptotics of the Calabi flow on X. This would provide a refinement of the results in [13, 14].

3.6.7 G-bundles and gauged maps

Let C be a smooth projective over $\mathbb{C}$ and let G be a connected reductive group. The notion of semistability for principal G-bundles over C was defined in [46]. A moduli space of semistable G-bundles on C was constructed in [47, 48] using GIT. The stack of semistable G-bundles $\mathrm{Bun}_G(C)^{\mathrm{ss}}$ is thus an open substack of the stack $\mathrm{Bun}_G(C)$ of all G-bundles that admits a good moduli space. A choice of norm on cocharacters of G induces a norm on $\mathrm{Bun}_G(C)$ as follows. Let T be a maximal torus of G with Weyl group W, then we have identifications

$$\begin{aligned}\mathrm{Grad}(\mathrm{Bun}_G(C)) &= \underline{\mathrm{Hom}}(B\mathbb{G}_{m,\mathbb{C}}, \underline{\mathrm{Hom}}(C, BG)) = \underline{\mathrm{Hom}}(C, \underline{\mathrm{Hom}}(B\mathbb{G}_{m,\mathbb{C}}, BG)) \\ &= \bigsqcup_{\lambda\in\Gamma^{\mathbb{Z}}(T)/W} \underline{\mathrm{Hom}}(L(\lambda), C) = \bigsqcup_{\lambda\in\Gamma^{\mathbb{Z}}(T)/W} \mathrm{Bun}_{L(\lambda)}(C)\end{aligned}$$

by [26, Theorem 1.4.8]. Therefore there is a natural map

$$\pi_0(\mathrm{Grad}(\mathrm{Bun}_G(C))) \to \pi_0(\mathrm{Grad}(BG)) = \Gamma^{\mathbb{Z}}(T)/W,$$

along which we can pullback the norm on BG to get a norm on cocharacters of $\mathrm{Bun}_G(C)$. This gives $\mathrm{Bun}_G(C)^{\mathrm{ss}}$ a natural structure of normed good moduli stack, and therefore defines the iterated balanced filtration for any semistable G-bundle. We expect the Yang-Mills flow for G-bundles [9] to be related to the iterated balanced filtration, in analogy with [23], which deals with the case $G = \mathrm{GL}_{n,\mathbb{C}}$.

The moduli stack of G-bundles is a special case of the general framework of gauged maps studied in [27]. For a projective-over-affine scheme X over $\mathbb{C}$ endowed with an action of G and $n \geq 0$, Halpern-Leistner and Fernandez Herrero define a stack $\mathcal{M}^G_n(X)$ parametrising families of Kontsevich stable maps from C to the quotient stack X/G. Taking $X = \operatorname{Spec}\mathbb{C}$ and $n = 0$ recovers the moduli stack of G-bundles on C. For a choice of numerical invariant μ, the semistable locus $\mathcal{M}^G_n(X)^{\mu\text{-ss}}$ admits a good moduli space. The numerical invariant μ depends on a choice of norm on cocharacters of G, that then gives a norm on graded points of $\mathcal{M}^G_n(X)$ similarly to the case of G-bundles [27, Definition 2.21]. Therefore $\mathcal{M}^G_n(X)^{\mu\text{-ss}}$ is a normed good moduli stack and the balancing stratification of $\mathcal{M}^G_n(X)^{\mu\text{-ss}}$ is defined. The construction in [27] works over a general noetherian base S of characteristic 0 with affine diagonal. In this setting one still gets a noetherian normed good moduli stack $\mathcal{M}^G_n(X)^{\mu\text{-ss}}$ with affine diagonal, and its balancing stratification is thus defined.

4 Chains of stacks

We introduce the formalism of *chains of stacks* (Definition 4.1.1) as a tool to compute the iterated balanced filtration. For every chain of stacks there is an associated sequential filtration (Definition 4.1.3). We give two constructions of chains. The first, the *balancing* *chain* (Construction 4.2.1), is very close to the balancing stratification and it computes the iterated balanced filtration. The second, the *torsor chain* (Construction 4.3.1) is more closely related to combinatorial versions of the iterated balanced filtration. The main theorem of this section states that the torsor chain also computes the iterated balanced filtration (Theorem 4.3.4). This fact will be used to relate the iterated balanced filtration to artinian lattices (Theorem 1.4.1) and to convex geometry (Theorem 5.2.16 and Corollary 5.2.18), giving ways to compute it that are combinatorial in nature.

4.1 Chains

Let k be a field. A *k-pointed stack* is an algebraic stack $\mathcal{X}$ together with a k-point $x\colon \operatorname{Spec} k \to \mathcal{X}$. k-Pointed stacks form a 2-category as follows. A morphism $(\mathcal{X}, x) \to (\mathcal{Y}, y)$ of k-pointed stacks is a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of stacks and a 2-isomorphism $\alpha\colon f\circ x \to y$. The composition of $(f,\alpha)\colon (\mathcal{X},x)\to(\mathcal{Y},y)$ and $(g,\beta)\colon(\mathcal{Y},y)\to(\mathcal{Z},z)$ is $(g\circ f, \beta\circ(1_g*\alpha))$, where $*$ denotes horizontal composition. If $(f,\alpha),(f',\alpha')\colon(\mathcal{X},x)\to(\mathcal{Y},y)$ are morphisms of k-pointed stacks, a 2-morphism $(f,\alpha)\to(f',\alpha')$ is a 2-morphism $\gamma\colon f\to f'$ such that

$$\begin{tikzcd} x\circ f \arrow[rr, "1_x*\gamma"] \arrow[dr, "\alpha"'] & & x\circ f' \arrow[dl, "\alpha'"] \\ & y & \end{tikzcd}$$

commutes.

Definition 4.1.1 (Chain)**.** A *chain* $(\mathcal{X}_n, x_n, \gamma_n, u_n)_{n\in\mathbb{N}}$ of k-pointed stacks is data:

1. For each $n\in\mathbb{N}$, a k-pointed normed stack $(\mathcal{X}_n, x_n)$, where $\mathcal{X}_n$ is of finite presentation over $\operatorname{Spec} k$, has affine diagonal, and admits a good moduli space $\mathcal{X}_n \to \operatorname{Spec} k$ which is $\operatorname{Spec} k$.
2. For each $n\in\mathbb{N}$, a $\mathbb{Q}$-filtration $\gamma_n \in \mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{X}_n, x_n)$ of x_n.
3. For each $n\in\mathbb{N}$, a representable, separated and norm-preserving morphism
$$u_n\colon (\mathcal{X}_{n+1}, x_{n+1}) \to (\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}_n), \operatorname{gr}\gamma_n).$$

We say that the chain $(\mathcal{X}_n, x_n, \gamma_n, u_n)_{n\in\mathbb{N}}$ is *bounded* if there is $N\in\mathbb{N}$ such that, for all $n\geq N$, we have that $\gamma_n = 0$ and u_n induces an isomorphism between $\mathcal{X}_{n+1}$ and $\mathcal{X}_n$, seen as the closed and open substack of $\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}_n)$ of "trivial gradings".

A *morphism* $f\colon (\mathcal{X}'_n, x'_n, \gamma'_n, u'_n) \to (\mathcal{X}_n, x_n, \gamma_n, u_n)$ *of chains* consists of morphisms $f_n\colon (\mathcal{X}'_n, x'_n)\to(\mathcal{X}_n, x_n)$ of pointed stacks, together with isomorphisms $f(\gamma'_n)\to\gamma_n$ of filtrations and 2-commutative squares

$$\begin{tikzcd} (\mathcal{X}'_{n+1}, x'_{n+1}) \arrow[r, "u'_n"] \arrow[d, "f_{n+1}"'] & (\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}'_n), \operatorname{gr}\gamma'_n) \arrow[d, "\operatorname{Grad}_{\mathbb{Q}}(f_n)"] \\ (\mathcal{X}_{n+1}, x_{n+1}) \arrow[r, "u_n"] & (\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}_n), \operatorname{gr}\gamma_n) \end{tikzcd}$$

of pointed stacks.

Suppose that $(\mathcal{X}_n, x_n, \gamma_n, u_n)_{n\in\mathbb{N}}$ is a bounded chain. For $n\in\mathbb{N}$, we define a map $c_n\colon \mathcal{X}_{n+1}\to \operatorname{Grad}^{n+1}_{\mathbb{Q}}(\mathcal{X}_0)$ by

$$c_n = \operatorname{Grad}^n_{\mathbb{Q}}(u_0)\circ\operatorname{Grad}^{n-1}_{\mathbb{Q}}(u_1)\circ\cdots\circ u_n.$$

By Proposition 2.2.19, c_n induces an injection

$$\mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{X}_{n+1}, x_{n+1}) \to \mathbb{Q}\text{-}\operatorname{Filt}(\operatorname{Grad}_{\mathbb{Q}}^{n+1}(\mathcal{X}_0), c_n(x_{n+1})).$$

Define $\lambda_{n+1} \in \mathbb{Q}\text{-}\operatorname{Filt}(\operatorname{Grad}_{\mathbb{Q}}^{n+1}(\mathcal{X}_0), c_n(x_{n+1}))$ to be the image of γ_{n+1} under this injection. Define also $\lambda_0 := \gamma_0 \in \mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{X}_0, x_0)$.

Lemma 4.1.2. *There is a canonical isomorphism $c_n(x_{n+1}) \simeq \operatorname{gr}\lambda_n$, for all $n \in \mathbb{N}$.*

Proof. For $n = 0$, $v_0(x_1) = u_0(x_1) \simeq \operatorname{gr}\gamma_0 = \operatorname{gr}\lambda_0$ is given by α_0. For $n > 0$, we have

$$\begin{aligned}\operatorname{gr}\lambda_n = \operatorname{gr}\left(\operatorname{Filt}_{\mathbb{Q}}(c_{n-1})(\gamma_n)\right) = \operatorname{Grad}_{\mathbb{Q}}(c_{n-1})(\operatorname{gr}\gamma_n) \simeq \\ \operatorname{Grad}_{\mathbb{Q}}(c_{n-1})\left(u_n(x_{n+1})\right) = c_n(x_{n+1}),\end{aligned}$$

as desired. □

Note that there is $N \in \mathbb{N}$ such that $\lambda_n = 0$ for $n \geq N$, because the chain is bounded. The lemma gives canonical isomorphisms

$$\mathbb{Q}\text{-}\operatorname{Filt}(\operatorname{Grad}_{\mathbb{Q}}^{n+1}(\mathcal{X}_0), c_n(x_{n+1})) \cong \mathbb{Q}\text{-}\operatorname{Filt}(\operatorname{Grad}_{\mathbb{Q}}^{n+1}(\mathcal{X}_0), \operatorname{gr}\lambda_n)$$

for all n. Therefore, by Remark 3.2.14, $(\lambda_n)_{n\in\mathbb{N}}$ defines an element of $\mathbb{Q}^{\infty}\text{-}\operatorname{Filt}(\mathcal{X}_0, x_0)$.

Definition 4.1.3. The $\mathbb{Q}^{\infty}$*-filtration associated to* the bounded chain $(\mathcal{X}_n, x_n, \gamma_n, u_n)_{n\in\mathbb{N}}$ is $(\lambda_n)_{n\in\mathbb{N}} \in \mathbb{Q}^{\infty}\text{-}\operatorname{Filt}(\mathcal{X}_0, x_0)$.

4.2 The balancing chain

We now construct a chain closely related to the balancing stratification.

Construction 4.2.1. Let $\mathcal{X}$ be a normed noetherian algebraic stack with affine diagonal and a good moduli space $\pi\colon \mathcal{X} \to X$. Let $x\colon \operatorname{Spec} k \to \mathcal{X}$ be a k-point, with k a field. We define, a chain $(\mathcal{X}_n, x_n, \gamma_n, u_n)_{n\in\mathbb{N}}$ over k as follows.

We set $(\mathcal{X}_0, x_0) = (\pi^{-1}(\pi(x)), x)$, which has good moduli space $\operatorname{Spec} k$. For $n \in \mathbb{N}$, assume $\mathcal{X}_n$ and x_n are defined. We now define $\mathcal{X}_{n+1}$, x_{n+1}, γ_n and u_n in terms of $\mathcal{X}_n$ and x_n. We consider two cases:

Case 1. The point x_n is closed in $\mathcal{X}_n$. We then define $\mathcal{X}_{n+1} = \mathcal{X}_n$, $x_{n+1} = x_n$, $\gamma_n = 0$ the trivial filtration in $\mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{X}_n, x_n)$, $u_n\colon (\mathcal{X}_{n+1}, x_{n+1}) \to (\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}_n), \operatorname{gr}\gamma_n)$ given by the "trivial grading" map.

Case 2. The point x_n is not closed in $\mathcal{X}_n$. Then we consider the blow-up $p\colon \mathcal{B} = \operatorname{Bl}_{\mathcal{X}_n^{\max}} \mathcal{X}_n \to \mathcal{X}_n$, where $\mathcal{X}_n^{\max}$ is the closed substack of points with maximal dimension stabiliser [19, Appendix C]. In fact, $|\mathcal{X}_n^{\max}|$ is a singleton consisting of the unique closed point of $|\mathcal{X}_n|$ [1, Proposition 9.1], which is different from x_n by assumption. Thus the point x_n lifts to a point of $\mathcal{B}$ that we still denote x_n.

By Theorem 2.6.3 and Example 2.6.4, $\mathcal{B}$ has Θ-stratification induced by the natural p-ample line bundle on $\mathcal{B}$ and the norm. Let $\mathcal{S}$ be the locally closed Θ-stratum containing x_n and let $\mathcal{Z}$ be its centre (Definition 2.5.1). Let $\gamma_n \in \mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{B}, x_n)$ be the Harder-Narasimhan filtration of x_n in $\mathcal{B}$ and let $x_{n+1} = \operatorname{gr}\gamma_n \in \mathcal{Z}(k)$. We identify $\mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{B}, x_n) = \mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{X}_n, x_n)$ by Proposition 2.2.18, and under this identification γ_n is actually the balanced filtration of x_n in $\mathcal{X}_n$ (Definition 3.1.6 and Proposition 3.1.2). By Theorem 2.6.3, $\mathcal{Z}$ has a good moduli space $\pi_{\mathcal{Z}}\colon \mathcal{Z} \to Z$. We set $\mathcal{X}_{n+1} = \pi_{\mathcal{Z}}^{-1}(\pi_{\mathcal{Z}}(x_{n+1}))$. We define u_n to be the composition

$$u_n\colon \mathcal{X}_{n+1} \to \mathcal{Z} \hookrightarrow \operatorname{Grad}_{\mathbb{Q}}(\mathcal{B}) \xrightarrow{\operatorname{Grad}_{\mathbb{Q}}(p)} \operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}_n),$$

which is representable and separated, since applying $\mathrm{Grad}_{\mathbb{Q}}$ preserves representability and separatedness, and the first two maps are immersions.

The stack $\mathcal{X}_{n+1}$ inherits a norm from $\mathcal{X}_n$ along the composition $\mathcal{X}_{n+1} \to \mathcal{Z} \to \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_n) \to \mathcal{X}_n$. By commutativity of

$$\begin{tikzcd} \mathrm{Filt}_{\mathbb{Q}}(\mathcal{B}) \arrow[r] \arrow[d, "\mathrm{gr}"'] & \mathrm{Filt}_{\mathbb{Q}}(\mathcal{X}_n) \arrow[d, "\mathrm{gr}"] \\ \mathrm{Grad}_{\mathbb{Q}}(\mathcal{B}) \arrow[r] & \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_n) \end{tikzcd}$$

we get a pointed morphism $u_n \colon (\mathcal{X}_{n+1}, x_{n+1}) \to (\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_n), \mathrm{gr}\,\gamma_n))$.

Definition 4.2.2. Let $\mathcal{X}$ be a normed noetherian good moduli stack with affine diagonal, let k be a field and let $x \in \mathcal{X}(k)$ be a k-point. The *balancing chain* of $(\mathcal{X}, x)$ is the chain $(\mathcal{X}_n, x_n, \gamma_n, u_n)_{n \in \mathbb{N}}$ of k-stacks constructed in Construction 4.2.1.

Remark 4.2.3. The following properties of the balancing chain are clear from Theorem 3.1.3:

1. For every $n \in \mathbb{N}$, we have $\gamma_n = 0$ if and only if x_n is closed in $\mathcal{X}_n$.
2. For every $n \in \mathbb{N}$, we have that $\mathrm{ev}_0(\gamma_n))$ is the unique closed point of $\mathcal{X}_n$.

Lemma 4.2.4. *Assume the setup of Definition 4.2.2. For every $n \in \mathbb{N}$ such that x_n is not closed in $\mathcal{X}_n$, we have that $z(\mathcal{X}_n) \geq z(\mathcal{X}_0) + n$, where $z(-)$ denotes central rank (Definition 3.4.1).*

Note as well that $z(\mathcal{X}_0) \geq z(\mathcal{X})$.

Proof. Let $n \in \mathbb{N}$ and suppose that x_n is not closed in $\mathcal{X}_n$. Then $z(\mathcal{X}_{n+1}) > z(\mathcal{X}_n)$ by Lemma 3.4.6. The results follows by induction. □

Corollary 4.2.5. *Assuming the setup of Definition 4.2.2, the balancing chain of $(\mathcal{X}, x)$ is bounded.*

Proof. If it was not bounded, then for every $n \in \mathbb{N}$ we would have that x_n is not closed in $\mathcal{X}_n$ and thus that $z(\mathcal{X}_n) \geq n$ by Lemma 4.2.4. This would contradict the bound $z(\mathcal{X}_n) \leq d(\mathcal{X}_0)$, where $d(-)$ denotes maximal stabiliser dimension (Definition 3.1.4). □

Proposition 4.2.6. *Let $\mathcal{X}$ be a normed noetherian good moduli stack with affine diagonal, and let $x \in \mathcal{X}(k)$ be a field-valued point. Then the sequential filtration of x associated to the balancing chain of $(\mathcal{X}, x)$ (Definitions 4.1.3 and 4.2.2) equals the iterated balanced filtration of x (Definition 3.5.8).*

Proof. Let $(\mathcal{X}_n, x_n, \gamma_n, u_n)_{n \in \mathbb{N}}$ be the balancing chain of $(\mathcal{X}, x)$, and let

$$\lambda_{\mathrm{bc}}(x) \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}_0, x_0) = \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$$

be its associated sequential filtration. Note that, by Proposition 3.5.7, the iterated balanced filtration $\lambda_{\mathrm{ib}}(x)$ of $(\mathcal{X}, x)$ equals that of $(\mathcal{X}_0, x)$.

We prove the statement by induction on $N(\mathcal{X}) = d(\mathcal{X}) - z(\mathcal{X})$. If $N(\mathcal{X}) = 0$, then x is closed in $\mathcal{X}_0$ and therefore $\lambda_{\mathrm{ib}}(x) = \lambda_{\mathrm{bc}}(x) = 0$. If x is not closed in $\mathcal{X}_0$ then, using the notation of Construction 4.2.1, $\gamma_0 \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}_0, x)$ is the balanced filtration of x (Definition 3.1.6). From Proposition 3.2.11, we get a map $\varphi \colon \mathbb{Q}^{\infty}\text{-}\mathrm{Filt}(\mathcal{X}_1, x_1) \to$

$\mathbb{Q}^\infty\text{-}\operatorname{Filt}(\operatorname{Bl}_{\mathcal{X}_0^{\max}} \mathcal{X}_0, x_0) \to \mathbb{Q}^\infty\text{-}\operatorname{Filt}(\mathcal{X}_0, x_0)$ that, in the notation of Remark 3.2.14, can be written as

$$(\lambda_0, \lambda_1, \ldots) \mapsto (\gamma_0, \lambda_0, \lambda_1, \ldots).$$

By construction of the balancing chain, we have $\lambda_{\mathrm{bc}}(x) = \varphi(\lambda_{\mathrm{bc}}(x_1))$, and by construction of the balancing stratification, we have $\lambda_{\mathrm{ib}}(x) = \varphi(\lambda_{\mathrm{ib}})(x_1)$. By Lemma 3.4.6, $N(\mathcal{X}_1) < N(\mathcal{X}_0)$. Therefore $\lambda_{\mathrm{ib}}(x_1) = \lambda_{\mathrm{bc}}(x_1)$ by induction, and hence $\lambda_{\mathrm{ib}}(x) = \lambda_{\mathrm{bc}}(x)$. $\square$

Remark 4.2.7. One could define chain similar to the balancing chain but where one replaces $\mathcal{X}_n$ with $\overline{\{x_n\}}$ (with reduced structure) at each step, and Proposition 4.2.6 would still be true by the same reasons.

4.3 The torsor chain

We introduce a second chain whose construction is similar to that of the balancing chain, but where at each step the exceptional divisor is replaced by the natural $\mathbb{G}_m$-torsor over it. Then we prove that this new chain also computes the iterated balanced filtration.

Construction 4.3.1. Let $\mathcal{X}$ be a normed noetherian algebraic stack with affine diagonal and with a good moduli space $\pi\colon \mathcal{X} \to X$. Let k be a field and let $x \in \mathcal{X}(k)$ be a k-point. We construct a chain $(\mathcal{Y}_n, y_n, \eta_n, v_n)_{n\in\mathbb{N}}$ inductively as follows.

We set $(\mathcal{Y}_0, y_0) = (\pi(\pi^{-1}(x)), x)$. Suppose that $(\mathcal{Y}_n, y_n)$ is defined. We define η_n, v_n and $(\mathcal{Y}_{n+1}, y_{n+1})$ in terms of $(\mathcal{Y}_n, y_n)$.

Case 1. The point y_n is in $\mathcal{Y}_n^{\max}$. In that case, we set $\eta_n = 0$, $(\mathcal{Y}_{n+1}, y_{n+1}) = (\mathcal{Y}_n, y_n)$ and $v_n\colon (\mathcal{Y}_{n+1}, y_{n+1}) \to (\operatorname{Grad}_{\mathbb{Q}}(\mathcal{Y}_n), \operatorname{gr}\eta_n)$ the "trivial grading" map.

Case 2. The point y_n is not in $\mathcal{Y}_n^{\max}$. Then y_n lifts uniquely to $\mathcal{B} := \operatorname{Bl}_{\mathcal{Y}_n^{\max}} \mathcal{Y}_n$, which is canonically endowed with a linear form on graded points coming from the exceptional divisor and the induced norm on graded points (Theorem 2.6.3 and Example 2.6.4). We let $\eta_n \in \mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{Y}_n, y_n)$ be the HN filtration of y_n in $\mathcal{B}$. Let $\mathcal{S}$ be the Θ-stratum of $\mathcal{B}$ containing y_n, and let $\mathcal{Z}$ be its centre. Let $\mathcal{E}$ be the exceptional divisor and $\mathcal{N} \to \mathcal{E}$ the natural $\mathbb{G}_{m,k}$-torsor, that is, $\mathcal{N}$ is the complement of the zero section inside the total space of the normal cone to $\mathcal{Y}_n^{\max} \to \mathcal{Y}_n$.

Lemma 4.3.2. *The "forget the grading" map $h\colon \mathcal{Z} \to \mathcal{B}$ factors through $\mathcal{E} \to \mathcal{B}$. As a consequence, the open immersion $\mathcal{Z} \to \operatorname{Grad}_{\mathbb{Q}}(\mathcal{B})$ factors through the closed immersion $\operatorname{Grad}_{\mathbb{Q}}(\mathcal{E}) \to \operatorname{Grad}_{\mathbb{Q}}(\mathcal{B})$. In particular, the induced map $\mathcal{Z} \to \operatorname{Grad}_{\mathbb{Q}}(\mathcal{E})$ is an open immersion.*

Proof. The centre $\mathcal{Z}$ carries a canonical rational $B\mathbb{G}_m$-action, which endows every quasi-coherent sheaf $\mathcal{M}$ on $\mathcal{Z}$ with a $\mathbb{Q}$-grading $\mathcal{M} = \bigoplus_{c\in\mathbb{Q}} \mathcal{M}_c$. To see this, let us assume for simplicity that the $B\mathbb{G}_m$-action is integral. Then for any map $f\colon T \to \mathcal{Z}$ there is an associated map $g\colon T \times B\mathbb{G}_m \to \mathcal{Z}$. The pullback $g^*\mathcal{M}$ is a $\mathbb{G}_m$-equivariant sheaf on T, that is, a $\mathbb{Z}$-graded sheaf on T, and the underlying sheaf is $(T \to T \times B\mathbb{G}_m)^* g^*\mathcal{M} = f^*\mathcal{M}$. Thus $f^*\mathcal{M}$ has a canonical $\mathbb{Z}$-grading for every f, and this gives the $\mathbb{Z}$-grading for $\mathcal{M}$ itself.

Let $\mathcal{L}$ be the ideal sheaf of $\mathcal{E}$. Since $\mathcal{Z}$ is the centre of a stratum, unstable with respect to the linear form $\langle -, \mathcal{L}\rangle$, we must have that the $\mathbb{Q}$-grading on $h^*\mathcal{L}$ is concentrated in degree -1. Indeed, we know that for every map $\operatorname{Spec} l \to \mathcal{Z}$, with l a field, the pullback $(\operatorname{Spec} l \times B\mathbb{G}_m \to \mathcal{Z})^* h^*\mathcal{L}$ is concentrated in degree -1. Thus by Nakayama's lemma, $(h^*\mathcal{L})_c = 0$ for all $c \in \mathbb{Q} \setminus \{-1\}$.

On the other hand, the stucture sheaf $\mathcal{O}_{\mathcal{Z}}$ is concentrated in degree 0. There is a map $\mathcal{L} \to \mathcal{O}_{\mathcal{B}}$ because $\mathcal{L}$ is an ideal sheaf. Pulling this map back along h, we get a homomorphism $h^*\mathcal{L} \to \mathcal{O}_{\mathcal{Z}}$ that must be 0 for degree reasons. Therefore h factors through $\mathcal{E} \to \mathcal{B}$. Since $\operatorname{Grad}_{\mathbb{Q}}(\mathcal{E}) = \mathcal{E} \times_{\mathcal{B}} \operatorname{Grad}_{\mathbb{Q}}(\mathcal{B})$, the lemma follows. $\square$

Let $\mathcal{M}$ be the fibre product

$$\begin{tikzcd} \mathcal{M} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{N} \arrow[d] \\ \mathcal{Z} \arrow[r] & \mathcal{E}, \end{tikzcd}$$

and let y_{n+1} be any k-point of $\mathcal{M}$ lying above $\operatorname{gr}\eta_n$. Any two choices of y_{n+1} are related by a unique $\mathbb{G}_{m,k}$-torsor automorphism of $\mathcal{M} \to \mathcal{Z}$. Now $\mathcal{M}$ has a good moduli space $\pi_{\mathcal{M}} \colon \mathcal{M} \to M$ because $\mathcal{M} \to \mathcal{Z}$ is affine and $\mathcal{Z}$ has a good moduli space. Finally, we let $\mathcal{Y}_{n+1} := \pi_{\mathcal{M}}^{-1}(\pi_{\mathcal{M}}(y_{n+1}))$ and $v_n \colon (\mathcal{Y}_{n+1}, y_{n+1}) \to (\mathcal{M}, y_{n+1}) \to (\mathcal{Z}, \operatorname{gr}\eta_n) \to (\operatorname{Grad}(\mathcal{X}_n), \operatorname{gr}\eta_n)$. We are abusively denoting η_n both the filtration of x_n in $\mathcal{X}_n$ and in $\mathcal{B}$, and by $\operatorname{gr}\eta_n$ the associated graded point in the two cases.

Definition 4.3.3. Let $\mathcal{X}$ be a normed noetherian good moduli stack with affine diagonal, let k be a field and let $x \in \mathcal{X}(k)$ be a k-point. The *torsor chain* of $(\mathcal{X}, x)$ is the chain $(\mathcal{Y}_n, y_n, \eta_n, v_n)_{n\in\mathbb{N}}$ of k-stacks from Construction 4.3.1.

Theorem 4.3.4. *Let k be a field and let $(\mathcal{X}, x)$ be a k-pointed normed noetherian good moduli stack with affine diagonal. Then the $\mathbb{Q}^\infty$-filtration associated to the torsor chain of $(\mathcal{X}, x)$ (Definitions 4.1.3 and 4.3.3) equals the iterated balanced filtration of $(\mathcal{X}, x)$ (Definition 3.5.8).*

Proof. Let $(\mathcal{X}_n, x_n, \gamma_n, u_n)_{n\in\mathbb{N}}$ be the balancing chain of $(\mathcal{X}, x)$ and let $(\mathcal{Y}_n, y_n, \eta_n, v_n)_{n\in\mathbb{N}}$ be the torsor chain. The iterated balanced filtration is well-behaved under field extension by Proposition 3.5.5, and the torsor chain also commutes with base field extension by a similar argument. Therefore, we may assume that k is algebraically closed.

By Corollary 2.1.4, $\mathcal{X}_0 = R_0/G_0$, where R_0 is an affine scheme and G_0 is the stabiliser of a k-point p_0 of R_0 which is also the unique closed G_0-orbit of R_0. Let q be the norm on graded points of $\mathcal{X}_0$. There is a closed immersion $BG_0 \to \mathcal{X}_0$ given by the point p_0, so BG_0 inherits a norm from $\mathcal{X}_0$ by pullback. On the other hand, the quotient presentation gives a representable map $\mathcal{X}_0 \to BG_0$, which then induces a new norm q' on $\mathcal{X}_0$ from the norm on BG_0. We claim that the balanced and the torsor chains do not depend on whether we endow $\mathcal{X}_0$ with q or with q'. Indeed, for any $\lambda \colon \Theta_k \to \mathcal{X}_0$ such that $\langle \lambda, \mathcal{X}_0^{\max} \rangle > 0$, we have that $\operatorname{ev}_0(\lambda)$ lies in $\mathcal{X}_0^{\max}$, and thus $\operatorname{gr}\lambda$ factors through $BG_0 \to \mathcal{X}_0$. Therefore $q(\lambda) = q'(\lambda)$. Since these are the only norms that show up in the definition of the Θ-stratification of $\mathcal{B}_0 = \operatorname{Bl}_{\mathcal{X}_0^{\max}} \mathcal{X}_0$, we have that q and q' define the same Θ-stratification on $\mathcal{B}_0$. If $\mathcal{Z}_0$ is the centre of an unstable stratum of $\mathcal{B}_0$, then for any filtration $\Theta_k \to \mathcal{Z}_0$, the composition $\Theta_k \to \mathcal{Z}_0 \to \mathcal{X}_0$ factors through $BG_0 \to \mathcal{X}_0$. Therefore q and q' induce the same norm on $\mathcal{Z}_0$ and, consequently, also on the $\mathbb{G}_m$-torsor $\mathcal{N}_0 \to \mathcal{Z}_0$ considered in the torsor chain. We may thus assume that $q = q'$ without loss of generality. The stack $\mathcal{X}_n$ can be written as $\mathcal{X}_n = R_n/G_n$, with R_n affine and G_n the stabiliser of a k-point p_n of R_n which is also the unique closed G_n-orbit; and the norm on $\mathcal{X}_n$ is induced by a norm on cocharacters of G_n. This last statement follows by considering the representable map $\mathcal{X}_n \to \mathcal{X}_0$ and the fact for $n = 0$.

We start by introducing some natural extra structure on the balancing chain. Let N the smallest natural number such that $\gamma_N = 0$. For every $n \le N$, there are line bundles $\mathcal{L}_{n,1}, \cdots, \mathcal{L}_{n,n}$ on $\mathcal{X}_n$ constructed inductively as follows. If $n < N$, then $\mathcal{L}_{n+1,i} = (\mathcal{X}_{n+1} \to \mathcal{X}_n)^*\mathcal{L}_{n,i}$ for all $i = 1, \dots, n$. The map $\mathcal{X}_{n+1} \to \mathcal{X}_n$ we are considering is the composition of $u_n \colon \mathcal{X}_{n+1} \to \operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}_n)$ and the “forget the grading” map $\operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}_n) \to \mathcal{X}_n$. Using the notation of Construction 4.2.1, we have a relatively ample line bundle $\mathcal{L}_{\mathcal{B}} = \mathcal{O}_{\mathcal{B}}(-\mathcal{E})$ on the blow-up $\mathcal{B} = \operatorname{Bl}_{\mathcal{X}_n^{\max}} \mathcal{X}_n$, and we let $\mathcal{L}_{n+1,n+1} = (\mathcal{X}_{n+1} \to \mathcal{B})^*\mathcal{L}_{\mathcal{B}}$, where the morphism considered is the composition $\mathcal{X}_{n+1} \to \mathcal{Z} \to \operatorname{Grad}_{\mathbb{Q}}(\mathcal{B}) \to \mathcal{B}$.

There is also a natural non-degenerate rational $B\mathbb{G}_m^n$-action on $\mathcal{X}_n$ for each $n \leq N$. Indeed, if this action has been constructed for $\mathcal{X}_n$, with $n < N$, we construct it for $\mathcal{X}_{n+1}$ as follows. Again, we use the notations of Case 2 in Construction 4.2.1. By Lemma 3.4.4, the rational $B\mathbb{G}_m^n$-action on $\mathcal{X}_n$ lifts canonically to a rational $B\mathbb{G}_m^n$-action on $\mathcal{B}$. Since $\mathcal{Z}$ is an open substack of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{B})$, it has a natural $B\mathbb{G}_m^n \times B\mathbb{G}_m$-action, and it is nondegenerate by the argument in the proof of Lemma 3.4.6. This action now restricts to the closed substack $\mathcal{X}_{n+1}$ of $\mathcal{Z}$.

Let T_n be a maximal torus of G_n. It follows from [26, Theorem 1.4.8] that the rational $B\mathbb{G}_m^n$-action on $\mathcal{X}_n$ is given by rational one-parameter subgroups $\beta_1, \dots, \beta_n$ of the centre $Z(G_n)$ that act trivially on R_n. Indeed, the $B\mathbb{G}_m^n$-action is given by an element $\beta \in \mathrm{Hom}(\mathbb{G}_{m,k}^n, T_n) \otimes_{\mathbb{Z}} \mathbb{Q}$ and a connected component C of the fixed point locus $R_n^{\beta,0}$ such that the induced "forget the grading" map $C/L_{G_n}(\beta) \subset \mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_n) \to \mathcal{X}_n$ is an isomorphism. Since R_n has a point fixed by G_n, it must be $L_{G_n}(\beta) = G_n$, and consequently $C = R_n$, from what we see that $(\beta_1, \dots, \beta_n) = \beta$ has the desired property.

The blow-up $\mathcal{B}_n = \mathrm{Bl}_{\mathcal{X}_n^{\max}} \mathcal{X}_n$can be written as $\mathcal{B}_n = B_n/G_n$, with B_n the blow-up of R_n along a closed subscheme $R_n^{\max}$ given by $\mathcal{X}_n^{\max}$. The centre of the locally closed Θ-stratum of $\mathcal{B}_n$ containing the lift of x_n to $\mathcal{B}_n$ is $\mathcal{Z}_n = Z_n/L_{G_n}(\beta_{n+1})$, where $\beta_{n+1} \in \Gamma^{\mathbb{Q}}(T_n)$ corresponds to $\mathrm{gr}\,\gamma_n$ and Z_n is an open subscheme of the fixed point locus $B_n^{\beta_{n+1},0}$. The group G_{n+1} is the stabiliser of a point of Z_n, so it is identified with a subgroup of $L_{G_n}(\beta_{n+1})$ containing $\beta_1, \dots, \beta_{n+1}$ in its centre. With this identifications we regard the β_i as independent of n.

Claim 4.3.5. For all $n < N$, for all $j \leq n$, we have the equalities $\langle \beta_j, \mathcal{L}_{n+1,n+1} \rangle = 0$ and $\langle \beta_{n+1}, \mathcal{L}_{n+1,n+1} \rangle = 1$. Therefore $\langle \beta_i, \mathcal{L}_{n,j} \rangle = \delta_{ij}$ for $n \leq N$ and $i \leq j \leq n$.

The first equality follows from the fact that the $\beta_1, \dots, \beta_n$ act trivially on R_n and thus also on the normal cone to $R_n^{\max}$. The second follows from the definition of γ_n as the minimiser of $\|-\|$ subject to $\langle \gamma_n, \mathcal{L}_{n+1,n+1} \rangle = 1$, and the fact that $\mathrm{gr}\,\gamma_n$ is given by β_{n+1}.

Claim 4.3.6. For all $n \leq N$ and all $0 \leq i \neq j \leq n$, we have $(\beta_i, \beta_j) = 0$.

We prove the claim by induction. Let $n < N$. By the Linear Recognition Theorem 2.6.5, a point $z \in (B_n)^{\beta_{n+1},0}$ belongs to Z_n if and only if it is semistable with respect to the shifted linear form, which in this case is

$$\ell_n = \langle -, \mathcal{L}_{\mathcal{B}_n} \rangle - \frac{1}{\|\beta_{n+1}\|^2} (\beta_{n+1}, -),$$

where $\mathcal{L}_{\mathcal{B}_n}$ is the relatively ample line bundle on $\mathcal{B}_n$. Since β_i for $i \leq n$ all fix z, we must have $\ell_n(\beta_i) \leq 0$ and $\ell_n(-\beta_i) \leq 0$, but $\ell_n(\beta_i) = \dfrac{1}{\|\beta_{n+1}\|^2}(\beta_{n+1}, \beta_i)$ and $\ell_n(-\beta_i) = -\dfrac{1}{\|\beta_{n+1}\|^2}(\beta_{n+1}, \beta_i)$, so $(\beta_{n+1}, \beta_i) = 0$. This proves the claim.

Let us denote $\mathcal{V}_n = \mathbb{A}(\mathcal{L}_{1,n}^\vee)^* \times_{\mathcal{X}_n} \cdots \times_{\mathcal{X}_n} \mathbb{A}(\mathcal{L}_{n,n}^\vee)^*$, where $\mathbb{A}(-)^*$ denotes total space minus zero section. There is a natural quotient presentation $\mathcal{V}_n = V_n/G_n$, where $V_n = \mathbb{A}(\mathcal{L}_{n,1}^\vee|_{R_n})^* \times_{R_n} \cdots \times_{R_n} \mathbb{A}(\mathcal{L}_{n,n}^\vee|_{R_n})^*$, which is a $\mathbb{G}_m^n$-torsor over R_n, and in particular carries a $\mathbb{G}_{m,k}^n$-action. Let $T_n' = \mathrm{im}(\beta_1, \dots, \beta_n) \subset Z(G_n)$. Note that this is well-defined even if the β_i are rational cocharacters and not necessarily integral.

Claim 4.3.7. The torus T_n' acts on V_n via a homomorphism $\delta_n \colon T_n' \to \mathbb{G}_{m,k}^n$. Moreover, δ_n is an isogeny.

Recall that the torus T_n' acts trivially on R_n. The $\mathbb{G}_m^n$-torsor $V_n/T_n' \to R_n/T_n' = R_n \times BT_n'$ is given by a map $r \colon R_n \times BT_n' \to B\mathbb{G}_{m,k}^n$, which is in turn given by a map $R_0 \to \underline{\mathrm{Hom}}(BT_n', B\mathbb{G}_{m,k}^n)$. We have the equality $\underline{\mathrm{Hom}}(BT_n', B\mathbb{G}_{m,k}^n) = \bigsqcup_{\alpha \in \mathrm{Hom}(T_n', \mathbb{G}_{m,k}^n)} B\mathbb{G}_m^n$

because T'_n is a split torus and by [26, Theorem 1.4.8]. Therefore, since R_n is connected, r corresponds to a pair $(R_0 \xrightarrow{o} B\mathbb{G}^n_{m,k}, \delta_n)$, where o corresponds to the $\mathbb{G}^n_m$-torsor $V_n \to R_0$ and $\delta_n \in \operatorname{Hom}(T', \mathbb{G}^n_{m,k})$. We recover r as the composition $R_0 \times BT' \xrightarrow{(o, B\delta_n)} B\mathbb{G}^n_{m,k} \times B\mathbb{G}^n_{m,k} \to B\mathbb{G}^n_{m,k}$, the last map being multiplication. The homomorphism δ_n induces a map $D(\delta_n)_{\mathbb{Q}} \colon \Gamma_{\mathbb{Q}}(\mathbb{G}^n_{m,k}) \to \Gamma_{\mathbb{Q}}(T'_n)$. If we take $\beta_1, \dots, \beta_n$ as a basis of $\Gamma_{\mathbb{Q}}(T'_n)$ and the standard basis of $\Gamma_{\mathbb{Q}}(\mathbb{G}^n_{m,k})$, then $D(\delta_n)_{\mathbb{Q}}$ is given by the matrix $\langle \beta_i, \mathcal{L}_{n,j} \rangle$, which we have shown is upper triangular with 1's in the diagonal. Therefore $D(\delta_n)_{\mathbb{Q}}$ is an isomorphism and δ_n is an isogeny.

Claim 4.3.8. Let $\overline{\mathfrak{X}}_n = R_n/(G_n/T'_n)$ and let $D_n = \ker \delta_n$. Then $V_n/(G_n/D_n) \cong \overline{\mathfrak{X}}_n$.

The group $(G_n/D_n)/(T'_n/D_n)$ acts on the stack $V_n/(T'_n/D_n)$ and there is a natural isomorphism

$$V_n/(G_n/D_n) \cong (V_n/(T'_n/D_n)) \,/\, ((G_n/D_n)/(T'_n/D_n)),$$

by [49, Remark 2.4]. Since $T'_n/D_n = \mathbb{G}^n_{m,k}$ and $Y_n \to R_n$ is a $\mathbb{G}^n_m$-torsor, we have $V_n/(T'_n/D_n) = R_n$. Noting that $(G_n/D_n)/(T'_n/D_n) = G_n/T'_n$, we get the desired isomorphism.

As a consequence of the claim, we see that the good moduli space of $\mathcal{V}_n$ is $\operatorname{Spec} k$.

Claim 4.3.9. Let $f_n \colon \mathcal{V}_n \to \mathfrak{X}_n$ be the $\mathbb{G}^n_m$-torsor map. Then $f_n^{-1}(\mathfrak{X}_n^{\max}) = \mathcal{V}_n^{\max}$ and $d(\mathfrak{X}_n) = d(\mathcal{V}_n) + n$.

Let p'_n be a k-point of V_n mapping to p_n along $\mathcal{V}_n \to \mathfrak{X}_n$. Necessarily $G_n p'_n$ is the unique closed orbit inside V_n. Let H_n be the stabiliser of p'_n. By [6, Theorem 10.4, (5) and (6)] together with the fact that the good moduli space of $\mathcal{V}_n$ is $\operatorname{Spec} k$, there is a locally closed H_n-equivariant subscheme S_n of V_n containing x'_n such that $S_n/H_n \to \mathcal{V}_n$ is an isomorphism. The map $S_n/(H_n/D_n) \to V_n/(G_n/D_n)$ must also be an isomorphism. Since S_n has a point fixed by H_n, the maximal dimension stabiliser locus is $\mathcal{V}_n^{\max} = S_n^{(H_n)_\circ}/H_n$, where $(H_n)_\circ$ is the reduced identity component of H_n (see [19, Appendix C]). Similarly, $(V_n/(G_n/D_n))^{\max} = (S_n)^{(H_n/D_n)_\circ}/(H_n/D_n)$. Since D_n acts trivially on S_n, we have $(S_n)^{(H_n/D_n)_\circ} = S_n^{(H_n)_\circ}$. This proves that, denoting $\rho_n \colon \mathcal{V}_n \to \overline{\mathfrak{X}}_n$ the obvious map, we have $\mathcal{V}_n^{\max} = \rho_n^{-1}(\mathfrak{X}_n^{\max})$. On the other hand, since R_n has a point fixed by G_n, we have $\mathfrak{X}_n^{\max} = R_n^{(G_n)_\circ}/G_n$. Since T'_n acts trivially on R_n, $\overline{\mathfrak{X}}_n^{\max} = R_n^{(G_n/T'_n)_\circ}/(G_n/T'_n) = R_n^{(G_n)_\circ}/(G_n/T'_n)$ so, denoting $q_n \colon \mathfrak{X}_n \to \overline{\mathfrak{X}}_n$, we have $q_n^{-1}(\overline{\mathfrak{X}}_n^{\max}) = \mathfrak{X}_n^{\max}$. Hence $f_n^{-1}(\mathfrak{X}_n^{\max}) = q_n^{-1}(\overline{\mathfrak{X}}_n^{\max}) = \rho_n^{-1}(\overline{\mathfrak{X}}_n^{\max}) = \mathcal{V}_n^{\max}$. For the last statement, just note that $d(\mathcal{V}_n) = d(\overline{\mathfrak{X}}_n)$ and that $d(\mathfrak{X}_n) = d(\overline{\mathfrak{X}}_n) + n$. This proves the claim.

Suppose that $n < N$. Let x'_n be a k-point of V_n mapping to x_n along $\mathcal{V}_n \to \mathfrak{X}_n$.

Claim 4.3.10. The balanced filtration of $(\mathcal{V}_n, x'_n)$ equals the balanced filtration of $(\mathfrak{X}_n, x_n)$ under the injection $\mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{V}_n, x'_n) \to \mathbb{Q}\text{-}\operatorname{Filt}(\mathfrak{X}_n, x_n)$.

We identify the balanced filtration of $(\mathfrak{X}_n, x_n)$ with $\beta_{n+1} \in \Gamma^{\mathbb{Q}}(G_n)$. Since β_{n+1} is orthogonal to the cocharacters in T'_n, by Claim 4.3.6, and since $\mathfrak{X}_n \to \overline{\mathfrak{X}}_n$ preserves the substacks of maximal stabiliser dimensions, the image $\overline{\beta}_{n+1} \in \Gamma^{\mathbb{Q}}(G_n/T'_n)$ of β_{n+1} inside G_n/T'_n is the balanced filtration of $(\overline{\mathfrak{X}}_n, x_n)$. Since $\rho_n \colon \mathcal{V}_n \to \overline{\mathfrak{X}}_n$ is a gerbe banded by the finite group D_n, $\mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{V}_n, x'_n) = \mathbb{Q}\text{-}\operatorname{Filt}(\overline{\mathfrak{X}}_n, x_n)$, and this equality identifies the balanced filtrations of $\mathcal{V}_n$ and $\overline{\mathfrak{X}}_n$ because $\rho_n^{-1}(\overline{\mathfrak{X}}_n^{\max}) = \mathcal{V}_n^{\max}$. Therefore β_{n+1} is also the balanced filtration of $(\mathcal{V}_n, x'_n)$.

Let $\mathcal{C}_n = \operatorname{Bl}_{\mathcal{V}_n^{\max}} \mathcal{V}_n = C_n/G_n$. Since $\mathcal{V}_n \to \mathfrak{X}_n$ is flat, being a $\mathbb{G}^n_m$-torsor, and since

$f_n^{-1}(\mathcal{X}_n^{\max}) = \mathcal{V}_n^{\max}$, we have that the blow-ups form a cartesian diagram.

$$\begin{tikzcd} \mathcal{C}_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{V}_n \arrow[d, "f_n"] \\ \mathcal{B}_n \arrow[r] & \mathcal{X}_n. \end{tikzcd}$$

Let $\mathcal{Z}_n'$ be the centre of the Θ-stratum of $\mathcal{C}_n$ containing the lift of x_n' to $\mathcal{C}_n$.

Claim 4.3.11. There is a natural cartesian square

$$\begin{tikzcd} \mathcal{Z}_n' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{V}_n \arrow[d, "f_n"] \\ \mathcal{Z}_n \arrow[r] & \mathcal{X}_n. \end{tikzcd}$$

We denote $\beta_{n+1} = \lambda$ for simplicity. As mentioned above, the centre $\mathcal{Z}_n$ is the semistable locus inside $\mathcal{B}_n^\lambda := B_n^{\lambda,0}/L_{G_n}(\lambda)$ for the shifted linear form

$$\langle -, \mathcal{L}\rangle - \frac{1}{\|\lambda\|^2}(\lambda, -),$$

where here $\mathcal{L} := \mathcal{L}_{\mathcal{B}_n}|_{\mathcal{B}_n^\lambda}$ is the standard relatively ample line bundle on $\mathcal{B}_n$, pulled back to $\mathcal{B}_n^\lambda$. Similarly, $\mathcal{Z}_n'$ is the semistable locus inside $\mathcal{C}_n^\lambda := C_n^{\lambda,0}/L_{G_n}(\lambda)$ for the form

$$\langle -, \mathcal{L}|_{\mathcal{C}_n^\lambda}\rangle - \frac{1}{\|\lambda\|^2}(\lambda, -),$$

since $\mathcal{L}|_{\mathcal{C}_n^\lambda}$ is the standard relatively ample line bundle on $\mathcal{C}_n$. First, note that the natural square

$$\begin{tikzcd} \mathcal{C}_n^\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{C}_n \arrow[d] \\ \mathcal{B}_n^\lambda \arrow[r] & \mathcal{B}_n \end{tikzcd}$$

is cartesian. This follows from cartesianity of the two squares

$$\begin{tikzcd} \mathcal{C}_n^\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{C}_n \arrow[d] \\ \overline{\mathcal{C}}_n^\lambda \arrow[r] & \overline{\mathcal{C}}_n \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{B}_n^\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{B}_n \arrow[d] \\ \overline{\mathcal{B}}_n^\lambda \arrow[r] & \overline{\mathcal{B}}_n, \end{tikzcd} \tag{8}$$

where $\overline{\mathcal{C}}_n = C_n/(G_n/D_n)$, $\overline{\mathcal{C}}_n^\lambda = C_n^{\lambda,0}/L_{G_n/D_n}(\lambda)$, $\overline{\mathcal{B}}_n = B_n/(G_n/T_n')$ and $\overline{\mathcal{B}}_n^\lambda = B_n^{\lambda,0}/L_{G_n/T_n'}(\lambda)$, together with the isomorphisms $\overline{\mathcal{C}}_n \cong \overline{\mathcal{B}}_n$ and $\overline{\mathcal{C}}_n^\lambda \cong \overline{\mathcal{B}}_n^\lambda$. From the compatibility of the standard relatively ample line bundles on $\mathcal{C}_n$, $\mathcal{B}_n$ and $\overline{\mathcal{C}}_n = \overline{\mathcal{B}}_n$ under pullbacks and from the shape of the shifted linear forms, it follows that $(\overline{\mathcal{C}}_n^\lambda)^{\mathrm{ss}} \times_{\overline{\mathcal{C}}_n} \mathcal{C}_n = (\mathcal{C}_n^\lambda)^{\mathrm{ss}}$ and $\overline{\mathcal{B}}_n^\lambda \times_{\overline{\mathcal{B}}_n} \mathcal{B}_n = (\mathcal{B}_n^\lambda)^{\mathrm{ss}}$, and therefore $\mathcal{Z}_n' = (\mathcal{C}_n^\lambda)^{\mathrm{ss}} = (\mathcal{B}_n^\lambda)^{\mathrm{ss}} \times_{\mathcal{B}_n} \mathcal{C}_n = \mathcal{Z}_n \times_{\mathcal{X}_n} \mathcal{V}_n$, as desired. This proves the claim.

We now show by induction that $(\mathcal{V}_n, x_n') \cong (\mathcal{Y}_n, y_n)$. For $n = 0$ there is nothing to prove. If $n < N$ and $(\mathcal{V}_n, x_n') \cong (\mathcal{Y}_n, y_n)$, then $\mathcal{Y}_{n+1}$ is constructed as follows. We take the standard relatively ample line bundle $\mathcal{L}_{\mathcal{C}_n}$ on $\mathcal{C}_n$ and let $\mathcal{M} = \mathbb{A}(\mathcal{L}_{\mathcal{C}_n}^\vee|_{\mathcal{Z}_n'})^*$. We choose a point y_{n+1} in $\mathcal{M}$ mapping to $z := \lim_{t\to 0}\lambda(t)x'_n \in \mathcal{Z}'_n$, and we let $(\mathcal{Y}_{n+1}, y_{n+1})$ be the fibre of the good moduli space of $\mathcal{N}$ containing y_{n+1}. Note that by the previous claim,

$$\mathcal{Z}'_n = \mathbb{A}(\mathcal{L}^\vee_{n,1}|_{\mathcal{Z}_n})^* \times_{\mathcal{Z}_n} \cdots \times_{\mathcal{Z}_n} \mathbb{A}(\mathcal{L}^\vee_{n,n}|_{\mathcal{Z}_n})^*,$$

so actually $\mathcal{M} = \mathbb{A}(\mathcal{L}^\vee_{n,1}|_{\mathcal{Z}_n})^* \times_{\mathcal{Z}_n} \cdots \times_{\mathcal{Z}_n} \mathbb{A}(\mathcal{L}^\vee_{n,n}|_{\mathcal{Z}_n})^* \times_{\mathcal{Z}_n} \mathbb{A}(\mathcal{L}^\vee_{\mathcal{B}_n}|_{\mathcal{Z}_n})^*$. The stack $\mathcal{X}_{n+1}$ is the fibre of the good moduli space of $\mathcal{Z}_n$ containing $\lim_{t\to 0}\lambda(t)x_n$, and from the definitions we have $\mathcal{V}_{n+1} = \mathcal{X}_{n+1} \times_{\mathcal{Z}_n} \mathcal{M}$. Since we have seen that $\mathcal{V}_{n+1}$ has for moduli space $\operatorname{Spec} k$, it must be the fibre of the good moduli space of $\mathcal{M}$. Therefore $\mathcal{V}_{n+1} \cong \mathcal{Y}_{n+1}$. By either choosing the lifts x'_{n+1} appropriately or applying a torsor automorphism of $\mathcal{Y}_n$, we can arrange that $y_{n+1} = x'_{n+1}$. We have seen that the balanced filtration of $\mathcal{V}_n$ maps to the balanced filtration of $\mathcal{X}_n$. Therefore the maps $\mathcal{Y}_n \to \mathcal{X}_n$ give a morphism of chains (the compatibility of link morphisms $\mathcal{Y}_{n+1} \to \operatorname{Grad}_{\mathbb{Q}}(\mathcal{Y}_n)$ and $\mathcal{X}_{n+1} \to \operatorname{Grad}_{\mathbb{Q}}(\mathcal{X}_n)$ naturally follows from the construction). Since $\mathcal{Y}_0 = \mathcal{X}_0$, the sequential filtration of the torsor chain equals the sequential filtration of the balancing chain, as we wanted to show. □

5 The iterated balanced filtration and convex geometry

Despite its seemingly convoluted definition in terms of repeated blow-ups and Θ-stratifications, we illustrate in this section how the iterated balanced filtration can be explicitly computed in terms of convex geometry and convex optimisation when the groups involved are diagonalisable. For an account on diagonalisable algebraic groups, we refer the reader to [43].

Suppose one is interested in the iterated balanced filtration of a geometric point $x\colon \operatorname{Spec} k \to \mathcal{X}$ in a normed good moduli stack $\mathcal{X}$. By taking the fibre of the good moduli space at x, we may assume that $\mathcal{X} \to \operatorname{Spec} k$ is the good moduli space. If y is the unique closed k-point of $\mathcal{X}$, and G is the stabiliser of y, then $\mathcal{X}$ is of the form $\mathcal{X} = \operatorname{Spec} A/G$ by Corollary 2.1.3. Since closed immersions have no effect in the iterated balanced filtration, we may assume after embedding $\operatorname{Spec} A$ in a representation of G that $\operatorname{Spec} A = V$ is a finite dimensional vector space on which G acts linearly, and the point x is given by some vector $x \in V$. From now we assume that G is diagonalisable (for example, $G = \mathbb{G}^n_{m,k}$ is a split torus). Then the structure of G-representation V is determined by a direct sum decomposition $V = \bigoplus_{\chi\in\Gamma_{\mathbb{Z}}(G)} V_\chi$, where G acts on V_χ by the character χ. The *state* of x (named after [36]) is the finite set $\Xi = \{\chi \in \Gamma_{\mathbb{Z}}(G) \mid\ p_\chi(x) \neq 0\}$, where $p_\chi\colon V \to V_\chi$ denotes the projection. Writing $V = V' \oplus V_0$ and considering the closed immersion $V'/G \cong (V' \times \{p_0(x)\})/G \to V/G$, we may assume that $0 \notin \Xi$. We may simplify the situation further by considering $\mathbb{A}^\Xi_k$ to be the product of $\#(\Xi)$ many copies of $\mathbb{A}^1_k$, endowed with the action of G via the characters in Ξ. There is a G-equivariant linear closed immersion $\mathbb{A}^\Xi \to V$ sending the point $(1,\dots,1)$ to x, so we may replace V/G and x with $\mathbb{A}^\Xi/G$ and $(1,\dots,1)$. This is the pointed stack *associated* to the state Ξ (Definition 5.2.1).

We will be able to determine the balanced filtration of x in terms of its state Ξ. To this aim, we develop a theory of polarised states (where, in addition to the set Ξ, we have the data of a character $\alpha\colon G \to \mathbb{G}_{m,k}$) purely in combinatorial terms. We introduce a notion of filtration (Definition 5.1.4) and $\mathbb{Q}^\infty$-filtration (Definition 5.1.6) for states, and an analogue of the balancing chain and of the iterated balanced filtration (Definition 5.1.20). We then define a functor (Definition 5.2.1) from a category of normed semistable polarised states to the category of pointed normed stacks over a field k and show that the functor maps the balancing chain of states to the torsor chain of stacks (Theorem 5.2.16), concluding that the iterated balanced filtration of the pointed stack coincides with that of the state

(Corollary 5.2.18). Computing the iterated balanced filtration of a state boils down to a simple convex optimisation problem.

5.1 Polarised states

We develop here a combinatorial analogue of the theory of iterated balanced filtrations.

Definition 5.1.1. A *polarised state* $\boldsymbol{\Xi}$ is a triple $\boldsymbol{\Xi} = (M, \Xi, \alpha)$ where
1. M is a finite type abelian group,
2. $\Xi \subset M \setminus \{0\}$ is a finite subset (the *state*), and
3. $\alpha \in M_{\mathbb{Q}}$ (the *polarisation*).

We denote $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$, $M^{\vee} = \mathrm{Hom}(M, \mathbb{Z})$ and $M^{\vee}_{\mathbb{Q}} = M^{\vee} \otimes_{\mathbb{Z}} \mathbb{Q}$. A *normed polarised state* is a polarised state $\boldsymbol{\Xi} = (M, \Xi, \alpha)$ together with a rational inner product on $M^{\vee}_{\mathbb{Q}}$.

Remark 5.1.2. For the theory of polarised states developed here, the rational numbers can be replaced by any subfield of $\mathbb{R}$, but this generality is not necessary for our purposes.

Let $\boldsymbol{\Xi} = (M, \Xi, \alpha)$ be a polarised state. We denote $\langle -, - \rangle \colon M^{\vee} \times M \to \mathbb{Z}$ the duality pairing. For $\lambda \in M^{\vee}_{\mathbb{Q}}$, we let $H_{\lambda} = \{\beta \in M_{\mathbb{Q}} \mid \langle \lambda, \beta \rangle \geq 0\}$ be the half-space defined by λ and $\partial H_{\lambda} = \{\beta \in M_{\mathbb{Q}} \mid \langle \lambda, \beta \rangle = 0\}$ the hyperplane defined by λ. We will need a few basic notions about convex geometry for which a sufficient source is [20]. A *cone* in $M_{\mathbb{Q}}$ is a subset of the form $H_{\lambda_1} \cap \cdots \cap H_{\lambda_n}$ for $\lambda_1, \ldots, \lambda_n \in M^{\vee}_{\mathbb{Q}}$ or, equivalently, of the form $(\mathbb{Q}_{\geq 0})\chi_1 + \cdots + (\mathbb{Q}_{\geq 0})\chi_k$ for $\chi_1, \ldots, \chi_k \in M_{\mathbb{Q}}$. We include the degenerate cases $\{0\}$ and $M^{\vee}_{\mathbb{Q}}$. If C is a cone, a *face* of C is a subset of the form $C \cap \partial H_{\lambda}$, where $\lambda \in M^{\vee}_{\mathbb{Q}}$ is such that $C \subset H_{\lambda}$.

Definition 5.1.3. We say that the polarised state $\boldsymbol{\Xi}$ is *semistable* if α is in the cone $\mathrm{cone}\,(\Xi)$ generated by Ξ inside $M_{\mathbb{Q}}$. We say that $\boldsymbol{\Xi}$ is *polystable* if it is semistable and the smallest face of $\mathrm{cone}\,(\Xi)$ containing α is $\mathrm{cone}\,(\Xi)$ itself (that is, α is in the relative interior of $\mathrm{cone}\,(\Xi)$).

We are abusing notation by denoting the image of Ξ inside $M_{\mathbb{Q}}$ also by Ξ. By $\mathrm{cone}\,(\Xi)$ we will always mean a subset of $M_{\mathbb{Q}}$.

Definition 5.1.4. Suppose that $\boldsymbol{\Xi}$ is semistable. The set of *rational filtrations* (or $\mathbb{Q}$-*filtrations*) of $\boldsymbol{\Xi}$ is

$$\mathbb{Q}\text{-}\mathrm{Filt}(\boldsymbol{\Xi}) \coloneqq \{\lambda \in N_{\mathbb{Q}} \mid \langle \lambda, \Xi \rangle \geq 0 \text{ and } \langle \lambda, \alpha \rangle = 0\}.$$

We are using the notation $\langle \lambda, \Xi \rangle \geq 0$ to mean that $\langle \lambda, \chi \rangle \geq 0$ holds for all $\chi \in \Xi$.

Definition 5.1.5. Suppose that $\boldsymbol{\Xi}$ is semistable and let $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\boldsymbol{\Xi})$. The *associated graded state* $\mathrm{Grad}_{\lambda}(\boldsymbol{\Xi})$ is the semistable polarised state $\mathrm{Grad}_{\lambda}(\boldsymbol{\Xi}) = (M, \Xi_{\lambda,0}, \alpha)$, where $\Xi_{\lambda,0} \coloneqq \Xi \cap \partial H_{\lambda}$.

Proof that $\mathrm{Grad}_{\lambda}(\boldsymbol{\Xi})$ *is semistable.* Since $\Xi \subset H_{\lambda}$, we have the equality $\mathrm{cone}(\Xi) \cap \partial H_{\lambda} = \mathrm{cone}(\Xi \cap \partial H_{\lambda})$. Therefore $\alpha \in \mathrm{cone}(\Xi_{\lambda,0})$. □

Definition 5.1.6 (Sequential filtrations of a state)**.** Let $\boldsymbol{\Xi} = (M, \Xi, \alpha)$ be a semistable polarised state. The set $\mathbb{Q}^{\infty}\text{-}\mathrm{Filt}(\boldsymbol{\Xi})$ of *sequential filtrations* (or $\mathbb{Q}^{\infty}$-*filtrations*) of $\boldsymbol{\Xi}$ is the set of those sequences $(\lambda_n)_{n \in \mathbb{N}}$ in $M^{\vee}_{\mathbb{Q}}$ satisfying
1. $\lambda_n = 0$ for $n >> 0$, and
2. $\lambda_0 \in \mathbb{Q}\text{-}\mathrm{Filt}(\boldsymbol{\Xi})$ and, for all $n \in \mathbb{N}_{>0}$, we have

$$\lambda_n \in \mathbb{Q}\text{-}\mathrm{Filt}\left(\mathrm{Grad}_{\lambda_{n-1}}\left(\cdots \mathrm{Grad}_{\lambda_1}\left(\mathrm{Grad}_{\lambda_0}(\boldsymbol{\Xi})\right)\cdots\right)\right).$$

Definition 5.1.7 (Morphism between states)**.** A *morphism* $\varphi\colon \mathbf{\Xi}_1 = (M_1, \Xi_1, \alpha_1) \to \mathbf{\Xi}_2 = (M_2, \Xi_2, \alpha_2)$ between semistable polarised states is a surjective homomorphism $\varphi\colon M_2 \to M_1$ such that

1. for all $\chi \in \Xi_2$, either $\varphi(\chi) \in \Xi_1$ or $\varphi(\chi) = 0$; and
2. $\alpha_2 \in \operatorname{cone}(\Xi_2 \cap \ker\varphi)$.

If $\mathbf{\Xi}_1$ and $\mathbf{\Xi}_2$ are normed, we say that φ is a morphism between normed semistable polarised states if the inner product on $(M_1)^\vee_{\mathbb{Q}}$ is the restriction of that on $(M_2)^\vee_{\mathbb{Q}}$ along the inclusion $(M_1)^\vee_{\mathbb{Q}} \hookrightarrow (M_2)^\vee_{\mathbb{Q}}$ defined by φ.

With the obvious composition and identity, semistable (normed) polarised states form a category.

Lemma 5.1.8. *Let $\varphi\colon \mathbf{\Xi}_1 = (M_1, \Xi_1, \alpha_1) \to \mathbf{\Xi}_2 = (M_2, \Xi_2, \alpha_2)$ be a morphism between semistable polarised states. Then the injection $\varphi^\vee_{\mathbb{Q}}\colon (M_1)^\vee_{\mathbb{Q}} \to (M_2)^\vee_{\mathbb{Q}}$ induces a map $\mathbb{Q}\text{-}\mathrm{Filt}(\mathbf{\Xi}_1) \to \mathbb{Q}\text{-}\mathrm{Filt}(\mathbf{\Xi}_2)$ between sets of rational filtrations.*

Proof. If $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathbf{\Xi}_1)$ and $\chi \in \Xi_2$, then $\langle \varphi^\vee_{\mathbb{Q}}(\lambda), \chi\rangle = \langle \lambda, \varphi(\chi)\rangle \geq 0$, since either $\varphi(\chi) = 0$ or $\varphi(\chi) \in \Xi_1$. Likewise, $\langle \varphi^\vee_{\mathbb{Q}}(\lambda), \alpha_2\rangle = \langle \lambda, \varphi(\alpha_2)\rangle = 0$, since $\varphi(\alpha_2) = 0$ in $(M_1)_{\mathbb{Q}}$. Therefore $\varphi^\vee_{\mathbb{Q}}(\lambda) \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathbf{\Xi}_2)$. □

In the situtation of Lemma 5.1.8, we will use the simpler notation $\varphi(\lambda) := \varphi^\vee_{\mathbb{Q}}(\lambda)$.

Proposition 5.1.9. *Let $\varphi\colon \mathbf{\Xi}_1 = (M_1, \Xi_1, \alpha_1) \to \mathbf{\Xi}_2 = (M_2, \Xi_2, \alpha_2)$ be a morphism between semistable polarised states and let $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathbf{\Xi}_1)$. The homomorphism $\varphi\colon M_2 \to M_1$ induces a map $\mathrm{Grad}_\lambda(\varphi)\colon \mathrm{Grad}_\lambda(\mathbf{\Xi}_1) \to \mathrm{Grad}_{\varphi(\lambda)}(\mathbf{\Xi}_2)$ between associated graded states.*

Proof. The first condition to be checked is that for all $\chi \in (\Xi_2)_{\lambda,0}$, we have $\varphi(\chi) = 0$ or $\varphi(\chi) \in (\Xi_1)_{\lambda,0}$. If $\varphi(\chi)$ is not 0, then $\varphi(\chi) \in \Xi_1$, and since $\langle \lambda, \varphi(\chi)\rangle = \langle \varphi(\lambda), \chi\rangle = 0$, we have indeed $\varphi(\chi) \in (\Xi_1)_{\lambda,0}$. For the second condition, we have

$$\begin{aligned}\alpha_2 \in \operatorname{cone}(\Xi_2 \cap \ker\varphi) \cap \partial H_{\varphi(\lambda)} &= \operatorname{cone}\left(\Xi_2 \cap \ker\varphi \cap \partial H_{\varphi(\lambda)}\right) \\ &= \operatorname{cone}\left((\Xi_2)_{\varphi(\lambda),0} \cap \ker\varphi\right),\end{aligned}$$

as desired. □

Definition 5.1.10 (Chain of states)**.** A *chain of normed semistable polarised states* is a sequence $(\mathbf{\Xi}_n, \lambda_n, u_n)_{n\in\mathbb{N}}$ where

1. for each $n \in \mathbb{N}$, $\mathbf{\Xi}_n$ is a normed semistable polarised state;
2. $\lambda_n \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathbf{\Xi}_n)$ for each $n \in \mathbb{N}$;
3. $u_n\colon \mathbf{\Xi}_{n+1} \to \mathrm{Grad}_{\lambda_n}(\mathbf{\Xi}_n)$ is a morphism of normed semistable polarised states; and
4. for all $n >> 0$, $\lambda_n = 0$ and u_n is an isomorphism (boundedness).

Suppose that $(\mathbf{\Xi}_n, \lambda_n, u_n)_{n\in\mathbb{N}}$ is a chain of normed semistable polarised states. We denote $c_n\colon \mathbf{\Xi}_{n+1} \to \mathrm{Grad}_{\lambda_n}(\cdots \mathrm{Grad}_{\lambda_1}(\mathrm{Grad}_{\lambda_0}(\mathbf{\Xi}_0))\cdots)$ the map defined by

$$c_n = u_n \circ \mathrm{Grad}_{\lambda_n}(u_{n-1}) \circ \cdots \circ \mathrm{Grad}_{\lambda_n}(\cdots \mathrm{Grad}_{\lambda_1}(\mathrm{Grad}_{\lambda_0}(u_0))\cdots).$$

We are abusively denoting by λ_n the image of λ_n along the relevant maps between sets of filtrations, but note as well that all these maps are injective. The sequence $(c_{n-1}(\lambda_n))_{n\in\mathbb{N}}$ (where $c_{-1}(\lambda_0) := \lambda_0$) is a $\mathbb{Q}^\infty$-filtration of $\mathbf{\Xi}_0$.

Definition 5.1.11 (Associated sequential filtration)**.** The *sequential filtration associated to the chain* $(\mathbf{\Xi}_n, \lambda_n, u_n)_{n\in\mathbb{N}}$ is $(c_{n-1}(\lambda_n))_{n\in\mathbb{N}} \in \mathbb{Q}^\infty\text{-}\mathrm{Filt}(\mathbf{\Xi}_0)$. We will simply denote it $(\lambda_n)_{n\in\mathbb{N}}$.

Definition 5.1.12 (Slice of a state)**.** Let $\mathbf{\Xi} = (M, \Xi, \alpha)$ be a semistable polarised state. Let F be the smallest face of $\operatorname{cone}(\Xi)$ containing α, let K be the subgroup of M generated by $F \cap \Xi$, let $M' = M/K$, let $q\colon M \to M'$ be the quotient map, and set $\Xi' = q(\Xi \setminus F)$. We define the *slice* of $\mathbf{\Xi}$ to be the state $\mathbf{\Xi}' = (M', \Xi', 0)$ together with the morphism $q\colon \mathbf{\Xi}' \to \mathbf{\Xi}$ given by $q\colon M \to M'$. If $\mathbf{\Xi}$ is normed, we regard $\mathbf{\Xi}'$ as a normed polarised state, where $(M')^\vee_\mathbb{Q}$ inherits an inner product along the inclusion $q^\vee_\mathbb{Q}\colon (M')^\vee_\mathbb{Q} \to M^\vee_\mathbb{Q}$.

Note that $\mathbf{\Xi}'$ is semistable, since 0 belongs to any cone. We are again abusing notation: the expression $F \cap \Xi$ denotes the subset of Ξ consisting of the elements χ whose image in $M_\mathbb{Q}$ is contained in F.

Proposition 5.1.13. *Let $\mathbf{\Xi} = (M, \Xi, \alpha)$ be a semistable polarised state and let $q\colon \mathbf{\Xi}' \to \mathbf{\Xi}$ be its slice. Then the map $\mathbb{Q}\text{-}\mathrm{Filt}(\mathbf{\Xi}') \to \mathbb{Q}\text{-}\mathrm{Filt}(\mathbf{\Xi})$ induced by q is a bijection.*

Proof. We use notation from Definition 5.1.12. Let $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathbf{\Xi})$. Then $\operatorname{cone}(\Xi) \cap \partial H_\lambda$ is a face of $\operatorname{cone}(\Xi)$ containing α, so $F \subset \operatorname{cone}(\Xi) \cap \partial H_\lambda$. In particular, $\langle \lambda, \chi \rangle = 0$ for all $\chi \in F \cap \Xi$. Therefore λ has a preimage λ' along $q^\vee_\mathbb{Q}\colon (M')^\vee_\mathbb{Q} \hookrightarrow M^\vee_\mathbb{Q}$, and $\langle \lambda', q(\chi) \rangle = \langle \lambda, \chi \rangle \geq 0$ for all $\chi \in \Xi \setminus F$, so $\lambda' \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathbf{\Xi}')$. Hence the map $\mathbb{Q}\text{-}\mathrm{Filt}(\mathbf{\Xi}') \to \mathbb{Q}\text{-}\mathrm{Filt}(\mathbf{\Xi})$ is surjective. Since it is also injective, it is a bijection. □

Definition 5.1.14. Let $\mathbf{\Xi} = (M, \Xi, \alpha)$ be a semistable polarised state and let $\mathbf{\Xi}' = (M', \Xi', 0)$ be its slice. Let $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\Xi) = \mathbb{Q}\text{-}\mathrm{Filt}(\Xi')$. We define the *complementedness* $\langle \lambda, \mathfrak{l} \rangle \in \mathbb{Q}_{\geq 0} \cup \{\infty\}$ of λ to be

$$\langle \lambda, \mathfrak{l} \rangle = \inf_{\chi \in \Xi'} \langle \lambda, \chi \rangle.$$

Proposition 5.1.15. *Let $\mathbf{\Xi} = (M, \Xi, \alpha)$ be a semistable normed polarised state. There is a unique element $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathbf{\Xi})$ such that*

1. *$\langle \lambda, \mathfrak{l} \rangle \geq 1$, and*
2. *for all $\gamma \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathbf{\Xi})$ such that $\langle \gamma, \mathfrak{l} \rangle \geq 1$, we have $\|\lambda\| \leq \|\gamma\|$.*

Proof. We may assume $\mathbf{\Xi} = \mathbf{\Xi}'$. If $\Xi = \varnothing$, then $\lambda = 0$ is the unique rational filtration satisfying the conditions. Otherwise, $P = \{\gamma \in M^\vee_\mathbb{R} \mid \langle \gamma, \chi \rangle \geq 1,\ \forall \chi \in \Xi\}$ is a nonempty closed convex set (actually, an intersection of translated half-spaces) inside $M^\vee_\mathbb{R} = M^\vee_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{R}$, so there is a unique element $\lambda \in P$ minimising the norm $\|-\|$. To see that $\lambda \in M^\vee_\mathbb{Q}$, note that λ lies in the relative interior of a face F of P. The affine space F generates is of the form $V_\mathbb{R} + v$, where V is a vector subspace of $M^\vee_\mathbb{Q}$, $V_\mathbb{R} = V \otimes_\mathbb{Q} \mathbb{R}$ and $v \in M^\vee_\mathbb{Q}$. Since λ is also the closest point to the origin in $V_\mathbb{R} + v$, we must have that $\lambda = v - p(v)$, where p is the orthogonal projection $p\colon M^\vee_\mathbb{R} \to V_\mathbb{R}$. Since the inner product on $M^\vee_\mathbb{Q}$ is rational, p is defined over $\mathbb{Q}$, and thus λ is rational. □

Definition 5.1.16 (Balanced filtration of a state)**.** Let $\mathbf{\Xi}$ be a semistable normed polarised state. The *balanced filtration* $\lambda_\mathrm{b}(\mathbf{\Xi})$ of $\mathbf{\Xi}$ is the unique $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathbf{\Xi})$ satisfying the conditions of Proposition 5.1.15.

Remark 5.1.17. For a normed semistable polarised state $\mathbf{\Xi} = (M, \Xi, \alpha)$, we have $\Xi' = \varnothing$ if and only if $\mathbf{\Xi}$ is polystable, if and only if the balanced filtration $\lambda_\mathrm{b}(\mathbf{\Xi}) = 0$.

Definition 5.1.18. Let $\mathbf{\Xi}$ be a normed semistable polarised state, and let $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathbf{\Xi})$ be such that $\langle \lambda, \mathfrak{l} \rangle \geq 1$. We define a normed polarised state $\Lambda_\lambda(\mathbf{\Xi})$ as follows. Let $\mathbf{\Xi}' = (M', \Xi', 0)$ be the slice of $\mathbf{\Xi}$, let

$$(\Xi')_{\lambda,1} = \{\chi \in M' \mid \langle \lambda, \chi \rangle = 1\} \subset M',$$

and let $\lambda^\vee$ be the unique element of $(M')_{\mathbb{Q}}$ satisfying $\langle \gamma, \lambda^\vee \rangle = (\gamma, \lambda)$ for all $\gamma \in (M')^\vee_{\mathbb{Q}}$, where $(-,-)$ denotes the inner product on $(M')^\vee_{\mathbb{Q}}$. Finally, we set

$$\Lambda_\lambda(\Xi) = (M', (\Xi')_{\lambda,1}, \lambda^\vee).$$

The following theorem is an analogue for states of Theorem 2.6.5 in the case of algebraic stacks and [24, Theorem 4.9] in the case of artinian lattices.

Theorem 5.1.19 (Recognition of the balanced filtration for states). *Let $\boldsymbol{\Xi}$ be a normed semistable polarised state. Then the balanced filtration of $\boldsymbol{\Xi}$ is the unique filtration $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\boldsymbol{\Xi})$ satisfying*

1. *$\langle \lambda, \mathfrak{l} \rangle \geq 1$ and*
2. *the polarised state $\Lambda_\lambda(\boldsymbol{\Xi})$ is semistable.*

Proof. We use the notation of the proof of Proposition 5.1.15. By definition, the balanced filtration λ is the unique element in P minimising the function $1/2\|-\|^2$, whose diferential at a point $\gamma \in (M')^\vee_{\mathbb{R}}$ is precisely $\gamma^\vee \in (M')_{\mathbb{R}}$. By the Karush-Kuhn-Tucker conditions, for $\gamma \in P$ we have $\gamma = \lambda$ if and only if there are numbers $u_\chi \geq 0$ for each $\chi \in \Xi'$ such that $\gamma^\vee = \sum_{\chi \in \Xi'} u_\chi \chi$ and $u_w = 0$ if $\langle \gamma, \chi \rangle > 1$. But this precisely means that $\gamma^\vee \in \mathrm{cone}\left((\Xi')_{\gamma,1}\right)$, i.e. that $\Lambda_\gamma(\boldsymbol{\Xi})$ is semistable. □

If λ is the balanced filtration of $\boldsymbol{\Xi}$, then $\mathrm{id}_{M'}$ defines a morphism $\Lambda_\lambda(\boldsymbol{\Xi}) \to \mathrm{Grad}_\lambda(\boldsymbol{\Xi}')$ (note that $(\Xi')_{\lambda,0} = \varnothing$), and the quotient map $M \to M'$ defines a morphism $\mathrm{Grad}_\lambda(\boldsymbol{\Xi}') \to \mathrm{Grad}_\lambda(\boldsymbol{\Xi})$. Therefore $\Lambda_\lambda(\boldsymbol{\Xi})$ is equipped with a canonical map $\Lambda_\lambda(\boldsymbol{\Xi}) \to \mathrm{Grad}_\lambda(\boldsymbol{\Xi})$.

Definition 5.1.20 (balancing chain of a state and the iterated balanced filtration). Let $\boldsymbol{\Xi}$ be a normed semistable polarised state. The *balancing chain* of $\boldsymbol{\Xi}$ is the chain $(\boldsymbol{\Xi}_n, \lambda_n, u_n)_{n \in \mathbb{N}}$ of normed semistable polarised states defined inductively as follows:

1. $\boldsymbol{\Xi}_0 \coloneqq \boldsymbol{\Xi}'$;
2. for every $n \in \mathbb{N}$, λ_n is the balanced filtration of $\boldsymbol{\Xi}_n$, $\boldsymbol{\Xi}_{n+1} \coloneqq (\Lambda_{\lambda_n}(\boldsymbol{\Xi}_n))'$, and $u_n \colon \boldsymbol{\Xi}_{n+1} \to \mathrm{Grad}_{\lambda_n}(\boldsymbol{\Xi}_n)$ is the composition of $(\Lambda_{\lambda_n}(\boldsymbol{\Xi}_n))' \to \Lambda_{\lambda_n}(\boldsymbol{\Xi}_n)$ and the canonical map $\Lambda_{\lambda_n}(\boldsymbol{\Xi}_n) \to \mathrm{Grad}_{\lambda_n}(\boldsymbol{\Xi}_n)$.

The *iterated balanced filtration* of $\boldsymbol{\Xi}$ is the sequential filtration $\lambda_{\mathrm{ib}}(\boldsymbol{\Xi}) \in \mathbb{Q}^\infty\text{-}\mathrm{Filt}(\boldsymbol{\Xi})$ associated to the balancing chain of $\boldsymbol{\Xi}$.

Proof that the balancing chain is well-defined. We need to check the boundedness condition in Definition 5.1.10. We observe that, denoting $\boldsymbol{\Xi} = (M, \Xi, \alpha)$, if $\alpha \neq 0$ and $\boldsymbol{\Xi}$ is not polystable, then $\#(\Xi') < \#(\Xi)$. Therefore, eventually $\lambda_n = 0$, and hence $\Xi'_n = \varnothing$ by Remark 5.1.17, from where the chain stabilises. □

Remark 5.1.21 (Torsion). The torsion of M does not affect the iterated balanced filtration, but we allow for any finite-type abelian group M in the definition of polarised state in order to make the correspondence with stacks cleaner. Even if we are only interested in the case of a torus action, other diagonalisable groups that are not tori will show up as stabilisers of points, so we need to consider that case too.

5.2 From states to good moduli stacks

We now define a functor from states to pointed stacks, and prove that both theories of iterated balanced filtrations coincide. We fix a field k for the rest of this section.

Definition 5.2.1 (Stack associated to a state)**.** Let $\mathbf{\Xi} = (M, \Xi, \alpha)$ be a (normed) polarised state. We define a k-pointed (normed) good moduli stack $(\mathcal{X}_{\mathbf{\Xi}}, x_{\mathbf{\Xi}})$ over k and a line bundle $\mathcal{L}_{\Xi}$ on $\mathcal{X}_{\mathbf{\Xi}}$ as follows.

First, we denote $G_{\mathbf{\Xi}} = D(M)$ the diagonalisable algebraic group over k Cartier dual to M. The group of characters $\Gamma_{\mathbb{Z}}(G_{\mathbf{\Xi}})$ of $G_{\mathbf{\Xi}}$ is identified with M. Let $\mathbb{A}_k^{\Xi}$ be the product of $\#(\Xi)$ many copies of $\mathbb{A}_k^1$. Consider the action of $G_{\mathbf{\Xi}}$ on $\mathbb{A}_k^{\Xi}$ through the characters in Ξ, that is, $g \cdot (x_\chi)_{\chi \in \Xi} = (\chi(g) x_\chi)_{\chi \in \Xi}$ for g in $G_{\mathbf{\Xi}}$ and $(x_\chi)_{\chi \in \Xi}$ in $\mathbb{A}_k^{\Xi}$. Let $\mathcal{X}_{\mathbf{\Xi}} := \mathbb{A}_k^{\Xi}/G_{\mathbf{\Xi}}$ and $x_{\mathbf{\Xi}} = (1, \cdots, 1) \in \mathbb{A}_k^{\Xi}(k)$. We also denote $x_{\mathbf{\Xi}}$ the composition $\operatorname{Spec} k \xrightarrow{x_{\mathbf{\Xi}}} \mathbb{A}_k^{\Xi} \to \mathcal{X}_{\mathbf{\Xi}}$. We identify $M_{\mathbb{Q}} = \operatorname{Pic}(BG_{\mathbf{\Xi}}) \otimes_{\mathbb{Z}} \mathbb{Q}$, and thus the polarisation α defines a rational line bundle $\mathcal{L}_{\mathbf{\Xi}} := \mathcal{O}_{\mathcal{X}_{\mathbf{\Xi}}}(\alpha) = (\mathcal{X}_{\mathbf{\Xi}} \to BG_{\mathbf{\Xi}})^* \alpha$ on $\mathcal{X}_{\mathbf{\Xi}}$. We denote $\ell_{\mathbf{\Xi}}$ the linear form on $\mathcal{X}_{\mathbf{\Xi}}$ associated to $\mathcal{L}_{\mathbf{\Xi}}$. If $\mathbf{\Xi}$ is normed, the data of the inner product on $M_{\mathbb{Q}}^{\vee}$ is equivalent to that of a norm on cocharacters of $G_{\mathbf{\Xi}}$. It thus defines a norm on graded points of $BG_{\mathbf{\Xi}}$ and also a norm on $\mathcal{X}_{\mathbf{\Xi}}$ by pullback along $\mathcal{X}_{\mathbf{\Xi}} \to BG_{\mathbf{\Xi}}$.

We fix a polarised state $\mathbf{\Xi} = (M, \Xi, \alpha)$, and denote $\mathbf{\Xi}^\circ = (M, \Xi, 0)$ the associated "unpolarised" state. We abbreviate $(\mathcal{X}, x) = (\mathcal{X}_{\mathbf{\Xi}}, x_{\mathbf{\Xi}})$ and $G = G_{\mathbf{\Xi}}$.

Proposition 5.2.2. *There is a canonical bijection*

$$\mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{X}, x) = \mathbb{Q}\text{-}\operatorname{Filt}(\mathbf{\Xi}^\circ).$$

Proof. By Remark 2.2.17, the set $\mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{X}, x)$ of rational filtrations of x in $\mathcal{X}$ is identified with the set of those rational cocharacters $\lambda \in M_{\mathbb{Q}}^{\vee} = \Gamma^{\mathbb{Q}}(G)$ such that $\lim_{t \to 0} \lambda(t) x$ exists in $\mathbb{A}_k^{\Xi}$. Since $\lambda(t)x = (t^{\langle \lambda, \chi \rangle})_{\chi \in \Xi}$, the limit exists precisely when $\langle \lambda, \chi \rangle \geq 0$ for all $\chi \in \Xi$. □

For $y = (y_\chi)_{\chi \in \Xi} \in \mathbb{A}^{\Xi}(k)$, we define the *state* of y to be the set $\Xi_y = \{\chi \in \Xi \mid y_\chi \neq 0\}$. If we let $\mathbf{\Xi}_y = (M, \Xi_y, \alpha)$, then we have a closed immersion $f \colon \mathcal{X}_{\mathbf{\Xi}_y} \to \mathcal{X}$ given by

$$f\left((z_\chi)_{\chi \in \Xi_y}\right)_\chi = \begin{cases} z_\chi y_\chi, & \chi \in \Xi_y \\ 0, & \text{else,} \end{cases}$$

and f maps $x_{\mathbf{\Xi}_y} = (1, \cdots, 1)$ to y.

Proposition 5.2.3. *Let $\lambda \in \mathbb{Q}\text{-}\operatorname{Filt}(\mathcal{X}, x)$. Then the state of $y = \lim_{t \to 0} \lambda(t) x$ is $\Xi_y = \Xi \cap \partial H_\lambda$.*

Proof. Again, $\lambda(t)x = (t^{\langle \lambda, \chi \rangle})_{\chi \in \Xi}$ and thus

$$y_\chi = \begin{cases} 0, & \langle \lambda, \chi \rangle > 0; \\ 1, & \langle \lambda, \chi \rangle = 0; \end{cases}$$

which implies the claim. □

The state also determines the stabiliser of x.

Proposition 5.2.4. *Let K be the subgroup of M generated by the elements of Ξ, and let $C = M/K$. Let $S = D(C)$ be the Cartier dual of C, which is equipped with an injection $S \to G$. Then S is the stabiliser group of x.*

Proof. The group G acts on $\mathbb{A}^{\Xi}$ via the characters $\chi \colon G \to \mathbb{G}_{m,k}$, $\chi \in \Xi$, each of which can be seen as the Cartier dual of the map $\mathbb{Z} \to \Gamma_{\mathbb{Z}}(G) \colon 1 \mapsto \chi$. If $\Xi = \{\chi_1, \ldots, \chi_n\}$, then the stabiliser of x is the kernel of $(\chi_1, \ldots, \chi_n) \colon G \to \mathbb{G}_m^n$, and its group of characters is, by Cartier duality, the cokernel of the map $\mathbb{Z}^n \to M \colon e_i \mapsto \chi_i$, which is C. □

Proposition 5.2.5. *The point x is semistable in $\mathcal{X}$ for the linear form $\ell_{\boldsymbol{\Xi}}$ if and only if the polarised state $\boldsymbol{\Xi}$ is semistable.*

Proof. It follows from Proposition 5.2.2, together with the fact that $\langle \lambda, \mathcal{O}_{\mathcal{X}}(\alpha)\rangle = -\langle \lambda, \alpha\rangle$ with our sign conventions (Remark 2.4.4), that x is semistable if and only if for all $\lambda \in M_{\mathbb{Q}}^{\vee}$ such that $\langle \lambda, \Xi\rangle \geq 0$ (that is, $\Xi \subset H_\lambda$) we have $\langle \lambda, \alpha\rangle \geq 0$ (that is, $\alpha \in H_\lambda$). The result now follows from this and the fact that cone (Ξ) is the intersection of those half-spaces H_λ such that $\Xi \subset H_\lambda$ (where we are again abusing notation by identifying Ξ with its image inside $M_{\mathbb{Q}}$). □

Proposition 5.2.6. *Suppose that x is semistable (equivalently, $\boldsymbol{\Xi}$ is semistable) and let $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$. Then the limit $y = \lim_{t\to 0} \lambda(t)x$ is semistable if and only if $\langle \lambda, \alpha\rangle = 0$.*

Proof. We have equalities $\partial H_\lambda \cap \mathrm{cone}\,(\Xi) = \mathrm{cone}\,(\partial H_\lambda \cap \Xi) = \mathrm{cone}\,(\Xi_y)$, the second of which follows from Proposition 5.2.3. The result follows from Proposition 5.2.5 applied to $\mathcal{X}_{\boldsymbol{\Xi}_y}$, which is a closed substack of $\mathcal{X}$ containing y. □

Note that $\mathcal{X}$ admits a good moduli space, and $\ell_{\boldsymbol{\Xi}}$ is trivially $\mathrm{id}_{\mathcal{X}}$-positive (Definition 2.6.1). Therefore Theorem 2.6.3 implies that the semistable locus $\mathcal{X}^{\mathrm{ss}} = \left(\mathbb{A}_k^{\Xi}\right)^{\mathrm{ss}}/G$ with respect to $\ell_{\boldsymbol{\Xi}}$ has a good moduli space $\pi\colon \mathcal{X}^{\mathrm{ss}} \to X$.

Proposition 5.2.7. *There is a canonical bijection $\mathbb{Q}\text{-}\mathrm{Filt}(\boldsymbol{\Xi}) \cong \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}^{ss}, x)$.*

Proof. $\mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}^{\mathrm{ss}}, x)$ is the subset of those $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}, x)$ such that $\lim_{t\to 0}\lambda(t)x$ is semistable, while $\mathbb{Q}\text{-}\mathrm{Filt}(\boldsymbol{\Xi})$ is the subset of those $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\boldsymbol{\Xi}^\circ)$ such that $\langle \lambda, \alpha\rangle = 0$. Thus the result follows from Propositions 5.2.2 and 5.2.6. □

We can also characterise polystability in terms of the state:

Proposition 5.2.8. *The point x is polystable inside $\mathcal{X}^{ss}$ if and only if the state $\boldsymbol{\Xi}$ is polystable.*

Proof. The point x being polystable is equivalent to it being semistable and, for all $\lambda \in M_{\mathbb{Q}}^{\vee}$ such that $\langle \lambda, \Xi\rangle \geq 0$ and $y = \lim_{t\to 0}\lambda(t)x$ is semistable, having $x = y$. By Propositions 5.2.3, 5.2.5 and 5.2.6, this condition is equivalent to having that $\boldsymbol{\Xi}$ is semistable and, for all $\lambda \in M_{\mathbb{Q}}^{\vee}$, having that the conditions $\Xi \subset H_\lambda$ and $\langle \lambda, \alpha\rangle = 0$ imply that $\Xi \subset \partial H_\lambda$. This means that the smallest face of cone (Ξ) containing x is cone (Ξ) itself, that is, that $\boldsymbol{\Xi}$ is polystable. □

From now, we do not fix a particular polarised state $\boldsymbol{\Xi}$ and we stop abbreviating $(\mathcal{X}, x) = (\mathcal{X}_{\boldsymbol{\Xi}}, x_{\boldsymbol{\Xi}})$ and $G = G_{\boldsymbol{\Xi}}$.

Definition 5.2.9. Let $\varphi\colon \boldsymbol{\Xi}_1 = (M_1, \Xi_1, \alpha_1) \to \boldsymbol{\Xi}_2 = (M_2, \Xi_2, \alpha_2)$ be a morphism between semistable polarised states. We define pointed morphisms $f_\varphi\colon (\mathcal{X}_{\boldsymbol{\Xi}_1}, x_{\boldsymbol{\Xi}_1}) \to (\mathcal{X}_{\boldsymbol{\Xi}_2}, x_{\boldsymbol{\Xi}_2})$ and $f_\varphi^{\mathrm{ss}}\colon (\mathcal{X}_{\boldsymbol{\Xi}_1}^{\mathrm{ss}}, x_{\boldsymbol{\Xi}_1}) \to (\mathcal{X}_{\boldsymbol{\Xi}_2}^{\mathrm{ss}}, x_{\boldsymbol{\Xi}_2})$ as follows.

First, the homomorphism $\varphi\colon M_2 \to M_1$ defines a Cartier dual group homomorphism $D(\varphi)\colon G_{\boldsymbol{\Xi}_1} \to G_{\boldsymbol{\Xi}_2}$. The k-algebra homomorphism

$$k[t_\chi;\ \chi \in \Xi_2] \to k[t_\psi;\ \psi \in \Xi_1]\colon \chi \mapsto \begin{cases} t_{\varphi(\chi)}, & \varphi(\chi) \in \Xi_1, \\ 1, & \varphi(\chi) = 0. \end{cases}$$

defines, after taking Spec, a $D(\varphi)$-equivariant map

$$\mathbb{A}_k^{\Xi_1} \to \mathbb{A}_k^{\Xi_2}\colon (y_\psi)_{\psi\in\Xi_1} \mapsto \left(\begin{cases} y_{\varphi(\chi)}, & \varphi(\chi) \in \Xi_1, \\ 1, & \varphi(\chi) = 0. \end{cases}\right)_{\chi\in\Xi_2}$$

sending $(1, \ldots, 1)$ to $(1, \ldots, 1)$, and thus a pointed morphism of stacks $f_\varphi \colon (\mathcal{X}_{\Xi_1}, x_{\Xi_1}) \to (\mathcal{X}_{\Xi_2}, x_{\Xi_2})$. For all geometric points $y = (y_\psi)_{\psi \in \Xi_1} \in \mathbb{A}_k^{\Xi_1}\left(\overline{k}\right)$, the state of $f_\varphi(y)$ contains all elements of $\Xi_2 \cap \ker\varphi$ and thus $f_\varphi(y)$ is semistable. Therefore f_φ restricts to a morphism $f_\varphi^{\mathrm{ss}} \colon (\mathcal{X}_{\Xi_1}^{\mathrm{ss}}, x_{\Xi_1}) \to (\mathcal{X}_{\Xi_2}^{\mathrm{ss}}, x_{\Xi_2})$.

The assignments $\varphi \mapsto f_\varphi$ and $\varphi \mapsto f_\varphi^{\mathrm{ss}}$ respect composition. If φ is a morphism of *normed* semistable polarised states, then f_φ and f_φ^{ss} are normed morphisms of stacks. Therefore, the assignment $\Xi \mapsto (\mathcal{X}_\Xi^{\mathrm{ss}}, x_\Xi)$ and $\varphi \mapsto f_\varphi^{\mathrm{ss}}$ defines a functor from the category of normed semistable polarised states to the category of k-pointed normed good moduli stacks with affine diagonal and finitely presented over k.

Proposition 5.2.10. *For Ξ a semistable polarised state and $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\Xi)$, there is a natural pointed isomorphism $(\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_\Xi^{\mathrm{ss}})_{\mathrm{gr}\,\lambda}, \mathrm{gr}\,\lambda) \cong \left(\mathcal{X}^{ss}_{\mathrm{Grad}_\lambda(\Xi)}, x_{\mathrm{Grad}_\lambda(\Xi)}\right)$. Here, $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_\Xi^{ss})_{\mathrm{gr}\,\lambda}$ is the connected component of $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_\Xi^{ss})$ containing $\mathrm{gr}\,\lambda$.*

Proof. By [26, Theorem 1.4.8], $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{X}_\Xi^{\mathrm{ss}})_{\mathrm{gr}\,\lambda} = \left(\mathbb{A}_k^\Xi\right)^{\lambda,0,\mathrm{ss}} / G_\Xi$, where $\left(\mathbb{A}_k^\Xi\right)^{\lambda,0}$ is the fixed point locus for the (rational) $\mathbb{G}_{m,k}$ action on $\mathbb{A}_k^\Xi$ given by λ and $\left(\mathbb{A}_k^\Xi\right)^{\lambda,0,\mathrm{ss}}$ is the semistable locus. On the other hand, looking at the weights one gets the equality $\left(\mathbb{A}_k^\Xi\right)^{\lambda,0} = \mathbb{A}_k^{\Xi_{\lambda,0}}$, and both $\mathrm{gr}\,\lambda$ and $x_{\mathrm{Grad}_\lambda(\Xi)}$ are the point $(1, \cdots, 1)$, giving the desired isomorphism of pointed stacks. □

Definition 5.2.11. Let $(\Xi_n, \lambda_n, u_n)_{n\in\mathbb{N}}$ be a chain of normed semistable polarised states. Applying the functor $\Xi \mapsto (\mathcal{X}_\Xi^{\mathrm{ss}}, x_\Xi)$ gives an *associated chain of k-stacks* $(\mathcal{X}_{\Xi_n}^{\mathrm{ss}}, x_{\Xi_n}, \lambda_n, h_n)$, where each h_n is the composition of

$$f_{u_n}^{\mathrm{ss}} \colon \left(\mathcal{X}_{\Xi_{n+1}}^{\mathrm{ss}}, x_{\Xi_{n+1}}\right) \to \left(\mathcal{X}^{\mathrm{ss}}_{\mathrm{Grad}_\lambda(\Xi_n)}, x_{\mathrm{Grad}_\lambda(\Xi_n)}\right),$$

the isomorphism $\left(\mathcal{X}^{\mathrm{ss}}_{\mathrm{Grad}_\lambda(\Xi_n)}, x_{\mathrm{Grad}_\lambda(\Xi_n)}\right) \xrightarrow{\sim} \left(\mathrm{Grad}_{\mathbb{Q}}\left(\mathcal{X}_{\Xi_n}^{\mathrm{ss}}\right)_{\mathrm{gr}\,\lambda}, \mathrm{gr}\,\lambda\right)$ from Proposition 5.2.10, and the open and closed immersion $\left(\mathrm{Grad}_{\mathbb{Q}}\left(\mathcal{X}_{\Xi_n}^{\mathrm{ss}}\right)_{\mathrm{gr}\,\lambda}, \mathrm{gr}\,\lambda\right) \to \left(\mathrm{Grad}_{\mathbb{Q}}\left(\mathcal{X}_{\Xi_n}^{\mathrm{ss}}\right), \mathrm{gr}\,\lambda\right)$.

Proposition 5.2.12. *Let Ξ be a semistable polarised state. Then there is a canonical bijection $\mathbb{Q}^\infty\text{-}\mathrm{Filt}(\Xi) \cong \mathbb{Q}^\infty\text{-}\mathrm{Filt}(\mathcal{X}_\Xi^{ss}, x_\Xi)$.*

Proof. The bijection follows from the description of $\mathbb{Q}^\infty\text{-}\mathrm{Filt}(\mathcal{X}_\Xi^{\mathrm{ss}}, x_\Xi)$ in Remark 3.2.14 and an iterated application of Propositions 5.2.7 and 5.2.10. □

Proposition 5.2.13. *Let Ξ be a semistable polarised state, and let $\Xi' \to \Xi$ be its slice. Then the associated morphism $\mathcal{X}_{\Xi'}^{ss} = \mathcal{X}_{\Xi'} \to \mathcal{X}_\Xi^{ss}$ identifies $\mathcal{X}_{\Xi'}$ with the fibre of the good moduli space $\mathcal{X}_\Xi^{ss} \to X_\Xi^{ss}$ containing x_Ξ.*

Proof. We use the notations of Definition 5.1.12. Let $\chi_1, \ldots, \chi_n$ be the different elements of Ξ and assume, after reordering, that $\{\chi_1, \ldots, \chi_l\} = F \cap \Xi$ (where F is the smallest face of $\mathrm{cone}(\Xi)$ containing α), the equality to be interpreted inside $M_{\mathbb{Q}}$ (that is, modulo torsion). The state of Ξ' is $\Xi' = q(\{\chi_{l+1}, \ldots, \chi_n\}) \subset M'$, where $q \colon M \to M'$ is the quotient map.

We remark that for any $\lambda \in M_{\mathbb{Q}}^\vee$ such that $\Xi \subset H_\lambda$ and $\partial H_\lambda \cap \mathrm{cone}(\Xi) = F$, the limit $y = \lim_{t\to 0} \lambda(t)x$ is polystable. Note also that y does not depend on the choice of λ. By Proposition 5.2.4 applied to Ξ_y, the stabiliser of y is $H \coloneqq G_{\Xi'}$.

We identify $\mathbb{A}_k^{\Xi} = \mathbb{A}_k^n$, the action of $G = G_{\boldsymbol{\Xi}}$ on $\mathbb{A}_k^n$ being via the characters $\chi_1, \dots, \chi_n$. The G-equivariant open subscheme $\mathbb{G}_{m,k}^l \times \mathbb{A}_k^{n-l} \subset (\mathbb{A}_k^n)^{\mathrm{ss}}$ is saturated with respect to the good moduli space $\mathcal{X}_{\boldsymbol{\Xi}}^{\mathrm{ss}} = (\mathbb{A}_k^n)^{\mathrm{ss}}/G \to (\mathbb{A}_k^n)^{\mathrm{ss}} /\!\!/ G = X_{\boldsymbol{\Xi}}^{\mathrm{ss}}$, and thus the fibre of $\mathcal{X}_{\boldsymbol{\Xi}}^{\mathrm{ss}} \to X_{\boldsymbol{\Xi}}^{\mathrm{ss}}$ containing $x_{\boldsymbol{\Xi}} = (1, \dots, 1)$ equals the fibre of the good moduli space $\left(\mathbb{G}_{m,k}^l \times \mathbb{A}_k^{n-l}\right)/G \to \left(\mathbb{G}_{m,k}^l \times \mathbb{A}_k^{n-l}\right) /\!\!/ G$ containing $(1, \dots, 1)$. Indeed, for every $\overline{k}$-point (a, b) in $\mathbb{G}_{m,k}^l \times \mathbb{A}_k^{n-l}$, $G_{\overline{k}}(a, 0)$ is the associated polystable orbit, and conversely if a semistable $\overline{k}$-point z in $\mathbb{A}_k^n$ has associated polystable orbit of the form $G_{\overline{k}}(a, 0)$, then it should lie in $\mathbb{G}_{m,k}^l \times \mathbb{A}_k^{n-l}$.

Consider the $(H \to G)$-equivariant map

$$h \colon \mathbb{A}_k^{n-l} \to \mathbb{G}_{m,k}^l \times \mathbb{A}_k^{n-l} \colon (z_1, \dots, z_{n-l}) \mapsto (1, \dots, 1, z_1, \dots, z_{n-l}),$$

and the associated morphism $\mathbb{A}_k^{n-l}/H \to \left(\mathbb{G}_{m,k}^l \times \mathbb{A}_k^{n-l}\right)/G$, which is the restriction on the codomain of the map $f_q^{\mathrm{ss}} \colon \mathcal{X}_{\boldsymbol{\Xi}'}^{\mathrm{ss}} \to \mathcal{X}_{\boldsymbol{\Xi}}^{\mathrm{ss}}$. Let I be the image of the homomorphism $G \to \mathbb{G}_{m,k}^l$ given by $\chi_1, \dots, \chi_k$. From a explicit computation of the ring of invariants it follows that $\left(\mathbb{G}_{m,k}^l \times \mathbb{A}_k^{n-l}\right)/G \to \left(\mathbb{G}_{m,k}^l \times \mathbb{A}_k^{n-l}\right) /\!\!/ G = \mathbb{G}_{m,k}^l/I$ is the good moduli space, and its fibre over $e \in \mathbb{G}_{m,k}^l/I(k)$ is $\left(I \times \mathbb{A}_k^{n-l}\right)/G$. To conclude, we note the isomorphism

$$\left(I \times \mathbb{A}_k^{n-l}\right)/G \cong \left((G/H) \times \mathbb{A}_k^{n-l}\right)/G \cong \mathbb{A}_k^{n-l}/H$$

induced by h. □

From Proposition 5.2.13 we get bijections

$$\mathbb{Q}\text{-}\mathrm{Filt}(\boldsymbol{\Xi}') \cong \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}_{\boldsymbol{\Xi}'}, x_{\boldsymbol{\Xi}'}) \cong \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}_{\boldsymbol{\Xi}}, x_{\boldsymbol{\Xi}}) \cong \mathbb{Q}\text{-}\mathrm{Filt}(\boldsymbol{\Xi})$$

between sets of filtrations, consistent with Proposition 5.1.13.

Proposition 5.2.14. *Let $\boldsymbol{\Xi}$ be a semistable polarised state and let $\lambda \in \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}_{\boldsymbol{\Xi}}, x_{\boldsymbol{\Xi}})$. Then the Kempf number $\langle \lambda, \mathcal{X}_{\boldsymbol{\Xi}'}^{\max} \rangle$ equals the complementedness of λ:*

$$\langle \lambda, \mathcal{X}_{\boldsymbol{\Xi}'}^{\max} \rangle = \langle \lambda, \mathfrak{l} \rangle.$$

Proof. Write $\mathcal{F} = \mathcal{X}_{\Xi'} = \mathbb{A}_k^m/H$ and $x = (1, \dots, 1)$, using notation from the proof of Proposition 5.2.13 and where $m = n - l$. The maximal stabiliser locus is $\mathcal{F}^{\max} = (\mathbb{A}_k^m)^{H_\circ}/H = \{0\}/H$, where $H_\circ = (H^\circ)_{\mathrm{red}}$ is the reduced identity component [19, Proposition C.5]. Let $r = \langle \lambda, \mathcal{F}^{\max} \rangle$. There is a cartesian square

$$\begin{tikzcd}
\operatorname{Spec}(k[t]/(t^r)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{A}_k^1 \arrow[d, "\lambda(t)x"'] & t \arrow[d, mapsto] \\
\{0\} \arrow[r] & \mathbb{A}_k^m & \left(t^{\langle \lambda, \chi \rangle}\right)_{\chi \in \Xi'} .
\end{tikzcd}$$

Taking global sections, we get a cocartesian square

$$\begin{tikzcd}
k[t]/(t^r) & k[t] \arrow[l] & t^{\langle \lambda, \chi \rangle} \\
k \arrow[u] \arrow[ur, phantom, "\ulcorner"] & k[t_\chi, \chi \in \Xi'] \arrow[l] \arrow[u] & t_\chi . \arrow[u, mapsto]
\end{tikzcd}$$

Therefore $k[t]/(t^r) = k[t]/(t^{\langle \lambda, \chi \rangle}, \chi \in \Xi')$ and thus $r = \inf\{\langle \lambda, \chi \rangle \mid \chi \in \Xi'\} = \langle \lambda, \mathfrak{l} \rangle$. □

Proposition 5.2.15. *Let Ξ be a normed semistable polarised state. Then the balanced filtration of $(\mathcal{X}^{ss}_{\Xi}, x_{\Xi})$ (Definition 3.1.6) equals, under the bijection $\mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{X}^{ss}_{\Xi}, x_{\Xi}) \cong \mathbb{Q}\text{-}\mathrm{Filt}(\Xi)$, the balanced filtration of Ξ (Definition 5.1.16).*

Proof. This follows directly from Propositions 5.2.13 and 5.2.14. □

Theorem 5.2.16. *Let Ξ be a normed semistable polarised state and let $(\Xi_n, \lambda_n, u_n)_{n\in\mathbb{N}}$ be its balancing chain (Definition 5.1.20). Then the chain of stacks associated to $(\Xi_n, \lambda_n, u_n)_{n\in\mathbb{N}}$ (Definition 5.2.11) is isomorphic to the torsor chain of $(\mathcal{X}^{ss}_{\Xi}, x_{\Xi})$ (Definition 4.3.3).*

Proof. Let $(\mathcal{Y}_n, y_n, \eta_n, v_n)_{n\in\mathbb{N}}$ be the torsor chain of $(\mathcal{X}^{ss}_{\Xi}, x_{\Xi})$. We will provide, for all $m \in \mathbb{N}$, isomorphisms

$$i_m\colon (\mathcal{X}^{ss}_{\Xi_m}, x_{\Xi_m}) \cong (\mathcal{Y}_m, y_m)$$

such that,

1. under the identification $\mathbb{Q}\text{-}\mathrm{Filt}(\Xi_m) = \mathbb{Q}\text{-}\mathrm{Filt}(\mathcal{Y}_m, y_m)$, we have $\eta_m = \lambda_m$; and
2. the square

$$\begin{tikzcd}
(\mathcal{X}^{ss}_{\Xi_{m+1}}, x_{\Xi_{m+1}}) \arrow[r, "f^{ss}_{u_m}"] \arrow[d, "i_{m+1}"'] & (\mathcal{X}^{ss}_{\mathrm{Grad}_{\lambda_m}(\Xi_m)}, x_{\mathrm{Grad}_{\lambda_m}(\Xi_m)}) \arrow[d] \\
(\mathcal{Y}_{m+1}, y_{m+1}) \arrow[r, "v_n"] & (\mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y}_m)_{\mathrm{gr}\,\eta_m}, \mathrm{gr}\,\eta_m)
\end{tikzcd} \tag{9}$$

commutes, where the arrow on the right comes from Proposition 5.2.10.

For $n = 0$, we have that y_0 is the fibre of the good moduli space of $\mathcal{X}^{ss}_{\Xi}$ containing x_{Ξ}, and that $\Xi_0 = \Xi'$. Therefore $(\mathcal{Y}_0, y_0) = (\mathcal{X}^{ss}_{\Xi_0}, x_{\Xi_0})$, by Proposition 5.2.13. Let $n \in \mathbb{N}$ and suppose that isomorphisms i_m as above have been provided in such a way that conditions (1) and (2) above hold for all $m < n$. Since i_n is an isomorphism, we have the equality $\eta_n = \lambda_n$ by Proposition 5.2.15, so condition (1) also holds for $m = n$.

If $\lambda_n = 0$, then Ξ_n and $(\mathcal{Y}_n, y_n)$ are polystable. Therefore $\Xi_{n+1} = \Xi_n$ and $(\mathcal{Y}_{n+1}, y_{n+1}) = (\mathcal{Y}_n, y_n)$, so there is nothing to prove.

Assume $\lambda_n \neq 0$. We freely use the notation of Case 2 in Construction 4.3.1. Let $\chi_1, \ldots, \chi_l$ be the different elements in Ξ_n, and let $G = G_{\Xi_n}$. We denote $V = k^l$ the G-representation given by the characters $\chi_1, \ldots, \chi_l$, so that $\mathbb{A}^{\Xi_0}_k = \mathbb{A}(V)$ (where $\mathbb{A}$ denotes *total space*). We have $\mathcal{Y}_n = \mathbb{A}(V)/G$, and $\mathcal{Y}^{\max}_n = \{0\}/G$. Therefore the relevant blow-up is $\mathcal{B} = (\mathrm{Bl}_0\, \mathbb{A}(V))/G$, the exceptional divisor is $\mathcal{E} = \mathbb{P}(V)/G$, and the $\mathbb{G}_m$-torsor over it is $\mathcal{N} = (\mathbb{A}(V) \setminus \{0\})/G$. We denote y_n also the unique lift of $y_n = (1, \ldots, 1)$ to $\mathrm{Bl}_0\, \mathbb{A}(V)$. The rational one-parameter subgroup λ_n has the property that $\langle \lambda_n, \chi_j \rangle \geq 1$ for all j and that equality holds for at least one j. Therefore, if we set $z = \lim_{t\to 0} \lambda_n(t) y_n$, the limit taken inside $\mathrm{Bl}_0\, \mathbb{A}(V)$, and if we write $z = [z_1, \ldots, z_l]$ in projective coordinates, noting that z lies on the exceptional divisor $\mathbb{P}(V)$, then we have $z_j = 0$ if $\langle \lambda_n, \chi_j \rangle > 1$ and $z_j = 1$ if $\langle \lambda_n, \chi_j \rangle = 1$. We denote $z^* = (z_1, \ldots, z_l)$ this lift of z to $\mathbb{A}(V)$.

The limit z lifts to the connected component $\overline{\mathcal{Z}}$ of $\mathrm{Grad}(\mathcal{B})$ containing $\mathrm{gr}\,\eta_n$. If we set $V_1 = \bigoplus_{\langle \lambda, \chi \rangle = 1} V_\chi$, where V_χ is the subrepresentation of V where G acts via the character χ, then $\overline{\mathcal{Z}} = \mathbb{P}(V_1)/G$. Thus $\mathrm{gr}\,\eta_n$ is identified with $z \in \mathbb{P}(V_1)(k)$, and the lift $z^* \in \mathbb{A}(V_1)$ to $\mathbb{A}(V_1)$ can be written in coordinates as $z = (1, \ldots, 1)$ when seen inside V_1. The centre $\mathcal{Z}$ of the Θ-stratum of $\mathcal{B}$ containing y_n is the semistable locus for the shifted linear form ℓ_c, by the Linear Recognition Theorem 2.6.5.

In this case, $c = \dfrac{1}{\|\lambda_n\|^2}$, and the shifted linear form is

$$\ell_c = \ell - \frac{1}{\|\lambda_n\|^2} \langle \lambda_n^\vee, - \rangle,$$

where ℓ is the linear form associated to the ample line bundle $\mathcal{O}_{\mathbb{P}(V_1)/G}(1)$. Let $x \in \mathbb{P}(V_1)(\bar{k})$ and let $x^* \in \mathbb{A}(V_1)(\bar{k})$ be a lift of x to $\mathbb{A}(V_1)$ with state Ξ_x. The point x is semistable for ℓ_c if for all $\gamma \in \Gamma^{\mathbb{Q}}(G)$ we have

$$0 \geq \ell_c(\gamma) = \min\langle \gamma, \Xi_x \rangle - \frac{1}{\|\lambda_x\|^2}(\gamma, \lambda_n),$$

which holds if and only if $0 \in \operatorname{conv}\left(\Xi_x - \frac{\lambda_n^\vee}{\|\lambda_n\|^2}\right)$. Since all elements of Ξ_x are in the hyperplane $\langle \lambda_n, - \rangle = 1$, the condition that 0 is in the convex hull of $\Xi_x - \frac{\lambda_n^\vee}{\|\lambda_n\|^2}$ is equivalent to the condition that $\lambda_n^\vee$ is in the cone generated by Ξ_x, by Lemma 5.2.17 below. This is in turn equivalent to the lift x^* of x to $\mathbb{A}(V_1)$ being semistable for the linear form given by $\lambda_n^\vee$, by Proposition 5.2.5. Therefore we have a cartesian square

$$\begin{tikzcd}
\mathcal{M} = \mathbb{A}(V_1)^{\mathrm{ss}(\lambda_n^\vee)}/G \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{N} = (\mathbb{A}(V) \setminus 0)/G \arrow[d] \\
\mathcal{Z} = \mathbb{P}(V_1)^{\mathrm{ss}(\ell_c)}/G \arrow[r] & \mathcal{E} = \mathbb{P}(V)/G,
\end{tikzcd}$$

using the notations of Construction 4.3.1. Just from the definitions, we see that $\mathcal{M} = \mathcal{X}^{\mathrm{ss}}_{\Lambda_{\lambda_n}(\boldsymbol{\Xi}_n)}$. We choose $y_{n+1} = z^*$ as the preimage of z along $\mathcal{M} \to \mathcal{Z}$ needed for the construction of the torsor chain. The stack $\mathcal{Y}_{n+1}$ is by definition the fibre of the good moduli space of $\mathcal{M}$, and hence by Proposition 5.2.13 we have the desired isomorphism $i_{n+1} \colon \mathcal{X}^{\mathrm{ss}}_{\boldsymbol{\Xi}_{n+1}} = \mathcal{X}^{\mathrm{ss}}_{\Lambda_{\lambda_n}(\boldsymbol{\Xi}_n)'} \cong \mathcal{Y}_{n+1}$, which sends $x_{\boldsymbol{\Xi}_{n+1}}$ to y_{n+1} by our choice of z^*. Both $\mathcal{X}^{\mathrm{ss}}_{\mathrm{Grad}_{\lambda_n}(\boldsymbol{\Xi}_n)}$ and $\mathrm{Grad}_{\mathbb{Q}}(\mathcal{Y}_n)_{\mathrm{gr}\,\eta_n}$ are naturally identified with BG, so the square (9) commutes for $m = n$. Since the torsor chain is bounded, repeating this process we eventually reach the case $\eta_n = 0$, getting the desired isomorphism of chains. □

In the proof of Theorem 5.2.16 we used the following fact in convex geometry.

Lemma 5.2.17. *Let N be a finite dimensional $\mathbb{Q}$-vector space endowed with a rational inner product $(-,-)$. Let $\Xi \subset N$ be a nonempty finite set and let $\gamma \in N$ be an element such that $(\gamma, \chi) = 1$ for all $\chi \in \Xi$. Then we have $0 \in \operatorname{conv}\left(\Xi - \frac{\gamma}{\|\gamma\|^2}\right)$ if and only if $\gamma \in \operatorname{cone}(\Xi)$.*

Proof. Each $\chi \in \Xi$ can be written as $\chi = \frac{\gamma}{\|\gamma\|^2} + \beta_\chi$ with $(\gamma, \beta_\chi) = 0$. Note that the condition $0 \in \operatorname{conv}\left(\Xi - \frac{\gamma}{\|\gamma\|^2}\right)$ is equivalent to $0 \in \operatorname{cone}\left(\Xi - \frac{\gamma}{\|\gamma\|^2}\right)$. If this is satisfied, then $0 = \sum_\chi c_\chi \beta_\chi$ with $c_\chi \geq 0$. After rescaling we may assume that $\sum c_\chi = \|\gamma\|^2$. Then

$$\gamma = \left(\sum c_\chi\right) \frac{\gamma}{\|\gamma\|^2} + \sum_\chi c_\chi \beta_\chi = \sum c_\chi \chi,$$

and thus $\gamma \in \operatorname{cone}(\Xi)$.

Conversely, if $\gamma = \sum_\chi c_\chi \chi$ with the $c_\chi \geq 0$, then after applying $(\gamma, -)$ to both sides we get $\|\gamma\|^2 = \sum_\chi c_\chi$, so

$$\gamma = \sum_\chi c_\chi \left(\frac{\gamma}{\|\gamma\|} + \beta_\chi\right) = \gamma + \sum_\chi c_\chi \beta_\chi$$

and thus $0 = \sum c_\chi \beta_\chi$ is in $\operatorname{cone}\left(\Xi - \frac{\gamma}{\|\gamma\|^2}\right)$. □

Corollary 5.2.18. *Let Ξ be a normed semistable polarised state. Then the iterated balanced filtration (Definition 5.1.20) of Ξ equals, under the bijection $\mathbb{Q}^\infty\text{-}\mathrm{Filt}(\Xi) \cong \mathbb{Q}^\infty\text{-}\mathrm{Filt}(\mathfrak{X}^{ss}_{\Xi}, x_\Xi)$, the iterated balanced filtration of $(\mathfrak{X}^{ss}_{\Xi}, x_\Xi)$ (Definition 3.5.8).*

Proof. By Theorem 5.2.16, the iterated balanced filtration of Ξ equals the sequential filtration associated to the torsor chain of $(\mathfrak{X}^{\mathrm{ss}}_{\Xi}, x_\Xi)$. The results then follows from Theorem 4.3.4. □

Example 5.2.19. Consider the normed polarised state

$$\Xi = \left(\mathbb{Z}^2, \{(1,0),(1,1)\}, 0\right),$$

where $(\mathbb{Z}^2)^\vee = \mathbb{Z}^2$ has the standard inner product ($(1,0),(0,1)$ is an orthonormal base). The associated stack over $\mathbb{C}$ is $\mathfrak{X}_\Xi = \mathbb{C}^2/(\mathbb{C}^\times)^2$, where $(\mathbb{C}^\times)^2$ acts by $(t_1,t_2)(v_1,v_2) = t_1v_1, t_1t_2v_2)$. The linearisation $\mathcal{L}_\Xi$ is trivial, so every point is semistable.

Let us compute the iterated balanced filtration of the state Ξ. We have that Ξ is its own slice $\Xi' = \Xi$, and the balanced filtration $\lambda_0 = (a,b)$ of Ξ is the minimiser of a^2+b^2 subject to the condition that $\langle \lambda_0, (1,0)\rangle = a \geq 1$ and $\langle \lambda_0, (1,1)\rangle = a+b \geq 1$. Therefore $\lambda_0 = (1,0)$. The iterated state is

$$\Lambda_{\lambda_0}(\Xi) = \left(\mathbb{Z}^2, \{(1,0),(1,1)\}, (1,0)\right),$$

whose slice is $\Xi_1 = \Lambda_{\lambda_0}(\Xi)' = (\mathbb{Z}(0,1), \{(0,1)\}, 0)$, and the balance filtration of Ξ_1 is $\lambda_1 = (0,1)$. Since $\Lambda_{\lambda_1}(\Xi_1)$ is polystable, the balancing chain of Ξ terminates here, and we conclude that the iterated balanced filtration of Ξ is $\lambda_0 = (1,0), \lambda_1 = (0,1)$. By Corollary 5.2.18, we deduce that the iterated balanced filtration of $x_\Xi = (1,1) \in \mathbb{C}^2/(\mathbb{C}^\times)^2$ is the sequence $(1,0),(0,1)$ in $\mathbb{Q}^2 = \Gamma^{\mathbb{Q}}((\mathbb{C}^\times)^2)$.

We now analyse Conjecture 1.5.1 in this case. We endow $\mathbb{C}^2$ with the standard hermitian metric. The associated Kempf-Ness potential for the point $(1,1) \in \mathbb{C}^2$ is

$$f\colon \mathbb{R}^2 \to \mathbb{R}\colon x \mapsto e^{x\cdot\binom{1}{0}} + e^{x\cdot\binom{1}{1}}$$

up to the addition of a constant, where we are identifying $\mathbb{R}^2 = i\,\mathrm{Lie}((S^1)^2) \xrightarrow[\sim]{exp} (\mathbb{C}^\times)^2/(S^1)^2$, the unit circle S^1 being the maximal compact subgroup of $\mathbb{C}^\times$. In this case, the exponential map is a global isometry between $\mathrm{Lie}((S^1)^2)$ and $(\mathbb{C}^\times)^2/(S^1)^2$. The gradient is

$$\nabla f(x) = e^{x_1}\begin{pmatrix}1\\0\end{pmatrix} + e^{x_1+x_2}\begin{pmatrix}1\\1\end{pmatrix}.$$

The equation for $h\colon (0,\infty) \to \mathbb{R}^2$ to be a flow line for $-\nabla f$ is

$$h'(t) = -e^{h_1(t)}\begin{pmatrix}1\\0\end{pmatrix} - e^{h_1(t)+h_2(t)}\begin{pmatrix}1\\1\end{pmatrix}.$$

We write $h(t) = -\log(t)\binom{1}{0} - \log\log(t)\binom{1}{0} + z(u)$, where $u = \log\log(t)$. The equation for z becomes

$$z'(u) = \begin{pmatrix} e^u(1-e^{z_1(u)}) - e^{z_1(u)+z_2(u)} \\ 1 - e^{z_1(u)+z_2(u)} \end{pmatrix}.$$

For $N_1 > 0$ real, we have that

1. if $z_2 = N_1$ and $z_1 \in (-N_1, N_1)$, then $z_2' < 0$.
2. if $z_2 = -N_1$ and $z_1 \in (-N_1, N_1)$, then $z_2' > 0$.

For $N_2 > 0$ big enough so that $-1 \leq \log(1 - e^{-N_2})$ and $u \geq 2$, we have that

1. if $z_1 = -N_2$ and $z_2 \in (-N_2 - 1, N_2 + 1)$, then $z_1' > 0$.
2. if $z_1 = N_2$ and $z_2 \in (-N_2 - 1, N_2 + 1)$, then $z_1' < 0$.

Therefore, an appropriate choice of N_1 and N_2 gives a rectangle that z cannot leave, because z' points inwards at the boundary. Therefore z is bounded when $t >> 0$. We have verified Conjecture 1.5.1 in this example.

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