On canonical filtrations of modular lattices

Draft

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Abstract

We give a new characterisation of the weight filtration of Haiden-Katzarkov-Kontsevich-Pandit for a normed artinian lattice as the minimiser of a certain norm function.

Contents

1 Introduction

In [4], Haiden-Katzarkov-Kontsevich-Pandit defined a canonical filtration $0 < a_0 < \cdots < a_n = 1$ of a normed artinian lattice L, together with labels $c_0 > \cdots > c_n$ where each $c_i = (c_i^{(0)}, c_i^{(1)}, \ldots) \in \mathbb{Q}^\infty$ is an eventually zero sequence of rational numbers. We call this filtration the *iterated HKKP filtration* of the lattice L. If Q is a quiver with central charge Z and E is a semistable representation of Q over the field $\mathbb{C}$ of complex numbers, then the lattice L of semistable subrepresentations of Q is artinian and naturally normed, so one may consider its iterated HKKP filtration.

Remarkably, this filtration describes the asymptotics of a certain natural gradient flow associated to the moduli problem. Let S be the Riemannian manifold of hermitian metrics on the representation E. The so called *Kempf potential* is a map $p\colon S \to \mathbb{R}$ that depends on the central charge Z. By the celebrated Kempf-Ness theorem [7], the algebraic notion of *semistability* of E can be detected from the potential p: the representation E

is semistable if and only if p is bounded below. Similarly, E is polystable if and only if p attains a minimum m, in which case the negative gradient flow of p starting at any point of S converges to m. The question arises of whether something more precise can be said about this flow in the case where E is strictly semistable. This is answered by the iterated HKKP filtration: in the piece of the filtration labelled by the sequence $(s_0, s_2, \dots)$ of rational numbers, asymptotically the gradient flow equals

$$t^{s_0} \log(t)^{s_1} \log(\log(t))^{s_2} \cdots$$

up to bounded terms.

In this paper, we give a new characterisation of the HKKP filtrations as the minimiser of the norm function among the set of paracomplemented filtrations (Theorem 6.0.22). The main idea is the systematic use of maximal distributive sublattices, which we see as the lattice analogue of maximal tori of algebraic groups. Our methods yield a simple proof of the existence of the HKKP filtration, and we get rationality of the filtration very naturally.

Along the way, we introduce a combinatorial gadget associated to the lattice, the *degeneration fan*, which is a formal fan in the sense of Halpern-Leistner [5]. For the lattice of subspaces of a vector space, the degeneration fan is the unprojectivised spherical building of GL_n.

2 Generalities

We recall from [2] and [4, Section 4] some basic definitions about lattices.

A *lattice* is a poset $(L, \leq)$ such that every two elements $a, b \in L$ have a least upper bound $a \wedge b$, called the *meet* of a and b, and a greatest lower bound $a \vee b$, called the *join* of a and b. If L is a lattice, a *sublattice* is a subset L' of L closed under taking meet and join. If L' is a sublattice of L, then L' is regarded as a lattice with the poset structure inherited from L. If $a \leq b$ are elements of L, then the *interval* $[a, b]$ is the sublattice of L consisting of those $x \in L$ with $a \leq x \leq b$.

Definition 2.0.1. A lattice L is said to be

1. *modular*, if for all $x, a, b \in L$ with $a \leq b$ we have

$$(x \wedge a) \vee b = (x \vee b) \wedge a;$$

2. *bound*, if it has a minimum, denoted 0, and a maximum, denoted 1;
3. *finite length*, if there exists $N \in \mathbb{N}$ such that for all chains $a_0 < a_1 < \cdots < a_n$ of elements of L, we have $n \leq N$;
4. *artinian*, if it is modular, of finite length, and nonempty;
5. *complemented*, if it is bound and for all $a \in L$ there is $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$;
6. *distributive*, if for all $a, b, c \in L$ we have $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$;
7. *complete*, if every subset of L has a least upper bound and a greatest lower bound;
8. *a finite boolean algebra*, if it is artinian, distributive and complemented.

The main example of modular lattice L is that of subobjects of an object E in an abelian category. In that case, L being complemented is equivalent to E being semisimple. Note that finite length lattices are complete and bound, and we use the notations 0 and 1 for the minimum and maximum of the lattice.

If L is a modular lattice, we denote $\sim$ the equivalence relation on intervals generated by $[a, a \vee b] \sim [a \wedge b, b]$ for $a, b \in L$. The modular law implies that two equivalent intervals of L are isomorphic as lattices.

Theorem 2.0.2 (Jordan-Hölder-Dedekind)**.** *Let L be an artinian lattice, and let $0 = a_0 < a_1 < \cdots < a_n = 1$ and $0 = b_0 < b_1 < \cdots < b_m = 1$ be two maximal chains in L. Then $n = m$ and there is a permutation σ of $\{1, \dots n\}$ such that $[a_{i-1}, a_i] \sim [b_{\sigma(i)-1}, b_i]$ for $i \in \{1, \dots n\}$.*

For a proof, see [6, Section 8.3]. We define the *length* of an artinian lattice L to be the length of any maximal chain in L.

For a bound lattice L and an element $x \in L$, we say that x is an *atom* if $0 < x$ and there is no $y \in L$ with $0 < y < x$. If L is a finite boolean algebra, then L has finitely many atoms $x_1, \dots, x_n$, and these determine L in the precise sense that the map $\{0,1\}^n \to L\colon (c_i) \mapsto c_1x_1 \vee \cdots \vee c_nx_n$ is an isomorphism of lattices, where we denote $0x = 0$ and $1x = x$.

3 Distributive artinian lattices and directed acyclic graphs

The following is a well-known property of distributive lattices.

Lemma 3.0.1. *Let D be a bound distributive lattice and let $a \in D$. Then the map $f\colon D \to [0, a] \times [a, 1]\colon x \mapsto (x \wedge a, x \vee a)$ is injective and preserves meets and joins.*

Proof. Suppose $x, y \in D$ are such that $x \wedge a = y \wedge a$ and $x \vee a = y \vee a$. Then

$$x = x \wedge (x \vee a) = x \wedge (y \vee a) = (x \wedge y) \vee (x \wedge a) = x \wedge y.$$

Thus $x \leq y$. By symmetry, $x = y$. The map f preserves meets and joins by distributivity. $\square$

It follows from the lemma that complements in bound distributive lattices are unique if they exist.

A *directed acyclic graph* Q is a pair (Q_0, Q_1) where Q_0 is a finite set, the set of *vertices*, and Q_1, the set of *arrows*, is a subset of $Q_0 \times Q_0$. The two projections $Q_0 \times Q_0 \to Q_0$ define source and target maps $s, t\colon Q_1 \to Q_0$. We require that Q has no loops, that is, there is no sequence of arrows $\alpha_1, \dots, \alpha_n \in Q_1$ with $t(\alpha_i) = s(\alpha_{i+1})$ for all $i = 1, \dots, n-1$ and $t(\alpha_n) = s(\alpha_1)$.

Finite posets are in correspondence to those directed acyclic graphs Q such that, for all $a, b, c \in Q_0$, if there are arrows $a \to b$ and $b \to c$, then there is also an arrow $a \to c$. We will just say that directed acyclic graphs satisfying this property are posets.

For any directed acyclic graph Q, a subset $R \subset Q_0$ is a *closed subgraph* if for all $\alpha \in Q_1$, we have the implication $s(\alpha) \in R \implies t(\alpha) \in R$. The lattice of closed subgraphs of Q is a distributive artinian lattice.In fact, all distributive artinian lattices arise in this way.

The following theorem is a convenient reformulation of Birkhoff duality.

Theorem 3.0.2. *Let D be a distributive artinian lattice and let $0 = a_0 < \cdots < a_n = 1$ be a maximal chain in D. We identify the product lattice $\prod_{i=1}^n [a_{i-1}, a_i]$ with the lattice 2^S of subsets of $S = \{1, \dots, n\}$. Then the map $f\colon D \to 2^S\colon x \mapsto ((x \vee a_{i-1}) \wedge a_i)_i$ is injective and preserves meets and joins. Moreover, there is a directed acyclic graph Q with set of vertices $Q_0 = S$ such that $f(D)$ is the lattice of closed subgraphs of Q. The graph Q can be taken to be a poset.*

Proof. Injectivity of f follows from Lemma 3.0.1 by induction on the length of D. Now, $f(D)$ can be seen as a topology on S. Since S is finite, for each $i \in S$ there is a smallest open set U_i containing i. We let

$$Q_1 = \{(i, j) \mid \ j \in U_i \setminus \{i\}\},$$

and denote Q the graph with set of vertices S and set of arrows Q_1, which is readily seen to be a poset.

Note that $U_i = U_i \cap f(a_i) = U_i \cap \{1, \ldots, i\}$, so for all $j \in U_i$ we have $j \leq i$. In particular, Q is acyclic. From the definition of Q it follows that a subset $R \subset S$ is a closed subgraph if and only if for all $i \in R$, we have $U_i \subset R$. That is, closed subgraphs are the open sets, which are the elements of $f(D)$. □

We recall that an element a of a bound lattice L is said to be *join-irreducible* if $a \neq 0$ and whenever $a = b \vee c$ we have $a = b$ or $a = c$.

Corollary 3.0.3 (Birkhoff duality)**.** *Let D be a distributive artinian lattice and let $\mathfrak{J}$ be the set of join-irreducible elements of D, then the map*

$$g\colon D \to 2^{\mathfrak{J}} : a \mapsto \{b \in \mathfrak{J} \mid \ b \leq a\}$$

is a lattice injection. Moreover, the cardinality of $\mathfrak{J}$ equals the length of D.

Proof. We use the notation from the proof of Theorem 3.0.2. We identify D with the sublattice $f(D)$ of 2^S. It is clear that the U_i are the join-irreducible elements of $f(D)$. Moreover, the bijection $S \to \mathfrak{J} : i \mapsto U_i$ yields a commuting triangle

$$\begin{tikzcd} D \arrow[rr, "f"] \arrow[dr, "g"'] & & 2^S \arrow[dl, "\sim"] \\ & 2^J & \end{tikzcd}$$

The result follows. □

Remark 3.0.4. Birkhoff duality thus provides a correspondence between distributive lattices of length n and posets of cardinality n.

Definition 3.0.5. Let D be a distributive artinian lattice with set of join-irreducibles $\mathfrak{J}$. The *finite boolean algebra* T_D associated to D is $T_D := 2^{\mathfrak{J}}$, the free boolean algebra on $\mathfrak{J}$. Corollary 3.0.3 provides a canonical injection $D \to T_D$.

4 Maximal distributive sublattices of artinian lattices

Let L be an artinian lattice. Maximal distributive sublattices of L are the analogue of maximal tori of reductive algebraic groups. We study here its basic properties.

Proposition 4.0.1. *Every distributive sublattice M of L is contained in a maximal distributive sublattice.*

Proof. This follows from Zorn's lemma, but here is an alternative argument. By applying [8, Lemma 1] repeatedly, we may assume that M contains a chain that is maximal in L. Then, by Theorem 3.0.2, there are no chains of distributive sublattices of L containing M of more than 2^n elements, where n is the length of L. □

Proposition 4.0.2. *Let M be a maximal distributive sublattice of L. Then M and L have the same length.*

Proof. This is [8, Corollary 2]. $\square$

Proposition 4.0.3. *Let M be a maximal distributive sublattice of L and let $a \leq b$ in M. Then $[a, b] \cap M$ is a maximal distributive sublattice of $[a, b]$.*

Proof. Let X be a maximal distributive sublattice of $[a, b]$ containing $[a, b] \cap M$. By [9, Corollary 5], the sublattice M' of L generated by M and X is distributive. Thus $M = M'$ and $X = M \cap [a, b]$. $\square$

Proposition 4.0.4. *Let M be a maximal distributive sublattice of L. Then L is complemented if and only if M is complemented.*

Note that in that case M is a finite boolean algebra.

Proof. Suppose M is complemented, and take $0 = a_0 < \cdots < a_l = 1$ a maximal chain in M. By repeatedly applying [4, Lemma 4.7], since each $[a_i, a_{i+1}]$ is complemented and each a_i has a complement in L, it follows that L is complemented.

Conversely, suppose that L is complemented, and let l be its length. For $l = 0, 1$ the result is trivial. If $l = 2$, take $0 < a < 1$ in M. If $M \setminus \{0, a, 1\} \neq \varnothing$, then any $b \in M \setminus \{0, a, 1\}$ is a complement of a. If $M = \{0, a, 1\}$ and b is a complement of a in L, then $\{0, a, b, 1\}$ is a distributive sublattice of L strictly containing M, a contradiction.

Suppose that $l > 2$ and let $0 < a_1 < a_2 < 1$ in M. Any interval in a complemented lattice is also complemented [2, Theorem 1.9.14], so in particular $[0, a_2]$ and $[a_2, 1]$ are complemented. By induction on the length, $[0, a_2] \cap M$ and $[a_2, 1] \cap M$ are complemented, and thus by [4, Lemma 4.7], M is complemented if and only if a_2 has a complement in M. Let b be the complement of a_2 in $[a_1, 1] \cap M$, which is complemented by induction hypothesis. Let c be the complement of a_1 in $[0, b] \cap M$, also complemented by induction. We have two bicartesian squares

$$\begin{tikzcd}
& & 1 \arrow[dl, dash] \arrow[dr, dash] & \\
& b \arrow[dl, dash] \arrow[dr, dash] & & a_2 \arrow[dl, dash] \\
c \arrow[dr, dash] & & a_1 \arrow[dl, dash] & \\
& 0 & &
\end{tikzcd}$$

so the concatenation is also bicartesian, that is, c is a complement of a_2 in M. $\square$

Corollary 4.0.5. *Let M be a maximal distributive sublattice of L and let $a \leq b$ in M. Then $[a, b] \cap M$ is complemented if and only if $[a, b]$ is complemented.*

Proof. This follows directly from propositions 4.0.4 and 4.0.3. $\square$

5 The degeneration fan of an artinian lattice

We recall Halpern-Leistner's notion degeneration fan [5, Definition 3.2.2], with slightly different conventions.

Let A be a subring of $\mathbb{R}$, endowed with the inherited order. We let Cone_A be the category that has

1. objects: $[n]$, where $n \in \mathbb{N}$,

2. morphisms: a map $[n] \to [m]$ is an A-linear homomorphism $\varphi\colon A^n \to A^m$ such that $\varphi(A^n_{\geq 0}) \subset A^m_{\geq 0}$.

We denote $\mathrm{Cone} := \mathrm{Cone}_{\mathbb{Z}}$. It is naturally a subcategory of $\mathrm{Cone}_{\mathbb{Q}}$ consisting of maps $\varphi\colon \mathbb{Q}^n \to \mathbb{Q}^m$ such that $\varphi(\mathbb{Z}^n) \subset \mathbb{Z}^m$.

Definition 5.0.1. An *A-linear formal fan* is a functor $\mathrm{Cone}_A^{\mathrm{op}} \to \mathbf{Set}$.

A $\mathbb{Z}$-linear formal fan will simply be called a *formal fan*, while by *rational formal fan* we mean a $\mathbb{Q}$-linear formal fan. A-linear formal fans form categories, the corresponding functor categories.

If $F_\bullet$ is a formal fan, then the multiplicative monoid $\mathbb{Z}_{>0}$ acts on each F_n: if $l \in \mathbb{Z}_{>0}$, multiplication by l gives a map $[n] \to [n]$, which in turns gives a map $F_k \to F_k$. We can localise this action by setting $F_\bullet^{\mathbb{Q}} := \varinjlim_{l \in \mathbb{Z}_{>0}} F_n$, which is the set of symbols a/l with $a \in F_n$ and $l \in \mathbb{Z}_{>0}$, and were we identify $a/l = a'/l'$ if there is $m \in \mathbb{Z}_{>0}$ such that $ml'a = mla'$. This gives a rational formal fan $F_\bullet^{\mathbb{Q}}$, that we will refer to as the *rational formal fan associated to F*.

We introduce the notion of degeneration fan of an artinian lattice, which is analogue to that of the degeneration fan of a point on a stack (see **??**). We fix an artinian lattice L for the rest of this subsection.

Definition 5.0.2. Let P be a lattice. A P-filtration of L is a map $F\colon P \to L\colon a \mapsto F_{\geq a}$ such that

1. There exists $a \in P$ such that $F_{\geq a} = 1$.
2. For $a \leq b$ in P, we have $F_{\geq b} \leq F_{\geq a}$.
3. For $a, b \in P$, we have $F_{\geq a \vee b} = (F_{\geq a}) \wedge (F_{\geq b})$.
4. There are $a_1, \dots, a_m \in P$ such that, for all $a \in P$, we have

$$F_{\geq a} = \bigvee_{a_i \geq a} F_{\geq a_i}.$$

5. The sublattice of L generated by $F(P)$ is distributive.

Remark 5.0.3. If F is a P-filtration of L and $a \in P$, then we define $F_{>a} = \bigvee_{b>a} F_{\geq b}$. We say that a is a *jump* of the filtration F if $F_{>a} < F_{\geq a}$. The set of jumps of F is denoted $\operatorname{supp} F$ and called the *support* of F. If $a_1, \dots, a_m \in P$ satisfy condition 4 in Definition 5.0.2, then $\operatorname{supp} F$ is a subset of $\{a_1, \dots, a_m\}$.

Remark 5.0.4. Note that condition 2 in Definition 5.0.2 actually follows from condition 4. On the other hand, for $P = \mathbb{Z}^n$ it is not hard to see that condition 4 follows from the other conditions. However, for $P = \mathbb{Q}$, condition 4 is necessary, since otherwise the jumps of F could be irrational.

Definition 5.0.5. Let P be a set and let L be an artinian lattice. A *P-grading* of L is a family $f = (x_i)_{i \in P}$ of elements of L indexed by P such that

1. $\bigvee_{i \in P} x_i = 1$; and
2. for all $i \in P$, we have $x_i \wedge x_i' = 0$, where $x_i' = \bigvee_{j \in P \setminus \{i\}} x_j$.

Lemma 5.0.6. *Let $M \subset L$ be a sublattice that is a finite boolean algebra with atoms $a_1, \dots, a_l$, and let $x \in L \setminus \{0\}$ such that for all $a \in M$ we have $a \wedge x = 0$. Then the sublattice M' of L generated by M and x is a finite boolean algebra with atoms $a_1, \dots, a_l, x$.*

Proof. If $a, b \in M$, then

$$(a \vee x) \wedge b \leq (a \vee x) \wedge (a \vee b) = (x \wedge (a \vee b)) \vee a = a,$$

so $a \wedge b \le (a \vee x) \wedge b \le a \wedge b$ and thus $(a \vee x) \wedge b = a \wedge b$. We also have

$$(a \vee x) \wedge (b \vee x) = ((a \vee x) \wedge b) \vee x = (a \wedge b) \vee x.$$

Let $c \in M$ be the maximal element of M. The formulas above show that the map $M \times \{0, x\} \to M' \colon (a, z) \mapsto a \vee z$ is an isomorphism of lattices with inverse $y \mapsto (y \wedge c, y \wedge x)$. The result follows. □

Lemma 5.0.7. *If P is a set and $g = (x_i)_{i \in P}$ is a grading of L, then the sublattice M of L generated by the x_i is a finite boolean algebra with set of atoms $\{x_i \mid i \in P, x_i \neq 0\}$. In particular, this set is finite.*

Proof. Let $a_1, \dots, a_l$ be different elements in $\{x_i \mid i \in P, x_i \neq 0\}$. We prove by induction on l that the sublattice N generated by $a_1, \dots, a_l$ is a finite boolean algebra with atoms $a_1, \dots, a_l$. It is clear for $l = 1$. For $l > 1$, let N' be the sublattice generated by $a_1, \dots, a_{l-1}$. If $a \in N'$, then $a \wedge a_l \le a'_l \wedge a_l = 0$, so $a \wedge a_l = 0$. The result then follows from Lemma 5.0.6. In particular, l is at most the length of L, so the set $\{x_i \mid i \in P, x_i \neq 0\}$ is finite and the lemma is proven by taking $a_1, \dots, a_l$ to be all the elements of this set. □

Proposition and Definition 5.0.8. Let P be a lattice and let $g = (x_i)_{i \in P}$ be a P-grading of L. Then the map $F^g \colon P \to L$ given by

$$F^g_{\ge c} = \bigvee_{i \ge c} x_i,$$

for $c \in P$, is a P-filtration. We call F^g the *P-filtration associated to g.*

Proof. Let $c_1, \dots, c_n$ be the elements of P such that $x_{c_i} \neq 0$. We denote $x_{c_i} = x_i$. Then $F^g_{\ge c_1 \wedge \dots \wedge c_n} = 1$. By Lemma 5.0.7, the sublattice of L generated by $F^g(P)$ lives inside a finite boolean algebra and it is thus distributive. Therefore,

$$F^g_{\ge a} \wedge F^g_{\ge b} = \left(\bigvee_{i \ge a} x_i \right) \wedge \left(\bigvee_{j \ge b} x_j \right) = \bigvee_{\substack{i \ge a \\ j \ge b}} x_i \wedge x_j = \bigvee_{i \ge a \vee b} x_i = F^g_{\ge a \vee b}.$$

Note as well that from the definition it follows that $F^g_{\ge c} = \bigvee_{c_i \ge c} F^g_{\ge c_i}$. Therefore F^g satisfies all conditions in Definition 5.0.2 □

Proposition 5.0.9. *Suppose that L is a finite boolean algebra, that P is a lattice, and that F is a P-filtration of L. Then there is a unique P-grading g of L such that $F^g = F$.*

Proof. For $a \le b$ in L, we denote b/a the unique complement of a in $[0, b]$. For a P-grading $g = (x_c)_{c \in P}$ of L, we have $x_c = F^g_{\ge c}/F^g_{> c}$. This proves uniqueness.

For existence, we define $x_c = F_{\ge c}/F_{> c}$ for all $c \in P$. The set $\{c \in P \mid x_c \neq 0\}$ is precisely the set of jumps of F. Call them $c_1, \dots, c_n$ and let $x_j = x_{c_j}$ for $j = 1, \dots, n$. We may assume, after a permutation of $\{1, \dots, n\}$, that $c_i \le c_j$ implies $j \le i$ for all $i, j = 1, \dots, n$. Let M_i be the sublattice of L generated by $x_1, \dots, x_i$. We prove by induction on i the following:

1. the lattice M_i is a finite boolean algebra with atoms $x_1, \dots, x_i$;
2. for all $j = 1, \dots, i$, we have $F_{\ge c_j} = \bigvee_{c_l \ge c_j} x_l$ (note that if $c_l \ge c_j$, then $l \le j$).

For $i = 1$, the first assertion is clear, and the second follows from the equality $F_{>c_1} = 0$. Suppose now that they hold for $i < n$. We have

$$x_{i+1} \wedge (x_1 \vee \cdots \vee x_i) = x_{i+1} \wedge \left(\bigvee_{j \leq i} F_{\geq c_j} \right) \leq F_{\geq c_{i+1}} \wedge \left(\bigvee_{j \leq i} F_{\geq c_j} \right) =$$
$$\bigvee_{j \leq i} F_{\geq c_j \vee c_{i+1}} = \bigvee_{j \leq i} \bigvee_{c_l \geq c_j \vee c_{i+1}} x_l = \bigvee_{c_l > c_{i+1}} x_l = F_{>c_{i+1}}.$$

Since $x_{i+1} \wedge F_{>c_{i+1}} = 0$, we actually have $x_{i+1} \wedge (x_1 \vee \cdots \vee x_i) = 0$. By Lemma 5.0.6 we have that M_{i+1} is a finite boolean algebra with atoms $x_1, \ldots, x_{i+1}$. Now

$$F_{\geq c_{i+1}} = x_{i+1} \vee F_{>c_{i+1}} = x_{i+1} \vee \bigvee_{c_l > c_{i+1}} x_l = \bigvee_{c_l \geq c_{i+1}} x_l.$$

It follows that $g = (x_c)_{c \in P}$ is a P-grading of L and that $F^g = F$. □

Proposition and Definition 5.0.10. Let P, P' be lattices, let $g \colon P \to P'$ be a map of posets and let F be a P-filtration of L. The map $g_* F \colon P' \to L$ defined by

$$(g_* F)_{\geq c} = \bigvee_{g(a) \geq c} F_{\geq a}$$

is a P'-filtration of L, called the *pushforward* of F along g.

Proof. Conditions 1 and 2 in the definition are straightforward to check. Let us check condition 3. We have, using distributivity and condition 3 for F,

$$(g_* F)_{\geq c} \wedge (g_* F)_{\geq d} = \left(\bigvee_{g(a) \geq c} F_{\geq a} \right) \wedge \left(\bigvee_{g(b) \geq d} F_{\geq d} \right) =$$
$$\bigvee_{\substack{g(a) \geq c \\ g(b) \geq d}} F_{\geq a \vee b} = \bigvee_{g(e) \geq c \vee d} F_{\geq e} = (g_* F)_{\geq c \vee d}.$$

The penultimate equality follows from the fact that

$$\{a \vee b \,|\, a, b \in P,\ g(a) \geq c,\ g(b) \geq d\} = \{e \in P \,|\, g(e) \geq c \vee d\}.$$

If $a_1, \ldots, a_m \in P$ satisfy condition 4 in Definition 5.0.2 for F, then $g(a_1), \ldots, g(a_m)$ satisfy the same condition for $g_* F$. Finally, the sublattice of L generated by $(g_* F)(P')$ is contained in that generated by $F(P)$, and it is thus distributive. □

From now on we fix a subring A of $\mathbb{R}$, endowed with the inherited order.

Definition 5.0.11. Let F be a A^n-filtration of L and let $h \colon A^l \to A^n$ be an A-linear order-preserving map[1]. Let $h^\vee \colon A^n \to A^l$ be the dual map of h. We define the *pullback of F along h* to be the A^l-filtration $h^* F \coloneqq (h^\vee)_* F$ of L.

If $a \in A_{\geq 0}$, multiplication by a defines an order-preserving map $h \colon A^n \to A^n$. We define the scalar multiplication $aF \coloneqq h^* F$.

From the definition we see that A^n-filtrations of L for different n are arranged in an A-linear formal fan (Definition 5.0.1).

[1]For a linear map $h \colon A^l \to A^n$, we have that h is order-preserving if and only if $h(A^l_{\geq 0}) \subset A^n_{\geq 0}$.

Definition 5.0.12. The *A-linear degeneration fan* $\mathbf{DF}^A(L)_\bullet$ of L is the A-linear formal fan defined as follows:

- For $n \in \mathbb{N}$, $\mathbf{DF}^A(L)_n$ is the set of A^n-filtrations of L.
- If $h\colon A^l \to A^n$ is a morphism in Cone_A, then the associated map $\mathbf{DF}^A(L)_n \to \mathbf{DF}^A(L)_l$ is given by $F \mapsto h^*F$.

We abbreviate $\mathbf{DF}(L)_\bullet := \mathbf{DF}^\mathbb{Z}(L)_\bullet$ and simply call it the *degeneration fan* of L.

The elements of $\mathbf{DF}(L)_1$ are called *integral filtrations* of L, and the elements of $\mathbf{DF}^\mathbb{Q}(L)_1$ are called *rational filtrations* of L. The reader will convince themselves that $\mathbf{DF}^\mathbb{Q}(L)_\bullet$ is actually the rational formal fan associated to $\mathbf{DF}(L)_\bullet$ as in subsection **??**.

Proposition 5.0.13. *Let k be a field, $d \in \mathbb{N}$, and let $\mathrm{pt}\colon \mathrm{Spec}\, k \to B\, GL(d)_k$ be the standard point, corresponding to the vector space k^d. Let L be the lattice of vector subspaces of k^d. Then there is a canonical isomorphism*

$$\mathbf{DF}(B\, GL(d)_k, \mathrm{pt})_\bullet \cong \mathbf{DF}(L)_\bullet$$

of formal fans.

Proof. We write $\mathbf{DF}(\mathrm{GL}(d)_k)_\bullet = \mathbf{DF}(B\,\mathrm{GL}(d)_k, \mathrm{pt})_\bullet$. The formal fan $\mathbf{DF}(\mathrm{GL}(d)_k)_\bullet$ can be explicitly described as

$$\mathbf{DF}(\mathrm{GL}(d)_k)_n = \mathrm{Hom}(\mathbb{G}_{m,k}^n, \mathrm{GL}(d)_k)/\sim$$

where for $\lambda \in \mathrm{Hom}(\mathbb{G}_{m,k}^n, \mathrm{GL}(d)_k)$, we set $\lambda \sim \lambda^g$ if $g \in P(\lambda)(k)$. This follows from [5, Theorem 1.4.8].

A homomorphism $\lambda\colon \mathbb{G}_{m,k}^n \to \mathrm{GL}(d)_k$ is the same data as grading $V := k^d = \bigoplus_{\chi \in \Gamma_\mathbb{Z}(\mathbb{G}_{m,k}^n)} V_\chi^\lambda$ of V by $\Gamma_\mathbb{Z}(\mathbb{G}_{m,k}^n)$, where each $V_\chi^\lambda \le V$. We identify $\Gamma_\mathbb{Z}(\mathbb{G}_{m,k}^n) = \mathbb{Z}^n$ canonically, and see it as a poset with product order. Thus there is a canonical bijection between the set of homomorphisms $\lambda\colon \mathbb{G}_{m,k}^n \to \mathrm{GL}(d)_k$ and the set of $\mathbb{Z}^n$-gradings of L. For such a λ, we have the associated filtration given by $F_{\ge c}^\lambda = \bigoplus_{\chi \in \mathbb{Z}^n} V_\chi^\lambda$, for $c \in \mathbb{Z}^n$. It is a $\mathbb{Z}^n$-filtration by Proposition and Definition 5.0.8.

Claim 5.0.14. For all $g \in \mathrm{GL}(d)(k)$, we have that $g \in P(\lambda)(k)$ if and only if for all $c \in \mathbb{Z}^n$, $g(F_{\ge c}^\lambda) \subset F_{\ge c}^\lambda$.

Indeed, $g \in P(\lambda)(k)$ if and only if $\lim_{t\to 0} \lambda(t) g \lambda(t)^{-1}$ exists. Write $g = (g_{\chi,\chi'})_{\chi,\chi' \in \mathbb{Z}^n}$, where $g_{\chi,\chi'}\colon V_{\chi'}^\lambda \to V_\chi^\lambda$. Then $(\lambda(t) g \lambda(t)^{-1})_{\chi,\chi'} = (\chi - \chi')(t) g_{\chi,\chi'}$. Thus $g \in P(\lambda)(k)$ if and only if for all $\chi, \chi' \in \mathbb{Z}^n$, if $g_{\chi,\chi'} \neq 0$ then $\chi \ge \chi'$. This condition is equivalent to

$$\forall \chi \in \mathbb{Z}^n,\ gV_\chi^\lambda \subset F_{\ge\chi}^\lambda,$$

which is in turn equivalent to

$$\forall \chi \in \mathbb{Z}^n,\ gF_{\le\chi}^\lambda \subset F_{\ge\chi}^\lambda.$$

This proves the claim. Note the equalities $V_\chi^{\lambda^g} = gV_\chi^\lambda$ and $F_{\ge\chi}^{\lambda^g} = gF_{\ge\chi}^\lambda$ for any $g \in \mathrm{GL}(d)(k)$. If $g \in P(\lambda)(k)$, then $F^{\lambda^g} = F^\lambda$. Thus the assignment $\lambda \mapsto F^\lambda$ gives a well-defined map $f_n\colon \mathbf{DF}(\mathrm{GL}(d)_k)_n \to \mathbf{DF}(L)_n$.

Suppose $\lambda_1, \lambda_2\colon \mathbb{G}_{m,k}^n \to \mathrm{GL}(d)_k$ are such that $F := F^{\lambda_1} = F^{\lambda_2}$. Let $\mathcal{C}$ be the set of jumps of F, and let $\chi_1, \dots, \chi_l$ be a labelling of the elements of $\mathcal{C}$ in such a way that $\mathcal{X}_i \le \mathcal{X}_j$ implies $i \le j$. Note that $\mathcal{C} = \{\chi \in \mathbb{Z}^n \mid V_\chi^{\lambda_i} \neq 0\}$ for $i = 1, 2$. Let $U_i = \sum_{j \ge i} F_{\ge\chi_j}$. We have $U_l \subsetneqq U_{l-1} \subsetneqq \cdots \subsetneqq U_1 = V$.

We define inductively an automorphism $g_i\colon U_i \to U_i$ such that $g_i(V_{\chi_i}^{\lambda_1}) = V_{\chi_i}^{\lambda_2}$ and $g_i|_{U_{i+1}} = g_{i+1}$. We set $g_l = \mathrm{id}_{U_1}$. Suppose g_i is defined, $i \geq 2$. We have two direct sum decompositions, $U_i = U_{i-1} \oplus V_{\chi_i}^{\lambda_1} = U_{i-1} \oplus V_{\chi_i}^{\lambda_2}$, so there is an isomorphism $o\colon V_{\chi_i}^{\lambda_1} \to V_{\chi_i}^{\lambda_2}$. We set $g_{i-1} = g_i \oplus o$. Now let $g = g_1$. We have $g(F_{\geq c}) = g\left(\sum_{\chi \geq c} V_\chi^{\lambda_1}\right) = \sum_{\chi \geq c} g\left(V_\chi^{\lambda_1}\right) = \sum_{\chi \geq c} V_\chi^{\lambda_2} = F_{\geq c}$. Thus $g \in P(\lambda_1)$ and $\lambda_2 = \lambda_1^g$, since $V_\chi^{\lambda_2} = gV_\chi^{\lambda_1}$ for all $\chi \in \mathbb{Z}^n$. This proves $f_n\colon \mathbf{DF}(\mathrm{GL}(d)_k)_n \to \mathbf{DF}(L)_n$ is injective.

Let $F \in \mathbf{DF}(L)_n$, and let M be a maximal distributive sublattice of L containing each $F_{\geq c}$ for $c \in \mathbb{Z}^n$. By Proposition 4.0.4, M is complemented. By Proposition 5.0.9, there is a grading λ of M, which is also a grading of L, such that $F = F^\lambda$. The grading λ can be seen as a map $\lambda\colon \mathbb{G}_{m,k}^n \to \mathrm{GL}(d)_k$, and thus $F = f_n(\lambda)$. The map f_n is therefore surjective.

It is only left to check naturality of the maps f_n. Let $h\colon \mathbb{Z}^l \to \mathbb{Z}^n$ be an order-preserving linear map and $\lambda\colon \mathbb{G}_{m,k}^n \to \mathrm{GL}(d)_k$. We denote $h^*(\lambda) := \lambda \circ D(h^\vee)$, where D denotes the Cartier dual map. This descends to give the pullback map $h^*\colon \mathbf{DF}(\mathrm{GL}(d)_k)_n \to \mathbf{DF}(\mathrm{GL}(d)_k)_l$. By Cartier duality, for $\chi \in \mathbb{Z}^n$, the torus $\mathbb{G}_{m,k}^l$ acts on V_χ^λ as $h^\vee(\chi)$, so $V_\alpha^{h^*\lambda} = \bigoplus_{h^\vee(\chi)=\alpha} V_\chi^\lambda$ for all $\alpha \in \mathbb{Z}^l$. Therefore for all $c \in \mathbb{Z}^l$ we have

$$F_{\geq c}^{h^*\lambda} = \bigoplus_{\substack{\chi \in \mathbb{Z}^n \\ h^\vee(\chi) \geq c}} V_\chi^\lambda = \sum_{\substack{\chi \in \mathbb{Z}^n \\ h^\vee(\chi) \geq c}} F_{\geq \chi}^\lambda = (h^* F^\lambda)_{\geq c},$$

as desired.

□

Definition 5.0.15. Let $F^{(1)}, \ldots, F^{(n)} \in \mathbf{DF}^A(L)_1$. We say that $F^{(1)}, \ldots, F^{(n)}$ *commute* if the sublattice of L they generate is distributive. In that case, we define the *box sum* by the formula

$$\left(F^{(1)} \boxplus \cdots \boxplus F^{(n)}\right)_{\geq d} = F_{\geq d_1}^{(1)} \wedge \cdots \wedge F_{\geq d_n}^{(n)},$$

for $d \in \mathbb{A}^n$.

Lemma 5.0.16. *For commuting A-filtrations $F^{(1)}, \ldots, F^{(n)} \in \mathbf{DF}^A(L)_1$, the box sum $F^{(1)} \boxplus \cdots \boxplus F^{(n)}$ is an A^n-filtration*

Proof. The box sum $H = F^{(1)} \boxplus \cdots \boxplus F^{(n)}$ satisfies all conditions of Definition 5.0.2 except perhaps condition 5. However, note that the sublattice generated by $H(A^n)$ is the same as that generated by $F^{(1)}(A) \cup \cdots \cup F^{(n)}(A)$, which is distributive by the commutativity assumption. □

Definition 5.0.17. Let $D_\bullet$ be an A-linear formal fan, and $F \in D_n$. The *vertices* $v_1 F, \ldots, v_n F \in D_1$ of F are the pullbacks of F along the maps $A \to A^n$ given by the standard basis of A^n.

Compare the following proposition with **??**.

Proposition 5.0.18. *For all $n \in \mathbb{Z}_{>0}$, the map $\varphi_n\colon \mathbf{DF}^A(L)_n \to \mathbf{DF}^A(L)_1^n\colon F \mapsto (v_1 F, \ldots, v_n F)$ is injective. It is bijective for $n = 2$.*

Moreover, $(F^{(1)}, \ldots, F^{(n)}) \in \mathbf{DF}^A(L)_1^n$ lie in the image of φ_n if and only if $F^{(1)}, \ldots, F^{(n)}$ commute. In that case $F^{(1)} \boxplus \cdots \boxplus F^{(n)}$ is the unique preimage.

Proof. Injectivity follows from the formula $F = v_1 F \boxplus \cdots \boxplus v_n F$, for $F \in \mathbf{DF}^A(L)_n$. Clearly, the $v_i F$ commute. Moreover, if $F^{(1)}, \ldots, F^{(n)}$ commute, then it follows from the definitions that $v_i\left(F^{(1)} \boxplus \cdots \boxplus F^{(n)}\right) = F^{(i)}$ for $i = 1, \ldots n$.

The sublattice of L generated by any two chains is distributive [2, Theorem III.7.9]. Therefore, any two filtrations $F, G \in \mathbf{DF}^A(L)$ commute, so φ_2 is surjective. $\square$

Remark 5.0.19. The statement that φ_2 is bijective can be interpreted as the degeneration fan $\mathbf{DF}^A(L)_\bullet$ being convex. The crucial fact here is that the sublattice generated by two chains is distributive. This can be seen as the lattice analogue of the fact that in a reductive algebraic group G, the intersection of any two parabolic subgroups contains a maximal torus of G.

Definition 5.0.20. Let $F_1, F_2 \in \mathbf{DF}^A(L)_1$. We define the *sum* of F_1 and F_2 as $F_1 + F_2 \coloneqq \binom{1}{1}^* (F_1 \boxplus F_2)$.

Remark 5.0.21. In principle, one could define the box sum $F^{(1)} \boxplus \cdots \boxplus F^{(n)} \in \mathbf{DF}^A(L)_n$ for commuting $F^{(i)} \in \mathbf{DF}^A(L)_{n_i}$ with $n = \sum_i n_i$, and use it to define an internal sum on each $\mathbf{DF}^A(L)_n$. We will not however make use of this construction.

The following computation readily follows from the definitions.

Proposition 5.0.22. *Let $F, G \in \mathbf{DF}^A(L)_1$. Then the sum $F + G$ is given by the formula*

$$(F+G)_{\geq c} = \bigvee_{a+b\geq c} F_{\geq a} \wedge G_{\geq b} = \bigvee_{a+b=c} F_{\geq a} \wedge G_{\geq b}, \quad c \in A.$$

Definition 5.0.23. We say that a formal fan $D_\bullet$ is *connected* if D_0 has a single element, that we denote 0. If $f\colon [n] \to [0]$ is the unique map between $[n]$ and $[0]$, we also denote $0 = f^* 0 \in D_n$.

Note that $\mathbf{DF}^A(L)_\bullet$ is connected. The zero n-filtration is given by, for $c \in A^n$,

$$0_{\geq c} = \begin{cases} 1, & c \leq 0^n \\ 0, & \text{else.} \end{cases}$$

Definition 5.0.24. If $F \in \mathbf{DF}^A(L)_1$ is a filtration, an *opposite* of F is an element $G \in \mathbf{DF}^A(L)_1$ such that $F + G = 0$.

Remark 5.0.25. A filtration may have several opposites, as the following example shows. Suppose that L is the lattice of vector subspaces of $\mathbb{C}^2$. Let F be the filtration given by $\mathbb{C}\langle(1,0)\rangle \leq \mathbb{C}^2$ and weights $1, 0$. Let $v \in \mathbb{C}^2 \setminus \mathbb{C}\langle(1,0)\rangle$ and let G_v be the filtration given by $\mathbb{C}\langle v\rangle \leq \mathbb{C}^2$ and weights $0, -1$. Then $F + G_v = 0$. In fact every opposite of F is of the form G_v for some $v \in \mathbb{C}^2 \setminus \mathbb{C}\langle(1,0)\rangle$.

The following proposition is the lattice analogue of **??**.

Proposition 5.0.26. *The artinian lattice L is complemented if and only if every element of $\mathbf{DF}^A(L)_1$ has an opposite.*

Proof. If L is complemented and F is an A-filtration of L, we can find a maximal distributive sublattice $D \subset L$ such that $F \in \mathbf{DF}^A(D)_1$. Note that D is complemented by Proposition 4.0.4 and thus a boolean algebra. If $x_1, \dots, x_n$ are the atoms of D, an A-grading g on D is a map $g\colon \{x_1, \dots, x_n\} \to A$, so the set of A-gradings of D is A^n. The map $A^n \to \mathbf{DF}^A(D)_1\colon g \mapsto F^g$ is a bijection by Proposition 5.0.9, and we have $F^{g_1+g_2} = F^{g_1} + F^{g_2}$ by Proposition 5.0.22. Thus every element of $\mathbf{DF}^A(D)_1$ has a (unique) opposite in $\mathbf{DF}^A(D)_1$, and in particular F has an opposite in $\mathbf{DF}^A(L)_1$.

Conversely, suppose that every filtration of L has an opposite, and let $x \in L$. Consider the filtration F of L given by

$$F_{\geq c} = \begin{cases} 0, & c > 1 \\ x, & 0 < c \leq 1 \\ 1, & c \leq 0. \end{cases}$$

Let $G \in \mathbf{DF}^A(L)_1$ be an opposite of F. It follows from Proposition 5.0.22 that $x \wedge G_{\geq 0} = 0$ and $x \vee G_{\geq 0} = 1$. Thus x has a complement in L. $\square$

Definition 5.0.27. Let $F, G \in \mathbf{DF}(L)_1$. We say $F \leq G$ if for all $c \in \mathbb{Z}$, we have $F_{\geq c} \leq G_{\geq c}$.

This defines a poset structure on $\mathbf{DF}(L)_1$.

Definition 5.0.28. A *formal fan with relation* is a pair $(D_\bullet, \leq)$ where $D_\bullet$ is a connected formal fan and $\leq$ is a relation on D_1.

If $(D_\bullet, \leq)$ is a formal fan with relation, we define the set

$$L_D = \{F \in D_1 \mid \ F \geq 0 \text{ and } \forall G \in D_1,\ 0 \leq 2G \leq F \implies G = 0\},$$

endowed with the relation $\leq$ inherited from D_1.

Definition 5.0.27 gives a formal fan with relation $(\mathbf{DF}(L)_\bullet, \leq)$.

Proposition 5.0.29. *Let $(D_\bullet, \leq)$ be a formal fan with relation. Suppose that there is an artinian lattice M and an isomorphism $\varphi\colon (D_\bullet, \leq) \to (\mathbf{DF}(M)_\bullet, \leq)$ of formal fans with relation. Then $(L_D, \leq)$ is an artinian lattice isomorphic to M. Moreover, there is canonical isomorphism $(\mathbf{DF}(L_D)_\bullet, \leq) \to (D_\bullet, \leq)$ of formal fans with relation, independent of M and φ.*

Proof. For every $a \in M$, we denote $f(a) \in \mathbf{DF}(M)_1$ the $\mathbb{Z}$-filtration of M given by

$$f(a)_{\geq c} = \begin{cases} 0, & c > 1, \\ a, & 0 < c \leq 1, \\ 1, & c \leq 0. \end{cases}$$

If $a \leq b$, then $f(a) \leq f(b)$, so the map $f\colon M \to \mathbf{DF}(M)_1\colon a \mapsto f(a)$ is an injection of posets.

Claim 5.0.30. The image of f is $L_{\mathbf{DF}(M)}$.

Proof of Claim. Let $f \in L_{\mathbf{DF}(M)}$ with jumps $c_0 < \ldots < c_n$ (in $\mathbb{Z}$). Since $F \geq 0$, we have $c_i \geq 0$ for all i. Let $a = F_{\geq c_n} > 0$. Then $rf(a) \leq F$ for all $r = 1, \ldots, c_n$, so c_n is 0 or 1. If $c_n = 0$, then $n = 0$ and $F = 0 = f(0)$. If $c_n = 1$, then $F = f(a)$. Thus $L_{\mathbf{DF}(M)} \subset f(M)$.

Let $a \in M$ and suppose there is $G \in \mathbf{DF}(M)_1$ with $0 \leq G$ and $2G \leq f(a)$ but $G \neq 0$. If $c_0 < \cdots < c_n$ are the jumps of G, then $c_n > 0$, since $G \neq 0$. Thus $2c_n$ is a jump of $2G$, but

$$0 \leq (2G)_{\geq 2c_n} \leq F_{\geq 2c_n} = 0,$$

since $2c_n > 1$. Therefore $(2G)_{\geq 2c_n} = 0$, contradicting that $2c_n$ is a jump of $2G$. It follows that $f(a) \in L_{\mathbf{DF}(M)}$. $\square$

Therefore f gives an isomorphism $M \cong L_{\mathbf{DF}(M)}$. Since φ induces an isomorphism $L_{\mathbf{DF}(M)} \cong L_D$, we have an isomorphism $M \cong L_D$ of sets with relation, and thus L_D is an artinian lattice.

Let $F \in \mathbf{DF}(M)_1$ be given by jumps $c_0 < \cdots < c_n$ in $\mathbb{Z}$ and a chain $1 > a_1 > \cdots > a_n > 0$ in M. Then $F = c_0 f(1) + (c_1 - c_0) f(a_1) + \cdots + (c_n - c_{n-1}) f(a_n)$ and the sum is associative because $f(1), f(a_1), \ldots, f(a_n)$ live inside a common distributive sublattice of M. We can use this observation to define an isomorphism $\alpha\colon \mathbf{DF}(L_D)_\bullet \to D_\bullet$ independently of φ.

Note that the map $v^{(n)}\colon D_n \to (D_1)^n$ that maps an element $F \in D_n$ to its vertices $(v_1 F, \ldots, v_n F) \in (D_1)^n$ is injective for all $n \in \mathbb{N}$ and bijective for $n = 2$ by Proposition 5.0.18. Therefore we can define sums and box sums of elements of D_1 as follows. We say that $F^{(1)}, \ldots, F^{(n)} \in D_1$ *commute* if $(F_1, \ldots, F_n)$ is in the image of $v^{(n)}$. In that case we denote $F^{(1)} \boxplus \cdots \boxplus F^{(n)}$ the unique preimage. We define the *sum* of $F, G \in D_1$ as $F + G \coloneqq \binom{1}{1}^* (F \boxplus G)$.

If $F \in \mathbf{DF}(L_D)_1$ is given by jumps $c_0 < \cdots < c_n$ in $\mathbb{Z}$ and a chain $1 > b_1 > \cdots > b_n > 0$ in L_D, we define $\alpha_1(F) = c_0 1 + (c_1 - c_0) b_1 + \cdots + (c_n - c_{n-1}) b_n$. The sum is associative because it is so in the case $D_\bullet = \mathbf{DF}(M)_\bullet$, and the map $\alpha_1\colon \mathbf{DF}(L_D)_1 \to D_1$ is an isomorphism since it is so in the case $D_\bullet = \mathbf{DF}(M)_\bullet$. We define $\alpha_n(F^{(1)} \boxplus \cdots \boxplus F^{(n)}) = \alpha_1(F^{(1)}) \boxplus \cdots \boxplus \alpha_1(F^{(n)})$. This is well defined and a bijection between $\mathbf{DF}(L_D)_n$ and D_n since it is so in the case $D_\bullet = \mathbf{DF}(M)_\bullet$. Thus $\alpha\colon (\mathbf{DF}(L_D)_\bullet, \leq) \to (D_\bullet, \leq)$ is an isomorphism of formal fans with relation and it is defined solely in terms of $(D_\bullet, \leq)$. □

6 Normed lattices and the HKKP filtration

In this section, we fix an artinian lattice L and a subring A of $\mathbb{R}$, endowed with the induced order.

Definition 6.0.1 (Complementedness)**.** Let $F \in \mathbf{DF}^A(L)_1$ be an A-filtration of L. We define the *complementedness* $\langle F, \mathfrak{l} \rangle$ of F to be

$$\langle F, \mathfrak{l} \rangle \coloneqq \sup\{ b \in A_{\geq 0} \mid \forall c \in A,\ [F_{\geq c+b}, F_{\geq c}] \text{ is complemented} \} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

The filtration F is said to be *paracomplemented* if $\langle F, \mathfrak{l} \rangle \geq 1$. We denote $\mathcal{B}_A(L)$ the set of all paracomplemented A-filtrations of L

Remark 6.0.2. The notion of paracomplementedness for $\mathbb{R}$-filtrations coincides with the original definition [4, Section 4.3].

Lemma 6.0.3. *Let D be a maximal distributive sublattice of L and let $F \in \mathbf{DF}^A(L)_1$ be a filtration that factors through D. If $\langle F, \mathfrak{l}_D \rangle$ denotes the complementedness of F seen as a filtration of D, then $\langle F, \mathfrak{l}_D \rangle = \langle F, \mathfrak{l} \rangle$.*

Proof. This follows directly from Corollary 4.0.5. □

As the lemma suggests, in order to understand complementedness for L it will be key to study first the distributive case. To that aim, we introduce some terminology.

Let M be a finite boolean algebra with atoms $x_1, \ldots, x_n$ and denote $\Gamma^A(M)$ the set of A-gradings of M. If $\lambda = (\lambda_i)_{i \in A}$ and $\mu = (\mu_i)_{i \in A}$ are A-gradings of M, we can consider the sum $\lambda + \mu = \left(\bigvee_{a+b=i} \lambda_a \wedge \mu_b\right)_{i \in A}$. If $a \in A$, we also have the scalar product $a\lambda = (\lambda_{i/a})_{i \in A}$. This makes $\Gamma^A(M)$ into a free A-module of rank n. To see this, note that an element λ of $\Gamma^A(M)$ is determined by assigning an element a_i^λ of A to each atom x_i, and the map $\Gamma^A(M) \to A^n\colon \lambda \mapsto (a_1^\lambda, \ldots, a_n^\lambda)$ is a bijection respecting the sum and scalar multiplication. This bijection gives a canonical basis of $\Gamma^A(M)$.

Definition 6.0.4. Let M be a finite boolean algebra. We denote $\Gamma^A(M)$ the A-module of A-gradings of M and $\Gamma_A(M)$ the dual module, which we call the module of A-*linear characters* of M.

Let D be a distributive artinian lattice, and consider it as a sublattice of its associated finite boolean algebra, $D \subset T_D$. Let $x_1, \ldots, x_n$ be the atoms of T_D, let $e_1, \ldots, e_n$ be the canonical basis of $\Gamma^A(T_D)$ and let $e_1^\vee, \ldots, e_n^\vee \in \Gamma_A(T_D)$ be the dual basis. We write $x_i \to x_j$ if $i \neq j$ and

$$\forall a \in D, \quad x_i \leq a \implies x_j \leq a$$

Note that this recovers the arrows of the directed acyclic graph described in the proof of Theorem 3.0.2.

Definition 6.0.5. The *state* of D is the finite subset

$$\Xi_D = \{e_j^\vee - e_i^\vee \mid i, j \in \{1, \ldots, n\},\ x_i \to x_j\}$$

of $\Gamma_{\mathbb{Z}}(T_D)$.

Remark 6.0.6. There is a canonical injection $\Gamma_{\mathbb{Z}}(T_D) \subset \Gamma_A(T_D)$, and thus we can see the state Ξ_D as a subset of $\Gamma_A(T_D)$ as well.

Every filtration of D can be seen as a filtration of T_D. Thus there is an injection $\mathbf{DF}^A(D)_1 \to \mathbf{DF}^A(T_D)_1 = \Gamma^A(T_D)$ through which we see $\mathbf{DF}^A(D)_1$ as a subset of $\Gamma^A(T_D)$.

Proposition 6.0.7. *Let D be a distributive artinian lattice. Via the canonical injection $DF^A(D)_1 \to \Gamma^A(T_D)$ we have the equality*

$$\mathbf{DF}^A(D)_1 = \{\lambda \in \Gamma^A(T_D) \mid \ \forall w \in \Xi_D,\ \langle \lambda, w \rangle \geq 0\}.$$

Proof. We see D as the lattice of closed subgraphs of the directed acyclic graph with vertices $x_1, \ldots, x_n$ and arrows $x_i \to x_j$ as in Theorem 3.0.2. If $\lambda \in \Gamma^A(T_D)$ is a grading and $c \in A$ we have $F^\lambda_{\geq c} = \bigvee_{\langle \lambda, e_i^\vee \rangle \geq c} x_i$. This is an element of D precisely if for each arrow $x_i \to x_j$ we have

$$\langle \lambda, e_i^\vee \rangle \geq c \implies \langle \lambda, e_j^\vee \rangle \geq c$$

This condition holds for all $c \in A$ if and only if $\langle \lambda, e_i^\vee \rangle \leq \langle \lambda, e_j^\vee \rangle$. Thus F^λ is a filtration of D precisely when $\langle \lambda, e_j^\vee - e_i^\vee \rangle \geq 0$ for all arrows $x_i \to x_j$. □

Proposition 6.0.8. *Let D be a distributive lattice and $F^\lambda \in \mathbf{DF}^A(D)_1$ a filtration of D, with $\lambda \in \Gamma^A(T_D)$. We have the following formula for the complementedness of F^λ:*

$$\langle F^\lambda, \mathfrak{l} \rangle = \inf\{\langle \lambda, w \rangle \mid \ w \in \Xi_D\}.$$

In particular, F^λ is paracomplemented if and only if for all $w \in \Xi_D$ we have $\langle \lambda, w \rangle \geq 1$.

Proof. Again we see D as the lattice of closed subgraphs of the directed acyclic graph Q with vertices $x_1, \ldots, x_n$ and arrows $x_i \to x_j$. For $b, c \in A$ with $b \geq 0$, the interval $[F_{\geq c+b}, F_{\geq c}]$ is isomorphic to the full subgraph of Q with set of vertices $\{x_i \mid \ c \leq \langle \lambda, e_i^\vee \rangle < c + b\}$. Thus $[F_{\geq c+b}, F_{\geq c}]$ is complemented if and only if there are no arrows $x_i \to x_j$ with $c \leq \langle \lambda, e_i^\vee \rangle, \langle \lambda, e_j^\vee \rangle < c+b$. It follows that $[F_{\geq c+b}, F_{\geq c}]$ is complemented for all $c \in A$ if and only if $\langle \lambda, e_j^\vee - e_i^\vee \rangle \geq b$ for all arrows $x_i \to x_j$, which can be seen by setting $c = \langle \lambda, e_i^\vee \rangle$. We thus have

$$b \leq \langle F^\lambda, \mathfrak{l} \rangle \iff b \leq \inf\{\langle \lambda, w \rangle \mid \ w \in \Xi_D\}.$$

Since this holds for all $b \geq 0$ in A, we have the desired equality. □

Corollary 6.0.9. *The set $\mathcal{B}_A(L)$ of paracomplemented A-filtrations is convex, in the sense that for all $F, G \in \mathcal{B}_A(L)$ and $t \in [0,1] \cap A$, we have $(1-t)F + tG \in \mathcal{B}_A(L)$.*

Proof. Take a maximal distributive sublattice D of L containing F and G. By Proposition 6.0.8, the complementedness $\langle -, \mathfrak{l} \rangle$ is a convex function on $\mathbf{DF}^A(D)_1$. The result follows. □

Definition 6.0.10. An A-valued *norm* X on L is the data of a nonnegative number $X([a,b]) \in A_{\geq 0}$ for every interval $[a,b]$ of L such that

1. $X([a,b]) = 0$ if and only if $a = b$;
2. $X([a,b]) = X([a,c]) + X([c,b])$ if $a \leq c \leq b$; and
3. $X([a,b]) = X([a',b'])$ if $[a,b] \sim [a',b']$.

From now on, we fix an A-valued norm X on L

Definition 6.0.11. For every filtration $F \in \mathbf{DF}^A(L)_1$, we define its *norm squared* $\|F\|^2$ to be

$$\|F\|^2 = \sum_{c \in A} c^2 X([F_{>c}, F_{\geq c}]).$$

Theorem 6.0.12. *Suppose A is a field. Then there is a unique paracomplemented filtration $F \in \mathcal{B}_A(L)$ minimising $\|-\|^2$ on the set $\mathcal{B}_A(L)$ of paracomplemented A-filtrations of L.*

Proof. Let D be a maximal distributive sublattice of L. We first study the problem of minimising $\|-\|^2$ in $\mathbf{DF}^A(D)_1$. We denote $x_1, \dots x_n$ the atoms of T_D and $e_1, \dots, e_n$ the standard basis of $\Gamma^A(T_D)$. The norm X on L induces norms on D and T_D that we still denote X. The norm on T_D gives an inner product $(-,-)$ on $\Gamma^A(T_D)$, whose matrix in the basis $e_1, \dots, e_n$ is $\operatorname{diag}(X([0,x_1]), \dots, X([0,x_n]))$. For $\lambda \in \Gamma^A(T_D)$ we have $\|\lambda\|^2 = \|F^\lambda\|^2$. Thus we are looking to minimise $\|\lambda\|^2$ for $\lambda \in R_A$, where $R_A = \{\lambda \in \Gamma^A(T_D) \mid \inf\langle \lambda, \Xi_D \rangle \geq 1\}$ is the set of those λ such that F^λ is paracomplemented. Since $R_\mathbb{R}$ is closed and convex and $\|-\|^2$ is strictly convex, there is a unique minimiser $\mu \in R_\mathbb{R}$ of $\|-\|^2$. Let $C = \{w \in \Xi_D \mid \langle \mu, w \rangle = 1\}$ and consider the affine subspace $V = \{\lambda \in \Gamma^\mathbb{R}(T_D) \mid \forall w \in C,\ \langle \lambda, w \rangle = 1\}$ of $\Gamma^\mathbb{R}(T_D)$. The minimiser μ is then the point in V that is closest to the origin. Note that V is defined over A, so actually we must have $\mu \in R_A$, since the inner product $(-,-)$ defining the norm is also defined over A and A is a field. We conclude that there is a unique paracomplemented $F \in \mathbf{DF}^A(D)_1$ minimising $\|-\|^2$.

From Theorem 3.0.2 we see that there are only finitely many isomorphism types of normed distributive lattices that appear as maximal distributive sublattices of L. Indeed, once we fix a maximal chain $0 = a_0 < \cdots < a_n = 1$ of L, distributive sublattices D of L containing the a_i are determined by the set of arrows of the corresponding directed acyclic graph with vertices $\{1, \dots, n\}$. The norm on D is determined by attaching the number $c_i = X([a_{i-1}, a_i]) \in A_{>0}$ to each vertex i. If we choose a different maximal chain $0 = b_0 < \cdots < b_n = 1$, by the theorem of Jordan-Hölder-Dedekind, we have $X([a_{i-1}, a_i]) = X([b_{\sigma(i)-1}, b_{\sigma(i)}])$ for a permutation σ of $\{1, \dots, n\}$. Therefore all maximal distributive sublattices D of L are isomorphic, as normed lattices, to the lattice of closed subgraphs of a directed acyclic graph with set of vertices $\{1, \dots, n\}$ and norm given by the numbers $c_1, \dots, c_n$. We conclude that the minimum of $\|-\|^2$ on $\mathcal{B}_A(L)$ is attained at some $F \in \mathcal{B}_A(L)$.

Suppose that there are two paracomplemented filtrations $F, G \in \mathcal{B}_A(L)$ minimising $\|-\|^2$. By [2, Theorem III.7.9], the lattice generated by the $F_{\geq c}$ and the $G_{\geq c}$ is distributive.

Therefore there is a maximal distributive sublattice D of L such that both F and G factor through D. We must have $F = G$ from uniqueness of the minimiser in the distributive case. $\square$

Definition 6.0.13. Suppose A is a field. Then the unique minimiser F of $\|-\|^2$ on $\mathcal{B}_A(L)$ is called the *HKKP weight filtration* (or simply the *HKKP filtration*) of the A-normed lattice (L, X).

Even though our definition of the weight filtration is a priori different from that given in [4], we will prove that the two definitions are equivalent. To that aim, we first study the notion of linear form on a lattice.

Definition 6.0.14. An *A-valued linear form* ℓ on L is the data of a number $\ell([a, b]) \in A$ for each interval $[a, b]$ of L such that:

1. $\ell([a, b]) = \ell([a, c]) + \ell([c, b])$ if $a \le c \le b$; and
2. $\ell([a, b]) = \ell([a', b'])$ if $[a, b] \sim [a', b']$.

If ℓ is an A-valued linear form on ℓ and $F \in \mathbf{DF}^A(L)_1$ is a filtration, we define the pairing

$$\langle F, \ell\rangle \coloneqq \sum_{c\in A} c\ \ell([F_{>c}, F_{\ge c}]).$$

The word *linear* in the definition is justified by the following property.

Lemma 6.0.15. *Let ℓ be an A-valued linear form on L. For $a, b \in A_{\ge 0}$ and $F, G \in \mathbf{DF}^A(L)_1$ we have*

$$\langle aF + bG, \ell\rangle = a\langle F, \ell\rangle + b\langle G, \ell\rangle.$$

Proof. By taking a distributive sublattice D of L through which both F and G factor, and subsequently taking the associated boolean algebra T_D, we may assume that $L = M$ is a finite boolean algebra. Then ℓ is given by an element $\alpha \in \Gamma_A(M)$, and for $\lambda \in \Gamma^A(M)$ we have the equality $\langle F^\lambda, \ell\rangle = \langle \lambda, \alpha\rangle$. In this context, linearity is clear. $\square$

Definition 6.0.16. Let ℓ be a linear form on L. We say that L is *semistable with respect to ℓ* if for all $F \in \mathbf{DF}^A(L)_1$ we have $\langle F, \ell\rangle \le 0$.

The following proposition relates the notion of semistability above with what in [4] is called *semistable of phase 0*.

Proposition 6.0.17. *Let ℓ be a linear form on L. Then L is semistable with respect to ℓ if and only if $\ell(L) = 0$ and for all $x \in L$ we have $\ell([0, x]) \le 0$.*

Proof. Suppose first that L is semistable with respect to L. Let $a \in A$ and consider the filtration F of L characterised by the fact that $F_{>a} = 0$, $F_{\ge a} = 1$. We have $\langle F, \ell\rangle = a\ell(L)$. Thus $a\ell(L) \le 0$ for all $a \in A$, which forces $\ell(L) = 0$.

Now let $x \in L$ and consider the filtration F with jumps 0 and 1 such that $F_{\ge 1} = x$ and $F_{\ge 0} = 1$. We have $\ell([0, x]) = \langle F, \ell\rangle \le 0$, as desired.

Conversely, suppose that $\ell(L) = 0$ and for all $x \in L$ we have $\ell([0, x]) \le 0$. Let $F \in \mathbf{DF}^A(L)_1$ be any filtration with jumps $c_1 < \ldots < c_n$. We have

$$\begin{aligned}\langle F, \ell\rangle &= \sum_{i=1}^{n} c_i\ell([F_{>c_i}, F_{\ge c_i}]) = \sum_{i=1}^{n-1} c_i\left(\ell([0, F_{\ge c_i}]) - \ell([0, F_{\ge c_{i+1}}])\right) + c_n\ell([0, F_{\ge c_n}]) = \\ &= \sum_{i=2}^{n} (c_i - c_{i-1})\ell([0, F_{\ge c_i}]) + c_1\ell([0, F_{\ge c_1}]) \le 0,\end{aligned}$$

since all $c_i - c_{i-1} > 0$ and $\ell([0, F_{\ge c_1}]) = \ell(L) = 0$. $\square$

If $L' \subset L$ is a sublattice and ℓ is a linear form on L, there is an induced linear form $\ell|_{L'}$ on L' given by $\ell|_{L'}([a,b]) = \ell([a,b])$ for $a \leq b$ in L'.

Definition 6.0.18. Let ℓ be a linear form on L and suppose that L is semistable with respect to L. We denote $L^{\mathrm{ss}(\ell)}$ (or just L^{ss} if ℓ is clear from the context) the sublattice of L consisting of those elements $x \in L$ such that $[0,x]$ is semistable with respect to $\ell|_{[0,x]}$.

Since in the definition L itself is assumed to be semistable, $x \in L$ is in L^{ss} if and only if $\ell([0,x]) = 0$. If $x, y \in L^{\mathrm{ss}}$, then

$$\ell([0, x \vee y]) = \ell([x, x \vee y]) = \ell([x \wedge y, y]) = -\ell([0, x \wedge y]),$$

therefore $\ell([0, x \vee y]) = \ell([0, x \wedge y]) = 0$, since both are ≤ 0. Thus L^{ss} is indeed a sublattice of L, and it contains 0 and 1.

In [4], the local structure of $\mathcal{B}_A(L)$ around a paracomplemented filtration F is described in terms of another lattice $\Lambda(F)$. We now recall how this works.

Definition 6.0.19. Let $F \in \mathbf{DF}^A(L)_1$ be an A-filtration of L. We define the *associated graded lattice* $\mathrm{Grad}_F(L)$ to be the product lattice

$$\mathrm{Grad}_F(L) \coloneqq \prod_{c \in A} [F_{>c}, F_{\geq c}].$$

The associated graded lattice is normed by setting

$$X\left([(x_c)_{c \in A}, (y_c)_{c \in A}]\right) = \sum_{c \in A} X([x_c, y_c]).$$

We still denote by X the norm on $\mathrm{Grad}_F(L)$. The lattice $\mathrm{Grad}_F(L)$ carries naturally a linear form $F^\vee$, defined by

$$F^\vee([(x_c)_{c \in A}, (y_c)_{c \in A}]) = \sum_{c \in A} cX([x_c, y_c]).$$

Suppose that D is a maximal distributive sublattice of L containing F. Then each $[F_{>c}, F_{\geq c}] \cap D$ is a maximal distributive sublattice of $[F_{>c}, F_{\geq c}]$ by Proposition 4.0.3, and thus $D_{\mathrm{gr}} = \prod_{c \in A} [F_{>c}, F_{\geq c}] \cap D$ is a maximal distributive sublattice of $\mathrm{Grad}_F(L)$. Note that D and D_{gr} have the same associated boolean algebra T_D. Suppose that G is an A-filtration of $\mathrm{Grad}_F(L)$ contained in D_{gr}. Then we have the formula

$$\langle G, F^\vee \rangle = (G, F), \tag{1}$$

where in the right hand side, G and F are seen as elements in $\Gamma^A(T_D)$ and $(-,-)$ is the inner product on $\Gamma^A(T_D)$ given by X.

Definition 6.0.20. Suppose that $F \in \mathcal{B}_A(L)$ is a paracomplemented A-filtration. We define the sublattice $\Lambda(F)$ of $\mathrm{Grad}_F(L)$ consisting of those $(x_c)_{c \in A} \in \mathrm{Grad}_F(L)F$ such that for each $c \in A$ the interval $[x_{c+1}, x_c]$ of L is complemented.

It is not obvious that $\Lambda(F)$ is actually a sublattice of $\mathrm{Grad}_F(L)$. This fact is [4, Proposition 4.5].

Lemma 6.0.21. *Let $F \in \mathcal{B}_A(L)$ be a paracomplemented A-filtration of L, and let D be a maximal distributive sublattice of L containing F. Then $D_{\Lambda(F)} \coloneqq D_{\mathrm{gr}} \cap \Lambda(F)$ is a maximal distributive sublattice of $\Lambda(F)$ with associated boolean algebra canonically isomorphic to T_D. Moreover, its state is*

$$\Xi_{D_{\Lambda(F)}} = \{w \in \Xi_D \mid \langle \lambda, w \rangle = 1\} \subset \Gamma_A(T_D),$$

where $\lambda \in \Gamma^A(T_D)$ is such that $F^\lambda = F$

Proof. Let D' be a maximal distributive sublattice of $\Lambda(F)$ containing $D_{\Lambda(F)}$. The projections $p_c\colon D' \to [F_{>c}, F_{\geq c}]$ are maps of lattices whose image contains $D \cap [F_{>c}, F_{\geq c}]$. Indeed, we have a section $s_c\colon D \cap [F_{>c}, F_{\geq c}] \to D_{\Lambda(F)} \subset D'$ defined by

$$s_c(x)_d = \begin{cases} F_{\geq d}, & d > c, \\ x, & d = c, \\ F_{>d}, & d < c. \end{cases}$$

Since $D \cap [F_{>c}, F_{\geq c}]$ is a maximal distributive sublattice of $[F_{>c}, F_{\geq c}]$ and $p_c(D')$ is distributive, we must have $p_c(D') = D \cap [F_{>c}, F_{\geq c}]$. Therefore $D' = D_{\Lambda(F)}$.

Denote F_D the filtration F seen as a filtration of D. From Corollary 4.0.5 it follows that $D_{\Lambda(F)} = \Lambda(F_D)$, that is $D_{\Lambda(F)}$ consists of the elements $(y_c)_{c\in A} \in D_{\mathrm{gr}}$ such that $[y_{c+1}, y_c] \cap D$ is complemented for all $c \in A$. We can regard D canonically as the lattice of closed subgraphs of a directed acyclic graph Q with vertices $\{1, \dots, n\}$. Then the filtration F_D is given by an element of $\lambda \in \Gamma^A(T_D)$ that we see as a number $\lambda_i \in A$ for each vertex i. The interval $[y_{c+1}, y_c] \cap D$ is complemented for all c precisely if there are no arrows $i \to j$ with $\lambda_j = c+1$, $\lambda_i = c$, with i in y_c and j not in y_{c+1}. This means precisely that y defines a closed subgraph Q' of the subgraph of Q consisting only of those arrows $\alpha\colon i \to j$ with $\lambda_j - \lambda_i = 1$. Note that if $w_\alpha \in \Xi_D$ is the weight corresponding to α then $\langle \lambda, w\rangle = \lambda_j - \lambda_i$. Hence the subgraph Q' of Q corresponds to the sublattice of T_D whose state is $\{w \in \Xi_D \mid \langle \lambda, w\rangle = 1\}$. □

The linear form $F^\vee$ on $\mathrm{Grad}_F(L)$ induces a linear form on $\Lambda(L)$ that we still denote by $F^\vee$.

Theorem 6.0.22. *Suppose A is a field and let $F \in \mathcal{B}_A(L)$ be a paracomplemented A-filtration of L. Then the following are equivalent:*

1. *The filtration F minimises the function $\|-\|^2$ on $\mathcal{B}_A(L)$.*
2. *The lattice $\Lambda(F)$ is semistable with respect to the linear form $-F^\vee$.*

Therefore, the HKKP filtration (Definition 6.0.13) equals the weight filtration *as defined by Haiden-Katzarkov-Kontsevich-Pandit ([3, Theorem 4.9]).*

Proof. The first condition is equivalent to, for every $G \in \mathcal{B}_A(L)$, the function

$$f(t) = \frac{1}{2}\|(1-t)F + tG\|^2, \quad t \in [0,1] \cap A,$$

having a local minimum at $t = 0$. This follows by convexity of $\mathcal{B}_A(L)$ and strict convexity of $\|-\|^2$, which can be seen by taking a maximal distributive sublattice D of L such that both F and G factor through D.

Claim 6.0.23. There is $H \in \mathbf{DF}^A(\Lambda(L))_1$ such that $f'(0) = \langle H, F^\vee\rangle$.

Indeed, take $\lambda, \mu \in \Gamma^A(T_D)$ such that $F = F^\lambda$ and $G = G^\mu$. Then $f(t) = \frac{1}{2}\|(1-t)\lambda + t\mu\|^2$, where now the norm is associated to the inner product $(-,-)$ in $\Gamma^A(T_D)$. Therefore $f'(0) = (\mu - \lambda, \lambda) = \langle H, F^\vee\rangle$, where H is the filtration of $T_D = D_{\mathrm{gr}} \subset \mathrm{Grad}_F(L)$ associated to $\mu - \lambda$. Using the description of $\Xi_{D_{\Lambda(F)}}$ from Lemma 6.0.21, we see that for all $w \in \Xi_{D_{\Lambda(F)}}$ we have $\langle \mu - \lambda, w\rangle = \langle \mu, w\rangle - 1 \geq 0$, since μ is a filtration of D. Hence H is a filtration of $\Lambda(F)$ by Proposition 6.0.7. This proves the claim.

If $\Lambda(F)$ is semistable with respect to $-F^\vee$, we have that $f'(0) \geq 0$, so 0 is a local minimum of f. Since this holds for all $G \in \mathcal{B}_A(L)$, we have that F minimises $\|-\|^2$ on $\mathcal{B}_A(L)$.

Conversely, suppose that F maximises $\|-\|^2$ on $\mathcal{B}_A(L)$, and let H be an A-filtration of $\Lambda(F)$. Take a maximal distributive sublattice D of L containing F such that H factors through $D_{\Lambda(F)}$, and let $\mu, \lambda \in \Gamma^A(T_D)$ correspond to H and F. Since $\langle \mu, w\rangle \geq 0$ for all $w \in \Xi_D$ such that $\langle \lambda, w\rangle = 1$, we have, for small enough $t \in A \cap (0, 1]$ that $\langle \lambda + t\mu, \Xi_D\rangle \geq 1$. Thus $\lambda + t\mu$ corresponds to a filtration G of $D \subset L$. Defining f as above, we see that $f'(0) = \langle tH, F^\vee\rangle$. Since f has a local minimum at 0, it must be $f'(0) \geq 0$ and thus $\langle H, F^\vee\rangle \geq 0$. This proves that $\Lambda(F)$ is semistable with respect to $-F^\vee$.

□

We can get more intuition about why Theorem 6.0.22 is true by interpreting $\Lambda(F)$ as describing $\mathcal{B}_A(L)$ locally around F, as we now explain.

Keep assuming A is a field. For $F \in \mathcal{B}_A(L)$ and $\varepsilon > 0$ a real number, we define the set $B(F, \varepsilon)$ as

$$B(F, \varepsilon) = \{H \in \mathcal{B}_A(L) \mid \forall c \in A,\ d(\operatorname{supp} F, c) \geq \varepsilon \implies H_{\geq c} = F_{\geq c}\}.$$

Recall that $\operatorname{supp} F$ denotes the support of F, which is defined as the set of its jumps.

Suppose now that

$$\varepsilon < \frac{1}{2} \inf\{|c - c'| \mid c, c' \in \operatorname{supp} F,\ c \neq c'\}.$$

If G is an A-filtration of $\Lambda(F)$ with support inside $(-\varepsilon, \varepsilon)$ we define a filtration $F + G$ of L as follows. Let $c_1 < \ldots < c_n$ be the jumps of F. For $t \in A$ with $|t| < \varepsilon$ and $i \in \{1, \ldots, n\}$ we set

$$(F + G)_{\geq c_i + t} = (G_{\geq t})_{c_i},$$

and for $c \in A$ with $d(\operatorname{supp} F, c) \geq \varepsilon$ we set $(F + G)_{\geq c} = F_{\geq c}$. This yields a well-defined element $F + G \in \mathbf{DF}^A(L)_1$.

The following result is [4, Proposition 4.8] for the case $A = \mathbb{R}$, but the same proof is valid for all subfields A of $\mathbb{R}$.

Proposition 6.0.24. *Suppose A is a field and let $F \in \mathcal{B}_A(L)$ be a paracomplemented A-filtration. For small enough $\varepsilon > 0$, the map $G \mapsto F + G$ establishes a bijection between the set of A-filtrations of $\Lambda(F)$ with support contained in $(-\varepsilon, \varepsilon)$ and $B(F, \varepsilon)$.*

Intuitively the proof of Theorem 6.0.22 goes as follows. By convexity, F minimises $\|-\|^2$ on $\mathcal{B}_A(L)$ if and only if F is a local minimum of $\|-\|^2$ on $\mathcal{B}_A(L)$. Since locally around F elements of $\mathcal{B}_A(L)$ are of the form $F + H$, with $H \in \mathbf{DF}^A(\Lambda(F))_1$, this is equivalent to the function

$$f(t) = \frac{1}{2}\|F + tH\|^2, \quad t \in [0, 1] \cap A$$

having a local minimum at 0, that is $f'(0) \geq 0$. Computing the derivative we get $f'(0) = \langle H, F^\vee\rangle$, from which the result follows. A concrete way to establish the computation of $f'(0)$ is by working with maximal distributive sublattices, which is what we do in the proof of Theorem 6.0.22.

7 The HKKP chain

In analogy with the notion of chain for stacks (Definition **??**), we define the concept of chain of lattices. The operation of taking the HKKP of a normed artinian lattice L can be iterated, and the result is best expressed as a chain of lattices. As a shadow of this procedure, we obtain a $\mathbb{Q}^\infty$-filtration of the original lattice L.

Definition 7.0.1. A *chain of lattices* is data $(L_n, F_n, c_n)_{n\in\mathbb{N}}$ where

1. for every $n \in \mathbb{N}$, L_n is an artinian lattice endowed with a rational norm;
2. $F_n \in \mathbf{DF}^{\mathbb{Q}}(L_n)_1$ is a $\mathbb{Q}$-filtration of L_n;
3. $c_n\colon L_{n+1} \hookrightarrow \mathrm{Grad}_{F_n}(L_n)$ is a sublattice of $\mathrm{Grad}_{F_n}(L_n)$ in such a way that the norm on L_{n+1} is the restriction along c_n of the norm on $\mathrm{Grad}_{F_n}(L_n)$, which is itself inherited from L_n.

Definition 7.0.2. Let L be an artinian lattice endowed with a rational norm. The *HKKP chain of* L is the chain $(L_n, F_n, c_n)_{n\in\mathbb{N}}$ defined inductively as follows:

1. $L_0 = L$ as normed lattices;
2. if $n \in \mathbb{N}$ and L_n is defined, we let $F_n \in \mathbf{DF}^{\mathbb{Q}}(L_n)_1$ be its HKKP filtration (Definition 6.0.13). We let $L_{n+1} = \Lambda(F_n)^{\mathrm{ss}(-F_n^\vee)}$ (see Definitions 6.0.18 and 6.0.20) and c_n to be the inclusion
$$c_n\colon L_{n+1} \hookrightarrow \Lambda(F_n) \hookrightarrow \mathrm{Grad}_{F_n}(L_n).$$
We endow L_{n+1} with the norm induced from $\mathrm{Grad}_{F_n}(L_n)$.

For every $n \in \mathbb{N}$, there is an inclusion

$$\iota\colon L_{n+1} \hookrightarrow \mathrm{Grad}^{n+1}_{F_n,\ldots,F_0}(L) := \mathrm{Grad}_{F_n}(\mathrm{Grad}_{F_{n-1}}(\cdots(\mathrm{Grad}_{F_0}(L))\cdots),$$

which allows us to see F_{n+1} as a $\mathbb{Q}$-filtration of $\mathrm{Grad}^{n+1}_{F_n,\ldots,F_0}$. The ι_n are constructed inductively by letting $\iota_0 = c_0$ and ι_{n+1} to be the composition

$$L_{n+1} \xrightarrow{c_{n+1}} \mathrm{Grad}_{F_{n+1}}(L_{n+1}) \xrightarrow{\mathrm{Grad}_{F_{n+1}}(\iota_{n+1})} \mathrm{Grad}_{F_{n+1}}(\mathrm{Grad}^{n+1}_{F_n,\ldots,F_0}(L)),$$

noting that $\mathrm{Grad}_{F_{n+1}}(\mathrm{Grad}^{n+1}_{F_n,\ldots,F_0}(L)) = \mathrm{Grad}^{n+2}_{F_{n+1},\ldots,F_0}(L)$.

The data of the F_n, seen as $\mathbb{Q}$-filtrations of $\mathrm{Grad}^{n+1}_{F_n,\ldots,F_0}(L)$, is the same as the data of a $\mathbb{Q}^\infty$-filtration of L in the sense of Definition 5.0.2. Here $\mathbb{Q}^\infty = \mathbb{Q}^{\oplus\mathbb{N}}$ with lexicographic order. Since $\mathbb{Q}^\infty$ is a totally ordered set, a $\mathbb{Q}^\infty$-filtration of L can be described in more concrete terms as a chain

$$0 < a_1 < a_2 < \cdots < a_n = 1$$

of elements of L together with a chain

$$c_1 > c_2 > \cdots > c_n$$

of elements of $\mathbb{Q}^\infty$.

Definition 7.0.3. The *iterated HKKP filtration* of L is the $\mathbb{Q}^\infty$-filtration of L determined by the F_n as above.

8 The case of nilpotent quiver representations

In this section, k is an algebraically closed field.

We start by recalling certain basic definitions and fixing notations. A *quiver* Q is a finite directed graph. It consists of a finite set Q_0 of *vertices*, a finite set Q_1 of *arrows* and *source* and *target* maps $s, t\colon Q_1 \to Q_0$. If $\alpha \in Q_1$, we write $\alpha\colon i \to j$ if $i = s(\alpha)$ and $j = t(\alpha)$. A *representation* E of Q over the field k consists of a vector space E_i for every vertex $i \in Q_0$ and a linear map $x_\alpha\colon E_i \to E_j$ for every arrow $\alpha\colon i \to j$ of Q. We will sometimes denote $E = ((E_i)_{i\in Q_0}, (x_\alpha)_{\alpha\in Q_1})$. We say that E is *finite dimensional* if every

E_i is. A morphism of representations $E \to E'$, where $E' = ((E'_i)_i, (x'_\alpha)_\alpha)$, is a family $(f_i)_{i\in Q_0}$ of maps $f_i \colon E_i \to E'_i$ such that $x'_\alpha \circ f_i = f_j \circ x_\alpha$ for every arrow $\alpha \colon i \to j$.

A *dimension vector* d for Q is the data of a number $d_i \in \mathbb{N}$ for every vertex i of Q. The *representation space* for the dimension vector d is

$$\mathrm{Rep}(Q,d) = \bigoplus_{\alpha\colon i\to j} \mathrm{Hom}(k^{d_i}, k^{d_j}).$$

The group $G(d) = \prod_{i\in Q_0} \mathrm{GL}_{d_i,k}$ acts on $\mathrm{Rep}(Q,d)$ by

$$(g_i)\cdot(x_\alpha)_\alpha = (g_j x_\alpha g_i^{-1})_{\alpha\colon i\to j}.$$

The quotient stack $\mathscr{R}ep(Q,d) \coloneqq \mathrm{Rep}(Q,d)/G(d)$ is referred to as the *moduli stack of representations of Q of dimension vector d.*

We now fix a quiver Q, a dimension vector d and we denote $G = G(d)$, $V = \mathrm{Rep}(Q,d)$ and $\mathcal{X} = \mathscr{R}ep(Q,d)$. If k is of positive characteristic, then we assume that G is a torus, that is $d_i = 0$ or 1 for every vertex i. In any case, G is linearly reductive and $\mathscr{R}ep(Q,d)$ has a good moduli space. We fix a k-point x of V, and, by slight abuse of notation, we denote the image of x under $V \to \mathcal{X}$ also by x. The point x gives a representation $E = ((k^{d_i})_i, (x_\alpha)_\alpha)$, and we will denote $L = L_E$ the lattice of subrepresentations of E, which is an artinian lattice 2.0.1.

Proposition 8.0.1. *There is a canonical isomorphism*

$$\mathbf{DF}(\mathcal{X},x)_\bullet \cong \mathbf{DF}(L)_\bullet$$

of formal fans.

Proof. By Proposition **??**, there is a injection $\mathbf{DF}(\mathcal{X},x)_\bullet \to \mathbf{DF}(BG,\mathrm{pt})_\bullet$ that identifies $\mathbf{DF}(\mathcal{X},x)_\bullet$ with a subfan of $\mathbf{DF}(BG,\mathrm{pt})_\bullet$ by

$$\mathbf{DF}(\mathcal{X},x)_n = \{\gamma \in \mathbf{DF}(BG,\mathrm{pt})_n \mid \lim \gamma x \text{ exists}\},$$

and where

$$\mathbf{DF}(BG,\mathrm{pt})_n = \mathrm{Hom}(\mathbb{G}^n_{m,k}, G)/\sim,$$

with $\gamma \sim \gamma^g$ if $g \in P(\gamma)(k)$.

Let Q° be the quiver with $Q^\circ_0 = Q_0$ and $Q^\circ_1 = \varnothing$. Then E defines a representation E° of Q° which is just the vector space $\bigoplus_{i\in Q_0} E_i$ endowed with its grading by the set Q_0. The lattice L_{E° of subrepresentations of E° is the product

$$L_{E^\circ} = \prod_{i\in Q_0} L_{E_i}$$

of the lattices of vector subspaces of each E_i. Therefore Proposition 5.0.13 gives a canonical isomorphism

$$\mathbf{DF}(BG,\mathrm{pt})_\bullet = \prod_{i\in Q_0} \mathbf{DF}(B\,\mathrm{GL}_{d_i,k},\mathrm{pt})_\bullet \cong \prod_{i\in Q_0} \mathbf{DF}(L_{E_i})_\bullet \cong \mathbf{DF}(L^\circ)_\bullet \colon \gamma \mapsto F^\gamma.$$

The injection of lattices $L \to L^\circ$ gives an injection of formal fans $\mathbf{DF}(L)_\bullet \to \mathbf{DF}(L^\circ)_\bullet$. Now if $\gamma \in \mathbf{DF}(BG,\mathrm{pt})_n$ is represented by a group homomorphism $\mathbb{G}^n_{m,k} \to G$ (that we also denote γ), then the associated filtration $F^\gamma \in \mathbf{DF}(L^\circ)_n$ is given by

$$(F^\gamma_{\geq c})_i = \bigoplus_{\substack{\chi\in\mathbb{Z}^n \\ \chi\geq c}} (E_i)^\gamma_\chi, \quad c \in \mathbb{Z}^n,$$

where $(E_i)^\gamma_\chi$ is the eigenspace of E_i where $\mathbb{G}^n_{m,k}$ acts, via γ, through the character $\chi \in \Gamma_{\mathbb{Z}}(\mathbb{G}^n_{m,k}) \cong \mathbb{Z}^n$. The direct sum decomposition of the E_i allows us to write x in coordinates $x = (x_{\alpha,\chi,\chi'})_{\alpha \in Q_1, \chi,\chi' \in \mathbb{Z}^n}$ where, if $\alpha\colon i \to j$, then $x_{\alpha,\chi,\chi'}\colon (E_i)^\gamma_{\chi'} \to (E_j)^\gamma_\chi$ is the corresponding component of x_α. For F^γ to be a filtration of E we need that

$$x_\alpha((E_i)^\gamma_{\chi'}) \subset \bigoplus_{\substack{\chi \in \mathbb{Z}^n \\ \chi \geq \chi'}} (E_j)^\gamma_\chi$$

for all $\alpha\colon i \to j$ and all $\chi' \in \mathbb{Z}^n$. This is equivalent to $x_{\alpha,\chi,\chi'}$ being 0 whenever $\chi < \chi'$, for all $\alpha \in Q_1$. On the other hand,

$$\gamma(t)x = ((\chi - \chi')(t)x_{\alpha,\chi,\chi'})_{\alpha,\chi,\chi'},$$

so $\lim \gamma x$ exists if and only if $x_{\alpha,\chi,\chi'} = 0$ whenever $\chi < \chi'$. Therefore $\mathbf{DF}(\mathcal{X}, x)_\bullet$ and $\mathbf{DF}(L)_\bullet$ are the same subfan under the isomorphism $\mathbf{DF}(BG, \mathrm{pt})_\bullet \cong \mathbf{DF}(L^\circ)_\bullet$, so we get an isomorphism between $\mathbf{DF}(\mathcal{X}, x)_\bullet$ and $\mathbf{DF}(L)_\bullet$, as desired. □

We denote $\mathbf{DF}(\mathcal{X}, x)_n \to \mathbf{DF}(L)_n\colon \gamma \mapsto F^\gamma$ the isomorphism above.

Definition 8.0.2. We say that a representation $M = ((M_i)_{i \in Q_0}, (y_\alpha)_{\alpha \in Q_1})$ of Q is null if $y_\alpha = 0$ for all $\alpha \in Q_1$.

Proposition and Definition 8.0.3. Let M be a finite dimensional representation of Q of dimension vector d, corresponding to the k-point y of $\mathscr{Rep}(Q, d)$. The following are equivalent:

1. There is a filtration of M whose associated graded is null.
2. We have $0 \in \overline{\{y\}}$ inside $\mathscr{Rep}(Q, d)$.

If these conditions hold we say that M is a *nilpotent* representation of Q.

Proof. Since $\mathscr{Rep}(Q, d)$ has a good moduli space, $0 \in \overline{\{y\}}$ if and only if there is $\lambda \in \mathbf{DF}(\mathscr{Rep}(Q, d), y)_1$ with $\mathrm{ev}_0\,\lambda = 0$ by [1, Lemma 3.24]. This gives a filtration F^λ of M whose associated graded representation is null, and conversely any such filtration gives a suitable λ. □

Proposition 8.0.4. *Let M be a nilpotent representation of Q of dimension vector d. Then:*

1. *Any subquotient of M is also nilpotent.*
2. *The representation M is semisimple if and only if M is null.*

Proof. A Jordan-Hölder filtration of a subquotient N of M can be extended to a Jordan-Hölder filtration of M, whose associated graded must be null by hypothesis. Thus the associated graded of a Jordan-Hölder filtration of N is also null.

The representation M is semisimple if it is isomorphic to the associated graded representation of a Jordan-Hölder filtration, which is the null representation by hypothesis. □

Proposition 8.0.5. *Suppose that E is nilpotent and let $\lambda \in \mathbf{DF}^{\mathbb{Q}}(\mathcal{X}, x)_1$. Then*

$$\langle \lambda, \mathcal{X}^{\max} \rangle = \langle F^\lambda, \mathfrak{l} \rangle,$$

*where $\langle \lambda, \mathcal{X}^{\max} \rangle$ denotes Kempf's intersection number (Definition **??**) and $\langle F^\lambda, \mathfrak{l} \rangle$ is the complementedness of F^λ as a filtration of the lattice L (Definition 6.0.1).*

Proof. It is enough to assume that $\lambda \in \mathbf{DF}(\mathcal{X}, x)_1$. We choose a cocharacter $\mathbb{G}_{m,k} \to G$ representing λ that we still denote by λ. Then λ induces a direct sum decomposition $\mathrm{Rep}(Q,d) = V = \bigoplus_{l\in\mathbb{Z}} V_l$. Let $p_l\colon V \to V_l$ be the induced projections and let $\Xi_{x,\lambda} = \{l \in \mathbb{Z} \mid\ p_l(x) \neq 0\}$. Note that $\Xi_{x,\lambda} \subset \mathbb{N}$ since $\lim \lambda x$ exists.

Claim 8.0.6. $\langle \lambda, \mathcal{X}^{\max} \rangle = \inf \Xi_{x,\lambda}$.

Proof. Note that $\mathcal{X}^{\max} = V^G/G$. Since G is linearly reductive, there is a unique splitting $V = V^G \oplus V'$, with V' a G-subrepresentation of V. Choose a basis $v_1, \ldots, v_m$ of V of eigenvectors for λ and $0 \leq r \leq r' \leq m$ such that $v_1, \ldots, v_r$ is a basis of V^G, $v_1, \ldots, v_{r'}$ is a basis of V_0 and the v_i with $i > r$ are in V'. Let $n_i \in \mathbb{Z}$ such that $v_i \in V_{n_i}$. Taking Spec of the cartesian square

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & V^G \arrow[d] \\ \mathbb{A}^1_k \arrow[r, "t\mapsto\lambda(t)x"] & V \end{tikzcd}$$

we get

$$\begin{tikzcd} k[t]/I \arrow[dr, phantom, "\ulcorner"] & k[v_1^\vee, \ldots, v_r^\vee] \arrow[l] \\ k[t] \arrow[u] & k[v_1^\vee, \ldots, v_m^\vee], \arrow[l, "x_it^{n_i} \leftarrow\!\shortmid v_i^\vee"'] \arrow[u] \end{tikzcd}$$

where I is the ideal generated by the t^{n_i} with $x_i \neq 0$ and $i > r$. Since $0 \in \overline{Gx}$, we have $x_i = 0$ for $i \leq r$. Therefore $I = (t^a \mid\ a \in \Xi_{x,\lambda}) = \left(t^{\inf \Xi_{x,\lambda}}\right)$ and the claim follows. $\square$

Claim 8.0.7. For any $a \in \mathbb{N}$, we have that $F^\lambda_{\geq c}/F^\lambda_{\geq c+a}$ is semisimple for all $c \in \mathbb{Z}$ if and only if $\inf \Xi_{x,\lambda} \geq a$.

Proof. This induces a direct sum decomposition $E_i = \bigoplus_{l\in\mathbb{Z}}(E_i)_l$ of every E_i, $i \in Q_0$. The subquotient $F^\lambda_{\geq c}/F^\lambda_{\geq c+a}$ can be written in coordinates as $(x_{\alpha,nm})_{\alpha, c\leq n,m<c+a}$ where, if $\alpha\colon i \to j$, then $x_{\alpha,nm}\colon (E_i)_m \to (E_j)_n$ is the corresponding component of $x_\alpha\colon E_i \to E_j$. Since it is nilpotent, the representation $F^\lambda_{\geq c}/F^\lambda_{\geq c+a}$ is semisimple if and only if $x_{\alpha,nm} = 0$ for all $\alpha \in Q_1$ and $c \leq n, m < c + a$. This holds for all $c \in \mathbb{Z}$ if and only if $x_{\alpha,nm} = 0$ whenever $|n - m| < a$. Note that $\lambda(t)x = (t^{n-m}x_{\alpha,nm})_{\alpha\in Q_1,\, n,m\in\mathbb{Z}}$. Therefore $\Xi_{x,\lambda} = \{n - m \mid\ \exists \alpha \in Q_1,\ x_{\alpha,nm} \neq 0\}$. The claim follows. $\square$

The complementedness of F^λ is the supremum of the set of those $a \in \mathbb{N}$ such that $F_{\geq c}/F_{\geq c+a}$ is semisimple for all $c \in \mathbb{Z}$. Thus $\inf \Xi_{x,\lambda} = \langle F^\lambda, \mathfrak{l}\rangle$ by the second claim, and the result follows from the first claim. $\square$

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