0.1 Implicit monotone schemes in 1D

In this section we discuss some stability properties of implicit schemes.

First we show that schemes written in a particular form satisfy the upwind range condition, meaning that the solution is always bounded by its neighbors in space-time as it travels along the characteristics.

Next we show that such schemes are also total variation non-increasing (TVNI), or total variation diminishing (TVD), as it is commonly referred to.

We conclude the section with the observation that if a scheme satisfies the implicit upwind range condition, then it also satisfies the TVD condition.

Both conditions can be used to construct implicit high-order monotone schemes. However, as we will see, the upwind range condition is much more straightforward to apply, only requiring the solution to be bounded between its neighbors. It can be much more cumbersome to require the TVD property of a scheme directly.

0.1.1 Upwind range condition for $c > 0$

Let us consider the implicit upwind scheme in 1 space dimension

$$\phi_i^{n+1} = \phi_i^n - c\left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right). \tag{1}$$

We assume, that the only unknown value is ϕ_i^{n+1}. To see why that statement holds, notice that the solution of the first order advection equation requires an inflow boundary condition. Thus, in the first cell, the value ϕ_0^{n+1} is known from the inflow BC, so the only unknown value is ϕ_1^{n+1}. Then we compute the next value and so on, until reaching the n-th cell, where ϕ_{i-1}^{n+1} is known and the only unknown value is ϕ_i^{n+1}.

In this section we want to show that schemes of the form of (1) satisfy the stability condition

$$\min\left(\phi_{i-1}^{n+1}, \phi_i^n\right) \le \phi_i^{n+1} \le \max\left(\phi_{i-1}^{n+1}, \phi_i^n\right), \tag{2}$$

for $c > 0$. We can distinguish three cases:

1. $\phi_i^n > \phi_{i-1}^{n+1}$:

$$\min\left(\phi_{i-1}^{n+1}, \phi_i^n\right) = \phi_{i-1}^{n+1}, \quad \max\left(\phi_{i-1}^{n+1}, \phi_i^n\right) = \phi_i^n.$$

We can rewrite (2) as

$$
\begin{aligned}
\phi_{i-1}^{n+1} \leq \phi_i^{n+1} &\leq \phi_i^n, \\
0 \leq \phi_i^{n+1} - \phi_{i-1}^{n+1} &\leq \phi_i^n - \phi_{i-1}^{n+1}, \\
0 \leq \frac{\phi_i^{n+1} - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} &\leq 1.
\end{aligned}
$$

2. $\phi_i^n < \phi_{i-1}^{n+1}$:

$$\min\left(\phi_{i-1}^{n+1}, \phi_i^n\right) = \phi_i^n, \quad \max\left(\phi_{i-1}^{n+1}, \phi_i^n\right) = \phi_{i-1}^{n+1}.$$

We can rewrite (2) as

$$
\begin{aligned}
\phi_i^n \leq \phi_i^{n+1} &\leq \phi_{i-1}^{n+1}, \\
\phi_i^n - \phi_{i-1}^{n+1} \leq \phi_i^{n+1} - \phi_{i-1}^{n+1} &\leq 0, \\
1 \geq \frac{\phi_i^{n+1} - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} &\geq 0.
\end{aligned}
$$

3. $\phi_i^n = \phi_{i-1}^{n+1}$, in which case we have

$$\phi_i^n = \phi_{i-1}^{n+1} = \phi_i^{n+1}.$$

Notice, that for the first two cases, where $\phi_i^n \neq \phi_{i-1}^{n+1}$ we got the same expression. Thus, we can use

$$0 \leq \frac{\phi_i^{n+1} - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} \leq 1 \tag{3}$$

instead of (2) to simplify the analysis. The condition above can be interpreted as the upwind range condition[7] or data compatibility condition [11] for implicit schemes. Notice that min and max is a combination of values from both time steps $n, n-1$. This distinguishes the implicit upwind range property from the explicit one. In the explicit case, the min and max are chosen only from the values at the current time step $n-1$. The idea is that the solution travels along characteristics and this value is always bounded by its neighbors in space-time.

To prove (2), first we solve (1) for the unknown ϕ_i^{n+1}.

$$\phi_i^{n+1} = \frac{1}{1+c}\phi_i^n + \frac{c}{1+c}\phi_{i-1}^{n+1}. \tag{4}$$

Notice that

$$\frac{1}{1+c} = \frac{1+c-c}{1+c} = 1 - \frac{c}{1+c}.$$

For convenience, let us denote

$$k = \frac{c}{1+c}. \tag{5}$$

Since $0 \leq c \leq 1 + c$, then

$$\begin{aligned}
&0 \leq c \leq 1 + c \rightarrow \\
&0 \leq \frac{c}{1+c} \leq 1 \rightarrow \\
&0 \leq k \leq 1.
\end{aligned} \tag{6}$$

Thus, we can rewrite (4) as

$$\phi_i^{n+1} = (1-k)\phi_i^n + k\phi_{i-1}^{n+1}, \tag{7}$$

where we can see that the solution ϕ_i^{n+1} is an interpolation between ϕ_i^n and ϕ_{i-1}^{n+1}, thus (2) holds.

Theorem 0.1.1. *Let us consider a numerical scheme of the form*

$$\phi_i^{n+1} = (1-k)\phi_i^n + k\phi_{i-1}^{n+1}. \tag{8}$$

Assume that ϕ_{i-1}^n is known. If $0 \leq k \leq 1$, then the solution satisfies

$$\min\left(\phi_{i-1}^{n+1}, \phi_i^n\right) \leq \phi_i^{n+1} \leq \max\left(\phi_{i-1}^{n+1}, \phi_i^n\right), \tag{9}$$

Proof. Let us begin with the case $\phi_i^n \neq \phi_{i-1}^{n+1}$, for which we have to show that

$$0 \leq \frac{\phi_i^{n+1} - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} \leq 1. \tag{10}$$

We substitute (8) to (10)

$$\begin{aligned}
&0 \leq \frac{(1-k)\phi_i^n + k\phi_{i-1}^{n+1} - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} \leq 1, \\
&0 \leq \frac{(1-k)\left(\phi_i^n - \phi_{i-1}^{n+1}\right)}{\phi_i^n - \phi_{i-1}^{n+1}} \leq 1, \\
&0 \leq 1 - k \leq 1, \\
&0 \leq k \leq 1,
\end{aligned}$$

which we assumed that holds. For the case of $\phi_i^n = \phi_{i-1}^{n+1}$ we get

$$\phi_i^{n+1} = (1-k)\phi_i^n + k\ \phi_i^n = \phi_i^n,$$

thus completing the proof. □

0.1.2 Upwind-range-condition for different values of c

In general

$$\begin{aligned} min \leq \phi_i^{n+1} \leq max, \\ 0 \leq \phi_i^{n+1} - min \leq max - min, \\ 0 \leq \frac{\phi_i^{n+1} - min}{max - min} \leq 1. \end{aligned} \tag{11}$$

We choose min and max to lie on the closest characteristics to ϕ_i^{n+1}. Thus, e.g., for $0 < c < 1$,

$$min = \min(\phi_{i-1}^n, \phi_i^n), \quad max = \max(\phi_{i-1}^n, \phi_i^n).$$

So, min and max is always chosen from some value on the left, and a value on the right. Let us rewrite

$$min = \min(\phi_{i,L}, \phi_{i,R}), \quad max = \max(\phi_{i,L}, \phi_{i,R}).$$

We can distinguish three cases:

1. $\phi_{i,R} > \phi_{i,L}$:

   $$\min(\phi_{i,L}, \phi_{i,R}) = \phi_{i,L}, \quad \max(\phi_{i,L}, \phi_{i,R}) = \phi_{i,R}.$$

   We can rewrite (2) as

   $$\begin{aligned} \phi_{i,L} \leq \phi_i^{n+1} \leq \phi_{i,R}, \\ 0 \leq \phi_i^{n+1} - \phi_{i,L} \leq \phi_{i,R} - \phi_{i,L}, \\ 0 \leq \frac{\phi_i^{n+1} - \phi_{i,L}}{\phi_{i,R} - \phi_{i,L}} \leq 1. \end{aligned}$$

2. $\phi_{i,R} < \phi_{i,L}$:

   $$\min(\phi_{i,L}, \phi_{i,R}) = \phi_{i,R}, \quad \max(\phi_{i,L}, \phi_{i,R}) = \phi_{i,L}.$$

   We can rewrite (2) as

   $$\begin{aligned} \phi_{i,L} \geq \phi_i^{n+1} \geq \phi_{i,R}, \\ 0 \geq \phi_i^{n+1} - \phi_{i,L} \geq \phi_{i,R} - \phi_{i,L}, \\ 0 \leq \frac{\phi_i^{n+1} - \phi_{i,L}}{\phi_{i,R} - \phi_{i,L}} \leq 1. \end{aligned}$$

3. $\phi_{i,R} = \phi_{i,L}$, in which case we have

   $$\phi_{i,R} = \phi_{i,L} = \phi_i^{n+1}.$$

Thus, in general, we can write the upwind-range-condition as

$$0 \le \frac{\phi_i^{n+1} - \phi_{i,L}}{\phi_{i,R} - \phi_{i,L}} \le 1. \tag{12}$$

For $0 < c < 1$, the URC reads as

$$0 \le \frac{\phi_i^{n+1} - \phi_{i-1}^n}{\phi_i^n - \phi_{i-1}^n} \le 1. \tag{13}$$

For $1 < c < 2$, the URC would be

$$0 \le \frac{\phi_i^{n+1} - \phi_{i-2}^n}{\phi_{i-1}^n - \phi_{i-2}^n} \le 1. \tag{14}$$

Next, we will show, that for Courant numbers $c > 0$, it is a sufficient condition for stability to satisfy

$$0 \le \frac{\phi_i^{n+1} - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1. \tag{15}$$

0.1.3 TVD condition

The total variation of a mesh function $\phi^n = \phi_i^n$ is defined as

$$TV(\phi^n) = \sum_{i=-\infty}^{\infty} |\phi_i^n - \phi_{i-1}^n|, \tag{16}$$

see, e.g., [12, 2, 10, 13, 3, 9, 6, 7, 5, 8, 4]. To prove the TVD property of the implicit upwind scheme (1), first we have to show that if the initial data has compact support, than the solution has this property.

Theorem 0.1.2. *If the initial data has compact support, thus $\phi_i^n \to 0$ as $i \to \infty$ or $i \to -\infty$, then the solution of a scheme written as (1) will also satisfy $\phi_i^{n+1} \to 0$ as $i \to \infty$ or $i \to -\infty$.*

Proof. Let us begin with proving that if $\phi_i^n \to 0$ as $i \to -\infty$, then $\phi_i^{n+1} \to 0$ as $i \to -\infty$.

Assume, that for any $\epsilon \in \mathbb{R}^+$, there exists $I_\epsilon \in \mathbb{Z}$ such that

$$0 < \phi_i^n < \epsilon \text{ for all } i < I_\epsilon. \tag{17}$$

To have a unique solution, an inflow boundary condition must be given. Let $\mathcal{C}_{I_\epsilon}$ represent the cell at the inflow boundary. Thus, we have to specify $\phi_{I_\epsilon-1}^{n+1}$. Since we attempt to solve the transport equation with positive speed $v > 0$, the boundary value $\phi^{n+1}_{I_\epsilon - 1}$ must be coming from the upstream direction. Thus, it must be bounded by the values

$$0 < \min_{i < I_\epsilon - 1} (\phi^n_i) < \phi^{n+1}_{I_\epsilon - 1} < \max_{i < I_\epsilon - 1} (\phi^n_i) < \epsilon,$$

since $I_\epsilon - 1 < I_\epsilon$. Thus,

$$\phi^{n+1}_i \to 0 \text{ as } i \to -\infty,$$

concluding the first part of the proof.

To finish, we want to show that if $\phi^n_i \to 0$ as $i \to \infty$, then $\phi^{n+1}_i \to 0$ as $i \to \infty$. Assume, that for any $\epsilon_1 \in \mathbb{R}^+$, there exists $I_{\epsilon_1} \in \mathbb{Z}$ such that

$$0 < \phi^n_i < \epsilon_1 \text{ for all } i \geq I_{\epsilon_1}. \tag{18}$$

Since

$$\min\left(\phi^{n+1}_{I_{\epsilon_1}-1}, \phi^n_{I_{\epsilon_1}}\right) \leq \phi^{n+1}_{I_{\epsilon_1}} \leq \max\left(\phi^{n+1}_{I_{\epsilon_1}-1}, \phi^n_{I_{\epsilon_1}}\right), \tag{19}$$

this implies that

$$|\phi^n_{I_{\epsilon_1}} - \phi^{n+1}_{I_{\epsilon_1}}| \leq |\phi^n_{I_{\epsilon_1}} - \phi^{n+1}_{I_{\epsilon_1}-1}|.$$

Now choose an $i > I_{\epsilon_2} > I_{\epsilon_1}$, such that for any $\epsilon_2 \in \mathbb{R}^+$

$$|\phi^n_i - \phi^{n+1}_i| < |\max_{i \geq I_{\epsilon_1}}(\phi^n_i) - \phi^{n+1}_{I_{\epsilon_2}}| < \epsilon_2.$$

Denoting $\epsilon = \epsilon_1 + \epsilon_2$ we conclude that there exists $I_\epsilon > I_{\epsilon_2} > I_{\epsilon_1}$ such that

$$|\phi^n_i - \phi^{n+1}_i| < \epsilon,$$

which implies that if $\phi^n_i \to 0$ as $i \to \infty$, then the solution will also $\phi^{n+1}_i \to 0$ as $i \to \infty$, thus completing the proof. $\square$

Theorem 0.1.3. *The solution of a numerical scheme written in the form*

$$\phi^{n+1}_i = \phi^n_i - c\left(\phi^{n+1}_i - \phi^{n+1}_{i-1}\right), \tag{20}$$

for $c > 0$ and data with compact support, such that $\phi^n_i \to 0$ as $i \to \infty$ or $i \to -\infty$, is total variation non-increasing

$$TV(\phi^{n+1}_i) \leq TV(\phi^n_i). \tag{21}$$

Proof. First we put the terms with $n+1$ to the left-hand side, then we write the scheme for cells $i, i-1$

$$\phi_i^{n+1} + c\left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right) = \phi_i^n, \tag{22}$$

$$\phi_{i-1}^{n+1} + c\left(\phi_{i-1}^{n+1} - \phi_{i-2}^{n+1}\right) = \phi_{i-1}^n. \tag{23}$$

Then we subtract (23) from (22) to get

$$\begin{aligned} \phi_i^{n+1} - \phi_{i-1}^{n+1} + c\left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right) - c\left(\phi_{i-1}^{n+1} + \phi_{i-2}^{n+1}\right) &= \phi_i^n - \phi_{i-1}^n, \\ (1+c)\left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right) - c\left(\phi_{i-1}^{n+1} - \phi_{i-2}^{n+1}\right) &= \phi_i^n - \phi_{i-1}^n. \end{aligned} \tag{24}$$

We take the absolute value of both sides

$$|(1+c)\left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right) - c\left(\phi_{i-1}^{n+1} - \phi_{i-2}^{n+1}\right)| = |\phi_i^n - \phi_{i-1}^n|. \tag{25}$$

Then we use the triangle inequality $|a| - |b| \leq |a-b|$ and $|c\ a| = |c||a|$ to obtain

$$(1+c)|\phi_i^n - \phi_{i-1}^n| - c|\phi_{i-1}^n - \phi_{i-2}^n| \leq |(1+c)\left(\phi_i^n - \phi_{i-1}^n\right) - c\left(\phi_{i-1}^n - \phi_{i-2}^n\right)|. \tag{26}$$

Thus, using (25)

$$(1+c)|\phi_i^{n+1} - \phi_{i-1}^{n+1}| - c|\phi_{i-1}^{n+1} - \phi_{i-2}^{n+1}| \leq |\phi_i^n - \phi_{i-1}^n|. \tag{27}$$

To complete the proof, we take the sum of both sides

$$\begin{aligned} \sum_{i=-\infty}^{\infty}(1+c)|\phi_i^{n+1} - \phi_{i-1}^{n+1}| - \sum_{i=-\infty}^{\infty} c|\phi_{i-1}^{n+1} - \phi_{i-2}^{n+1}| &\leq \sum_{i=-\infty}^{\infty}|\phi_i^n - \phi_{i-1}^n|, \\ (1+c)\sum_{i=-\infty}^{\infty}|\phi_i^{n+1} - \phi_{i-1}^{n+1}| - c\sum_{i=-\infty}^{\infty}|\phi_{i-1}^{n+1} - \phi_{i-2}^{n+1}| &\leq \sum_{i=-\infty}^{\infty}|\phi_i^n - \phi_{i-1}^n|. \end{aligned} \tag{28}$$

We assumed that the initial data has compact support, which implies that the solution also have this property, meaning that

$$\sum_{i=-\infty}^{\infty}|\phi_{i-1}^{n+1} - \phi_{i-2}^{n+1}| = \sum_{i=-\infty}^{\infty}|\phi_i^{n+1} - \phi_{i-1}^{n+1}| = TV(\phi^{n+1}). \tag{29}$$

Thus, using (28) we conclude

$$\begin{aligned} (1+c)TV(\phi^{n+1}) - c\ TV(\phi^{n+1}) &\leq TV(\phi^n), \\ TV(\phi^{n+1}) &\leq TV(\phi^n). \end{aligned} \tag{30}$$

$\square$

0.1.4 Conclusion

Since we can transform any scheme written in the form of (1) to a scheme written as (7) and vice versa, then it follows that if a scheme satisfies the implicit upwind range property (2) it is also TVD.

0.1.5 High-order schemes based on the implicit upwind scheme

We can write a general second order scheme as

$$\begin{aligned} \phi_i^{n+1} &= \phi_i^n - c\left(\phi_i^{n+1} + \sigma_i \frac{1+c}{2}\Delta x - \phi_{i-1}^{n+1} - \sigma_{i-1}\frac{1+c}{2}\Delta x\right) \\ &= \phi_i^n - c\left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right) - c\left(\sigma_i - \sigma_{i-1}\right)\frac{1+c}{2}\Delta x, \end{aligned} \tag{31}$$

where σ_i and σ_{i-1} are slopes of a linear reconstruction in cells i and $i-1$ respectively. We can solve (31) for the new cell average

$$\phi_i^{n+1} = \frac{\phi_i^n + c\ \phi_{i-1}^{n+1}}{1+c} - \frac{c}{2}\left(\sigma_i - \sigma_{i-1}\right)\Delta x. \tag{32}$$

Substituting (32) to

$$0 \le \frac{\phi_i^{n+1} - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1$$

we get

$$\begin{aligned} 0 &\le \frac{\frac{\phi_i^n + c\ \phi_{i-1}^{n+1}}{1+c} - \frac{c}{2}\left(\sigma_i - \sigma_{i-1}\right)\Delta x - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1, \\ 0 &\le \frac{\frac{\phi_i^n - \phi_{i-1}^{n+1}}{1+c} - \frac{c}{2}\left(\sigma_i - \sigma_{i-1}\right)\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1, \\ 0 &\le \frac{1}{1+c} - \frac{c}{2}\frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} + \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1. \end{aligned} \tag{33}$$

We can derive sufficient conditions for the slope σ_i by considering

$$\begin{aligned} 0 &\le \frac{\phi_{i-1}^{n+1} - \sigma_{i-1}\frac{1+c}{2}\Delta x - \phi_{i-1}^{n-1}}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1, \\ 0 &\le \frac{1+c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1, \\ 0 &\le \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \le \frac{c}{1+c}. \end{aligned} \tag{34}$$

Thus,

$$
\begin{aligned}
0 &\le \frac{1}{1+c} - \frac{c}{2}\frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1 - \frac{c}{1+c}, \\
0 &\le \frac{1}{1+c} - \frac{c}{2}\frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \le \frac{1}{1+c}, \\
0 &\le \frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \le \frac{2}{c}\frac{1}{1+c},
\end{aligned} \tag{35}
$$

yielding

$$
|\sigma_i| \le \frac{2}{c}\frac{|\phi_i^n - \phi_{i-1}^{n+1}|}{(1+c)\Delta x}. \tag{36}
$$

Remark. *We can derive the schemes from [1] by writing the slope as a convex combination*

$$
\sigma_i = a\ \frac{\phi_{i+1}^n - \phi_i^n}{\Delta x} + b\ \frac{\phi_i^n - \phi_i^{n+1}}{c\ \Delta x} \tag{37}
$$

Let us write the slope as a linear combination

$$
\begin{aligned}
\sigma_i &= a\ \frac{\phi_{i+1}^n - \phi_i^n}{\Delta x} + b\ \frac{\phi_i^n - \phi_i^{n+1}}{c\ \Delta x}, \\
&= (a\ R_i + b)\,\frac{\phi_i^n - \phi_i^{n+1}}{c\ \Delta x}, \\
&= \Psi(R_i)\frac{\phi_i^n - \phi_i^{n+1}}{c\ \Delta x},
\end{aligned} \tag{38}
$$

where

$$
R_i = \frac{c\ \left(\phi_{i+1}^n - \phi_i^n\right)}{\phi_i^n - \phi_i^{n+1}} \tag{39}
$$

is the smoothness indicator.

$$
\begin{aligned}
0 &\le \frac{\phi_i^n - \phi_{i-1}^{n+1} - c\left(\Psi(R_i)\frac{\phi_i^n - \phi_i^{n+1}}{c\ \Delta x} - \Psi(R_{i-1})\frac{\phi_{i-1}^n - \phi_{i-1}^{n+1}}{c\ \Delta x}\right)\frac{1+c}{2}\Delta x}{(1+c)\left(\phi_{i-1}^n - \phi_{i-1}^{n+1}\right)} \le 1, \\
0 &\le \frac{\phi_i^n - \phi_{i-1}^{n+1} - c\left(\Psi(R_i)\frac{\phi_i^n - \phi_i^{n+1}}{c\ \Delta x} - \Psi(R_{i-1})\frac{\phi_{i-1}^n - \phi_{i-1}^{n+1}}{c\ \Delta x}\right)\frac{1+c}{2}\Delta x}{(1+c)\left(\phi_{i-1}^n - \phi_{i-1}^{n+1}\right)} \le 1, \\
0 &\le \frac{\phi_i^n - \phi_{i-1}^{n+1}}{(1+c)\left(\phi_{i-1}^n - \phi_{i-1}^{n+1}\right)} - \frac{1}{2}\left(\Psi(R_i)\frac{\phi_i^n - \phi_i^{n+1}}{\phi_{i-1}^n - \phi_{i-1}^{n+1}} - \Psi(R_{i-1})\right) \le 1, \\
0 &\le \frac{\phi_i^n - \phi_{i-1}^n + \phi_{i-1}^n - \phi_{i-1}^{n+1}}{(1+c)\left(\phi_{i-1}^n - \phi_{i-1}^{n+1}\right)} - \frac{1}{2}\left(\Psi(R_i)\frac{\phi_i^n - \phi_i^{n+1}}{\phi_{i-1}^n - \phi_{i-1}^{n+1}} - \Psi(R_{i-1})\right) \le 1, \\
0 &\le \frac{\frac{R_{i-1}}{c}\left(\phi_{i-1}^n - \phi_{i-1}^{n+1}\right) + \phi_{i-1}^n - \phi_{i-1}^{n+1}}{(1+c)\left(\phi_{i-1}^n - \phi_{i-1}^{n+1}\right)} - \frac{1}{2}\left(\Psi(R_i)\frac{\phi_i^n - \phi_i^{n+1}}{\phi_{i-1}^n - \phi_{i-1}^{n+1}} - \Psi(R_{i-1})\right) \le 1, \\
0 &\le \frac{1 + R_{i-1}}{c(1+c)} - \frac{1}{2}\left(\Psi(R_i)\frac{\phi_i^n - \phi_i^{n+1}}{\phi_{i-1}^n - \phi_{i-1}^{n+1}} - \Psi(R_{i-1})\right) \le 1,
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
&0 \leq \frac{\phi_{i-1}^{n+1} + \Psi(R_{i-1}) \frac{\phi_{i-1}^{n} - \phi_{i-1}^{n+1}}{c\ \Delta x} \frac{1+c}{2} \Delta x - \phi_{i-1}^{n+1}}{\phi_{i-1}^{n} - \phi_{i-1}^{n+1}} \leq 1, \\
&0 \leq \Psi(R_{i-1}) \frac{1+c}{2c} \leq 1, \\
&0 \leq \Psi(R_{i-1}) \leq \frac{2c}{1+c},
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
0 \leq \frac{1+R_{i-1}}{c(1+c)} - \frac{1}{2} \Psi(R_i) \frac{\phi_i^n - \phi_i^{n+1}}{\phi_{i-1}^n - \phi_{i-1}^{n+1}} &\leq 1 - \frac{c}{1+c}, \\
-\frac{1+R_{i-1}}{c(1+c)} \leq -\frac{1}{2} \Psi(R_i) \frac{\phi_i^n - \phi_i^{n+1}}{\phi_{i-1}^n - \phi_{i-1}^{n+1}} &\leq \frac{1}{1+c} - \frac{1+R_{i-1}}{c(1+c)}, \\
\frac{2\,(1+R_{i-1})}{c(1+c)} \geq \Psi(R_i) \frac{\phi_i^n - \phi_i^{n+1}}{\phi_{i-1}^n - \phi_{i-1}^{n+1}} &\geq \frac{1+R_{i-1}-c}{c(1+c)},
\end{aligned}
\tag{42}
$$

$$\begin{aligned}
\sigma_i &= a\ \frac{\phi^n_{i+1}-\phi^{n+1}_i}{(1+c)\Delta x} + b\ \frac{\phi^n_i-\phi^{n+1}_i}{c\ \Delta x},\\
&= (a+b\ r_i)\ \frac{\phi^n_{i+1}-\phi^{n+1}_i}{(1+c)\ \Delta x},\\
&= \Psi(r_i)\frac{\phi^n_{i+1}-\phi^{n+1}_i}{(1+c)\ \Delta x},
\end{aligned} \tag{43}$$

where

$$r_i = \frac{1+c}{c}\frac{\phi^n_i-\phi^{n+1}_i}{\phi^n_{i+1}-\phi^{n+1}_i} \tag{44}$$

$$\begin{aligned}
0 &\le \frac{\phi^n_i-\phi^{n+1}_{i-1}-c\,(\sigma_i-\sigma_{i-1})\,\frac{1+c}{2}\Delta x}{(1+c)\left(\phi^n_{i-1}-\phi^{n+1}_{i-1}\right)} \le 1,\\
0 &\le \frac{\phi^n_i-\phi^{n+1}_{i-1}-c\left(\Psi(r_i)\frac{\phi^n_{i+1}-\phi^{n+1}_i}{(1+c)\ \Delta x}-\Psi(r_{i-1})\frac{\phi^n_i-\phi^{n+1}_{i-1}}{(1+c)\ \Delta x}\right)\frac{1+c}{2}\Delta x}{(1+c)\left(\phi^n_{i-1}-\phi^{n+1}_{i-1}\right)} \le 1,\\
0 &\le \frac{\phi^n_i-\phi^{n+1}_{i-1}-\frac{c}{2}\left(\Psi(r_i)\frac{\phi^n_{i+1}-\phi^{n+1}_i}{\phi^n_i-\phi^{n+1}_{i-1}}-\Psi(r_{i-1})\right)\left(\phi^n_i-\phi^{n+1}_{i-1}\right)}{(1+c)\left(\phi^n_{i-1}-\phi^{n+1}_{i-1}\right)} \le 1,\\
0 &\le \frac{\left(\phi^n_i-\phi^{n+1}_{i-1}\right)\left(1-\frac{c}{2}\left(\Psi(r_i)\frac{\phi^n_{i+1}-\phi^{n+1}_i}{\phi^n_i-\phi^{n+1}_{i-1}}-\Psi(r_{i-1})\right)\right)}{(1+c)\left(\phi^n_{i-1}-\phi^{n+1}_{i-1}\right)} \le 1,\\
0 &\le \frac{1-\frac{c}{2}\left(\Psi(r_i)\frac{\phi^n_{i+1}-\phi^{n+1}_i}{\phi^n_i-\phi^{n+1}_{i-1}}-\Psi(r_{i-1})\right)}{c\ r_{i-1}} \le 1,
\end{aligned} \tag{45}$$

$$\begin{aligned}
0 &\le \frac{\phi^{n+1}_{i-1}+\Psi(r_{i-1})\frac{\phi^n_i-\phi^{n+1}_{i-1}}{(1+c)\ \Delta x}\frac{1+c}{2}\Delta x-\phi^{n+1}_{i-1}}{\phi^n_i-\phi^{n+1}_{i-1}} \le 1,\\
0 &\le \Psi(r_{i-1}) \le 2
\end{aligned} \tag{46}$$

$$\begin{aligned}
0 &\le \frac{\phi^{n+1}_{i-1}+\sigma_i\frac{1+c}{2}\Delta x-\phi^{n+1}_{i-1}}{\phi^n_i-\phi^{n+1}_{i-1}} \le 1,\\
0 &\le \Psi(r_{i-1}) \le 2
\end{aligned} \tag{47}$$

0.2 Reconstruct-evolve-average interpretation of implicit schemes

It is well known that explicit finite volume schemes can be interpreted as a reconstruct-evolve-average(REA) algorithm [9, 8, 12]:

1. Reconstruct a piecewise polynomial function $\tilde{\phi}^n_i(x, t_n)$ for all $x \in \mathcal{C}_i$ from the known cell averages

2. Evolve the function to obtain $\tilde{\phi}_i^n(x, t_{n+1})$

3. Compute the average of this evolved function over each cell to obtain the new cell averages
$$\phi_i^{n+1} = \frac{1}{\Delta x} \int_{\mathcal{C}_i} \tilde{\phi}_i^n(x, t_{n+1}) \, \mathrm{d}x \, .$$

This interpretation can help us to construct high-order schemes. Many flux and slope limiters can be designed using the fact that if the reconstruction process does not generate new extrema, then evolving and averaging operations will neither, meaning that our numerical solution will be oscillation free.

In this section we would like to show that implicit schemes can also be interpreted similarly by considering that the values travel along characteristics. Thus, the values in the new time step can be interpreted as values in the current time step.

In this section we restrict our discussion to the simplest problem of advection with positive constant speed $v > 0$.

0.2.1 Courant numbers < 1

0.2.1.1 Explicit case

We start with the most straightforward piecewise constant reconstruction:

1. Reconstruct a piecewise constant function $\tilde{\phi}_i^n(x, t_n)$ from the known cell averages ϕ_i^n:
$$\tilde{\phi}_i^n(x, t_n) = \phi_i^n \text{ for } x \in \mathcal{C}_i = (x_{i-1/2}, x_{i+1/2}).$$

2. Evolve the function using the exact solution of the advection equation
$$\tilde{\phi}_i^n(x, t_{n+1}) = \tilde{\phi}_i^n(x - v\Delta t, t_n),$$
meaning, that
$$\tilde{\phi}_i^n(x, t_{n+1}) = \phi_i^n \text{ for } x \in \mathcal{C}_i + v\Delta t.$$

3. Compute the average of this evolved function over each cell to obtain the new cell averages
$$\phi_i^{n+1} = \frac{1}{\Delta x} \int_{\mathcal{C}_i} \tilde{\phi}_i^n(x, t_{n+1}) \, \mathrm{d}x = \frac{1}{\Delta x} \left(\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{\phi}_i^n(x, t_{n+1}) \, \mathrm{d}x \right),$$

To evaluate the integral, we need to know, by what values it the region $\mathcal{C}_i$ occupied at time t_{n+1}. If the advection speed v over a time step Δt is smaller than a cell width Δx, thus

$$0 \leq v\Delta t \leq \Delta x, \tag{48}$$

then

$$\tilde{\phi}_i^n(x, t_{n+1}) = \begin{cases} \phi_{i-1}^n & \text{for } x \in (x_{i-1/2}, x_{i-1/2} + v\Delta t) \\ \phi_i^n & \text{for } x \in (x_{i-1/2} + v\Delta t, x_{i+1/2}). \end{cases} \tag{49}$$

$$\begin{aligned}
\phi_i^{n+1} &= \frac{1}{\Delta x}\left(\int_{x_{i-1/2}}^{x_{i-1/2}+v\Delta t} \phi_{i-1}^n \,\mathrm{d}x + \int_{x_{i-1/2}+v\Delta t}^{x_{i+1/2}} \phi_i^n \,\mathrm{d}x\right), \\
&= \frac{1}{\Delta x}\left(\phi_{i-1}^n \int_{x_{i-1/2}}^{x_{i-1/2}+v\Delta t} \mathrm{d}x + \phi_i^n \int_{x_{i-1/2}+v\Delta t}^{x_{i+1/2}} \mathrm{d}x\right), \\
&= \frac{1}{\Delta x}\left(\phi_{i-1}^n \int_{x_{i-1/2}}^{x_{i-1/2}+v\Delta t} \mathrm{d}x + \phi_i^n \int_{x_{i-1/2}+v\Delta t}^{x_{i+1/2}} \mathrm{d}x\right), \\
&= \frac{v\Delta t}{\Delta x}\phi_{i-1}^n + \left(1 - \frac{v\Delta t}{\Delta x}\right)\phi_i^n \\
&= c\, \phi_{i-1}^n + (1 - c)\, \phi_i^n.
\end{aligned}$$

$$\begin{aligned}
&0 \leq v\Delta t \leq \Delta x \\
&0 \leq \frac{v\Delta t}{\Delta x} \leq 1, \\
&0 \leq c \leq 1,
\end{aligned} \tag{50}$$

where c is the Courant number, the algorithm yields the first order explicit upwind scheme. The new cell averages can be computed using the above described algorithm as

$$\begin{aligned}
\phi_i^{n+1} &= \frac{1}{\Delta x}\left(v\Delta t\ \phi_{i-1}^n + (\Delta x - v\Delta t)\, \phi_i^n\right) \\
&= \frac{v\Delta t}{\Delta x}\phi_{i-1}^n + \left(1 - \frac{v\Delta t}{\Delta x}\right)\phi_i^n \\
&= c\, \phi_{i-1}^n + (1 - c)\, \phi_i^n.
\end{aligned} \tag{51}$$

We can see, that if the CFL condition (50) holds, the new cell average is a linear interpolation between ϕ_i^{n-1} and ϕ_i^n.

0.2.1.2 Implicit case

Let us first look at the implicit upwind scheme

$$\phi_i^{n+1} = \phi_i^n - c\left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right). \tag{52}$$

Solving for the new cell average ϕ_i^{n+1} we get

$$
\begin{aligned}
\phi_{i-1}^{n,*} &= \left(1 - \frac{c}{1+c}\right)\phi_{i-1}^{n+1} + \frac{c}{1+c}\phi_i^n,\\
\phi_i^{n+1} &= \frac{c}{1+c}\phi_{i-1}^{n+1} + \left(1 - \frac{c}{1+c}\right)\phi_i^n.
\end{aligned}
$$

Using the fact that if we start the computation at the inflow boundary, where an inflow boundary condition must be specified, reaching the n-th cell, ϕ_i^n and ϕ_{i-1}^{n+1} will be known. Thus, using these values, we obtain the new cell average ϕ_i^{n+1}.

How can this equation be interpreted as an REA algorithm?

Assume constant reconstruction in the cell $\mathcal{C}_i$ at the current time step t_n as before

$$\tilde{\phi}_i^n(x, t_n) = \phi_i^n \text{ for } x \in \mathcal{C}_i,$$

and a constant reconstruction in the cell $\mathcal{C}_{i-1}$ at the new time step t_{n+1} as

$$\tilde{\phi}_{i-1}^{n+1}(x, t_{n+1}) = \phi_{i-1}^{n+1} \text{ for } x \in \mathcal{C}_{i-1}.$$

Using the exact solution of the advection equation, we evolve the value from the current time step

$$\tilde{\phi}_i^n(x, t_{n+1}) = \tilde{\phi}_i^n(x - v\Delta t, t_n).$$

To obtain the new cell average, we need to evaluate the integral

$$\phi_i^{n+1} = \frac{1}{\Delta x}\int_{\mathcal{C}_i} \tilde{\phi}_i^n(x, t_{n+1})\,\mathrm{d}x\,.$$

In the explicit case, we computed the new average using the values ϕ_{i-1}^n, ϕ_i^n. But now, instead of ϕ_{i-1}^n we have ϕ_{i-1}^{n+1}. What we can make is to compute an estimated value $\phi_{i-1}^{n,*}$ to compute the new cell average as in (51)

$$\phi_i^{n+1} = c\,\phi_{i-1}^{n,*} + (1-c)\,\phi_i^n. \tag{53}$$

We can express the cell average $\phi_{i-1}^{n,*}$ using $\phi_{i-1}^{n+1}, \phi_i^{n+1}$ by averaging in cell $\mathcal{C}_{i-1}$

$$\phi_{i-1}^{n,*} = (1-c)\,\phi_{i-1}^{n+1} + c\,\phi_i^{n+1}. \tag{54}$$

Thus, we can solve the 2 equations above for 2 unknowns $\phi_{i-1}^{n,*}, \phi_i^{n+1}$ to get

$$
\begin{aligned}
\phi_{i-1}^{n,*} &= \left(1 - \frac{c}{1+c}\right)\phi_{i-1}^{n+1} + \frac{c}{1+c}\phi_i^n,\\
\phi_i^{n+1} &= \frac{c}{1+c}\phi_{i-1}^{n+1} + \left(1 - \frac{c}{1+c}\right)\phi_i^n,
\end{aligned}
$$

yielding the same result for the new cell average as solving the equation directly. This shows that the REA interpretation of the implicit upwind scheme (52) is correct. Also notice that both values are linear interpolations between the known values $\phi_{i-1}^{n+1}, \phi_i^n$. Thus, to solve the above equation (52),we don't need an estimated value for ϕ_{i-1}^n. However, interpreting the solution in this manner can help us to deepen our understanding of the behavior of the solutions. Also, in later sections, it will be a great help for us to develop implicit high-resolution schemes.

Bibliography

[1] Peter Frolkovič and Michal Žeravỳ. "High resolution compact implicit numerical scheme for conservation laws". In: *Applied Mathematics and Computation* 442 (2023), p. 127720. URL: https://www.researchgate.net/profile/Peter-Frolkovic/publication/365886997_High_resolution_compact_implicit_numerical_scheme_for_conservation_laws/links/63888ddf2c563722f2297b03/High-resolution-compact-implicit-numerical-scheme-for-conservation-laws.pdf.

[2] Amiram Harten, Peter D Lax, and Bram van Leer. "On upstream differencing and Godunov-type schemes for hyperbolic conservation laws". In: *SIAM review* 25.1 (1983), pp. 35–61.

[3] Charles Hirsch. *Numerical computation of internal and external flows, vol. 2: Computational methods for inviscid and viscous flows*. J. Wiley & sons, 1990. URL: https://soaneemrana.org/onewebmedia/NUMERICAL%20COMPUTATION%20OF%20INTERNAL%20&%20EXTERNAL%20FLOWS%20BY%20C.%20HIRSEH%20(VOL.-2).pdf.

[4] Charles Hirsch. *Numerical computation of internal and external flows: The fundamentals of computational fluid dynamics*. Elsevier, 2007. URL: https://cfd.spbstu.ru/agarbaruk/doc/Hirsch%20C.%20Numerical%20Computation%20of%20Internal%20and%20External%20Flows.Volume1-Fundamentals%20of%20Computational%20Fluid%20Dynamics(Elsevier,2nd%20edn,2007).pdf.

[5] Klaus A Hoffmann and Steve T Chiang. "Computational fluid dynamics Volume I". In: *Engineering education system* (2000). URL: https://sharif.ir/~moosavi/Hoffmann_v1.pdf.

[6] Hrvoje Jasak. “Error analysis and estimation for the finite volume method with applications to fluid flows.” In: (1996).

[7] Culbert B Laney. *Computational gasdynamics*. Cambridge university press, 1998. URL: https://aerocastle.files.wordpress.com/2012/10/laney.pdf.

[8] Randall J LeVeque. *Finite volume methods for hyperbolic problems*. Vol. 31. Cambridge university press, 2002. URL: http://www.tevza.org/home/course/modelling-II_2016/books/Leveque%20-%20Finite%20Volume%20Methods%20for%20Hyperbolic%20Equations.pdf.

[9] Randall J LeVeque. *Numerical methods for conservation laws*. Vol. 214. Springer, 1992. URL: http://tevza.org/home/course/modelling-II_2016/books/Leveque%20-%20Numerical%20Methods%20for%20Conservation%20Laws.pdf.

[10] Peter K Sweby. “High resolution schemes using flux limiters for hyperbolic conservation laws”. In: *SIAM journal on numerical analysis* 21.5 (1984), pp. 995–1011.

[11] Eleuterio F Toro. *Riemann Solvers and Numerical Methods for Fluid Dynamics: A Practical Introduction*. Springer Science & Business Media, 2009. URL: https://www.academia.edu/8844903/Riemann_Solvers_and_Numerical_Methods_for_Fluid_Dynamics_Third_Edition.

[12] Bram Van Leer. “Towards the ultimate conservative difference scheme. IV. A new approach to numerical convection”. In: *Journal of computational physics* 23.3 (1977), pp. 276–299.

[13] HC Yee, RF Warming, and A Harten. “Implicit total variation diminishing (TVD) schemes for steady-state calculations”. In: *Journal of Computational Physics* 57.3 (1985), pp. 327–360. URL: https://ntrs.nasa.gov/api/citations/19830014814/downloads/19830014814.pdf.