0.1 High-order implicit schemes based polynomial reconstruction

We are interested in constructing implicit finite-volume schemes that can be solved efficiently. Thus, e.g., yields a similar system of equations with a bidiagonal matrix as in the case of implicit-upwind scheme, where we can simply compute the solution one by one, starting at the inflow boundary. As a guidance, let us use our knowledge of the solutions of the advection equation. In that case we know that the solution is the initial values translating along characteristics, see Figure 1.

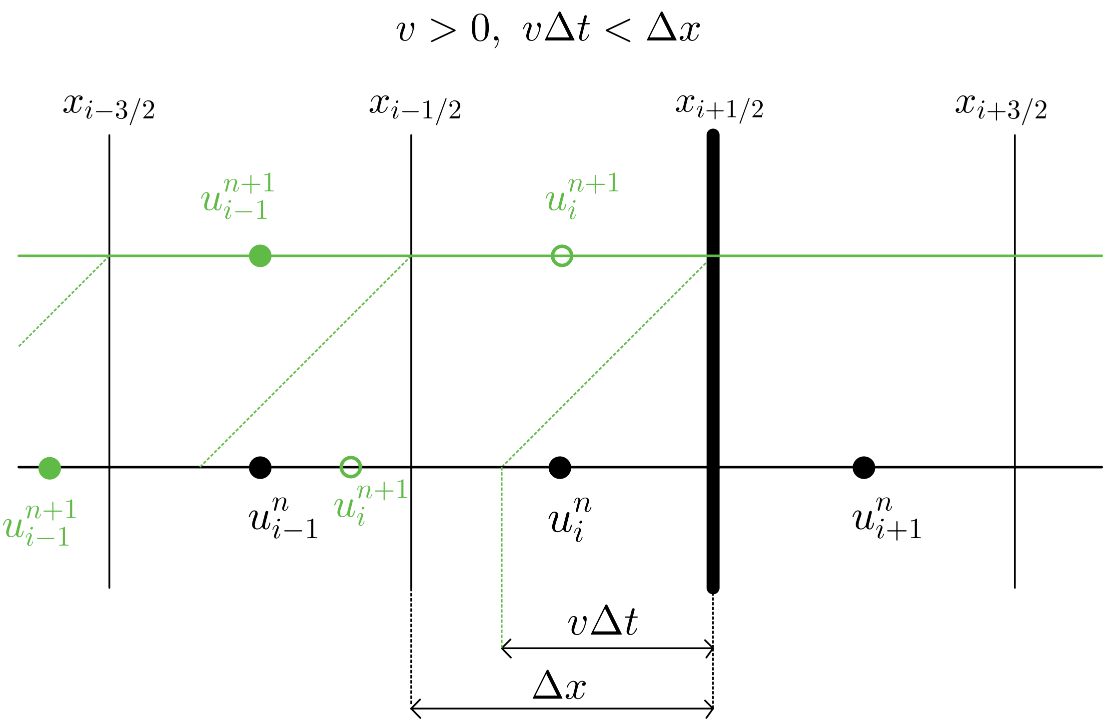

Figure 1: Averages in $x - t$ coordinates

The finite-volume schemes differ only in their definition of the numerical fluxes, thus, how can it predict the flow of a quantity through a cell face in a given time interval. Our goal is to compute the flux through the face at $x_{i+1/2}$ using the known cell averages. For convenience, we shift to space coordinates. This way we can see the connection directly between well established explicit schemes based on polynomial reconstruction [15, 10] and our implicit schemes. We could equivalently shift to time coordinates, as it was done in, e.g., [1, 3]. We are interested in compact schemes in a sense that to compute the time-average of the flux through the face at $x_{i+1/2}$ we want to use the stencil $i-1, i, i+1$.

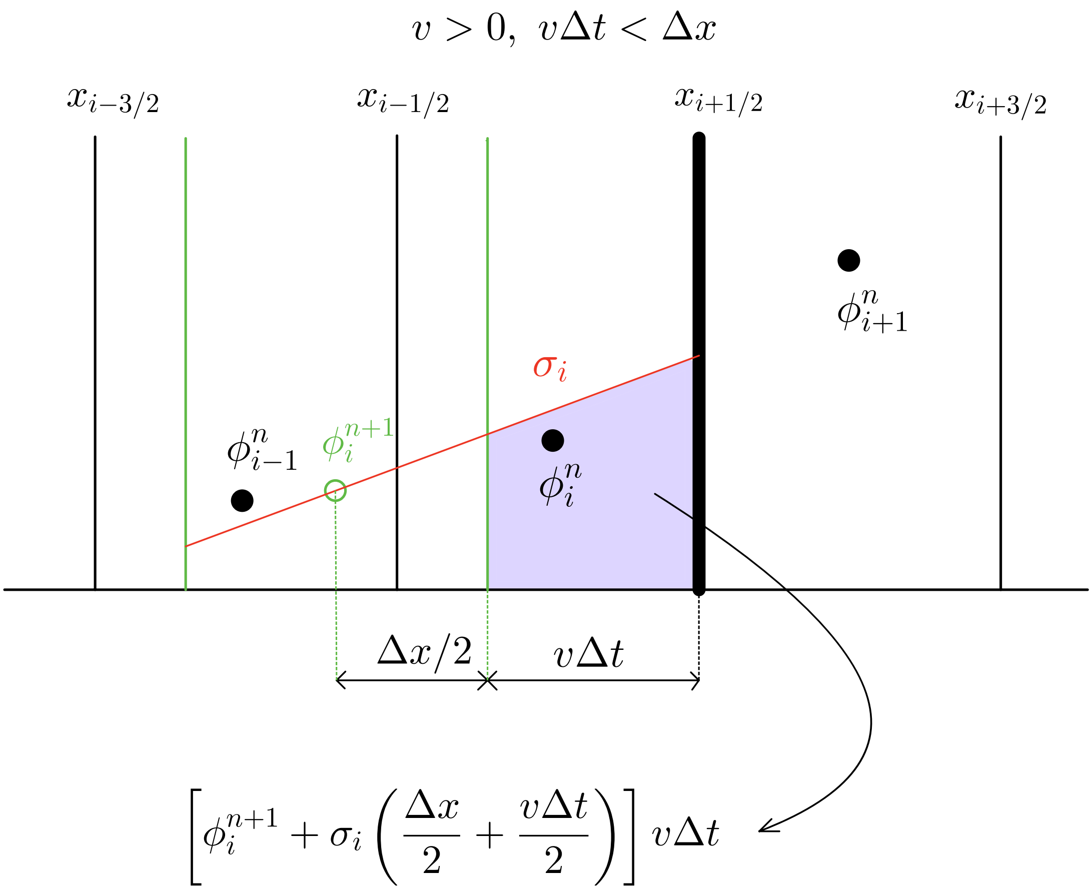

Figure 2: Linear reconstruction with slope σ

Let us begin with a linear reconstruction yielding second-order finite volume schemes, see Figure 2. For $v > 0$, the quantity that flow through the face at $x_{i+1/2}$ is simply the shaded area

$$v\Delta t\left[\phi_i^{n+1} + \sigma_i\left(\frac{\Delta x}{2} + \frac{v\Delta t}{2}\right)\right],$$

where σ_i is the slope. Thus, the full update of the cell average is

$$\begin{aligned}\phi_i^{n+1}\Delta x = \phi_i^n\Delta x - v\Delta t&\left[\phi_i^{n+1} + \sigma_i\left(\frac{\Delta x}{2} + \frac{v\Delta t}{2}\right)\right]\\ + v\Delta t&\left[\phi_{i-1}^{n+1} + \sigma_{i-1}\left(\frac{\Delta x}{2} + \frac{v\Delta t}{2}\right)\right].\end{aligned} \tag{1}$$

Dividing by the cell width Δx and further simplifying we can write a general second order scheme as

$$
\begin{aligned}
\phi_i^{n+1} &= \phi_i^n - c\left(\phi_i^{n+1} + \sigma_i \Delta x \frac{1+c}{2} - \phi_{i-1}^{n+1} - \sigma_{i-1}\Delta x \frac{1+c}{2}\right) \\
&= \phi_i^n - c\left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right) - c\left(\sigma_i - \sigma_{i-1}\right)\Delta x \frac{1+c}{2},
\end{aligned} \tag{2}
$$

where $c = \frac{v\Delta t}{\Delta x}$ is the Courant number. Different choices for the slope σ_i yield different schemes studied in previous works, see, e.g., [4, 5, 11].

In particular, if our linear reconstruction satisfies the integral equations

$$
\begin{aligned}
\frac{1}{\Delta x}\int_{x_{i+1/2}}^{x_{i+3/2}} \phi_i^{n+1} + \sigma_i\left(x - (x_i - v\Delta t)\right) \mathrm{d}x = \phi_{i+1}^n, \\
\frac{1}{\Delta x}\int_{x_{i-1/2}-v\Delta t}^{x_{i+1/2}-v\Delta t} \phi_i^{n+1} + \sigma_i\left(x - (x_i - v\Delta t)\right) \mathrm{d}x = \phi_i^{n+1},
\end{aligned} \tag{3}
$$

we get the slope

$$
\sigma_i = \frac{\phi_{i+1}^n - \phi_i^{n+1}}{(1+c)\Delta x}. \tag{4}
$$

Substituting (4) to (2) yields the scheme

$$
\begin{aligned}
\phi_i^{n+1} &= \phi_i^n - c\left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right) - c\left(\sigma_i - \sigma_{i-1}\right)\Delta x \frac{1+c}{2}, \\
&= \phi_i^n - c\left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right) - c\left(\frac{\phi_{i+1}^n - \phi_i^{n+1}}{(1+c)\Delta x} - \frac{\phi_i^n - \phi_{i-1}^{n+1}}{(1+c)\Delta x}\right)\Delta x \frac{1+c}{2}, \\
&= \phi_i^n - c\left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right) - \frac{c}{2}\left(\phi_{i+1}^n - \phi_i^{n+1} - \left(\phi_i^n - \phi_{i-1}^{n+1}\right)\right), \\
&= \phi_i^n - c\left(\frac{\phi_{i+1}^n + \phi_i^{n+1}}{2} - \frac{\phi_i^n + \phi_{i-1}^{n+1}}{2}\right),
\end{aligned} \tag{5}
$$

which one can recognize is the IIOE scheme for linear advection with constant speed in 1D [11, 4, 8].

0.2 Stabilization of implicit schemes

In this section we construct stabilized schemes by requiring the updated cell average to lie between values $\phi_{i-1}^{n+1}, \phi_i^n$, thus

$$
\min\left(\phi_{i-1}^{n+1}, \phi_i^n\right) \le \phi_i^{n+1} \le \max\left(\phi_{i-1}^{n+1}, \phi_i^n\right), \tag{6}
$$

for $c > 0$. We can distinguish three cases:

1. $\phi_i^n > \phi_{i-1}^{n+1}$:

$$\min\left(\phi_{i-1}^{n+1}, \phi_i^n\right) = \phi_{i-1}^{n+1}, \quad \max\left(\phi_{i-1}^{n+1}, \phi_i^n\right) = \phi_i^n.$$

   We can rewrite (6) as

$$\begin{aligned} \phi_{i-1}^{n+1} \leq \phi_i^{n+1} \leq \phi_i^n, \\ 0 \leq \phi_i^{n+1} - \phi_{i-1}^{n+1} \leq \phi_i^n - \phi_{i-1}^{n+1}, \\ 0 \leq \frac{\phi_i^{n+1} - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} \leq 1. \end{aligned}$$

2. $\phi_i^n < \phi_{i-1}^{n+1}$:

$$\min\left(\phi_{i-1}^{n+1}, \phi_i^n\right) = \phi_i^n, \quad \max\left(\phi_{i-1}^{n+1}, \phi_i^n\right) = \phi_{i-1}^{n+1}.$$

   We can rewrite (6) as

$$\begin{aligned} \phi_i^n \leq \phi_i^{n+1} \leq \phi_{i-1}^{n+1}, \\ \phi_i^n - \phi_{i-1}^{n+1} \leq \phi_i^{n+1} - \phi_{i-1}^{n+1} \leq 0, \\ 1 \geq \frac{\phi_i^{n+1} - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} \geq 0. \end{aligned}$$

3. $\phi_i^n = \phi_{i-1}^{n+1}$, in which case we have

$$\phi_i^n = \phi_{i-1}^{n+1} = \phi_i^{n+1}.$$

Notice, that for the first two cases, where $\phi_i^n \neq \phi_{i-1}^{n+1}$ we can simply require

$$0 \leq \frac{\phi_i^{n+1} - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} \leq 1 \tag{7}$$

instead of (6) to simplify the analysis. The condition above can be interpreted as the upwind range condition[9], data compatibility condition [14], upwind monotonic property [7], local boundedness property [12, 13] for implicit schemes. The idea is that the solution travels along characteristics and this value is always bounded by its neighbors in space-time. In the case of well established explicit schemes, requiring a scheme to satisfy the upwind-range property is equivalent to requiring a scheme to be TVD [13]. However, as it was also pointed out in [13], this condition is a more straightforward local condition, which makes it also more straightforward to extend to more complicated equation also in higher dimensions, than the TVD condition. Not to mention the results of Notice that min and max is a combination of values from both time steps $n, n+1$. This distinguishes the implicit upwind range property from the explicit one. In the explicit case, the min and max are chosen only from the values at the current time step n.

First, we show that if a scheme satisfies the upwind-range-condition, then it is also total variation non-increasing, or TVD. In order to show this, first notice that if the new cell average ϕ_i^{n+1} lies between the values $\phi_{i-1}^{n+1}, \phi_i^n$, then it can be written as a convex combination of the two

$$\phi_i^{n+1} = k_i \phi_i^n + (1-k_i)\phi_{i-1}^{n+1}, \tag{8}$$

for some $0 \leq k_i \leq 1$.

If $k_i = 0$, then

$$\phi_i^{n+1} = \phi_{i-1}^{n+1},$$

thus, if ϕ_{i-1}^{n+1}, the update certainly not changes the total variation.

For $k_i > 0$, we can recast the above convex combination to a conservation form

$$\begin{aligned}
\phi_i^{n+1} &= k_i \phi_i^n + (1-k_i)\phi_{i-1}^{n+1}, \\
\frac{1}{k_i}\phi_i^{n+1} &= \phi_i^n + \frac{1-k_i}{k_i}\phi_{i-1}^{n+1}, \\
\frac{1}{k_i}\phi_i^{n+1} + \phi_i^{n+1} - \phi_i^{n+1} &= \phi_i^n + \frac{1-k_i}{k_i}\phi_{i-1}^{n+1}, \\
\phi_i^{n+1} + \frac{1-k_i}{k_i}\phi_i^{n+1} &= \phi_i^n + \frac{1-k_i}{k_i}\phi_{i-1}^{n+1}, \\
\phi_i^{n+1} &= \phi_i^n - \frac{1-k_i}{k_i}\left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right).
\end{aligned} \tag{9}$$

Notice that if $0 < k_i \leq 1$, then $\frac{1-k_i}{k_i} > 0$, which is sufficient for a scheme to be TVD, see, e.g., [2, 5].

Let us use some definition, that will make some of our formulations simpler, where we want to enforce bounds, see, e.g., [7, 6].

Let $I(z_1, \ldots, z_k)$ be the smallest closed interval containing $z_1, \ldots, z_k$, thus,

$$I(z_1, \ldots, z_k) = [\min(z_1, \ldots, z_k), \max(z_1, \ldots, z_k)]. \tag{10}$$

Let the median of three numbers be the one, which lies between the other two. In order to define the median function, first we need a definition of the minmod function of two variables(see, e.g., [6]):

$$\operatorname{minmod}(a,b) = \begin{cases} \operatorname{sgn}(a)\min(|a|,|b|) & \text{if } ab > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Then, the median function of 3 variables, as in [7, 6], can be defined

$$\begin{aligned} \operatorname{median}(a,b,c) &= a + \operatorname{minmod}(b-a, c-a) \\ &= b + \operatorname{minmod}(a-b, c-b) . \end{aligned} \tag{12}$$

It is important to observe that the $\operatorname{median}(x,y,z)$ lies in the interval defined by any other two of the three arguments, e.g.,

$$\operatorname{median}(x,y,z) \in I(y,z), \text{ or } \operatorname{median}(x,y,z) \in I(x,z). \tag{13}$$

Also, $\operatorname{median}(z_1,z_2,z_3,z_4,z_5)$ lies in the interval defined by any three of the five arguments, e.g.,

$$\operatorname{median}(z_1,\dots,z_5) \in I(z_1,z_4,z_3). \tag{14}$$

Using the definitions above, the upwind-range condition (7) can be written in a convenient way by requiring

$$\phi_i^{n+1} \in I(\phi_{i-1}^{n+1}, \phi_i^n). \tag{15}$$

0.2.1 Second-order implicit TVD schemes

Now we describe how we can construct schemes satisfying the upwind-range condition by choosing the slope properly. We can solve

$$\phi_i^{n+1} = \phi_i^n - c\left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right) - c\left(\sigma_i - \sigma_{i-1}\right)\Delta x \frac{1+c}{2}, \tag{16}$$

for the new cell average to obtain

$$\phi_i^{n+1} = \frac{\phi_i^n + c\ \phi_{i-1}^{n+1}}{1+c} - \frac{c}{2}\left(\sigma_i - \sigma_{i-1}\right)\Delta x. \tag{17}$$

Remark. *Notice, that in the schemes we describe, the slope σ_i can also contain terms involving ϕ_i^{n+1}, which we ignore in the first analysis. We will deal with that case later.*

Substituting to the upwind-range condition (7) we get

$$\begin{aligned} 0 &\le \frac{\phi_i^{n+1} - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1, \\ 0 &\le \frac{\frac{\phi_i^n + c\ \phi_{i-1}^{n+1}}{1+c} - \frac{c}{2}\left(\sigma_i - \sigma_{i-1}\right)\Delta x - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1, \\ 0 &\le \frac{\frac{\phi_i^n - \phi_{i-1}^{n+1}}{1+c} - \frac{c}{2}\left(\sigma_i - \sigma_{i-1}\right)\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1, \\ 0 &\le \frac{1}{1+c} - \frac{c}{2}\frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} + \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1. \end{aligned} \tag{18}$$

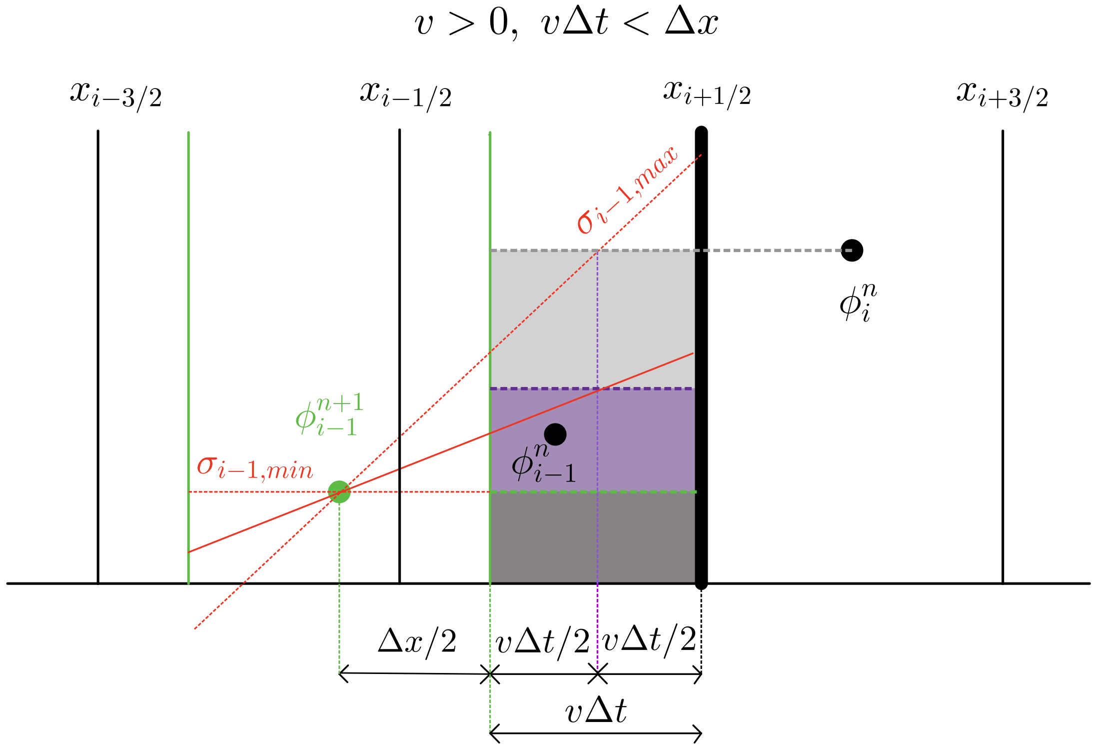

Figure 3: Bounds for the slope

We require that the outflow be at most the downwind outflow, and at least the implicit-upwind value, see, Figure 3. Thus, we write

$$\begin{aligned} v\Delta t \min(\phi_{i-1}^{n+1}, \phi_i^n) \le v\Delta t \left[\phi_{i-1}^{n+1} + \sigma_{i-1}\left(\frac{\Delta x}{2} + \frac{v\Delta t}{2}\right)\right] \le v\Delta t \max(\phi_{i-1}^{n+1}, \phi_i^n), \\ \min(\phi_{i-1}^{n+1}, \phi_i^n) \le \phi_{i-1}^{n+1} + \sigma_{i-1}\Delta x \frac{1+c}{2} \le \max(\phi_{i-1}^{n+1}, \phi_i^n). \end{aligned} \tag{19}$$

Notice, that this is the same upwind-range condition, as for the cell average (7). Thus, we can write

$$\begin{aligned} 0 \le \frac{\phi_{i-1}^{n+1} + \sigma_{i-1}\frac{1+c}{2}\Delta x - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1, \\ 0 \le \frac{1+c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1, \\ 0 \le \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \le \frac{c}{1+c}, \\ 0 \le \frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \le \frac{2}{1+c}. \end{aligned} \tag{20}$$

If the slope σ_{i-1} is available, then σ_i has to satisfy

$$
\begin{gathered}
0 \leq \frac{1}{1+c} - \frac{c}{2}\frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} + \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \leq 1, \\
-\left(\frac{1}{1+c} + \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}}\right) \leq -\frac{c}{2}\frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \leq 1 - \left(\frac{1}{1+c} + \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}}\right), \\
-\left(\frac{1}{1+c} + \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}}\right) \leq -\frac{c}{2}\frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \leq \frac{c}{1+c} - \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}}, \\
\frac{1}{1+c} + \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \geq \frac{c}{2}\frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \geq \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} - \frac{c}{1+c}, \\
\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} - \frac{2}{1+c} \leq \frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \leq \frac{2}{c(1+c)} + \frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}}.
\end{gathered}
\tag{21}
$$

Also, from (19), we have

$$
0 \leq \frac{c}{2}\frac{\sigma_i \Delta x}{\phi_{i+1}^n - \phi_i^{n+1}} \leq \frac{c}{1+c}. \tag{22}
$$

Let us define the intervals

$$
I_1 = I\left(\sigma_{i-1} - 2\frac{\phi_i^n - \phi_{i-1}^{n+1}}{(1+c)\Delta x},\ \sigma_{i-1} + \frac{2}{c}\frac{\phi_i^n - \phi_{i-1}^{n+1}}{(1+c)\Delta x}\right),\ I_2 = I\left(0,\ 2\frac{\phi_{i+1}^n - \phi_i^{n+1}}{(1+c)\Delta x}\right), \tag{23}
$$

then the slope must satisfy

$$
\sigma_i \in I_1 \cap I_2. \tag{24}
$$

Using (20) and (21), we can see that

$$
\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} - \frac{2}{1+c} \leq 0 \leq \frac{2}{c(1+c)} + \frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}}, \tag{25}
$$

which implies that $0 \in I_1$.

Now we discuss how to compute a particular slope. Let us have a slope from a high-order reconstruction of our choice denoted as σ_i^{HO}. We want to use this slope as much as possible. So, for example, if

$$
\sigma_i^{HO} \in I_1 \cap I_2 \Rightarrow \sigma_i = \sigma_i^{HO}, \tag{26}
$$

otherwise the slope takes one of the bounds of the interval $I_1 \cap I_2$. Let

$$
\sigma^{(1)} = \text{median}\left(\sigma_{i-1} - 2\frac{\phi_i^n - \phi_{i-1}^{n+1}}{(1+c)\Delta x},\ \sigma_{i-1} + \frac{2}{c}\frac{\phi_i^n - \phi_{i-1}^{n+1}}{(1+c)\Delta x},\ 2\frac{\phi_{i+1}^n - \phi_i^{n+1}}{(1+c)\Delta x}\right), \tag{27}
$$

then the slope of our reconstruction is

$$
\sigma_i = \text{minmod}\left(\sigma_i^{HO}, \sigma^{(1)}\right). \tag{28}
$$

Proof. The above equation (28) can be equivalently written as

$$\sigma_i = \text{median}\left(0, \sigma_i^{HO}, \sigma^{(1)}\right),$$

which implies, using a property of the median function, that

$$\sigma_i \in I(0, \sigma^{(1)}).$$

We can see, that both

$$\sigma^{(1)} \in I_1 \quad \text{and} \quad 0 \in I_1,$$

thus, we conclude that $\sigma_i \in I_1$. ($\sigma_i \in I_1$!!!) Again, using a property of the median function, we have

$$\sigma^{(1)} \in I\left(\sigma_{i-1} - 2\frac{\phi_i^n - \phi_{i-1}^{n+1}}{(1+c)\Delta x},\ 2\frac{\phi_{i+1}^n - \phi_i^{n+1}}{(1+c)\Delta x}\right)$$

or

$$\sigma^{(1)} \in I\left(\sigma_{i-1} + \frac{2}{c}\frac{\phi_i^n - \phi_{i-1}^{n+1}}{(1+c)\Delta x},\ 2\frac{\phi_{i+1}^n - \phi_i^{n+1}}{(1+c)\Delta x}\right)$$

□

Using (18) and (19), we can also derive sufficient condition for the slope σ_i

$$\begin{aligned} 0 \le \frac{1}{1+c} - \frac{c}{2}\frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} &\le 1 - \frac{c}{1+c}, \\ 0 \le \frac{1}{1+c} - \frac{c}{2}\frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} &\le \frac{1}{1+c}, \\ 0 \le \frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} &\le \frac{2}{c}\frac{1}{1+c}, \end{aligned} \tag{29}$$

so the slope must lie in

$$\sigma_i \in I\left(0, \frac{2}{c}\frac{\phi_i^n - \phi_{i-1}^{n+1}}{1+c}\right) \cap I\left(0,\ 2\frac{\phi_{i+1}^n - \phi_i^{n+1}}{(1+c)\Delta x}\right) \tag{30}$$

Remark. *We can derive similar schemes appearing in [5] by writing the slope as a convex combination*

$$\begin{aligned} \sigma_i &= (1-\omega_i)\frac{\phi_{i+1}^n - \phi_i^{n+1}}{(1+c)\Delta x} + \omega_i \frac{\phi_i^n - \phi_{i-1}^{n+1}}{(1+c)\Delta x}, \\ &= (1 - \omega_i + \omega_i r_i)\,\frac{\phi_{i+1}^n - \phi_i^{n+1}}{(1+c)\Delta x}, \\ &= \Psi_i \frac{\phi_{i+1}^n - \phi_i^{n+1}}{(1+c)\Delta x}, \end{aligned} \tag{31}$$

where

$$0 \le \omega_i \le 1, \quad \Psi_i = \Psi_i(r_i) = (1 - \omega_i + \omega_i r_i), \quad r_i = \frac{\phi_i^n - \phi_{i-1}^{n+1}}{\phi_{i+1}^n - \phi_i^{n+1}}. \tag{32}$$

Substituting the new form of the slopes we get

$$\begin{aligned}
&0 \le \frac{1}{1+c} - \frac{c}{2}\frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} + \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1,\\
&0 \le \frac{1}{1+c} - \frac{c}{2}\frac{\Psi_i \frac{\phi_{i+1}^n - \phi_i^{n+1}}{(1+c)\Delta x}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} + \frac{c}{2}\frac{\Psi_{i-1}\frac{\phi_i^n - \phi_{i-1}^{n+1}}{(1+c)\Delta x}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1,\\
&0 \le \frac{1}{1+c} - \frac{c}{2(1+c)}\frac{\Psi_i}{r_i} + \frac{c}{2(1+c)}\Psi_{i-1} \le 1.
\end{aligned} \tag{33}$$

Also, we consider similar bounds as before (20)

$$\begin{aligned}
&0 \le \frac{\phi_{i-1}^{n+1} + \sigma_{i-1}\frac{1+c}{2}\Delta x - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1,\\
&0 \le \frac{\phi_{i-1}^{n+1} + \Psi_{i-1}\frac{\phi_i^n - \phi_{i-1}^{n+1}}{(1+c)\Delta x}\frac{1+c}{2}\Delta x - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1,\\
&0 \le \Psi_{i-1} \le 2.
\end{aligned} \tag{34}$$

If Ψ_{i-1} *is available, then*

$$\begin{aligned}
-\left(\frac{1}{1+c} + \frac{c}{2(1+c)}\Psi_{i-1}\right) &\le -\frac{c}{2(1+c)}\frac{\Psi_i}{r_i} \le 1 - \left(\frac{1}{1+c} + \frac{c}{2(1+c)}\Psi_{i-1}\right),\\
-\left(\frac{1}{1+c} + \frac{c}{2(1+c)}\Psi_{i-1}\right) &\le -\frac{c}{2(1+c)}\frac{\Psi_i}{r_i} \le \frac{c}{1+c} - \frac{c}{2(1+c)}\Psi_{i-1},\\
\frac{1}{1+c} + \frac{c}{2(1+c)}\Psi_{i-1} &\ge \frac{c}{2(1+c)}\frac{\Psi_i}{r_i} \ge \frac{c}{2(1+c)}\Psi_{i-1} - \frac{c}{1+c},\\
\frac{2}{c} + \Psi_{i-1} &\ge \frac{\Psi_i}{r_i} \ge \Psi_{i-1} - 2,\\
\Psi_{i-1} - 2 &\le \frac{\Psi_i}{r_i} \le \frac{2}{c} + \Psi_{i-1}.
\end{aligned} \tag{35}$$

Thus,

$$\Psi_i \in I\left(r_i(\Psi_{i-1} - 2),\ r_i\left(\frac{2}{c} + \Psi_{i-1}\right)\right) \text{ for } r_i \ne 0. \tag{36}$$

This condition, however, differs a little from the one appearing in [5], where they require

$$-1 \le \Psi_{i-1} \le 2. \tag{37}$$

It is also possible to derive a sufficient condition for Ψ_i. *Using* (33) *and* (34), *it is sufficient for the limiter to satisfy*

$$\begin{aligned}
&0 \le \frac{1}{1+c} - \frac{c}{2(1+c)}\frac{\Psi_i}{r_i} \le 1 - \frac{c}{1+c},\\
&0 \le \frac{1}{1+c} - \frac{c}{2(1+c)}\frac{\Psi_i}{r_i} \le \frac{1}{1+c},\\
&0 \le \frac{\Psi_i}{r_i} \le \frac{2}{c}.
\end{aligned} \tag{38}$$

$$0 \le \Psi_i \le \min\left(2, \frac{2r_i}{c}\right), \quad \textit{for } r_i > 0, \quad \textit{and } \Psi_i = 0 \quad \textit{for } r_i \le 0. \tag{39}$$

0.2.2 A compact, third-order accurate, semi-implicit piecewise-parabolic method

Following a philosophy as in [15], we can compute the average-flux at the face $i+1/2$ using a piecewise-parabolic reconstruction from the cell averages $\phi^n_{i+1}, \phi^n_i, \phi^{n+1}_i$. A second order polynomial can be written in the form

$$p(x) = c_0 + c_1 x + c_2 x^2. \tag{40}$$

In order to compute the three unknown coefficients do this, we require the reconstruction to satisfy the average equations in the appropriate intervals:

$$\begin{aligned}
\int_{x_{i+1/2}}^{x_{i+3/2}} p(x)\,\mathrm{d}x &= \phi^n_{i+1}, \\
\int_{x_{i-1/2}}^{x_{i+1/2}} p(x)\,\mathrm{d}x &= \phi^n_i, \\
\int_{x_{i-1/2}-c\Delta x}^{x_{i+1/2}-c\Delta x} p(x)\,\mathrm{d}x &= \phi^{n+1}_i.
\end{aligned} \tag{41}$$

For convenience, we can make a change of variables

$$\xi = \frac{x - x_{i-1/2}}{\Delta x}, \tag{42}$$

and write the polynomial in a different form

$$p(\xi) = c_0 + c_1\xi + c_2\xi^2. \tag{43}$$

The integral equations become

$$\begin{aligned}
\int_1^2 p(\xi)\,\mathrm{d}\xi &= \phi^n_{i+1}, \\
\int_0^1 p(\xi)\,\mathrm{d}\xi &= \phi^n_i, \\
\int_{-c}^{1-c} p(\xi)\,\mathrm{d}\xi &= \phi^{n+1}_i.
\end{aligned} \tag{44}$$

Solving the equations yields

$$\begin{aligned}
c_0 &= \phi^{n+1}_i + \frac{1-3c}{6(1+c)}\left(\phi^n_{i+1} - \phi^n_i\right) + \frac{-2+3c+3c^2}{3(1+c)}\frac{\phi^n_i - \phi^{n+1}_i}{c}, \\
c_1 &= \frac{c-1}{1+c}\left(\phi^n_{i+1} - \phi^n_i\right) + \frac{2}{1+c}\frac{\phi^n_i - \phi^{n+1}_i}{c}, \\
c_2 &= \frac{1}{1+c}\left(\phi^n_{i+1} - \phi^n_i\right) - \frac{1}{1+c}\frac{\phi^n_i - \phi^{n+1}_i}{c}.
\end{aligned} \tag{45}$$

To get the average flux through the face $x = x_{i+1/2}$, or $\xi = 1$, we integrate

$$\begin{aligned}
\int_{x_{i+1/2}-c\Delta x}^{x_{i+1/2}} p(x)\,\mathrm{d}x = \int_{1-c}^{1} p(\xi)\,\mathrm{d}\xi &= c\,c_0 + \frac{c(2-c)}{2}c_1 + \frac{c(3-3c+c^2)}{3}c_2 \\
&= c\left(\phi_i^{n+1} + \frac{1-c}{6}\left(\phi_{i+1}^n - \phi_i^n\right) + \frac{1+2c}{3}\frac{\phi_i^n - \phi_i^{n+1}}{c}\right) \\
&= \frac{c-1}{3}\phi_i^{n+1} + \frac{2+3c+c^2}{6}\phi_i^n - \frac{c(c-1)}{6}\phi_{i+1}^n.
\end{aligned} \tag{46}$$

Also, evaluating the flux on the inflow face yields the system of equations

$$-\frac{1-c}{3}\phi_{i-1}^{n+1} + \frac{2+c}{3}\phi_i^{n+1} = \frac{(1+c)(2+c)}{6}\phi_{i-1}^n + \frac{(1-c)(2+c)}{3}\phi_i^n - \frac{c(1-c)}{6}\phi_{i+1}^n \tag{47}$$

The choice for the slope

$$\sigma_i = \frac{1-c}{6}\frac{\phi_{i+1}^n - \phi_i^n}{\Delta x} + \frac{1+2c}{3}\frac{\phi_i^n - \phi_i^{n+1}}{c\Delta x} \tag{48}$$

is the slope for a piecewise-parabolic reconstruction, thus, resulting in a third-order method.

$$\begin{aligned}
0 &\le \frac{1}{1+c} - \frac{c}{2}\frac{\sigma_i\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} + \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1, \\
0 &\le \frac{1}{1+c} - \frac{c(1-c)}{12}\frac{\phi_{i+1}^n - \phi_i^n}{\phi_i^n - \phi_{i-1}^{n+1}} - \frac{1+2c}{6}\frac{\phi_i^n - \phi_i^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} + \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1,
\end{aligned} \tag{49}$$

$$\begin{gathered}
\frac{c(1-c)}{12}\frac{\phi_{i+1}^n - \phi_i^n}{\phi_i^n - \phi_{i-1}^{n+1}} + \frac{1+2c}{6}\frac{\phi_i^n - \phi_i^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}}, \\
\frac{c(1-c)(\phi_{i+1}^n - \phi_i^n) + 2(1+2c)(\phi_i^n - \phi_i^{n+1})}{12(\phi_i^n - \phi_{i-1}^{n+1})},
\end{gathered} \tag{50}$$

0.2.3 High-order correction formulation

Any high-order method discussed earlier can be written in the form

$$\phi_i^{n+1} = \phi_i^n - c\left(\phi_i^{n+1} + \delta\phi_{i+1/2} - \phi_{i-1}^{n+1} - \delta\phi_{i-1/2}\right), \tag{51}$$

where $\delta\phi_{i+1/2}$ is a high-order correction term for the unconditionally stable first-order implicit upwind flux. For stability, we want to ensure the upwind-range-condition (7)

$$0 \le \frac{\phi_i^{n+1} - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1$$

is satisfied. First we can make some simplifications by substituting (51) for the new cell average ϕ_i^{n+1}

$$\begin{aligned}
\frac{\phi_i^{n+1}-\phi_{i-1}^{n+1}}{\phi_i^n-\phi_{i-1}^{n+1}} &= \frac{\phi_i^n - c\left(\phi_i^{n+1}+\delta\phi_{i+1/2}-\phi_{i-1}^{n+1}-\delta\phi_{i-1/2}\right)-\phi_{i-1}^{n+1}}{\phi_i^n-\phi_{i-1}^{n+1}}, \\
\frac{\phi_i^{n+1}-\phi_{i-1}^{n+1}}{\phi_i^n-\phi_{i-1}^{n+1}} &= 1 - c\frac{\phi_i^{n+1}-\phi_{i-1}^{n+1}}{\phi_i^n-\phi_{i-1}^{n+1}} - c\frac{\delta\phi_{i+1/2}-\delta\phi_{i-1/2}}{\phi_i^n-\phi_{i-1}^{n+1}} \\
\frac{\phi_i^{n+1}-\phi_{i-1}^{n+1}}{\phi_i^n-\phi_{i-1}^{n+1}} &= \frac{1}{1+c} - \frac{c}{1+c}\frac{\delta\phi_{i+1/2}-\delta\phi_{i-1/2}}{\phi_i^n-\phi_{i-1}^{n+1}}
\end{aligned} \tag{52}$$

Substituting to the URC (7) we get

$$\begin{aligned}
0 \leq \frac{1}{1+c} - \frac{c}{1+c}\frac{\delta\phi_{i+1/2}-\delta\phi_{i-1/2}}{\phi_i^n-\phi_{i-1}^{n+1}} &\leq 1, \\
-\frac{1}{1+c} \leq -\frac{c}{1+c}\frac{\delta\phi_{i+1/2}-\delta\phi_{i-1/2}}{\phi_i^n-\phi_{i-1}^{n+1}} &\leq 1-\frac{1}{1+c}, \\
-\frac{1}{1+c} \leq -\frac{c}{1+c}\frac{\delta\phi_{i+1/2}-\delta\phi_{i-1/2}}{\phi_i^n-\phi_{i-1}^{n+1}} &\leq \frac{c}{1+c}, \\
-1 \leq -c\frac{\delta\phi_{i+1/2}-\delta\phi_{i-1/2}}{\phi_i^n-\phi_{i-1}^{n+1}} &\leq c, \\
\frac{1}{c} \geq \frac{\delta\phi_{i+1/2}-\delta\phi_{i-1/2}}{\phi_i^n-\phi_{i-1}^{n+1}} &\geq -1, \\
-1 \leq \frac{\delta\phi_{i+1/2}-\delta\phi_{i-1/2}}{\phi_i^n-\phi_{i-1}^{n+1}} &\leq \frac{1}{c}.
\end{aligned} \tag{53}$$

Let $\delta\phi_{i+1/2}^{HO}$ be a high-order correction term of our choice. We want to use it whenever possible. We can write the boundedness of the correction term elegantly using the median function [7]. Thus, if $\delta\phi_{i-1/2}$ is available, we define the correction term as

$$\delta\phi_{i+1/2} = \text{median}\left(\delta\phi_{i+1/2}^{HO}, \delta\phi_{i-1/2} + \frac{\phi_i^n-\phi_{i-1}^{n+1}}{c}, \delta\phi_{i-1/2} - (\phi_i^n-\phi_{i-1}^{n+1})\right). \tag{54}$$

This choice ensures the required boundedness for stability while choosing the high-order correction whenever it yields a stable flux. Notice, however, that to compute a correction term $\delta\phi_{i+1/2}$, we also need $\delta\phi_{i-1/2}$, which might not be available to us. If we bound the correction term $\delta\phi_{i-1/2}$, we can also derive simpler sufficient condition for $\delta\phi_{i+1/2}$.We can, e.g., bound the average flux $\phi_{i-1}^{n+1}+\delta\phi_{i-1/2}$ as in earlier examples, the bound takes a simpler form

$$\begin{aligned}
0 \leq \frac{\phi_{i-1}^{n+1}+\delta\phi_{i-1/2}-\phi_{i-1}^{n+1}}{\phi_i^n-\phi_{i-1}^{n+1}} &\leq 1, \\
0 \leq \frac{\delta\phi_{i-1/2}}{\phi_i^n-\phi_{i-1}^{n+1}} &\leq 1 \text{ for all faces,}
\end{aligned} \tag{55}$$

then

$$
\begin{aligned}
-1 \leq \frac{\delta\phi_{i+1/2} - \delta\phi_{i-1/2}}{\phi_i^n - \phi_{i-1}^{n+1}} &\leq \frac{1}{c}, \\
-1 \leq \frac{\delta\phi_{i+1/2}}{\phi_i^n - \phi_{i-1}^{n+1}} - \frac{\delta\phi_{i-1/2}}{\phi_i^n - \phi_{i-1}^{n+1}} &\leq \frac{1}{c}, \\
\frac{\delta\phi_{i-1/2}}{\phi_i^n - \phi_{i-1}^{n+1}} - 1 \leq \frac{\delta\phi_{i+1/2}}{\phi_i^n - \phi_{i-1}^{n+1}} &\leq \frac{1}{c} + \frac{\delta\phi_{i-1/2}}{\phi_i^n - \phi_{i-1}^{n+1}}, \\
\frac{\delta\phi_{i-1/2}}{\phi_i^n - \phi_{i-1}^{n+1}} - 1 \leq 0 \leq \frac{\delta\phi_{i+1/2}}{\phi_i^n - \phi_{i-1}^{n+1}} &\leq \frac{1}{c} \leq \frac{1}{c} + \frac{\delta\phi_{i-1/2}}{\phi_i^n - \phi_{i-1}^{n+1}}.
\end{aligned}
\tag{56}
$$

Thus, for $\phi_i^n - \phi_{i-1}^{n+1} > 0$

$$0 \leq \delta\phi_{i+1/2} \leq \frac{\phi_i^n - \phi_{i-1}^{n+1}}{c}, \tag{57}$$

and for $\phi_i^n - \phi_{i-1}^{n+1} < 0$

$$0 \geq \delta\phi_{i+1/2} \geq \frac{\phi_i^n - \phi_{i-1}^{n+1}}{c}. \tag{58}$$

The choice

$$\delta\phi_{i+1/2} = \mathrm{minmod}\left(\delta\phi_{i+1/2}^{HO}, \frac{\phi_i^n - \phi_{i-1}^{n+1}}{c}\right) \tag{59}$$

gives a sufficient condition for the high-order correction term.

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