### 0.0.1 High-order schemes based on the implicit upwind scheme

We can write a general second order scheme as

$$\begin{aligned}\phi_i^{n+1} &= \phi_i^n - c\left(\phi_i^{n+1} + \sigma_i \frac{1+c}{2}\Delta x - \phi_{i-1}^{n+1} - \sigma_{i-1}\frac{1+c}{2}\Delta x\right) \\ &= \phi_i^n - c\left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right) - c\left(\sigma_i - \sigma_{i-1}\right)\frac{1+c}{2}\Delta x,\end{aligned} \tag{1}$$

where $\sigma_i$ and $\sigma_{i-1}$ are slopes of a linear reconstruction in cells $i$ and $i-1$ respectively. We can solve (1) for the new cell average

$$\phi_i^{n+1} = \frac{\phi_i^n + c\ \phi_{i-1}^{n+1}}{1+c} - \frac{c}{2}\left(\sigma_i - \sigma_{i-1}\right)\Delta x. \tag{2}$$

Substituting to the upwind range condition

$$0 \le \frac{\phi_i^{n+1} - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1$$

we get

$$\begin{aligned}0 &\le \frac{\frac{\phi_i^n + c\ \phi_{i-1}^{n+1}}{1+c} - \frac{c}{2}\left(\sigma_i - \sigma_{i-1}\right)\Delta x - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1, \\ 0 &\le \frac{\frac{\phi_i^n - \phi_{i-1}^{n+1}}{1+c} - \frac{c}{2}\left(\sigma_i - \sigma_{i-1}\right)\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1, \\ 0 &\le \frac{1}{1+c} - \frac{c}{2}\frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} + \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1.\end{aligned} \tag{3}$$

We require that the outflow be at most the downwind outflow, by considering

$$\begin{aligned}0 \le \frac{\phi_{i-1}^{n+1} + \sigma_{i-1}\frac{1+c}{2}\Delta x - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} &\le 1, \\ 0 \le \frac{1+c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} &\le 1, \\ 0 \le \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} &\le \frac{c}{1+c}, \\ |\sigma_{i-1}| &\le 2\frac{\phi_i^n - \phi_{i-1}^{n+1}}{(1+c)\Delta x}.\end{aligned} \tag{4}$$

If the slope $\sigma_{i-1}$ is available, then $\sigma_i$ has to satisfy

$$\begin{aligned}
0 \leq \frac{1}{1+c} - \frac{c}{2}\frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} + \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \leq 1, \\
-\left(\frac{1}{1+c} + \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}}\right) \leq -\frac{c}{2}\frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \leq 1 - \left(\frac{1}{1+c} + \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}}\right), \\
-\left(\frac{1}{1+c} + \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}}\right) \leq -\frac{c}{2}\frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \leq \frac{c}{1+c} - \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}}, \\
\frac{1}{1+c} + \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \geq \frac{c}{2}\frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \geq \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} - \frac{c}{1+c}, \\
\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} - \frac{2}{1+c} \leq \frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \leq \frac{2}{c(1+c)} + \frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}}, \\
\left|\sigma_{i-1} - 2\frac{\phi_i^n - \phi_{i-1}^{n+1}}{(1+c)\Delta x}\right| \leq |\sigma_i| \leq \left|\frac{2}{c}\frac{\phi_i^n - \phi_{i-1}^{n+1}}{(1+c)\Delta x} + \sigma_{i-1}\right|.
\end{aligned} \tag{5}$$

From (4) we have

$$\begin{aligned}
|\sigma_i| \leq 2\frac{|\phi_{i+1}^n - \phi_i^{n+1}|}{(1+c)\Delta x}, \\
\left|\sigma_{i-1} - 2\frac{\phi_i^n - \phi_{i-1}^{n+1}}{(1+c)\Delta x}\right| \leq |\sigma_i| \leq \min\left(\left|\frac{2}{c}\frac{\phi_i^n - \phi_{i-1}^{n+1}}{(1+c)\Delta x} + \sigma_{i-1}\right|, 2\frac{|\phi_{i+1}^n - \phi_i^{n+1}|}{(1+c)\Delta x}\right).
\end{aligned} \tag{6}$$

Using (3) and (4), we can derive sufficient condition for the slope $\sigma_i$

$$\begin{aligned}
0 \leq \frac{1}{1+c} - \frac{c}{2}\frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \leq 1 - \frac{c}{1+c}, \\
0 \leq \frac{1}{1+c} - \frac{c}{2}\frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \leq \frac{1}{1+c}, \\
0 \leq \frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \leq \frac{2}{c}\frac{1}{1+c}, \\
|\sigma_i| \leq \frac{2}{c}\frac{|\phi_i^n - \phi_{i-1}^{n+1}|}{(1+c)\Delta x}.
\end{aligned} \tag{7}$$

$$|\sigma_i| \leq \min\left(\frac{2}{c}\frac{|\phi_i^n - \phi_{i-1}^{n+1}|}{(1+c)\Delta x}, 2\frac{|\phi_{i+1}^n - \phi_i^{n+1}|}{(1+c)\Delta x}\right). \tag{8}$$

**Remark.** *We can derive similar schemes appearing in [1] by writing the slope as a convex combination*

$$\begin{aligned}
\sigma_i &= (1-\omega_i)\frac{\phi_{i+1}^n - \phi_i^{n+1}}{(1+c)\Delta x} + \omega_i\frac{\phi_i^n - \phi_{i-1}^{n+1}}{(1+c)\Delta x}, \\
&= (1-\omega_i+\omega_i r_i)\,\frac{\phi_{i+1}^n - \phi_i^{n+1}}{(1+c)\Delta x}, \\
&= \Psi_i\frac{\phi_{i+1}^n - \phi_i^{n+1}}{(1+c)\Delta x},
\end{aligned} \tag{9}$$

*where*

$$0 \leq \omega_i \leq 1, \quad \Psi_i = \Psi_i(r_i) = (1-\omega_i+\omega_i r_i)\,, \quad r_i = \frac{\phi_i^n - \phi_{i-1}^{n+1}}{\phi_{i+1}^n - \phi_i^{n+1}}. \tag{10}$$

*Substituting the new form of the slopes we get*

$$
\begin{aligned}
&0 \leq \frac{1}{1+c} - \frac{c}{2}\frac{\sigma_i \Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} + \frac{c}{2}\frac{\sigma_{i-1}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \leq 1,\\
&0 \leq \frac{1}{1+c} - \frac{c}{2}\frac{\Psi_i \frac{\phi_{i+1}^n - \phi_i^{n+1}}{(1+c)\Delta x}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} + \frac{c}{2}\frac{\Psi_{i-1}\frac{\phi_i^n - \phi_{i-1}^{n+1}}{(1+c)\Delta x}\Delta x}{\phi_i^n - \phi_{i-1}^{n+1}} \leq 1,\\
&0 \leq \frac{1}{1+c} - \frac{c}{2(1+c)}\frac{\Psi_i}{r_i} + \frac{c}{2(1+c)}\Psi_{i-1} \leq 1.
\end{aligned} \tag{11}
$$

*Also, we consider*

$$
\begin{aligned}
&0 \leq \frac{\phi_{i-1}^{n+1} + \sigma_{i-1}\frac{1+c}{2}\Delta x - \phi_{i-1-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} \leq 1,\\
&0 \leq \frac{\phi_{i-1}^{n+1} + \Psi_{i-1}\frac{\phi_i^n - \phi_{i-1}^{n+1}}{(1+c)\Delta x}\frac{1+c}{2}\Delta x - \phi_{i-1-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} \leq 1,\\
&0 \leq \Psi_{i-1} \leq 2.
\end{aligned} \tag{12}
$$

*If* $\Psi_{i-1}$ *is available, then*

$$
\begin{aligned}
-\left(\frac{1}{1+c} + \frac{c}{2(1+c)}\Psi_{i-1}\right) &\leq -\frac{c}{2(1+c)}\frac{\Psi_i}{r_i} \leq 1 - \left(\frac{1}{1+c} + \frac{c}{2(1+c)}\Psi_{i-1}\right),\\
-\left(\frac{1}{1+c} + \frac{c}{2(1+c)}\Psi_{i-1}\right) &\leq -\frac{c}{2(1+c)}\frac{\Psi_i}{r_i} \leq \frac{c}{1+c} - \frac{c}{2(1+c)}\Psi_{i-1},\\
\frac{1}{1+c} + \frac{c}{2(1+c)}\Psi_{i-1} &\geq \frac{c}{2(1+c)}\frac{\Psi_i}{r_i} \geq \frac{c}{2(1+c)}\Psi_{i-1} - \frac{c}{1+c},\\
\frac{2}{c} + \Psi_{i-1} &\geq \frac{\Psi_i}{r_i} \geq \Psi_{i-1} - 2,\\
\Psi_{i-1} - 2 &\leq \frac{\Psi_i}{r_i} \leq \frac{2}{c} + \Psi_{i-1},\\
0 &\leq \frac{\Psi_i}{r_i} \leq \frac{2}{c} + \Psi_{i-1},
\end{aligned} \tag{13}
$$

*since* $\Psi_{i-1} - 2 \leq 0$ *from* (12). *Thus,*

$$
0 \leq \Psi_i \leq \min\left(2, r_i\left(\frac{2}{c} + \Psi_{i-1}\right)\right) \ \textit{for}\ r_i > 0, \tag{14}
$$

*and*

$$
\Psi_i = 0 \ \textit{for}\ r_i = 0. \tag{15}
$$

*This condition, however, differs a little from the one appearing in [1], where it was derived using an ENO reconstruction.*

*It is also possible to derive a sufficient condition for* $\Psi_i$. *Using* (11) *and* (12), *it is* *sufficient to satisfy*

$$\begin{aligned}
&0 \leq \frac{1}{1+c} - \frac{c}{2(1+c)}\frac{\Psi_i}{r_i} \leq 1 - \frac{c}{1+c}, \\
&0 \leq \frac{1}{1+c} - \frac{c}{2(1+c)}\frac{\Psi_i}{r_i} \leq \frac{1}{1+c}, \qquad (16) \\
&0 \leq \frac{\Psi_i}{r_i} \leq \frac{2}{c}.
\end{aligned}$$

$$0 \leq \Psi_i \leq \min\left(2, \frac{2r_i}{c}\right), \quad \textit{for } r_i > 0, \quad \textit{and } \Psi_i = 0 \quad \textit{for } r_i \leq 0. \qquad (17)$$

# Bibliography

[1] Peter Frolkovič and Michal Žeravý. "High resolution compact implicit numerical scheme for conservation laws". In: *Applied Mathematics and Computation* 442 (2023), p. 127720. URL: https://www.researchgate.net/profile/Peter-Frolkovic/publication/365886997_High_resolution_compact_implicit_numerical_scheme_for_conservation_laws/links/63888ddf2c563722f2297b03/High-resolution-compact-implicit-numerical-scheme-for-conservation-laws.pdf.