0.1 Implicit monotone schemes in 1D

In this section we discuss some stability properties of implicit schemes.

First we show that schemes written in a particular form satisfy the upwind range condition, meaning that the solution is always bounded by its neighbors in space-time as it travels along the characteristics.

Next we show that such schemes are also total variation non-increasing (TVNI), or total variation diminishing (TVD), as it is commonly referred to.

We conclude the section with the observation that if a scheme satisfies the implicit upwind range condition, then it also satisfies the TVD condition.

Both conditions can be used to construct implicit high-order monotone schemes. However, as we will see, the upwind range condition is much more straightforward to apply, only requiring the solution to be bounded between its neighbors. It can be much more cumbersome to require the TVD property of a scheme directly.

0.1.1 Upwind range condition for $c > 0$

Let us consider the implicit upwind scheme in 1 space dimension

$$\phi_i^{n+1} = \phi_i^n - c\left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right). \tag{1}$$

We assume, that the only unknown value is ϕ_i^{n+1}. To see why that statement holds, notice that the solution of the first order advection equation requires an inflow boundary condition. Thus, in the first cell, the value ϕ_0^{n+1} is known from the inflow BC, so the only unknown value is ϕ_1^{n+1}. Then we compute the next value and so on, until reaching the n-th cell, where ϕ_{i-1}^{n+1} is known and the only unknown value is ϕ_i^{n+1}.

In this section we want to show that schemes of the form of (1) satisfy the stability condition

$$\min\left(\phi_{i-1}^{n+1}, \phi_i^n\right) \leq \phi_i^{n+1} \leq \max\left(\phi_{i-1}^{n+1}, \phi_i^n\right), \tag{2}$$

for $c > 0$. We can distinguish three cases:

1. $\phi_i^n > \phi_{i-1}^{n+1}$:

$$\min\left(\phi_{i-1}^{n+1}, \phi_i^n\right) = \phi_{i-1}^{n+1}, \quad \max\left(\phi_{i-1}^{n+1}, \phi_i^n\right) = \phi_i^n.$$

We can rewrite (2) as

$$\begin{aligned}\phi_{i-1}^{n+1} \le \phi_i^{n+1} &\le \phi_i^n,\\ 0 \le \phi_i^{n+1} - \phi_{i-1}^{n+1} &\le \phi_i^n - \phi_{i-1}^{n+1},\\ 0 \le \frac{\phi_i^{n+1} - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} &\le 1.\end{aligned}$$

2. $\phi_i^n < \phi_{i-1}^{n+1}$:

$$\min\left(\phi_{i-1}^{n+1}, \phi_i^n\right) = \phi_i^n, \quad \max\left(\phi_{i-1}^{n+1}, \phi_i^n\right) = \phi_{i-1}^{n+1}.$$

We can rewrite (2) as

$$\begin{aligned}\phi_i^n \le \phi_i^{n+1} &\le \phi_{i-1}^{n+1},\\ \phi_i^n - \phi_{i-1}^{n+1} \le \phi_i^{n+1} - \phi_{i-1}^{n+1} &\le 0,\\ 1 \ge \frac{\phi_i^{n+1} - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} &\ge 0.\end{aligned}$$

3. $\phi_i^n = \phi_{i-1}^{n+1}$, in which case we have

$$\phi_i^n = \phi_{i-1}^{n+1} = \phi_i^{n+1}.$$

Notice, that for the first two cases, where $\phi_i^n \neq \phi_{i-1}^{n+1}$ we got the same expression. Thus, we can use

$$0 \le \frac{\phi_i^{n+1} - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} \le 1 \tag{3}$$

instead of (2) to simplify the analysis. The condition above can be interpreted as the upwind range condition[6] or data compatibility condition [10] for implicit schemes. Notice that min and max is a combination of values from both time steps $n, n-1$. This distinguishes the implicit upwind range property from the explicit one. In the explicit case, the min and max are chosen only from the values at the current time step $n-1$. The idea is that the solution travels along characteristics and this value is always bounded by its neighbors in space-time.

To prove (2), first we solve (1) for the unknown ϕ_i^{n+1}.

$$\phi_i^{n+1} = \frac{1}{1+c}\phi_i^n + \frac{c}{1+c}\phi_{i-1}^{n+1}. \tag{4}$$

Notice that

$$\frac{1}{1+c} = \frac{1+c-c}{1+c} = 1 - \frac{c}{1+c}.$$

For convenience, let us denote

$$k = \frac{c}{1+c}. \tag{5}$$

Since $0 \leq c \leq 1 + c$, then

$$\begin{aligned} &0 \leq c \leq 1 + c \rightarrow \\ &0 \leq \frac{c}{1+c} \leq 1 \rightarrow \\ &0 \leq k \leq 1. \end{aligned} \tag{6}$$

Thus, we can rewrite (4) as

$$\phi_i^{n+1} = (1-k)\phi_i^n + k\phi_{i-1}^{n+1}, \tag{7}$$

where we can see that the solution ϕ_i^{n+1} is an interpolation between ϕ_i^n and ϕ_{i-1}^{n+1}, thus (2) holds.

Theorem 0.1.1. *Let us consider a numerical scheme of the form*

$$\phi_i^{n+1} = (1-k)\phi_i^n + k\phi_{i-1}^{n+1}. \tag{8}$$

Assume that ϕ_{i-1}^n is known. If $0 \leq k \leq 1$, then the solution satisfies

$$\min\left(\phi_{i-1}^{n+1}, \phi_i^n\right) \leq \phi_i^{n+1} \leq \max\left(\phi_{i-1}^{n+1}, \phi_i^n\right), \tag{9}$$

Proof. Let us begin with the case $\phi_i^n \neq \phi_{i-1}^{n+1}$, for which we have to show that

$$0 \leq \frac{\phi_i^{n+1} - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} \leq 1. \tag{10}$$

We substitute (8) to (10)

$$\begin{aligned} &0 \leq \frac{(1-k)\phi_i^n + k\phi_{i-1}^{n+1} - \phi_{i-1}^{n+1}}{\phi_i^n - \phi_{i-1}^{n+1}} \leq 1, \\ &0 \leq \frac{(1-k)\left(\phi_i^n - \phi_{i-1}^{n+1}\right)}{\phi_i^n - \phi_{i-1}^{n+1}} \leq 1, \\ &0 \leq 1 - k \leq 1, \\ &0 \leq k \leq 1, \end{aligned}$$

which we assumed that holds. For the case of $\phi_i^n = \phi_{i-1}^{n+1}$ we get

$$\phi_i^{n+1} = (1-k)\phi_i^n + k\ \phi_i^n = \phi_i^n,$$

thus completing the proof. □

0.1.2 Upwind-range-condition for different values of c

In general

$$\begin{aligned} min \leq \phi_i^{n+1} \leq max, \\ 0 \leq \phi_i^{n+1} - min \leq max - min, \\ 0 \leq \frac{\phi_i^{n+1} - min}{max - min} \leq 1. \end{aligned} \tag{11}$$

We choose min and max to lie on the closest characteristics to ϕ_i^{n+1}. Thus, e.g., for $0 < c < 1$,

$$min = \min(\phi_{i-1}^n, \phi_i^n), \quad max = \max(\phi_{i-1}^n, \phi_i^n).$$

So, min and max is always chosen from some value on the left, and a value on the right. Let us rewrite

$$min = \min(\phi_{i,L}, \phi_{i,R}), \quad max = \max(\phi_{i,L}, \phi_{i,R}).$$

We can distinguish three cases:

1. $\phi_{i,R} > \phi_{i,L}$:

   $$\min(\phi_{i,L}, \phi_{i,R}) = \phi_{i,L}, \quad \max(\phi_{i,L}, \phi_{i,R}) = \phi_{i,R}.$$

   We can rewrite (2) as

   $$\begin{aligned} \phi_{i,L} \leq \phi_i^{n+1} \leq \phi_{i,R}, \\ 0 \leq \phi_i^{n+1} - \phi_{i,L} \leq \phi_{i,R} - \phi_{i,L}, \\ 0 \leq \frac{\phi_i^{n+1} - \phi_{i,L}}{\phi_{i,R} - \phi_{i,L}} \leq 1. \end{aligned}$$

2. $\phi_{i,R} < \phi_{i,L}$:

   $$\min(\phi_{i,L}, \phi_{i,R}) = \phi_{i,R}, \quad \max(\phi_{i,L}, \phi_{i,R}) = \phi_{i,L}.$$

   We can rewrite (2) as

   $$\begin{aligned} \phi_{i,L} \geq \phi_i^{n+1} \geq \phi_{i,R}, \\ 0 \geq \phi_i^{n+1} - \phi_{i,L} \geq \phi_{i,R} - \phi_{i,L}, \\ 0 \leq \frac{\phi_i^{n+1} - \phi_{i,L}}{\phi_{i,R} - \phi_{i,L}} \leq 1. \end{aligned}$$

3. $\phi_{i,R} = \phi_{i,L}$, in which case we have

   $$\phi_{i,R} = \phi_{i,L} = \phi_i^{n+1}.$$

Thus, in general, we can write the upwind-range-condition as

$$0 \le \frac{\phi_i^{n+1} - \phi_{i,L}}{\phi_{i,R} - \phi_{i,L}} \le 1. \tag{12}$$

For $0 < c < 1$, the URC reads as

$$0 \le \frac{\phi_i^{n+1} - \phi_{i-1}^n}{\phi_i^n - \phi_{i-1}^n} \le 1. \tag{13}$$

For $1 < c < 2$, the URC would be

$$0 \le \frac{\phi_i^{n+1} - \phi_{i-2}^n}{\phi_{i-1}^n - \phi_{i-2}^n} \le 1. \tag{14}$$

However, we want to use the values in the closest neighborhood of i. Instead of ϕ_{i-2}^n, we can use the value from the new time level ϕ_{i-1}^{n+1},

$$0 \le \frac{\phi_i^{n+1} - \phi_{i-1}^{n+1}}{\phi_{i-1}^n - \phi_{i-1}^{n+1}} \le 1, \tag{15}$$

which turns out to be a sufficient condition for Courant numbers $1 < c$.

For these larger Courant numbers, we can write a general second order scheme as

$$\begin{aligned} \phi_i^{n+1} &= \phi_i^n - c\left(\phi_i^{n+1} + \sigma_i \frac{1+c}{2}\Delta x - \phi_{i-1}^{n+1} - \sigma_{i-1}\frac{1+c}{2}\Delta x\right) \\ &= \phi_i^n - c\left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right) - c\left(\sigma_i - \sigma_{i-1}\right)\frac{1+c}{2}\Delta x, \end{aligned} \tag{16}$$

where σ_i and σ_{i-1} are slopes of a linear reconstruction in cells i and $i-1$ respectively. We can solve (16) for the new cell average

$$\phi_i^{n+1} = \frac{\phi_i^n + c\ \phi_{i-1}^{n+1} - c\left(\sigma_i - \sigma_{i-1}\right)\frac{1+c}{2}\Delta x}{1+c}. \tag{17}$$

Substituting (17) to (15) we get

$$\begin{aligned} 0 &\le \frac{\frac{\phi_i^n + c\ \phi_{i-1}^{n+1} - c(\sigma_i - \sigma_{i-1})\frac{1+c}{2}\Delta x}{1+c} - \phi_{i-1}^{n+1}}{\phi_{i-1}^n - \phi_{i-1}^{n+1}} \le 1, \\ 0 &\le \frac{\phi_i^n + c\ \phi_{i-1}^{n+1} - c\left(\sigma_i - \sigma_{i-1}\right)\frac{1+c}{2}\Delta x - (1+c)\phi_{i-1}^{n+1}}{(1+c)\left(\phi_{i-1}^n - \phi_{i-1}^{n+1}\right)} \le 1, \\ 0 &\le \frac{\phi_i^n - \phi_{i-1}^{n+1} - c\left(\sigma_i - \sigma_{i-1}\right)\frac{1+c}{2}\Delta x}{(1+c)\left(\phi_{i-1}^n - \phi_{i-1}^{n+1}\right)} \le 1. \end{aligned} \tag{18}$$

Let us write the slope as a linear combination

$$\begin{aligned} \sigma_i &= a\ \frac{\phi_{i+1}^n - \phi_i^n}{\Delta x} + b\ \frac{\phi_i^n - \phi_i^{n+1}}{c\ \Delta x}, \\ &= (a\ R_i + b)\ \frac{\phi_i^n - \phi_i^{n+1}}{c\ \Delta x}, \\ &= \Psi(R_i)\frac{\phi_i^n - \phi_i^{n+1}}{c\ \Delta x}, \end{aligned} \tag{19}$$

where

$$R_i = \frac{c\ \left(\phi_{i+1}^n - \phi_i^n\right)}{\phi_i^n - \phi_i^{n+1}} \tag{20}$$

is the smoothness indicator.

$$\begin{aligned}
0 &\leq \frac{\phi_i^n - \phi_{i-1}^n + \phi_{i-1}^n - \phi_{i-1}^{n+1} - c\left(\sigma_i - \sigma_{i-1}\right)\frac{1+c}{2}\Delta x}{(1+c)\left(\phi_{i-1}^n - \phi_{i-1}^{n+1}\right)} \leq 1, \\
0 &\leq \frac{1}{1+c}\left(\frac{\phi_i^n - \phi_{i-1}^n}{\phi_{i-1}^n - \phi_{i-1}^{n+1}} + 1\right) - \frac{c\Delta x}{2}\frac{\sigma_i - \sigma_{i-1}}{\phi_{i-1}^n - \phi_{i-1}^{n+1}} \leq 1, \\
0 &\leq \frac{1}{1+c}\left(\frac{R_{i-1}}{c} + 1\right) - \frac{c\Delta x}{2}\frac{\sigma_i - \sigma_{i-1}}{\phi_{i-1}^n - \phi_{i-1}^{n+1}} \leq 1, \\
0 &\leq \frac{1}{1+c}\left(\frac{R_{i-1}}{c} + 1\right) - \frac{c\Delta x}{2}\frac{\Psi(R_i)\frac{\phi_i^n - \phi_i^{n+1}}{c\ \Delta x} - \Psi(R_{i-1})\frac{\phi_{i-1}^n - \phi_{i-1}^{n+1}}{c\ \Delta x}}{\phi_{i-1}^n - \phi_{i-1}^{n+1}} \leq 1, \\
0 &\leq \frac{1}{1+c}\left(\frac{R_{i-1}}{c} + 1\right) - \frac{1}{2}\left(\Psi(R_i)\frac{\phi_i^n - \phi_i^{n+1}}{\phi_{i-1}^n - \phi_{i-1}^{n+1}} - \Psi(R_{i-1})\right) \leq 1,
\end{aligned} \tag{21}$$

$$\begin{aligned}
0 &\leq \frac{\phi_i^n - \phi_{i-1}^{n+1} - c\left(\sigma_i - \sigma_{i-1}\right)\frac{1+c}{2}\Delta x}{(1+c)\left(\phi_{i-1}^n - \phi_{i-1}^{n+1}\right)} \leq 1, \\
0 &\leq \frac{\phi_i^n - \phi_{i-1}^{n+1} - c\left(\Psi(R_i)\frac{\phi_i^n - \phi_i^{n+1}}{c\ \Delta x} - \Psi(R_{i-1})\frac{\phi_{i-1}^n - \phi_{i-1}^{n+1}}{c\ \Delta x}\right)\frac{1+c}{2}\Delta x}{(1+c)\left(\phi_{i-1}^n - \phi_{i-1}^{n+1}\right)} \leq 1, \\
0 &\leq \frac{\phi_i^n - \phi_{i-1}^{n+1} - \frac{1+c}{2}\Psi(R_i)\left(\phi_i^n - \phi_i^{n+1}\right) - \frac{1+c}{2}\Psi(R_{i-1})\left(\phi_{i-1}^n - \phi_{i-1}^{n+1}\right)}{(1+c)\left(\phi_{i-1}^n - \phi_{i-1}^{n+1}\right)} \leq 1,
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\phi_i^n - \phi_{i-1}^{n+1} - \frac{1+c}{2}\Psi(R_i)\left(\phi_i^n - \phi_i^{n+1}\right) \\
&\phi_i^n - \phi_{i-1}^n + \phi_{i-1}^n - \phi_{i-1}^{n+1} - \frac{1+c}{2}\Psi(R_i)\left(\phi_i^n - \phi_i^{n+1}\right) \\
&\left(\phi_i^n - \phi_{i-1}^n\right) + \left(\phi_{i-1}^n - \phi_{i-1}^{n+1}\right) - \frac{1+c}{2}\Psi(R_i)\frac{c\ \left(\phi_{i+1}^n - \phi_i^n\right)}{R_i}
\end{aligned} \tag{23}$$

$$\frac{R_i}{R_{i-1}} = \frac{c\ \left(\phi_{i+1}^n - \phi_i^n\right)}{\phi_i^n - \phi_i^{n+1}}\frac{\phi_{i-1}^n - \phi_{i-1}^{n+1}}{c\ \left(\phi_i^n - \phi_{i-1}^n\right)} \tag{24}$$

0.1.3 TVD condition

The total variation of a mesh function $\phi^n = \phi_i^n$ is defined as

$$TV(\phi^n) = \sum_{i=-\infty}^{\infty} |\phi_i^n - \phi_{i-1}^n|, \tag{25}$$

see, e.g., [11, 1, 9, 12, 2, 8, 5, 6, 4, 7, 3]. To prove the TVD property of the implicit upwind scheme (1), first we have to show that if the initial data has compact support, than the solution has this property.

Theorem 0.1.2. *If the initial data has compact support, thus $\phi_i^n \to 0$ as $i \to \infty$ or $i \to -\infty$, then the solution of a scheme written as (1) will also satisfy $\phi_i^{n+1} \to 0$ as $i \to \infty$ or $i \to -\infty$.*

Proof. Let us begin with proving that if $\phi_i^n \to 0$ as $i \to -\infty$, then $\phi_i^{n+1} \to 0$ as $i \to -\infty$.

Assume, that for any $\epsilon \in \mathbb{R}^+$, there exists $I_\epsilon \in \mathbb{Z}$ such that

$$0 < \phi_i^n < \epsilon \text{ for all } i < I_\epsilon. \tag{26}$$

To have a unique solution, an inflow boundary condition must be given. Let $\mathcal{C}_{I_\epsilon}$ represent the cell at the inflow boundary. Thus, we have to specify $\phi_{I_\epsilon-1}^{n+1}$. Since we attempt to solve the transport equation with positive speed $v > 0$, the boundary value $\phi_{I_\epsilon-1}^{n+1}$ must be coming from the upstream direction. Thus, it must be bounded by the values

$$0 < \min_{i<I_\epsilon-1}(\phi_i^n) < \phi_{I_\epsilon-1}^{n+1} < \max_{i<I_\epsilon-1}(\phi_i^n) < \epsilon,$$

since $I_\epsilon - 1 < I_\epsilon$. Thus,

$$\phi_i^{n+1} \to 0 \text{ as } i \to -\infty,$$

concluding the first part of the proof.

To finish, we want to show that if $\phi_i^n \to 0$ as $i \to \infty$, then $\phi_i^{n+1} \to 0$ as $i \to \infty$. Assume, that for any $\epsilon_1 \in \mathbb{R}^+$, there exists $I_{\epsilon_1} \in \mathbb{Z}$ such that

$$0 < \phi_i^n < \epsilon_1 \text{ for all } i \geq I_{\epsilon_1}. \tag{27}$$

Since

$$\min\left(\phi_{I_{\epsilon_1}-1}^{n+1}, \phi_{I_{\epsilon_1}}^n\right) \leq \phi_{I_{\epsilon_1}}^{n+1} \leq \max\left(\phi_{I_{\epsilon_1}-1}^{n+1}, \phi_{I_{\epsilon_1}}^n\right), \tag{28}$$

this implies that

$$|\phi_{I_{\epsilon_1}}^n - \phi_{I_{\epsilon_1}}^{n+1}| \leq |\phi_{I_{\epsilon_1}}^n - \phi_{I_{\epsilon_1}-1}^{n+1}|.$$

Now choose an $i > I_{\epsilon_2} > I_{\epsilon_1}$, such that for any $\epsilon_2 \in \mathbb{R}^+$

$$|\phi_i^n - \phi_i^{n+1}| < |\max_{i \geq I_{\epsilon_1}}(\phi_i^n) - \phi_{I_{\epsilon_2}}^{n+1}| < \epsilon_2.$$

Denoting $\epsilon = \epsilon_1 + \epsilon_2$ we conclude that there exists $I_\epsilon > I_{\epsilon_2} > I_{\epsilon_1}$ such that

$$|\phi_i^n - \phi_i^{n+1}| < \epsilon,$$

which implies that if $\phi_i^n \to 0$ as $i \to \infty$, then the solution will also $\phi_i^{n+1} \to 0$ as $i \to \infty$, thus completing the proof. □

Theorem 0.1.3. *The solution of a numerical scheme written in the form*

$$\phi_i^{n+1} = \phi_i^n - c\left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right), \tag{29}$$

for $c > 0$ and data with compact support, such that $\phi_i^n \to 0$ as $i \to \infty$ or $i \to -\infty$, is total variation non-increasing

$$TV(\phi_i^{n+1}) \leq TV(\phi_i^n). \tag{30}$$

Proof. First we put the terms with $n+1$ to the left-hand side, then we write the scheme for cells $i, i-1$

$$\phi_i^{n+1} + c\left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right) = \phi_i^n, \tag{31}$$

$$\phi_{i-1}^{n+1} + c\left(\phi_{i-1}^{n+1} - \phi_{i-2}^{n+1}\right) = \phi_{i-1}^n. \tag{32}$$

Then we subtract (32) from (31) to get

$$\begin{aligned} \phi_i^{n+1} - \phi_{i-1}^{n+1} + c\left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right) - c\left(\phi_{i-1}^{n+1} + \phi_{i-2}^{n+1}\right) &= \phi_i^n - \phi_{i-1}^n, \\ (1+c)\left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right) - c\left(\phi_{i-1}^{n+1} - \phi_{i-2}^{n+1}\right) &= \phi_i^n - \phi_{i-1}^n. \end{aligned} \tag{33}$$

We take the absolute value of both sides

$$|(1+c)\left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right) - c\left(\phi_{i-1}^{n+1} - \phi_{i-2}^{n+1}\right)| = |\phi_i^n - \phi_{i-1}^n|. \tag{34}$$

Then we use the triangle inequality $|a| - |b| \leq |a - b|$ and $|c\ a| = |c||a|$ to obtain

$$(1+c)|\phi_i^n - \phi_{i-1}^n| - c|\phi_{i-1}^n - \phi_{i-2}^n| \leq |(1+c)\left(\phi_i^n - \phi_{i-1}^n\right) - c\left(\phi_{i-1}^n - \phi_{i-2}^n\right)|. \tag{35}$$

Thus, using (34)

$$(1+c)|\phi_i^{n+1} - \phi_{i-1}^{n+1}| - c|\phi_{i-1}^{n+1} - \phi_{i-2}^{n+1}| \leq |\phi_i^n - \phi_{i-1}^n|. \tag{36}$$

To complete the proof, we take the sum of both sides

$$\begin{aligned} \sum_{i=-\infty}^{\infty} (1+c)|\phi_i^{n+1} - \phi_{i-1}^{n+1}| - \sum_{i=-\infty}^{\infty} c|\phi_{i-1}^{n+1} - \phi_{i-2}^{n+1}| &\leq \sum_{i=-\infty}^{\infty} |\phi_i^n - \phi_{i-1}^n|, \\ (1+c)\sum_{i=-\infty}^{\infty} |\phi_i^{n+1} - \phi_{i-1}^{n+1}| - c\sum_{i=-\infty}^{\infty} |\phi_{i-1}^{n+1} - \phi_{i-2}^{n+1}| &\leq \sum_{i=-\infty}^{\infty} |\phi_i^n - \phi_{i-1}^n|. \end{aligned} \tag{37}$$

We assumed that the initial data has compact support, which implies that the solution also have this property, meaning that

$$\sum_{i=-\infty}^{\infty} |\phi_{i-1}^{n+1} - \phi_{i-2}^{n+1}| = \sum_{i=-\infty}^{\infty} |\phi_i^{n+1} - \phi_{i-1}^{n+1}| = TV(\phi^{n+1}). \tag{38}$$

Thus, using (37) we conclude

$$\begin{aligned} (1+c)TV(\phi^{n+1}) - c\,TV(\phi^{n+1}) &\le TV(\phi^n), \\ TV(\phi^{n+1}) &\le TV(\phi^n). \end{aligned} \tag{39}$$

□

0.1.4 Conclusion

Since we can transform any scheme written in the form of (1) to a scheme written as (7) and vice versa, then it follows that if a scheme satisfies the implicit upwind range property (2) it is also TVD.