

$$\begin{aligned}
& \frac{|\Omega_p|}{\Delta t} (\phi_p^n - \phi_p^{n-1}) + \sum_{f \in \mathcal{F}_p^-} (\phi_q^n + \Psi_{pf}^- \mathcal{D}_{pf}^- \phi) a_{pf} \\
& + \sum_{f \in \mathcal{F}_p^+} (\phi_p^n + \Psi_{pf}^+ \mathcal{D}_{pf}^+ \phi) a_{pf} \\
& + \sum_{b \in \mathcal{B}_p^-} (\phi_e(\mathbf{x}_{pb}, t^n) + \Psi_{pb}^- \mathcal{D}_{pb}^- \phi) a_{pb} \\
& + \sum_{b \in \mathcal{B}_p^+} (\phi_p^n + \Psi_{pb}^+ \mathcal{D}_{pb}^+ \phi) a_{pb} = 0.
\end{aligned} \tag{1}$$

For an internal cell

$$\begin{aligned}
& \frac{|\Omega_p|}{\Delta t} (\phi_p^n - \phi_p^{n-1}) + \sum_{f \in \mathcal{F}_p^-} \phi_q^n a_{pf} + \sum_{f \in \mathcal{F}_p^+} \phi_p^n a_{pf} \\
& + \sum_{f \in \mathcal{F}_p^-} \Psi_f \mathcal{D}_{pf}^- \phi a_{pf} + \sum_{f \in \mathcal{F}_p^+} \Psi_f \mathcal{D}_{pf}^+ \phi a_{pf} = 0.
\end{aligned} \tag{2}$$

We split the antidiffusive fluxes to positive and negative values. Then we can write the sum of antidiffusive fluxes as

$$\begin{aligned}
& \sum_{f \in \mathcal{F}_p^-} \Psi_f \mathcal{D}_{pf}^- \phi a_{pf} + \sum_{f \in \mathcal{F}_p^+} \Psi_f \mathcal{D}_{pf}^+ \phi a_{pf} = \\
& \sum_{f \in \mathcal{F}_p^+} \Psi_f \max(\mathcal{D}_{pf}^+ \phi a_{pf}, 0) + \sum_{f \in \mathcal{F}_p^+} \Psi_f \min(\mathcal{D}_{pf}^+ \phi a_{pf}, 0) \\
& + \sum_{f \in \mathcal{F}_p^-} \Psi_f \max(\mathcal{D}_{pf}^- \phi a_{pf}, 0) + \sum_{f \in \mathcal{F}_p^-} \Psi_f \min(\mathcal{D}_{pf}^- \phi a_{pf}, 0).
\end{aligned} \tag{3}$$

Since the limiter is positive $\Psi_f \geq 0$, we can write

$$\begin{aligned}
& \sum_{f \in \mathcal{F}_p^+} \Psi_f \max(\mathcal{D}_{pf}^+ \phi a_{pf}, 0) + \sum_{f \in \mathcal{F}_p^-} \Psi_f \max(\mathcal{D}_{pf}^- \phi a_{pf}, 0) = \\
& \Psi_p^M \left(\sum_{f \in \mathcal{F}_p^+} \max(\mathcal{D}_{pf}^+ \phi a_{pf}, 0) + \sum_{f \in \mathcal{F}_p^-} \max(\mathcal{D}_{pf}^- \phi a_{pf}, 0) \right),
\end{aligned} \tag{4}$$

where $\Psi_p^M \geq 0$. Similarly,

$$\begin{aligned}
& \sum_{f \in \mathcal{F}_p^+} \Psi_f \min(\mathcal{D}_{pf}^+ \phi a_{pf}, 0) + \sum_{f \in \mathcal{F}_p^-} \Psi_f \min(\mathcal{D}_{pf}^- \phi a_{pf}, 0) = \\
& \Psi_p^N \left(\sum_{f \in \mathcal{F}_p^+} \min(\mathcal{D}_{pf}^+ \phi a_{pf}, 0) + \sum_{f \in \mathcal{F}_p^-} \min(\mathcal{D}_{pf}^- \phi a_{pf}, 0) \right),
\end{aligned} \tag{5}$$

where $\Psi_p^N \geq 0$. We denote

$$\begin{aligned} M_p &= \sum_{f \in \mathcal{F}_p^+} \max(\mathcal{D}_{pf}^+ \phi a_{pf}, 0) + \sum_{f \in \mathcal{F}_p^-} \max(\mathcal{D}_{pf}^- \phi a_{pf}, 0), \\ N_p &= \sum_{f \in \mathcal{F}_p^+} \min(\mathcal{D}_{pf}^+ \phi a_{pf}, 0) + \sum_{f \in \mathcal{F}_p^-} \min(\mathcal{D}_{pf}^- \phi a_{pf}, 0) \end{aligned} \quad (6)$$

we can equivalently write

$$\frac{|\Omega_p|}{\Delta t} (\phi_p^n - \phi_p^{n-1}) + \sum_{f \in \mathcal{F}_p^-} \phi_q^n a_{pf} + \sum_{f \in \mathcal{F}_p^+} \phi_p^n a_{pf} + \Psi_p^M M_p + \Psi_p^N N_p = 0. \quad (7)$$

After finding the single value Ψ_p^M , we should find a way how do we distribute it among the faces to get the most accurate solution.

$$\phi_p^n - \phi_p^{n-1} + \lambda_p \sum_{f \in \mathcal{F}_p^-} \phi_q^n a_{pf} + \lambda_p \sum_{f \in \mathcal{F}_p^+} \phi_p^n a_{pf} + \lambda_p \Psi_p^M M_p + \lambda_p \Psi_p^N N_p = 0. \quad (8)$$

We want

$$\min \leq \phi_p^{n-1} - \lambda_p \sum_{f \in \mathcal{F}_p^-} \phi_q^n a_{pf} - \lambda_p \sum_{f \in \mathcal{F}_p^+} \phi_p^n a_{pf} - \lambda_p \Psi_p^M M_p - \lambda_p \Psi_p^N N_p \leq \max$$

Since $N_p \leq \Psi_p^N N_p \leq 0$ we can write the sufficient conditions

$$\begin{aligned} \phi_p^{n-1} - \lambda_p \sum_{f \in \mathcal{F}_p^-} \phi_q^n a_{pf} - \lambda_p \sum_{f \in \mathcal{F}_p^+} \phi_p^n a_{pf} - \lambda_p \Psi_p^M M_p - \lambda_p N_p &\leq \max \\ \min &\leq \phi_p^{n-1} - \lambda_p \sum_{f \in \mathcal{F}_p^-} \phi_q^n a_{pf} - \lambda_p \sum_{f \in \mathcal{F}_p^+} \phi_p^n a_{pf} - \lambda_p \Psi_p^M M_p. \end{aligned}$$

Thus,

$$\phi_p^{n-1} - \lambda_p \sum_{f \in \mathcal{F}_p^-} \phi_q^n a_{pf} - \lambda_p \sum_{f \in \mathcal{F}_p^+} \phi_p^n a_{pf} - \lambda_p N_p - \max \leq \lambda_p \Psi_p^M M_p$$

and

$$\lambda_p \Psi_p^M M_p \leq \phi_p^{n-1} - \lambda_p \sum_{f \in \mathcal{F}_p^-} \phi_q^n a_{pf} - \lambda_p \sum_{f \in \mathcal{F}_p^+} \phi_p^n a_{pf} - \min.$$

It is possible to bound Ψ_p^M if

$$\begin{aligned} \phi_p^{n-1} - \lambda_p \sum_{f \in \mathcal{F}_p^-} \phi_q^n a_{pf} - \lambda_p \sum_{f \in \mathcal{F}_p^+} \phi_p^n a_{pf} - \lambda_p N_p - \max &\leq \phi_p^{n-1} - \lambda_p \sum_{f \in \mathcal{F}_p^-} \phi_q^n a_{pf} - \lambda_p \sum_{f \in \mathcal{F}_p^+} \phi_p^n a_{pf} - \min \\ &- \lambda_p N_p \leq \max - \min \end{aligned}$$

We want

$$\min \leq \phi_p^{n-1} - \lambda_p \sum_{f \in \mathcal{F}_p^-} \phi_q^n a_{pf} - \lambda_p \sum_{f \in \mathcal{F}_p^+} \phi_p^n a_{pf} - \alpha_p (\lambda_p M_p + \lambda_p N_p) \leq \max$$