0.0.1 Interpreting implicit schemes as a reconstruction in the current time step

Explicit finite volume schemes can be interpreted as a reconstruct-evolve-average(REA) algorithm [2, 1, 3]:

1. Reconstruct a piecewise polynomial function $\tilde{\phi}_i^n(x, t_n)$ for all $x \in \mathcal{C}_i$ from the known cell averages

2. Evolve the function to obtain $\tilde{\phi}_i^n(x, t_{n+1})$

3. Compute the average of this evolved function over each cell to obtain the new cell averages

$$\phi_i^{n+1} = \frac{1}{\Delta x} \int_{p_i} \tilde{\phi}_i^n(x, t_{n+1}) \, \mathrm{d}x \,.$$

Meaning, we reconstruct the values in a given cell using the cell averages. For example, the explicit upwind scheme can be interpreted as a piecewise constant reconstruction in the current time step. Then, we evolve this reconstruction and compute the cell averages in the new time step. We repeat this process until we reach the desired time.

This interpretation can help construct high-order schemes, where instead of constant reconstruction we use linear, parabolic, etc., depending on the desired accuracy. Also, if we make sure that our reconstruction does not generate new extrema, then evolving and averaging operations will neither, meaning that our numerical solution will be oscillation free.

In this section we would like to show that implicit schemes can also be interpreted similarly by considering that the values travel along characteristics. Thus, the values in the new time step can be interpreted as values in the current time step.

In this section we restrict our discussion to the simplest problem of advection with positive constant speed $v > 0$.

We start with the most straightforward piecewise constant reconstruction:

1. Reconstruct a piecewise constant function $\tilde{\phi}_i^n(x, t_n)$ for all $x \in \mathcal{C}_i$ from the known cell averages ϕ_i^n. Thus,

$$\tilde{\phi}_i^n(x, t_n) = \phi_i^n \text{ for all } x \in \mathcal{C}_i.$$

2. Evolve the function using the exact solution of the advection equation

$$\tilde{\phi}_i^n(x, t_{n+1}) = \tilde{\phi}_i^n(x - v\Delta t, t_n)$$

3. Compute the average of this evolved function over each cell to obtain the new cell averages

$$\phi_i^{n+1} = \frac{1}{\Delta x} \int_{\mathcal{C}_i} \tilde{\phi}_i^n(x, t_{n+1}) \, \mathrm{d}x \,,$$

or, equivalently

$$\phi_i^{n+1} = \frac{1}{\Delta x} \int_{\mathcal{C}_i - v\Delta t} \tilde{\phi}_i^n(x, t_n) \, \mathrm{d}x \,.$$

If the advection with a speed v over a time step Δt is smaller than a cell width Δx, thus the time step is restricted by

$$0 \leq v\Delta t \leq \Delta x$$

$$0 \leq \frac{v\Delta t}{\Delta x} \leq 1, \tag{1}$$

$$0 \leq c \leq 1,$$

where c is the Courant number, the algorithm yields the first order explicit upwind scheme. The new cell averages are computed as

$$\begin{aligned}
\phi_i^{n+1} &= \frac{1}{\Delta x} \left(v\Delta t \, \phi_{i-1}^n + (\Delta x - v\Delta t) \, \phi_i^n \right) \\
&= \frac{v\Delta t}{\Delta x} \phi_{i-1}^n + \left(1 - \frac{v\Delta t}{\Delta x} \right) \phi_i^n \tag{2} \\
&= c \, \phi_{i-1}^n + (1 - c) \, \phi_i^n.
\end{aligned}$$

We can see, that if the CFL condition (1) holds, the new cell average is a linear interpolation between ϕ_i^{n-1} and ϕ_i^n.

Let us compare this with the implicit upwind scheme

$$\phi_i^{n+1} = \phi_i^n - c \left(\phi_i^{n+1} - \phi_{i-1}^{n+1} \right). \tag{3}$$

We assume that in a cell the values ϕ_i^n and ϕ_{i-1}^{n+1} are known, and we want to compute the new cell average ϕ_i^{n+1}.

Assume constant reconstruction in the cell $\mathcal{C}_i$ at the current time step t_n as before

$$\tilde{\phi}_i^n(x, t_n) = \phi_i^n \text{ for } x \in \mathcal{C}_i,$$

and a constant reconstruction in the cell $\mathcal{C}_{i-1}$ at the new time step t_{n+1} as

$$\tilde{\phi}_{i-1}^{n+1}(x, t_{n+1}) = \phi_{i-1}^{n+1} \text{ for } x \in \mathcal{C}_{i-1}.$$

Using the exact solution of the advection equation, we can project the reconstruction at time t_{n+1} to the current time t_n, by evolving it back in time as

$$\tilde{\phi}_{i-1}^{n+1}(x, t_n) = \tilde{\phi}_{i-1}^{n+1}(x + v\Delta t, t_{n+1}) \text{ for } x \in \mathcal{C}_{i-1} - v\Delta t.$$

In the explicit case, we computed the new average using the values ϕ_{i-1}^n, ϕ_i^n. But now, instead of ϕ_{i-1}^n we have ϕ_{i-1}^{n+1}. What we can make is to compute an estimated value $\phi_{i-1}^{n,*}$ to compute the new cell average as in (2)

$$\phi_i^{n+1} = c \; \phi_{i-1}^{n,*} + (1 - c) \; \phi_i^n. \tag{4}$$

We can express the cell average $\phi_{i-1}^{n,*}$ using $\phi_{i-1}^{n+1}, \phi_i^{n+1}$ by averaging in cell $\mathcal{C}_{i-1}$

$$\phi_{i-1}^{n,*} = (1 - c) \, \phi_{i-1}^{n+1} + c \; \phi_i^{n+1}. \tag{5}$$

Thus, we can solve the 2 equations above for 2 unknowns $\phi_{i-1}^{n,*}, \phi_i^{n+1}$. This yields

$$\phi_{i-1}^{n,*} = \left(1 - \frac{c}{1 + c}\right) \phi_{i-1}^{n+1} + \frac{c}{1 + c}\phi_i^n,$$

$$\phi_i^{n+1} = \frac{c}{1 + c}\phi_{i-1}^{n+1} + \left(1 - \frac{c}{1 + c}\right) \phi_i^n,$$

yielding the same result for the new cell average as solving the equation directly. This shows that the REA interpretation of the scheme (**??**) is correct. Also notice that both values are linear interpolations between the known values $\phi_{i-1}^{n+1}, \phi_i^n$.