

## 0.1 Implicit monotone schemes in 1D

In this section we discuss some stability properties of implicit schemes.

First we show that schemes written in a particular form satisfy the upwind range condition, meaning that the solution is always bounded by its neighbors in space-time as it travels along the characteristics.

Next we show that such schemes are also total variation non-increasing (TVNI), or total variation diminishing (TVD), as it is commonly referred to.

We conclude the section with the observation that if a scheme satisfies the implicit upwind range condition, then it also satisfies the TVD condition.

Both conditions can be used to construct implicit high-order monotone schemes. However, as we will see, the upwind range condition is much more straightforward to apply, only requiring the solution to be bounded between its neighbors. It can be much more cumbersome to require the TVD property of a scheme directly.

### 0.1.1 Upwind range condition for implicit schemes

Let us consider the implicit upwind scheme in 1 space dimension

$$\phi_i^n = \phi_i^{n-1} - c (\phi_i^n - \phi_{i-1}^n). \quad (1)$$

We assume, that the only unknown value is  $\phi_i^n$ . To see why that statement holds, notice that the solution of the first order advection equation requires an inflow boundary condition. Thus, in the first cell, the value  $\phi_0^n$  is known from the inflow BC, so the only unknown value is  $\phi_1^n$ . Then we compute the next value and so on, until reaching the  $n$ -th cell, where  $\phi_{i-1}^n$  is known and the only unknown value is  $\phi_i^n$ .

In this section we want to show that for Courant numbers  $c > 0$ , schemes written in the form of (1) satisfy the stability condition

$$\min(\phi_{i-1}^n, \phi_i^{n-1}) \leq \phi_i^n \leq \max(\phi_{i-1}^n, \phi_i^{n-1}), \quad (2)$$

which can be interpreted as the upwind range condition[6] or data compatibility condition [10] for implicit schemes. Notice that min and max is a combination of values from both time steps  $n, n - 1$ . This distinguishes the implicit upwind range property from the explicit one. In the explicit case, the min and max are chosen only from

the values at the current time step  $n - 1$ . The idea is that the solution travels along characteristics and this value is always bounded by its neighbors in space-time.

To prove (2), first we solve (1) for the unknown  $\phi_i^n$ .

$$\phi_i^n = \frac{1}{1+c} \phi_i^{n-1} + \frac{c}{1+c} \phi_{i-1}^n. \quad (3)$$

Notice that

$$\frac{1}{1+c} = \frac{1+c-c}{1+c} = 1 - \frac{c}{1+c}.$$

For convenience, let us denote

$$k = \frac{c}{1+c}. \quad (4)$$

Since  $0 \leq c \leq 1+c$ , then

$$\begin{aligned} 0 &\leq c \leq 1+c \rightarrow \\ 0 &\leq \frac{c}{1+c} \leq 1 \rightarrow \\ 0 &\leq k \leq 1. \end{aligned} \quad (5)$$

Thus, we can rewrite (3) as

$$\phi_i^n = (1-k) \phi_i^{n-1} + k \phi_{i-1}^n, \quad (6)$$

where we can see that the solution  $\phi_i^n$  is a linear interpolation between  $\phi_i^{n-1}$  and  $\phi_{i-1}^n$ , thus (2) holds.

**Theorem 1.** *Let us consider a numerical scheme written in the form*

$$\phi_i^n = (1-k) \phi_i^{n-1} + k \phi_{i-1}^n. \quad (7)$$

*Assume that  $\phi_{i-1}^n$  is known. If the coefficient is bounded  $0 \leq k \leq 1$ , then the solution satisfies*

$$\min(\phi_{i-1}^n, \phi_i^{n-1}) \leq \phi_i^n \leq \max(\phi_{i-1}^n, \phi_i^{n-1}), \quad (8)$$

*Proof.* Without the loss of generality, let us begin with the case where the data satisfies

$$\phi_{i-1}^n \leq \phi_i^{n-1}. \quad (9)$$

Then we can write

$$\begin{aligned} \min(\phi_{i-1}^n, \phi_i^{n-1}) &= \phi_{i-1}^n \\ \max(\phi_{i-1}^n, \phi_i^{n-1}) &= \phi_i^{n-1}. \end{aligned} \quad (10)$$

Thus, in the first case we want to show that

$$\phi_{i-1}^n \leq \phi_i^n \leq \phi_i^{n-1}. \quad (11)$$

We start by proving that  $\phi_i^n \leq \phi_i^{n-1}$ .

Using (9) and the fact that  $0 \leq k \leq 1$

$$\begin{aligned} \phi_{i-1}^n &\leq \phi_i^{n-1}, \\ k\phi_{i-1}^n &\leq k\phi_i^{n-1}, \end{aligned}$$

where  $0 \leq k \leq 1$ . Then we use (7) to express  $k\phi_{i-1}^n$

$$\begin{aligned} \phi_i^n - (1-k)\phi_i^{n-1} &\leq k\phi_i^{n-1}, \\ \phi_i^n - \phi_i^{n-1} &\leq 0, \\ \phi_i^n &\leq \phi_i^{n-1}. \end{aligned}$$

It remains to show that  $\phi_{i-1}^n \leq \phi_i^n$ .

$$\begin{aligned} \phi_{i-1}^n &\leq \phi_i^{n-1}, \\ (1-k)\phi_{i-1}^n &\leq (1-k)\phi_i^{n-1}. \end{aligned}$$

Again, using (7) to express  $(1-k)\phi_i^{n-1}$

$$\begin{aligned} (1-k)\phi_{i-1}^n &\leq \phi_i^n - k\phi_{i-1}^n, \\ \phi_{i-1}^n &\leq \phi_i^n. \end{aligned}$$

Thus,

$$\phi_{i-1}^n \leq \phi_i^n \leq \phi_i^{n-1}.$$

The proof for the other case where the data is such that  $\phi_{i-1}^n \geq \phi_i^{n-1}$  is straightforward.  $\square$

### 0.1.2 TVD condition

The total variation of a mesh function  $\phi^n = \phi_i^n$  is defined as

$$TV(\phi^n) = \sum_{i=-\infty}^{\infty} |\phi_i^n - \phi_{i-1}^n|, \quad (12)$$

see, e.g., [11, 1, 9, 12, 2, 8, 5, 6, 4, 7, 3].

**Theorem 2.** *The solution of a numerical scheme written in the form*

$$\phi_i^n = \phi_i^{n-1} - c (\phi_i^n - \phi_{i-1}^n), \quad (13)$$

for  $c > 0$  and data with compact support, such that  $\phi_i^{n-1} \rightarrow 0$  as  $i \rightarrow \infty$  or  $i \rightarrow -\infty$ , is total variation non-increasing

$$TV(\phi_i^{n-1}) \geq TV(\phi_i^n). \quad (14)$$

*Proof.* We start by writing the scheme for cells  $i, i-1$

$$\phi_i^n = \phi_i^{n-1} - c (\phi_i^n - \phi_{i-1}^n), \quad (15)$$

$$\phi_{i-1}^n = \phi_{i-1}^{n-1} - c (\phi_{i-1}^n - \phi_{i-2}^n). \quad (16)$$

Then we subtract (16) from (15) to get

$$\begin{aligned} \phi_i^n - \phi_{i-1}^n &= \phi_i^{n-1} - \phi_{i-1}^{n-1} - c (\phi_i^n - \phi_{i-1}^n - \phi_{i-1}^n + \phi_{i-2}^n), \\ \phi_i^n - \phi_{i-1}^n + c (\phi_i^n - \phi_{i-1}^n - \phi_{i-1}^n + \phi_{i-2}^n) &= \phi_i^{n-1} - \phi_{i-1}^{n-1}, \\ (1+c) (\phi_i^n - \phi_{i-1}^n) - c (\phi_{i-1}^n - \phi_{i-2}^n) &= \phi_i^{n-1} - \phi_{i-1}^{n-1}. \end{aligned} \quad (17)$$

We take the absolute value

$$|(1+c) (\phi_i^n - \phi_{i-1}^n) - c (\phi_{i-1}^n - \phi_{i-2}^n)| = |\phi_i^{n-1} - \phi_{i-1}^{n-1}|. \quad (18)$$

Then we use the triangle inequality  $|a-b| \geq |a| - |b|$  and  $|c a| = |c| |a|$

$$|(1+c) (\phi_i^n - \phi_{i-1}^n) - c (\phi_{i-1}^n - \phi_{i-2}^n)| \geq (1+c) |\phi_i^n - \phi_{i-1}^n| - c |\phi_{i-1}^n - \phi_{i-2}^n|, \quad (19)$$

Thus, using (18)

$$|\phi_i^{n-1} - \phi_{i-1}^{n-1}| \geq (1+c) |\phi_i^n - \phi_{i-1}^n| - c |\phi_{i-1}^n - \phi_{i-2}^n|. \quad (20)$$

To complete the proof, we take the sum of both sides

$$\begin{aligned} \sum_{i=-\infty}^{\infty} |\phi_i^{n-1} - \phi_{i-1}^{n-1}| &\geq \sum_{i=-\infty}^{\infty} (1+c) |\phi_i^n - \phi_{i-1}^n| - \sum_{i=-\infty}^{\infty} c |\phi_{i-1}^n - \phi_{i-2}^n|, \\ \sum_{i=-\infty}^{\infty} |\phi_i^{n-1} - \phi_{i-1}^{n-1}| &\geq (1+c) \sum_{i=-\infty}^{\infty} |\phi_i^n - \phi_{i-1}^n| - c \sum_{i=-\infty}^{\infty} |\phi_{i-1}^n - \phi_{i-2}^n|. \end{aligned} \quad (21)$$

To complete the proof, we use the fact that the data has compact support and the solution  $\phi^n$  satisfies the upwind range condition (2)

$$\begin{aligned} \min(\phi_{i-1}^n, \phi_i^{n-1}) &\leq \phi_i^n \leq \max(\phi_{i-1}^n, \phi_i^{n-1}) \text{ for all } i \in (-\infty, \infty), \\ \phi_{-\infty, \infty}^{n-1} = 0, \rightarrow \phi_{-\infty, \infty}^n = 0 &\rightarrow \sum_{i=-\infty}^{\infty} |\phi_{i-1}^n - \phi_{i-2}^n| = \sum_{i=-\infty}^{\infty} |\phi_i^n - \phi_{i-1}^n| = TV(\phi^n). \end{aligned} \quad (22)$$

Thus, using (21) we conclude

$$\begin{aligned} TV(\phi^{n-1}) &\geq (1+c)TV(\phi^n) - c TV(\phi^n), \\ TV(\phi^{n-1}) &\geq TV(\phi^n). \end{aligned} \tag{23}$$

□

### 0.1.3 Conclusion

Since we can transform a scheme written in the form of (1) to a scheme written as (6) and vice versa, then it follows that if a scheme satisfies the implicit upwind range property (2) it is also TVD.