0.1 Reconstruct-evolve-average interpretation of implicit schemes

It is well known that explicit finite volume schemes can be interpreted as a reconstruct-evolve-average(REA) algorithm [2, 1, 3]:

1. Reconstruct a piecewise polynomial function $\tilde{\phi}_i^n(x, t_n)$ for all $x \in \mathcal{C}_i$ from the known cell averages

2. Evolve the function to obtain $\tilde{\phi}_i^n(x, t_{n+1})$

3. Compute the average of this evolved function over each cell to obtain the new cell averages

$$\phi_i^{n+1} = \frac{1}{\Delta x} \int_{\mathcal{C}_i} \tilde{\phi}_i^n(x, t_{n+1}) \, \mathrm{d}x \,.$$

This interpretation can help us to construct high-order schemes. Many flux and slope limiters can be designed using the fact that if the reconstruction process does not generate new extrema, then evolving and averaging operations will neither, meaning that our numerical solution will be oscillation free.

In this section we would like to show that implicit schemes can also be interpreted similarly by considering that the values travel along characteristics. Thus, the values in the new time step can be interpreted as values in the current time step.

In this section we restrict our discussion to the simplest problem of advection with positive constant speed $v > 0$.

0.1.1 Courant numbers < 1

0.1.1.1 Explicit case

We start with the most straightforward piecewise constant reconstruction:

1. Reconstruct a piecewise constant function $\tilde{\phi}_i^n(x, t_n)$ from the known cell averages ϕ_i^n:

$$\tilde{\phi}_i^n(x, t_n) = \phi_i^n \text{ for } x \in \mathcal{C}_i = (x_{i-1/2}, x_{i+1/2}).$$

2. Evolve the function using the exact solution of the advection equation

$$\tilde{\phi}_i^n(x, t_{n+1}) = \tilde{\phi}_i^n(x - v\Delta t, t_n),$$

meaning, that

$$\tilde{\phi}_i^n(x, t_{n+1}) = \phi_i^n \text{ for } x \in \mathcal{C}_i + v\Delta t.$$

3. Compute the average of this evolved function over each cell to obtain the new cell averages

$$\phi_i^{n+1} = \frac{1}{\Delta x} \int_{\mathcal{C}_i} \tilde{\phi}_i^n(x, t_{n+1}) \, dx = \frac{1}{\Delta x} \left(\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{\phi}_i^n(x, t_{n+1}) \, dx \right),$$

To evaluate the integral, we need to know, by what values it the region $\mathcal{C}_i$ occupied at time t_{n+1}. If the advection speed v over a time step Δt is smaller than a cell width Δx, thus

$$0 \le v\Delta t \le \Delta x, \tag{1}$$

then

$$\tilde{\phi}_i^n(x, t_{n+1}) = \begin{cases} \phi_{i-1}^n & \text{for } x \in (x_{i-1/2}, x_{i-1/2} + v\Delta t) \\ \phi_i^n & \text{for } x \in (x_{i-1/2} + v\Delta t, x_{i+1/2}). \end{cases} \tag{2}$$

$$\phi_i^{n+1} = \frac{1}{\Delta x} \left(\int_{x_{i-1/2}}^{x_{i-1/2}+v\Delta t} \phi_{i-1}^n \, dx + \int_{x_{i-1/2}+v\Delta t}^{x_{i+1/2}} \phi_i^n \, dx \right),$$

$$= \frac{1}{\Delta x} \left(\phi_{i-1}^n \int_{x_{i-1/2}}^{x_{i-1/2}+v\Delta t} dx + \phi_i^n \int_{x_{i-1/2}+v\Delta t}^{x_{i+1/2}} dx \right),$$

$$= \frac{1}{\Delta x} \left(\phi_{i-1}^n \int_{x_{i-1/2}}^{x_{i-1/2}+v\Delta t} dx + \phi_i^n \int_{x_{i-1/2}+v\Delta t}^{x_{i+1/2}} dx \right),$$

$$= \frac{v\Delta t}{\Delta x} \phi_{i-1}^n + \left(1 - \frac{v\Delta t}{\Delta x} \right) \phi_i^n$$

$$= c \, \phi_{i-1}^n + (1 - c) \, \phi_i^n.$$

$$0 \le v\Delta t \le \Delta x$$

$$0 \le \frac{v\Delta t}{\Delta x} \le 1, \tag{3}$$

$$0 \le c \le 1,$$

where c is the Courant number, the algorithm yields the first order explicit upwind scheme. The new cell averages can be computed using the above described algorithm as

$$\phi_i^{n+1} = \frac{1}{\Delta x} \left(v\Delta t \, \phi_{i-1}^n + (\Delta x - v\Delta t) \, \phi_i^n \right)$$

$$= \frac{v\Delta t}{\Delta x} \phi_{i-1}^n + \left(1 - \frac{v\Delta t}{\Delta x} \right) \phi_i^n \tag{4}$$

$$= c \, \phi_{i-1}^n + (1 - c) \, \phi_i^n.$$

We can see, that if the CFL condition (3) holds, the new cell average is a linear interpolation between ϕ_i^{n-1} and ϕ_i^n.

0.1.1.2 Implicit case

Let us first look at the implicit upwind scheme

$$\phi_i^{n+1} = \phi_i^n - c\left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right). \tag{5}$$

Solving for the new cell average ϕ_i^{n+1} we get

$$\phi_{i-1}^{n,*} = \left(1 - \frac{c}{1+c}\right)\phi_{i-1}^{n+1} + \frac{c}{1+c}\phi_i^n,$$

$$\phi_i^{n+1} = \frac{c}{1+c}\phi_{i-1}^{n+1} + \left(1 - \frac{c}{1+c}\right)\phi_i^n.$$

Using the fact that if we start the computation at the inflow boundary, where an inflow boundary condition must be specified, reaching the n-th cell, ϕ_i^n and ϕ_{i-1}^{n+1} will be known. Thus, using these values, we obtain the new cell average ϕ_i^{n+1}.

How can this equation be interpreted as an REA algorithm?

Assume constant reconstruction in the cell $\mathcal{C}_i$ at the current time step t_n as before

$$\tilde{\phi}_i^n(x, t_n) = \phi_i^n \text{ for } x \in \mathcal{C}_i,$$

and a constant reconstruction in the cell $\mathcal{C}_{i-1}$ at the new time step t_{n+1} as

$$\tilde{\phi}_{i-1}^{n+1}(x, t_{n+1}) = \phi_{i-1}^{n+1} \text{ for } x \in \mathcal{C}_{i-1}.$$

Using the exact solution of the advection equation, we evolve the value from the current time step

$$\tilde{\phi}_i^n(x, t_{n+1}) = \tilde{\phi}_i^n(x - v\Delta t, t_n).$$

To obtain the new cell average, we need to evaluate the integral

$$\phi_i^{n+1} = \frac{1}{\Delta x}\int_{\mathcal{C}_i} \tilde{\phi}_i^n(x, t_{n+1})\,\mathrm{d}x.$$

In the explicit case, we computed the new average using the values ϕ_{i-1}^n, ϕ_i^n. But now, instead of ϕ_{i-1}^n we have ϕ_{i-1}^{n+1}. What we can make is to compute an estimated value $\phi_{i-1}^{n,*}$ to compute the new cell average as in (4)

$$\phi_i^{n+1} = c\,\phi_{i-1}^{n,*} + (1-c)\,\phi_i^n. \tag{6}$$

We can express the cell average $\phi_{i-1}^{n,*}$ using $\phi_{i-1}^{n+1}, \phi_i^{n+1}$ by averaging in cell $\mathcal{C}_{i-1}$

$$\phi_{i-1}^{n,*} = (1-c)\,\phi_{i-1}^{n+1} + c\,\phi_i^{n+1}. \tag{7}$$

Thus, we can solve the 2 equations above for 2 unknowns $\phi_{i-1}^{n,*}, \phi_i^{n+1}$ to get

$$\phi_{i-1}^{n,*} = \left(1 - \frac{c}{1+c}\right)\phi_{i-1}^{n+1} + \frac{c}{1+c}\phi_i^n,$$
$$\phi_i^{n+1} = \frac{c}{1+c}\phi_{i-1}^{n+1} + \left(1 - \frac{c}{1+c}\right)\phi_i^n,$$

yielding the same result for the new cell average as solving the equation directly. This shows that the REA interpretation of the implicit upwind scheme (5) is correct. Also notice that both values are linear interpolations between the known values $\phi_{i-1}^{n+1}, \phi_i^n$. Thus, to solve the above equation (5),we don't need an estimated value for ϕ_{i-1}^n. However, interpreting the solution in this manner can help us to deepen our understanding of the behavior of the solutions. Also, in later sections, it will be a great help for us to develop implicit high-resolution schemes.