0.1 Implicit monotone schemes in 1D

In this section we discuss some stability properties of implicit schemes.

First we show that schemes written in a particular form satisfy the upwind range condition, meaning that the solution is always bounded by its neighbors in space-time as it travels along the characteristics.

Next we show that such schemes are also total variation non-increasing (TVNI), or total variation diminishing (TVD), as it is commonly referred to.

We conclude the section with the observation that if a scheme satisfies the implicit upwind range condition, then it also satisfies the TVD condition.

Both conditions can be used to construct implicit high-order monotone schemes. However, as we will see, the upwind range condition is much more straightforward to apply, only requiring the solution to be bounded between its neighbors. It can be much more cumbersome to require the TVD property of a scheme directly.

0.1.1 Upwind range condition for implicit schemes

Let us consider the implicit upwind scheme in 1 space dimension

$$\phi_i^n = \phi_i^{n-1} - c\left(\phi_i^n - \phi_{i-1}^n\right). \tag{1}$$

We assume, that the only unknown value is ϕ_i^n. To see why that statement holds, notice that the solution of the first order advection equation requires an inflow boundary condition. Thus, in the first cell, the value ϕ_0^n is known from the inflow BC, so the only unknown value is ϕ_1^n. Then we compute the next value and so on, until reaching the n-th cell, where ϕ_{i-1}^n is known and the only unknown value is ϕ_i^n.

In this section we want to show that for Courant numbers $c > 0$, schemes written in the form of (1) satisfy the stability condition

$$\min\left(\phi_{i-1}^n, \phi_i^{n-1}\right) \le \phi_i^n \le \max\left(\phi_{i-1}^n, \phi_i^{n-1}\right), \tag{2}$$

which can be interpreted as the upwind range condition[6] or data compatibility condition [10] for implicit schemes. Notice that min and max is a combination of values from both time steps $n, n-1$. This distinguishes the implicit upwind range property from the explicit one. In the explicit case, the min and max are chosen only from the values at the current time step $n-1$. The idea is that the solution travels along characteristics and this value is always bounded by its neighbors in space-time.

To prove (2), first we solve (1) for the unknown ϕ_i^n.

$$\phi_i^n = \frac{1}{1+c}\phi_i^{n-1} + \frac{c}{1+c}\phi_{i-1}^n. \tag{3}$$

Notice that

$$\frac{1}{1+c} = \frac{1+c-c}{1+c} = 1 - \frac{c}{1+c}.$$

For convenience, let us denote

$$k = \frac{c}{1+c}. \tag{4}$$

Since $0 \leq c \leq 1+c$, then

$$\begin{aligned} &0 \leq c \leq 1+c \rightarrow \\ &0 \leq \frac{c}{1+c} \leq 1 \rightarrow \\ &0 \leq k \leq 1. \end{aligned} \tag{5}$$

Thus, we can rewrite (3) as

$$\phi_i^n = (1-k)\phi_i^{n-1} + k\phi_{i-1}^n, \tag{6}$$

where we can see that the solution ϕ_i^n is a linear interpolation between ϕ_i^{n-1} and ϕ_{i-1}^n, thus (2) holds.

Theorem 1. *Let us consider a numerical scheme written in the form*

$$\phi_i^n = (1-k)\phi_i^{n-1} + k\phi_{i-1}^n. \tag{7}$$

Assume that ϕ_{i-1}^n is known. If the coefficient is bounded $0 \leq k \leq 1$, then the solution satisfies

$$\min\left(\phi_{i-1}^n, \phi_i^{n-1}\right) \leq \phi_i^n \leq \max\left(\phi_{i-1}^n, \phi_i^{n-1}\right), \tag{8}$$

Proof. Without the loss of generality, let us begin with the case where the data satisfies

$$\phi_{i-1}^n \leq \phi_i^{n-1}. \tag{9}$$

Then we can write

$$\begin{aligned} \min\left(\phi_{i-1}^n, \phi_i^{n-1}\right) &= \phi_{i-1}^n \\ \max\left(\phi_{i-1}^n, \phi_i^{n-1}\right) &= \phi_i^{n-1}. \end{aligned} \tag{10}$$

Thus, in the first case we want to show that

$$\phi_{i-1}^n \le \phi_i^n \le \phi_i^{n-1}. \tag{11}$$

We start by proving that $\phi_i^n \le \phi_i^{n-1}$.

Using (9) and the fact that $0 \le k \le 1$

$$\phi_{i-1}^n \le \phi_i^{n-1},$$
$$k\phi_{i-1}^n \le k\phi_i^{n-1},$$

where $0 \le k \le 1$. Then we use (7) to express $k\phi_{i-1}^n$

$$\phi_i^n - (1-k)\phi_i^{n-1} \le k\phi_i^{n-1},$$
$$\phi_i^n - \phi_i^{n-1} \le 0,$$
$$\phi_i^n \le \phi_i^{n-1}.$$

It remains to show that $\phi_{i-1}^n \le \phi_i^n$.

$$\phi_{i-1}^n \le \phi_i^{n-1},$$
$$(1-k)\phi_{i-1}^n \le (1-k)\phi_i^{n-1}.$$

Again, using (7) to express $(1-k)\phi_i^{n-1}$

$$(1-k)\phi_{i-1}^n \le \phi_i^n - k\phi_{i-1}^n,$$
$$\phi_{i-1}^n \le \phi_i^n.$$

Thus,

$$\phi_{i-1}^n \le \phi_i^n \le \phi_i^{n-1}.$$

The proof for the other case where the data is such that $\phi_{i-1}^n \ge \phi_i^{n-1}$ is straightforward.

□

0.1.2 TVD condition

The total variation of a mesh function $\phi^n = \phi_i^n$ is defined as

$$TV(\phi^n) = \sum_{i=-\infty}^{\infty} |\phi_i^n - \phi_{i-1}^n|, \tag{12}$$

see, e.g., [11, 1, 9, 12, 2, 8, 5, 6, 4, 7, 3].

Theorem 2. *The solution of a numerical scheme written in the form*

$$\phi_i^n = \phi_i^{n-1} - c\left(\phi_i^n - \phi_{i-1}^n\right), \tag{13}$$

for $c > 0$ and data with compact support, such that $\phi_i^{n-1} \to 0$ as $i \to \infty$ or $i \to -\infty$, is total variation non-increasing

$$TV(\phi_i^{n-1}) \geq TV(\phi_i^n). \tag{14}$$

Proof. We start by writing the scheme for cells $i, i-1$

$$\phi_i^n = \phi_i^{n-1} - c\left(\phi_i^n - \phi_{i-1}^n\right), \tag{15}$$

$$\phi_{i-1}^n = \phi_{i-1}^{n-1} - c\left(\phi_{i-1}^n - \phi_{i-2}^n\right). \tag{16}$$

Then we subtract (16) from (15) to get

$$\begin{aligned}
\phi_i^n - \phi_{i-1}^n &= \phi_i^{n-1} - \phi_{i-1}^{n-1} - c\left(\phi_i^n - \phi_{i-1}^n - \phi_{i-1}^n + \phi_{i-2}^n\right), \\
\phi_i^n - \phi_{i-1}^n + c\left(\phi_i^n - \phi_{i-1}^n - \phi_{i-1}^n + \phi_{i-2}^n\right) &= \phi_i^{n-1} - \phi_{i-1}^{n-1}, \\
(1+c)\left(\phi_i^n - \phi_{i-1}^n\right) - c\left(\phi_{i-1}^n - \phi_{i-2}^n\right) &= \phi_i^{n-1} - \phi_{i-1}^{n-1}.
\end{aligned} \tag{17}$$

We take the absolute value

$$|(1+c)\left(\phi_i^n - \phi_{i-1}^n\right) - c\left(\phi_{i-1}^n - \phi_{i-2}^n\right)| = |\phi_i^{n-1} - \phi_{i-1}^{n-1}|. \tag{18}$$

Then we use the triangle inequality $|a-b| \geq |a| - |b|$ and $|c\, a| = |c||a|$

$$|(1+c)\left(\phi_i^n - \phi_{i-1}^n\right) - c\left(\phi_{i-1}^n - \phi_{i-2}^n\right)| \geq (1+c)|\phi_i^n - \phi_{i-1}^n| - c|\phi_{i-1}^n - \phi_{i-2}^n|, \tag{19}$$

Thus, using (18)

$$|\phi_i^{n-1} - \phi_{i-1}^{n-1}| \geq (1+c)|\phi_i^n - \phi_{i-1}^n| - c|\phi_{i-1}^n - \phi_{i-2}^n|. \tag{20}$$

To complete the proof, we take the sum of both sides

$$\begin{aligned}
\sum_{i=-\infty}^{\infty} |\phi_i^{n-1} - \phi_{i-1}^{n-1}| &\geq \sum_{i=-\infty}^{\infty} (1+c)|\phi_i^n - \phi_{i-1}^n| - \sum_{i=-\infty}^{\infty} c|\phi_{i-1}^n - \phi_{i-2}^n|, \\
\sum_{i=-\infty}^{\infty} |\phi_i^{n-1} - \phi_{i-1}^{n-1}| &\geq (1+c)\sum_{i=-\infty}^{\infty} |\phi_i^n - \phi_{i-1}^n| - c\sum_{i=-\infty}^{\infty} |\phi_{i-1}^n - \phi_{i-2}^n|.
\end{aligned} \tag{21}$$

To complete the proof, we use the fact that the data has compact support and the solution ϕ^n satisfies the upwind range condition (2)

$$\begin{aligned}
\min\left(\phi_{i-1}^n, \phi_i^{n-1}\right) \leq \phi_i^n \leq \max\left(\phi_{i-1}^n, \phi_i^{n-1}\right) \text{ for all } i \in (-\infty, \infty), \\
\phi_{-\infty,\infty}^{n-1} = 0, \to \phi_{-\infty,\infty}^n = 0 \to \sum_{i=-\infty}^{\infty} |\phi_{i-1}^n - \phi_{i-2}^n| = \sum_{i=-\infty}^{\infty} |\phi_i^n - \phi_{i-1}^n| = TV(\phi^n).
\end{aligned} \tag{22}$$

Thus, using (21) we conclude

$$\begin{aligned} TV(\phi^{n-1}) &\geq (1+c)TV(\phi^{n}) - c\,TV(\phi^{n}), \\ TV(\phi^{n-1}) &\geq TV(\phi^{n}). \end{aligned} \tag{23}$$

□

0.1.3 Conclusion

Since we can transform any scheme written in the form of (1) to a scheme written as (6) and vice versa, then it follows that if a scheme satisfies the implicit upwind range property (2) it is also TVD.

0.1.4 Interpreting implicit schemes as a reconstruction in the current time step

Explicit finite volume schemes can be interpreted as a reconstruct-evolve-average (REA) algorithm [8, 7, 11]:

1. Reconstruct a piecewise polynomial function $\tilde{\phi}_i^n(x, t_n)$ for all $x \in \mathcal{C}_i$ from the known cell averages
2. Evolve the function to obtain $\tilde{\phi}_i^n(x, t_{n+1})$
3. Compute the average of this evolved function over each cell to obtain the new cell averages
$$\phi_i^{n+1} = \frac{1}{\Delta x}\int_{p_i} \tilde{\phi}_i^n(x, t_{n+1})\,\mathrm{d}x\,.$$

Meaning, we reconstruct the values in a given cell using the cell averages. For example, the explicit upwind scheme can be interpreted as a piecewise constant reconstruction in the current time step. Then, we evolve this reconstruction and compute the cell averages in the new time step. We repeat this process until we reach the desired time.

This interpretation can help construct high-order schemes, where instead of constant reconstruction we use linear, parabolic, etc., depending on the desired accuracy. Also, if we make sure that our reconstruction does not generate new extrema, then evolving and averaging operations will neither, meaning that our numerical solution will be oscillation free.

In this section we would like to show that implicit schemes can also be interpreted similarly by considering that the values travel along characteristics. Thus, the values in the new time step can be interpreted as values in the current time step.

In this section we restrict our discussion to the simplest problem of advection with positive constant speed $v > 0$.

We start with the most straightforward piecewise constant reconstruction:

1. Reconstruct a piecewise constant function $\tilde{\phi}_i^n(x, t_n)$ for all $x \in \mathcal{C}_i$ from the known cell averages ϕ_i^n. Thus,
$$\tilde{\phi}_i^n(x, t_n) = \phi_i^n \text{ for all } x \in \mathcal{C}_i.$$

2. Evolve the function using the exact solution of the advection equation
$$\tilde{\phi}_i^n(x, t_{n+1}) = \tilde{\phi}_i^n(x - v\Delta t, t_n)$$

3. Compute the average of this evolved function over each cell to obtain the new cell averages
$$\phi_i^{n+1} = \frac{1}{\Delta x} \int_{\mathcal{C}_i} \tilde{\phi}_i^n(x, t_{n+1}) \, \mathrm{d}x \,,$$
or, equivalently
$$\phi_i^{n+1} = \frac{1}{\Delta x} \int_{\mathcal{C}_i - v\Delta t} \tilde{\phi}_i^n(x, t_n) \, \mathrm{d}x \,.$$

If the advection with a speed v over a time step Δt is smaller than a cell width Δx, thus the time step is restricted by

$$\begin{aligned} 0 &\le v\Delta t \le \Delta x \\ 0 &\le \frac{v\Delta t}{\Delta x} \le 1, \\ 0 &\le c \le 1, \end{aligned} \tag{24}$$

where c is the Courant number, the algorithm yields the first order explicit upwind scheme. The new cell averages are computed as

$$\begin{aligned} \phi_i^{n+1} &= \frac{1}{\Delta x} \left(v\Delta t \; \phi_{i-1}^n + (\Delta x - v\Delta t) \, \phi_i^n \right) \\ &= \frac{v\Delta t}{\Delta x} \phi_{i-1}^n + \left(1 - \frac{v\Delta t}{\Delta x} \right) \phi_i^n \\ &= c \; \phi_{i-1}^n + (1 - c) \, \phi_i^n. \end{aligned} \tag{25}$$

We can see, that if the CFL condition (24) holds, the new cell average is a linear interpolation between ϕ_i^{n-1} and ϕ_i^n.

Let us compare this with the implicit upwind scheme. We assume that in a cell the values ϕ_i^n and ϕ_{i-1}^{n+1} are known, and we want to compute the new cell average ϕ_i^{n+1}.

Assume constant reconstruction in the cell $\mathcal{C}_i$ at the current time step t_n as before

$$\tilde{\phi}_i^n(x, t_n) = \phi_i^n \text{ for } x \in \mathcal{C}_i,$$

and a constant reconstruction in the cell $\mathcal{C}_{i-1}$ at the new time step t_{n+1} as

$$\tilde{\phi}_{i-1}^{n+1}(x, t_{n+1}) = \phi_{i-1}^{n+1} \text{ for } x \in \mathcal{C}_{i-1}.$$

Using the exact solution of the advection equation, we can project the reconstruction at time t_{n+1} to the current time t_n, by evolving it back in time as

$$\tilde{\phi}_{i-1}^{n+1}(x, t_n) = \tilde{\phi}_{i-1}^{n+1}(x + v\Delta t, t_{n+1}) \text{ for } x \in \mathcal{C}_{i-1} - v\Delta t.$$

In the explicit case, we computed the new average using the values ϕ_{i-1}^n, ϕ_i^n. But now, instead of ϕ_{i-1}^n we have ϕ_{i-1}^{n+1}. What we can make is to compute an estimated value $\phi_{i-1}^{n,*}$ to compute the new cell average as in (25)

$$\phi_i^{n+1} = c\, \phi_{i-1}^{n,*} + (1-c)\, \phi_i^n. \tag{26}$$

We can express the cell average $\phi_{i-1}^{n,*}$ using $\phi_{i-1}^{n+1}, \phi_i^{n+1}$ by averaging in cell $\mathcal{C}_{i-1}$

$$\phi_{i-1}^{n,*} = (1-c)\, \phi_{i-1}^{n+1} + c\, \phi_i^{n+1}. \tag{27}$$

Thus, we can solve the 2 equations above for 2 unknowns $\phi_{i-1}^{n,*}, \phi_i^{n+1}$. This yields

$$\begin{aligned}
\phi_{i-1}^{n,*} &= \left(1 - \frac{c}{1+c}\right) \phi_{i-1}^{n+1} + \frac{c}{1+c} \phi_i^n, \\
\phi_i^{n+1} &= \frac{c}{1+c} \phi_{i-1}^{n+1} + \left(1 - \frac{c}{1+c}\right) \phi_i^n,
\end{aligned}$$

yielding the same result for the new cell average as solving the equation directly. Also notice that both values are linear interpolations between the known values $\phi_{i-1}^{n+1}, \phi_i^n$.