$$\phi_{p,vnbd}^{min} \le \phi_p^{n-1} - \lambda_p \sum_{f\in\mathcal{F}_p^-} \phi_{pf}^{n-1/2} a_{pf} - \lambda_p \sum_{f\in\mathcal{F}_p^+} \phi_{pf}^{n-1/2} a_{pf} \le \phi_{p,vnbd}^{max}$$

For $a_{ph} \ge 0$

$$\phi_{p,vnbd}^{min} \le \phi_p^{n-1} - \lambda_p \sum_{f\in\mathcal{F}_p^-} \phi_{pf}^{n-1/2} a_{pf} - \lambda_p \sum_{\substack{f\in\mathcal{F}_p^+,\\ f\neq h}} \phi_{pf}^{n-1/2} a_{pf} - \lambda_p \phi_{ph}^{n-1/2} a_{ph} \le \phi_{p,vnbd}^{max}$$

$$\phi_{p,vnbd}^{min} - \phi_p^{n-1} + \lambda_p \sum_{f\in\mathcal{F}_p^-} \phi_{pf}^{n-1/2} a_{pf} + \lambda_p \sum_{\substack{f\in\mathcal{F}_p^+,\\ f\neq h}} \phi_{pf}^{n-1/2} a_{pf} \le -\lambda_p \phi_{ph}^{n-1/2} a_{ph}$$

$$\phi_{p,vnbd}^{max} - \phi_p^{n-1} + \lambda_p \sum_{f\in\mathcal{F}_p^-} \phi_{pf}^{n-1/2} a_{pf} + \lambda_p \sum_{\substack{f\in\mathcal{F}_p^+,\\ f\neq h}} \phi_{pf}^{n-1/2} a_{pf} \ge -\lambda_p \phi_{ph}^{n-1/2} a_{ph}$$

$$\frac{\phi_p^{n-1} - \phi_{p,vnbd}^{min}}{\lambda_p} - \sum_{f\in\mathcal{F}_p^-} \phi_{pf}^{n-1/2} a_{pf} - \sum_{\substack{f\in\mathcal{F}_p^+,\\ f\neq h}} \phi_{pf}^{n-1/2} a_{pf} \ge \phi_{ph}^{n-1/2} a_{ph}$$

$$\frac{\phi_p^{n-1} - \phi_{p,vnbd}^{max}}{\lambda_p} - \sum_{f\in\mathcal{F}_p^-} \phi_{pf}^{n-1/2} a_{pf} - \sum_{\substack{f\in\mathcal{F}_p^+,\\ f\neq h}} \phi_{pf}^{n-1/2} a_{pf} \le \phi_{ph}^{n-1/2} a_{ph}$$

0.1 Version 1

Assume

$$0 \le \phi_{p,vnbd}^{min} \le \phi_{pf}^{n-1/2} \le \phi_{p,vnbd}^{max} \le 1.$$

Then

$$\phi_{p,vnbd}^{min} \sum_{f\in\mathcal{F}_p^+} a_{pf} \le \sum_{f\in\mathcal{F}_p^+} \phi_{pf}^{n-1/2} a_{pf} \le \phi_{p,vnbd}^{max} \sum_{f\in\mathcal{F}_p^+} a_{pf},$$

$$\phi_{p,vnbd}^{min} \sum_{\substack{f\in\mathcal{F}_p^+,\\ f\neq h}} a_{pf} \le \sum_{\substack{f\in\mathcal{F}_p^+,\\ f\neq h}} \phi_{pf}^{n-1/2} a_{pf} \le \phi_{p,vnbd}^{max} \sum_{\substack{f\in\mathcal{F}_p^+,\\ f\neq h}} a_{pf}$$

and

$$\phi_{p,vnbd}^{min} \sum_{f\in\mathcal{F}_p^-} a_{pf} \ge \sum_{f\in\mathcal{F}_p^-} \phi_{pf}^{n-1/2} a_{pf} \ge \phi_{p,vnbd}^{max} \sum_{f\in\mathcal{F}_p^-} a_{pf}.$$

Thus,

$$\frac{\phi_p^{n-1} - \phi_{p,vnbd}^{min}}{\lambda_p} - \sum_{f\in\mathcal{F}_p^-} \phi_{pf}^{n-1/2} a_{pf} - \sum_{\substack{f\in\mathcal{F}_p^+,\\ f\neq h}} \phi_{pf}^{n-1/2} a_{pf} \ge$$

$$\ge \frac{\phi_p^{n-1} - \phi_{p,vnbd}^{min}}{\lambda_p} - \phi_{p,vnbd}^{min} \sum_{f\in\mathcal{F}_p^-} a_{pf} - \phi_{p,vnbd}^{max} \sum_{\substack{f\in\mathcal{F}_p^+,\\ f\neq h}} a_{pf} \ge \phi_{ph}^{n-1/2} a_{ph}.$$

Similarly,

$$\frac{\phi_p^{n-1} - \phi_{p,vnbd}^{max}}{\lambda_p} - \sum_{f \in \mathcal{F}_p^-} \phi_{pf}^{n-1/2} a_{pf} - \sum_{\substack{f \in \mathcal{F}_p^+,\\ f \neq h}} \phi_{pf}^{n-1/2} a_{pf} \leq$$

$$\leq \frac{\phi_p^{n-1} - \phi_{p,vnbd}^{max}}{\lambda_p} - \phi_{p,vnbd}^{max} \sum_{f \in \mathcal{F}_p^-} a_{pf} - \phi_{p,vnbd}^{min} \sum_{\substack{f \in \mathcal{F}_p^+,\\ f \neq h}} a_{pf} \leq \phi_{ph}^{n-1/2} a_{ph}.$$

$$\frac{\phi_p^{n-1} - \phi_{p,vnbd}^{min}}{\lambda_p} - \phi_{p,vnbd}^{min} \sum_{f \in \mathcal{F}_p^-} a_{pf} - \phi_{p,vnbd}^{max} \sum_{\substack{f \in \mathcal{F}_p^+,\\ f \neq h}} a_{pf} \geq \frac{\phi_p^{n-1} - \phi_{p,vnbd}^{max}}{\lambda_p} - \phi_{p,vnbd}^{max} \sum_{f \in \mathcal{F}_p^-} a_{pf} - \phi_{p,vnbd}^{min} \sum_{\substack{f \in \mathcal{F}_p^+,\\ f \neq h}} a_{pf}$$

$$0 \leq \frac{\phi_{p,vnbd}^{max} - \phi_{p,vnbd}^{min}}{\lambda_p} + \left(\phi_{p,vnbd}^{max} - \phi_{p,vnbd}^{max}\right) \sum_{f \in \mathcal{F}_p^-} a_{pf} - \left(\phi_{p,vnbd}^{max} - \phi_{p,vnbd}^{max}\right) \sum_{\substack{f \in \mathcal{F}_p^+,\\ f \neq h}} a_{pf}$$

0.2 Version 2

Let $\min(\phi_p^{n-1}, \phi_q^{n-1}) = min_{pq}^{n-1}$ and $\max(\phi_p^{n-1}, \phi_q^{n-1}) = max_{pq}^{n-1}$. Assume

$$0 \leq min_{pq}^{n-1} \leq \phi_{pf}^{n-1/2} \leq max_{pq}^{n-1} \leq 1.$$

Then

$$\sum_{f \in \mathcal{F}_p^+} min_{pq}^{n-1} a_{pf} \leq \sum_{f \in \mathcal{F}_p^+} \phi_{pf}^{n-1/2} a_{pf} \leq \sum_{f \in \mathcal{F}_p^+} max_{pq}^{n-1} a_{pf},$$

and

$$\sum_{f \in \mathcal{F}_p^-} max_{pq}^{n-1} a_{pf} \leq \sum_{f \in \mathcal{F}_p^-} \phi_{pf}^{n-1/2} a_{pf} \leq \sum_{f \in \mathcal{F}_p^-} min_{pq}^{n-1} a_{pf}.$$

$$\frac{\phi_p^{n-1} - \phi_{p,vnbd}^{min}}{\lambda_p} - \sum_{f \in \mathcal{F}_p^-} \phi_{pf}^{n-1/2} a_{pf} - \sum_{\substack{f \in \mathcal{F}_p^+,\\ f \neq h}} \phi_{pf}^{n-1/2} a_{pf} \geq$$

$$\geq \frac{\phi_p^{n-1} - \phi_{p,vnbd}^{min}}{\lambda_p} - \sum_{f \in \mathcal{F}_p^-} min_{pq}^{n-1} a_{pf} - \sum_{\substack{f \in \mathcal{F}_p^+,\\ f \neq h}} max_{pq}^{n-1} a_{pf} \geq \phi_{ph}^{n-1/2} a_{ph}.$$

Similarly,

$$\frac{\phi_p^{n-1} - \phi_{p,vnbd}^{max}}{\lambda_p} - \sum_{f \in \mathcal{F}_p^-} \phi_{pf}^{n-1/2} a_{pf} - \sum_{\substack{f \in \mathcal{F}_p^+,\\ f \neq h}} \phi_{pf}^{n-1/2} a_{pf} \leq$$

$$\leq \frac{\phi_p^{n-1} - \phi_{p,vnbd}^{max}}{\lambda_p} - \sum_{f \in \mathcal{F}_p^-} max_{pq}^{n-1} a_{pf} - \sum_{\substack{f \in \mathcal{F}_p^+,\\ f \neq h}} min_{pq}^{n-1} a_{pf} \leq \phi_{ph}^{n-1/2} a_{ph}.$$

0.3 Version 3

$$\begin{aligned}\frac{\phi_p^{n-1}-\phi_{p,vnbd}^{min}}{\lambda_p} &- \sum_{f\in\mathcal{F}_p^-}\left(\phi_q^{n,k-1}+\Psi_f^{k-1}\mathcal{D}_{pf}^{-,k-1}\phi\right)a_{pf}\\ &- \sum_{\substack{f\in\mathcal{F}_p^+,\\ f\neq h}}\left(\phi_p^{n,k-1}+\Psi_f^{k-1}\mathcal{D}_{pf}^{+,k-1}\phi\right)a_{pf} \geq \left(\phi_p^{n,k-1}+\Psi_h^{k}\mathcal{D}_{ph}^{+,k-1}\phi\right)a_{ph},\\ \frac{\phi_p^{n-1}-\phi_{p,vnbd}^{max}}{\lambda_p} &- \sum_{f\in\mathcal{F}_p^-}\left(\phi_q^{n,k-1}+\Psi_f^{k-1}\mathcal{D}_{pf}^{-,k-1}\phi\right)a_{pf}\\ &- \sum_{\substack{f\in\mathcal{F}_p^+,\\ f\neq h}}\left(\phi_p^{n,k-1}+\Psi_f^{k-1}\mathcal{D}_{pf}^{+,k-1}\phi\right)a_{pf} \leq \left(\phi_p^{n,k-1}+\Psi_h^{k}\mathcal{D}_{ph}^{+,k-1}\phi\right)a_{ph}.\end{aligned}$$

We want to maximize Ψ at each face.

$$\sum_{\substack{f\in\mathcal{F}_p^+,\\ f\neq h}}\left(\phi_p^{n,k-1}+\Psi_f^{k-1}\mathcal{D}_{pf}^{+,k-1}\phi\right)a_{pf} = \sum_{f\in\mathcal{F}_p^+}\left(\phi_p^{n,k-1}+\Psi_f^{k-1}\mathcal{D}_{pf}^{+,k-1}\phi\right)a_{pf} - \left(\phi_p^{n,k-1}+\Psi_h^{k-1}\mathcal{D}_{ph}^{+,k-1}\phi\right)a_{ph}.$$

Substituting we get

$$\begin{aligned}\frac{\phi_p^{n-1}-\phi_{p,vnbd}^{min}}{\lambda_p} &- \sum_{f\in\mathcal{F}_p^-}\left(\phi_q^{n,k-1}+\Psi_f^{k-1}\mathcal{D}_{pf}^{-,k-1}\phi\right)a_{pf}\\ &- \sum_{f\in\mathcal{F}_p^+}\left(\phi_p^{n,k-1}+\Psi_f^{k-1}\mathcal{D}_{pf}^{+,k-1}\phi\right)a_{pf}\\ &+ \left(\phi_p^{n,k-1}+\Psi_h^{k-1}\mathcal{D}_{ph}^{+,k-1}\phi\right)a_{ph} \geq \left(\phi_p^{n,k-1}+\Psi_h^{k}\mathcal{D}_{ph}^{+,k-1}\phi\right)a_{ph},\end{aligned}$$

$$\begin{aligned}\frac{\phi_p^{n-1}-\phi_{p,vnbd}^{min}}{\lambda_p} &- \sum_{f\in\mathcal{F}_p^-}\left(\phi_q^{n,k-1}+\Psi_f^{k-1}\mathcal{D}_{pf}^{-,k-1}\phi\right)a_{pf}\\ &- \sum_{f\in\mathcal{F}_p^+}\left(\phi_p^{n,k-1}+\Psi_f^{k-1}\mathcal{D}_{pf}^{+,k-1}\phi\right)a_{pf}\\ &+ \Psi_h^{k-1}\mathcal{D}_{ph}^{+,k-1}\phi\ a_{ph} \geq \Psi_h^{k}\mathcal{D}_{ph}^{+,k-1}\phi\ a_{ph},\end{aligned}$$

$$\begin{aligned}\Psi_h^k = \max\Bigg(\frac{\phi_p^{n-1}-\phi_{p,vnbd}^{min}}{\lambda_p\mathcal{D}_{ph}^{+,k-1}\phi\ a_{ph}} &- \frac{\sum_{f\in\mathcal{F}_p^-}\left(\phi_q^{n,k-1}+\Psi_f^{k-1}\mathcal{D}_{pf}^{-,k-1}\phi\right)a_{pf}}{\mathcal{D}_{ph}^{+,k-1}\phi\ a_{ph}}\\ &- \frac{\sum_{f\in\mathcal{F}_p^+}\left(\phi_p^{n,k-1}+\Psi_f^{k-1}\mathcal{D}_{pf}^{+,k-1}\phi\right)a_{pf}}{\mathcal{D}_{ph}^{+,k-1}\phi\ a_{ph}} + \Psi_h^{k-1},\Bigg)\end{aligned}$$

0.4 Version 4

First we make an estimate for the new cell value, using the maximal possible value for the limiter at outflow faces.

$$\phi^n_{est} = \phi^{n-1}_p - \lambda_p \sum_{f\in\mathcal{F}^-_p} \left(\phi^{n,k-1}_q + \Psi^{k-1}_f \mathcal{D}^{-,k-1}_{pf}\phi\right) a_{pf} - \lambda_p \sum_{f\in\mathcal{F}^+_p} \left(\phi^{n,k-1}_p + \Psi_{max} \mathcal{D}^{+,k-1}_{pf}\phi\right) a_{pf}$$

Then we compute a bounded value

$$\phi^n_{bounded} = \min\left(\phi^{max}_{p,vnbd}, \max\left(\phi^{min}_{p,vnbd}, \phi^n_{est}\right)\right) \tag{1}$$

By assuming

$$\phi^n_{bounded} = \phi^{n-1}_p - \lambda_p \sum_{f\in\mathcal{F}^-_p} \left(\phi^{n,k-1}_q + \Psi^{k-1}_f \mathcal{D}^{-,k-1}_{pf}\phi\right) a_{pf} - \lambda_p \sum_{f\in\mathcal{F}^+_p} \left(\phi^{n,k-1}_p + \Psi^{k}_f \mathcal{D}^{+,k-1}_{pf}\phi\right) a_{pf} \tag{2}$$

we can compute the sum

$$\begin{aligned} S_p &= \sum_{f\in\mathcal{F}^+_p} \Psi^k_f \mathcal{D}^{+,k-1}_{pf}\phi \; a_{pf} \\ &= \frac{\phi^{n-1}_p - \phi^n_{bounded}}{\lambda_p} - \sum_{f\in\mathcal{F}^-_p} \left(\phi^{n,k-1}_q + \Psi^{k-1}_f \mathcal{D}^{-,k-1}_{pf}\phi\right) a_{pf} - \sum_{f\in\mathcal{F}^+_p} \phi^{n,k-1}_p a_{pf}. \end{aligned} \tag{3}$$

We want to be as close to the second order scheme as possible. Thus, we want to minimize

$$\left(\sum_{f\in\mathcal{F}^+_p} \Psi^k_f \mathcal{D}^{+,k-1}_{pf}\phi \; a_{pf} - \Psi_{max} \sum_{f\in\mathcal{F}^+_p} \mathcal{D}^{+,k-1}_{pf}\phi \; a_{pf}\right)^2 = \left(\sum_{f\in\mathcal{F}^+_p} \left(\Psi^k_f - \Psi_{max}\right) \mathcal{D}^{+,k-1}_{pf}\phi \; a_{pf}\right)^2.$$

This, however, is not good, because we already fixed the sum to S_p!!!

Be careful, that

$$\left(\sum_{f\in\mathcal{F}^+_p} \left(\Psi^k_f - \Psi_{max}\right) \mathcal{D}^{+,k-1}_{pf}\phi \; a_{pf}\right)^2 \neq \sum_{f\in\mathcal{F}^+_p} \left(\Psi^k_f - \Psi_{max}\right)^2 \left(\mathcal{D}^{+,k-1}_{pf}\phi \; a_{pf}\right)^2$$

!!!!!!

Minimizing

$$\sum_{f\in\mathcal{F}^+_p} \left(\Psi^k_f - \Psi_{max}\right)^2 \left(\mathcal{D}^{+,k-1}_{pf}\phi \; a_{pf}\right)^2 \tag{4}$$

seems reasonable.

$$\mathcal{L}(..., \Psi_f^k, ..., \lambda) = \sum_{f\in\mathcal{F}_p^+} \left(\Psi_f^k - \Psi_{max}\right)^2 \left(\mathcal{D}_{pf}^{+,k-1}\phi\ a_{pf}\right)^2 \\ -\lambda\left(\sum_{f\in\mathcal{F}_p^+} \Psi_f^k \mathcal{D}_{pf}^{+,k-1}\phi\ a_{pf} - S_p\right) \tag{5}$$

$$\frac{\partial\mathcal{L}}{\partial\Psi_f^k} = 0$$

$$2\left(\Psi_f^k - \Psi_{max}\right)\left(\mathcal{D}_{pf}^{+,k-1}\phi\ a_{pf}\right)^2 - \lambda\mathcal{D}_{pf}^{+,k-1}\phi\ a_{pf} = 0$$

$$\Psi_f^k = \Psi_{max} + \frac{\lambda}{2\mathcal{D}_{pf}^{+,k-1}\phi\ a_{pf}}$$

We know that

$$\begin{aligned} S_p &= \sum_{f\in\mathcal{F}_p^+} \Psi_f^k \mathcal{D}_{pf}^{+,k-1}\phi\ a_{pf} \\ &= \sum_{f\in\mathcal{F}_p^+} \left(\Psi_{max} + \frac{\lambda}{2\mathcal{D}_{pf}^{+,k-1}\phi\ a_{pf}}\right)\mathcal{D}_{pf}^{+,k-1}\phi\ a_{pf} \\ &= \sum_{f\in\mathcal{F}_p^+} \Psi_{max}\mathcal{D}_{pf}^{+,k-1}\phi\ a_{pf} + \frac{\lambda}{2}n_{pos}, \end{aligned} \tag{6}$$

where $n_{pos} = |\mathcal{F}_p^+|$. We can compute λ

$$\lambda = \frac{2}{n_{pos}}\left(S_p - \Psi_{max}\sum_{f\in\mathcal{F}_p^+}\mathcal{D}_{pf}^{+,k-1}\phi\ a_{pf}\right) \tag{7}$$

$$\Psi_f^k = \max\left(\min\left(\Psi_{max} + \frac{\lambda}{2\mathcal{D}_{pf}^{+,k-1}\phi\ a_{pf}}, \Psi_{max}\right), 0\right) \tag{8}$$

Another choice would be to minimize

$$\sum_{f\in\mathcal{F}_p^+}\left(\Psi_f^k - \Psi_{max}\right)^2$$

$$\frac{\partial\mathcal{L}}{\partial\Psi_f^k} = 0$$

$$2\left(\Psi_f^k - \Psi_{max}\right) - \lambda\mathcal{D}_{pf}^{+,k-1}\phi\ a_{pf} = 0$$

$$\Psi_f^k = \Psi_{max} + \frac{\lambda}{2}\mathcal{D}_{pf}^{+,k-1}\phi\ a_{pf}$$

We know that

$$
\begin{aligned}
S_p &= \sum_{f\in\mathcal{F}_p^+} \Psi_f^k \mathcal{D}_{pf}^{+,k-1}\phi\ a_{pf} \\
&= \sum_{f\in\mathcal{F}_p^+} \left(\Psi_{max} + \frac{\lambda}{2}\mathcal{D}_{pf}^{+,k-1}\phi\ a_{pf}\right)\mathcal{D}_{pf}^{+,k-1}\phi\ a_{pf} \\
&= \sum_{f\in\mathcal{F}_p^+} \Psi_{max}\mathcal{D}_{pf}^{+,k-1}\phi\ a_{pf} + \frac{\lambda}{2}\sum_{f\in\mathcal{F}_p^+}\left(\mathcal{D}_{pf}^{+,k-1}\phi\ a_{pf}\right)^2,
\end{aligned} \tag{9}
$$

$$
\lambda = \frac{2}{\sum_{f\in\mathcal{F}_p^+}\left(\mathcal{D}_{pf}^{+,k-1}\phi\ a_{pf}\right)^2}\left(S_p - \Psi_{max}\sum_{f\in\mathcal{F}_p^+}\mathcal{D}_{pf}^{+,k-1}\phi\ a_{pf}\right) \tag{10}
$$

$$
\Psi_f^k = \max\left(\min\left(\Psi_{max} + \frac{\lambda}{2}\mathcal{D}_{pf}^{+,k-1}\phi\ a_{pf}, \Psi_{max}\right), 0\right) \tag{11}
$$

0.5 Version 5

$$
\phi_{p,vnbd}^{min}\sum_{f\in\mathcal{F}_p^-} a_{pf} \geq \sum_{f\in\mathcal{F}_p^-}\phi_{pf}^{n-1/2}a_{pf} \geq \phi_{p,vnbd}^{max}\sum_{f\in\mathcal{F}_p^-} a_{pf}.
$$

$$
\phi_{p,vnbd}^{min} \leq \phi_p^{n-1} - \lambda_p\sum_{f\in\mathcal{F}_p^-}\phi_{pf}^{n-1/2}a_{pf} - \lambda_p\sum_{f\in\mathcal{F}_p^+}\left(\phi_p^{n,k-1} + \Psi_f\mathcal{D}_{pf}^{+,k-1}\phi\right)a_{pf} \leq \phi_{p,vnbd}^{max}
$$

$$
\phi_{p,vnbd}^{min} \leq \phi_p^{n-1} - \lambda_p\phi_{p,vnbd}^{min}\sum_{f\in\mathcal{F}_p^-} a_{pf} - \lambda_p\sum_{f\in\mathcal{F}_p^+}\left(\phi_p^{n,k-1} + \Psi_f\mathcal{D}_{pf}^{+,k-1}\phi\right)a_{pf}
$$

$$
\phi_{p,vnbd}^{min}\left(1 + \lambda_p\sum_{f\in\mathcal{F}_p^-} a_{pf}\right) \leq \phi_p^{n-1} - \lambda_p\sum_{f\in\mathcal{F}_p^+}\left(\phi_p^{n,k-1} + \Psi_f^k\mathcal{D}_{pf}^{+,k-1}\phi\right)a_{pf}
$$

$$
\phi_{p,vnbd}^{min}\left(1 + \lambda_p\sum_{f\in\mathcal{F}_p^-} a_{pf}\right) \leq \phi_{p,vnbd}^{max}\left(1 + \lambda_p\sum_{f\in\mathcal{F}_p^-} a_{pf}\right)
$$

Only holds if

$$
1 + \lambda_p\sum_{f\in\mathcal{F}_p^-} a_{pf} > 0.
$$