$$
\begin{aligned}
\frac{|\Omega_p|}{\Delta t}(\phi_p^n - \phi_p^{n-1}) &+ \sum_{f\in\mathcal{F}_p^-} \left(\phi_q^n + \Psi_{pf}^-\mathcal{D}_{pf}^-\phi\right) a_{pf} \\
&+ \sum_{f\in\mathcal{F}_p^+} \left(\phi_p^n + \Psi_{pf}^+\mathcal{D}_{pf}^+\phi\right) a_{pf} \\
&+ \sum_{b\in\mathcal{B}_p^-} \left(\phi_e(\boldsymbol{x}_{pb}, t^n) + \Psi_{pb}^-\mathcal{D}_{pb}^-\phi\right) a_{pb} \\
&+ \sum_{b\in\mathcal{B}_p^+} \left(\phi_p^n + \Psi_{pb}^+\mathcal{D}_{pb}^+\phi\right) a_{pb} = 0.
\end{aligned} \tag{1}
$$

For an internal cell

$$
\begin{aligned}
\frac{|\Omega_p|}{\Delta t}(\phi_p^n - \phi_p^{n-1}) + \sum_{f\in\mathcal{F}_p^-} \phi_q^n a_{pf} + \sum_{f\in\mathcal{F}_p^+} \phi_p^n a_{pf} \\
+ \sum_{f\in\mathcal{F}_p^-} \Psi_f\mathcal{D}_{pf}^-\phi\ a_{pf} + \sum_{f\in\mathcal{F}_p^+} \Psi_f\mathcal{D}_{pf}^+\phi\ a_{pf} = 0.
\end{aligned} \tag{2}
$$

We split the antidiffusive fluxes to positive and negative values. Then we can write the sum of antidiffusive fluxes as

$$
\begin{aligned}
&\sum_{f\in\mathcal{F}_p^-} \Psi_f\mathcal{D}_{pf}^-\phi\ a_{pf} + \sum_{f\in\mathcal{F}_p^+} \Psi_f\mathcal{D}_{pf}^+\phi\ a_{pf} = \\
&\sum_{f\in\mathcal{F}_p^+} \Psi_f \max\left(\mathcal{D}_{pf}^+\phi\ a_{pf}, 0\right) + \sum_{f\in\mathcal{F}_p^+} \Psi_f \min\left(\mathcal{D}_{pf}^+\phi\ a_{pf}, 0\right) \\
+&\sum_{f\in\mathcal{F}_p^-} \Psi_f \max\left(\mathcal{D}_{pf}^-\phi\ a_{pf}, 0\right) + \sum_{f\in\mathcal{F}_p^-} \Psi_f \min\left(\mathcal{D}_{pf}^-\phi\ a_{pf}, 0\right).
\end{aligned} \tag{3}
$$

Since the limiter is positive $\Psi_f \geq 0$,, we can write

$$
\begin{aligned}
&\sum_{f\in\mathcal{F}_p^+} \Psi_f \max\left(\mathcal{D}_{pf}^+\phi\ a_{pf}, 0\right) + \sum_{f\in\mathcal{F}_p^-} \Psi_f \max\left(\mathcal{D}_{pf}^-\phi\ a_{pf}, 0\right) = \\
&\Psi_p^M \left(\sum_{f\in\mathcal{F}_p^+} \max\left(\mathcal{D}_{pf}^+\phi\ a_{pf}, 0\right) + \sum_{f\in\mathcal{F}_p^-} \max\left(\mathcal{D}_{pf}^-\phi\ a_{pf}, 0\right) \right),
\end{aligned} \tag{4}
$$

where $\Psi_p^M \geq 0$. Similarly

$$
\begin{aligned}
&\sum_{f\in\mathcal{F}_p^+} \Psi_f \min\left(\mathcal{D}_{pf}^+\phi\ a_{pf}, 0\right) + \sum_{f\in\mathcal{F}_p^-} \Psi_f \min\left(\mathcal{D}_{pf}^-\phi\ a_{pf}, 0\right) = \\
&\Psi_p^N \left(\sum_{f\in\mathcal{F}_p^+} \min\left(\mathcal{D}_{pf}^+\phi\ a_{pf}, 0\right) + \sum_{f\in\mathcal{F}_p^-} \min\left(\mathcal{D}_{pf}^-\phi\ a_{pf}, 0\right) \right),
\end{aligned} \tag{5}
$$

where $\Psi_p^N \geq 0$. We denote

$$M_p = \sum_{f\in\mathcal{F}_p^+} \max\left(\mathcal{D}_{pf}^+\phi\ a_{pf}, 0\right) + \sum_{f\in\mathcal{F}_p^-} \max\left(\mathcal{D}_{pf}^-\phi\ a_{pf}, 0\right),$$
$$N_p = \sum_{f\in\mathcal{F}_p^+} \min\left(\mathcal{D}_{pf}^+\phi\ a_{pf}, 0\right) + \sum_{f\in\mathcal{F}_p^-} \min\left(\mathcal{D}_{pf}^-\phi\ a_{pf}, 0\right) \tag{6}$$

we can equivalently write

$$\frac{|\Omega_p|}{\Delta t}(\phi_p^n - \phi_p^{n-1}) + \sum_{f\in\mathcal{F}_p^-} \phi_q^n a_{pf} + \sum_{f\in\mathcal{F}_p^+} \phi_p^n a_{pf} + \Psi_p^M M_p + \Psi_p^N N_p = 0. \tag{7}$$

After finding the single value Ψ_p^M, we should find a way how do we distribute it among the faces to get the most accurate solution.

$$\phi_p^n - \phi_p^{n-1} + \lambda_p \sum_{f\in\mathcal{F}_p^-} \phi_q^n a_{pf} + \lambda_p \sum_{f\in\mathcal{F}_p^+} \phi_p^n a_{pf} + \lambda_p \Psi_p^M M_p + \lambda_p \Psi_p^N N_p = 0. \tag{8}$$

We want

$$min \leq \phi_p^{n-1} - \lambda_p \sum_{f\in\mathcal{F}_p^-} \phi_q^n a_{pf} - \lambda_p \sum_{f\in\mathcal{F}_p^+} \phi_p^n a_{pf} - \lambda_p \Psi_p^M M_p - \lambda_p \Psi_p^N N_p \leq max$$

Since $N_p \leq \Psi_p^N N_p \leq 0$ we can write the sufficient conditions

$$\phi_p^{n-1} - \lambda_p \sum_{f\in\mathcal{F}_p^-} \phi_q^n a_{pf} - \lambda_p \sum_{f\in\mathcal{F}_p^+} \phi_p^n a_{pf} - \lambda_p \Psi_p^M M_p - \lambda_p N_p \leq max$$

$$min \leq \phi_p^{n-1} - \lambda_p \sum_{f\in\mathcal{F}_p^-} \phi_q^n a_{pf} - \lambda_p \sum_{f\in\mathcal{F}_p^+} \phi_p^n a_{pf} - \lambda_p \Psi_p^M M_p.$$

Thus

$$\phi_p^{n-1} - \lambda_p \sum_{f\in\mathcal{F}_p^-} \phi_q^n a_{pf} - \lambda_p \sum_{f\in\mathcal{F}_p^+} \phi_p^n a_{pf} - \lambda_p N_p - max \leq \lambda_p \Psi_p^M M_p$$

and

$$\lambda_p \Psi_p^M M_p \leq \phi_p^{n-1} - \lambda_p \sum_{f\in\mathcal{F}_p^-} \phi_q^n a_{pf} - \lambda_p \sum_{f\in\mathcal{F}_p^+} \phi_p^n a_{pf} - min.$$

It is possible to bound Ψ_p^M if

$$\phi_p^{n-1} - \lambda_p \sum_{f\in\mathcal{F}_p^-} \phi_q^n a_{pf} - \lambda_p \sum_{f\in\mathcal{F}_p^+} \phi_p^n a_{pf} - \lambda_p N_p - max \leq \phi_p^{n-1} - \lambda_p \sum_{f\in\mathcal{F}_p^-} \phi_q^n a_{pf} - \lambda_p \sum_{f\in\mathcal{F}_p^+} \phi_p^n a_{pf} - min$$

$$-\lambda_p N_p \leq max - min$$

We want

$$min \leq \phi_p^{n-1} - \lambda_p \sum_{f \in \mathcal{F}_p^-} \phi_q^n a_{pf} - \lambda_p \sum_{f \in \mathcal{F}_p^+} \phi_p^n a_{pf} - \alpha_p \left(\lambda_p M_p + \lambda_p N_p\right) \leq max$$