EMBEDDABILITY OF ABSTRACT PSEUDO-EINSTEIN CAUCHY-RIEMANN MANIFOLDS

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Abstract. Andrzej Trautman and Ivor Robinson's studies of solutions to Einstein's Equations on Lorentzian manifolds led Trautman to present the following conjecture. A 3-dimensional Cauchy Riemann manifold is locally embeddable if and only if it admits locally a closed, non-vanishing section of its canonical bundle. This writing sample will provide background to an ongoing thesis project on Trautman's conjecture. While many approaches to this conjecture use 4-dimensional Lorentzian Einstein manifolds, we will make use of 3-dimensional pseudo-Einstein structures, which are closely connected to Kähler-Einstein metrics in 4 real dimensions. The aim of the present piece is to articulate the link that the pseudo-Einstein condition has with closed sections of the canonical bundle and with local embeddability of Cauchy-Riemann manifolds.

1. Definitions

Cauchy-Riemann (CR) structures arise on real hypersurfaces in $\mathbb{C}^{n+1}$. The maximal complex subspace of $\mathbb{C}^{n+1}$ is denoted by H, and J is the restriction of the complex structure of $\mathbb{C}^{n+1}$ to H. A *CR manifold* M is given by the triple (M, H, J). H from before is a contact distribution, and the induced complex structure $J : H \to H$ is a bundle endomorphism such that $J \circ J = -\mathrm{Id}_H$. We denote the i-eigenspace of J by $T^{1,0} \subseteq \mathbb{C} \otimes H$, and we require $[T^{1,0}, T^{1,0}] \subseteq T^{1,0}$, which by definition means that $T^{1,0}$ is *formally integrable*. More precisely, we are stipulating that if $X, Y \in \Gamma(T^{1,0})$, then $[X, Y] \in \Gamma(T^{1,0})$. We say a CR manifold M of dimension $2n+1$ is *embeddable* if we can find $n+1$ functions, $z^1, ..., z^{n+1}$ on M that satisfy the tangential Cauchy-Riemann equations, and are functionally independent (meaning $dz^1 \wedge dz^2 \wedge ... \wedge dz^{n+1}$ is nowhere zero). If we seek a local embedding, then these conditions need only hold in a neighborhood of each point. A complex-valued q-form, ζ is of the type $(q, 0)$ if $\overline{T^{1,0}} \lrcorner \zeta = 0$. The complex line bundle of $(n+1, 0)$ forms is the *canonical bundle*, denoted K_M. A *contact form*, denoted by θ, is any non-vanishing section of the annhilator of H. Fixing a contact form on M yields a *pseudohermitian manifold*, given by (M, H, J, θ). Fixing a choice of contact form determines a unique vector field T (the *Reeb vector field*), satisfying $\theta(T) = 1$ and $T \lrcorner d\theta = 0$. An *admissible coframe* is a set of $(1, 0)$-forms $\{\theta^\alpha\}_{\alpha=1}^{\alpha=n}$ whose restrictions to the complexified tangent bundle form a basis for the complexified cotangent bundle and which satisfy $\theta^\alpha(T) = 0$ for all $\alpha = 1, ..., n$. The dual frame for the complexified tangent bundle is denoted $\{T, W_\alpha, W_{\bar{\alpha}}\}_{\alpha=1}^{\alpha=n}$. Given a fixed contact form, the *Levi form* is the hermitian bilinear form $L_\theta(U, \overline{V}) = -id\theta(U, \overline{V})$.

An expression for $d\theta$ is given in the structure equations for the Tanaka-Webster connection and torsion forms, respectively denoted by ${\omega_\alpha}^\beta$ and $\tau_\beta = A_{\beta\alpha}\theta^\alpha$ [5], [7]. The first equation is

$$d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}} \tag{1}$$

where $h_{\alpha\bar{\beta}}$ is a hermitian matrix. The form of this expression comes from θ being forced to be real, as well as the formal integrability of $T^{1,0}$. The integrability condition is equivalent to the expressions for $d\theta$ and $d\theta^\beta$ being 0 modulo θ and θ^σ. We thus have the following expression for $d\theta^\beta$

$$d\theta^\beta = \theta^\alpha \wedge {\omega_\alpha}^\beta + \theta \wedge \tau^\beta \tag{2}$$

where we may additionally require

$$\omega_{\alpha\bar{\beta}} + \omega_{\bar{\beta}\alpha} = dh_{\alpha\bar{\beta}}. \tag{3}$$

We also require $A_{\beta\alpha}$ to be symmetric in that $A_{\beta\alpha} = A_{\alpha\beta}$. Since $dh_{\alpha\bar{\beta}} = 0$, (3) tells us that $\omega_{\alpha\bar{\beta}}$ is purely imaginary. Computing the exterior derivative of (2) and combining with (1) gives the following expression for the exterior derivative of the connection form, which involves the Ricci tensor

$$d{\omega_\alpha}^\alpha = R_{\rho\bar{\sigma}}\theta^\rho \wedge \theta^{\bar{\sigma}} + {W_\alpha}^\alpha{}_\gamma\theta^\gamma \wedge \theta - {W_{\bar{\alpha}}}^{\bar{\alpha}}{}_{\bar{\gamma}}\theta^{\bar{\gamma}} \wedge \theta. \tag{4}$$

The pseudo-Einstein condition, which relates the Ricci tensor and the Levi form is as follows [1]

$$R_{\alpha\bar{\beta}} - \frac{1}{n}Rh_{\alpha\bar{\beta}} = 0, \qquad n \geq 2 \tag{5}$$

$$\nabla_\alpha R - i\nabla^\beta A_{\alpha\beta} = 0, \qquad n = 1. \tag{6}$$

We may note that (6) is a consequence of (5).

2. Pseudo-Einstein Structures and Canonical Bundles

The first aim of this section will be to show that closed, non-zero sections of the canonical bundle are closely tied to pseudo-Einstein contact forms. Throughout this paper, M will be a 3-dimensional (CR) manifold. This means that M is $(2n + 1)$-dimensional with $n = 1$. Since we work in 3 dimensions, we will make use of the $n = 1$ pseudo-Einstein condition, (6).

We begin by recalling the volume normalization condition for sections of the canonical bundle. In 3 dimensions, we say that θ is *volume-normalized* with respect to a $(2,0)$-form ζ if

$$\theta \wedge d\theta = i\theta \wedge (T \lrcorner \zeta) \wedge (T \lrcorner \bar{\zeta}). \tag{7}$$

Lemma 2.1. [3] *Given any smooth non-vanishing (2,0) form ζ on M, there exists a contact form θ that can be volume-normalized with respect to ζ.*

Proof. In 3 dimensions, we write our admissible coframe as $\{\theta, \theta^1, \theta^{\bar{1}}\}$. We may write an arbitrary $(2,0)$-form as $\zeta = \theta \wedge \theta^1$. Then recalling the properties of θ on the Reeb vector field T, we can write

$$T \lrcorner \zeta = \theta^1.$$

We now recall structure equation (1) in 3 dimensions

$$d\theta = ih_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}}.$$

Substituting the expression for $T \lrcorner \zeta$ in the above gives

$$d\theta = ih_{1\bar{1}}(T \lrcorner \zeta) \wedge (T \lrcorner \bar{\zeta})$$

Now simply wedging both sides of this equation with θ (possibly scaled by a smooth positive function) gives

$$\theta \wedge d\theta = i\theta \wedge (T \lrcorner \zeta) \wedge (T \lrcorner \bar{\zeta}).$$

This is exactly the volume-normalization condition (1). $\square$

We now move to show that a pseudo-Einstein contact form is equivalent to a particular 1-form being closed.

Lemma 2.2. [1] *On M, θ is pseudo-Einstein if and only if the 1-form ${\omega_1}^1 + iR\theta$ is closed* .

Proof. The two sided implication can be shown directly by computing $d({\omega_1}^1 + iR\theta)$. From structure equation (4) with $n = 1$ we have

$$d{\omega_1}^1 = Rh_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}} + \nabla^1 A_{11}\theta^1 \wedge \theta - \nabla^{\bar{1}} A_{\bar{1}\bar{1}}\theta^{\bar{1}} \wedge \theta.$$

It follows that

$$\begin{aligned} d({\omega_1}^1 + iR\theta) &= Rh_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}} + \nabla^1 A_{11}\theta^1 \wedge \theta - \nabla^{\bar{1}} A_{\bar{1}\bar{1}}\theta^{\bar{1}} \wedge \theta \\ &\quad - Rh_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}} + i\nabla_1 R\theta^1 \wedge \theta + i\nabla_{\bar{1}} R\theta^{\bar{1}} \wedge \theta \\ &= i\nabla_1 R\theta^1 \wedge \theta + i\nabla_{\bar{1}} R\theta^{\bar{1}} \wedge \theta + \nabla^1 A_{11}\theta^1 \wedge \theta - \nabla^{\bar{1}} A_{\bar{1}\bar{1}}\theta^{\bar{1}} \wedge \theta \\ &= 2i\mathrm{Re}\,(\nabla_1 R\theta^1 \wedge \theta) - 2i\mathrm{Re}(i\nabla^1 A_{11}\theta^1 \wedge \theta) \\ &= 2i(\mathrm{Re}\,(\nabla_1 R - i\nabla^1 A_{11})\theta^1 \wedge \theta). \end{aligned}$$

The last expression contains the pseudo-Einstein condition (6), and hence is equal to zero. Thus $d({\omega_1}^1 + iR\theta) = 0$, giving our result.

□

We can use this lemma to show that a pseudo-Einstein contact form is locally volume normalized with respect to a closed section of the canonical bundle.

Theorem 2.3. [1] *θ is pseudo-Einstein if and only if, for every point $p \in M$, there exists a neighborhood $U \subseteq M$ containing p, where θ is volume-normalized with respect to a closed section of the canonical bundle.*

Proof. We begin by choosing a coframe where $h_{1\bar{1}} = \delta_{1\bar{1}}$. We can write the structure equation (1) as

$$d\theta = i\theta^1 \wedge \theta^{\bar{1}}. \tag{8}$$

Now, for $p \in U \subseteq M$, let θ be volume-normalized with respect to $\zeta \in K_M$. Since $\{\theta, \theta^1\}$ is a frame for $(1,0)$-forms and ζ is a $(2,0)$-form, we may write $\zeta = \lambda\theta \wedge \theta^1$ for some $\lambda \in C^\infty(M, \mathbb{C})$. Plugging ζ into the volume normalization condition (7) tells us that $|\lambda| = 1$, which importantly tells us that λ is not 0. This means that we can write $\zeta = \theta \wedge \theta^1$ by redefining θ^1 as $\lambda\theta^1$.

Computing the exterior derivative of ζ gives

$$d\zeta = d\theta \wedge \theta^1 - \theta \wedge d\theta^1.$$

Using structure equations (1) and (2), we have

$$\begin{aligned} d\zeta &= (i\theta^1 \wedge \theta^{\bar{1}}) \wedge \theta^1 - \theta \wedge (\theta^1 \wedge {\omega_1}^1 + \theta \wedge \tau^\beta) \\ &= -\theta \wedge \theta^1 \wedge {\omega_1}^1 \\ &= -{\omega_1}^1 \wedge \zeta. \end{aligned}$$

Since ζ was closed we have that ${\omega_1}^1$ is a (1, 0)-form. Recall from (3) that ${\omega_1}^1$ is purely imaginary. This allows us to write ${\omega_1}^1 = iu\theta$ for some $u \in C^\infty(M, \mathbb{C})$.

Taking the exterior derivative of ${\omega_1}^1$ gives

$$d{\omega_1}^1 = -u\theta^1 \wedge \theta^{\bar{1}} + i((\nabla_1 u)\theta^1 + (\nabla_{\bar{1}} u)\theta^{\bar{1}}) \wedge \theta.$$

Comparing this equation with structure equation (4) tells us that $-u = R$ and $i\nabla_1 u = \nabla^1 A_{11}$ which in turn means that $\nabla_1 R - i\nabla^1 A_{11} = 0$, making θ pseudo-Einstein.

Now assume that θ is pseudo-Einstein, and take our old coframe $\{\theta, \theta^1, \theta^{\bar{1}}\}$. Using (2), we can define a section of the canonical bundle,

$$\zeta_0 = \theta \wedge \theta^1.$$

Using our previous computation of $d\zeta$, we know that

$$d\zeta_0 = -{\omega_1}^1 \wedge \zeta_0.$$

By Lemma 2.2, the 1-form ${\omega_1}^1 + iR\theta$ is closed and, so for some function ϕ we may write

$${\omega_1}^1 + iR\theta = id\phi.$$

We can take ϕ to be real since ${\omega_1}^1$ is purely imaginary. This allows us to write

$$d(e^{i\phi}\zeta_0) = e^{i\phi}({\omega_1}^1 + iR\theta) \wedge \zeta_0 - e^{i\phi}({\omega_1}^1) \wedge \zeta_0 = 0.$$

Since θ is volume-normalized with respect to $e^{i\phi}\zeta_0$, we have our closed section of the canonical bundle.

□

Now, by considering Lemma 2.1 together with Theorem 2.3, we arrive at the following.

Theorem 2.4. [4] *If M admits a closed, non-vanishing (2, 0)-form, then M admits a pseudo-Einstein structure. Conversely, if M admits a pseudo-Einstein structure, then in a neighborhood of every point, M admits a closed non-vanishing (2, 0)-form.*

Now note two important facts. First, the canonical bundle K_M of a CR manifold M embedded in $\mathbb{C}^2$ is equal to the canonical bundle of $\mathbb{C}^2$ restricted to M, and $dz_1 \wedge dz_2$ is a closed section of the canonical bundle of $\mathbb{C}^2$.

Second, we recall that the exterior derivative is a natural operator. Thus if we pull back the section $dz_1 \wedge dz_2$ via the inclusion mapping to M, then the section $\iota^*(dz_1 \wedge dz_2) = \zeta \in K_M$ is still closed.

This tells us that any CR manifold embedded in $\mathbb{C}^2$ admits a closed section of its canonical bundle. We are led to the following corollary.

Corollary 2.5. [4] *If M is CR embeddable, then M admits a pseudo-Einstein structure.*

This reasoning allows us to conclude that locally, closed sections of the canonical bundle are equivalent to the admittance of a pseudo-Einstein structure. We also see that CR embeddability implies the admittance of a pseudo-Einstein structure. These facts provide the basis to posit that the pseudo-Einstein condition will be key to proving Trautman's conjecture. The next phase of this project will be to connect the pseudo-Einstein condition with the boundary condition for a Kähler-Einstein extension problem on the pseudoconvex side of CR 3-dimensional manifolds.

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