EMBEDDABILITY OF ABSTRACT PSEUDO-EINSTEIN CAUCHY-RIEMANN MANIFOLDS

IBRAHIM HAJAR

ABSTRACT. Andrzej Trautman and Ivor Robinson's studies of solutions to Einstein and Maxwell's equations on Lorentzian manifolds led Trautman to present the following conjecture. A 3-dimensional Cauchy-Riemann manifold is locally embeddable if and only if it admits locally a closed, non-vanishing section of its canonical bundle. Many approaches to this conjecture use 4-dimensional Lorentzian Einstein manifolds and their close relation, in the algebraically special case, to 3-dimensional CR structures. We will make use of 3-dimensional pseudo-Einstein structures, which are closely connected to Kähler-Einstein metrics in 4 real dimensions. The aim of this thesis will be to provide context and background to Trautman's conjecture coming from both geometry and physics, as well as to articulate the link that the pseudo-Einstein condition has with closed sections of the canonical bundle and with local embeddability of Cauchy-Riemann manifolds.

1. DEFINITIONS

Let M be a real $(2n + 1)$-dimensional C^∞ manifold with tangent bundle TM. Let $H \subseteq TM$ be a rank $2n$ maximally non-integrable subbundle, meaning that H is given as the kernel of a nowhere vanishing 1-form θ which satisfies $\theta \wedge (d\theta)^n \neq 0$. Such an H is called a *contact structure*, and θ a *contact form* (note that we permit θ to be scaled by a smooth, positive non-vanishing function). The *complexification* of TM is denoted by $\mathbb{C}TM = TM \otimes \mathbb{C} = \{v_1 + iv_2 : v_1, v_2 \in TM, \pi(v_1) = \pi(v_2)\}$ where $\pi : TM \to M$ is the natural projection. A *Cauchy-Riemann (CR) structure* on H is specified by a bundle endomorphism $J : H \to H$, satisfying $J \circ J = -\text{Id}_H$. The Nijenhuis tensor of J must also vanish, which means that for any vector fields, X and Y in the space of smooth sections of the contact distribution $\Gamma(H)$, we have

(1) $[JX, Y] + [X, JY] \in \Gamma(H)$

(2) $N_J(X, Y) = [X, Y] - [JX, JY] + J([JX, Y] + [X, JY]) = 0$

Note that (1) ensures that the Nijenhuis tensor, N_J, is well-defined and (2) is the vanishing condition (one can check that N_J is also C^∞-linear and thus tensorial). The data (M, H, J) collectively define a (nondegenerate) CR $(2n + 1)$-manifold. A framing for H_p

may always be chosen such that J_p may is given by the constant matrix $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ for any point $p \in M$. Here we clearly see that J_p has eigenvalues i and $-i$, leading to the decomposition $\mathbb{C}H_p = T_p^{1,0} \oplus T_p^{0,1}$, where $T_p^{1,0} \subseteq \mathbb{C}TM_p$ is the i-eigenspace of J_p, and $T_p^{0,1} = \overline{T_p^{1,0}}$ is the $-i$-eigenspace. Both of the eigenspaces have complex dimension n. The eigenspace decomposition depends smoothly on the point p. We note that we can also define a CR structure by specifying the bundle $T^{1,0}$ and asking that it satisfy the conditions

$$(3) \qquad\qquad\qquad T_p^{1,0} \cap \overline{T_p^{1,0}} = \{0\} \ \ \forall p \in M$$

$$(4) \qquad\qquad\qquad X, Y \in \Gamma(T^{1,0}) \Rightarrow [X, Y] \in \Gamma(T^{1,0}).$$

The formal integrability condition (4) is equivalent to conditions (1) and (2). We will again denote $T^{0,1} = \overline{T^{1,0}}$. We also note that in three dimensions ($n = 1$), $T^{1,0}$ is a complex line bundle, and so condition (4) is automatically satisfied. If equations (3) and (4) hold, then we define H by requiring

$$(5) \qquad\qquad\qquad T^{1,0} \oplus T^{0,1} = \mathbb{C}H,$$

where the $\oplus$ in equation (5) denotes the *Whitney sum*, which amounts to a direct sum over each point of M of the fibers of $T^{1,0}$ and $T^{0,1}$. We also define the complex structure J to act as multiplication by i on $T^{1,0}$ and multiplication by -1 on $T^{0,1}$. That is,

$$(6) \qquad\qquad\qquad J(X) = iX \ \ \forall X \in T^{1,0}$$

$$(7) \qquad\qquad\qquad J(Y) = -iY \ \ \forall Y \in T^{0,1}.$$

Thus, conditions (3) and (4) regarding $T^{1,0}$ define a rank $2n$ distribution H and complex structure J which satisfy the necessary conditions of (1) and (2). The distribution H will be made a contact distribution by requiring that the Levi form (to be defined shortly) does not vanish. A CR structure which is described by the data (M, H, J) can be described equivalently by the data $(M, T^{1,0})$. A real hypersurface M embedded in $\mathbb{C}^{2n+1}$ inherits a CR structure by setting $T^{1,0}M = \mathbb{C}TM \cap (T^{1,0}\mathbb{C})$.

A CR manifold M of dimension $2n + 1$ is said to be *CR embeddable* if there exists an embedding such that the induced CR structure on M is the same as the original CR structure.

Fixing, in addition, a contact form θ for H yields a *pseudohermitian manifold*, (M, H, J, θ). A fixed contact form also determines a unique vector field, T, the *Reeb vector field*, which is defined by being transverse to H and being the dual to θ, i.e. $\theta(T) = 1$. As we will see below, we also have $T \lrcorner \, d\theta = 0$. In defining a local framing for the complexified tangent

bundle, we endeavor to reflect the bundles which endowed the space with a CR structure. Toward this end, we take a framing for $T^{1,0}$ given by Z_α. By conjugacy, we take the framing of $T^{0,1}$ given by $Z_{\bar\alpha}$. We take T to be the final direction. Our local framing is then given by $\{T, Z_\alpha, Z_{\bar\alpha}\}$. We can then define an *admissible coframe* as simply the dual to our frame. An admissible coframe will be denoted by $\{\theta, \theta^\alpha, \theta^{\bar\alpha}\}$. The *Levi form* is a measure of the non-integrability of the contact distribution. Given a fixed contact form, we can define the Levi form as $L_\theta(U_p, \overline{V_p}) = -id\theta(U_p, \overline{V_p})$, for $U_p, V_p \in \mathbb{C}H$. L_θ is clearly hermitian and bilinear. The conditions which define an admissible coframe, as well as the reality of θ and the vanishing of θ on the Reeb vector field tell us how to express $d\theta$. We have

$$d\theta = ih_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta} \tag{8}$$

where $h_{\alpha\bar\beta}$ is a hermitian matrix. Equation (8) is the first of the Tanaka-Webster structure equations for a pseudohermitian structure [14]. The formal integrability of $T^{1,0}$ additionally requires that the expression for $d\theta^\beta$ be 0 modulo θ and θ^σ. An expression for $d\theta^\beta$ is given in the structure equations for the Tanaka-Webster connection and torsion forms, respectively denoted by $\omega_\alpha{}^\beta$ and $\tau_\beta = A_{\beta\alpha}\theta^\alpha$ [21], [24]. The matrix $h_{\alpha\bar\beta}$ will be used to raise and lower indices. For example, $\omega_\alpha{}^{\bar\beta} = \omega_\alpha{}^\delta h_{\delta\bar\beta}$. We thus have the following expression for $d\theta^\beta$

$$d\theta^\beta = \theta^\alpha \wedge \omega_\alpha{}^\beta + \theta \wedge \tau^\beta \tag{9}$$

where $\omega_\alpha{}^\beta$ and τ_β are uniquely determined by the requirement that

$$\omega_{\alpha\bar\beta} + \omega_{\bar\beta\alpha} = dh_{\alpha\bar\beta}, \tag{10}$$

and the requirement that $A_{\beta\alpha}$ be symmetric in that $A_{\beta\alpha} = A_{\alpha\beta}$. The indices α and β range from 1 to n (where again we work in dimension $2n+1$). Note that if $h_{\alpha\bar\beta} = \delta_{\alpha\bar\beta}$ then $dh_{\alpha\bar\beta} = 0$, which would make $\omega_{\alpha\bar\beta}$ purely imaginary. Computing the exterior derivative of (10) and combining with (9) gives the following expression for the exterior derivative of the connection form (we present the following modulo θ)

$$d\omega_\alpha{}^\beta - \omega_\alpha{}^\gamma \wedge \omega_\gamma{}^\beta = R_\alpha{}^\beta{}_{\rho\bar\sigma}\theta^\rho \wedge \theta^{\bar\sigma} + i(\theta_\alpha \wedge \tau^\beta - \tau_\alpha \wedge \theta^\beta), \tag{11}$$

where $R_\alpha{}^\beta{}_{\rho\bar\sigma}$ indicates components of the pseudohermitian curvature tensor. We also have the contraction (also given modulo θ)

$$d\omega_\alpha{}^\alpha = R_{\rho\bar\sigma}\theta^\rho \wedge \theta^{\bar\sigma}, \tag{12}$$

where $R_{\rho\bar\sigma}$ denotes the pseudohermitian Ricci curvature tensor, which is the trace of the pseudohermitian curvature tensor. The connection given by equations (9) and (10) induces a unique covariant derivative, denoted by ∇, which will be used in the following.

A contact form θ is said to be pseudo-Einstein condition, if the following conditions (depending on dimension) hold [3] [14]

$$(13) \qquad\qquad R_{\alpha\bar\beta} - \frac{1}{n} R h_{\alpha\bar\beta} = 0, \qquad n \geq 2$$

$$(14) \qquad\qquad \nabla_\alpha R - i\nabla^\beta A_{\alpha\beta} = 0, \qquad n = 1.$$

We may note that in dimensions higher than 3, a generalization of (14) holds as a consequence of (13) with an application of the second Bianchi Identity, given in its contracted form by

$$(15) \qquad\qquad R_{,\gamma} - R_{\gamma\bar\sigma,}{}^{\bar\sigma} = i(n-1)A_{\alpha\gamma,}{}^\alpha$$

where indices placed after commas denote a direction in which a covariant derivative has been taken.

Finally, we term the complex bundle of $(n+1, 0)$ forms the *canonical bundle*, denoted K_M. In 3 dimensions, K_M is a complex line bundle of $(2,0)$-forms. Given an admisssible coframe, $\{\theta, \theta^1, \theta^{\bar 1}\}$ a frame for the canonical bundle is given by $\theta \wedge \theta^1$. Then a local section of the canonical bundle, ζ, can be written as $\zeta = y\theta \wedge \theta^1$ where y is a smooth complex-valued function.

2. Examples

Here we will work two examples, involving the unit 3-sphere and the Heisenberg group, to demonstrate a computation of the Tanaka-Webster connection coefficients.

Example 2.1. [5]

Beginning with the definition of the unit 3-sphere, we have the defining function $u = 1 - |z|^2 - |w|^2$. The standard choice of contact form is $\theta = i\partial u = i(zd\bar z + wd\bar w)$ where we have used the fact that $du = \partial u + \bar\partial u = 0$ on S^3. Noting that, on TS^3, $zd\bar z + \bar z dz + wd\bar w + \bar w dw = 0 = du$ allows us to write $d\theta = i(dz \wedge d\bar z + dw \wedge d\bar w)$. A global framing for $T^{1,0}$ is given by $Z_1 = \{\bar w \frac{\partial}{\partial z} - \bar z \frac{\partial}{\partial w}\}$, which in turn determines θ^1 to be $wdz - zdw$. We may note that these definitions of θ and θ^1 fulfill structure equation (8). Noting that $d\theta^1 = 2(dw \wedge dz)$ and plugging into structure equation (9) reveals that $\omega_1{}^1 = 2(zd\bar z + wd\bar w) = -2i\theta$. Finally computing the exterior derivative of $\omega_1{}^1$ gives $d\omega_1{}^1 = 2\theta^1 \wedge \theta^{\bar 1}$. Comparing with structure equation (12) tells us that $R = 2$.

Example 2.2. [6]

We now turn to the Heisenberg group in general dimension, defined as $H = \{z \in \mathbb{C}^n : Im(z) = \sum_{\alpha=1}^{n-1} |z_\alpha|^2\}$. A global framing for $T^{1,0}$ is given by $Z_\alpha = \frac{\partial}{\partial z_\alpha} + 2i\overline{z_\alpha}\frac{\partial}{\partial z_n}$. A framing for $T^{0,1}$ is given by $Z_{\bar\alpha} = \frac{\partial}{\partial \bar z_\alpha} + 2iz_\alpha \frac{\partial}{\partial \bar z_n}$, while a framing for T is given by

$Z = \frac{1}{2}(\frac{\partial}{\partial z_n} + \frac{\partial}{\partial \overline{z_n}})$. We take the dual of these vector fields while also understanding that the coframe elements must annihilate their non-dual counterparts. Accordingly, we obtain $\theta = dz_n + d\overline{z_n} - i(2z_\alpha dz_\alpha + 2\overline{z_\alpha}d\overline{z_\alpha})$ and $\theta^\alpha = dz_\alpha$. $\theta^{\bar{\alpha}}$ is found by taking $\overline{\theta^\alpha}$. We can then easily see that $d\theta^\alpha = d\theta^{\bar{\alpha}} = 0$. Plugging this into structure equation (11) tells us that $\omega_\alpha{}^\beta = \tau^\alpha = 0$, given the independence of our coframe. Since $\omega_\alpha{}^\alpha = 0$, $\omega_\alpha{}^\alpha$ is clearly closed, which tells us that $R_\alpha{}^\beta{}_{\rho\bar{\sigma}} = 0$ by the same structure equation.

3. Background to Trautman's Conjecture: CR Structures 'Behind the Scenes' of Maxwell and Einstein Fields

As this quotation from [8] in the section title suggests, there exist pathways which allow one to move between 3-dimensional CR structures and 4-dimensional Lorentz spacetimes. To link a spacetime to a 3-dimensional CR structure, we will construct a conformal class of Lorentz metrics from the elements of an admissible coframe. First, recall that the definition of a local coframe (in 3 dimensions, the index α only takes the value 1), $\{\theta, \theta^1, \theta^{\bar{1}}\}$, stipulates that θ be real, and θ^1 be complex. The definition of a coframe also requires $\{\theta, \theta^1, \theta^{\bar{1}}\}$ to form a basis for the (complexified) cotangent bundle $\mathbb{C}T^*M$, at each point of M. Now let us abstract from our coframe an equivalence class of pairs of 1-forms, $[\theta, \theta^1]$. Then a CR structure is a 3-dimensional real manifold M together with this equivalence class of pairs of 1-forms. The equivalence relation is simply given by the requirements of the coframe. Namely, this means permitting scaling and requiring reality of θ, while θ^1 and $\theta^{\bar{1}}$ are complex and restrict to give a frame for $T^{1,0}$ and $T^{0,1}$ respectively. The equivalence relation is given by:

$$(\theta, \theta^1) \equiv (\theta', \theta^{1'}) \qquad \Longleftrightarrow \qquad \exists f \neq 0 \in C^\infty(M),\ h \neq 0,\ p \in C^\infty(\mathbb{C}M)$$

such that

$$(16) \qquad \theta' = f\theta, \qquad \theta^{1'} = h\theta^1 + p\theta, \qquad \overline{\theta^{1'}} = \bar{h}\bar{\theta}^{\bar{1}} + \bar{p}\theta.$$

We can thus identify our CR structure as $(M, [\theta, \theta^1])$. Then, we construct the product $\mathcal{M} = M \times \mathbb{R}$. Parametrize the $\mathbb{R}$ direction by a function r, such that $\partial_r \neq 0$. Set $k = \partial_r$, so that k is tangent to the $\mathbb{R}$ direction. $\mathcal{M}$ can then be equipped with a class of metrics, $[g]$. For real functions, $P \neq 0$ and H and a complex function on $\mathcal{M}$, these metrics are of the form

$$(17) \qquad g = 2P^2[\theta^1\bar{\theta}^{\bar{1}} + \theta(dr + W\theta^1 + \overline{W}\bar{\theta}^{\bar{1}} + H\theta)],$$

where products denote a symmetrized tensor product, e.g. $\theta^1\bar{\theta}^{\bar{1}} = \frac{1}{2}(\theta^1 \otimes \bar{\theta}^{\bar{1}} + \bar{\theta}^{\bar{1}} \otimes \theta^1)$. Note that we can see that g is an indefinite metric because $\partial_r \neq 0$ is a null direction. Now, if we replaced $(\theta, \theta^1, \bar{\theta}^{\bar{1}})$ by a different element of the equivalence class, $(\theta', \theta^{1'}) \in [\theta, \theta^1]$,

then the new metric $g' \in [g]$ can be rewritten in terms of $(\theta, \theta^1, \bar{\theta}^1)$ with only the functions P, H, W and the parameter r changed. The transformation from g to g' can be captured by the relation

$$g' = \sigma^2 g + 2g(k)\varphi \tag{18}$$

where $\sigma \neq 0$ is a real function and φ a 1-form on $\mathcal{M}$. Together, (15) and (16) define a class of Lorentz metrics, $[g]$ adapted to the CR structure $(M, [\theta, \theta^1])$. This defines a class of *lifts* from a 3-dimensional CR structure to a spacetime.

Now, we will traverse the path from 4-dimensional spacetimes to 3-dimensional CR structures, the path usually walked by physicists searching for solutions to Maxwell's and Einstein's equations. These solutions exist in algebraically special spacetimes, and as noted above, "impose a complex structure" on their spacetimes, uncovering an underlying CR structure. The spacetimes we work with will be *Robinson manifolds*, named after Ivor Robinson who worked heavily on null solutions to Maxwell's equations in the 1950s. Robinson manifolds are Lorentz manifolds equipped with a particular structure on their complexified tangent bundles, called an *integrable $\mathcal{N}$-structure*. We will need to develop a few more definitions in order to make sense of this structural condition. Firstly, a vector subspace $N \subset V$ is *maximally totally null* if $N^\perp = N$. An *$\mathcal{N}$-structure* on a Lorentzian manifold of dimension $2n$ $(n \geq 2)$ is a complex subbundle of the complexified tangent bundle, $\mathcal{N} \subseteq \mathbb{C}TM$ such that every fiber of $\mathcal{N}$ is maximally totally null. The requirement that a Robinson manifold be of even dimension is suggestive of an underlying complex structure, so let us make this precise. Let $\mathcal{M}$ be a 4-dimensional Robinson manifold. Let $K \subset T\mathcal{M}$ be the null line bundle defined by the condition that $\mathcal{N} \cap \overline{\mathcal{N}} = \mathbb{C} \otimes K$, where $\mathcal{N}$ denotes the N-structure on $\mathcal{M}$. The Frobenius Theorem guarantees that $\mathcal{M}$ is foliated by a 3-dimensional family of null curves tangent to K. The formal integrability of $\mathcal{N}$ implies that these curves are indeed geodesics. We can then construct the 3-dimensional manifold, M, which is the leaf space of the foliation of $\mathcal{M}$, meaning that M consists of the null geodesics tangent to K. We must also construct the quotient space $\mathcal{N}/(\mathbb{C} \otimes K)$. This complex line bundle descends to $T^{1,0}$ on the quotient manifold M. We now have all the necessary data, namely $(M, T^{1,0})$, to describe a CR structure.

The connections between CR and Robinson structures has been related to the theory of electromagnetism, and in particular to null solutions of Maxwell's equations. We will close this section by developing this relation, culminating in some key results.

The vector form of Maxwell's equations and the *Faraday tensor* which will follow, are defined in flat Minkowski space. Maxwell's equations are given in vector form as

(19) $$\nabla \cdot E = \rho$$

(20) $$\nabla \times B = \frac{1}{c}(\frac{\partial E}{\partial t} + j)$$

(21) $$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$$

(22) $$\nabla \cdot B = 0,$$

where E is the electric field, B is the magnetic field, ρ is the charge density, j is the current density, and c is the speed of light. One often introduces the Faraday tensor, which subsumes both the electric and magnetic fields. The Faraday tensor is defined as

(23) $$F = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}.$$

As a two-form, we have

(24) $$\mathbf{F} = (E_x dx + E_y dy + E_z dz) \wedge dt + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$$

and the Hodge dual

(25) $$*\mathbf{F} = -E_x dy \wedge dz - E_y dz \wedge dx - E_z dx \wedge dy + (B_z dz + B_y dy + B_x dx) \wedge dt.$$

Now we will broaden the reach of our electromagnetic theory to curved spacetimes, by formulating the following two equations. The Hodge dual operator $*$ now denotes $*_g$ where g is the relevant metric. We now reduce the four vector equations above to the following two:

(26) $$d\mathbf{F} = 0$$

(27) $$d*\mathbf{F} = *\mathbf{j},$$

where $\mathbf{j}$ denotes the 4-current. We call $\mathbf{F}$ a *Maxwell field*. $\mathbf{F}$ is said to be *null* if, in addition to (26) and (27), we have $\mathbf{F} \wedge \mathbf{F} = 0$ and $\mathbf{F} \wedge *\mathbf{F} = 0$. Further simplification is achieved by defining a self-dual Maxwell field, $\widetilde{\mathbf{F}} = \mathbf{F} - i*\mathbf{F}$. Then (26) and (27) are condensed to

(28) $$d\widetilde{\mathbf{F}} = 0,$$

and the null condition is simply

(29) $$\widetilde{\mathbf{F}} \wedge \widetilde{\mathbf{F}} = 0.$$

Note that $\widetilde{\mathbf{F}}$ is a $(2,0)$-form and thus a section of the canonical bundle. A CR structure, $(M, [\theta, \theta^1])$ is said to admit a null aligned Maxwell field [8] if there exists such a $(2,0)$-form, $\widetilde{\mathbf{F}}$ defined as above and satisfying (28) and (29). Note that this is equivalent to admitting a closed, non-vanishing section of the canonical bundle, $\zeta = y\theta \wedge \theta^1$ (for y a smooth complex-valued function) such that $d\zeta = 0$ but $\zeta \neq 0$.

We will now present two results which tie local CR embeddability, spacetime lifts, and Einstein conditions together. These theorems will be presented without proof because they are not explicitly part of the approach to the present problem in this paper, but give a sense of the sort of fascinating work which has been done at the nexus of Maxwell's equations, general relativity, and CR geometry. The first of these theorems is due to C. Denson Hill, Jerzy Lewandowski, and Pawel Nurowski in [8].

Theorem 3.1. [8] *If M is a sufficiently smooth strictly pseudoconvex 3-dimensional CR manifold, then M is locally CR embeddable as a hypersurface in $\mathbb{C}$ if and only if*

   (1) M admits a lift to a spacetime whose complexified Ricci tensor vanishes on the corresponding distribution of α-planes (where the distribution of α-planes is defined as the set of vector fields X on $M \times \mathbb{R}$ such that $X \lrcorner \widetilde{\boldsymbol{F}} = 0$)

   (2) and M admits a nontrivial null aligned Maxwell field associated to the CR structure.

The distribution of α-planes mentioned above is the $\mathcal{N}$-structure corresponding to a Robinson manifold. We also know that condition (2) is equivalent to the admittance of a nonvanishing and closed section of the canonical bundle, by the discussion above. Note that like Trautman's conjecture, the result above deals with smooth, and not analytic manifolds. In fact, Trautman's conjecture (Conjectures 4.3 and 4.4 below), is equivalent to Theorem 3.1 with condition (1) removed. Specifically, we hope to show that starting from condition (2), there exists a choice of a spacetime lift such that condition (1) holds.

The next theorem is originally due to Lewandowski, Nurowski, and Jacek Tafel in [15]. Nurowski and Trautman restated it in [18] as follows:

Theorem 3.2. [18] *If a CR structure M admits a lift to an Einstein-Robinson spacetime, then M is locally embeddable.*

Here an Einstein-Robinson spacetime means a Robinson manifold for which the underlying metric satisfies the vacuum Einstein equations. Theorem 3.2 indicates that there is a deep connection between the Einstein equations and the Cauchy-Riemann equations, and that furthermore the Einstein equations partially characterize CR embeddability. This result can be contrasted with the first statement of Trautman's conjecture at the

end of the next section, which replaces the Einstein condition with the requirement that the Robinson spacetime admits a null solution to Maxwell's equations.

4. Robinson's Hopes and Trautman's Conjecture

In 1960, Ivor Robinson theorized that every null solution of Maxwell's equations defined a corresponding shearfree null geodesic (SNG) congruence and that conversely, every SNG congruence supports a null Maxwell field. In 1985, Jacek Tafel revisited this Robinson Theorem. Tafel confirmed that null Maxwell fields do always determine a SNG congruence, however it is not always possible to solve for a null Maxwell field starting from SNG congruences [20]. Tafel's examination of Robinson's work can be summarized in two lemmas.

Lemma 4.1. [20] *Every null Maxwell field determines a SNG congruence.*

Proof. We begin with a self-dual Maxwell field, defined earlier and denoted by $\widetilde{\mathbf{F}}$. The null condition (29) implies that there is a 1-form κ such that

$$(30) \qquad \kappa \wedge \widetilde{\mathbf{F}} = 0.$$

Equation (30) means that κ (with scaling permitted) is null. Additionally, equation (30) implies that $\widetilde{\mathbf{F}}$ is simple, i.e. of the form $\widetilde{\mathbf{F}} = \alpha \wedge \beta$ for 1-forms α and β. In fact, we can determine that there must be a null vector within the span of $\{\alpha_p, \beta_p\}$ at every point $p \in M$. Given this, we can write $\widetilde{\mathbf{F}}$ as

$$(31) \qquad \widetilde{\mathbf{F}} = y\kappa \wedge \alpha$$

for y a non-vanishing complex function and α a complex 1-form with the permissible transformation

$$(32) \qquad \alpha \mapsto \alpha' = B\alpha + C\kappa,$$

for $B \neq 0$ and C complex functions. The spacetime metric, g, can be written in the form

$$(33) \qquad g = \kappa\omega + p\alpha\bar{\alpha},$$

for p a real positive function and ω a real 1-form such that the set $\{\omega, \alpha, \bar{\alpha}, \kappa\}$ is linearly independent. Now since equation (30) is equivalent to $k \lrcorner \widetilde{\mathbf{F}} = 0$, we can use the identity

$$(34) \qquad \mathcal{L}_k \widetilde{\mathbf{F}} = k \lrcorner d\widetilde{\mathbf{F}} + d(k \lrcorner \widetilde{\mathbf{F}})$$

to show that (30) implies

$$(35) \qquad \mathcal{L}_k \widetilde{\mathbf{F}} = 0.$$

Equations (31) and (34) together imply

$$\kappa \wedge \mathcal{L}_k \kappa = 0 \tag{36}$$

$$\kappa \wedge \alpha \wedge \mathcal{L}_k \alpha = 0, \tag{37}$$

where (36) tells us that the flow generated k consists of geodesics. Equation (37) follows by noting that α is analogous to θ^1 and then wedging with α. Using the form of the metric given in (33), equations (36) and (37) are together equivalent to

$$\mathcal{L}_k g = ag + \kappa\gamma \tag{38}$$

for a some function, and γ a 1-form. Equation (38) gives us a geodesic shearfree null congruence. The nullity follows simply because κ was null. The congruence is shearfree because we see that the flow generated by k acts by conformal transformations on the screen space, $k^\perp/k$, of the contact distribution. We see that the congruence consists of geodesics because when we consider the contraction of (38) with k (and also use the fact that κ is null as well as identity (34)), we see that for the flow lines of κ (where the flow lines are denoted by φ) we have

$$\nabla_{\varphi'}\varphi' \propto \varphi' \tag{39}$$

which fits the geodesic equation for unparameterized curves. $\qquad\square$

Lemma 4.2. [20] [19] [11] *Every SNG congruence need not determine a null Maxwell field.*

Proof. The reasoning given here restates Tafel's reasoning by combining results of Jean-Pierre Rosay [19] and Howard Jacobowitz [11].

It can be shown that a spacetime admits a SNG congruence if and only if the metric g can be given in the form of (33). Now if we consider null Maxwell fields, they must be of the form (31), allowing us to write Maxwelll's equations as

$$d(y\kappa \wedge \alpha) = 0. \tag{40}$$

Performing this computation yields $dy + yd(\kappa \wedge \alpha) = 0$, which can be rewritten as

$$\overline{\partial_b}y + f(\kappa, \alpha)y = 0. \tag{41}$$

The operator $\overline{\partial_b}$ is defined as

$$\overline{\partial_b}y_p : T_p^{0,1}M \to \mathbb{C} \tag{42}$$

$$\overline{\partial_b}y(X) = Xy \ \ \forall X \in T_p^{0,1}M. \tag{43}$$

Now we note that the coefficient $f(\kappa, \alpha)$ may not always be analytic. Because of this, equation (41) falls into a class of partial differential equations studied by Hans Lewy [16].

Work by Rosay [19], Jacobowitz, and Francois Treves [12] has shown that equations of the form (41) may not always be solvable. Thus while it is true that every null Maxwell field does determine a SNG congruence, the converse is not always true. □

Continuing in the search for special solutions to Maxwell's equations, Trautman put forth a conjecture which made use of the CR 3-spaces associated to the curved spacetimes of general relativity. The conjecture can be stated in physical terms which manifest its heritage in Robinson's Theorem:

Conjecture 4.3. *If a Robinson manifold admits a nowhere vanishing null solution of Maxwell's equations, then the associated Cauchy-Riemann space is embeddable.*

The approach to Trautman's conjecture presented in this paper will function only in CR 3-spaces, for which a reframing of Trautman's conjecture will be more convenient. This formulation substitutes a closed, non-zero (2, 0)-form in the canonical bundle for a null Maxwell field. As it is presented in [22], the conjecture is stated as follows:

Conjecture 4.4. *A 3-dimensional CR manifold, M locally admits a closed, non-vanishing section, ζ of its canonical bundle if and only if if M is locally embeddable.*

We note that being locally CR embeddable clearly implies the admittance of a closed non-vanishing section of the canonical bundle. Thus most of the work lies in proving the reverse implication. The approach to this we are suggesting is to make use of the pseudo-Einstein condition, as detailed in the following section.

5. Pseudo-Einstein Structures and Canonical Bundles

The first aim of this section will be to show that closed, non-zero sections of the canonical bundle are closely tied to pseudo-Einstein contact forms. Throughout this paper, M will be a 3-dimensional (CR) manifold. This means that M is $(2n+1)$-dimensional with $n = 1$. Since we work in 3 dimensions, we will make use of the $n = 1$ pseudo-Einstein condition, (14). We begin by recalling the volume normalization condition for sections of the canonical bundle. In 3 dimensions, we say that θ is *volume-normalized* with respect to a $(2,0)$-form ζ if

$$(44) \qquad \theta \wedge d\theta = i\theta \wedge (T \lrcorner \zeta) \wedge (T \lrcorner \bar{\zeta}).$$

Consider the following lemma.

Lemma 5.1. [13] *Given any smooth non-vanishing (2,0) form ζ on M, there exists a contact form θ that can be volume-normalized with respect to ζ.*

Proof. In 3 dimensions, we write our admissible coframe as $\{\theta, \theta^1, \theta^{\bar{1}}\}$. We may write an arbitrary $(2,0)$-form as $\zeta = \theta \wedge \theta^1$. Then recalling the properties of θ on the Reeb vector field T, we can write

$$T \lrcorner \zeta = \theta^1.$$

We now recall the first structure equation (8) in 3 dimensions

$$d\theta = ih_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}}.$$

Substituting the expression for $T \lrcorner \zeta$ in the above gives

$$d\theta = ih_{1\bar{1}}(T \lrcorner \zeta) \wedge (T \lrcorner \bar{\zeta})$$

Now simply wedging both sides of this equation with θ (possibly scaled by a smooth positive function) gives

$$\theta \wedge d\theta = i\theta \wedge (T \lrcorner \zeta) \wedge (T \lrcorner \bar{\zeta}).$$

This is exactly the volume-normalization condition (44). $\qquad\square$

We now move to show that a pseudo-Einstein contact form is equivalent to a particular 1-form being closed.

Lemma 5.2. [3] *On M, θ is pseudo-Einstein if and only if the 1-form $\omega_1{}^1 + iR\theta$ is closed.*

Proof. The two sided implication can be shown directly by computing $d(\omega_1{}^1 + iR\theta)$. From the relevant structure equation (12) with $n = 1$ we have

$$d\omega_1{}^1 = Rh_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}} + \nabla^1 A_{11}\theta^1 \wedge \theta - \nabla^{\bar{1}}A_{\bar{1}\bar{1}}\theta^{\bar{1}} \wedge \theta.$$

It follows that

$$\begin{aligned}
d(\omega_1{}^1 + iR\theta) &= Rh_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}} + \nabla^1 A_{11}\theta^1 \wedge \theta - \nabla^{\bar{1}}A_{\bar{1}\bar{1}}\theta^{\bar{1}} \wedge \theta \\
&\quad - Rh_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}} + i\nabla_1 R\theta^1 \wedge \theta + i\nabla_{\bar{1}}R\theta^{\bar{1}} \wedge \theta \\
&= i\nabla_1 R\theta^1 \wedge \theta + i\nabla_{\bar{1}}R\theta^{\bar{1}} \wedge \theta + \nabla^1 A_{11}\theta^1 \wedge \theta - \nabla^{\bar{1}}A_{\bar{1}\bar{1}}\theta^{\bar{1}} \wedge \theta \\
&= 2i\mathrm{Re}\,(\nabla_1 R\theta^1 \wedge \theta) - 2i\mathrm{Re}(i\nabla^1 A_{11}\theta^1 \wedge \theta) \\
&= 2i(\mathrm{Re}\,(\nabla_1 R - i\nabla^1 A_{11})\theta^1 \wedge \theta).
\end{aligned}$$

The last expression contains the pseudo-Einstein condition (14), and hence is equal to zero. Thus $d(\omega_1{}^1 + iR\theta) = 0$, giving our result. $\qquad\square$

Note that Hirachi has proven a stronger result, which says that an admissible coframe can be chosen such that $\omega_1{}^1 + iR\theta$ is itself 0 [10].

We can use Lemma 5.2 to show that a pseudo-Einstein contact form is locally volume normalized with respect to a closed section of the canonical bundle.

Theorem 5.3. [3] *θ is pseudo-Einstein if and only if, for every point $p \in M$, there exists a neighborhood $U \subseteq M$ containing p, where $θ$ is volume-normalized with respect to a closed section of the canonical bundle.*

Proof. We begin by choosing a coframe where $h_{1\bar{1}} = \delta_{1\bar{1}}$. We can write structure equation (8) as

$$(45) \qquad d\theta = i\theta^1 \wedge \theta^{\bar{1}}.$$

Now, for $p \in U \subseteq M$, let θ be volume-normalized with respect to a section of the canonical bundle, ζ. Since $\{\theta, \theta^1\}$ is a frame for $(1,0)$-forms and ζ is a $(2,0)$-form, we may write $\zeta = \lambda\theta \wedge \theta^1$ for some $\lambda \in C^\infty(M, \mathbb{C})$. Plugging ζ into the volume normalization condition (44) tells us that $|\lambda| = 1$, which importantly tells us that λ is not 0. This means that we can write $\zeta = \theta \wedge \theta^1$ by redefining θ^1 as $\lambda\theta^1$. Computing the exterior derivative of ζ gives

$$d\zeta = d\theta \wedge \theta^1 - \theta \wedge d\theta^1.$$

Using structure equations (8) and (9), we have

$$\begin{aligned}
d\zeta &= (i\theta^1 \wedge \theta^{\bar{1}}) \wedge \theta^1 - \theta \wedge (\theta^1 \wedge \omega_1{}^1 + \theta \wedge \tau^\beta) \\
&= -\theta \wedge \theta^1 \wedge \omega_1{}^1 \\
&= -\omega_1{}^1 \wedge \zeta.
\end{aligned}$$

Since ζ was closed we have that $\omega_1{}^1$ is a $(1, 0)$-form. Recall from (10) that $\omega_1{}^1$ is purely imaginary. This allows us to write $\omega_1{}^1 = iu\theta$ for some $u \in C^\infty(M, \mathbb{C})$. Taking the exterior derivative of $\omega_1{}^1$ gives

$$d\omega_1{}^1 = -u\theta^1 \wedge \theta^{\bar{1}} + i((\nabla_1 u)\theta^1 + (\nabla_{\bar{1}} u)\theta^{\bar{1}}) \wedge \theta.$$

Comparing this equation with structure equation (12) tells us that $-u = R$ and $i\nabla_1 u = \nabla^1 A_{11}$ which in turn means that $\nabla_1 R - i\nabla^1 A_{11} = 0$, making θ pseudo-Einstein.

Now to prove the converse implication, we assume that θ is pseudo-Einstein, and take our old coframe $\{\theta, \theta^1, \theta^{\bar{1}}\}$. We can define a section of the canonical bundle,

$$\zeta_0 = \theta \wedge \theta^1.$$

Using our previous computation of $d\zeta$, we know that

$$d\zeta_0 = -\omega_1{}^1 \wedge \zeta_0.$$

By Lemma 5.2, the 1-form $\omega_1{}^1 + iR\theta$ is closed and, so for some function ϕ we may write

$$\omega_1{}^1 + iR\theta = id\phi.$$

We can take ϕ to be real since $\omega_1{}^1$ is purely imaginary. This allows us to write

$$d(e^{i\phi}\zeta_0) = e^{i\phi}(\omega_1{}^1 + iR\theta) \wedge \zeta_0 - e^{i\phi}(\omega_1{}^1) \wedge \zeta_0 = 0.$$

Since θ is volume-normalized with respect to $e^{i\phi}\zeta_0$, we have our closed section of the canonical bundle. $\square$

Now, by considering Lemma 5.1 together with Theorem 5.3, we arrive at the following.

Theorem 5.4. [3] [14] *If M admits a closed, non-vanishing (2, 0)-form, then M admits a pseudo-Einstein structure. Conversely, if M admits a pseudo-Einstein structure, then in a neighborhood of every point, M admits a closed non-vanishing (2, 0)-form.*

Now note two important facts. First, the canonical bundle K_M of a CR manifold M embedded in $\mathbb{C}^2$ is equal to the canonical bundle of $\mathbb{C}^2$ restricted to M, and $dz_1 \wedge dz_2$ is a closed section of the canonical bundle of $\mathbb{C}^2$. Second, we recall that the exterior derivative is a natural operator. Thus if we pull back the section $dz_1 \wedge dz_2$ via the inclusion mapping to M, then the section $\iota^*(dz_1 \wedge dz_2) = \zeta$ is still closed.

This tells us that any CR manifold embedded in $\mathbb{C}^2$ admits a closed section of its canonical bundle. We are led to the following corollary.

Corollary 5.5. *If M is CR embeddable, then M admits a pseudo-Einstein structure.*

6. Conclusion

The work of Ivor Robinson and Andrzej Trautman has shed light on the intricate and surprising links between electromagnetic theory, general relativity, and CR geometry. The work of Roger Penrose on twistor theory was significantly influenced by Robinson's studies of SNG congruences.

While much of the work on Trautman's conjecture has been conducted with an eye toward the 4-dimensional Lorentz setting of general relativity, the approach put forth in this paper functions in 3 dimensions, and makes use of the pseudo-Einstein condition. Our reasoning allows us to conclude that locally, closed sections of the canonical bundle are equivalent to the admittance of a pseudo-Einstein structure. We also see that CR embeddability implies the admittance of a pseudo-Einstein structure. These findings provide the basis to posit that the pseudo-Einstein condition will be key to proving Trautman's conjecture. Let us denote our 3-dimensional CR manifold by M and a closed, non-vanishing section of the canonical bundle by ζ, so that we can summarize these conclusions in the following schematic:

$$(46) \quad \exists\, \zeta \neq 0 \in K_M \text{ closed} \iff M \text{ is pseudo-Einstein} \impliedby M \text{ is locally CR embeddable.}$$

The remaining implication would be to show that if M is pseudo-Einstein, then M is locally CR embeddable.

The next phase of this project will explore Fefferman defining functions coming from the complex Monge-Ampère equation and the arising Kähler-Einstein metrics [7]. We hope to connect the pseudo-Einstein condition with the boundary condition for a Kähler-Einstein extension problem on the pseudoconvex side of CR 3-dimensional manifolds.

References

[1] Olivier Biquard and Marc Herzlich. A Burns-Epstein invariant for ACHE 4-manifolds. *Duke Mathematical Journal*, 126(1):53 – 100, 2005.

[2] John S. Bland. Contact geometry and CR structures on S3. *Acta Mathematica*, 172(1):1 – 49, 1994.

[3] Jeffrey S. Case and Paul Yang. A Paneitz-type operator for CR pluriharmonic functions. *Bull. Inst. Math. Acad. Sin. (N.S.)*, 8(3):285–322, 2013.

[4] David W. Catlin and Sanghyun Cho. Extension of CR structures on three dimensional compact pseudoconvex CR manifolds. *Mathematische Annalen*, 334(2):253–280, 2006.

[5] Sean N. Curry and Peter Ebenfelt. Bounded strictly pseudoconvex domains in $\mathbb{C}^2$ with obstruction flat boundary II, 2018.

[6] S. Dragomir and G. Tomassini. *Differential Geometry and Analysis on CR Manifolds*. Progress in Mathematics. Birkhäuser Boston, 2007.

[7] Charles L. Fefferman. Monge-Ampere equations, the Bergman kernel, and geometry of pseudoconvex domains. *Annals of Mathematics*, 103(3):395–416, 1976.

[8] C. Denson Hill, Jerzy Lewandowski, and Pawel Nurowski. Einstein's equations and the embedding of 3-dimensional CR manifolds, 2008.

[9] Kengo Hirachi. Q-prime curvature on CR manifolds. *Differential Geom. Appl.*, 33(suppl.):213–245, 2014.

[10] Kengo Hirachi, Taiji Marugame, and Yoshihiko Matsumoto. Variation of total Q-prime curvature on CR manifolds. *Advances in Mathematics*, 306:1333–1376, 2017.

[11] Howard Jacobowitz. The canonical bundle and realizable CR hypersurfaces. *Pacific Journal of Mathematics*, 127(1):91 – 101, 1987.

[12] Howard Jacobowitz and Francois Treves. Non-realizable CR structures. *Inventiones mathematicae*, 66:231–250, 1982.

[13] John M. Lee. The Fefferman Metric and Pseudohermitian Invariants. *Trans. Amer. Math. Soc.*, 296(1):411–429, 1986.

[14] John M. Lee. Pseudo-Einstein structures on CR manifolds. *Amer. J. Math.*, 110(1):157–178, 1988.

[15] J Lewandowski, P Nurowski, and J Tafel. Einsteins equations and realizability of CR manifolds. *Classical and Quantum Gravity*, 7(11):L241–L246, nov 1990.

[16] Hans Lewy. An example of a smooth linear partial differential equation without solution. *Annals of Mathematics*, 66(1):155–158, 1957.

[17] L. Nirenberg. On a problem of Hans Lewy. *Russian Mathematical Surveys*, 459:224–234, 1975.

[18] Pawel Nurowski and Andrzej Trautman. Robinson manifolds as the Lorentzian analogs of Hermite manifolds. *Differential Geometry and its Applications*, 17, 02 2002.

[19] Jean-Pierre Rosay. New examples of non-locally embeddable CR structures (with no non-constant CR distributions). *Annales de l'Institut Fourier*, 39(3):811–823, 1989.

[20] J. Tafel. On the Robinson theorem and shearfree geodesic null congruences. *Letters in Mathematical Physics*, 10:33–39, 1985.

[21] Noboru Tanaka. *A differential geometric study on strongly pseudo-convex manifolds*. Kinokuniya Book-Store Co., Ltd., Tokyo, 1975. Lectures in Mathematics, Department of Mathematics, Kyoto University, No. 9.

[22] Andrzej Trautman. On complex structures in physics. In *On Einstein's path (New York, 1996)*, pages 487–501. Springer, New York, 1999.

[23] Andrzej Trautman. Robinson manifolds and Cauchy–Riemann spaces. *Classical and Quantum Gravity*, 19(2):R1–R10, jan 2002.

[24] S. M. Webster. On the transformation group of a real hypersurface. *Trans. Amer. Math. Soc.*, 231(1):179–190, 1977.

[25] S. M. Webster. Pseudo-Hermitian structures on a real hypersurface. *J. Differential Geometry*, 13(1):25–41, 1978.

Department of Mathematics, University of California, San Diego

Email address: ihajar@ucsd.edu