Advanced Algorithms Ex.2

Deadline: June 27

1 Tightness of Analysis

Metric TSP

Consider the following instance for the Metric TSP problem:

- Construct a graph $G = (V, E, w)$ which is a path of $n = 2k+1$ vertices with edge weight 1 (i.e., for each $1 \leq i < n : (v_i, v_{i+1}) \in E$ and $w(v_i, v_{i+1}) = 1$).
- For an $\varepsilon > 0$, augment G by adding edges with weights $1 + \varepsilon$ connecting vertices two steps apart (i.e., for each $1 \leq i < n - 1 : (v_i, v_{i+2}) \in E$ and $w(v_i, v_{i+2}) = 1 + \varepsilon$).
- Let $G' = (V, E)$ with metric d defined by shortest paths among each pair of vertices over G, i.e., $d(u, v) = \delta_G(u, v)$.

1. Show and compare the results of the two algorithms we discussed in class ("twice MST" and Christofides's) to the optimal solution's cost.
2. Describe a function $f(n)$ such that if $\varepsilon = f(n)$, then the analysis of Christofides's algorithm becomes tight as n grows.

Set Cover Problem

Consider the greedy algorithm described in class for the Set Cover problem, and show a family of instances (example for general n) such that the analysis is tight (the size of the greedy solution is $\Omega(|\mathrm{OPT}| \cdot \log \mathrm{n})$.

Hint: Design an instance where $|\mathrm{OPT}| = 2$ *and the greedy always pick other sets than* OPT

2 Approximations Algorithms & FPTAS

Bin Packing

Consider the following algorithm. Given an item, pack it in the last opened bin, if impossible, open a new bin. Denote the result of this scheme by ALG, and show that $|\mathrm{ALG}| \leq 2 \cdot |\mathrm{OPT}| - 1$.

Show that the analysis is tight.

One more Bin Packing for Dessert

Recall the rounding process we used in the bin-packing algorithm in class, for the case of items of size at least δ (where we created groups $G_1, \ldots, G_{\lceil \frac{n}{k} \rceil}$, deleted G_1 and made all the items in group G_i to equal the maximum). For input U, we created rounded input U' and proved $|\mathrm{OPT}(U')| \le |\mathrm{OPT}(U)| \le |\mathrm{OPT}(U')| + k$.

Suppose that instead we use a rounding process somewhat similar to the one used in the Knapsack problem for creating new instance U': Specifically, for each item $u \in U$, round the weight w_u to the nearest multiple of $\frac{\delta}{k}$ (for some parameter k that will be determined later). That is, for $u \in U$ with $w_u \in (\frac{i-1}{k}\delta, \frac{i}{k}\delta]$, create an item $u' \in U'$ with new weight of $\frac{i}{k}\delta$.

1. Prove $|\mathrm{OPT}(U)| \le |\mathrm{OPT}(U')|$.
2. What is the best upper bound you can prove on $|\mathrm{OPT}(U')|$ (and what k will you choose)?

   Can you use it to get $1 + \Theta(\varepsilon)$ approximation for $|\mathrm{OPT}(U)|$?

an alternative version for the question:

Recall the Bin-packing problem where all the items are of weight at least δ. Consider the following rounding algorithm: given

a set of items U, construct a new problem on a set of items U', where for every $u \in U$, we add an item u' to U' by rounding up to the closest multiple of $\frac{\delta}{k}$. That it if $w_u \in (\frac{i-1}{k} \cdot \delta, \frac{i}{k} \cdot \delta]$,

then we set $w_{u'} = \frac{i}{k} \cdot \delta$.

1. Prove that $|\mathrm{Opt}(U)| \le |\mathrm{Opt}(U')|$.
2. Prove the best upper bound you can on $|\mathrm{Opt}(U')|$. Specifically, show that $|\mathrm{Opt}(U')| \le t \cdot |\mathrm{Opt}(U)| + O(1)$ for t as small as you can (you can choose k).

   Can you get arbitrarily good approximation? Specifically, can you prove that for every $\varepsilon > 0$, there is a choice of k such that

   $|\mathrm{Opt}(U')| \le (1 + \varepsilon) \cdot |\mathrm{Opt}(U)| + O(1)$?