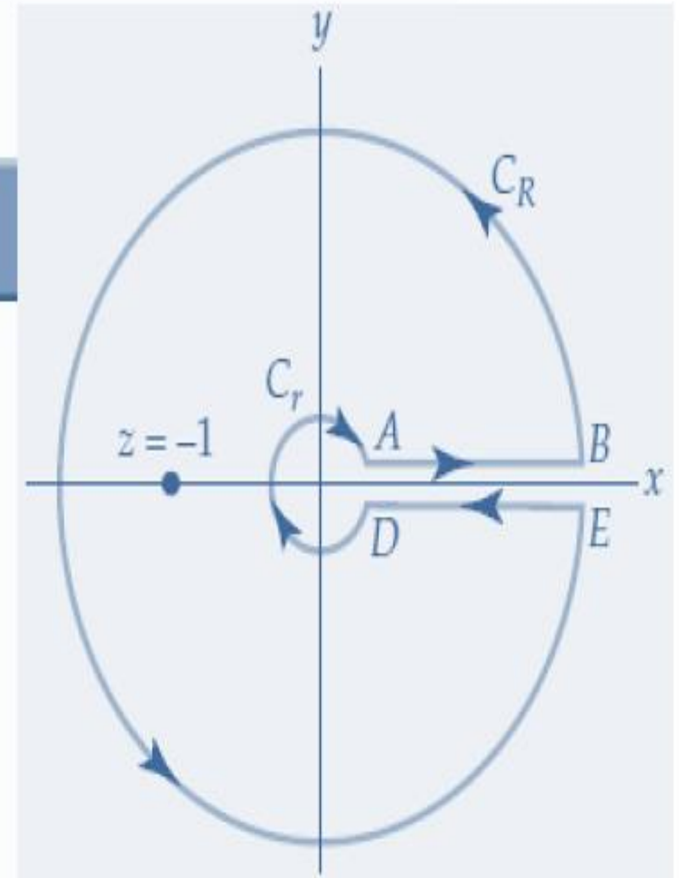
Complex Integration



Cauchy's Integral Formula

Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0,

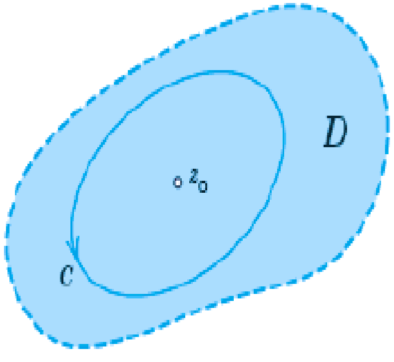
$$\oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0) \tag{1}$$

(Cauchy's integral formula)

the integration being taken counterclockwise. Alternatively (for representing $f(z_0)$ by a contour integral, divide (1) by $2\pi i$),

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz \tag{1*}$$

(Cauchy's integral formula).



Examples:

1. Evaluate $\oint_C \frac{z^2-4z+4}{z+i}\,dz$, where C is the circle $|z|=2$.

Solution First, we identify $f(z)=z^2-4z+4$ and $z_0=-i$ as a point within the circle C. Next, we observe that f is analytic at all points within and on the contour C. Thus, by the Cauchy integral formula (1) we obtain

$$\oint_C \frac{z^2-4z+4}{z+i}\,dz = 2\pi i f(-i) = 2\pi i(3+4i) = \pi(-8+6i).$$

2. Evaluate $\oint_C \frac{z^3-6}{2z-i}dz$, where C is any contour enclosing $z_0=\frac{i}{2}$.

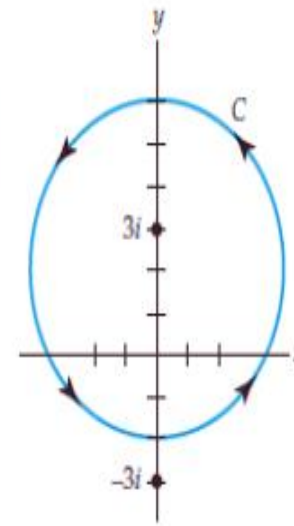
$$\oint_C \frac{z^3-6}{2z-i}dz = \oint_C \frac{\frac{1}{2}z^3-3}{z-\frac{1}{2}i}dz$$

$$= 2\pi i[\tfrac{1}{2}z^3-3]|_{z=i/2}$$

$$= \frac{\pi}{8} - 6\pi i$$

Examples:

3. Evaluate $\oint_C \frac{z}{z^2+9}\,dz$, where C is the circle $|z-2i|=4$.



Solution By factoring the denominator as $z^2+9=(z-3i)(z+3i)$ we see that $3i$ is the only point within the closed contour C at which the integrand fails to be analytic. Then by rewriting the integrand as

$$\frac{z}{z^2+9}=\frac{\left.\dfrac{z}{z+3i}\right\}f(z)}{z-3i},$$

we can identify $f(z)=z/(z+3i)$. The function f is analytic at all points within and on the contour C. Hence, from Cauchy's integral formula (1) we have

$$\oint_C \frac{z}{z^2+9}\,dz=\int_C \frac{\dfrac{z}{z+3i}}{z-3i}\,dz=2\pi i\,f(3i)=2\pi i\frac{3i}{6i}=\pi i.$$

Cauchy's integral formula for derivatives.

If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, which are then also analytic functions in D. The values of these derivatives at a point z_0 in D are given by the formulas

$$(1') \qquad f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2}\, dz$$

$$(1'') \qquad f''(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^3}\, dz$$

and in general

$$(1) \qquad \boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz} \qquad (n = 1, 2, \cdots);$$

here C is any simple closed path in D that encloses z_0 and whose full interior belongs to D; and we integrate counterclockwise around C.

Example:

Evaluate $\oint_C \frac{z+1}{z^4+2iz^3}\,dz$, where C is the circle $|z|=1$.

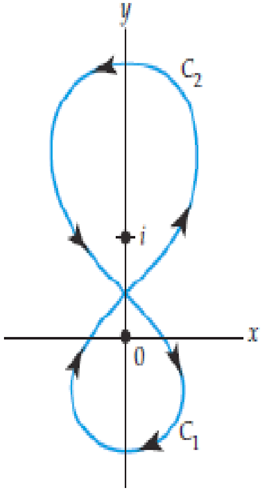
Solution Inspection of the integrand shows that it is not analytic at $z=0$ and $z=-2i$, but only $z=0$ lies within the closed contour. By writing the integrand as

$$\frac{z+1}{z^4+2iz^3}=\frac{\dfrac{z+1}{z+2i}}{z^3}$$

we can identify, $z_0=0$, $n=2$, and $f(z)=(z+1)/(z+2i)$. The quotient rule gives $f''(z)=(2-4i)/(z+2i)^3$and so $f''(0)=(2i-1)/4i$. Hence we find

$$\oint_C \frac{z+1}{z^4+4z^3}\,dz=\frac{2\pi i}{2!}f''(0)=-\frac{\pi}{4}+\frac{\pi}{2}i.$$

Practice Questions:



- Evaluate $\int_C \frac{z^3+3}{z(z-i)^2}dz$, where C is the contour shown in figure.

- Evaluate $\int_C \frac{e^z}{(z-1)^2(z^3+4)}dz$, for any positive oriented contour C for which 1 is interior to C while $\pm 2i$ lie outside the contour.

- Evaluate $\int_C \frac{z^2+1}{z^2-1}dz$, where the contour C is: (i) $|z-1| = 1$, (ii) $|z+1| = 1$.

- Evaluate $\int_C \frac{1}{z^2-4}dz$, where the contour C is: (i) $|z| = 1$, (ii) $|z-1| = 2$, (i) $|z+1| = 2$.