Functional Iteration

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Abstract

In this brief project, we delve into the mathematical process of functional iteration, focusing on $y = \frac{x+2}{x+1}$ and the associated sequence s_n. Through mathematical experimentation, we investigate the convergence behavior of the iterations, the influence of the starting value on sequence behavior, and the identification and classification of fixed points. Building on these findings, we propose hypotheses, subsequently testing it against new data.

1 Introduction and main results

Functional iteration is the mathematical process where a numerical sequence arises from the repeated evaluation of a function $y = f(x)$, using the results of y at one stage as the argument x of the next. For example, consider the following function:

$$y = \frac{x+2}{x+1}.$$

Functional iteration involves starting with an initial value x_0, and then applying the function iteratively, thus obtaining the sequence of values $x_0, x_1, x_2, \ldots$. In this example, if you choose $x = 2$, you get $y = \frac{4}{3} \approx 1.3333$, then $x = \frac{4}{3}$ gives $y = \frac{10}{7} \approx 1.429$, and so on, producing the sequence s_n defined by letting

$$s_1 = 2 \quad \text{and} \quad s_{n+1} = \frac{s_n + 2}{s_n + 1} \quad \text{for} \quad n = 1, 2, 3, \ldots.$$

We will now analyse what this sequence converges to, and what this looks like diagrammatically using R.

Using the following code in R, we get the following cobweb plot (Figure 1).

```r
# our starting value in this example is 2, we will explore what will
happen when we change this value, e.g s <- 5, or s <- -5
s <- 2
for (i in 1:9) {
  x <- s[i]
  y <- (x + 2) / (x + 1)
  s <- c(s, y)
}
plot(s)

dev.new() # new figure

# plot y = x and y = (x + 2) / (x + 1)

U <- seq(min(s) - 0.1, max(s) + 0.1, by = 0.01)
plot(U, U, type = "l", xlab = "x", ylab = "y")
lines(U, (U + 2) / (U + 1))

# draw cobweb plot on the same figure

z <- rep(NA, 18)
```

```r
w <- rep(NA, 18)
for (i in 1:9) {
  z[2 * i - 1] <- s[i]
  w[2 * i - 1] <- s[i]
  z[2 * i] <- s[i]
  w[2 * i] <- s[i + 1]
}
lines(z, w)
```

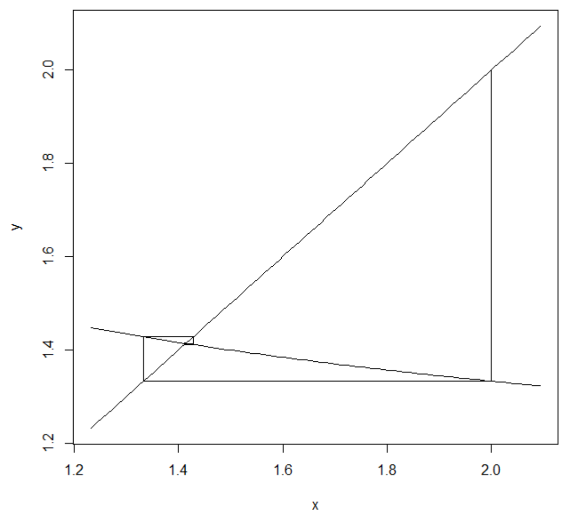


Figure 1: Cobweb plot for $s = 2$

The cobweb plot depicts the curves $y = x$ and $y = f(x)$, along with the straight lines from (s_n, s_n) to (s_n, s_{n+1}) and from (s_n, s_{n+1}) to (s_{n+1}, s_{n+1}). The significance lies in the last plot, providing insights into whether the sequence converges. In a cobweb plot, convergence to a limit L can be observed by examining the trajectory's behavior through iterations. Questions arise: Is there a repeated pattern? Are the spirals tightening? Is the trajectory consistently approaching a single point? These are indicators of convergence. If (s_n) converges to a limit L, then (s_{n+1}) has the same limit, and therefore

$$L = \frac{L+2}{L+1},$$

which we can solve for L. In this case, $L = \sqrt{2}$.

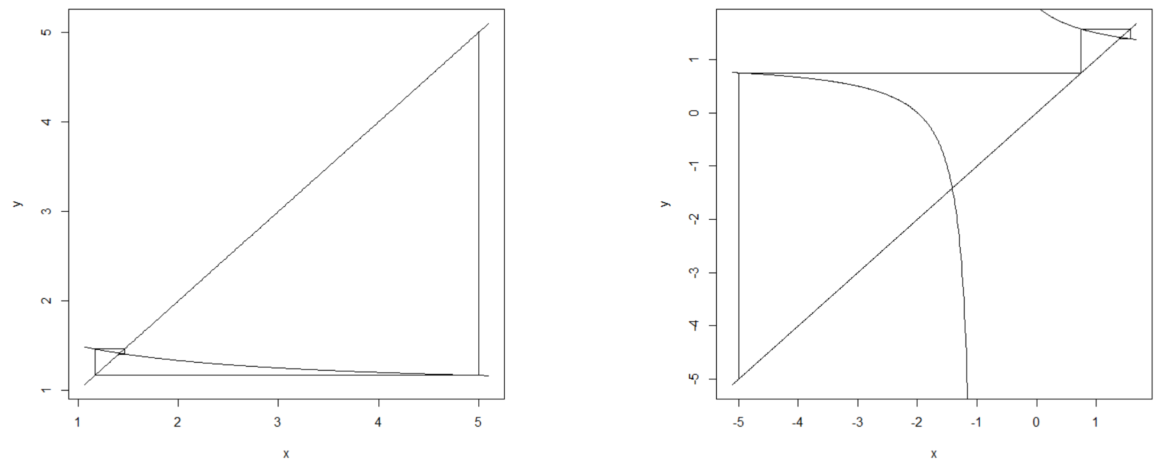
Alternatively, this result could be obtained by performing the iteration process 13 times. For example, on any calculator, setting ans = 2 and executing (ans+2)/(ans+1) it would take 13 executions to reach our desired value.

A solution of the equation $p = f(p)$ is called a fixed point of the function f. A limit point l of the iteration must also be a fixed point, but a fixed point is not necessarily a limit point. There may be one limit point, several (reached from different starting points), or none at all. Similarly, there may be one fixed point, more than one, or none.

If p is a fixed point and you start the sequence with $s_1 = p$, then the sequence will remain at p. For example, $\left(\frac{\sqrt{2}+2}{\sqrt{2}+1}\right) = \sqrt{2}$, and therefore $\sqrt{2}$ is a fixed point.

A fixed point p is stable (or attracting) if, starting from any value sufficiently close to p, the sequence converges to p. A fixed point p is unstable (or repelling) if the sequence diverges when starting from any value sufficiently close, but not equal, to p. For the example where $s = 2$, the sequence values were 1.3333, 1.429, and 1.412 (3dp), getting closer to $\sqrt{2} \approx 1.414$ (3dp), indicating that our fixed point is stable.

Now, we will analyse the starting points of 5 and -5 using R to generate their cobweb plots.



(a) Cobweb plot for $s = 5$ (b) Cobweb plot for $s = -5$

Figure 2: Cobweb plots for different values of s

From this, we can see that when you perform iterative calculations starting from different initial values ($x_0 = 5$ and $x_0 = -5$), the sequence of values $x_0, x_1, x_2, \ldots$ eventually converges to the fixed point $\sqrt{2}$. However, we can clearly see that the cobweb plots look drastically different, demonstrating that different starting points affect sequential behavior.

Due to the different starting points, the trajectory of the sequence in the cobweb plot will follow different paths. This can be explained as follows:

For (a) $x_0 = 5$, the initial value is relatively far from the fixed point, and the trajectory in the cobweb plot might show a more pronounced initial divergence before converging to $\sqrt{2}$.

For (b) $x_0 = -5$, the initial value is also relatively far from the fixed point but on the opposite side.

The key factor is the stability of the fixed point $\sqrt{2}$, which influences how quickly and smoothly the sequence converges to it.

However, the convergent rate, due to the initial values, change. Note, when the iteration is carried out 13 times, no matter the input, the output will be the fixed point. We must analyse convergence rate by using another route, for example, take the fixed point to be 1.41(2dp). When $x = 2$, the 3rd value of the sequence is 1.41(2dp). When $x = -5$, the 5th value is 1.41, meaning it took 2 further iterations to reach the desired amount. We therefore conclude that different starting points have different convergent rates.

2 Special Cases

- When $x = -1$, the denominator of the function is 0. Division by 0 is undefined, and hence we cannot divide by 0. Any attempts, either on R (which will give us an error and no cobweb plot) or on a calculator, will simply not have an output and give an error code. If we take $x = -1.5$, the second value will be -1, and again, an error will be outputted.
- When $x = -2$, the numerator is 0. However, the next iterative value will be 2, and the sequence will eventually converge to $\sqrt{2}$ as normal.
- Any other negative value, where -1 is not part of the sequence, will converge to the fixed point.

3 Analysis and Questions

We will now answer the following questions:

1. Why does this sequence converge? If it didn't, why not?

The reason our function converges is that it exhibits stable behavior around a fixed point. We solved the equation to find the limit and stated that the limit point of the iteration must also be a fixed point. You could also analyse the behavior of a function through brute force on a calculator, performing multiple numerical iterations with different initial values (we showed 3 examples), and they would all converge to the same fixed point, even if they use different routes to get to the fixed point. Sequences will therefore not converge if we start the iteration from certain initial values, that may lead to divergence or undefined outputs, or if the sequence demonstrates periodic orbits or chaotic behaviour.

2. How does this behavior depend on various parameters?

The convergence behavior of an iterative function depends on various parameters, and understanding these dependencies is crucial in analysing the stability and dynamics of the system. The starting point can significantly impact the convergence. Different initial conditions may lead to different convergence patterns, such as convergence to a fixed point, periodic orbit, or chaotic behavior. The function form is also very important. Some functions converge to a fixed point for any initial condition, while others may exhibit complex behavior, such as oscillations, periodic orbits, or chaos.

For example, consider the function $\frac{X-2}{X}$. The values of s_n change signs numerous times, oscillate drastically, and do not converge to a single value. The key factor here is the presence of the term X in the denominator, which introduces oscillations and prevents the sequence from settling down to a stable value.

4 Hypotheses

From analysing the first function, we make the following hypotheses and will test these against a new function.

Hypothesis 1: Regardless of the initial value, the iterative function exhibits stable convergence to a fixed point.

Hypothesis 2: Initial values change convergent rate.

Hypothesis 3: The iterative function is sensitive to initial conditions, and variations in the starting point can lead to different convergence patterns (cobweb plots).

Consider the following function:

$$y = \frac{x-2}{x-7}$$

First, let's use a starting point where x does not equal 7. For example, let $x_0 = 1$. Using the definition of s_n stated before, we get:

$$s_n = 1, \frac{1}{6}, \frac{11}{41}, \ldots, 4-\sqrt{14} \quad \left(4-\sqrt{14} = 0.2583426132\right)$$

Now, to test Hypothesis 1, take another x_0, for example, -10, and see whether s_n converges to $4-\sqrt{14}$:

$$s_n = -10, \frac{12}{17}, \frac{22}{107}, \ldots, 4-\sqrt{14}$$

Despite using different starting points, both initial values lead the sequence to diverge to the fixed point $4-\sqrt{14}$. Therefore, we accept Hypothesis 1: Regardless of the initial value, the iterative function will converge to a stable point.

We will now also check if $L = 4-\sqrt{14}$:

$$L = \frac{L-2}{L-7}$$

$$L^2 - 7L = L - 2$$

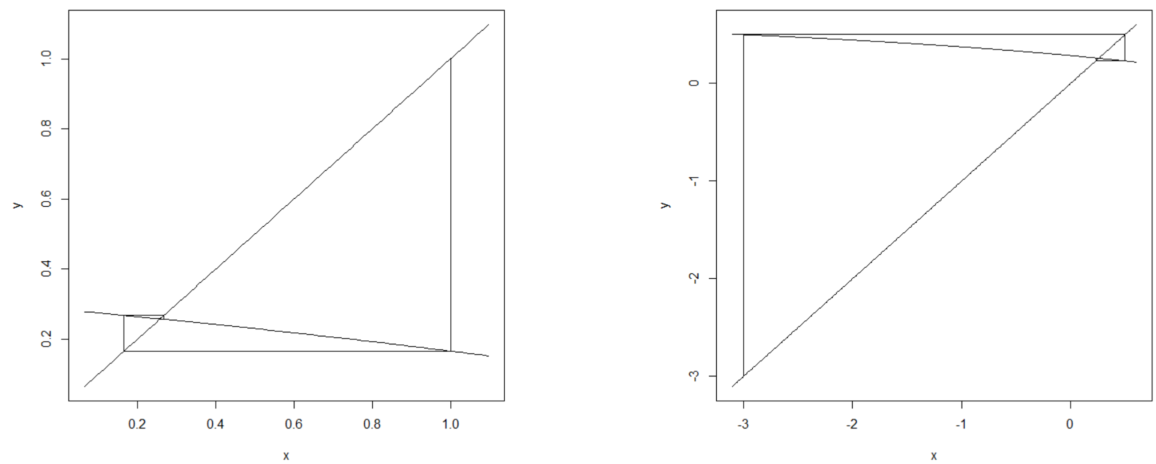
$$L^2 - 8L + 2 = 0$$
$$L = 4 \pm \sqrt{14}$$

We take $L = 4 - \sqrt{14}$. This choice is often based on the stability of the fixed points. In some cases, one of the roots might be stable (attracting), while the other is unstable (repelling).

To test Hypothesis 2 and analyse the rate of convergence, let $L = 0.26$. We will now see how many iterations it takes to reach this value using different starting points. We will use the following initial values: $x_0 = 1$ and $x_0 = -20$. Using brute force on a calculator, when $x_0 = 1$, it took 3 iterations to reach 0.26 (2dp). When $x_0 = -20$, it took 4 iterations, demonstrating different rates of convergence. Therefore, we accept Hypothesis 2: Initial values do change convergent rates.

To test our third and final hypothesis, we will analyse 2 different starting points and see how this changes the convergence pattern using cobweb plots.

Our new initial values will be $x_0 = 1$ and $x_0 = -3$. Inputting these values into an updated version of our R code for the new function, we get the following cobweb plots:



(a) Cobweb plot for $s = 1$

(b) Cobweb plot for $s = -3$

Figure 3: Cobweb plots for different values of s

From Figure 3, comparing plot a and b, we can clearly see the convergence patterns are extremely different. The reason for this is different initial values lead to different trajectories. While the initial value of 1 would need to decrease in value, -3 would need to increase to reach the fixed point. -3 is also further away from the fixed point, hence having to take a different convergent pattern to reach the fixed point. Thus, we also accept Hypothesis 3: Different initial conditions can lead to different convergent patterns.

5 Conclusion and Biblography

Our overall findings emphasise the importance and dependence of the convergence behavior on various parameters, including the initial value and the stability of fixed points. Several hypotheses were formulated and tested to understand the dynamics of functional iteration.

Dmitry Korshunov: MATH240 Individual Project: Functional Iteration
Lancaster University: MATH114 Notes Extract 6.1 - 6.3
Iterated Functions: Tom Davis
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