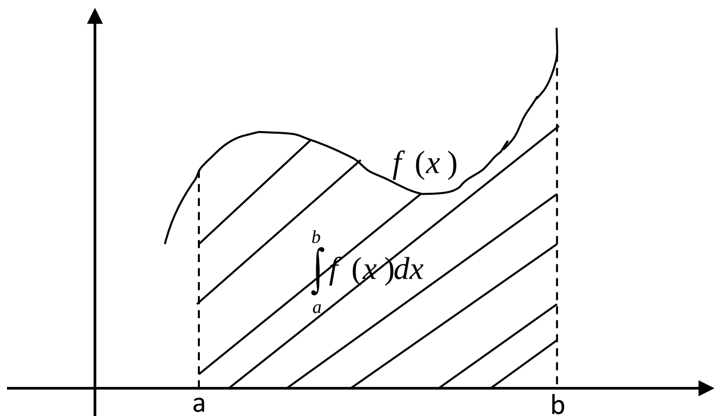
Chapter Two

Numerical Integration

❶ We wish to approximate the definite integral:

$$I = \int_a^b f(x)dx \longrightarrow (1)$$

Where $f(x)$ is a continuous function in $[a,b]$, a < b and **a**, **b** are finite real numbers.



$f(x)$ is called the integrand.

a = lower limit.

b = upper limit.

Definition: Integration is the process of measuring area under a function plotted on a graph.

There are a number of techniques for numerical integration.

❷ **Newton – Cotes Formulas:**

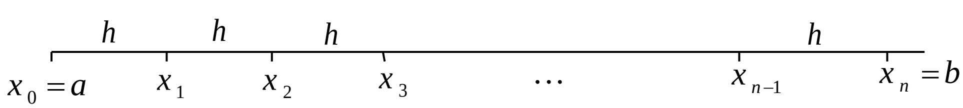
These are a group of formulas that approximate the definite integral (1), using polynomial interpolation.

■ **General Description:**

(i) Subdivide the interval $(b-a)$ into n ($n \geq 1$) equal parts each of length

$h = \dfrac{b-a}{n}$ which is called the step size, using the (n+1) points:

$x_0 = a, x_1, x_2, \ldots, x_n = b$



$$x_0 = a \quad , \quad x_1 = a+h \, , x_2 = a+2h \; , \; x_3 = a+3h \, , \ldots , x_i = a+ih \, , \ldots , x_n = a+nh = b$$

(ii) Evaluate the function f(x) at the division points to get the data function:

x	$x_0=a$	x_1	x_2	$\cdots$	x_{n-1}	$x_n=b$
f(x)	$f(x_0)$	$f(x_1)$	$f(x_2)$	$\cdots$	$f(x_{n-1})$	$f(x_n)$

(iii) Replace the function to be integrated f(x), by a polynomial $p_n(x)$.

(iv) Integrate $p_n(x)$ and take $\int_a^b p(x)dx$ to be the required approximation to $\int_a^b f(x)dx$, and use Lagrange interpolating polynomial for this purpose.

Hence, $f(x) \approx p(x) \Rightarrow \int_a^b f(x)dx \approx \int_a^b p(x)dx = \int_a^b \left(\sum_{i=1}^{n} L_i(x)\,\mathrm{f}(\mathrm{x}_i) \right) dx$

$$= \sum_{i=0}^{n} \left(\int_a^b L_i(x)dx \right) \mathrm{f}(\mathrm{x}_i)$$

$$\boxed{\int_a^b f(x)dx \approx \sum_{i=0}^{n} A_i f(x_i)} \longrightarrow (2)$$

This is called **Newton- Cotes formula of degree n** for approximating the definite integral $\int_a^b f(x)dx$, and A_i are called the weighting coefficients (the weights) and are given by $A_i = \int_a^b L_i(x)dx$.

❸ **Result:** Newton – Cotes formula of degree n is given by:

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

Where : $$\boxed{A_i = \frac{(-1)^{n-i}\,h}{i!(n-i)!} \int_0^n \frac{u(u-1)(u-2)\cdots(\mathrm{u}-\mathrm{n})}{u-i}\,du}$$

are called the weights. They derived from Lagrange polynomials, which means they depend only on the x_i and not on the function $f(x)$.

Proof:

∗ Since $L_i(x) = \dfrac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$

$$\because, \quad A_i = \int_a^b L_i(x)\,dx.$$

$$A_i \int_a^b \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}dx \;---\to (1)$$

∗ Now, let: $\boxed{x = a+uh} \quad \Rightarrow \quad dx = h\,du$

$x = a \quad \Rightarrow \quad u = 0 \quad and$

$x = b \quad \Rightarrow \quad u = n \quad (b = a+uh \quad \Rightarrow b-a = uh \Rightarrow uh = nh \quad \Rightarrow u = n).$

∗ $a = x_0 \quad \Rightarrow \quad (x-x_0) = uh.$

$(x-x_1) = (x-x_0)-(x_1-x_0) = uh-h = (u-1)h.$

$(x-x_2) = (u-2)h.$

$\vdots$

$(x-x_i) = (\mathrm{u}-(\mathrm{i}-1))\,\mathrm{h}.$

$(x-x_{i+1}) = (\mathrm{u}-(\mathrm{i}+1))\,\mathrm{h}.$

$\vdots$

$(x-x_n) = (u-n)h.$

$$\therefore, \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(x-x_i)} = \frac{uh.(u-1)h.(u-2)h\cdots(u-n)h}{(u-i)h}$$

$$= \frac{u(u-1)(u-2)\cdots(u-n)h^n}{(u-i)} \quad ---\to (\mathrm{I})$$

∗ $(x_i-x_0) = ih.$

$(x_i-x_1) = (x_i-x_0)-(x_1-x_0) = \mathrm{ih}-\mathrm{h} = (\mathrm{i}-1)\,\mathrm{h}.$

$(x_i-x_2) = (x_i-x_0)-(x_2-x_0) = \mathrm{ih}-2\,\mathrm{h} = (\mathrm{i}-2)\,\mathrm{h}.$

$\vdots$

$(x_i-x_{i-1}) = (x_i-x_0)-(x_{i-1}-x_0) = \mathrm{ih}-(\mathrm{i}-1)\,\mathrm{h} = (\mathrm{i}-(\mathrm{i}-1))h = h.$

$\therefore, \; (x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1}) = ih.(i-1)h.(\mathrm{i}-2)\,\mathrm{h}\cdots 2h.h = i\,!h^i \quad ---\to (II)$

∗ $(x_i-x_{i+1}) = (x_i-x_0)-(x_{i+1}-x_0) = ih-(i+1)h = (i-(i+1))h = -h.$

$(x_i-x_{i+2}) = (x_i-x_0)-(x_{i+2}-x_0) = ih-(i+2)h = (i-(i+2))h = -2h.$

$\vdots$

$(x_i-x_n) = (x_i-x_0)-(x_n-x_0) = \mathrm{ih}-\mathrm{nh} = (\mathrm{i}-\mathrm{n})\,\mathrm{h} = -(\mathrm{n}-\mathrm{i})\,\mathrm{h}.$

$$(x_i - x_{i+1})(x_i - x_{i+2})\cdots(x_i - x_n) = (-\mathrm{h})(-2\,\mathrm{h})(-3\mathrm{h})\cdots(-(\mathrm{n}-\mathrm{i})\,\mathrm{h})$$
$$= (-1)^{n-i}(n-i)!h^{n-i} \quad \longrightarrow (III)$$

From (I) , (II) , (III), we get:

$$L_i(x) = \frac{u(u-1)(u-2)\cdots(u-n)h^n}{(u-i)} \times \frac{1}{[i!h^i][(-1)^{n-i}(n-i)!h^{n-i}]}$$
$$= \frac{(-1)^{n-i}}{i!(n-i)!}\left(\frac{u(u-1)(u-2)\cdots(u-n)}{(u-i)}\right)$$

$$\therefore\ , \quad A_i = \frac{(-1)^{n-i}h}{i!(n-i)!}\int_0^n \frac{u(u-1)(u-2)\cdots(\mathrm{u}-\mathrm{n})}{u-i}\,du$$

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❹ The Trapezoid Method(Linear):

The trapezoid method is the Newton – Cotes method of degree one, that is n = 1 (two points).

$$\int_a^b f(x)dx \approx \sum_{i=0}^{1} A_i f(x_i) = \mathrm{A}_0 f(x_0) + \mathrm{A}_1 f(x_1) \quad \dashrightarrow (1)$$

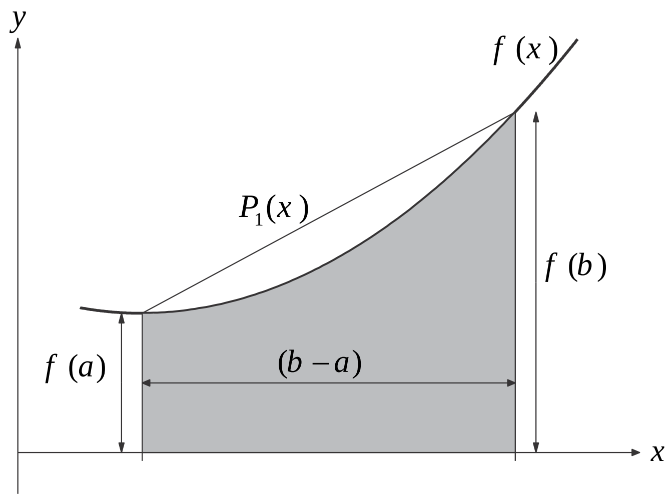
$$* \quad \mathrm{A}_0 = \frac{(-1)^{1-0}h}{0!(1-0)!}\int_0^1 \frac{u(u-1)}{u-0}\,du = -h\int_0^1 (u-1)du = -h\left[\frac{u^2}{2} - u\right|_0^1$$
$$= -h\left[\frac{1}{2} - 1\right] = \frac{h}{2} \quad \Rightarrow \boxed{A_0 = \frac{h}{2}}$$

$$* \quad \mathrm{A}_1 = \frac{(-1)^{1-1}h}{1!(1-1)!}\int_0^1 \frac{u(u-1)}{u-1}\,du = h\int_0^1 u\,du = h\left[\frac{u^2}{2}\right|_0^1$$
$$= h\left[\frac{1}{2} - 0\right] = \frac{h}{2} \quad \Rightarrow \boxed{A_1 = \frac{h}{2}}$$

$$\int_a^b f(x)dx \approx \frac{h}{2}f(x_0) + \frac{h}{2}f(x_1) = \frac{h}{2}\left[f(x_0) + f(x_1)\right]$$

Therefore, $\boxed{\int_a^b f(x)dx \approx \frac{(b-a)}{2}\left[f(a) + f(b)\right]}$

This is called the **Simple** trapezoid Rule.



Simple Trapezoid Method

The trapezoid rule can also be derived from geometry. From the above figure, the area under the curve $P_1(x)$ is a trapezoid, so:

$$\int_a^b f(x)dx \approx Area\ of\ \ trapezoid\,.$$

$$= \frac{1}{2}\ \text{[sum of parallel sides]} \times \text{(height)}.$$

$$= \frac{1}{2}[f(a)+f(b)]\times(b-a) = (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

$$= \frac{(b-a)}{2}[f(a)+f(b)].$$

If $f(x)$ is itself a polynomial of degree one, then the trapezoid rule will give the exact value of $\int_a^b f(x)dx$, in this case we say its degree of precision is one.

Examples:

(1) Using the trapezoid rule approximate the integral: $\int_0^1 \frac{dx}{x+1}$ and find the error.

Solution:

$$\because,\ f(x)=\frac{1}{x+1}\quad,\quad a=0\ ,\ b=1$$

$$and\quad \int_a^b f(x)dx \approx \frac{(b-a)}{2}\left[f(a)+f(b)\right].$$

$$\Rightarrow \int_0^1 \frac{1}{x+1}dx \approx \frac{(1-0)}{2}\left[f(0)+f(1)\right]=\frac{1}{2}\left[\frac{1}{0+1}+\frac{1}{1+1}\right]$$

$$=\frac{1}{2}\left[1+\frac{1}{2}\right]=\frac{1}{2}\left(\frac{3}{2}\right)=\frac{3}{4}=0.7500$$

$$*\quad \therefore,\quad \boxed{I_t \approx 0.7500}$$

$$*\quad I_{exact}=\left[\ln(x+1)\right|_0^1=\ln(1+1)-\ln(1+0)=\ln 2-\ln 1=\ln 2-0=\ln 2$$

$$\therefore,\quad \boxed{I_{exact}=0.6931}$$

$$*E_t=\left|I_t-I_{exact}\right|=\left|0.7500-0.6931\right|=0.0569 \Longrightarrow \boxed{E_t=0.0569}$$

(2) Using the trapezium method approximate the integral:

$$I=\int_1^3 (x^3-2x^2+7x-5)dx$$

and find the absolute error.

<u>Solution:</u>

$$\because,\ f(x)=x^3-2x^2+7x-5\quad,\quad a=1\ ,\ b=3$$

$$and\quad \int_a^b f(x)dx \approx \frac{(b-a)}{2}\left[f(a)+f(b)\right].$$

$$\Rightarrow \int_1^3 (x^3-2x^2+7x-5)dx \approx \frac{(3-1)}{2}\left[f(1)+f(3)\right]$$

$$f(1)=(1)^3-2(1)^2+7(1)-5=1\quad \&\quad f(3)=(3)^3-2(3)^2+7(3)-5=25$$

$$\Rightarrow \int_1^3 (x^3-2x^2+7x-5)dx \approx \frac{(3-1)}{2}\left[1+25\right]=26$$

$$\therefore,\quad \boxed{I_t \approx 26}$$

* $I_{exact} = \left[\frac{x^4}{4} - \frac{2}{3}x^3 + \frac{7}{2}x^2 - 5x \right|_1^3 = 20\frac{2}{3}.$

* $E_t = \left| I_t - I_{exact} \right| = \left| 26 - 20\frac{2}{3} \right| = 5\frac{1}{3}.$

(3) Use the trapezium rule to calculate approximately the value of $\int_1^2 f(x)dx$

where $f(x)$ is given by the data:

x	1	2
f(x)	3	4

Solution:

$$\because, \quad \int_a^b f(x)dx \approx \frac{(b-a)}{2}\left[f(a) + f(b)\right].$$

$$\int_1^2 f(x)dx \approx \frac{(2-1)}{2}\left[f(1) + f(2)\right] = \frac{1}{2}\left[3+4\right] = \frac{7}{2} = 3.5000$$

(4) Approximate each of the following integrals using the simple trapezium rule: $(a) \quad I = \int_0^{\pi/4} \sin x dx \qquad (b) \quad I = \int_1^2 e^{-x^2/2} dx$

Solution:

$$(a) \quad I = \int_0^{\pi/4} \sin x dx \approx \frac{(b-a)}{2}\left[\sin a + \sin b\right] = \frac{1}{2}\left(\frac{\pi}{4} - 0\right)\left[0 + \frac{1}{\sqrt{2}}\right] = \frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\pi}{8\sqrt{2}} = 0.27768$$

$$(b) \quad I = \int_1^2 e^{-x^2/2} dx \approx \frac{(b-a)}{2}\left[e^{-a^2/2} + e^{-b^2/2}\right] = \frac{1}{2}(2-1)\left[e^{-1/2} + e^{-2}\right] = 0.37093$$

Note: One deficiency in the simple trapezoid rule is that there is nothing we can do to improve it. The general trapezoid method uses the simple trapezoid in away that allows us to improve the accuracy of the answer we get.

Exercise: Approximate the following integrals by the simple trapezoid method:

$(i)\ \int_1^5 \sqrt{x}\,dx \quad [ans: 6.47214]$ $\qquad (ii)\ \int_1^2 \ln x\,dx \quad [ans: 0.34657]$

$(iii)\ \int_0^1 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \quad [ans: 0.3206].$

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❺ Simpson's Rule (Parabolic):

Simpson's rule is the Newton – Cotes method of degree two, that is n = 2 (three points).

$$\int_a^b f(x)dx \approx \sum_{i=0}^{2} A_i f(x_i) = \mathrm{A}_0 f(x_0) + \mathrm{A}_1 f(x_1) + \mathrm{A}_2 f(x_2) \quad --\rightarrow (1)$$

$$* \quad \mathrm{A}_0 = \frac{(-1)^{2-0} h}{0!(2-0)!}\int_0^2 \frac{u(u-1)(\mathrm{u}-2)}{u-0}du = \frac{h}{2}\int_0^2 (u^2-3u+2)du = \frac{h}{2}\left[\frac{u^3}{3} - \frac{3u^2}{2} + 2u\right]_0^2$$

$$= \frac{h}{2}\left[\frac{8}{3} - \frac{12}{2} + 4\right] = \frac{h}{2}\left[\frac{8}{3} - 2\right] = \frac{h}{2}\left[\frac{2}{3}\right] \Rightarrow \boxed{A_0 = \frac{h}{3}}$$

$$* \quad \mathrm{A}_1 = \frac{(-1)^{2-1} h}{1!(2-1)!}\int_0^2 \frac{u(u-1)(\mathrm{u}-2)}{(u-1)}du = -h\int_0^2 (u^2-2u)du = -h\left[\frac{u^3}{3} - u^2\right]_0^2$$

$$= -h\left[\frac{8}{3} - 4\right] = -h\left[\frac{-4}{3}\right] \quad \Rightarrow \boxed{A_1 = \frac{4h}{3}}$$

$$* \quad \mathrm{A}_2 = \frac{(-1)^{2-2} h}{2!(2-2)!}\int_0^2 \frac{u(u-1)(\mathrm{u}-2)}{(u-2)}du = \frac{h}{2}\int_0^2 (u^2-u)du = \frac{h}{2}\left[\frac{u^3}{3} - \frac{u^2}{2}\right]_0^2$$

$$= \frac{h}{2}\left[\frac{8}{3} - 2\right] = \frac{h}{2}\left[\frac{8-6}{3}\right] = \frac{h}{3}.$$

$$\therefore, \quad \boxed{A_2 = \frac{h}{3}}$$

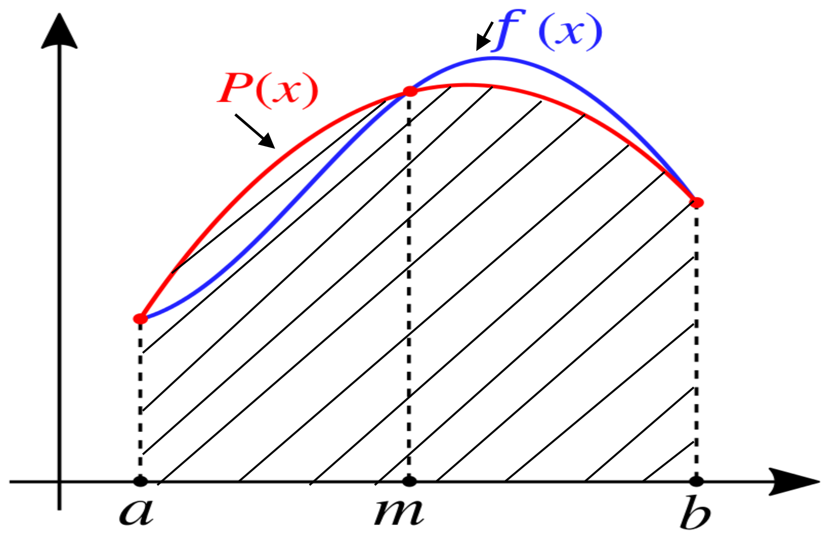
Therefore, the formula (1) implies:

$$\int_a^b f(x)dx \approx \frac{h}{3}f(x_0) + \frac{4h}{3}f(x_1) + \frac{h}{3}f(x_2) = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

$$\because, \ h = \frac{b-a}{n} = \frac{b-a}{2} \quad , \quad x_0 = a, x_1 = \frac{a+b}{2}, \ x_2 = b.$$

$$\boxed{\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]}$$

This is called the simple Simpson's method where the step size $h=\dfrac{b-a}{2}$.



Examples:

(1) Use Simpson's method to approximate the integral: $I=\int_0^2 \dfrac{dx}{x^2+1}$.

Solution:

$$* \because \quad , f(x)=\frac{1}{x^2+1} \quad , \quad a=0 \, , \; b=2 \Rightarrow \frac{a+b}{2}=\frac{0+2}{2}=1$$

$$\Rightarrow \int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]=\frac{2-0}{6}[f(0)+4f(1)+f(2)]$$

$$=\frac{1}{3}\left[1+2+\frac{1}{5}\right]=\frac{1}{3}\left[\frac{16}{5}\right]=\frac{16}{15}=1.0667$$

$$\therefore , \quad \boxed{I_S \approx 1.0667}$$

$$* \quad I_{exact}=\int_0^2 \frac{dx}{x^2+1}=\left[\tan^{-1}x\right|_0^2=\tan^{-1}2-\tan^{-1}0=\tan^{-1}2=1.1071$$

$$\therefore , \quad \boxed{I_{exact}=1.1071}$$

$* \quad E_S = |I_{exact} - I_S| = |1.1071 - 1.0667| = 0.0404$

(2) Apply Simpson's rule to the integral: $I = \int_1^3 (4x^3 - 3x^2 + x - 5)dx$, evaluate the integral exactly and find the actual error.

Solution:

$* \because \;, f(x) = 4x^3 - 3x^2 + x - 5 \quad , \quad a = 1,\; b = 3 \Rightarrow \dfrac{a+b}{2} = \dfrac{1+3}{2} = 2$

$$\Rightarrow \int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] = \frac{3-1}{6}\left[f(1) + 4f(2) + f(3)\right]$$

$$= \frac{1}{3}\left[-3 + 4(17) + 79\right] = \frac{1}{3}(144) = 48$$

$\therefore, \boxed{I_S \approx 48}$

$* \; I_{exact} = \left[\dfrac{4x^4}{4} - \dfrac{3x^3}{3} + \dfrac{x^2}{2} - 5x\right]_1^3 = 43.5 - (-4.5) = 48 \quad , \boxed{\therefore \;, I_{exact} = 48}$

$* \quad E_S = |I_{exact} - I_S| = |48 - 48| = 0$

<u>**Remark:**</u> If $f(x)$ is a polynomial of $\deg \le 3$, then Simpson's rule will return the exact value of the $\int_a^b f(x)dx$. Thus, Simpson's rule is more accurate than the trapezoid rule.

(3) Use Simpson's rule to calculate approximately the value of $\int_2^4 f(x)dx$ where $f(x)$ is given by the table:

x	2	3	4
$f(x)$	6	4.5	2

Solution:

Since: $\displaystyle \int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

$$\Longrightarrow \int_2^4 f(x)dx \approx \frac{4-2}{6}\left[f(2)+4f(3)+f(4)\right]=\frac{1}{3}\left[6+4(4.5)+2\right]$$

$$=\frac{1}{3}\left[8+18\right]=\frac{26}{3}=8.6667$$

Exercise:

(1) Using a single application of Simpson's method calculate approximately the values of the following integrals:

(1) $\int_1^2 6\sqrt{x+1}dx \quad [ans: 9.4708]$ (2) $\int_{-2}^{-1} 2e^{-x^2}dx \quad [ans: 0.26926]$

(3) $\int_0^{0.5} \frac{5}{x^3+1}dx \quad [ans: 2.4281]$ (4) $\int_2^3 \left(\frac{1}{2}x^3+4\right)dx \quad [ans: 12.125]$

(5) $\int_{-1}^{-0.5} 4\ln(x+10)dx \quad [ans: 4.4490]$ (6) $\int_0^1 (\sin x+2\cos x)dx \quad [ans: 2.1434]$

which of these approximations is exact ?

(2) Use Simpson's rule to calculate approximately the value of $\int_0^2 f(x)dx$

where $f(x)$ is given by the table:

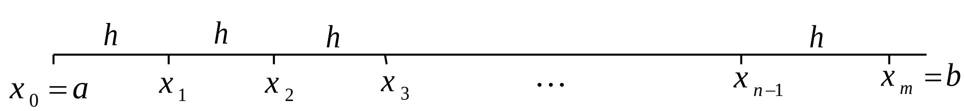
x	0	1	2
$f(x)$	3	4.1	3.8

[ans: 7.7333]

❻ The General Trapezoid Method:

In this method we subdivide the interval [a, b] into m ($m>1$) equally spaced subintervals by the points:

$$x_0=a, x_1, x_2, \ldots, x_m=b$$

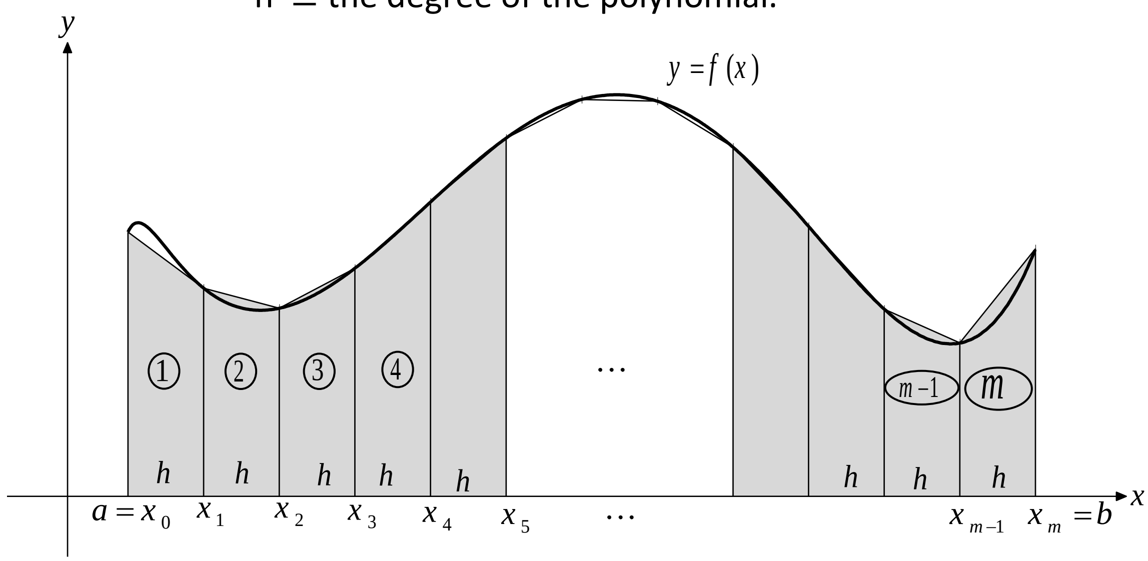


$$x_0=a \quad , \quad x_1=a+h \;, x_2=a+2h \;, \; x_3=a+3h \;, \ldots, x_i=a+ih \;, \ldots, x_n=a+mh=b$$

$$or \quad x_i=a+ih \;, \; i=0,1,2,3,\cdots,m \quad , \quad h=\frac{b-a}{m}$$

Note that: m ≡ number of subintervals.

n ≡ the degree of the polynomial.



The General Trapezoid

$$\int_a^b f(x)dx = \int_{x_0}^{x_m} f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{m-1}}^{x_m} f(x)dx$$

Now, approximating each integral by the simple trapezoid method, that is:

$\displaystyle\int_{x_{i-1}}^{x_i} f(x)dx \approx \frac{h}{2}\left[f(x_{i-1}) + f(x_i)\right]$, we get:

$$\int_a^b f(x)dx \approx \frac{h}{2}\left[f(x_0)+f(x_1)\right] + \frac{h}{2}\left[f(x_1)+f(x_2)\right] + \cdots + \frac{h}{2}\left[f(x_{m-1})+f(x_m)\right]$$

$$= \frac{h}{2}\left[f(x_0)+f(x_1)+f(x_1)+f(x_2)+\cdots+f(x_{m-1})+f(x_m)\right]$$

$$= \frac{h}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+\cdots+2f(x_{m-1})+f(x_m)\right]$$

$$= \frac{(b-a)}{2m}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+\cdots+2f(x_{m-1})+f(x_m)\right]$$

∴ ,

$$\boxed{\begin{aligned} \int_a^b f(x)dx &\approx \frac{(b-a)}{2m}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+\cdots+2f(x_{m-1})+f(x_m)\right] \\ &= h\left[\frac{1}{2}\left(f(x_0)+f(x_m)\right)+\sum_{i=1}^{m-1} f(x_i)\right] = \frac{h}{2}\left[f(a)+f(b)+2\sum_{i=1}^{m-1} f(x_i)\right] \end{aligned}}$$

This is called the general trapezoid (composite) trapezoid method. If m = 1 in the above formula, we get the simple rule that is:

$$\int_{x_0=a}^{x_1=b} f(x)dx \approx \frac{(x_1 - x_0)}{2}\left[f(x_0) + f(x_1)\right]$$

Examples:

(1) Use the general trapezoid method approximate the integral:

$$I = \int_1^6 x^2 dx \quad with\ m = 5.$$

Solution:

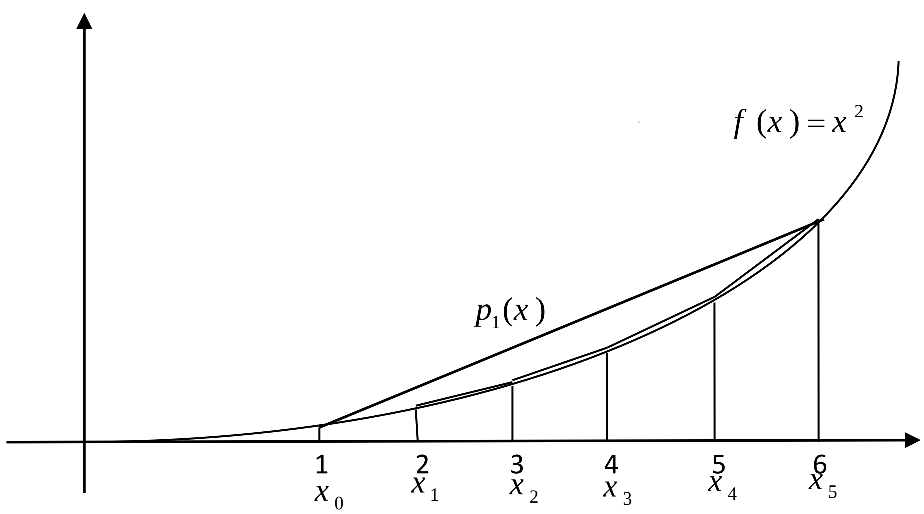
∗ The general method is given by:

$$\int_a^b f(x)dx \approx \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{m-1}) + f(x_m)\right] \longrightarrow (1)$$

∗ a = 1, b = 6 , m = 5 $\longrightarrow$ (2)

∗ $h = \dfrac{b-a}{m} = \dfrac{6-1}{5} = \dfrac{5}{5} = 1 \quad \rightarrow (3)$

where h is the interval length (step size).



$$f(x) = x^2$$

i	x_i	$f(x_i)$	w_i	$w_i f(x_i)$
0	1	1	1	01
1	2	4	2	08
2	3	9	2	18
3	4	16	2	32
4	5	25	2	50
5	6	36	1	36
$\sum$				145

$\longrightarrow$ (4)

Substitute (2) , (3) and (4) in (1) to get:

$$* \quad I = \int_1^6 x^2 dx \approx \frac{1}{2}[145] = \frac{145}{2} = \boxed{72.5000}$$

$$* \quad I_{exact} = \int_1^6 x^2 dx = \left[\frac{x^3}{3}\right|_1^6 = \frac{1}{3}[216-1] = \frac{215}{3} = \boxed{71.6667}$$

$*$ By simple trapezoid rule (m = 1):

$$\int_1^6 x^2 dx \approx \frac{(6-1)}{2}[f(1)+f(6)] = \frac{5}{2}[1+36] = \frac{5}{2}\times 37 = \boxed{92.5000}$$

<u>Note:</u> As m increases, so h deceases, and the accuracy improves.

(2) Use the general trapezoid method approximate the constant π by the integral: $I = \int_0^1 \frac{4}{x^2+1} dx$, $with\ m = 4$ (with four intervals).

<u>Solution:</u>

$*$ The general method is given by:

$$\int_a^b f(x)dx \approx \frac{h}{2}[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+\cdots+2f(x_{m-1})+f(x_m)] \longrightarrow (1)$$

$*$ a = 0, b = 1 , m = 4 $\longrightarrow$ (2)

$$* \quad h = \frac{b-a}{m} = \frac{1-0}{4} = \frac{1}{4} \rightarrow (3)$$

i	x_i	$f(x_i)$	w_i	$w_i f(x_i)$
0	0	4	1	4.0000
1	1/4	64/17	2	7.5294
2	1/2	16/5	2	6.4000
3	3/4	64/25	2	5.1200
4	1	2	2	2.0000
$\sum$				25.0494

$\rightarrow (4)$

Substitute (2), (3) and (4) in (1) to get:

$$* \quad I = \int_1^6 x^2 dx \approx \frac{0.25}{2}[25.0494] = 0.125 \times 25.0494 = 3.131175 \approx 3.1312$$

(3) Using 4 subintervals in the composite trapezoid rule and working to 6 decimal places approximate: $I = \int_0^2 \cosh x dx$.

Solution:

$$* \quad a = 0 \quad , \quad b = 2 \quad , \quad m = 4 \quad \rightarrow (1)$$

$$* \quad h = \frac{b-a}{m} = \frac{2-0}{4} = \frac{1}{2} = 0.5 \quad \rightarrow (2)$$

x_i	$f(x_i) = \cosh(x_i)$
0	1.000000
0.5	1.127626
1	1.543081
1.5	2.352410
2	3.762196

$*$ The general method is given by:

$$\int_a^b f(x)dx \approx \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{m-1}) + f(x_m)]$$

$$\int_0^2 \cosh x dx \approx \frac{h}{2}[f(x_0) + f(x_4) + 2\{f(x_1) + f(x_2) + f(x_3)\}]$$

$$= \frac{0.5}{2}[1 + 3.762196 + 2\{1.127626 + 1.543081 + 2.352410\}] = 3.452107$$

(4) The following points were found empirically:

x	2.1	2.4	2.7	3.0	3.3	3.6
y	3.2	2.7	2.9	3.5	4.1	5.2

Use the general trapezoid rule estimate $\int_{2.1}^{3.6} y dx$.

Solution:

By inspection we see that h = 0.3

$$\int_a^b f(x)dx \approx \frac{h}{2}[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+\cdots+2f(x_{m-1})+f(x_m)]$$

$$\int_{2.1}^{3.6} y dx \approx \frac{0.3}{2}[3.2+2(2.7)+2(2.9)+2(3.5)+2(4.1)+5.2] \approx 5.22$$

Exercise:

Using the general trapezoid method approximate:

(1) $I = \int_1^2 \ln x dx \quad , \quad with\ m = 4. \qquad [ans : 0.383700].$

(2) $I = \int_1^5 \sqrt{1+x^2}dx \quad , \quad with\ m = 8 \quad [ans : 12.76]$

■ **The Algorithm:**

Assuming that the function $f(x)$ is declared, the following algorithm describes the general trapezoid rule.

∗ Read (a, b, m) **%** input: a, b and number of subintervals m.

$* h := \dfrac{b-a}{m}$ % *step size*.

$*$ $S := 0$ % *initial value of* I.

$*$ *for* $i := 1$ *to* $m-1$ *do* % *sum of* $(m-1)$ *terms of* S.

  $*$ $x := a + ih$

  $*$ $S := S + f(x)$

end

$*$ $S := f(a) + f(b) + 2 \times S$.

$*$ $write\left(\dfrac{h}{2} \times S\right)$ % multiplying by the scale factor $\dfrac{h}{2}$.

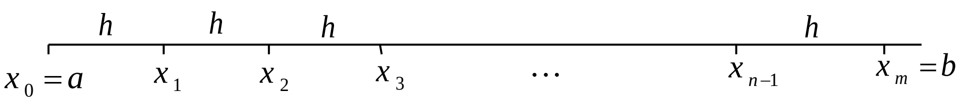
The output is: $S\ \left(\approx \int_a^b f(x)dx\right)$

Exercise: Write a matlab m-file which implements the general trapezoid rule.

7 The General Simpson's Method:

In this method we subdivide the interval [a, b] into m ($m>1$) equally spaced subintervals where m is an even number by the (m+1) points:

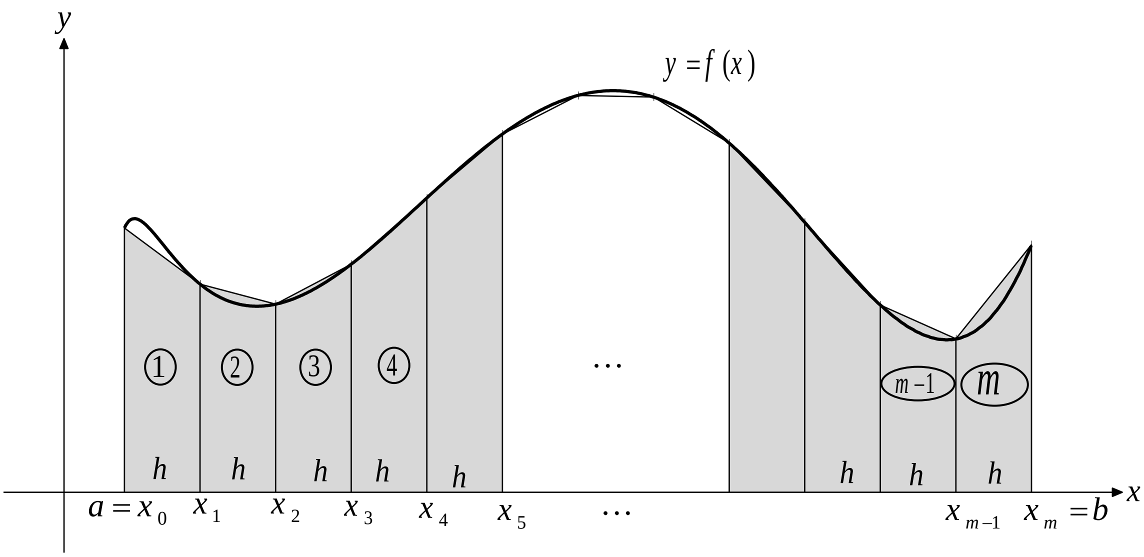
$$x_0=a, x_1, x_2, \dots, x_m=b$$



$$x_0=a \quad , \quad x_1=a+h, x_2=a+2h \ , \ x_3=a+3h, \dots, x_i=a+ih, \dots, x_n=a+mh=b$$

$$or \quad x_i=a+ih\ ,\ i=0,1,2,3,\cdots,m \quad , \quad h=\frac{b-a}{m}$$

Note that: m ≡ number of subintervals. n ≡ the degree of the polynomial.



The General Simpson's

$$\int_a^b f(x)dx = \int_{x_0}^{x_m} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots + \int_{x_{2i-2}}^{x_{2i}} f(x)dx + \cdots + \int_{x_{m-2}}^{x_m} f(x)dx$$

Now, approximating each integral by the simple Simpson's method, that is:

$$\int_{x_{2i-2}}^{x_{2i}} f(x)dx \approx \frac{h}{3}\left[f(x_{2i-2})+4f(x_{2i-1})+\mathrm{f}(\mathrm{x}_{2i})\right]$$

$$\Rightarrow \int_a^b f(x)dx \approx \frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right]+\frac{h}{3}\left[f(x_2)+4f(x_3)+f(x_4)\right]+\cdots$$

$$+\frac{h}{3}\left[f(x_{m-2})+4f(x_{m-1})+f(x_m)\right]$$

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)+f(x_2)+4f(x_3)+f(x_4)+\cdots+f(x_{m-2})+4f(x_{m-1})+f(x_m)\right]$$

$$=\frac{h}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots+2f(x_{m-2})+4f(x_{m-1})+f(x_m)\right]$$

$$=\frac{(b-a)}{3m}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots+2f(x_{m-2})+4f(x_{m-1})+f(x_m)\right]$$

$$=\frac{(b-a)}{3m}\left[f(x_0)+f(x_m)+4\{f(x_1)+f(x_3)+\cdots+f(x_{m-1})\}+2\{f(x_2)+f(x_4)+\cdots+f(x_{m-2})\right]$$

$$\boxed{\begin{aligned}\int_a^b f(x)dx &\approx \frac{(b-a)}{3m}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+4f(x_{m-1})+f(x_m)\right]\\ &=\frac{(b-a)}{3m}\left[f(x_0)+f(x_m)+4\{f(x_1)+f(x_3)+\cdots+f(x_{m-1})\}+2\{f(x_2)+f(x_4)+\cdots+f(x_{m-2})\right]\\ &=\frac{h}{3}\left[first+last+4\times odds+2\times evens\right]\end{aligned}}$$

Where ‘odds’ and ‘even’ refer to the subscript on x in the expression $f(x)$. But beware! This rule applies only if we write $x_0=a$: if we write $x_1=a$, then ‘odds’ become ‘evens’ and ‘evens’ become ‘odds’. This is the general Simpson’s method.

<u>Remarks:</u>

Note that Simpson’s method requires no more data than does the trapezoid method, both require the values of $f(x)$ at (m+1) equally spaced points. However, Simpson’s method treat the data by different weights, this will produce a much better approximations to the integral of $f(x)$.

<u>Examples:</u>

(1) Use the general Simpson’s method approximate the constant π by the integral: $I=\int_0^1 \frac{4}{x^2+1}dx$, $with\ m=4$ (with four intervals).

Solution:

∗ The general Simpson's method is given by:

$$\int_a^b f(x)dx \approx \frac{(b-a)}{3m}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+4f(x_{m-1})+f(x_m)\right] \rightarrow (1)$$

∗ a = 0, b = 1 , m = 4 ⟶ (2)

$$* \quad h=\frac{b-a}{m}=\frac{1-0}{4}=\frac{1}{4} \rightarrow (3)$$

i	x_i	$f(x_i)$	w_i	$w_i f(x_i)$
0	0	4	1	4.0000
1	1/4	64/17	4	15.0588
2	1/2	16/5	2	06.4000
3	3/4	64/25	4	10.2400
4	1	2	1	02.0000
$\sum$				37.6988

Substitute (2), (3) and (4) in (1) to get:

$$* \quad I=\int_1^6 x^2dx \approx \frac{0.25}{3}[37.6988]=\frac{9.4247}{3}=3.1415666667 \approx 3.1415$$

Note: Simpson's method is more accurate than the trapezoid method.

(2) Using 4 subintervals in the composite Simpson's rule approximate:

$$I=\int_1^2 \ln x dx$$

Solution:

$$* \quad a=1 \quad , \quad b=2 \quad , \quad m=4 \quad \rightarrow (1)$$

$$* \quad h=\frac{b-a}{m}=\frac{2-1}{4}=\frac{1}{4}=0.25 \quad \rightarrow (2)$$

There will be five x-values and the results are tabulated below to 6 decimal places:

x_i	$f(x_i) = \ln x$
1.00	0.000000
1.25	0.223144
1.50	0.405465
1.75	0.559616
2.00	0.693147

$\rightarrow (3)$

$$I = \int_1^2 \ln x dx \approx \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$

$$= \frac{0.25}{3}\left[0 + 4(0.223144) + 2(0.40546) + 4(0.559616) + 0.693147\right]$$

$$\approx 0.386260 \quad to\ 6\, d.p$$

(3) Calculate approximately the value of $\int_0^1 f(x)dx$, where $f(x)$ is given by the following table with four intervals:

x	0	0.25	0.5	0.75	1
f(x)	101	103.1	102.8	102.1	101.8

Solution:

By inspection we see that h = 0.25

$$I = \int_0^1 f(x)dx \approx \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$

$$= \frac{0.25}{3}\left[101 + 4\times 103.1 + 2\times 102.8 + 4\times 102.1 + 101.8\right] = 0.0833\times 1229.2 = 102.39236$$

Exercise:

(1) Use the composite Simpson's method to approximate the following integrals:

$(i)\ \int_1^2 \sqrt[3]{x}\, dx \quad , \quad m = 4$ $\qquad (ii)\ \int_0^{1.2} (2 + e^x)dx \ , \quad m = 6$

$(iii)\ \int_0^3 \frac{dx}{x^2 + 1} \quad , \quad m = 4$ $\qquad (iv)\ \int_0^{1.6} \frac{1}{\sqrt{x} + 1}dx \quad , \quad m = 8$ $\qquad (v)\ \int_{-5}^{5} \frac{1}{4}x^2 dx \ , \quad m = 2$

(2) Write an algorithm to describe the general Simpson's method.

(3) Calculate approximately the value of $\int_1^5 4e^{-x}\,dx$ using Simpson's rule with four intervals. Calculate the integral exactly and find the error in your approximation.

CHAPTER THREE

Solving the nonlinear equation (The Root-Finding Problem)

❶ All nonlinear equations can be written in the general form:

$$f(x) = 0 \qquad \rightarrow (1)$$

Where x is a single variable, and the function $f(x)$ is a given real algebraic or nonalgebraic" transcendental" (trigonometric, exponential, and logarithmic).

Examples:

$$\left.\begin{array}{ll} (i) & x^4 + x^3 - x^2 - 2x - 2 = 0 \\ (ii) & \sqrt{x} = 1 + x \quad \Rightarrow \quad \sqrt{x} - x - 1 = 0 \\ (iii) & x^3 - 5x^2 + x + 9 = 0 \\ (iv) & x^2 - 100x + 1 = 0 \end{array}\right\} algebraic$$

$$\left.\begin{array}{ll} (v) & x^2 = e^{-x} \quad \Rightarrow \quad x^2 - e^{-x} = 0 \\ (vi) & x + \ln x^2 = 1 \quad \Rightarrow \quad x + \ln x^2 - 1 = 0 \\ (vii) & \cos^2 x = x^{2/3} \end{array}\right\} nonalgeraic\ equations.$$

❷ **Roots (zeros):**

① **Definition:** A number c, real or complex is called a root of the equation $f(x) = 0$ or a zero of the function $f(x)$ if $f(c) = 0$.

Or:

A number c, real or complex is called a root of the equation $f(x) = 0$ if $|f(c)| < \varepsilon \ \Rightarrow \ -\varepsilon < f(c) < \varepsilon$, where $\varepsilon > 0$ is a given tolerance. Note the inequality determines an interval instead of a point.

② Graphically , the real roots of $f(x) = 0$ occur when the graph of $y = f(x)$ crosses the x axis (x intercepts) or touches it, and the roots of $f(x) = g(x)$ occur at points where the graphs intersect.

③ Consider the polynomial equation: $f(x)=a_n x^n + a_{n-1}x^{n-1}+\cdots+a_1 x + a_0 = 0$,

Then:

(i) It has only n roots (f(x) has n zeros, crosses the x axis n times) and they can be : distinct, multiple, complex.

(ii) If f(x) is divisible by $(x-\alpha)$, then α is a root of the equation.

(iii) If $f(a)$ and $f(b)$ have different signs, then at least one root of $f(x)=0$ lies between a and b.

(iv) Every equation of odd degree has at least one real root.

(v) Complex roots occur in pairs, and irrational roots also occur in pairs.

❸ The Intermediate Value Theorem For Continuous Functions:

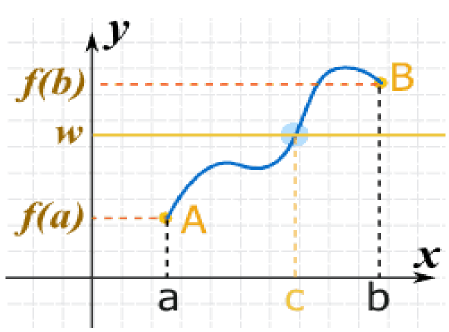
Although the graph is useful to us in locating the roots, a computer can not see the graph and look for x- axis crossings as we can. We have the following theorem.

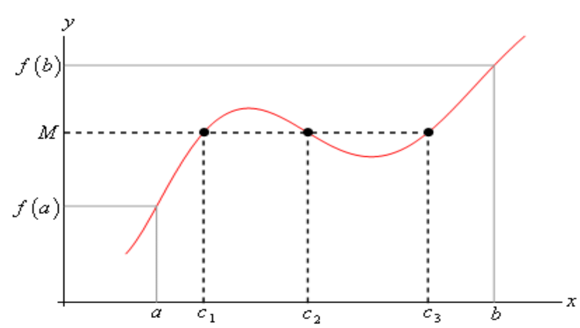
Remark:

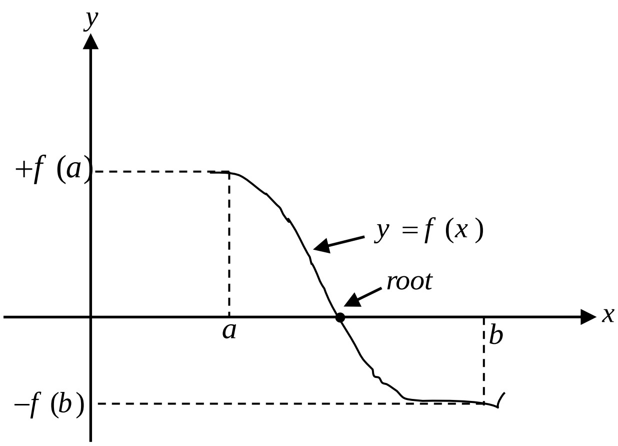
$$If \quad ab<0 \quad \Rightarrow \begin{cases} either & a>0 \ \& \ b<0. \\ or & a<0 \ \& \ b>0. \end{cases}$$

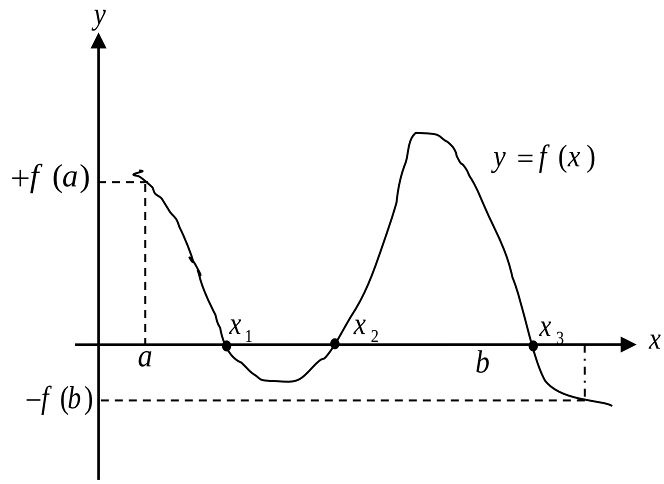
$$\Rightarrow \ f(a)f(b)<0 \ \Rightarrow \ f(a)<0 \, and \, f(b)>0. \quad or \quad f(a)>0 \, and \, f(b)<0.$$

Theorem: If $f(x)$ is a continuous function in $[a,b]$ and $f(a)f(b)<0$ (the function has opposite signs), then the equation $f(x)=0$ has **at least** one root in the interval (a,b) .









Examples: Use the intermediate Value Theorem; show that the given equations have roots in the given interval:

(1) $x^4+x^3-x^2-2x-2=0$, $on\ [1,2]$ (2) $x^2-6x+8=0\ on\ [0,3]$

(3) $x^3+4x-13\sqrt{2}=0\ on\ (-4,10)$. (4) $\sqrt{x}=1-x\ on\ [0,1]$

Solution:

(1) $Let\quad f(x)=x^4+x^3-x^2-2x-2$. f(x) is a continuous function on [1,2].

$f(1)=1+1-1-2-2=-3<0 \quad \& \quad f(2)=2^4+2^3-2^2-4-2=24-10=14>0$

$\because f(1)f(2)<0 \quad \Rightarrow \quad \exists\ a\ root\ in\ (1,2)$.

(2) $Let:\ f(x)=x^2-6x+8$. f(x) is a continuous function on [0,3], and

$f(0)=0-0+8=8>0 \quad \& \quad f(3)=9-18+8=-9+8=-1<0$

$\because f(0)f(3)<0 \quad \Rightarrow \quad \exists\ a\ root\ in\ (0,3)$.

(3) $Let\quad f(x)=x^3+4x-13\sqrt{2}$. $f(x)$ is a continuous function on [-4,10]

and $f(-4)=-80-13\sqrt{2}<0 \quad \& \quad f(10)=1040-13\sqrt{2}>0$

By the IVT ⟹ ∃ a root in (-4, 10), that is there is a number c in (-4, 10)

such that $f(c)=0$.

(4) $\sqrt{x}=1-x \quad \Rightarrow \quad \sqrt{x}+x-1=0 \quad \Rightarrow \quad let\quad f(x)=\sqrt{x}+x-1$

f(x) is continuous for all $x>0$,so in [0,1] and $f(0)=0+0-1=-1<0$,

$f(x)=1+1-1=1>0$.

$\because, \quad f(0)f(1) < 0 \quad \Rightarrow \quad \exists\, a\ root\ \ in\ (0,1).$

(5) Find the intervals that contain the roots of the equation $3x^2 - 10x + 5 = 0$.

Solution:

$Let: \ f(x) = 3x^2 - 10x + 5$

x	$f(x) = 3x^2 - 10x + 5$
0	+5
1	-2
2	-3
3	+2

Thus, roots occur in the intervals [0, 1] and [2, 3].

(6) Show that the equation: $x + \sin^2\left(\frac{x}{3}\right) = 8$ has a root in $(-\infty, \infty)$.

Solution:

Let $f(x) = x + \sin^2\left(\frac{x}{3}\right) - 8$. We show that $f(x) = 0$ has at least one solution in the given interval. Note that f is continuous on [0, 20] and that : $f(0) = -8 < 0 \, while \ f(20) = 12 + \sin(4) > 0$. So, by the IVT there is a root in (0,20) and hence in $(-\infty, \infty)$

❹ Root Finding Methods(Iterative Methods):

① The Bisection Method (Binary search):

- The Bisection Method is a *successive* approximation method that narrows down an interval that contains a root of the function *f(x)*.
- The Bisection Method is *given* an initial interval [*a*..*b*] that contains a root (We can use the property sign of *f(a)* ≠ sign of *f(b)* to find such

  an initial interval)

- The Bisection Method will *cut the interval* into 2 halves and check which half interval contains a root of the function.
- The Bisection Method will keep *cut the interval* in halves until the resulting interval is extremely small

The root is then *approximately equal* to *any value* in the final (very small) interval.

$I_0 = [a_0, b_0] = [a, b] \equiv$ Initial interval.

$$
\begin{aligned}
I_1 &= [a_1, b_1] \\
I_2 &= [a_2, b_2] \\
&\vdots \\
I_n &= [a_n, b_n]
\end{aligned}
$$

I_{n+1} is obtained from I_n by the following steps:

$$find\ c_n = \frac{a_n + b_n}{2},\ n = 0, 1, 2, \ldots$$

$$\text{if } f(a_n) f(c_n) < 0, then$$

$$\left[\begin{array}{ll} a_{n+1} := a_n & \{store\ a_n\ in\ a_{n+1}\} \\ b_{n+1} := c_n & \{store\ c_n\ in\ b_{n+1}\} \end{array}\right.$$

$$else$$

$$\left[\begin{array}{ll} a_{n+1} := c_n & \{store\ c_n\ in\ a_{n+1}\} \\ b_{n+1} := b_n & \{store\ b_n\ in\ b_{n+1}\} \end{array}\right.$$

The sequence is terminated at I_n if $(b_n - a_n) < \varepsilon, where\ \varepsilon > 0$ is a given small tolerance. Thus, $c_n = (a_n + b_n)/2$ could be taken as a good approximation of the root c. So, we have the following algorithm:

$*$ $read\,(a,b,eps)$ {required : f(a) f(b) < 0 , eps : accuracy of root}

$*$ repeat

$c := (a+b)/2$

if $f(a)\,f(b) < 0$, *then*

$b := c$

else

$a := c$

end

$*$ *until* $(b-a) < eps$

$*$ $write\,(c)$

2 Examples:

(1) By using 5 iterations of the bisection method find the root of the equation

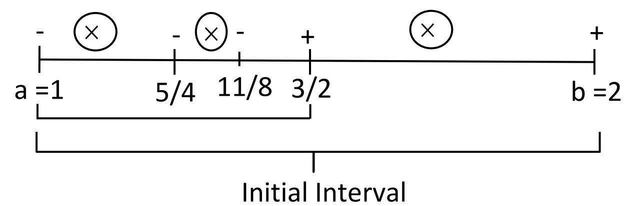
$x^2 - 2 = 0 \; in \; [1,2]$.

Solution:

$* \; \because , \quad x^2 - 2 = 0 \quad \Rightarrow \quad let \;\; f(x) = x^2 - 2$

$f(1) = 1^2 - 2 = -1\,(-ve) \quad \& \quad f(2) = 2^2 - 2 = 2\,(+ve)$

$*$ Start:



n	a	b	$c = (a+b)/2$	$f(a)$	$f(c) = c^2 - 2$	Comment
1	1	2	3/2	-	+	b:=c
2	1	3/2	5/4	-	-	a:=c
3	5/4	3/2	11/8	-	-	a:=c
4	11/8	3/2	23/16	-	+	b:=c
5	11/8	23/16	45/32			

$*$ ∴, the root = $\dfrac{45}{32} \approx \boxed{1.40625}$

(2) By using 4 iterations of the bisection method find the root of the equation

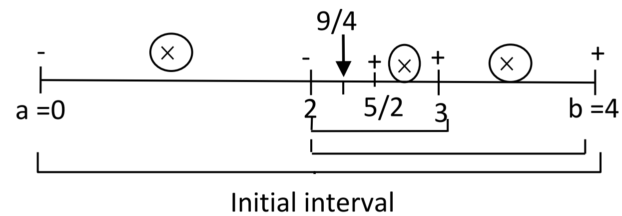
$x^2 - 5 = 0 \; in \; [0,4]$.

Solution:

$* \quad \because, \quad x^2 - 5 = 0 \quad \Rightarrow \quad let \;\; f(x) = x^2 - 5$

$f(2) = 0^2 - 5 = -5\,(-ve) \quad \& \quad f(4) = 4^2 - 5 = 11\,(+ve)$

$*$ Start (initial interval):



n	a	b	$c=(a+b)/2$	$f(a)$	$f(c)=c^2-5$	Comment
1	0	4	2	-	-	a:=c
2	2	4	3	-	+	b:=c
3	2	3	5/2	-	+	b:=c
4	2	5/2	9/4			

$*$ ∴, the root = $\dfrac{9}{4} = 2.25 \quad (\sqrt{5} \approx 2.23)$

---

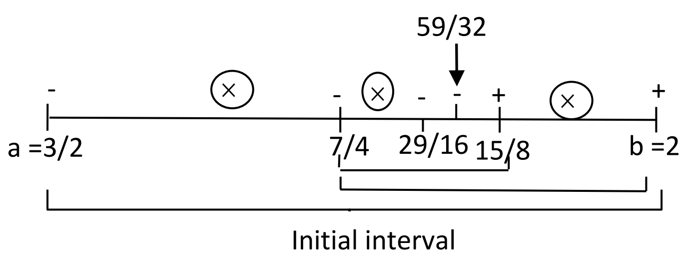
(3) Use five iterations of the bisection method find a positive root of the equation $x^4 - x - 10 = 0$.

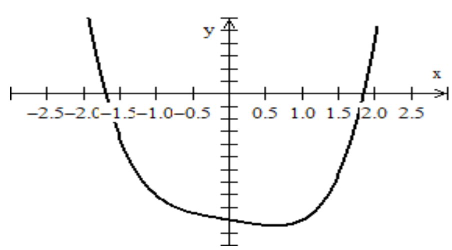
Solution:

$* \quad \because, \quad x^4 - x - 10 = 0 \quad \Rightarrow \quad let \;\; f(x) = x^4 - x - 10$

From the graph of the function and by the IVT, we let:

$a = 1.5 \quad \Rightarrow \quad f(1.5) = (1.5)^4 - 1.5 - 10 = -ve \quad \& \quad b = 2 \quad \Rightarrow f(2) = 16 - 2 - 10 = +ve$





$$f(x) = x^4 - x - 10$$

n	a	b	$c = (a+b)/2$	$f(a)$	$f(c)$	Comment
1	3/2	2	7/4	-	-	a:=c
2	7/4	2	15/8	-	+	b:=c
3	7/4	15/8	29/16	-	-	a:=c
4	29/16	15/8	59/32	-	-	a:=c
5	59/32	15/8	119/64			

$*$ $\therefore$, the root $\approx \dfrac{119}{64} = 1.859375 \approx 1.86$

---

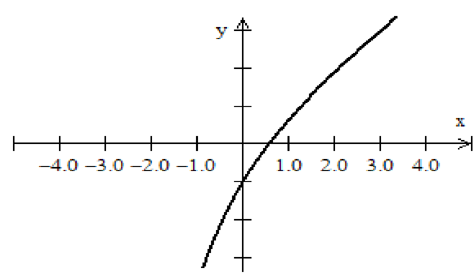
(4) Using four iterations of the bisection method, find the root of the equation : $x - e^{-x} = 0.$

<u>Solution:</u>

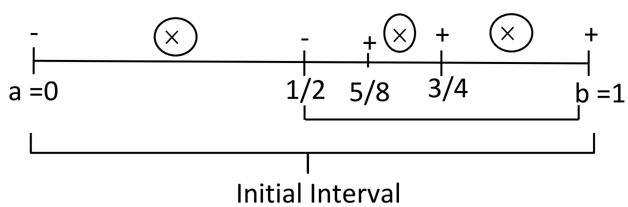
$* \quad \because, \quad x - e^{-x} = 0 \quad \Rightarrow \quad let \;\; f(x) = x - e^{-x}.$

From the graph of the function and by the IVT, we let:

$a = 0 \quad \Rightarrow \quad f(0) = 0 - e^{0} = -1\,(-ve) \quad \& \quad b = 1 \quad \Rightarrow f(1) = 1 - e^{-1} = 0.632 \quad (+ve)$



$*$ start (initial interval):



$$f(x) = x - e^{-x}$$

n	a	b	$c=(a+b)/2$	$f(a)$	$f(c)$	Comment
1	0	1	1/2	-	-	a:=c
2	1/2	1	3/4	-	+	b:=c
3	1/2	3/4	5/8	-	+	a:=c
4	1/2	5/8	9/16			

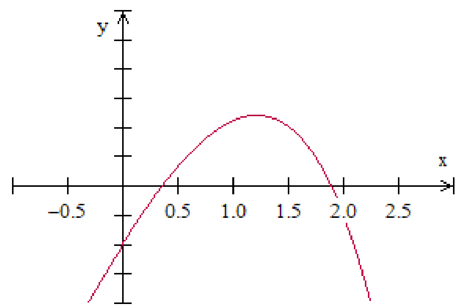
$*$ $\therefore$, the root $= \dfrac{9}{16} = 0.5625 \approx 0.562$

(5) Using four iterations of the bisection method, find a root of:

$$3x + \sin x - e^{x} = 0.$$

<u>Solution:</u>

From the graph of this equation, it has two roots one in $[0,1/2]$ and the other in $[3/2,2]$. Consider the interval $[0,1/2]$.

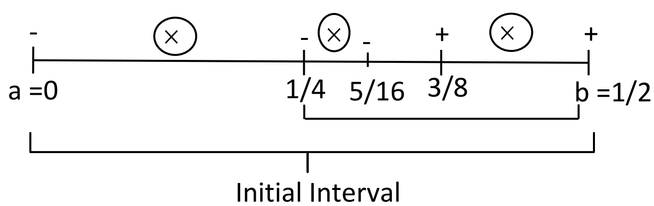


$*$ $\quad Let \quad f(x) = 3x + \sin x - e^{x}.$

$f(0) = 0 + 0 - e^{0} = -1 \quad (-ve).$

$f(1/2) = \dfrac{3}{2} + \sin(1/2) - e^{1/2} = 0.33 \quad (+ve).$

$*$ start:



n	a	b	$c=(a+b)/2$	$f(a)$	$f(c)$	Comment
1	0	1/2	1/4	-	-	a:=c
2	1/4	1/2	3/8	-	+	b:=c
3	1/4	3/8	5/16	-	-	a:=c
4	5/16	3/8	11/32			

$*$ $\therefore$, the root $= \dfrac{11}{32} = 0.34375 \approx 0.34$

Remarks:

(1) The bisection method is too inefficient for hand computation, it is well suited to electronic computers.

(2) When an interval contains more than one root, the bisection method can find one of them. When an interval contains a singularity, the bisection method converges to that singularity.

(3) **Advantages of the BM:**

i) It is certain: the root MUST be inside the range.

ii) It is ALWAYS converges to the root.

iii) Each iteration improves the estimate, and only requires evaluation of $f(x)$ and not its derivatives.

(4) **Disadvantages of the BM:**

i) Slow convergence, but it is not a serious problem, because of the speed of the modern computers.

ii) The bisection method is extremely wasteful of information. In any iteration, if [a, b] represents the interval containing the solution, the values $f(a)\, and\, f(b)$ are known ,but are not used, only the signs are of interest.

Exercise:

(1) Use the bisection method to find, as accurately as possible all real roots for each of the following equations:

$(i)\quad x^3 - x - 1 = 0 \qquad ans: 1.3247$ $\qquad (ii)\quad x^3 - x^2 - x - 1 = 0 \qquad ans: 1.8393$

$(iii)\quad x^2 - 5 = 0 \qquad ans: \pm 2.2361$ $\qquad (iv)\quad x^2 = e^{-x}$

$(v)\quad x^3 = 3 \qquad ans: 1.4422$ $\qquad (vi)\quad \ln|x| = \sin x$

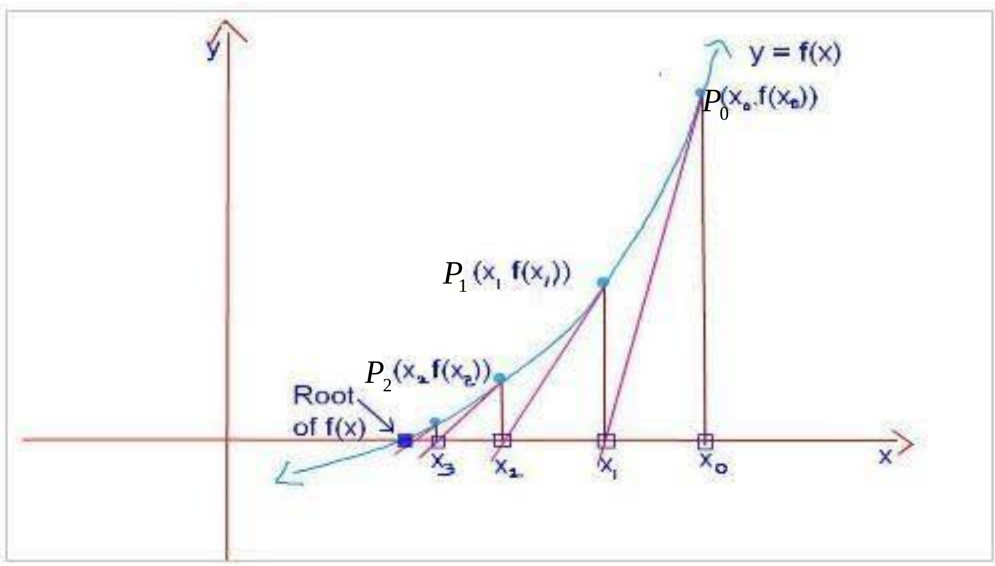
$(vii)\quad x^3 - 3x + 3 = 0 \quad ans: -2.1038$

(2) Use four iterations of the bisection method to approximate:

$(i)\quad \sqrt{14} \qquad ans: 3.7417$ $\qquad (ii)\quad \sqrt{21} \qquad ans: 4.5826$ $\qquad (iii)\quad \sqrt[3]{2}$

❺ Newton – Raphson's Method (The tangent method):

① Newton-Raphson's or Newton method is one of the most efficient, popular and widely used method for finding a root of an equation. Let $f(x)=0$ be the equation to be solved. Suppose that the function $f(x)$ is differentiable, which implies that the graph of f has a definite slope at each point and hence a unique tangent line. Let x_0 is our initial guess or approximation of the root r.



From the figure, the slope of the line through the points $(x_1,0)\ \&\ (x_0,f(x_0))$ is equal to:

$$m=\frac{0-f(x_0)}{x_1-x_0}=\frac{-f(x_0)}{x_1-x_0}\quad\rightarrow(I)$$

Moreover, the slope also equals: $m=\left.\frac{df}{dx}\right|_{x=x_0}=f'(x_0)\quad\rightarrow(II)$

$$\therefore,(I)\ \&\ (II)\ \Rightarrow\ f'(x_0)=\frac{-f(x_0)}{x_1-x_0}\ \Rightarrow\ x_1-x_0=\frac{-f(x_0)}{f'(x_0)}$$

$$\Rightarrow\ x_1=x_0-\frac{f(x_0)}{f'(x_0)}$$

Iterating this, yields the general term:

$$\boxed{x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}\ ,\ for\ n=0,1,2,3,\ldots}$$

Provided $f'(x) \neq 0$ (the tangent line must be not horizontal). The formula is called Newton – Raphson formula, can be easily applied via a computer program. The sequence is terminated at $x_n = r$ when $|f(x_n)| < \varepsilon, \varepsilon > 0$ is a given tolerance. We give the following algorithm: Assuming both $f(x)$ *and* $f'(x)$ are declared.

$*$ $read(x, \varepsilon)$ {x is the initial guess}.

$*$ *repeat*

$$x := x - \frac{f(x)}{f'(x)}$$

$*$ *until* $|f(x)| < \varepsilon$

$*$ $write(x)$

■ **Summary:** To approximate roots of an equation by N-R method, we have the following steps:

i) Make an educated (not blined) guess: x_0 .

ii) x_0 takes the value of x_n .

iii) Substitute x_0 in: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

② **Examples:**

(1) Use three iterations (steps) of Newton- Raphson's method to find the root of the equation $x^2 - 2 = 0$ starting with $x_0 = 2$.

Solution:

$*$ $\because, x^2 - 2 = 0 \Rightarrow f(x) = x^2 - 2 \Rightarrow f'(x) = 2x$.

$*$ $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

$$= x_n - \frac{x_n^2 - 2}{2x_n} = \frac{2x_n^2 - (x_n^2 - 2)}{2x_n} = \frac{x_n^2 + 2}{2x_n}$$

$$= \frac{1}{2}\left[x_n + \frac{2}{x_n}\right]$$

n	x_n	$x_{n+1}=\frac{1}{2}\left[x_n+\frac{2}{x_n}\right]$
0	2	$x_1=\frac{1}{2}\left[2+\frac{2}{2}\right]=\frac{3}{2}$
1	3/2	$x_2=\frac{1}{2}\left[\frac{3}{2}+\frac{2}{3/2}\right]=\frac{17}{12}=1.4166667$
2	17/12	$x_3=\frac{1}{2}\left[\frac{17}{12}+\frac{2}{17/12}\right]=\frac{577}{408}=1.4142157$
3	577/408	

The root = $\frac{577}{408}=1.4142157$

(2) Using two iterations of the N-R method, find the real root of the equation: $x^3+5x-3=0$.

Solution:

$*\quad Let:\ f(x)=x^3+5x-3.$

$let\quad a=0\ \Rightarrow\ f(0)=0+0-3=-ve$

$b=1\ \Rightarrow\ f(1)=1+5-3=+ve$

Take the initial guess to be: $x_0=0$

$*\quad f'(x)=3x^2+5$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}.$$

$$=x_n-\frac{x_n^3+5x_n-3}{3x_n^2+5}=\frac{3x_n^3+5x_n-x_n^3-5x_n+3}{3x_n^2+5}=\frac{2x_n^3+3}{3x_n^2+5}.$$

$\therefore$, $\boxed{x_{n+1}=\frac{2x_n^3+3}{3x_n^2+5}}$ for n = 0 ,1 2.

$*\quad n=0\ \Rightarrow\ x_1=\frac{2(0)^3+3}{3(0)^2+5}=\frac{3}{5}=0.6$

$n=1\ \Rightarrow\ x_2=\frac{2(0.6)^3+3}{3(0.6)^2+5}=\frac{3.432}{6.08}=0.5644$

$$n=2 \quad \Rightarrow \quad x_3=\frac{2(0.5644)^3+3}{3(0.5644)^2+5}=\frac{3.35957}{5.95564}=0.5640$$

(3) Use three iterations of Newton-Raphson's method to find the real root of the equation: $\ln x=\sin x$, starting at $x_0=2$.

Solution:

$$* \quad \ln x=\sin x \quad \Rightarrow \quad \ln x-\sin x=0$$

$$\Rightarrow \quad f(x)=\ln x-\sin x \quad \& \quad f'(x)=\frac{1}{x}-\cos x=\frac{1-x\cos x}{x}$$

$$* \quad x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} \quad \text{for } n=0,1,2. \quad x_0=2.$$

$$\Rightarrow \quad x_1=x_0-\frac{f(x_0)}{f'(x_0)}=2-\frac{f(2)}{f'(2)}=2.235934$$

$$\Rightarrow \quad x_2=x_1-\frac{f(x_1)}{f'(x_1)}=2.235934-\frac{f(2.235934)}{f'(2.235934)}=2.21186$$

$$\Rightarrow \quad x_3=x_2-\frac{f(x_2)}{f'(x_2)}=2.21186-\frac{f(2.21186)}{f'(2.21186)}=2.219107$$

Exercise:

(1) Use Newton-Raphson 's method to find an iterative formula for the $\sqrt{a}$, where $a \in \mathbb{R}^+$.

(2) Solve : $x^2-5x+6=0$ by Newton-Raphson method with the initial guess $x_0=1$.

(3) Use one iteration of Newton-Raphson method to find the real root of $x^3-x+3=0$ Starting from $x_0=-2$.