Variants of the penalty method for contact problems
Formulations unifying Nitsche and penalty methods

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- The unilateral contact problem

- Nitsche formulation of the contact problem

- Penalty method and its variants for the contact problem

- Numerical analysis and simulation

✓ A simple and well-known contact problem: unilateral contact problem.

✓ This problem plays an important role in industry or in biomechanics.

✓ Contact conditions are nonlinear and are under inequalities form.

✓ Contact phenomena modeling poses many difficulties.

✓ In general, their simulation is done by the Finite Element Method.

✓ Easy to implement in practice and precise from a theoretical point of view.

✓ Techniques which make it possible to effectively incorporate boundary conditions by finite elements are:

  ★ mixed methods, the penalty method,

  ★ Nitsche's method, stabilized methods.

✓ We will see here two techniques: Nitsche method and penalty method.
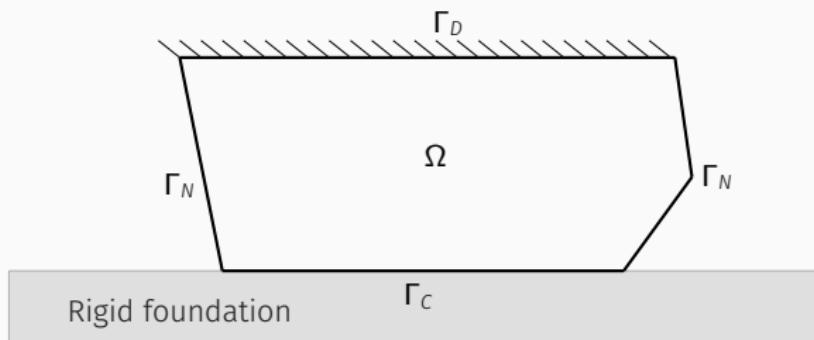
Goal: New penalty methods for contact:

  ★ Formulations asymptotically equivalent to the Nitsche method.

  ★ Others more stable and better suited to large linear systems.

The unilateral contact problem

✓ To model an elastic body of reference configuration $\Omega \subset \mathbb{R}^d$, $d = 2, 3$:

★ The domain Ω is open, bounded, connected and non-empty.

★ Small elastic transformations (small deformations).

✓ Let $\boldsymbol{\sigma} = (\sigma_{i,j}), 1 \leq i, j \leq d$, be the stress tensor field,

✓ $\nabla\cdot$ is the divergence operator for tensor-valued functions,

✓ $\boldsymbol{u} : \Omega \rightarrow \mathbb{R}^d$ is the displacement field of the elastic body,

✓ $\varepsilon(\boldsymbol{u})$ represents the linearized strain tensor field,

$$\varepsilon(\boldsymbol{u}) = \frac{1}{2}\left(\nabla u + \nabla u^t\right),$$

✓ $\boldsymbol{C} = (C_{ijkl})$ is the fourth order elasticity tensor, uniformly bounded:

⋆ symmetrical,

$$C_{ijkl} = C_{klij} = C_{jikl}, \; \forall \, 1 \leq i, j, l, k \leq d,$$

⋆ elliptical, i.e. there exists $\alpha > 0$ such as

$$(\boldsymbol{C}(x)\varepsilon : \varepsilon) \geq \alpha\varepsilon : \varepsilon, \; \forall \, x \in \Omega.$$

✓ Find the displacements field $u : \Omega \rightarrow \mathbb{R}^d$, of the elastic body Ω,

✓ satisfying the linear elasticity equations, as well as the boundary conditions of Dirichlet, Neumann and contact:

$$-\nabla \cdot \sigma(u) = f, \qquad \text{in } \Omega, \qquad (1)$$

$$\sigma(u) = C : \varepsilon(u), \qquad \text{in } \Omega,$$

$$u = 0, \qquad \text{on } \Gamma_D, \qquad (2)$$

$$\sigma(u)n = g, \qquad \text{on } \Gamma_N, \qquad (3)$$

$$u \cdot n \leq 0, \quad \sigma_n(u) \leq 0, \quad (u \cdot n)\sigma_n(u) = 0, \qquad \text{on } \Gamma_C, \qquad (4)$$

$$\sigma_t(u) = 0, \qquad \text{on } \Gamma_C. \qquad (5)$$

✓ The total mechanical energy of the elastic body is defined by

$$J(\boldsymbol{u}) := \frac{1}{2}a(\boldsymbol{u}, \boldsymbol{u}) - L(\boldsymbol{u}),$$

✓ where the bilinear $a(\cdot, \cdot)$ and linear $L(\cdot)$ forms are defined by

$$a(\boldsymbol{u}, \boldsymbol{v}) := \int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \ \forall\, \boldsymbol{u}, \boldsymbol{v} \in V,$$

$$L(\boldsymbol{v}) := \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}\, dx + \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{v}\, ds, \ \forall\, \boldsymbol{v} \in V.$$

✓ We introduce the Hilbert space V of admissible displacements [3]

$$V := \left\{ \boldsymbol{v} \in \left(H^1(\Omega)\right)^d : \boldsymbol{v} = \boldsymbol{0} \ \text{on} \ \Gamma_D \right\}.$$

✓ The convex cone K of admissible displacements (satisfying the non-penetration on the contact boundary Γ_C):

$$K := \left\{ \boldsymbol{v} \in V : \boldsymbol{v} \cdot \boldsymbol{n} \leq 0 \ \text{on} \ \Gamma_C \right\}.$$

✓ The problem (without friction) of unilateral contact [5] consists in:

$$\begin{cases} \text{Find } \boldsymbol{u} \in \boldsymbol{K}, \text{ such that,} \\ J(\boldsymbol{u}) = \min_{\boldsymbol{v} \in \boldsymbol{K}} J(\boldsymbol{v}) \end{cases} \quad (6)$$

This is a minimization problem under inequality constraints.

✓ $J(\cdot)$ being Fréchet-differentiable, we have the variational inequality [5, 4]

$$\langle J'(\boldsymbol{u}), \boldsymbol{v} - \boldsymbol{u} \rangle \geq 0, \ \forall \boldsymbol{v} \in \boldsymbol{K}.$$

✓ Any solution $\boldsymbol{u}$ of problem (6), is also a solution of problem [5]

$$\begin{cases} \text{Find } \boldsymbol{u} \in \boldsymbol{K}, \text{ such that,} \\ a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) \geq L(\boldsymbol{v} - \boldsymbol{u}), \ \forall \boldsymbol{v} \in \boldsymbol{K}. \end{cases} \quad (7)$$

Nitsche formulation of the contact problem

✓ Nitsche's method is based on the reformulation of the conditions (4):

$$\boldsymbol{\sigma}_n(\boldsymbol{u}) = -\frac{1}{\gamma}\Big[\boldsymbol{u}\cdot\boldsymbol{n} - \gamma\boldsymbol{\sigma}_n(\boldsymbol{u})\Big]^+ \quad \text{on } \Gamma_c, \ \forall \, \gamma > 0.$$

✓ The unilateral contact problem is now written as:

$$\begin{cases} \text{Find } \boldsymbol{u} \in V_N, \text{ such that,} \\ J_N(\boldsymbol{u}) = \min_{\boldsymbol{v}\in V_N} J_N(\boldsymbol{v}) \end{cases} \tag{8}$$

This is a minimization problem «without» inequality constraints.

✓ The functional $J_N(\cdot)$ is defined as follows:

$$J_N(\boldsymbol{u}) := J(\boldsymbol{u}) - \frac{1}{2}\int_{\Gamma_c} \gamma \left(\boldsymbol{\sigma}_n(\boldsymbol{u})\right)^2 ds$$

$$+ \frac{1}{2}\int_{\Gamma_c} \frac{1}{\gamma}\left(\Big[\boldsymbol{u}\cdot\boldsymbol{n} - \gamma\boldsymbol{\sigma}_n(\boldsymbol{u})\Big]^+\right)^2 ds,$$

✓ where the (more regular) space V_N is defined as follows:

$$V_N = \left\{ v \in V : \nabla v_{|\Gamma_C} \in \left(L^2(\Gamma_C) \right)^{d \times d} \right\}.$$

✓ Then, the first-order optimality condition is written in the following:

$$\langle J'_N(u), v \rangle = 0, \ \forall \ v \in V_N.$$

✓ The solution u of Problem (8) is a solution of the variational problem

$$\begin{cases} \text{Find } u \in V_N, \text{ such that,} \\ a(u,v) - \displaystyle\int_{\Gamma_C} \gamma \sigma_n(u) \sigma_n(v) ds \\ \qquad + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma} \left[u \cdot n - \gamma \sigma_n(u) \right]^+ \left(v \cdot n - \gamma \sigma_n(v) \right) ds = L(v), \end{cases} \tag{9}$$

for all $v \in V_N$.

✓ Variants of formulation (9) have been proposed in the form [1]:

$$
\begin{cases}
\text{Find } \boldsymbol{u} \in V_N, \text{ such that,} \\
a(\boldsymbol{u}, \boldsymbol{v}) - \theta \int_{\Gamma_C} \gamma \boldsymbol{\sigma}_n(\boldsymbol{u}) \boldsymbol{\sigma}_n(\boldsymbol{v}) ds \\
\qquad + \int_{\Gamma_C} \frac{1}{\gamma} \big[\boldsymbol{u} \cdot \boldsymbol{n} - \gamma \boldsymbol{\sigma}_n(\boldsymbol{u}) \big]^+ \Big(\boldsymbol{v} \cdot \boldsymbol{n} - \theta \gamma \boldsymbol{\sigma}_n(\boldsymbol{v}) \Big) ds = L(\boldsymbol{v}),
\end{cases}
\tag{10}
$$

for all $\boldsymbol{v} \in V_N$, where the parameter θ takes the values $\theta = -1, 0, 1$.

✓ $\theta = 1$, symmetrical formulation but the coercivity is from a bound $\gamma_0 > 0$.

✓ $\theta = -1$, anti-symmetric formulation but coercive for all $\gamma > 0$.

✓ $\theta = 0$, the standard penalty method which is coercive for all $\gamma > 0$.

✓ All of these Nitsche variants are consistent.

Penalty method and its variants for the contact problem

✓ Recall: The unilateral contact (frictionless) problem consists of:

$$\begin{cases} \text{Find } \boldsymbol{u} \in K \text{ such that,} \\ J(\boldsymbol{u}) = \min_{\boldsymbol{v} \in K} J(\boldsymbol{v}) \end{cases} \tag{11}$$

✓ To penalize the constraint $\boldsymbol{v} \cdot \boldsymbol{n} \leq 0$, we introduce the functional $J_\varepsilon(\cdot)$

$$J_\varepsilon(\boldsymbol{v}) = J(\boldsymbol{v}) + \frac{1}{2\varepsilon} \left\| [\boldsymbol{v} \cdot \boldsymbol{n}]^+ \right\|_{0,\Gamma_C}^2, \ \ \forall \, \boldsymbol{v} \in V. \tag{12}$$

✓ The penalized contact problem consists of:

$$\begin{cases} \text{Find } \boldsymbol{u}_\varepsilon \in V \text{ such that,} \\ J_\varepsilon(\boldsymbol{u}_\varepsilon) = \min_{\boldsymbol{v} \in V} J_\varepsilon(\boldsymbol{v}) \end{cases} \tag{13}$$

✓ The first-order optimality condition of problem (13) is written as [2]

$$\begin{cases} \text{Find } \boldsymbol{u}_\varepsilon \in \boldsymbol{V}, \text{ such that,} \\ a(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) + \dfrac{1}{\varepsilon} \displaystyle\int_{\Gamma_C} \left[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n}\right]^+ (\boldsymbol{v} \cdot \boldsymbol{n}) ds = L(\boldsymbol{v}), \ \forall \, \boldsymbol{v} \in \boldsymbol{V}. \end{cases} \qquad (14)$$

✓ From (14), we have the penalized contact force $\boldsymbol{\sigma}_n(\boldsymbol{u}_\varepsilon)$ given by

$$\boldsymbol{\sigma}_n(\boldsymbol{u}_\varepsilon) = -\frac{1}{\varepsilon} \left[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n}\right]^+, \ \text{ on } \Gamma_C, \ \forall \, \varepsilon > 0. \qquad (15)$$

The penalty method is equivalent to reformulating the contact condition in (4) under the form in (15).

The penalized contact force in (15) is equivalent to the following:

$$\boldsymbol{\sigma}_n(\boldsymbol{u}_\varepsilon) = -\frac{1}{2\varepsilon}\Big[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \varepsilon\boldsymbol{\sigma}_n(\boldsymbol{u}_\varepsilon)\Big]^+, \quad \text{on } \Gamma_C, \ \forall \, \varepsilon > 0. \qquad (16)$$

Preuve :

See paper …

✓ With respect to the penalized contact force (16), we have the variants:

$$
\begin{cases}
\text{Find } \boldsymbol{u}_\varepsilon \in \boldsymbol{V}_N, \text{ such that,} \\
a(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) - \theta \displaystyle\int_{\Gamma_C} \varepsilon \boldsymbol{\sigma}_n(\boldsymbol{u}_\varepsilon) \boldsymbol{\sigma}_n(\boldsymbol{v}) ds \\
\qquad + \displaystyle\int_{\Gamma_C} \frac{1}{2\varepsilon} \Big[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \varepsilon \boldsymbol{\sigma}_n(\boldsymbol{u}_\varepsilon) \Big]^{+} \Big(\boldsymbol{v} \cdot \boldsymbol{n} - \theta \varepsilon \boldsymbol{\sigma}_n(\boldsymbol{v}) \Big) ds = L(\boldsymbol{v})
\end{cases}
\qquad (17)
$$

for all $\boldsymbol{v} \in \boldsymbol{V}_N$, where the parameter θ takes the values $\theta = -1, 0, 1$.

✓ Penalization is asymptotically equivalent to Nitsche (i.e. si $\varepsilon \to 0$).

✓ The penalty method is asymptotically consistent.

✓ The penalty variant $\theta = -1$ from (17), is better conditioned than the standard one [1].

Lemme

The penalized contact force in (15) is equivalent to the following:

$$\boldsymbol{\sigma}_n(\boldsymbol{u}_\varepsilon) = -\frac{2}{\varepsilon}\left[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} + \frac{\varepsilon}{2}\boldsymbol{\sigma}_n(\boldsymbol{u}_\varepsilon)\right]^+, \quad \text{on } \Gamma_C, \, \forall \, \varepsilon > 0. \qquad (18)$$

Preuve :

See paper ...

✓ With respect to the penalized contact force (18), we have these variants:

$$
\begin{cases}
\text{Find } \boldsymbol{u}_\varepsilon \in \boldsymbol{V}_N, \text{ such that,} \\[2mm]
a(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) + \theta \displaystyle\int_{\Gamma_C} \frac{\varepsilon}{2} \boldsymbol{\sigma}_n(\boldsymbol{u}_\varepsilon) \boldsymbol{\sigma}_n(\boldsymbol{v}) ds \\[4mm]
\qquad + \displaystyle\int_{\Gamma_C} \frac{2}{\varepsilon} \Big[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} + \frac{\varepsilon}{2} \boldsymbol{\sigma}_n(\boldsymbol{u}_\varepsilon)\Big]^+ \Big(\boldsymbol{v} \cdot \boldsymbol{n} + \theta \frac{\varepsilon}{2} \boldsymbol{\sigma}_n(\boldsymbol{v})\Big) ds = L(\boldsymbol{v}),
\end{cases}
\tag{19}
$$

for all $\boldsymbol{v} \in \boldsymbol{V}_N$, where the parameter θ takes the values $\theta = -1, 0, 1$.

✓ For $\theta = 1$, the formulation (19) is both symmetric and coercive:

- ⋆ This formulation is better conditioned than the standard one.

- ⋆ It is preferable in solving large scale systems.

✓ We have a coercive formulation for any $\theta \in \left]3 - \sqrt{8},\ 3 + \sqrt{8}\right[$.

Théorème

1. Problem (19) admits a unique solution, for all $\varepsilon > 0$ and for all $\theta \in \left]3 - \sqrt{8}, 3 + \sqrt{8}\right[$.

2. There exists $\varepsilon_0 > 0$ such that for $\varepsilon > \varepsilon_0$, problem (19) admits a unique solution for all $\theta \in \mathbb{R}\setminus\left]3 - \sqrt{8}, 3 + \sqrt{8}\right[$.

Preuve :
See paper ...

✓ To prove the existence and uniqueness of the solution, we need:

  ⋆ Korn's inequality.

  ⋆ Brezis' argument for M-like and pseudo-monotonic operators.

Théorème

Let $\Omega \subset \mathbb{R}^d, d = 2, 3$, be a bounded domain and $\boldsymbol{u}, \boldsymbol{u}_\varepsilon \in \left(H^{\frac{3}{2}+\nu}(\Omega)\right)^d$, $0 < \nu \leq \frac{1}{2}$, the solutions of problems (7) and (19), respectively. We obtain the a priori error estimate

$$\|\boldsymbol{u} - \boldsymbol{u}_\varepsilon\|_{1,\Omega} + \sqrt{\varepsilon}\left\|\boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon}\left[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} + \frac{\varepsilon}{2}\boldsymbol{\sigma}_n(\boldsymbol{u}_\varepsilon)\right]^+\right\|_{0,\Gamma_C} \leq c\varepsilon^{\frac{1}{2}+\nu}\|\boldsymbol{u}\|_{\frac{3}{2}+\nu,\Omega}$$

for all $\varepsilon > 0$, where $c > 0$ is a constant, independent of ε and of $\boldsymbol{u}$.

Preuve :
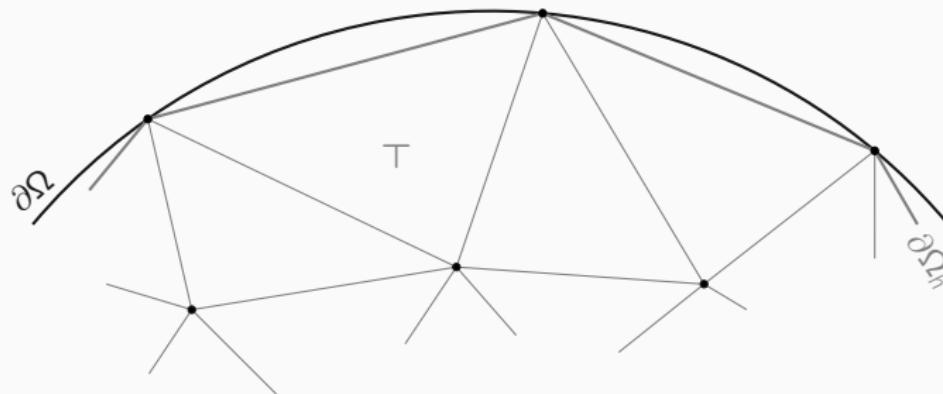
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Numerical analysis and simulation

✓ For the finite element approximation, we mesh the domain

$$\overline{\Omega}_h := \bigcup_{\top \in \mathcal{T}_h} \top, \qquad (20)$$

✓ At each point, we define a piecewise polynomial ϕ_i of degree $k = 1, 2$.

✓ We thus introduce the finite dimensional space V_h (P_k finite element)

$$V_h = Vect\{\phi_i\} \subset H^1(\Omega) \quad \text{et} \quad dim(V_h) = N.$$

✓ The finite element problem is to find $\boldsymbol{u}_{\varepsilon h} \in (V_h)^d$ such that

$$a(\boldsymbol{u}_{\varepsilon h}, \boldsymbol{v}_h) + \theta \int_{\Gamma_C} \frac{\varepsilon}{2} \boldsymbol{\sigma}_n(\boldsymbol{u}_{\varepsilon h}) \boldsymbol{\sigma}_n(\boldsymbol{v}_h) \, d\boldsymbol{s}$$

$$+ \frac{2}{\varepsilon} \int_{\Gamma_C} \left[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} + \frac{\varepsilon}{2} \boldsymbol{\sigma}_n(\boldsymbol{u}_{\varepsilon h}) \right]^+ \left(\boldsymbol{v}_h \cdot \boldsymbol{n} + \theta \frac{\varepsilon}{2} \boldsymbol{\sigma}_n(\boldsymbol{v}_h) \right) d\boldsymbol{s} = L(\boldsymbol{v}_h),$$

for all $\boldsymbol{v} \in (V_h)^d$.

Théorème
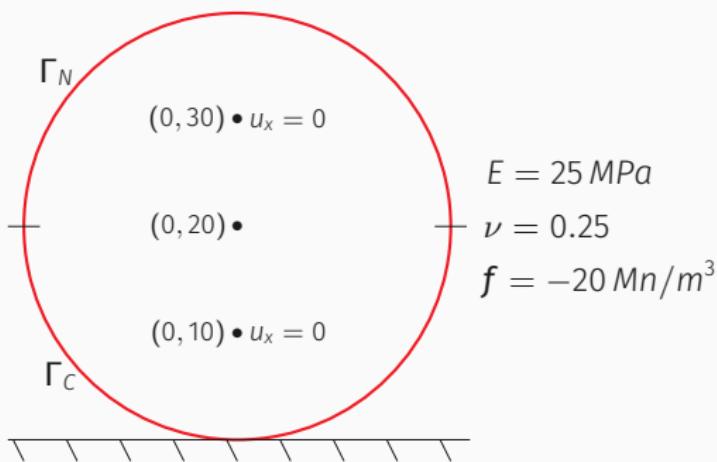
Under the conditions of the previous theorem, if the solution $\boldsymbol{u}$ of the problem (7) belongs to $\left(H^{\frac{3}{2}+\nu}(\Omega) \right)^d$, with $0 < \nu \leq k - \frac{1}{2}$ and $k = 1, 2$ is the degree of the finite element method, then for any $\varepsilon > 0$ we obtain

$$\|\boldsymbol{u} - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega} + \left(\sqrt{\varepsilon} - \sqrt{h} \right) \left\| \boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon} \left[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} + \frac{\varepsilon}{2} \boldsymbol{\sigma}_n(\boldsymbol{u}_{\varepsilon h}) \right]^+ \right\|_{0,\Gamma_C}$$

$$\leq c \left(h^{\frac{1}{2}+\nu} + \varepsilon^{\frac{1}{2}+\nu} \right) \|\boldsymbol{u}\|_{\frac{3}{2}+\nu,\Omega},$$

where $c > 0$ is a constant independent of $\boldsymbol{u}$, the regularization parameter ε and the mesh size h.
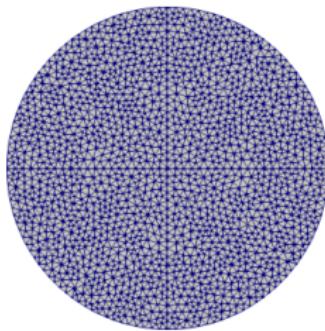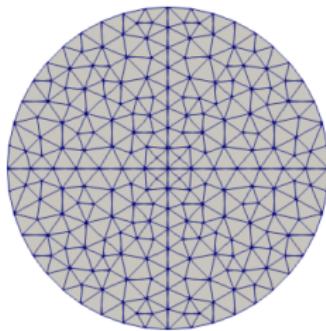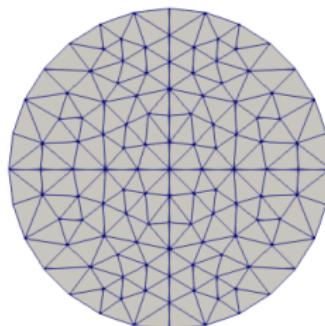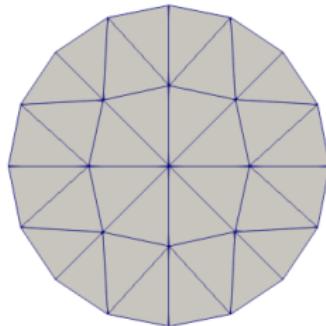
✓ Illustration of the contact problem by the Hertz model (in $\mathbb{R}^2$).

✓ Ω is a disc of radius $20\,cm$, centered at $(0, 20)$ where the contact area is
$$\Gamma_C = \{(x, y) \mid x^2 + y^2 = 20^2 \text{ et } y < 20 \text{ cm}\}$$

Γ_N

$(0, 30) \bullet u_x = 0$

$E = 25\,MPa$

$(0, 20) \bullet$

$\nu = 0.25$

$f = -20\,Mn/m^3$

$(0, 10) \bullet u_x = 0$

Γ_C

✓ E is the Young's modulus, ν is the Poisson coefficient of the material.
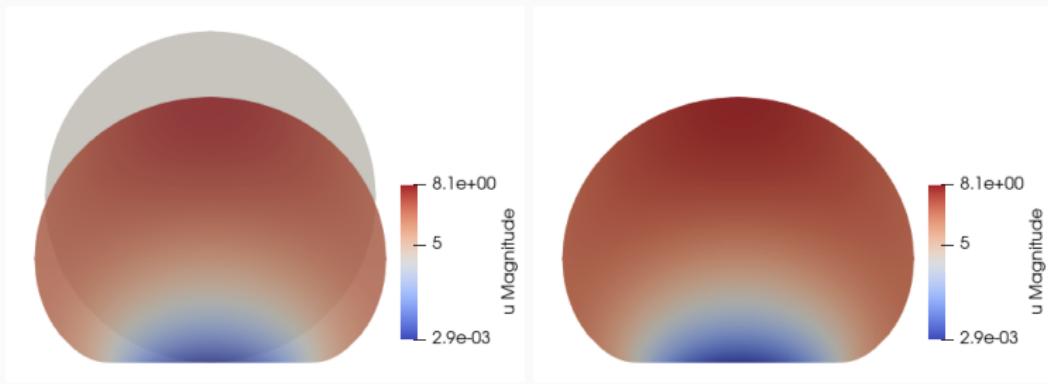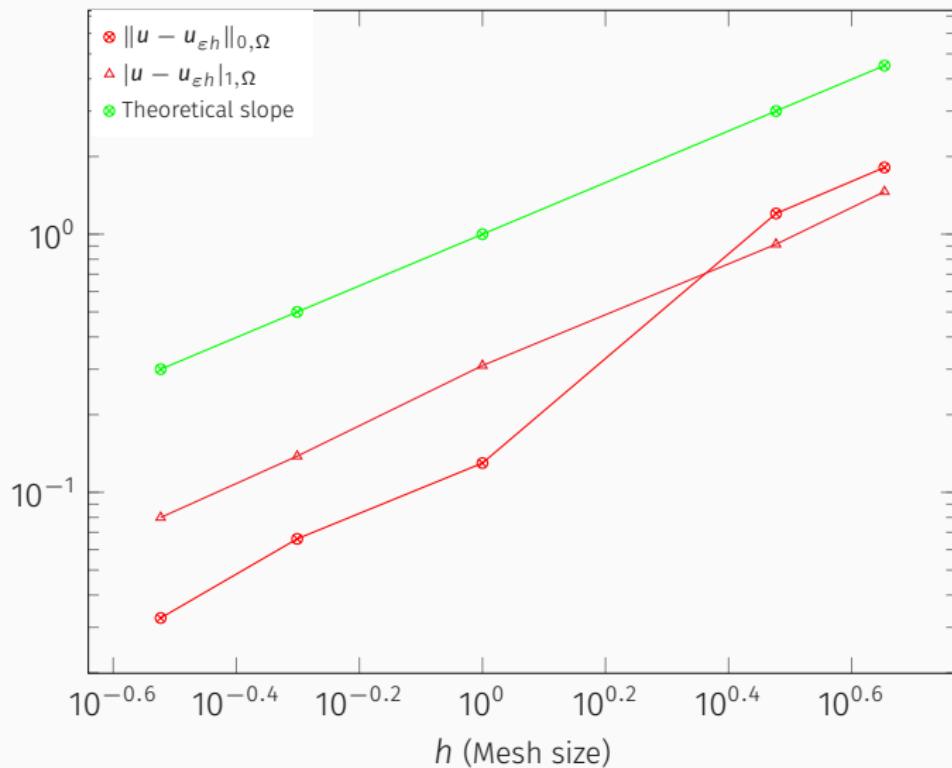
✓ At each mesh, the numerical solution is compared with that of reference obtained by a finest mesh ($h = 0.13$) and with isoparametric elements.
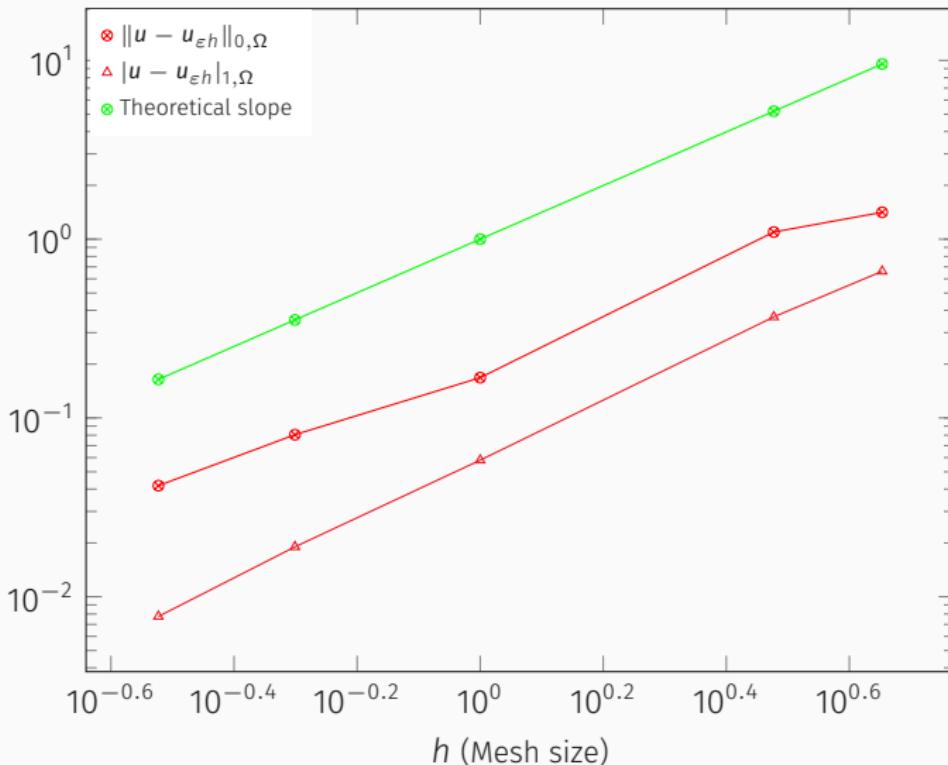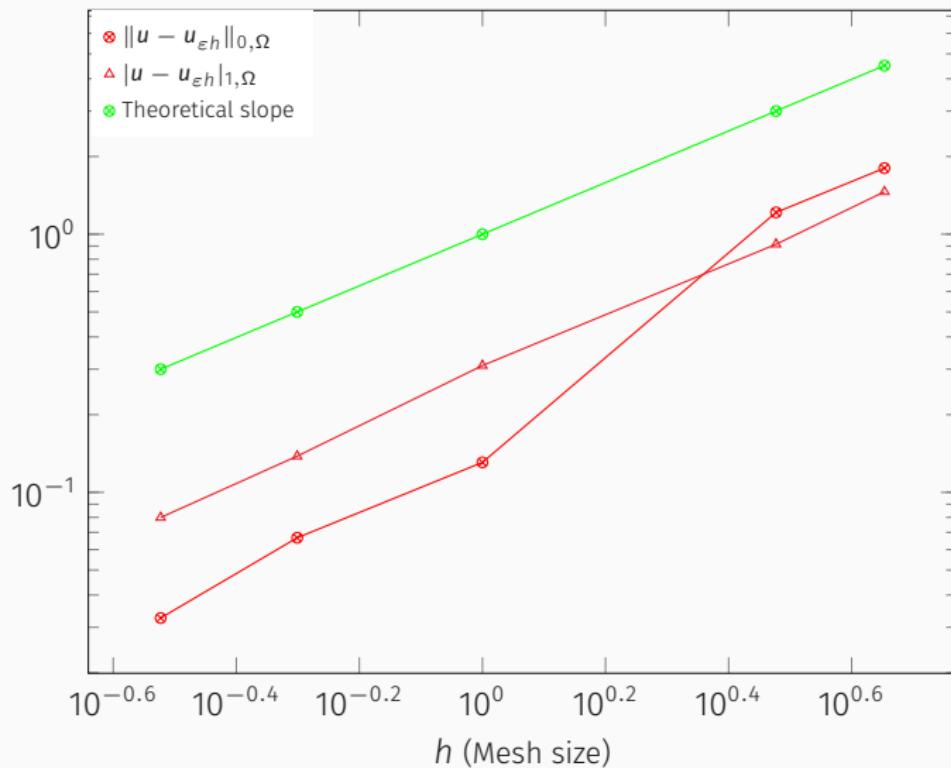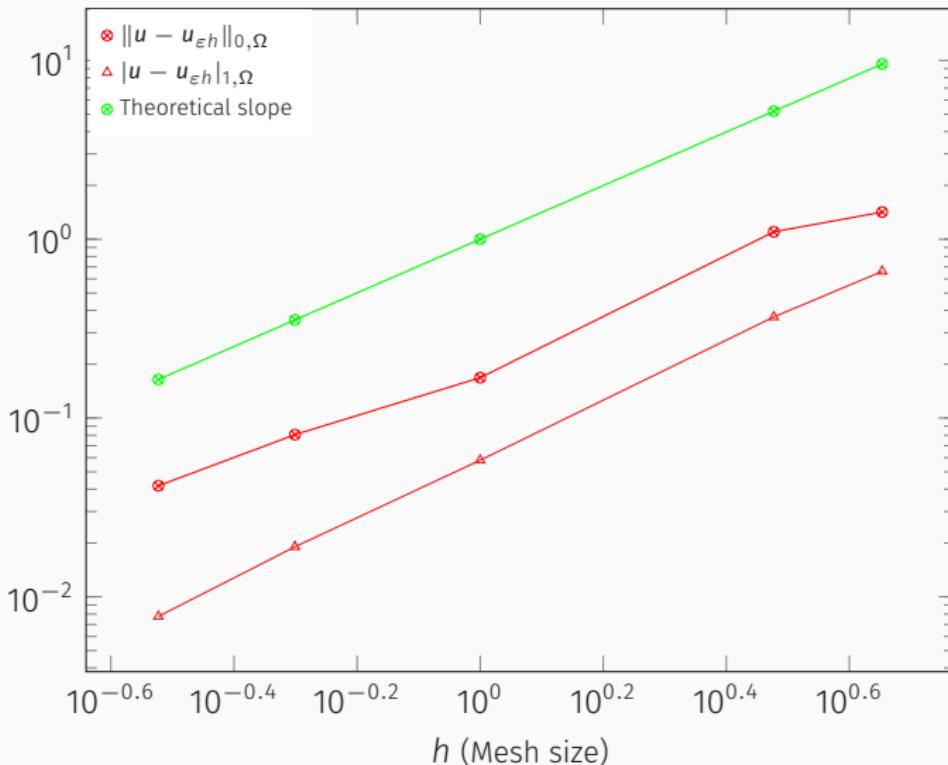
P_1 Finite element with $\varepsilon = h$

✓ P_1 Finite element with $\varepsilon = h$

✓ Analysis and simulation of the contact in large deformation.

✓ Interface problems with compatible and non-compatible meshes.

✓ Multi-body contact problems:

 ★ Contact in small deformations.

 ★ Contact in large deformations.

 ★ quasi-incompressible materials case.

✓ Deformable-deformable contact.

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Thank you for your attention! Questions or comments?