Variants of the penalty method for contact problems

Formulations unifying Nitsche and penalty methods

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- The unilateral contact problem
- Nitsche formulation of the contact problem
- Penalty method and its variants for the contact problem
- Numerical analysis and simulation

- ✓ A simple and well-known contact problem: unilateral contact problem.
- ✓ This problem plays an important role in industry or in biomechanics.
- ✓ Contact conditions are nonlinear and are under inequalities form.
- ✓ Contact phenomena modeling poses many difficulties.
- ✓ In general, their simulation is done by the Finite Element Method.
- ✓ Easy to implement in practice and precise from a theoretical point of view.

✓ Techniques which make it possible to effectively incorporate boundary conditions by finite elements are:

- ⋆ mixed methods, the penalty method,
- ⋆ Nitsche's method, stabilized methods.

✓ We will see here two techniques: Nitsche method and penalty method.

Goal: New penalty methods for contact:

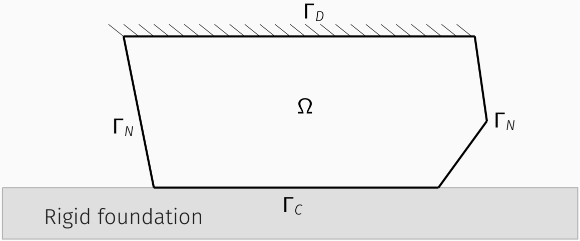
- ⋆ Formulations asymptotically equivalent to the Nitsche method.
- ⋆ Others more stable and better suited to large linear systems.

The unilateral contact problem

Unilateral contact problem (continuous form)

✓ To model an elastic body of reference configuration $\Omega \subset \mathbb{R}^d$, $d = 2, 3$:

- ⋆ The domain Ω is open, bounded, connected and non-empty.



- ⋆ Small elastic transformations (small deformations).

✓ Let $\boldsymbol{\sigma} = (\sigma_{i,j}), 1 \leq i, j \leq d$, be the stress tensor field,

✓ $\nabla \cdot$ is the divergence operator for tensor-valued functions,

✓ $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ is the displacement field of the elastic body,

✓ $\boldsymbol{\varepsilon}(\boldsymbol{u})$ represents the linearized strain tensor field,

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^t\right),$$

✓ $\boldsymbol{C} = (C_{ijkl})$ is the fourth order elasticity tensor, uniformly bounded:

- ⋆ symmetrical,

$$C_{ijkl} = C_{klij} = C_{jikl}, \; \forall\, 1 \leq i,\, j,\, l,\, k \leq d,$$

- ⋆ elliptical, i.e. there exists $\alpha > 0$ such as

$$(\boldsymbol{C}(\boldsymbol{x})\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}) \geq \alpha \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}, \; \forall\, \boldsymbol{x} \in \Omega.$$

✓ Find the displacements field $\boldsymbol{u} : \Omega \to \mathbb{R}^d$, of the elastic body Ω,

✓ satisfying the linear elasticity equations:

$$-\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{f}, \qquad \text{in } \Omega, \tag{1}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{C} : \boldsymbol{\varepsilon}(\boldsymbol{u}), \qquad \text{in } \Omega,$$

✓ as well as the boundary conditions of Dirichlet, Neumann and contact:

$$\boldsymbol{u} = \boldsymbol{0}, \qquad \text{on } \Gamma_D, \tag{2}$$

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{g}, \qquad \text{on } \Gamma_N, \tag{3}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} \leq 0, \quad \boldsymbol{\sigma}_n(\boldsymbol{u}) \leq 0, \quad (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{\sigma}_n(\boldsymbol{u}) = 0, \qquad \text{on } \Gamma_C, \tag{4}$$

$$\boldsymbol{\sigma}_t(\boldsymbol{u}) = 0, \qquad \text{on } \Gamma_C. \tag{5}$$

Variational formulation of the contact problem

✓ The total mechanical energy of the elastic body is defined by

$$J(\boldsymbol{u}) := \frac{1}{2} a(\boldsymbol{u}, \boldsymbol{u}) - L(\boldsymbol{u}),$$

✓ where the bilinear $a(\cdot, \cdot)$ and linear $L(\cdot)$ forms are defined by

$$a(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \ \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V},$$

$$L(\boldsymbol{v}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx + \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{v}\, ds, \ \forall\, \boldsymbol{v} \in \boldsymbol{V}.$$

✓ We introduce the Hilbert space $\boldsymbol{V}$ of admissible displacements [3]

$$\boldsymbol{V} := \left\{ \boldsymbol{v} \in \left(H^1(\Omega) \right)^d : \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \ \Gamma_D \right\}.$$

✓ The convex cone K of admissible displacements (satisfying the non-penetration on the contact boundary Γ_C):

$$K := \left\{ \boldsymbol{v} \in \boldsymbol{V} : \boldsymbol{v} \cdot \boldsymbol{n} \leq 0 \ \text{ on } \ \Gamma_C \right\}.$$

✓ The problem (without friction) of unilateral contact [5] consists in:

$$\begin{cases} \text{Find } \boldsymbol{u} \in K, \ \text{such that,} \\ J(\boldsymbol{u}) = \min_{\boldsymbol{v} \in K} J(\boldsymbol{v}) \end{cases} \tag{6}$$

This is a minimization problem under inequality constraints.

✓ $J(\cdot)$ being Fréchet-differentiable, we have the variational inequality [5, 4]

$$\langle J'(\boldsymbol{u}), \boldsymbol{v} - \boldsymbol{u} \rangle \geq 0, \ \forall \, \boldsymbol{v} \in K.$$

✓ Any solution $\boldsymbol{u}$ of problem (6), is also a solution of problem [5]

$$\begin{cases} \text{Find } \boldsymbol{u} \in K, \ \text{such that,} \\ a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) \geq L(\boldsymbol{v} - \boldsymbol{u}), \ \forall \, \boldsymbol{v} \in K. \end{cases} \tag{7}$$

Nitsche formulation of the contact problem

Variational formulation of the problem

✓ Nitsche's method is based on the reformulation of the conditions in (4):

$$\boldsymbol{\sigma}_n(\boldsymbol{u}) = -\frac{1}{\gamma}\left[\boldsymbol{u}\cdot\boldsymbol{n} - \gamma\boldsymbol{\sigma}_n(\boldsymbol{u})\right]^+ \quad \text{on } \Gamma_C,\ \forall\,\gamma > 0.$$

✓ The unilateral contact problem is now written as:

$$\begin{cases} \text{Find } \boldsymbol{u} \in V_N, \text{ such that,} \\ J_N(\boldsymbol{u}) = \min\limits_{\boldsymbol{v}\in V_N} J_N(\boldsymbol{v}) \end{cases} \tag{8}$$

This is a minimization problem «without» inequality constraints.

✓ The functional $J_N(\cdot)$ is defined as follows:

$$\begin{aligned} J_N(\boldsymbol{u}) := J(\boldsymbol{u}) &- \frac{1}{2}\int_{\Gamma_C} \gamma\,(\boldsymbol{\sigma}_n(\boldsymbol{u}))^2\, ds \\ &+ \frac{1}{2}\int_{\Gamma_C} \frac{1}{\gamma}\left(\left[\boldsymbol{u}\cdot\boldsymbol{n} - \gamma\boldsymbol{\sigma}_n(\boldsymbol{u})\right]^+\right)^2 ds, \end{aligned}$$

✓ where the (more regular) space $\boldsymbol{V}_N$ is defined as follows:

$$\boldsymbol{V}_N = \left\{ \boldsymbol{v} \in \boldsymbol{V} : \nabla \boldsymbol{v}_{|\Gamma_C} \in \left(L^2(\Gamma_C) \right)^{d \times d} \right\}.$$

✓ Then, the first-order optimality condition is written in the following:

$$\left\langle J_N'(\boldsymbol{u}), \boldsymbol{v} \right\rangle = 0, \ \forall \, \boldsymbol{v} \in \boldsymbol{V}_N.$$

✓ The solution $\boldsymbol{u}$ of Problem (8) is a solution of the variational problem

$$\begin{cases} \text{Find } \boldsymbol{u} \in \boldsymbol{V}_N, \ \text{such that,} \\ a(\boldsymbol{u}, \boldsymbol{v}) - \displaystyle\int_{\Gamma_C} \gamma \boldsymbol{\sigma}_n(\boldsymbol{u}) \boldsymbol{\sigma}_n(\boldsymbol{v}) ds \\ \qquad\qquad + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma} \left[\boldsymbol{u} \cdot \boldsymbol{n} - \gamma \boldsymbol{\sigma}_n(\boldsymbol{u}) \right]^+ \left(\boldsymbol{v} \cdot \boldsymbol{n} - \gamma \boldsymbol{\sigma}_n(\boldsymbol{v}) \right) ds = L(\boldsymbol{v}), \end{cases} \tag{9}$$

for all $\boldsymbol{v} \in \boldsymbol{V}_N$.

- ✓ Variants of formulation (9) have been proposed in the form [1]:

$$\begin{cases} \text{Find } \boldsymbol{u} \in \boldsymbol{V}_N, \text{ such that,} \\ a(\boldsymbol{u},\boldsymbol{v}) - \theta \displaystyle\int_{\Gamma_C} \gamma \boldsymbol{\sigma}_n(\boldsymbol{u}) \boldsymbol{\sigma}_n(\boldsymbol{v}) ds \\ \qquad + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma} \left[\boldsymbol{u}\cdot\boldsymbol{n} - \gamma \boldsymbol{\sigma}_n(\boldsymbol{u})\right]^{+} \Big(\boldsymbol{v}\cdot\boldsymbol{n} - \theta\gamma\boldsymbol{\sigma}_n(\boldsymbol{v})\Big) ds = L(\boldsymbol{v}), \end{cases} \tag{10}$$

  for all $\boldsymbol{v} \in \boldsymbol{V}_N$, where the parameter θ takes the values $\theta = -1, 0, 1$.

- ✓ $\theta = 1$, symmetrical formulation but coercivity obtained from a $\gamma_0 > 0$.
- ✓ $\theta = -1$, anti-symmetric formulation but coercive for all $\gamma > 0$.
- ✓ $\theta = 0$, the standard penalty method which is coercive for all $\gamma > 0$.
- ✓ All of these Nitsche variants are consistent.

Penalty method and its variants for the contact problem

Penalized formulation of the contact problem

✓ Recall: The unilateral contact (frictionless) problem consists of:

$$\begin{cases} \text{Find } \boldsymbol{u} \in K \text{ such that,} \\ J(\boldsymbol{u}) = \min_{\boldsymbol{v} \in K} J(\boldsymbol{v}) \end{cases} \tag{11}$$

✓ To penalize the constraint $\boldsymbol{v} \cdot \boldsymbol{n} \leq 0$, we introduce the functional $J_\varepsilon(\cdot)$

$$J_\varepsilon(\boldsymbol{v}) = J(\boldsymbol{v}) + \frac{1}{2\varepsilon} \left\| [\boldsymbol{v} \cdot \boldsymbol{n}]^+ \right\|^2_{0,\Gamma_C}, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}. \tag{12}$$

✓ The penalized contact problem consists of:

$$\begin{cases} \text{Find } \boldsymbol{u}_\varepsilon \in \boldsymbol{V} \text{ such that,} \\ J_\varepsilon(\boldsymbol{u}_\varepsilon) = \min_{\boldsymbol{v} \in \boldsymbol{V}} J_\varepsilon(\boldsymbol{v}) \end{cases} \tag{13}$$

✓ The first-order optimality condition of problem (13) is written as [2]

$$\begin{cases} \text{Find } \boldsymbol{u}_\varepsilon \in \boldsymbol{V}, \text{ such that,} \\ a(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) + \frac{1}{\varepsilon} \int_{\Gamma_C} [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n}]^+ (\boldsymbol{v} \cdot \boldsymbol{n}) ds = L(\boldsymbol{v}), \ \forall\, \boldsymbol{v} \in \boldsymbol{V}. \end{cases} \tag{14}$$

✓ From (14), we have the penalized contact force $\boldsymbol{\sigma}_n(\boldsymbol{u}_\varepsilon)$ given by

$$\boldsymbol{\sigma}_n(\boldsymbol{u}_\varepsilon) = -\frac{1}{\varepsilon} [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n}]^+, \ \text{ on } \ \Gamma_C, \ \forall\, \varepsilon > 0. \tag{15}$$

The penalty method is equivalent to reformulating the contact condition in (4) under the form in (15).

Lemme

The penalized contact force in (15) is equivalent to the following:

$$\boldsymbol{\sigma}_n(\boldsymbol{u}_\varepsilon) = -\frac{1}{2\varepsilon}\left[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \varepsilon \boldsymbol{\sigma}_n(\boldsymbol{u}_\varepsilon)\right]^+, \quad \text{on} \quad \Gamma_C, \; \forall\, \varepsilon > 0. \tag{16}$$

Preuve :

See paper ... $\square$

✓ With respect to the penalized contact force (16), we have the variants:

$$\begin{cases} \text{Find } \boldsymbol{u}_\varepsilon \in \boldsymbol{V}_N, \text{ such that,} \\ a(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) - \theta \displaystyle\int_{\Gamma_C} \varepsilon \boldsymbol{\sigma}_n(\boldsymbol{u}_\varepsilon)\boldsymbol{\sigma}_n(\boldsymbol{v}) ds \\ \qquad + \displaystyle\int_{\Gamma_C} \frac{1}{2\varepsilon} \Big[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \varepsilon \boldsymbol{\sigma}_n(\boldsymbol{u}_\varepsilon)\Big]^+ \Big(\boldsymbol{v} \cdot \boldsymbol{n} - \theta\varepsilon\boldsymbol{\sigma}_n(\boldsymbol{v})\Big) ds = L(\boldsymbol{v}) \end{cases} \tag{17}$$

for all $\boldsymbol{v} \in \boldsymbol{V}_N$, where the parameter θ takes the values $\theta = -1, 0, 1$.

✓ Penalization is asymptotically equivalent to Nitsche (i.e. si $\varepsilon \to 0$).

✓ The penalty method is asymptotically consistent.

✓ The penalty variant $\theta = -1$ from (17), is better conditioned than the standard one [1].

Lemme

The penalized contact force in (15) is equivalent to the following:

$$\boldsymbol{\sigma}_n(\boldsymbol{u}_\varepsilon) = -\frac{2}{\varepsilon}\left[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} + \frac{\varepsilon}{2}\boldsymbol{\sigma}_n(\boldsymbol{u}_\varepsilon)\right]^+, \quad \text{on } \Gamma_C, \ \forall\, \varepsilon > 0. \tag{18}$$

Preuve :

See paper ... $\square$

✓ With respect to the penalized contact force (18), we have these variants:

$$\begin{cases} \text{Find } \boldsymbol{u}_\varepsilon \in \boldsymbol{V}_N, \text{ such that,} \\ a(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) + \theta \displaystyle\int_{\Gamma_C} \frac{\varepsilon}{2} \boldsymbol{\sigma}_n(\boldsymbol{u}_\varepsilon) \boldsymbol{\sigma}_n(\boldsymbol{v}) ds \\ \qquad\qquad + \displaystyle\int_{\Gamma_C} \frac{2}{\varepsilon} \left[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} + \frac{\varepsilon}{2} \boldsymbol{\sigma}_n(\boldsymbol{u}_\varepsilon) \right]^+ \left(\boldsymbol{v} \cdot \boldsymbol{n} + \theta \frac{\varepsilon}{2} \boldsymbol{\sigma}_n(\boldsymbol{v}) \right) ds = L(\boldsymbol{v}), \end{cases} \tag{19}$$

for all $\boldsymbol{v} \in \boldsymbol{V}_N$, where the parameter θ takes the values $\theta = -1, 0, 1$.

✓ For $\theta = 1$, the formulation (19) is both symmetric and coercive:

- ⋆ This formulation is better conditioned than the standard one.
- ⋆ It is preferable in solving large scale systems.

✓ We have a coercive formulation for any $\theta \in \left]3 - \sqrt{8},\ 3 + \sqrt{8}\right[$.

✓ In the following, we will focus on the formulation (19).

Théorème

1. Problem (19) admits a unique solution, for all $\varepsilon > 0$ and for all $\theta \in \left]3-\sqrt{8}, 3+\sqrt{8}\right[$.

2. There exists $\varepsilon_0 > 0$ such that for $\varepsilon > \varepsilon_0$, problem (19) admits a unique solution for all $\theta \in \mathbb{R} \setminus \left]3-\sqrt{8}, 3+\sqrt{8}\right[$.

Preuve :

See paper ... □

✓ To prove the existence and uniqueness of the solution, we need:

- ⋆ Korn's inequality.
- ⋆ Brezis' argument for M-like and pseudo-monotonic operators.

Théorème

Let $\Omega \subset \mathbb{R}^d, d = 2, 3$, be a bounded domain and $\boldsymbol{u}, \boldsymbol{u}_\varepsilon \in \left(H^{\frac{3}{2}+\nu}(\Omega)\right)^d$, $0 < \nu \leq \frac{1}{2}$, the solutions of problems (7) and (19), respectively. We obtain the a priori error estimate

$$\|\boldsymbol{u} - \boldsymbol{u}_\varepsilon\|_{1,\Omega} + \sqrt{\varepsilon}\left\|\boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon}\left[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} + \frac{\varepsilon}{2}\boldsymbol{\sigma}_n(\boldsymbol{u}_\varepsilon)\right]^+\right\|_{0,\Gamma_C} \leq c\varepsilon^{\frac{1}{2}+\nu}\|\boldsymbol{u}\|_{\frac{3}{2}+\nu,\Omega}$$

for all $\varepsilon > 0$, where $c > 0$ is a constant, independent of ε and of $\boldsymbol{u}$.

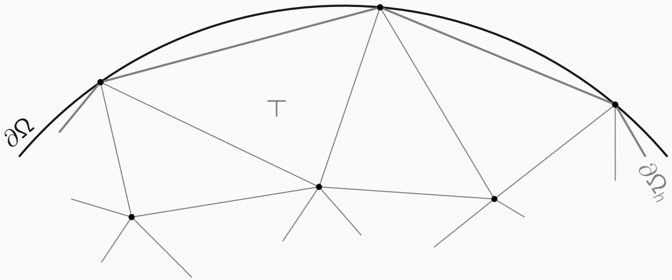
Preuve :

See paper ... □

Numerical analysis and simulation

✓ For the finite element approximation, we mesh the domain

$$\overline{\Omega}_h := \bigcup_{T \in \mathcal{T}_h} T, \tag{20}$$



✓ At each point, we define a piecewise polynomial ϕ_i of degree $k = 1, 2$.

✓ We thus introduce the finite dimensional space V_h (P_k finite element)

$$V_h = Vect\{\phi_i\} \subset H^1(\Omega) \text{ et } dim(V_h) = N.$$

✓ The finite element problem is to find $\boldsymbol{u}_{\varepsilon h} \in (V_h)^d$ such that

$$a(\boldsymbol{u}_{\varepsilon h}, \boldsymbol{v}_h) + \theta \int_{\Gamma_C} \frac{\varepsilon}{2} \boldsymbol{\sigma}_n(\boldsymbol{u}_{\varepsilon h}) \boldsymbol{\sigma}_n(\boldsymbol{v}_h)\, d\boldsymbol{s}$$
$$+ \frac{2}{\varepsilon} \int_{\Gamma_C} \left[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} + \frac{\varepsilon}{2} \boldsymbol{\sigma}_n(\boldsymbol{u}_{\varepsilon h})\right]^+ \left(\boldsymbol{v}_h \cdot \boldsymbol{n} + \theta \frac{\varepsilon}{2} \boldsymbol{\sigma}_n(\boldsymbol{v}_h)\right) d\boldsymbol{s} = L(\boldsymbol{v}_h),$$

for all $\boldsymbol{v} \in (V_h)^d$.

Théorème

Under the conditions of the previous theorem, if the solution $\boldsymbol{u}$ of the problem (7) belongs to $\left(H^{\frac{3}{2}+\nu}(\Omega)\right)^d$, with $0 < \nu \leq k - \frac{1}{2}$ and $k = 1, 2$ is the degree of the finite element method, then for any $\varepsilon > 0$ we obtain

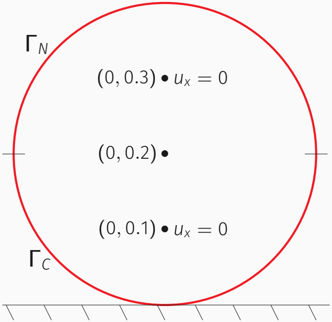
$$\|\boldsymbol{u} - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega} + \left(\sqrt{\varepsilon} - \sqrt{h}\right) \left\| \boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} + \frac{\varepsilon}{2} \boldsymbol{\sigma}_n(\boldsymbol{u}_{\varepsilon h})\right]^+ \right\|_{0,\Gamma_C}$$
$$\leq c \left(h^{\frac{1}{2}+\nu} + \varepsilon^{\frac{1}{2}+\nu}\right) \|\boldsymbol{u}\|_{\frac{3}{2}+\nu,\Omega},$$

where $c > 0$ is a constant independent of $\boldsymbol{u}$, the regularization parameter ε and the mesh size h.

Numerical simulations - Hertz model

- ✓ Illustration of the contact problem by the Hertz model (in $\mathbb{R}^2$).
- ✓ Ω is a disc of radius $0.2\,m$, centered at $(0, 0.2)$ where the contact area is

$$\Gamma_C = \{(x, y) \mid x^2 + (y - 0.2)^2 = 0.2^2 \text{ et } y < 0.2\}$$



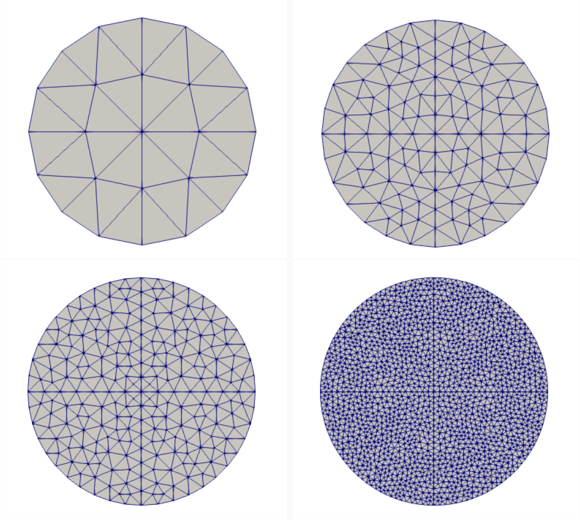
$E = 25\,MPa$

$\nu = 0.25$

$\boldsymbol{f} = -20\,Mn/m^3$

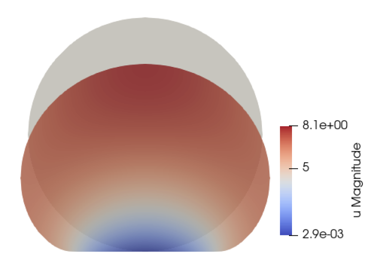
- ✓ E is the Young's modulus, ν is the Poisson coefficient of the material.

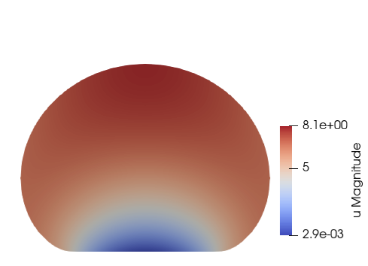
✓ Mesh Sequence of size: $h = 0.045, 0.03, 0.01, 0.005, 0.003$ (m)



✓ At each mesh, the numerical solution is compared with that of reference obtained by a finest mesh ($h = 0.0013$ m) and with isoparametric elements.

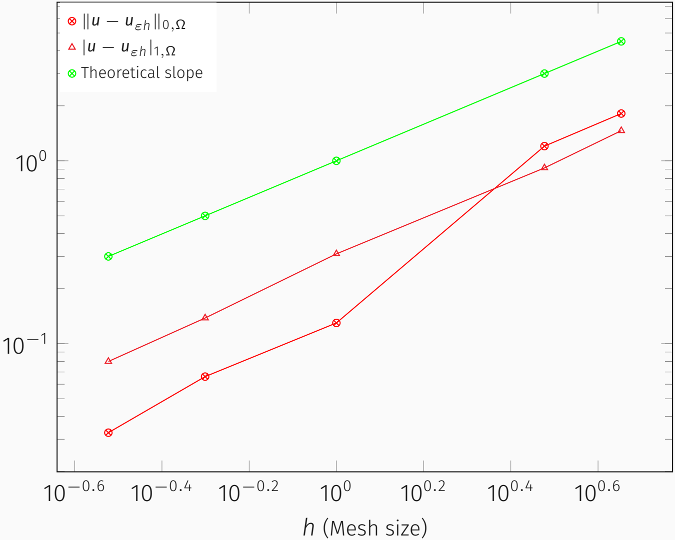
✓ Solution Visualization



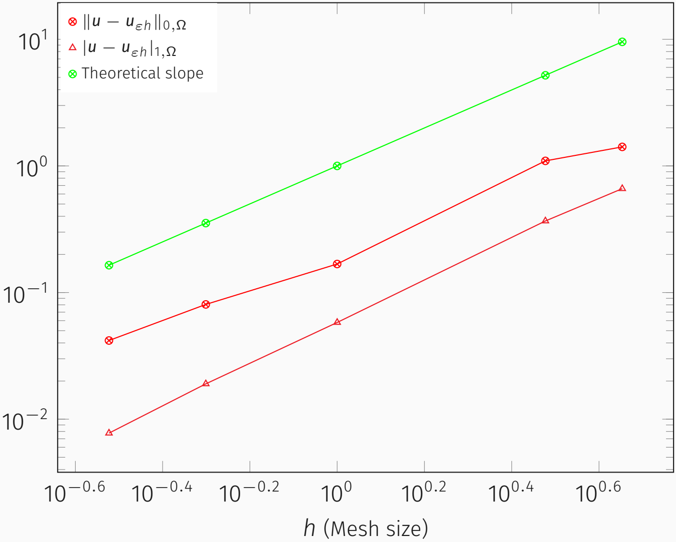


Numerical results ($\theta = 1$) - Hertz model

✓ P_1 Finite element with $\varepsilon = h$

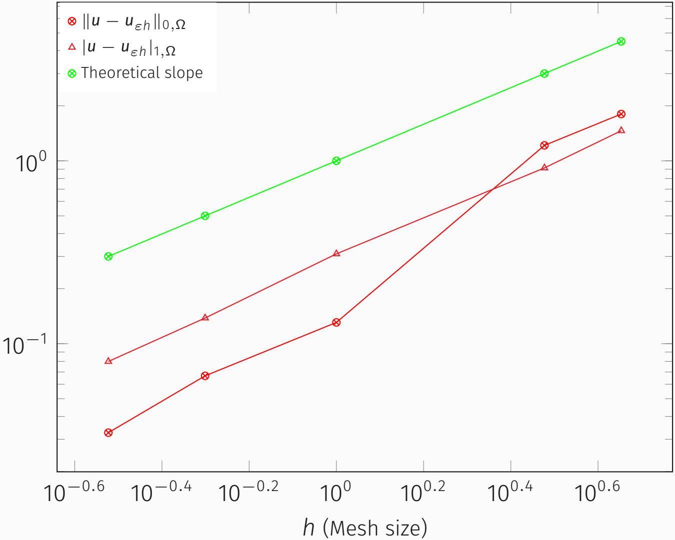


✓ P_2 Finite element with $\varepsilon = h$

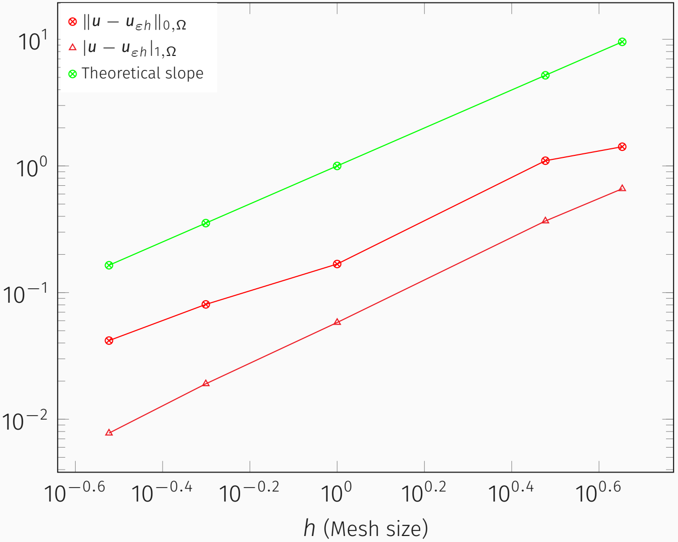


Numerical results ($\theta = -1$) - Hertz model

✓ P_1 Finite element with $\varepsilon = h$



✓ P_2 Finite element with $\varepsilon = h$



Prospects

- ✓ Analysis and simulation of the contact in large deformation.
- ✓ Interface problems with compatible and non-compatible meshes.
- ✓ Multi-body contact problems:
  - ⋆ Contact in small deformations.
  - ⋆ Contact in large deformations.
  - ⋆ quasi-incompressible materials case.
- ✓ Deformable-deformable contact.

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Thank you for your attention! Questions or comments?