CENG4513
MODELLING & SILMULATION

LECTURE #3

Randomness

Randomness in simulation is often modeled by using random variables and probability distributions.

Generating randomness requires algorithms that permit sequences of numbers to act as the underlying source of randomness within the model.

Randomness

- Randomness is essential to simulations having non-deterministic inputs. If input values are not random, then they are biased, leading to biased outputs. Thus, randomness is an assurance that uncertainty is fairly represented.
- Probability distributions are based on independent, random values. For example, we might say that an input parameter is "normally distributed with mean x and variance v." In modeling such a distribution, we have to be able to produce random values that are normally distributed with mean x and variance v.

Randomness

Clearly, within the context of computer simulation, it might be best to rely on algorithms; however, if an algorithm is used to generate the random numbers, then they cannot be truly random. For this reason, the random numbers that are used in computer simulation are called *pseudorandom*.

The algorithms that produce pseudorandom numbers are called random number generators.

Probability Review

Probability

- Probability is a measure of how likely it is for an event to happen.
- We name a probability with a number from 0 to 1.
- If an event is certain to happen, then the probability of the event is 1.
- If an event is certain not to happen, then the probability of the event is 0.

Probability

- According to Wikipedia
  - ***Probability*** *is a way of expressing knowledge or belief that an event will occur or has occurred.*
- Is widely used in mathematics, statistics, finance, gambling, science, and philosophy to draw conclusions about the likelihood of potential events and the underlying mechanics of complex systems.

Probability – What is an Event?

- In probability theory, an event is a set of outcomes (a subset of the sample space) to which a probability is assigned.
- Typically, when the sample space is finite, any subset of the sample space is an event (i.e. all elements of the power set of the sample space are defined as events).

Probability – Fundamentals

- We measure the probability for Random Events
  - How likely an event would occur
- The set of all possible events is called Sample Space
- In each experiment, an event may occur with a certain probability (Probability Measure)
- Example:
  - Tossing a dice with 6 faces
  - The sample space is {1, 2, 3, 4, 5, 6}
  - Getting the Event « 2 » in on experiment has a probability 1/6

Probability – Fundamentals

Example:

- A single card is pulled (out of 52 cards).
  - Possible Events
    - having a red card (P=1/2);
    - having a Jack (P= 1/13);
- Two true 6-sided dice are used to consider the event where the sum of the up faces is 10.

  $P = 3 / 36 = 1/12$

Probability – Fundamentals

- The probability of every set of possible events is between 0 and 1, inclusive.
- The probability of the whole set of outcomes is 1.
  - Sum of all probability is equal to one
  - Example for a dice: P(1)+P(2)+P(3)+ P(4)+P(5)+P(6)=1
- If A and B are two events with no common outcomes, then the probability of their union is the sum of their probabilities.
  - Event E1={1},
  - Event E2 ={6}
  - P(E1UE2)=P(E1)+P(E2)

Probability – Complementary Event

- Complementary Event of A is: not(A)
- P(A)=1-P(not A)
- The probability that event A will not happen is 1-P(A).
- Example
  - Event E1={1}
  - Probability to get a value different from {1} is 1-P(E1).

Probability – Joint Events

- Event Union (U = OR)   Event Intersection (∩= AND)
- Joint Probability (A ∩ B)
  - The probability of two events in conjunction. It is the probability of both events together.

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

- Independent Events
  - Two events A and B are independent if

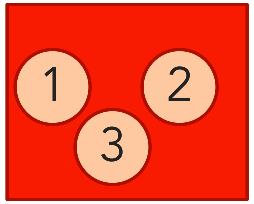
$$p(A \cap B) = p(A) \cdot p(B)$$

Probability – Joint Events

Example

E1: Drawing Ball 1 P(E1): 1/3
E2: Drawing Ball 2 P(E2):1/3
E3: Drawing Ball 3 P(E3): 1/3

$$p(A \cap B) = p(A) \cdot p(B)$$



Case 1: Drawing **with** replacement of the ball
The second draw is **independent** of the first draw

$$p(E1 \cap E2) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9} = p(E1) \cdot p(E2)$$

Case 2: Drawing **without** replacement of the ball
The second draw is **dependent** on the first draw

$$p(E1 \cap E2) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6} \neq p(E1) \cdot p(E2)$$

Probability – Conditional Probability

- **Conditional Probability p(AlB)** is the probability of some event A, given the occurrence of some other event B.

$$p(A \mid B) = \frac{p(A \cap B)}{p(B)}$$

$$p(B \mid A) = \frac{p(B \cap A)}{P(A)}$$

- If A and *B* are independent,

  then $p(A \mid B) = p(A)$ and $p(B \mid A) = p(B)$

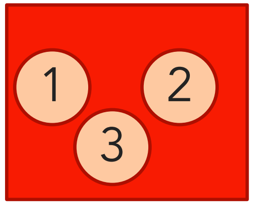
  - If A and B are independent, the conditional probability of *A*, given *B* is simply the individual probability of *A* alone; same for B given A.
- p(A) is the prior probability;
- p(A|B) is called a posterior probability.
- Once you know B is true, the universe you care about shrinks to B.

Probability – Conditional Probability

Example

E1: Drawing Ball 1 P(E1): 1/3
E2: Drawing Ball 2 P(E2):1/3
E3: Drawing Ball 3 P(E3): 1/3

$$p(A \mid B) = \frac{p(A \cap B)}{p(B)}$$



Case 1: Drawing **with** replacement of the ball
The second draw is **independent** of the first draw

$$p(E1 \mid E2) = \frac{1}{3}$$

$$p(E1 \mid E2) = \frac{p(E1 \cap E2)}{p(E2)} = \frac{\frac{1}{3} \cdot \frac{1}{3}}{\frac{1}{3}} = \frac{1}{3}$$

Case 2: Drawing **without** replacement of the ball
The second draw is **dependent** on the first draw

$$p(E1 \mid E2) = \frac{1}{2}$$

$$p(E1 \mid E2) = \frac{p(E1 \cap E2)}{p(E2)} = \frac{\frac{1}{3} \cdot \frac{1}{2}}{\frac{1}{3}} = \frac{1}{2}$$

Probability

- In probability theory, ***the law of total probability*** means the prior probability of A, P(A), is equal to the expected value of the posterior probability of A.
- That is, for any random variable N,

$$p(A) = E\left[p(A \mid N)\right]$$

  - where $p(A|N)$ is the conditional probability of A given N.

Probability

- Law of alternatives: The term *law of total probability* is sometimes taken to mean the law of alternatives, which is a special case of the law of total probability applying to discrete random variables.
- if { B_n : n = 1, 2, 3, ... } is a ***finite*** partition of a probability space and each set B_n is measurable, then for any event A we have

$$p(A) = \sum_n p(A \cap B_n)$$

or, alternatively, (using Rule of Conditional Probability)

$$p(A) = \sum_n p(A \mid B_n) \cdot p(B)$$

Probability

Example:

Sample Space $S = \{1,2,3,4,5,6,7\}$

Partitions $B1 = \{1,5\}$ $B2 = \{2,3,6\}$ $B3 = \{4,7\}$

Event $A = \{3\}$

Law of Total Probability

$$p(A) = p(\{3\}) = p(A \cap B_1) + p(A \cap B_2) + p(A \cap B_3)$$

$$= 0 + p(\{3\} \cap \{2,3,6\}) + 0 = p(\{3\})$$

Probability – Random Variable

- **Random Variable** is also known as **stochastic variable**.
  - random variable is defined as a quantity whose values are random and to which a **probability distribution** is assigned.

Examples:

- The number of packets that arrives to the destination
- The waiting time of a customer in a queue
- The number of cars that enters the parking each hour
- The number of students that succeed in the exam

Probability – Probability Distribution

- The probability distribution of a discrete random variable is a list of probabilities associated with each of its possible values.
- It is also sometimes called the probability function or the probability mass function (PMF) for discrete random variable.

Probability – Probability Mass Function

- Formally
  - the probability distribution or probability mass function (PMF) of a discrete random variable X is a function that gives the probability p(xi) that the random variable equals some value xi, for each value xi:

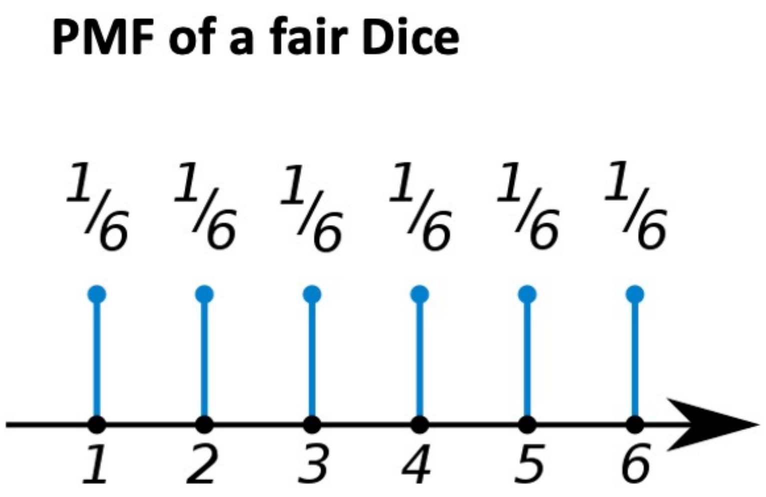
$$p(x_i) = P(X = x_i)$$

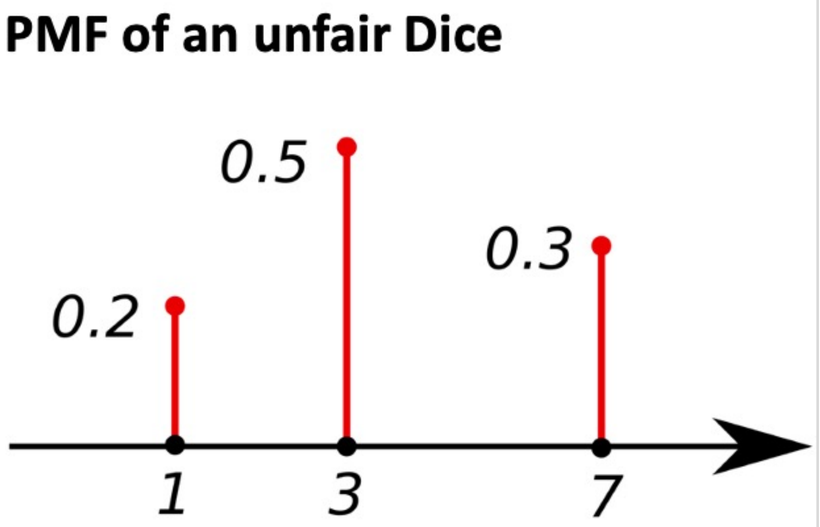
  - It satisfies the following conditions:

$$0 \le p(x_i) \le 1$$

$$\sum_i p(x_i) = 1$$

Probability – Probability Distribution





Probability – Continuous Random Variable

- A continuous random variable is one which takes an infinite number of possible values.
- Continuous random variables are usually measurements.
- Examples include height, weight, the amount of sugar in an orange, the time required to run a mile.

Probability – Continuous Random Variable

- For the case of continuous variables, we do not want to ask what the probability of "1/6" is, because the answer is always 0...
- Rather, we ask what is the probability that the value is in the interval (a,b).
- So for continuous variables, we care about the derivative of the distribution function at a point (that's the derivative of an integral). This is called a probability density function (PDF).
- The probability that a random variable has a value in a set A is the integral of the p.d.f. over that set A.

Probability – Probability Density Function (PDF)

- The Probability Density Function (PDF) of a continuous random variable is a function that can be integrated to obtain the probability that the random variable takes a value in a given interval.
- More formally, the probability density function, f(x), of a continuous random variable X is the derivative of the cumulative distribution function F(x):

$$f(x) = \frac{d}{dx} F(x)$$

- Since F(x)=P(X≤x), it follows that:

$$F(b) - F(a) = P(a \le X \le b) = \int_a^b f(x) \cdot dx$$

Probability – Cumulative Distribution Function (CDF)

- The Cumulative Distribution Function (CDF) is a function giving the probability that the random variable X is less than or equal to x, for every value x.
- Formally
  - the cumulative distribution function F(x) is defined to be:

$$\forall -\infty < x < +\infty,$$
$$F(x) = P(X \le x)$$

Probability – Cumulative Distribution Function (CDF)

- For a **discrete random variable**, the cumulative distribution function is found by summing up the probabilities as in the example below.

$$\forall -\infty < x < +\infty,$$

$$F(x) = P(X \le x) = \sum_{x_i \le x} P(X = x_i) = \sum_{x_i \le x} p(x_i)$$

- For a **continuous random variable**, the cumulative distribution function is the integral of its **probability density function f(x)**.

$$F(a) - F(b) = P(a \le X \le b) = \int_a^b f(x) \cdot dx$$

Probability – Cumulative Distribution Function (CDF)

- Example
  - Discrete case: Suppose a random variable X has the following probability mass function p(xi):

xi	0	1	2	3	4	5
$p(xi)$	1/32	5/32	10/32	10/32	5/32	1/32

- The cumulative distribution function F(x) is then:

xi	0	1	2	3	4	5
$F(xi)$	1/32	6/32	16/32	26/32	31/32	32/32

Probability – Cumulative Distribution Function (CDF)

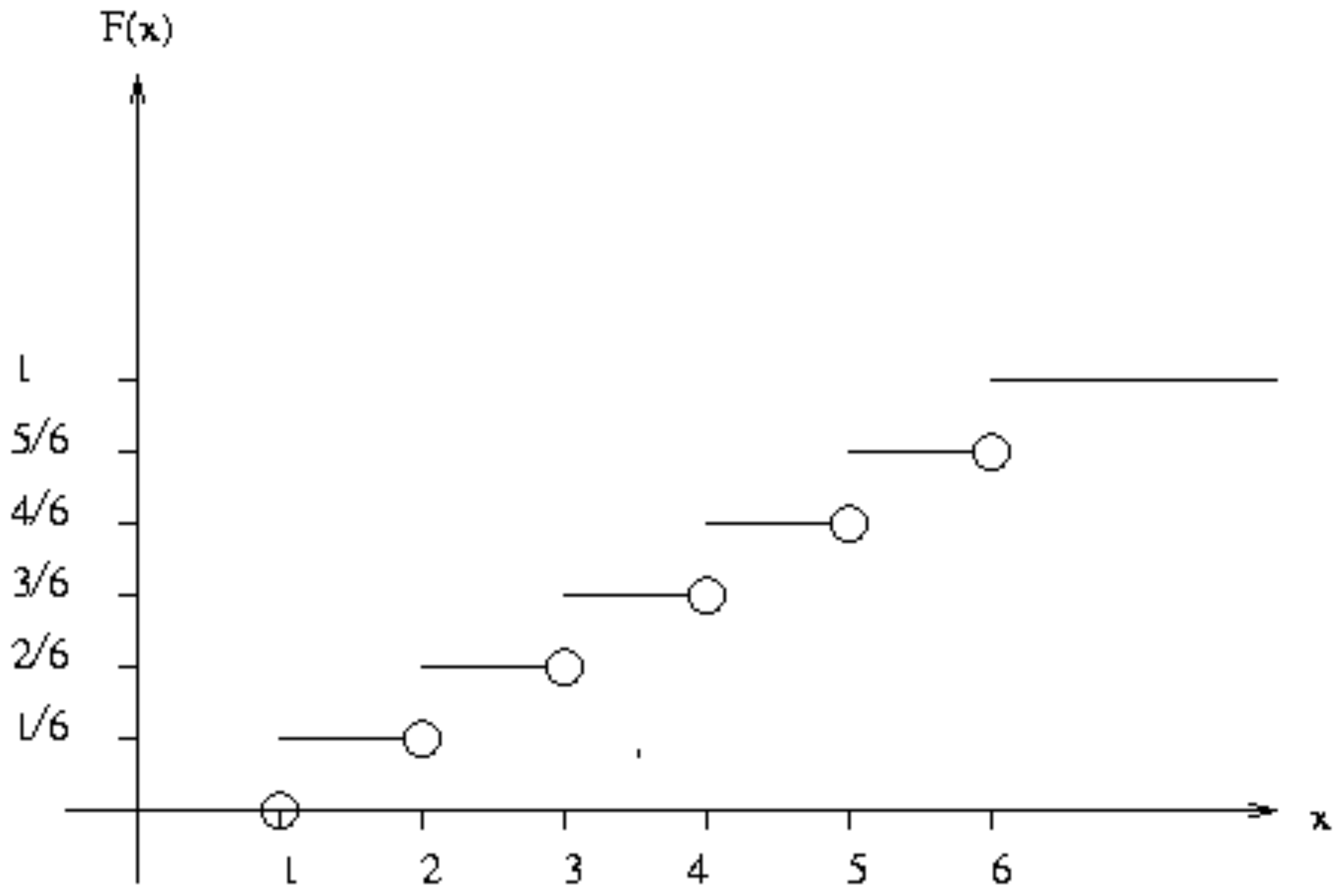


Figure 1.4: Fair die: Graph of the distribution function.

Probability – Mean (Expected Value)

Expectation of discrete random variable X

$$\mu_X = E(X) = \sum_{i=1}^{n} x_i \cdot p(x_i)$$

Expectation of continuous random variable X

$$\mu_X = E(X) = \int_{-\infty}^{+\infty} x \cdot f(x)\,dx$$

Probability – Mean (Expected Value)

- When a die is thrown, each of the possible faces 1, 2, 3, 4, 5, 6 (the xi's) has a probability of 1/6 (the p(xi)'s) of showing. The expected value of the face showing is therefore:

$$\mu = E(X) = (1 \times 1/6) + (2 \times 1/6) + (3 \times 1/6) + (4 \times 1/6) + (5 \times 1/6) + (6 \times 1/6) = 3.5$$

- Notice that, in this case, E(X) is 3.5, which is not a possible value of X.

Probability - Variance

- The **variance** is a measure of the '***spread***' of a distribution about its average value.

- Variance is symbolized by V(X) or Var(X) or $\boldsymbol{\sigma}^2$.
  - The **mean** is a way to describe the location of a distribution,
  - the **variance** is a way to capture its scale or degree of being spread out. The unit of variance is the square of the unit of the original variable.

Probability - Variance

- The Variance of the random variable X is defined as:

$$V(X) = \sigma_X^2 = E\left(X - E(X)\right)^2 = E\left(X^2\right) - E(X)^2$$

where E(X) is the expected value of the random variable X.

- The standard deviation is defined as the square root of the variance, i.e.:

$$\sigma_X = \sqrt{\sigma_X^2} = \sqrt{V(X)} = s$$

Probability - Coefficient of Variation

- The Coefficient of Variance of the random variable X is defined as:

$$CV(X) = \frac{V(X)}{E(X)} = \frac{\sigma_X}{\mu_X}$$