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MC 6461
Section 010

MATH 115 - Linear Algebra   Sep 7/2017

Chapter 1 - Euclidean Vector Spaces

Vectors: Magnitude, direction supported on a line $\mathbb{R}^n$

↳ $\mathbb{R}^2$ is set of all vectors in form $\begin{pmatrix}x_1\\x_2\end{pmatrix}$ ← real # (components)

↳ $\mathbb{R}^n$ " " $\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} = \vec{x}$ (curved arrow), $\mathbb{R}^n$: $\in \mathbb{R}^n$

Duality b/w pts + $\vec{v}$ in $\mathbb{R}^2$, $\mathbb{R}^n$ (can get pt $\rightleftarrows \vec{v}$)

Refer to gr12 Calc notes for geometric rep, operations, etc...

Adding vectors:

↳ Can add vectors in $\mathbb{R}^3$ by using parallelogram method on plane containing vectors.

Zero vector:

$\vec{0} = \begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix} \in \mathbb{R}^n$ → initial = terminal pt

↳ supported on ∞ lines, no direction

Theorem:

Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$

↳ $\vec{x} + \vec{y} \in \mathbb{R}^n$ "closure under addition"

↳ $\vec{0} + \vec{x} = \vec{x}$

↳ $t\vec{x} \in \mathbb{R}^n$

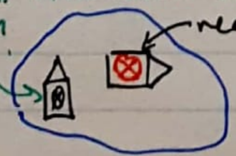
Lecture 2 - Sep 12, 2017:

Fixing the origin:

Vectors always anchored on origin for linear algebra.

↳ In physics, often anchored to point where force is applied - "Free vectors"

↳ Also used in local coordinate systems



"Global coordinate system" - origin must be independent of your movement

Lines in $\mathbb{R}^n$:

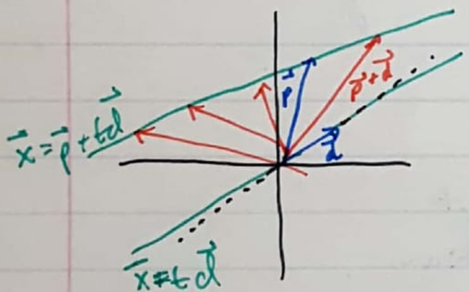
1) Directed line segments

- Let $P, Q \in \mathbb{R}^n$. $\overrightarrow{PQ} = \vec{q} - \vec{p}$ (second - first) ← tech not true...
- Origin translated to be at P. ↳ b/c this (origin location)

Equation of a Line in $\mathbb{R}^n$

- Let $\vec{p}, \vec{d}$ be vectors in $\mathbb{R}^n$, $\vec{d} \neq \vec{0}$, has direction vector $\vec{d}$
- $\{\vec{p} + t\vec{d} \mid t \in \mathbb{R}\} \to \vec{x} = \vec{p} + t\vec{d}, t \in \mathbb{R}$ (just direction vectors)
- $/\!/^{el}$ to line through origin w/ vector eqn $\vec{x} = t\vec{d}$, $t \in \mathbb{R}$

↳ Just a translation by $\vec{p}$.



← all lie on this line

// parallel

★ Old notes: eqns of lines
↳ sym, normal, para

Sym. eqn:
↳ solve for x_1 / x_2: $\frac{d_2}{d_1} x_1 + (p_2 - \frac{d_2}{d_1})$ $\quad y = mx + b$

↳ If $d_1 = 0$, $d_2 \neq 0$ — get a vertical line

$d_2 = 0$, $d_1 \neq 0$ — get a horiz. line

★ In $\mathbb{R}^n$, where $x_n = a_{n-1}x_{n-1} + \dots + a_1 x_1$ w/ real coefficients is never the eqn of a line when $n \geq 3$

Two vectors $\vec{u}$ and $\vec{v}$ are $/\!/^{el}$ if one is $\vec{0}$ or sc. multiples.

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Subspaces in $\mathbb{R}^n$:

A non-empty subset S is a subspace of $\mathbb{R}^n$ if for any $\vec{x}, \vec{y} \in S$ and any $t \in \mathbb{R}$, the following holds:

$$\left.\begin{array}{l} 1)\ \vec{x}+\vec{y} \in S \\ 2)\ t\vec{x} \in S \end{array}\right\} \text{closure under addition/scalar multi.}$$

Properties:

a) $\vec{0}$ is always in any subspace (let $t=0$)

b) $\{\vec{0}\}$ is subspace of $\mathbb{R}^n$ "trivial subspace"

c) $\mathbb{R}^n$ is subspace of itself

↳ Euclidean vector space: set $\mathbb{R}^n$ for $n \geq 1$

Example:

1) Let $S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle/\; x_1 + x_2 - 8x_3 = 4 \right\}$ (/ = such that)

Not a subspace of $\mathbb{R}^4$

↳ if $\vec{0} \in S$, right eqn $\neq$ true.

2) Let $T = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle/\; x_1 x_2 + x_3^2 = 0 \right\}$ ← multi coords and sq. = unlikely to be subspace.

$\vec{0} \in T$, but not subspace of $\mathbb{R}^3$.

↳ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in T$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in T$ but sum $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \notin T$ (eqn doesn't work)

3) Let $U = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle/\; x_1 + 4x_2 = 0 \right\}$

✓ Subspace of $\mathbb{R}^2$

$$\left.\begin{array}{l} ✓ ↳ \vec{0} \in U \\ ✓ ↳ \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} : (x_1+y_1) + 4(x_2+y_2) \\ \qquad = \underbrace{(x_1 + 4x_2)}_{0} + \underbrace{(y_1 + 4y_2)}_{0} \quad \text{since} \in U \\ ✓ ↳ t\vec{x} = \begin{pmatrix} tx_1 \\ tx_2 \end{pmatrix} \quad t(x_1 + 4x_2) = 0 \\ \qquad t(0) = 0 ✓ \end{array}\right\} \text{3 conditions}$$

Spanning Sets:

Let $\vec{v}_1 \ldots \vec{v}_k$ be k vectors in $\mathbb{R}^n$. Then,

S is all possible lin combos of $\vec{v}_{1 \ldots k}$ $S = \{t_1\vec{v}_1 \ldots t_k\vec{v}_k \mid t_1 \ldots t_k \in \mathbb{R}\} \subset \mathbb{R}^n$ *Try to prove this

is a subspace of $\mathbb{R}^n$

↳ set of all linear combinations is a subspace!

Let $B = \{\vec{v}_1 \ldots \vec{v}_k\}$ be set of k vectors in $\mathbb{R}^n$

Let $S = \{t_1\vec{v}_1 \ldots t_k\vec{v}_k \mid t_1 \ldots t_k \in \mathbb{R}\} \subset \mathbb{R}^n$

↳ The subspace S is "spanned" by B

↳ B "spans" S or is a "spanning set" of S

$\}$ $S = \text{span}\{\vec{v}_1 \ldots \vec{v}_k\} = \text{span } B$

Example:

1) A line through the origin in $\mathbb{R}^n$, $\vec{x} = t\vec{v}$ $(\vec{v} \neq \vec{0})$ is a subspace of $\mathbb{R}^n$ ($k=1$ from prev. theorem) spanned by any given direction vector $(\vec{v})$.

↳ Because $k=1$, so in $S = \{t_1\vec{v}_1 \mid t_1 \in \mathbb{R}\}$

Theorem:

Let $\vec{v}_1 \ldots \vec{v}_k$ be k vectors in $\mathbb{R}^n$. If $\vec{v}_k$ can be expressed as a linear combination of $\vec{v}_1 \ldots \vec{v}_{k-1}$ then $\text{span}\{\vec{v}_1 \ldots \vec{v}_k\} = \text{span}\{\vec{v}_1 \ldots \vec{v}_{k-1}\}$

↳ If you can rep. a $\vec{v}$ in the set as a lin comb. of two other vectors in the set, you can omit it. (same thing (=))

Linear Independance

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set $\{\vec{v}_1 \ldots \vec{v}_k \in \mathbb{R}^n\}$ is linearly independent if the only solution to

$$\vec{0} = t_1\vec{v}_1 \ldots t_k\vec{v}_k$$

is the trivial solution $t_{1 \ldots k} = 0$. else $\{$ linearly dependant $\}$

↳ Complicated way of saying no linear combinations within the set.

Theorem:
If the set of vectors $\vec{v}_1 \ldots \vec{v}_k$ contains $\vec{0}$, linearly dependent.
↳ Can have any t for the $\vec{0}$ and the rest 0.

Bases:
Subspace $\{\vec{v}_1 \ldots \vec{v}_k\}$ in $\mathbb{R}^n$
$\{\vec{v}_1 \ldots \vec{v}_k\}$ is a basis for subspace if:
→ linearly independent (no redundant info)
→ Spanning set for S.

Example:
1) "The standard basis" in $\mathbb{R}^n$ Just unit vectors
→ $\vec{e}_1 \ldots \vec{e}_n$ where $\vec{e}_i \in \mathbb{R}^n$ is the vector whose i^{th} component is 1 and all the others are 0.
→ So..

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Big\} n-1^{th} \rightarrow \vec{e}_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ i^{th} \\ \vdots \\ 0 \end{pmatrix}$$

All component 0, i^{th} component 1.

→ 1) Linearly independent.

$$t_1\vec{e}_1 + \ldots + t_n\vec{e}_n = \vec{0}$$

adding gives you $= \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}$ ∴ All $t_1 \ldots t_n$ must be 0 for $\vec{0}$.

2) Spanning set for S

$\vec{v} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$ can be written as l. comb of $\vec{e}_1 \ldots \vec{e}_n$

$$\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = v_1\vec{e}_1 + \vec{v}_2\vec{e}_2 \ldots \vec{v}_n\vec{e}_n$$

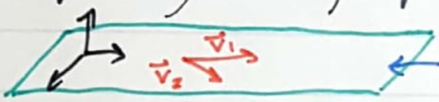
∴ $\vec{v} \in \text{span}\{\vec{e}_1 \ldots \vec{e}_n\}$

Surfaces in Higher Dimensions:

Let $\vec{v}_1, \vec{v}_2, \vec{p} \in \mathbb{R}^n$ with $\{\vec{v}_1, \vec{v}_2\}$ linearly independent. Then,

$$\vec{x} = \vec{p} + t_1\vec{v}_1 + t_2\vec{v}_2 \quad , t_1, t_2 \in \mathbb{R}$$

is a plane passing through $\vec{p}$. (translated by $\vec{p}$)



← span of the vectors is the plane

Hyperplanes:

Let $\{\vec{v}_1 \dots \vec{v}_{n-1}\}$ be linearly independent. Let $\vec{p} \in \mathbb{R}^n$

$\therefore$, $\vec{x} = \vec{p} + t_1\vec{v}_1 + \dots t_{n-1}\vec{v}_{n-1}$

is a hyperplane passing through $\vec{p}$. (4 dimensions? 3 d. vectors)

Length and Dot Product:

"Dot Product" / Scalar Product / Standard Inner Product

$\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 \dots x_ny_n$ (get a scalar)

Properties:

1) $\vec{x} \cdot \vec{x} \geq 0$. only 0 when $\vec{x} = \vec{0}$
2) $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$
3) $\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$
4) $t\vec{x} \cdot \vec{y} = t(\vec{x} \cdot \vec{y}) = \vec{x} \cdot (t\vec{y})$

Length:

$|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x_1^2 \dots + x_n^2}$ "norm" $\qquad \cos\theta = \dfrac{\vec{x} \cdot \vec{y}}{|\vec{x}||\vec{y}|}$

Properties:

1) $|t \cdot \vec{x}| = |t||\vec{x}|$
2) $|\vec{x} \cdot \vec{y}| \leq |\vec{x}||\vec{y}|$ Cauchy-Schwarz.
3) $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$ Δ ineq.

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Scaler Eqn of A Hyperplane:

Hyperplanes are given by: in $\mathbb{R}^n$...

$$\vec{x} = \vec{p} + t_1\vec{v}_1 \ldots t_{n-1}\vec{v}_{n-1} \quad t \in \mathbb{R}$$

where $\{\vec{v}_1 \ldots {}_{n-1}\}$ is linearly independent.

These hyperplanes have normals. ⇒ $\vec{w}$ n coords $\begin{pmatrix} \vdots \\ n \end{pmatrix}$

- ↳ Non zero
- ↳ Given any $P(x,y,z)$, ignore this
- ↳ Any vector in plane is $\perp$ to normal of plane.

Another point x is only in plane iff $\vec{px} \cdot \vec{n} = 0$.

Scalar equation:

$$m_1x_1 \ldots m_nx_n = d$$

or... $Ax + By + Cz = D$ for $n = 3$.

If...

$n = 2$? Hyperplane is a line
$n = 3$? Hyperplane is a plane
$n > 3$? Hyperplane is a hyperplane.

} One less than n.

Example:

1) $2x_1 + x_2 + 4x_3 = 2$

$2(x_1 - 1) + x_2 + 4x_3 = 0$, or...

$$\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} x_1 - 1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

because $\vec{px}$ is on the plane!

$$\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \cdot \left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] = 0$$

$\therefore \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is on the hyperplane.

Projection and Minimum Distance:

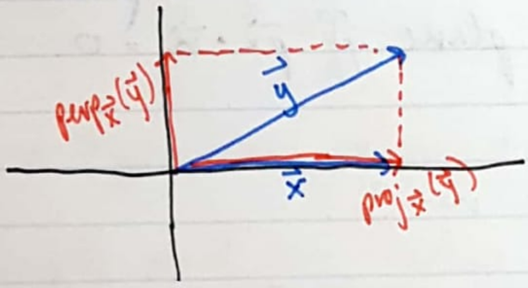
Projections:

$$\text{proj}_{\vec{x}}(\vec{y}) = \left(\frac{\vec{y}\cdot\vec{x}}{\|\vec{x}\|^2}\right)\vec{x}$$

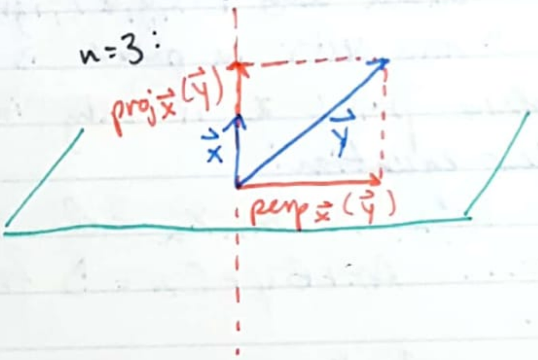
Projection of $\vec{y}$ onto perpendicular of $\vec{x}$:

$$\text{perp}_{\vec{x}}(\vec{y}) = \vec{y} - \text{proj}_{\vec{x}}(\vec{y})$$

When $n=2$:



$n=3$:



Example:

1) $\vec{x} = \begin{pmatrix}1\\1\end{pmatrix}, \vec{y} = \begin{pmatrix}3\\2\end{pmatrix}$

$$\text{proj}_{\vec{x}}(\vec{y}) = 3\sqrt{2}\begin{pmatrix}1\\1\end{pmatrix} \qquad \text{perp}_{\vec{x}}(\vec{y}) = \begin{pmatrix}3\\2\end{pmatrix} - 3\sqrt{2}\begin{pmatrix}1\\1\end{pmatrix}$$

Properties of Projections:

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$\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n, \vec{x} \neq \vec{0}$

1) $\text{proj}_{\vec{x}}(\vec{y}+\vec{z}) = \text{proj}_{\vec{x}}(\vec{y}) + \text{proj}_{\vec{x}}(\vec{z})$
2) $\text{proj}_{\vec{x}}(t\vec{y}) = t\,\text{proj}_{\vec{x}}(\vec{y})$
3) $\text{proj}_{\vec{x}}(\text{proj}_{\vec{x}}(\vec{y})) = \text{proj}_{\vec{x}}(\vec{y})$

Minimum Distance:

Distance btwn point and line/hyperplane is min. distance w̄ closest point! ← S is point on line/hyperplane

$\|\text{perp}_{\vec{d}}(\vec{SQ})\|$ → Point Q to line w̄ dir. $\vec{v}: \vec{d}$

$\|\text{proj}_{\vec{m}}(\vec{SQ})\|$ → Point Q to hyperplane w̄ normal $\vec{v}: \vec{m}$

Fundamental Example:

Let $Q = (Q_1, Q_2, Q_3) \in \mathbb{R}^3$, $H: m_1 x_1 + m_2 x_2 + m_3 x_3 + d = 0$ (hyperplane)

Distance from H to Q:

watch! +D if all on one side

$$\frac{|Ax_1 + Bx_2 + Cx_3 (+D)|}{\sqrt{A^2 + B^2 + C^2}} \quad \vec{n} = (A, B, C) \quad P = (x_1, x_2, x_3)$$

Assume $m_1 \neq 0$. Then $S_1 = (\frac{d}{m_1}, 0, 0)$ is on H. Therefore, distance is:

$$\|\text{proj}_{\vec{m}}(\vec{SQ})\| = \left\| \frac{(\vec{q} - \vec{s}) \cdot \vec{m}}{\|\vec{m}\|^2} \vec{m} \right\|$$

$$= \frac{|(q_1 - \frac{d}{m_1}) m_1 + q_2 m_2 + q_3 m_3|}{\sqrt{m_1^2 + m_2^2 + m_3^2}}$$

= original formula!

Cross Product:

$\vec{u} \times \vec{v}$ is $\perp$ to both $\vec{u}$ and $\vec{v}$ (dot prod = 0)

1) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
2) $\vec{u} \times \vec{u} = \vec{0}$
3) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
4) $t(\vec{u} \times \vec{v}) = (t\vec{u}) \times \vec{v} = \vec{u} \times (t\vec{v})$
5) $(\vec{u} \times \vec{v}) \times \vec{w} \neq \vec{u} \times (\vec{v} \times \vec{w})$ ORDER MATTERS!

Proof of 3): Let $\vec{a} = \vec{v} + \vec{w}$.

$$\vec{u} \times \vec{a} = u_2(v_3 + w_3) - u_3(v_2 + w_2) \text{ } \} \text{ first comp. of } \times$$

$$= u_2 v_3 + u_2 w_3 - u_3 v_2 - u_3 w_2$$

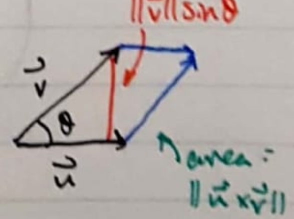
$$= \underbrace{(u_2 v_3 - u_3 v_2)}_{\vec{u} \times \vec{v} \text{ comp.}} + \underbrace{(u_2 w_3 - u_3 w_2)}_{\vec{u} \times \vec{w} \text{ comp.}}$$

And repeat.

Applications:

$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta$ * $\theta \in [0, \pi]$

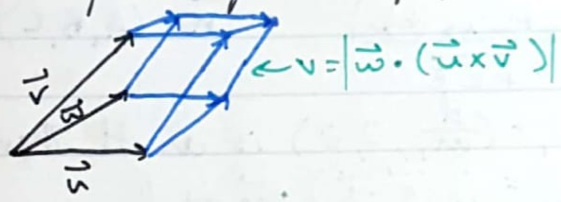
= area of parallelogram spanned by $\vec{u}$ and $\vec{v}$



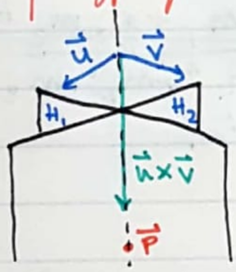
Triple Scalar Product:

$\vec{w} \cdot (\vec{u} \times \vec{v})$

→ Volume of the parallelepiped $\bar{w}$ adj sides $\vec{u}, \vec{w}, \vec{v}$.



Intersection of Hyperplanes:



Let $\vec{u}$ be normal of H_1, $\vec{v}$ for H_2.

The line of int. has dir. vector $\vec{u} \times \vec{v}$.

Example:

1) $H_1: x_1 + x_2 + x_3 = 1 \quad H_2: x_1 - x_2 + x_3 = 0$

$\vec{n}_1 \times \vec{n}_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \quad \vec{P} = (\tfrac{1}{2}, \tfrac{1}{2}, 0) \quad \vec{x} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, t \in \mathbb{R}$

Chapter 2: Systems of Linear Equations:

Linear Eqns:

A system of m linear equations in n variables $x_1 \dots x_n$ is a set of m eqns:

$$\begin{cases} a_{11} x_1 \cdots a_{1n} x_n = b_1 \\ \vdots \\ a_{m1} x_1 \cdots a_{mn} x_n = b_m \end{cases}$$

The coefficient matrix is...

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

The augmented matrix is...

$$\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_n \end{array}\right) = [A \mid \vec{b}]$$

Sep 20, 2017

Matrices:

$$\begin{array}{c c} & \\ i^{th}\text{ row} & \begin{pmatrix} & \vdots & \\ \cdots & a_{ij} & \cdots \\ & \vdots & \end{pmatrix} \\ & j^{th}\text{ col} \end{array}$$

$1 \le i \le m$ equations

$1 \le j \le n$ degree

Row Echelon Form:

Elementary row operations:

1) Multiply by non-zero constant
2) Switch rows
3) Add multiples of rows ★ Can't do $2R_4 - R_5$

Two matrices are row equivalent if one can be obtained by row reducing the other. $A \sim B$

A matrix is in row echelon form if:

1) All 0s at bottom row (don't need row of 0s tho)
2) First # in upper row is ← of first # in lower row

Example:

1) $A: \begin{pmatrix} 1 & 12 & -22 \\ 0 & 2 & 3 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{pmatrix}$ ✓ REF $B: \begin{pmatrix} 1 & 12 & -22 \\ 0 & 2 & 3 \\ 0 & 8 & 4 \end{pmatrix}$ ✗ Not REF

Gaussian Elimination:

$$\begin{pmatrix} 0 & 0 & 1 & 3 \\ 0 & 8 & -1 & 10 \\ 0 & 0 & -3 & 9 \\ 0 & 16 & 4 & -7 \end{pmatrix}$$

1) Identify first non-zero column. Switch rows so that top entry in this column is non zero

first non-zero column

needs to be at top (first non-zero column value)

PIVOT

$$\begin{pmatrix} 0 & 0 & 1 & 3 \\ 0 & 8 & -1 & 10 \\ 0 & 0 & -3 & 9 \\ 0 & 16 & 4 & -7 \end{pmatrix} \Big| R_1 \updownarrow R_2 \sim \begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -3 & 9 \\ 0 & 16 & 4 & -7 \end{pmatrix}$$

2) Use ERO to make all entries below pivot = 0.

$$\begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -3 & 9 \\ 0 & 16 & 4 & -7 \end{pmatrix} \Big| R_4 - 2R_1 \sim \begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -3 & 9 \\ 0 & 0 & 6 & -27 \end{pmatrix}$$

should be 0!

3) Consider all columns right of pivot. Becomes submatrix. Repeat 1+2 for this submatrix w/ all rows below last pivot.

$$\begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -3 & 9 \\ 0 & 0 & 6 & -27 \end{pmatrix} \Big| R_3 + 3R_2 \text{ then } \Big| R_4 - 6R_2 \sim \begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & -45 \end{pmatrix}$$

new pivot

need these 0 now

4) Repeat step 3 until row echelon form.

$$\begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & -45 \end{pmatrix} \Big| R_4 + 5R_3 \sim \begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

new pivot

(would still be REF w/o last row)

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Determining Solutions using Matrices: Sep 21, 2017

Consistent - 1+ solutions

Inconsistent - 0 solutions

↳ If last row is $[0 \ldots 0 \mid c]$, $c \neq 0$

If consistent...

↳ # pivots = # variables? Unique Solution. $[0 \ldots c \mid c]$

↳ # pivots < # variables? Infinite solutions. $[0 \ldots 0 \mid 0]$

Example:

1) $\left(\begin{array}{cccc|c} 0 & 8 & -1 & 10 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$

inconsistent

2) $\left(\begin{array}{cccc|c} 0 & 8 & -1 & 10 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$ 3 pivots, 4 variables

$\therefore \infty$ solutions

consistent

3) $\left(\begin{array}{ccc|c} 8 & -1 & 10 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 9 & 1 \end{array}\right)$ 3 pivots, 3 variables

$\therefore$ 1 solution

Reduced REF (RREF):

1) REF
2) All leading entries = 1.
3) Leading 1's have all 0s after (in the column)

↳ Any matrix has a equivalent, unique matrix in RREF

REF ←

RREF ←

$\begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -7 \end{pmatrix}$

Example:

1) $\begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ RREF

$\frac{1}{8}R_1$, then $\frac{1}{9}R_3$

then $R_1 + \frac{1}{8}R_2$

then $R_1 + \left(\frac{-13}{8}\right)R_3$

then $R_2 - 3R_3$

(Do in 1 step tho)

Homogeneity and Rank:

Homogeneous if right side of all eqns $= 0$.

$$\begin{cases} 2x_1 - x_2 = 0 \\ x_1 + 3x_2 - 4x_3 = 0 \end{cases} \rightarrow \vec{x} = \vec{0} \text{ is a trivial solution.}$$

Rank ($\text{rank}(M)$) is # of leading 1's in it's RREF.

Theorem:

1) System is consistent iff rank(coefficient matrix) = rank(augmented)
2) If consistent, parameters needed to express solutions is #variables − rank(coefficient)

Example:

1) $$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 - 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} -x_3 = x_1 + x_2 \\ -3x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_2 = -x_1 \\ x_3 = 0 \end{cases}$$

The set of solutions is $\left\{ \begin{pmatrix} x_1 \\ -x_1 \\ 0 \end{pmatrix} \middle| \, x_1 \in \mathbb{R} \right\}$

Applications:

1) Balancing Chemical Reactions

→ Manganese (Mn)

$$x_1 MnO_4^- + x_2 H_2C_2O_4 + x_3 H^+ \rightarrow x_4 Mn^{2+} + x_5 CO_2 + x_6 H_2O$$

stoichiometric coefficients

$$\begin{aligned} C:&\ 2x_2 &= x_5 \\ Mn:&\ x_1 &= x_4 \\ O:&\ 4x_1 + 4x_2 &= 2x_5 + x_6 \\ H:&\ 2x_2 + x_3 &= 2x_6 \\ \text{Charges}:&\ -x_1 + x_3 &= 2x_4 \end{aligned}$$

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{5}{8} & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{5}{4} & 0 \end{array}\right) \Bigg\} \begin{array}{l} \text{Rank} = 5 \\ \text{Vars} = 6. \end{array}$$

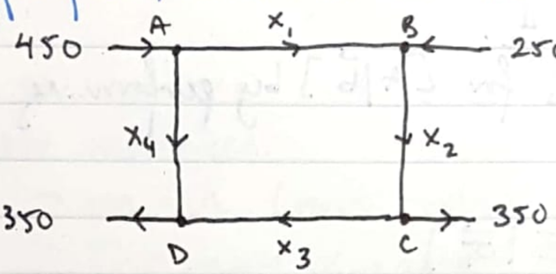
Get $x_1 = \_\, x_6$

Let $x_6 = 8$

Get coefficients

Application to Information Theory Sep 28, 2017.

Traffic flow / communication network.



Each node (ABCD) = antenna.
Each → = one way com. channel
↳ capacity $x_1 x_2 \ldots _4$: info/sec
↳ unit: Bits! bit/s

*Applications week 2.

This assumes eq. state (No info gained/lost)

$$
\begin{aligned}
A:&\ 450 = x_1 + x_4 \\
B:&\ 250 + x_1 = x_2 \\
C:&\ x_2 = x_3 + 350 \\
D:&\ x_3 + x_4 = 350
\end{aligned}
\quad\Bigg\}\quad
\begin{aligned}
x_1 + x_4 &= 450 \\
-x_1 + x_2 &= 250 \\
x_2 - x_3 &= 350 \\
x_3 + x_4 &= 350.
\end{aligned}
$$

$$
\left(\begin{array}{cccc|c}
1 & 0 & 0 & 1 & 450 \\
-1 & 1 & 0 & 0 & 250 \\
0 & 1 & -1 & 0 & 350 \\
0 & 0 & 1 & 1 & 350
\end{array}\right)
\sim
\left(\begin{array}{cccc|c}
1 & 0 & 0 & 1 & 450 \\
0 & 1 & 0 & 1 & 700 \\
0 & 0 & 1 & 1 & 350 \\
0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

consistent ∞ soln.

Can be expressed as #vars − rank = 4 − 3 = 1 parameter needed.

Back substitution:

$$
\begin{aligned}
x_1 + x_4 &= 450 \\
x_2 + x_4 &= 700 \\
x_3 + x_4 &= 350.
\end{aligned}
\quad\Bigg\}\quad
\begin{aligned}
x_1 &= 450 - x_4 \quad x_4 \in \mathbb{R} \\
x_2 &= 700 - x_4 \\
x_3 &= 350 - x_4.
\end{aligned}
$$

All vars need +! can't have − info

Physical Solutions:

$$
\begin{pmatrix} 450 - t \\ 700 - t \\ 350 - t \\ t \end{pmatrix}, \quad 0 \le t \le 350
$$

When does a m eq. × n vars matrix have a solution no matter what the constant terms are? $[A|\vec{b}]$, $\vec{b} \in \mathbb{R}^m$

Chapter 3: Matrices

2 matrices are equal when all entries are same.

Special Matrices:

Square matrices:

↳ $m = n$ (rows = cols).

↳ The main diagonal:

$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ A_{ii}, $1 \le i \le n$

↳ Upper triangular / Lower triangular

↳ All entries below main diagonal are zeroes.

$\begin{pmatrix} 1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 4 & 5 & 2 \end{pmatrix}$

↳ Diagonal:

→ Both upper and lower triangular

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $diag(d_1, d_2, d_3 \ldots d_n)$

Vectors:

Sep 27, 2017.

↳ $n \times 1$ matrices

Operations on Matrices:

Let A, B be $m \times n$ matrices, s be a scalar.

$(A+B)_{ij} = A_{ij} + B_{ij}$ *both need to be $n \times n$

$(sA)_{ij} = s(A_{ij})$

Theorem:

Let A, B, C be $m \times n$ matrices. $s, t \in \mathbb{R}$.

1) Closure under addition:
   $A+B$ is $m \times n$ matrix.
2) Comm. of addition:
   $A+B = B+A$
3) Assoc. of addition:
   $(A+B)+C = A+(B+C)$
4) Zero Matrix: $\mathbb{O}_{m \times n}$ Only 0 entries.
   $A + \mathbb{O}_{m \times n} = A$.
5) $-A$.
   $A + (-A) = \mathbb{O}_{m \times n}$
6) Closure under scaler multiplication:
   sA is $m \times n$.
7) Assoc. of sc. multi:
   $s(tA) = (st)A$
8) Distributivity:
   $(s+t)A = sA + tA$
9) Dist. 2:
   $s(A+B) = sA + sB$

Transposition:

Let A be $\underline{m \times n}$ [ ]. A^T is a $\underline{n \times m}$ [ ]

$$A_{ij} = A^T_{ji}$$

$$\underset{A}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}} \qquad \underset{A^T}{\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}}$$

Rows become cols, cols become rows

Rows are transposes of vectors.

Properties of Transposition:

Let A, B be $m\times n$ [ ]. Let $s \in \mathbb{R}$.

1) $(A^T)^T = A$. Involution
2) $(A+B)^T = A^T + B^T$
3) $(sA)^T = s(A^T)$

Proof of 1):

$((A^T)^T)_{ij} = (A^T)_{ji} = A_{ij}$

Proof of 2):

$((A+B)^T)_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = A^T + B^T$

Proof of 3):

$((sA)^T)_{ij} = (sA)_{ji} = s(A_{ji}) = s(A^T)$

Multiplication of Matrices:

Let $a_1 \dots a_n \in \mathbb{R}$. 1) $a_1 + a_2 \dots + a_n = \sum_{k=1}^{n} a_k$, and...

2) $a_1 \times a_2 \times a_n = \prod_{k=1}^{n} a_k$

Let B be $\boxed{m\times n}$ [ ] $\bar{w}$ rows:

$(b_{11}, \dots, b_{1n}) = \vec{b}_1^T$

$\vdots \qquad \vdots$

$(b_{m1}, \dots b_{mn}) = \vec{b}_m^T$

$(m\times \underline{n}\,[\,])(\underline{n}\times p[\,]) = (m\times p[\,])$

must be same!

Let A be $\boxed{n\times p}$ [ ] $\bar{w}$ columns

$$\vec{a}_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} \dots \vec{a}_p = \begin{pmatrix} a_{1p} \\ \vdots \\ a_{np} \end{pmatrix}$$

Then, BA is the $\boxed{m\times p}$ matrix whose ij^{th} entry is:

$$(BA)_{ij} = \vec{b}_i \cdot \vec{a}_j = \sum_{k=1}^{n} (B_{ik})(A_{kj})$$

Examples:

$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & -3 & -4 \end{pmatrix}$ [2×4] $\quad B = \begin{pmatrix} -2 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -2 \end{pmatrix}$ [4×2]

$D = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ [2×1] $\quad C = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$ [2×2] $\quad E = (1, 0)$ [1×0]

1) AB works
2) BA works
3) CD works
4) DC doesn't work
5) EC works
6) CE doesn't work.

Sep 28, 2017

$AB: \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & -3 & -4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -2 \end{pmatrix}$

$\boxed{\vec{a}_i \cdot \vec{b}_j} : \begin{pmatrix} (1)(-2) + (2)(-1) + (3)(0) + (4)(1) & (1)(1) + (2)(0) + 3(-1) + 4(-2) \\ (-1)(-2) + (-2)(-1) + (-3)0 + (-4)(1) & (-1)(1) + (-2)0 + (-3)(-1) + (-4)(-2) \end{pmatrix}$

for $1 \le i \le m$:
$\llcorner$ for $1 \le j \le n$:

$= \begin{pmatrix} 0 & -10 \\ 0 & 10 \end{pmatrix}$

Properties: (assumes product is defined)

$AB = AC$ doesn't imply $B = C$

$AB \neq BA$ (necessarily)

$A(B+C) = AB + AC$

$t(AB) = (tA)B = A(tB)$

$\star$ $(AB)^T = B^T A^T$ Careful. Let A m×n[ ] and B n×p[ ], AB defined

$\hookrightarrow A^T$ — n×m, B^T — p×n $1 \le i \le p$ and $1 \le j \le m$

$((AB)^T)_{ij} = AB_{ji} = \sum_{k=1}^{m} (B)_{ki} (A)_{jk} = \sum_{k=1}^{m} (B^T)_{ik} (A^T)_{jk} = (B^T A^T)_{ij}$

Let $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ $[n\times 1]$ $\quad \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$ $[n\times 1]$

$$(\vec{x}^T)\cdot\vec{y} = (x_1, x_2 \ldots x_n)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = x_1y_1 + \ldots + x_ny_n = \vec{x}\cdot\vec{y}$$

$[1\times n]\cdot[n\times 1]$

Definition:

Identity matrix: $n\times n$ $I_n = \mathrm{diag}(1, \ldots 1_n) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

↳ If no ambiguity, $I_n = I$.

↳ $I_m A = A I_m = A$.

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

Inverse Matrices:

Let A be $n\times n$ □. If there exists an unique $n\times n$ □ B such that $\boxed{AB = BA = I_n}$, then A is invertible with inverse B. (A^{-1})

↳ Assume not unique. $BA = AB = I = CA = AC = I$.

↳ $B = BI = B(AC) = (BA)C = IC = C$ $\therefore$ unique.

↳ If $AB = I_N$, then $BA = I_n$ and $B = A^{-1}$

↳ Any invertible matrix of size n has max rank n.

Theorem:

Assume A and B are invertible of size n. Let $t \in \mathbb{R}$, $t \neq 0$.

need to state

1) tA is invertible and $(tA)^{-1} = \frac{1}{t}\cdot A^{-1}$

2) AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

3) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Never use A^{-1} before justifying A is invertible.

1) $(tA)\cdot(\frac{1}{t})(A^{-1}) = (t\cdot\frac{1}{t})(A\cdot A^{-1}) = I$

2) $(AB)(B^{-1}A^{-1}) = A[(B)(B^{-1})]A^{-1} = I$

3) $(A^T)\cdot(A^{-1})^T = (A^{-1}A)^T = I^T = I$.

Invertibility: Sep 24, 2017

If A [ ] B [ ] such that $AB = I$, then this does NOT imply that A is invertibe w inverse B.

→ Only holds iff A, B are □. ✱

Example:

1) Consider $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$ $\boxed{\text{BUT}}$ $BA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq I_2$

Proposition:

Let A, B $m \times n$ [ ] where $\forall \vec{x} \in \mathbb{R}^n$:

$A\vec{x} = B\vec{x}$, Then $A = B$

Proof:

Consider $\vec{e}_i$, $1 \leq i \leq n$:

$\vec{e}_i$: $\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ ← i^{th} entry is given. To avoid ambig, $\vec{e}^i = \vec{e}_i$

$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{1m} & \cdots & a_{mn} \end{bmatrix}$, Then $\underbrace{A\vec{e}^i}_{[m \times n] \cdot [n \times 1]} \in \mathbb{R}^m$ ← [m × 1] vector

$= \begin{pmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_m^T \end{pmatrix}$ the j^{th} component is: $\vec{a}_j \cdot \vec{e}^i = (a_{j1} \cdots a_{jn}) \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ ← ith

$= a_{j1}(0) + \ldots + a_{ji}(1) + \ldots + a_{jn}(0)$

$= a_{ji}$ ← This shows:

$A\vec{e}^i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ji} \\ \vdots \\ a_{mi} \end{pmatrix} = i^{th}$ column in A. Means all colums for A, B must be equal (i can be any col #).

Finding the Inverse:

Consider n eqns, n vars system.

$a_{11}x_1 + \dots a_{1n}x_n = b_1$
$\vdots \qquad \vdots$
$a_{n1}x_1 + \dots a_{nn}x_n = b_n$

Let A be coef. [ ], $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{R}^n$

Then the system can be rewritten as $A\vec{x} = \vec{b}$.

If A^{-1} exists, $\vec{x} = A^{-1}\vec{b}$

Inverse Matrix Algorithm:

Let A be □

Reduce the matrix [A|I] so A is in RREF.

* 1) If the left block (A) is not I, then A is not invertible.
2) If the form is [I|B], then B is the inverse and A^{-1} exists.

Example:

1) Let $A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$.

$$\left(\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right) \underset{1R_2 - \frac{2}{3}R_1}{\sim} \left(\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 0 & \frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right) \underset{1\frac{1}{3}R_1}{\sim} \left(\begin{array}{cc|cc} 1 & \frac{4}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right) \underset{1R_1 - 4R_2}{\sim}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 3 & -4 \\ 0 & \frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right) \underset{1R_2 \cdot 3}{\sim} \left(\begin{array}{cc|cc} 1 & 0 & 3 & -4 \\ 0 & 1 & -3 & 3 \end{array}\right)$$

$= I$ ∴, A is invertible and is the inverse.

Say "from the I.M.A" A is invertible and $A^{-1} = B$.

Properties of Inverse Matrixes:

Oct 3, 2017

Let A $m \times m$ □. Then, the following statements are equiv.

If one is true, all are true.

i) A is invertible
ii) A has rank m
iii) The RREF of A is I_m
iv) Given any $\vec{b} \in \mathbb{R}^m$, $A\vec{x} = b$ is consistent and has a unique solution
v) Columns of A from linearly independent vectors.

Proof of iv) $\Rightarrow$ v):

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{m1} & \cdots & a_{mm} \end{pmatrix} = [\vec{a}_1 \ldots \vec{a}_m]$$

where $\vec{a}_i$ is i^{th} col in A.

Let $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{R}^m$

Consider $A\vec{x} = \vec{0}$:

$$\left.\begin{matrix} a_{11} x_1 \quad \cdots = 0 \\ \vdots \\ a_{m1} x_1 \quad \cdots = 0 \end{matrix}\right\} = \sum_{i=1}^{n} \vec{a}_i x_i = \vec{0}$$

linear independence, opitimized

came from $\vec{a}_1$

Example:

1) Let A be a □ where $A^3 = I$. Show A is invertible and determine it's inverse.

$A \times A \times A = I$ Let $D = A^2$

then $AD = I$, where AD are □ (multi is defined).

$\therefore$ The inverse of A is A^2.

2) Let A, B □. Show if A and B are invertible $\Leftrightarrow$ AB is invertible.

$\Rightarrow$ Assume A, B invertible. Then:

$$(AB)(B^{-1}A^{-1}) = (A)(BB^{-1})(A^{-1}) = AIA^{-1} = AA^{-1} = I.$$

Since $B^{-1}A^{-1}$ invertible, $(AB)^{-1}$ is invertible $\bar{w}$ inverse $B^{-1}A^{-1}$!

$\Leftarrow$ Assume AB invertible. Then, $\exists\, \square\, C \ni (AB)C = I$, $C(AB) = I$.
Since $(AB)C = I$, $A(BC) = I$, so A is invertible.
Same w/ B. $A^{-1} = BC$, $B^{-1} = CA$.

Linear Mappings:

Functions are mappings/transformations. If f has dom U, codom V:

maps to

$$f : U \longrightarrow V$$
$$x \longmapsto f(x)$$

transforms

$\Big\}$ $f : x \in U \longmapsto f(x) \in V$.

$\hookrightarrow$ Range is attainable codomain.

$f: \begin{array}{l} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^2 \end{array}$ $\qquad$ $g: \begin{array}{l} [-1,1] \to \mathbb{R} \\ x \mapsto x^2 \end{array}$ $\quad$ Two dif. fcns.

	dom	cod	range
f	$\mathbb{R}$	$\mathbb{R}$	$[0,\infty]$
g	$[-1,1]$	$\mathbb{R}$	$[0,1]$

Matrix Mappings:

Let A be $m \times n$ [ ].

$$f_A : \mathbb{R}^n \to \mathbb{R}^m$$
$$\underbrace{\underset{[n\times 1]}{\vec{x}} \longmapsto \underset{[m\times n]}{A\vec{x}}}_{\text{need to highlight that } A\vec{x} \text{ exists!}}$$

$\Big\{$ Definition of product of 2 matrices shows that matrix mapping f_A is welldefined.

VERY IMPORTANT

Example:

1) Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$. Then $f_A(\vec{x}) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \\ 5x_1 + 6x_2 \end{pmatrix}$ $\mathbb{R}^3$!

Given $m, n \geq 1$, mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ is <u>linear</u> if
1) $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$, $f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y})$
2) $\forall t \in \mathbb{R}$, $\forall \vec{x} \in \mathbb{R}^n$, $f(t\vec{x}) = t f(\vec{x})$.

A linear operator is a linear mapping whose domain = codomain
$\hookrightarrow \mathbb{R}^n \to \mathbb{R}^n$ $\qquad$ ✱ $[0,1] \longrightarrow \mathbb{R}$ not linear.

Oct 4, 2017

Linearity Continued:

The following mappings are linear:

a) Given $\vec{v} \in \mathbb{R}^n$, $\text{proj}_{\vec{v}} : \mathbb{R}^n \to \mathbb{R}^n$

$\vec{x} \mapsto \text{proj}_{\vec{v}}(\vec{x})$ ← and perp

→ B/C $\text{proj}_{\vec{v}}(\vec{x}+\vec{y}) = \text{proj}_{\vec{v}}(\vec{x}) + \text{proj}_{\vec{v}}(\vec{y})$
$\text{proj}_{\vec{v}}(t\vec{x}) = t\,\text{proj}_{\vec{v}}(\vec{x})$ } $\vec{v} \neq \vec{0}$

b) Given $\vec{v} \in \mathbb{R}^n$, $S_{\vec{v}} : \mathbb{R}^n \to \mathbb{R}$

$\vec{x} \mapsto \vec{v}\cdot\vec{x}$. ← and cross product $\mathbb{R}^3$ ← !

→ Same properties for dot product.

Very similar properties to subspaces.

Not Linear:

a) $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$

$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \sqrt{x_1^2 + \dots x_n^2}$

If linear, $\|\vec{x}+\vec{y}\| = \|\vec{x}\| + \|\vec{y}\|$ ← △ ineq, tho

→ $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

→ Give counter example w $\vec{e}_1, \vec{e}_2$.

1) Any linear mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ always maps $\vec{0}_n \to \vec{0}_m$ ($t = 0$)

2) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Let $\vec{x}_1, \dots \vec{x}_k \in \mathbb{R}^n$, $t_1, \dots t_k \in \mathbb{R}$,

→ $f(t_1\vec{x}_1 + \dots t_k\vec{x}_k) = t_1 f(\vec{x}_1) + \dots + t_k f(\vec{x}_k)$

→ $f\left(\sum_{i=1}^{k} t_i\vec{x}_i\right) = \sum_{i=1}^{k} t_i f(\vec{x}_i)$

→ mapping of lin. comb is lin comb. of mapping.

Proof of 2: $k=3$

$f(t_1\vec{x}_1 + t_2\vec{x}_2 + t_3\vec{x}_3) = f(t_1\vec{x}_1 + t_2\vec{x}_2) + f(t_3\vec{x}_3)$

Just example not really proof. { $= f(t_1\vec{x}_1) + f(t_2\vec{x}_2) + f(t_3\vec{x}_3)$

$= t_1 f(\vec{x}_1) + t_2 f(\vec{x}_2) + t_3 f(\vec{x}_3)$

Theorem:

Let $A = m \times n [\,]$, then,

$f_A : \mathbb{R}^n \to \mathbb{R}^m$
$\vec{x} \mapsto A\vec{x}$ is linear..

Proof: Let $\vec{x}, \vec{y} \in \mathbb{R}^n$, $t \in \mathbb{R}$.

1) $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$
2) $A(t\vec{x}) = t(A\vec{x})$.

Yet Another Theorem:

Let $A = m \times n [\,]$, mapping $f_A : \mathbb{R}^n \to \mathbb{R}^m$.

→ f_A is uniquely determined by the values it takes on the standard basis $(\vec{e}_1, \dots \vec{e}_n) \in \mathbb{R}^n$.

→ $f_A(\vec{x}) = x_1 f_A(\vec{e}_1) + \dots x_n f_A(\vec{e}_n)$, or

$\sum_{k=1}^{n} x_k f_A(\vec{e}_k)$. components of $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

→ Proof of above:

$\vec{x} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \dots + x_n \vec{e}_n$, so

$f_A(\vec{x}) = x_i f_A(\vec{e}_i)$.

iff for $\mathbb{R}^1$

Consider $y = ax$. $\left.\begin{array}{l} f : \mathbb{R} \to \mathbb{R} \\ x \mapsto ax \end{array}\right\}$ is linear! That's why it's called linear ö.

→ Conversely, if $g : \mathbb{R} \to \mathbb{R}$ is linear, $\exists\, a \in \mathbb{R}$ (depends on g), where $g(t) = at$, $t \in \mathbb{R}$.

$g(t) = g(t \times 1) = t\, g(1) = g(1)\, t$

Let $a = g(1)$

$f: \mathbb{R} \to \mathbb{R}$ linear $\Leftrightarrow \exists \alpha \in \mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto \alpha x$ $(\alpha = f(1))$ Oct 5, 2017.

So... ↳ $f: \mathbb{R}^n \to \mathbb{R}^m$ linear $\Leftrightarrow \exists A(m\times n)$, $f_A: \mathbb{R}^n \to \mathbb{R}^m$, $\vec{x} \mapsto A\vec{x}$

both ways!

Theorem:

Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then, can be represented as a matrix mapping where $[L]_{m\times n}$:

$$[L] = [L(\vec{e}_1), L(\vec{e}_2), L(\vec{e}_3), \dots, L(\vec{e}_n)]$$

$$L: \mathbb{R}^n \to \mathbb{R}^m$$
$$\vec{e}_i \mapsto L(\vec{e}_i)$$
$$\vec{x} \mapsto L(\vec{x}) = [L]\vec{x}.$$

Proof:

Let $\vec{x} \in \mathbb{R}^n$ be:

$$\vec{x} = x_1 \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \dots + x_n \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} = \sum_{i=0}^{n} x_i \vec{e}_i.$$

By linearity: $L(\vec{x}) = L\left(\sum_{i=1}^{n} x_i \vec{e}_i\right) = \sum_{i=1}^{n} x_i L(\vec{e}_i)$

$\therefore$, $[L]\vec{x} = L(\vec{x})$.

↳ From this we can conclude linearity $\Leftrightarrow$ matrix mapping

Example:

1) Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$, then $C_{\vec{v}}: \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{x} \mapsto \vec{v} \times \vec{x} = [C_{\vec{v}}]\vec{x}$. is linear

Only works for linear mappings.

$$C_{\vec{v}}(\vec{e}_1) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \vec{e}_1 = \begin{pmatrix} 0 \\ v_3 \\ -v_2 \end{pmatrix}$$

$$C_{\vec{v}}(\vec{e}_2) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \vec{e}_2 = \begin{pmatrix} -v_3 \\ 0 \\ v_1 \end{pmatrix}$$

$$C_{\vec{v}}(\vec{e}_3) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \vec{e}_3 = \begin{pmatrix} v_2 \\ -v_1 \\ 0 \end{pmatrix}$$

thus, $[C_{\vec{v}}] = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$

$\therefore$, $\vec{v} \times \vec{x} = [C_{\vec{v}}]\vec{x}$

2) $S_{\vec{v}} : \mathbb{R}^n \to \mathbb{R}$ where $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$

$\vec{x} \longmapsto \vec{v} \cdot \vec{x}$

Let $1 \le i \le n$. $S_{\vec{v}}(\vec{e}_i) = \vec{v} \cdot \vec{e}_i = v_i$

$\therefore [S_{\vec{v}}] = [v_1, v_2 \ldots v_n] = \vec{v}^T$.

$\therefore$ For all $\vec{x} \in \mathbb{R}^n$, $\vec{v} \cdot \vec{x} = [v_1, \ldots, v_n]\vec{x} = \vec{v}^T \vec{x}$.

3) $\text{proj}_{\vec{v}} \quad \mathbb{R}^n \to \mathbb{R}^n$ where $\vec{v} \neq \vec{0}$.

$\vec{x} \longmapsto \text{proj}_{\vec{v}}(\vec{x})$

$$\text{proj}_{\vec{v}}(\vec{e}_i) = \frac{\vec{e}_i \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{1}{\|\vec{v}\|^2}\begin{pmatrix} v_i v_1 \\ \vdots \\ v_i v_n \end{pmatrix}$$

$$\therefore [\text{proj}_{\vec{v}}] = \frac{1}{\|\vec{v}\|^2}\begin{pmatrix} v_1 v_1 & \cdots & v_n v_n \\ \vdots & & \vdots \\ v_1 v_n & \cdots & v_n v_n \end{pmatrix}$$

$[\text{proj}_{\vec{v}}]_{ij} = \dfrac{v_i v_j}{\|\vec{v}\|^2}$ *Try: prove $[\text{proj}_{\vec{v}}] = \dfrac{1}{\|\vec{v}\|^2}\vec{v}(\vec{v}^T)$

Definitions:

① Let $L : \mathbb{R}^n \to \mathbb{R}^m$, $M : \mathbb{R}^n \to \mathbb{R}^m$ be linear mappings.

$L+M : \mathbb{R}^n \to \mathbb{R}^m$ ← sum of each mapping (matrices)

$\vec{x} \longmapsto (L+M)\vec{x} = L(\vec{x}) + M(\vec{x})$

② Let $t \in \mathbb{R}$.

$tL : \mathbb{R}^n \to \mathbb{R}^m$

$\vec{x} \longmapsto (tL)(\vec{x}) = tL(\vec{x})$

③ Let $N : \mathbb{R}^m \to \mathbb{R}^p$ be another linear mapping.

$N \circ L : \mathbb{R}^m \to \mathbb{R}^p$

$\vec{x} \longmapsto (N \circ L)(\vec{x}) = N(L(\vec{x}))$

} $\mathbb{R}^m$ must be same w/ L. (like Matrix multi)

*Watch: $\mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \xrightarrow{N} \mathbb{R}^p$

These 3 end up still being linear. *Try: Prove.

THE ANSWER TO ALL YOUR PROBLEMS

Proposition:

1) Matrix sum of lin. maps = sum of indv. matrices.
$[L+M] = [L] + [M]$

2) Matrix of sc. multiple of a lin map = sc. multiple of matrix.
$[tL] = t[L]$

Proof:

1) Let $\vec{x} \in \mathbb{R}^n$: $[L+M]\vec{x} = (L+M)\vec{x} = L(\vec{x}) + M(\vec{x}) = [L]\vec{x} + [M]\vec{x}$

↳ Implies $([L]+[M])\vec{x} = [L+M]\vec{x}$, QED.

↳ Try 2).

Theorem:

Let $L: \mathbb{R}^n \to \mathbb{R}^m$, $N: \mathbb{R}^m \to \mathbb{R}^p$ be two linear mappings.

$[N \circ L] = [N][L]$ ← THIS IS WHY MATRIX MULTI EXISTS!!

↳ Not defined when $[N][L]$ is not defined!!!

Proof:

Let $[L]_{m \times n}$, $[N]_{p \times m}$ where $[N][L]$ is well defined.

Let $\vec{x} \in \mathbb{R}^n$: $[N \circ L](\vec{x}) = (N \circ L)\vec{x} = N(L(x))$
$= N([L]\vec{x}) = [N]([L]\vec{x})$
$= ([N][L])\vec{x}$

Example:

1) Let $L: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \xrightarrow{\mathbb{R}^2 \to \mathbb{R}^2} \begin{pmatrix} 2x_1 + 3x_2 \\ x_1 - x_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$N: \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2 \longmapsto \begin{pmatrix} y_1 + y_2 \\ -3y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

THE ANSWER TO ALL YOUR PROBLEMS

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$= N([L]\vec{x}) = [N]([L]\vec{x})$
$= ([N][L])\vec{x}$

Example:

1) Let $L: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \overset{\mathbb{R}^2 \to \mathbb{R}^2}{\longmapsto} \begin{pmatrix} 2x_1 + 3x_2 \\ x_1 - x_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$N: \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2 \longmapsto \begin{pmatrix} y_1 + y_2 \\ -3y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$f_A: \mathbb{R}^n \to \mathbb{R}^m$, $\vec{x} \mapsto A\vec{x}$; $f_B: \mathbb{R}^m \to \mathbb{R}^p$, $\vec{y} \mapsto B\vec{y}$ then $\left\{ f_B \circ f_A = f_{BA}: \mathbb{R}^n \to \mathbb{R}^p, \ \vec{x} \mapsto (BA)\vec{x} \right.$

So... → Next page
↳ Must first check that both are linear.

Oct 12, 2017

Let $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. Then...

→ $N(\vec{a}+\vec{b}) = N(a_1+b_1, a_2+b_2) = \begin{pmatrix} a_1+a_2 \\ -3a_2 \end{pmatrix} + \begin{pmatrix} b_1+b_2 \\ -3b_2 \end{pmatrix}$

$= N(\vec{a}) + N(\vec{b})$ ←

→ $N(t\vec{a}) = N(ta_1, ta_2) = t\begin{pmatrix} a_1+a_2 \\ -3a_2 \end{pmatrix}$

$= tN(\vec{a})$ ← therefore, N is linear.

Same thing w/ L, both are linear.

[N][L] gives you:

$$\begin{bmatrix} 3 & 2 \\ -3 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = [N \circ L]\vec{x}$$

Notation:

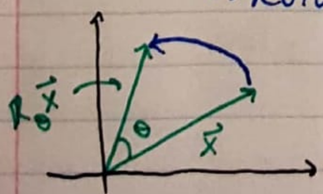
$L(\vec{x}) = L(x_1, x_2, \dots, x_n)$. (Horizontal for convenience) MIDTERM LINE

Applications:

Week 1: Coordinate System
Week 2: Networks and chemical reactions } Also on midterm
Week 3 & 4: Excercise
Week 4 & 5: x

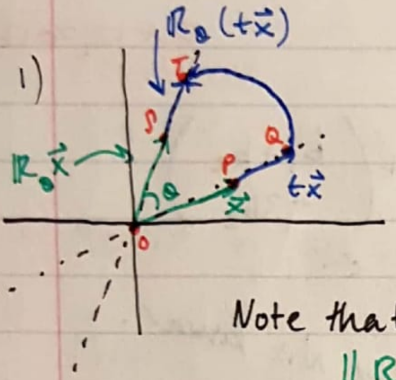
Linear Geometric Transformations: Rotations in the Plane:

Let $\theta \in \mathbb{R}$. The rotation in the plane $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is the mapping which maps $\vec{0} \to \vec{0}$ and $\vec{x}$ to $R_\theta(\vec{x})$.
↳ Rotates vector $\vec{x}$ counter-c'wise θ.



Is $R_\theta(\vec{x})$ linear?

1)



Since O,P,Q are collinear, so are their images O,S,T.

↳ ∴, $R_\theta(\vec{x})$ and $R_\theta(t\vec{x})$ are collinear.

↳ $\exists s \in \mathbb{R} \mid R_\theta(t\vec{x}) = s \cdot R_\theta(\vec{x})$ ← assumes $\neq \vec{0}$.

Note that rotations preserve length.

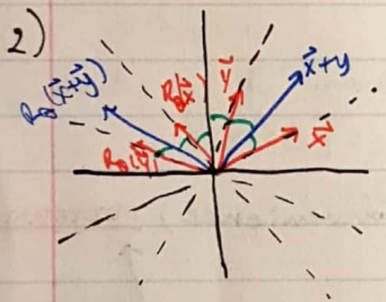
$\|R_\theta(t\vec{x})\| = \|t\vec{x}\| = |t| \cdot \|\vec{x}\|$ } b/c $\vec{x} \neq 0$

So: $\|s \cdot R_\theta(\vec{x})\| = |s| \cdot \|R_\theta \vec{x}\| = |s| \cdot \|\vec{x}\|$

∴ $|s| = |t|$.

Since rotation preserves orientation, $s = t$ QED. $R_\theta(t\vec{x}) = tR_\theta(\vec{x})$

2)



Rotations also preserve angles. ✱

↳ Since angles are same, shape of $\parallel$ogram is conserved. ✱ length

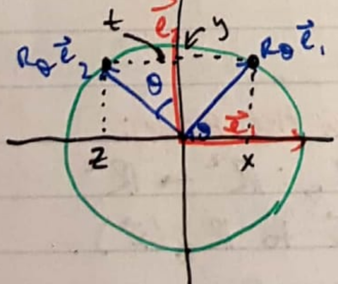
∴ $\|R_\theta(\vec{x}+\vec{y})\| = \|R_\theta(\vec{x})\| + \|R_\theta(\vec{y})\|$. ∴ linear.

But is θ same? Looong proof, but true.

Std Matrix of R_θ:

Let $[R_\theta] = \begin{pmatrix} x & z \\ y & t \end{pmatrix}$

$R_\theta(\vec{e}_1) = \begin{pmatrix} x \\ y \end{pmatrix}$ $\qquad$ $R_\theta(\vec{e}_2) = \begin{pmatrix} z \\ t \end{pmatrix}$



$x = \cos\theta$

$y = \sin\theta$

$z = \cos(\theta + \frac{\pi}{2}) = -\sin\theta$

$t = (\sin(\theta + \frac{\pi}{2})) = \cos\theta$

} R_θ is linear

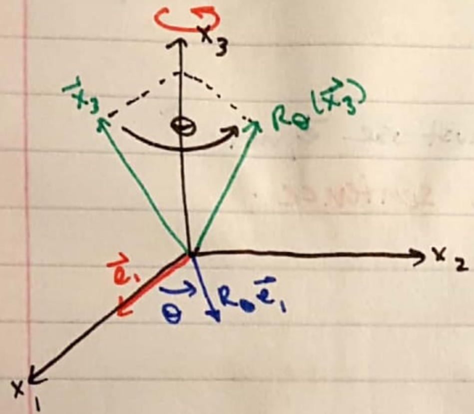
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Rotations in $\mathbb{R}^3$:

$R_\theta : \mathbb{R}^3 \to \mathbb{R}^3$

1) $R_\theta(\vec{0}) = \vec{0}$
2) Any vector on x_3 axis is mapped to itself.
3) Any vector not on x_3 axis to $R_\theta(\vec{x})$

→ counterclockwise around x_3 axis



$x_1 - x_2$ plane is invariant (stays in that plane)

Theorem:

$$[R_{\theta_3}] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection in Hyperplane $\mathbb{R}^3$:

Oct 25, 2017

Let H be a hyperplane through origin in $\mathbb{R}^n$.
Let $\vec{m}$ be normal to H. $\vec{m} \cdot \vec{x} = d$ for some $d \in \mathbb{R}$.

→ Since $\vec{0} \in H$, $\vec{m} \cdot \vec{0} = \vec{0}$, so $d = 0$.

→ $H: \vec{m} \cdot \vec{x} = 0$.

→ The reflection (or "across") H is the transformation:

$\mathrm{refl}_{\vec{m}} : \mathbb{R}^n \to \mathbb{R}^n$

$\vec{x} \mapsto \vec{x} - 2\,\mathrm{proj}_{\vec{m}}(\vec{x}) = Id - 2\,\mathrm{proj}_{\vec{m}}$

$\vec{x} \mapsto \vec{x} = [I]\vec{x}$

Sum of linear maps, ∴ refl. is linear mapping

Std Matrix of Refl:

$$[\text{refl}_{\vec{m}}] = [\text{Id} - 2\text{proj}_{\vec{m}}]$$
$$= [\text{Id}] - 2[\text{proj}_{\vec{m}}]$$
$$= I - 2[\text{proj}_{\vec{m}}]$$

Where $[\text{proj}_{\vec{m}}] = \dfrac{(\vec{m})(\vec{m})^T}{\|\vec{m}\|^2}$ (don't need to remember)
→ But need to be able to prove.

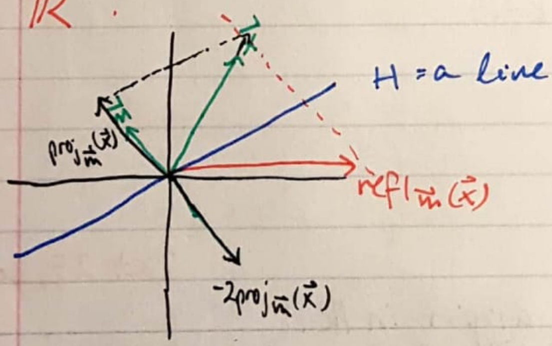
Example:

1) $\text{refl}_{\vec{m}} : \mathbb{R}^3 \to \mathbb{R}^3$ where $\vec{m} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$ Must use $\vec{e}$, not ↑

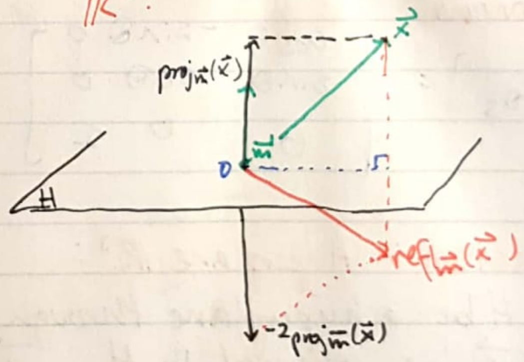
* Let $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \ldots \in \mathbb{R}^3$ Need this sentence.

$\text{refl}_{\vec{e}_1} = \begin{pmatrix} 21/29 \\ 12/29 \\ -16/29 \end{pmatrix}$ etc.

$\mathbb{R}^2$:



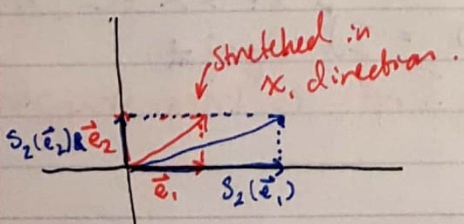
$\mathbb{R}^3$:



Other Transformations:

x_1 dir: Stretch: $t > 0$. $S_t : \mathbb{R}^2 \to \mathbb{R}^2$ $[S_t] = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$. "Shrink" if $t < 1$.

x_2 dir: Stretch: $t > 0$. $S'_t : \mathbb{R}^2 \to \mathbb{R}^2$ $[S'_t] = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$ "Shrink" if $t < 1$.



$t>0.\ T_t: \mathbb{R}^2 \to \mathbb{R}^2$ where $[T_t] = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$

$\rightsquigarrow$ Contraction if $t<1$
$\rightsquigarrow$ Dilation if $t>1$ $\Big\}\ \vec{x} \mapsto t\vec{x}$

Shears:

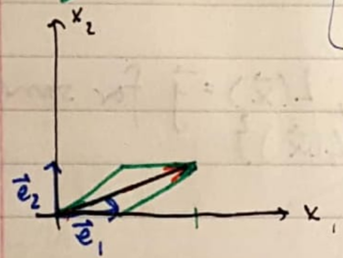
Oct 26, 2017

In x_1 dir by s: $\Sigma_s: \mathbb{R}^2 \to \mathbb{R}^2$

$[\Sigma_s] = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ $\quad x_2: \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$

$s=2$:

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$



Inverse Linear Mappings:

L. map ($n \to n$) invertible if $\exists$ map $\mathbb{R}^n \to \mathbb{R}^n$ where

$L \circ M = M \circ L = Id$

Proof that M is also linear:

$(L\circ M)(\vec{x}) = \vec{x} = (M \circ L)(\vec{x})$

$\vec{x} + \vec{y} = (L\circ M)(\vec{x}) + (L \circ M)(\vec{y})$

$= L\left(M(\vec{x}) + M(\vec{y})\right)$

So, $M(\vec{x}+\vec{y}) = M\left(L\left(M(\vec{x}) + M(\vec{y})\right)\right) = (M\circ L)(M(\vec{x}) + M(\vec{y}))$

$= M(\vec{x}) + M(\vec{y})$

And...

$t(\vec{x}) = t\left(L(M(\vec{x}))\right) = L(t(M(\vec{x})))$

So, $M(t(\vec{x})) = (M \circ L)(t M(\vec{x})) = tM(\vec{x})$. $\quad \therefore$ linear

M is inverse of L iff [L] is inverse of [M]

↳ Means inverse mapping is unique

Example:

1) Consider $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

R_θ is invertible with inverse $R_{-\theta}$.

$$[R_\theta][R_{-\theta}] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Mapping Ranges:

$L: \mathbb{R}^m \to \mathbb{R}^n$. Range(L): All vectors $\vec{y} \in \mathbb{R}^n$, $L(\vec{x}) = \vec{y}$ for some $\vec{x}$.

↳ $\text{Range}(L) = \{\vec{y} \in \mathbb{R}^n : \exists \vec{x} \in \mathbb{R}^m \ni \vec{y} = L(\vec{x})\}$

Null Space:

Null(L): All vectors mapped to $\vec{0}$.

↳ $\text{Null}(L) = \{\vec{x} \in \mathbb{R}^m : L(\vec{x}) = 0\}$ * never empty ($\vec{0} \to \vec{0}$)

Column Space:

A : m×n[ ] w/ columns $\vec{a}_1, \ldots \vec{a}_n \in \mathbb{R}^m$.

↳ Column space of a is subspace ($\mathbb{R}^m$) spanned by columns of A:

$\text{span}\{\vec{a}_1, \ldots \vec{a}_n\} = \{t_1\vec{a}_1 + t_2\vec{a}_2 + \ldots + t_n\vec{a}_n : t_{1\ldots n} \in \mathbb{R}\}$.

Linear Mapping Stuff: $L: \mathbb{R}^n \to \mathbb{R}^n$ [L] = A. Inv. Matrix/Mapping theorem

i) A is invertible

ii) Rank(A) = n

iii) RREF = I

iv) $\forall \vec{b} \in \mathbb{R}^n$, $A\vec{x} = b$ consistent w/ unique

v) Cols in A are lin. independent

vi) Col Space(A) = $\mathbb{R}^n$

vii) L is invertible linear mapping

viii) $\text{Null}(L) = \{\vec{0}\}$ $L(\vec{x}) = 0 \Rightarrow \vec{x} = 0$.

ix): $\text{Range}(L) = \mathbb{R}^n$: $\forall \vec{y} \in \mathbb{R}^n, \exists \vec{x} \in \mathbb{R}^n, \vec{y} = L(\vec{x})$

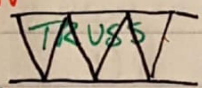
Invertibility Examples: Oct 27, 2017

1) Prove $proj_{\vec{v}} : \mathbb{R}^n \to \mathbb{R}^n$ is not invertible.
∵ More than one vector perp to $\vec{x}$, making Null more than $\{\vec{0}\}$. ∴ not invertible. $\vec{w} \cdot \vec{v} = 0$ but $\vec{w} \neq 0$.

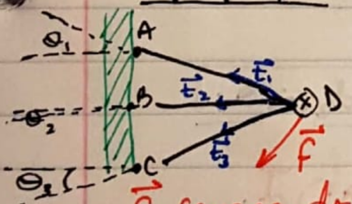
2) Stretch by $t > 0 : \mathbb{R}^2 \to \mathbb{R}^2$
$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$ invertible w/ inverse $\begin{pmatrix} \frac{1}{t} & 0 \\ 0 & 1 \end{pmatrix}$

3) Refl is inverse of itself.

Applications: Pin Jointed Planar Trusses Oct 27/31, 2017



Consider: 3 rigid members, 4 pinjoints ABCD.



- Free to rotate at the pin-joints
- Fastened to wall at ABC.

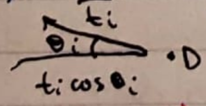
$\vec{F}$ causes displacement $\vec{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ at D. $\vec{d}$ rel. to $\vec{f}$? $\vec{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$

A few assumptions:

- ⇝ Weight is negligible wrt $\vec{f}$
- ⇝ $\vec{f}$ causes compression force (contract / dilate) on each member, t_i = compression where:

$$t_i = \begin{cases} \|\vec{t}_i\| & \text{if contraction} \\ -\|\vec{t}_i\| & \text{if dilation} \end{cases}$$

1) Resolution of forces at D.



$$\begin{cases} \text{Horizontally: } f_1 = t_1\cos\theta_1 + t_2\cos\theta_2 + t_3\cos\theta_3 \\ \text{Vertically: } f_2 = t_1\sin\theta_1 + t_2\sin\theta_2 + t_3\sin\theta_3 \end{cases}$$

$$\vec{f} = A\vec{t} : \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 & \cos\theta_2 & \cos\theta_3 \\ \sin\theta_1 & \sin\theta_2 & \sin\theta_3 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

2) Relate t_i to e_i, (extension).

Let e_i (a length) be the compression/extension due to t_i (force).

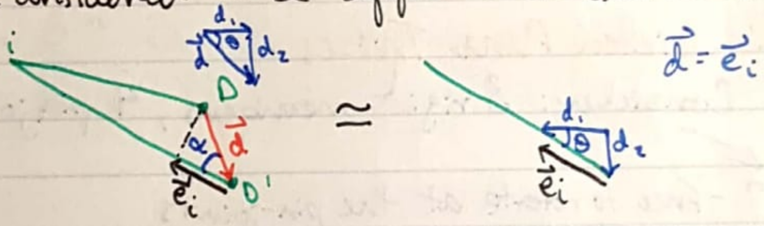
Hooke's Law: $e = kt$ (backwards from physics) b/c ~~k~~ $= k^{-1}$

k^{-1} is spring constant.

$\vec{e} = K\vec{t}$: $\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$ Assume $k_{1,2,3} \neq 0$.

Thus, K is invertible w/ inverse $\begin{pmatrix} k_1^{-1} & 0 & 0 \\ 0 & k_2^{-1} & 0 \\ 0 & 0 & k_3^{-1} \end{pmatrix}$. so $\vec{t} = K^{-1}\vec{e}$.

3) Relate $\vec{e}$ to $\vec{d}$. Assume $\vec{d}$ is small enough so, in first approximation, can be considered to be supported on each member.



Assumes α is very small. $\boxed{e_i = d_1 \cos\theta_i + d_2 \sin\theta_i}$

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ \cos\theta_2 & \sin\theta_2 \\ \cos\theta_3 & \sin\theta_3 \end{pmatrix}}_{A^T} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$ so $\vec{e} = (A^T)\vec{d}$.

Conclusion:

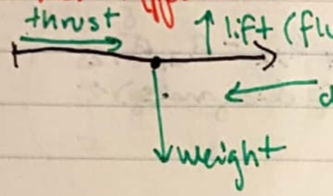
$\vec{f} = A\vec{t} = A(K^{-1}\vec{e}) = AK^{-1}A^T\vec{d}$ Let $B = AK^{-1}A^T$

So $\vec{f} = B\vec{d}$ where $B = AK^{-1}A^T$ (2×2).

If B is invertible, $\boxed{\vec{d} = B^{-1}\vec{f}}$.

Another Application: Stabilizing Flight:

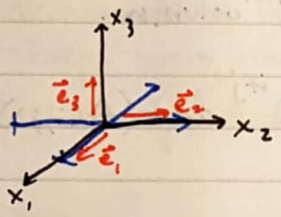
(sideways plane)



↑ lift (fluid flowing past surface of body exerts lift).

← drag (air resistance)

To take off, thrust > drag so lift can take over weight.



- Nose @ tip of $\vec{e}_2$ (initially).

To keep constant direction:

1) A <u>yaw</u> through θ_1. Rotation in x_1, x_2 plane, fixing x_3 axis.

$$Y_{\theta_1} = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2) A <u>pitch</u> through θ_2. Rotation in x_2, x_3 plane, fixing x_1 axis.

$$P_{\theta_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_2 & -\sin\theta_2 \\ 0 & \sin\theta_2 & \cos\theta_2 \end{pmatrix}$$

3) A <u>roll</u> through θ_3. Rotation in x_1, x_3 plane, fixing x_2 axis.

$$R_{\theta_3} = \begin{pmatrix} \cos\theta_3 & 0 & -\sin\theta_3 \\ 0 & 1 & 0 \\ \sin\theta_3 & 0 & \cos\theta_3 \end{pmatrix}$$

The "wrong" nose position $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\vec{x} = (R_{\theta_3} P_{\theta_2} Y_{\theta_1})\vec{e}_2$$ Let $T(\theta_1, \theta_2, \theta_3)$ = That thing.

($\vec{e}_2$ is "correct" position)

If it's invertible, autopilot applies the <u>inverse transformation</u> T^{-1}!

↳ Each rotation is invertible, so T is too. Each inverse is the transpose.

$$T(\theta_1, \theta_2, \theta_3)^{-1} = Y_{\theta_1}^T P_{\theta_2}^T R_{\theta_3}^T$$
$$= (R_{\theta_3} P_{\theta_2} Y_{\theta_1})^T$$

Determinants (2x2):

Nov 1, 2017

2x2

$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ Determinant: $\det(A) = a_{11}a_{22} - a_{21}a_{12}$

(product of diagonals) ↑

$\det(a) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

3x3

$a = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ Let $A(i,j)$ be 2x2 submatrix obtained by removing i^{th} row, j^{th} column.

The cofactor of a_{ij} is then: $c_{ij} = (-1)^{i+j} \det(A(i,j))$

Example:

1) Let $A = \begin{array}{c} \begin{array}{ccc} & & 3 \end{array} \\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \; 3 \end{array}$

$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5 - 8 = -3$

$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} = 9 - 7 \times 3 = -12$

Determinants (3x3):

$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

Example:

1) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \underset{a_{11}}{1}\underbrace{(-1)^{1+1}\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}}_{c_{11}} + \underset{a_{12}}{2}\underbrace{(-1)^{1+2}\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}}_{c_{12}} + \underset{a_{13}}{3}\underbrace{(-1)^{1+3}\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}}_{c_{13}} = 0$

Generalized Determinants (nxn):

$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ $\quad c_{ij} = (-1)^{i+j} \det A(i,j)$

$\det A = a_{11}c_{11} + a_{12}c_{12} + \ldots + a_{1n}c_{1n}$

$= \sum_{k=1}^{n} a_{1k}c_{1k}$

Example:

1) $A = \begin{pmatrix} 1 & 3 & 6 & 10 \\ 2 & 5 & 9 & 13 \\ 4 & 8 & 12 & 15 \\ 7 & 11 & 14 & 16 \end{pmatrix}$ $\det A = 1(-1)^{1+1}\begin{vmatrix} 5 & 9 & 13 \\ 8 & 12 & 15 \\ 11 & 14 & 16 \end{vmatrix} + 3(-1)^{1+2}\begin{vmatrix} 2 & 9 & 13 \\ 4 & 12 & 15 \\ 7 & 14 & 16 \end{vmatrix} + 6(-1)^{1+3}\begin{vmatrix} 2 & 5 & 13 \\ 4 & 8 & 15 \\ 7 & 11 & 16 \end{vmatrix}$

$+ 10(-1)^{1+4}\begin{vmatrix} 2 & 5 & 9 \\ 4 & 8 & 12 \\ 7 & 11 & 14 \end{vmatrix} = 6$

Things to Note:

1) Determinant of nxn [ ] is a linear combination of [ ] $n-1 \times n-1$.
   → keeps going recursively to show it can be expressed as lin comb. of $|2\times2|$ det
2) Cofactor expansion of first row.

Theorem:

Let A be nxn [ ]:

$$\begin{array}{cc} \begin{array}{c} j^{th}\text{ column} \end{array} & \\ \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} & \begin{array}{l} \\ \\ i^{th}\text{ row} \\ \\ \\ \end{array} \end{array}$$

can use any row or column i, j.

The determinant can be obtained by cofactor expansion by any i, j.

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots a_{in}C_{in}$$
$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots a_{nj}C_{nj}$$

} can use any row/col

→ Can choose most $\varnothing$ row/col.

Diagonal/Triangular Matrix Determinants: Nov 2, 2017.

Let A $n \times n$ [] be diagonal or upper/lower Δ.

Then $\det A$ = product of diagonal entries.

$$\begin{vmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{vmatrix} = \begin{vmatrix} a_1 & & \\ & \ddots & \\ 0 & & a_n \end{vmatrix} = \begin{vmatrix} a_1 & & 0 \\ & \ddots & \\ & & a_n \end{vmatrix}$$

$$= \prod_{i=1}^{n} a_i = a_1 a_2 \dots a_n.$$

Proof: (Start w/ upper Δ)

Base case: $n=2$ $\quad \begin{vmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{vmatrix} = a_{11}a_{22} - 0 \times a_{12} = a_{11}a_{22}$

IH: Assume $n \geq 3$: by cofactor expansion along first (last) row:

$$\Delta = \begin{vmatrix} a_{11} & & & \times \\ & \ddots & & \vdots \\ 0 & & & \times \\ 0 & \dots & 0 & a_{nn} \end{vmatrix} = a_{nn}(-1)^{n+n} \begin{vmatrix} a_{11} & & \\ & \ddots & \\ 0 & & a_{n-1,n-1} \end{vmatrix}$$

By IH: $\begin{vmatrix} a_{11} & & \\ & \ddots & \\ 0 & & a_{n-1,n-1} \end{vmatrix} = a_{11} \cdots a_{n-1,n-1}$

So $\Delta = (a_{nn})(1)(a_{11} \dots a_{n-1})$. QED.

(Similar for lower Δ and $\swarrow$). Would be last col or row for lower Δ

Properties of the Determinant:

Let A, B be $n \times n$ [], $r \in \mathbb{N}$. Assume $B = A$ but one row is $\times$ by r.

- $\rightarrow$ All zero column/row $\Rightarrow$ determinant $= 0$.
- $\rightarrow$ $\det A = \det A^T$
- $\rightarrow$ $\det B = r(\det A)$
- $\rightarrow$ $\det rA = r^n \det A$ (means det map not linear)

If you swap two rows in A to get B:

- $\rightarrow$ $\det B = -\det A$

If a has 2 identical rows/columns:

- $\rightarrow$ $\det A = 0$

If you get B by adding multiples of rows of A:
→ det B = det A

Reducing to Upper Triangular:
1) Swapping rows/colums changes sign
2) Adding rows ($R_1 = R_1 + 2R_2$) doesn't change det.

*Indicate row/column used for cofactor expansion with →

$$\begin{array}{r} \rightarrow \\ \\ \\ \end{array}\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Example:

1)
$$\begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ -1 & 0 & 3 & 1 \\ 3 & 1 & 2 & 0 \end{vmatrix}_{R_3+R_1} = \begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 2 & 4 & 1 \\ 3 & 1 & 2 & 0 \end{vmatrix}_{R_4-3R_1} = \begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & -5 & -1 & 0 \end{vmatrix}_{C_2-5C_3} =$$

$$\begin{vmatrix} 1 & -3 & 1 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & -18 & 4 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix}_{R_3-9R_2} = \begin{vmatrix} 1 & -3 & 1 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -5 & -8 \\ 0 & 0 & -1 & 0 \end{vmatrix}_{\boxed{C_3 \leftrightarrow C_4}} = - \begin{vmatrix} 1 & -3 & 0 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -8 & -5 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 16.$$