SE 212

Syntax:

Sept 11, 2018

- $\rightarrow \models$ "entails" (semantics)
- $\rightarrow \vdash$ "proves" (proof theory)
  - $\rightarrow \vdash \Rightarrow \models$ "soundness"
  - $\rightarrow \models \Rightarrow \vdash$ "completeness"

Well Formed Formulas:

1) Prop symbols + consts are formulas. (prime propositions)
2) If P, Q formulas, then

   $\neg P \quad P \wedge Q \quad P \vee Q \quad P \Rightarrow Q \quad P \Leftrightarrow Q$

   are formulas (compound propositions)

   $\neg > \wedge > \vee > \Rightarrow > \Leftrightarrow$

Example:

1) $((\neg p) \wedge (\neg q)) \vee r$
2) $(p \Rightarrow (q \wedge r) \vee x) \Leftrightarrow (y \vee r)$

$a \Rightarrow b \Rightarrow c$ means $a \Rightarrow (b \Rightarrow c)$ (right assoc.)

Terms:

$P \wedge Q$; P, Q are conjuncts
$P \vee Q$; P, Q are disjuncts

Example:

1) It is neither snowing nor cold.

   c = snowing d = cold : $\neg(c \vee d)$

2) It rains unless I take an umbrella.

   $\equiv r \vee u$

   i) will it rain?

   $\neg u \Rightarrow r$

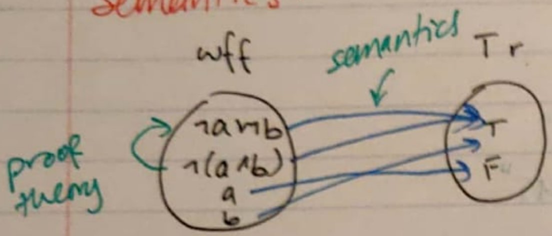
   $\neg(\neg u) \vee r$

   $u \vee r$

   ii) Will it not rain?

   can't garrantee.

Semantics:



Boolean Valuation: function from wff → Tr

Example:

1) $[p] = F$ $[q] = F$ $[r] = T$ what is $[(p \Rightarrow q) \wedge r]$?

$[(p \Rightarrow q) \wedge r]$
$= [p \Rightarrow q]$ AND $[r]$
$= ([p]$ IMP $[q])$ AND $[r]$
$= (F$ IMP $F)$ AND T $= T$.

Satisfiability:

P is satisfiable if there is a bool. val. where $[P] = T$.

Tautologies:

$[P] = T$ for all bool. values.

$\models P$

Logical Imp:

$P \models Q$ iff $\forall$ bool. values, $[P] = T \Rightarrow [Q] = T$

$P \models Q$ iff $\models P \Rightarrow Q$

$P \Rightarrow Q$ is a tautology.

Example:

1) $p \Rightarrow q, q \Rightarrow p \overset{?}{\models} q \wedge p$

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$q \wedge p$
F	F	$\boxed{T}$	$\boxed{T}$	$\boxed{F}$ ×
F	T	T	F	F
T	F	F	T	F
T	T	$\boxed{T}$	$\boxed{T}$	$\boxed{T}$ ✓

∴ ⊭ not entailment

Contradiction:

A is a contr. if $[A] = F$ $\forall$ bool. valuations.

Contingent Formula:

Not tautology and not contradiction (mix of T and F)

Relationships:

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Contingent $\Rightarrow$ Satisfiable

satisfiable $\Rightarrow$ can be contingent or taut.

Not satisfiable $\Rightarrow$ contradiction

P is a taut. $\Leftrightarrow$ $\neg P$ is a contradiction

Equivalence:

Logical eq: $\Leftrightarrow$ same for all bool. vals.

$P \Leftrightarrow Q$ iff $\vDash P \Leftrightarrow Q$

"material eq."

Consistency:

Set of formulas are consistent if

↳ There is a bool. valuation that maps each to T.

↳ To check, AND all of the formulas together and check satisfyability.

False Imp. Anything:

If premises aren't consistent, anything is provable.

Transformational Proofs:

$((c \Rightarrow d) \wedge (b \vee a)) \wedge \neg(b \vee a) \Leftrightarrow$ false?

$\Leftrightarrow (c \Rightarrow d) \wedge ((b \vee a) \wedge \neg(b \vee a))$

$\Leftrightarrow (c \Rightarrow d) \wedge (\text{false})$

$\Leftrightarrow$ false.

Refer to summary page.

Example:

1) $\neg(\neg p \vee \neg(r \vee s)) \Leftrightarrow (p \wedge r) \vee (p \wedge s)$

can start on either side.

$\neg(\neg p \vee \neg(r \vee s))$

$\Leftrightarrow \neg\neg p \wedge \neg\neg(r \vee s)$ by dm

$\Leftrightarrow p \wedge (r \vee s)$ by neg * 2

$\Leftrightarrow (p \wedge r) \vee (p \wedge s)$ by distr.

2) $p \vee (p \wedge q) \Leftrightarrow p$

$p \vee (p \wedge q)$

$\Leftrightarrow (p \wedge \text{true}) \vee (p \wedge q)$ by simp'

$\Leftrightarrow p \wedge (\text{true} \vee q)$ by distr

$\Leftrightarrow p \wedge \text{true}$ by simp'

$\Leftrightarrow p$

3) $\neg((p \wedge q) \Rightarrow p) \Leftrightarrow$ false

$\neg((p \wedge q) \Rightarrow p)$

$\Leftrightarrow \neg(\neg(p \wedge q) \vee p)$ by imp

$\Leftrightarrow \neg\neg(p \wedge q) \wedge \neg p$ by dm

$\Leftrightarrow (p \wedge q) \wedge \neg p$ by neg

$\Leftrightarrow (q \wedge p) \wedge \neg p$ by comm

$\Leftrightarrow q \wedge (p \wedge \neg p)$ by assoc

Don't need to show

$\Leftrightarrow q \wedge \text{false}$ by contr

$\Leftrightarrow$ false by simp'

4) $\neg true \Leftrightarrow false$

$\neg true$

$\Leftrightarrow \neg(q \vee \neg q)$ by lem

$\Leftrightarrow (\neg q) \wedge \neg(\neg q)$ by dm

$\Leftrightarrow false$ by contr.

5) $p \wedge (\neg(\neg q \wedge \neg p) \vee p) \Leftrightarrow p$

$p \wedge (\neg(\neg q \wedge \neg p) \vee p)$

$\Leftrightarrow p \wedge (\boxed{\neg\neg q \vee \neg\neg p)} \vee p)$ by dm

$\Leftrightarrow p$ by simp2

$\Leftrightarrow$:

1) If $P \Leftrightarrow Q$ can be proved, then $P \Leftrightarrow Q$ soundness

2) If $P \Leftrightarrow Q$, then $P \Leftrightarrow Q$ can be proved completeness

↳ Remove rules to eliminate completeness, add to elim. soundness.

P is a tautology : $P \Leftrightarrow true$.

$P \Leftrightarrow Q$: Find a bool. val. where $P \neq Q$

↳ NOT same as $\models \neg(P \Leftrightarrow Q)$ → means $P \Leftrightarrow Q$ is contr

Example:

1) To prove if a section of code is unreachable, conjunct all of the conditions necessary and try to prove if $P \Leftrightarrow false$.

↳ To prove reachable, provide a B.V. for True.

Normal Forms:

CNF - conjunction of disjunctions - pos

DNF - disjunction of conjunctions - sop

Example:

1) $(p \wedge q \wedge r) \vee (\neg q \wedge \neg r)$ DNF

2) $\neg(p \Rightarrow q) \wedge (r \Leftrightarrow q)$ neither can't have $\Rightarrow$, $\Leftrightarrow$

3) $(p \wedge q) \vee \neg q \vee \neg r$ DNF

4) $(p \vee q) \wedge \neg r$ CNF

5) $p \vee q$ BOTH

6) $\neg(p \wedge q) \vee r$ Neither

must be single literal (no $\neg(p \wedge q)$)

Example: Sept 18, 2018

1) $\neg(b \wedge \neg a) \Leftrightarrow \neg a \Rightarrow \neg b$

$\neg(b \wedge \neg a)$

$\Leftrightarrow \neg b \wedge \neg\neg a$ by dm

$\Leftrightarrow \neg b \wedge a$ by neg

$\Leftrightarrow b \Rightarrow a$ by imp

$\Leftrightarrow \neg a \Rightarrow \neg b$ by contrapos

Normal Forms:

DNF good for satisfiability checking.

Converting to NF:

1) Remove all $\Rightarrow$, $\Leftrightarrow$

2) Remove negations / push them in

3) Dist. laws for $\wedge$ and $\vee$

4) Simplify until no repeated literals.

Example:

1) CNF of $\neg((p \vee \neg q) \wedge \neg r)$

$\Leftrightarrow \neg(p \vee \neg q) \vee \neg\neg r$ by dm

$\Leftrightarrow \neg(p \vee \neg q) \vee r$ by neg

$\Leftrightarrow (\neg p \wedge \neg\neg q) \vee r$ by dm

$\Leftrightarrow \neg p \wedge q \vee r$ by neg.

$\Leftrightarrow (\neg p \vee r) \wedge (q \vee r)$ by dist

Natural Deduction:

Argument: premises $\vdash$ conclusion.

Deductive Reasoning: Truth of premises imply (guarantee) truth of conclusion

Inductive Reasoning: Conclude general new knowledge.

Natural Deduction: Rules to go from premises $\rightarrow$ concl.

↳ Forward proof

↳ Gerhard Gentzen

Inference Rules:

Conjunction: and-introduction.

$$\frac{P \quad Q}{P \wedge Q} \text{ and\_i}$$

and-elimination

$$\frac{P \wedge Q}{P} \quad \text{or} \quad \frac{P \wedge Q}{Q} \quad \text{and\_e}$$

Rule must apply to whole line (not subpart)

↑ refer to website for other rules

Example:

1) $a \wedge b, c \vdash b \wedge c$

1) $a \wedge b$ premise
2) c premise
3) b by and_e on 1
4) $b \wedge c$ by and_i on 2, 3

2) $a, \neg\neg(b \wedge c) \vdash \neg\neg a \wedge c$

1) a premise
2) $\neg\neg(b \wedge c)$ premise
3) $b \wedge c$ by not_not_e on 2)
4) c by and_e on 3)
5) $\neg\neg a$ by not_not_i
6) $\neg\neg a \wedge c$ by and_i on 4, 5

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3) $a, a \Rightarrow b, b \Leftrightarrow c \vdash b \wedge c$

1 mark
1) a premise
2) $a \Rightarrow b$ premise
3) $b \Leftrightarrow c$ premise
4) b by imp-e on 1, 2
5) $b \Rightarrow c$ by iff-e on 3
6) c by imp-e on 4, 5
7) $b \wedge c$ by and-i on 4, 6

Rules:

$$\frac{P \Leftrightarrow Q \quad P}{Q} \text{ iff\_mp}$$

$$\frac{Q \quad P \Rightarrow \neg Q}{\neg P} \text{ imp\_e}$$

$$\frac{P \Rightarrow Q \quad P}{Q} \text{ imp-e modus ponens}$$

$$\frac{P \Rightarrow Q \quad \neg Q}{\neg P} \text{ modus tollens}$$

4) $a \wedge b \vdash a \vee c$
1) $a \wedge b$ premise
2) a from and-e on 1
3) $a \vee c$ from or-i on 2

5) $\neg\neg a \Leftrightarrow b \wedge c, a \vdash b \vee s$
1) $\neg\neg a \Leftrightarrow b \wedge c$ premise
2) a premise
3) $\neg\neg a$ by not_not_i on 2
4) $b \wedge c$ by iff-mp on 1)
5) b by and-e on 4
6) $b \vee s$ by or-i on 5

can't do this on line 1! must match whole formula.

6) 1) $a \Rightarrow (b \Rightarrow c)$ premise
2) b premice
3) c

× WRONG

7) $\neg a \Leftrightarrow \neg b, \neg b \wedge a \vdash c$
1) $\neg a \Leftrightarrow \neg b$ premise
2) $\neg b \wedge a$ premise
3) $\neg b$ by and_e on 2
4) $\neg a$ by iff-mp on 1, 2
5) a by and_e on 2
6) c by not_e on 4, 5

8) $b \Rightarrow a, a \Rightarrow b, b \Leftrightarrow c, a \vdash (a \Leftrightarrow c) \wedge \neg\neg(b \vee c)$

1) $b \Rightarrow a$ premise
2) $a \Rightarrow b$ "
3) $b \Leftrightarrow c$ "
4) a "
5) $a \Leftrightarrow b$ by iff-i on 1, 2
6) $a \Leftrightarrow c$ by trans
7) b by imp-e on 2, 4
8) $b \vee c$ by or-i on 7
9) $\neg\neg(b \vee c)$ by not_not-i on 8
10) $a \Leftrightarrow c \wedge \neg\neg(b \vee c)$ by and-i.

trans:

$$\frac{\begin{array}{c} a \Leftrightarrow b \\ b \Leftrightarrow c \end{array}}{a \Leftrightarrow c}$$

Subproofs:

assume R {

...

Q

}

$$\frac{}{R \Rightarrow Q} \text{ imp-i}$$

Example:

1) $b \Rightarrow c \vdash \neg c \Rightarrow \neg b$

1) $b \Rightarrow c$ premise
2) assume $\neg c$ {
3) $\neg b$ by imp-e on 1, 2
}
4) $\neg c \Rightarrow \neg b$ by imp-i on 2-3.

2)
1) $a \Rightarrow b \Rightarrow c$ premise
2) b premise
3) assume a {
4) $b \Rightarrow c$ by imp-e on 1, 3
5) c by imp-e on 2, 3
}
6) $a \Rightarrow c$ by imp-i on 3-5.

Example: Sept 20, 2018

1) $\neg q, p \Rightarrow (q \Leftrightarrow \neg r), p \wedge \neg r \vdash r$ hint: left side could be contradiction.

1) $\neg q$ premise
2) $p \Rightarrow (q \Leftrightarrow \neg r)$ premise
3) $p \wedge \neg r$ premise
4) p by and_e on 3
5) $\neg r$ by and_e on 3
6) $q \Leftrightarrow \neg r$ by imp_e on 2, 4
7) q by iff_mp on 6, 5
8) r by not_e on 1, 7

2) $\vdash (c \Rightarrow d) \Rightarrow (\neg c \Rightarrow \neg b) \Rightarrow (b \Rightarrow d)$

1) assume $c \Rightarrow d$ {
2) assume $\neg c \Rightarrow \neg b$ {
3) assume b {
4) $\neg\neg c$ by imp_e on 2, 3
5) c by not_not_e on 4
6) d by imp_e on 1, 5
}
7) $b \Rightarrow d$ by imp_i on 3-6
}
8) $(\neg c \Rightarrow \neg b) \Rightarrow (b \Rightarrow d)$ by imp_i on 2-7
}
9) $(c \Rightarrow d) \Rightarrow (\neg c \Rightarrow \neg b) \Rightarrow (b \Rightarrow d)$ by imp_i on 1-8

3) $\vdash b \Rightarrow b$

1) assume b {}
2) $b \Rightarrow b$ by imp_i on 1-1

Proof by Contradiction (RAA):

disprove R {

...

false ← need false

}

$$\frac{3}{\neg R}\ \text{raa}$$

not_i

Example:

1) $a \Rightarrow b, \neg b \vdash \neg a$

1) $a \Rightarrow b$ premise
2) $\neg b$ premise
3) disprove a {
   4) b by imp_e on 1,3
   5) false by not_e on 2,4
   }
6) $\neg a$ by raa on 3-5

2) $b \Rightarrow c, b \Rightarrow \neg c \vdash \neg b$

1) $b \Rightarrow c$ premise
2) $b \Rightarrow \neg c$ premise
3) disprove b {
   4) c by imp_e on 1,3
   5) $\neg c$ by imp_e on 2,3
   6) false by not_e on 4,5
   }
7) $\neg b$ by raa on 3-6

More on Natural Deduction:

$P_1, \ldots, P_n \vdash Q$ in ND. - sound:

$P_1, \ldots, P_N \models Q$

$P_1 \wedge \ldots \wedge P_N \models Q$

$\models P_1 \wedge \ldots \wedge P_N \Rightarrow Q$

$P_1 \wedge \ldots \wedge P_N \Rightarrow Q \Leftrightarrow$ true } completeness

$P_1 \wedge \ldots \wedge P_N \Rightarrow Q \Leftrightarrow$ true

can change between proof systems.

Example:

1) $p \wedge \neg q \Rightarrow r, \neg r, p, \vdash q$

1) $p \wedge \neg q \Rightarrow r$ premise
2) $\neg r$ "
3) p "
4) $\neg(p \wedge \neg q)$ by imp_e on 1, 2
5) disprove $\neg q$ {
    6) $p \wedge \neg q$ by and_i on 3, 5
    7) false by not_e on 4, 6
}
8) q by raa on 5-7

Case Analysis:

$P \vee R$

case P {
    ⋮
    Q
}

case R {
    ⋮
    Q
}

$$\frac{}{Q.}\text{ cases}$$

Example: $p \vee q, \neg p \vdash q$

1) $p \vee q$ premise
2) $\neg p$ premise
3) case p {
    4) q by not_e on 2, 3
}
5) case q {}
6) q by cases 1, 3-4, 5-5

Disjunctive Syllogism:

$$\frac{P \lor Q \quad \neg P}{Q}\ \text{or\_e} \qquad \frac{P \lor Q \quad \neg Q}{P}\ \text{or\_e}$$

Example:

1) $a, \neg b \Leftrightarrow a, c \lor b, c \Rightarrow b \vdash false$
   1), 2), 3), 4) premises
   5) $\neg b$ by iff-mp on 1, 2
   6) c by or_e on 3, 5
   7) b by imp_e on 4, 6
   8) false by not_e on 5, 7

2) $\vdash p \lor \neg p$
   1) disprove $\neg(p \lor \neg p)$ {
      2) disprove p {
         3) $p \lor \neg p$ by or_i on 2
         4) false by not_e on 1, 3
      }
      5) $\neg p$ by raa on 2-4
      6) $p \lor \neg p$ by or_i on 5
      7) false by not_e on 1, 6
   }
   8) $p \lor \neg p$ by raa on 1-7

To prove	Possible Goals	premise	Possible goals
$A \land B$	Both A and B	$A \land B$	and_e to A, B
$A \lor B$	Either A or B	$A \lor B$	cases, or_e
$A \Rightarrow B$	assume A, prove B	$A \Rightarrow B$	Prove A, then imp_e
A	$\neg A$ then contradict	$\neg A$	Prove A, use not_e
		?	LEM + cases

3) $\neg a \vee \neg b \vdash \neg(a \wedge b)$ *try other direction

1) $\neg a \vee \neg b$ premise
2) disprove $a \wedge b$ {
   3) a by and_e on 2
   4) b by and_e on 2
   5) case $\neg a$ {
      6) false by not_e on 3,5
   }
   7) case $\neg b$ {
      8) false by not_e on 4,7
   }
   9) false by cases on 1, 5-6, 7-8
}
10) $\neg(a \wedge b)$ by raa on 2-9

4) $\neg(a \vee b) \vdash \neg a \wedge \neg b$

1) $\neg(a \vee b)$ premise
2) disprove a {
   3) $a \vee b$ by or_i on 2
   4) false by not_e on 1,3
}
5) $\neg a$ by raa on 2-4
6) disprove b {
   7) $a \vee b$ by or_i on 6
   8) false by not_e on 1,7
}
9) $\neg b$ by raa on 6-8
10) $\neg a \wedge \neg b$ by and_i on 5,9

Example: Sept 25, 2018

1) $b \Rightarrow c \vdash \neg b \vee c$

1) $b \Rightarrow c$ premise
2) $b \vee \neg b$ by lem
3) case b {
    4) c by imp_e on 1,3
    5) $\neg b \vee c$ by or_i on 4)
}
6) case $\neg b$ {
    7) $\neg b \vee c$ by or_i
}
8) $\neg b \vee c$ by cases on 2, 3-5, 6-7

Semantic Tableaux:

Tree: all ways formulas can be true

Example:

1) Prove $b \wedge c$, d, $\neg(c \wedge d)$ is inconsistent.

1) $b \wedge c$
2) d
3) $\neg(c \wedge d)$
| and_nb on 1
4) b
5) c
not_and_br on 3
6) $\neg c$ — closed on 5,6
7) $\neg d$ — closed on 2,7

OR

1) $b \wedge c$
2) d
3) $\neg(c \wedge d)$
not_and_br on 3
4) $\neg c$
| and_nb on 1)
6) b
7) c
closed on 4,7
5) $\neg d$
closed on 2,5

close if P and $\neg P$ appear on path root → leaf.

2) 1) $r \Rightarrow s$
2) $s \Rightarrow \neg h$
3) r
4) h

imp-br on 1

5) $\neg r$ — Closed on 3,5

6) s

imp-br on 2

7) $\neg s$ — closed on 6,7

8) $\neg h$ — closed on 4,8

∴ inconsistent set of formulas.

(non-branch first = shorter proofs)

Proving validity:

Premises + ¬conclusion = inconsistent.

→ proof by contradiction

Example:

1) $\vdash b \vee \neg b$

1) $\neg(b \vee \neg b)$

| not-or-nb on 1

2) $\neg b$

3) $\neg\neg b$

| not-not-nb on 3

4) b

Closed on 2,4

2) $r, p \Leftrightarrow q \vdash q$

1) p
2) $p \Leftrightarrow q$
3) $\neg q$

iff-br on 2

4) $p \wedge q$

| and-nbr 4

6) p

7) q

closed 3,7

5) $\neg p \wedge \neg q$

| and-nb on 5

8) $\neg p$

9) $\neg q$

closed 1,8

Air Traffic Example:

Vertical Separation:

Result	Condition
1000	a_{290}
2000	$\neg a_{290} \wedge \neg(a_{450} \wedge bsup)$
4000	$a_{450} \wedge bsup$

a_{290}: Flight A below FL290

a_{450}: Flight A above FL450

b_{sup}: Flight B supersonic

Check if disj of all is a tautology:

$a_{290} \vee (a_{450} \wedge bsup) \vee (\neg a_{290} \wedge \neg(a_{450} \wedge bsup))$

take out neg by demg, then lem.

by nat. deduction:

$\vdash (a_{290}) \vee (a_{450} \wedge bsup) \vee (\neg a_{290} \wedge \neg(a_{450} \wedge bsup))$

1) $(a_{450} \wedge bsup) \vee \neg(a_{450} \wedge bsup)$

2) case $a_{450} \wedge bsup$ {

3) concl by or-i on 2

}

4) case $\neg(a_{450} \wedge b_{sup})$ {

5) $a_{290} \vee \neg a_{290}$ by lem

6) case a_{290} {

7) concl by or-i on 6

}

8) case $\neg a_{290}$ {

9) $\neg a_{290} \wedge \neg(a_{450} \wedge bsup)$ by and-i on 4, 8

10) concl by or-i on 9

}

11) concl by cases 5, 6-7, 8-10

}

12) concl by cases 1, 2-3, 4-11

Check only one result for every valuation.

$\neg(A \wedge B)$
$\neg(A \wedge C)$ } no 2 can be true at same time.
$\neg(B \wedge C)$

Need an env. assumption: $\neg(a_{290} \wedge a_{450})$

$\neg(a_{290} \wedge a_{450}) \vdash \neg(a_{290} \wedge (a_{450} \wedge b_{sup}))$

1) $\neg(a_{290} \wedge a_{450})$

2) $\neg\neg(a_{290} \wedge a_{450} \wedge b_{sup})$

| not-not-nb on 2

3) $a_{290} \wedge a_{450} \wedge b_{sup}$

| and-nb on 3.

4) $a_{290} \wedge a_{450}$ 5) b_{sup}

closed on 1,4

Predicate logic:

Sept 27, 2018

"First order logic" - Gottlob Frege.

Unary Predicates:

- Ravi is an adult

adult(Ravi)

adult(x) means x is an adult.

Multiple argument Predicates:

- Barbara plays the piano.

plays(Barbara, piano)

plays(x, y) means x plays y.

- 10 is greater than 5.

$10 > 5$.

Quantifiers:

$\forall$ forall $\exists$ exists $\forall x \cdot$ likes(x, Fridays)

($\neg\forall$) not everything ($\neg\exists$) Nothing such that

(not something)

- Something is rotten in the state of Denmark
  $\exists x \cdot rotten(x) \wedge in(x, Denmark)$
- Everyone is a tall adult
  $\forall x \cdot adult(x) \wedge tall(x)$
- Every child likes Mickey Mouse
  $\forall x \cdot child(x) \Rightarrow likes(x, MM)$

Functions:

1) Mary's age is less than (20)
   $\exists x \cdot age(Mary, x) \wedge x < 20$: $age(Mary) < 20$
   allows multiple x's T/F — returns value

2) Eunsuk was born north of Toronto
   northof(birthplace (Eunsuk), Toronto)
   } Functions only have one returned value. Not T/F

3) For all x, if "input" has the value x, then "output" has val x^2.
   $\forall x \cdot value(\text{"input"}) = x \Rightarrow value(\text{"output"}) = x^2$

WFF:

predicates p, q
Variables x, y, z
functions k, g, h
constants c, d

1) $p(\underbrace{k(x,y,\underbrace{g(z)}_{term})}_{term}, \underbrace{h(x)}_{term}) \wedge q(\underbrace{x, c}_{terms})$ yep.
2) $\forall x \cdot \exists y \cdot p(c,d)$ ✓ yep
3) $\forall x \cdot \forall x \cdot p(x,y)$ ✓ yep
4) $\exists x \cdot k(x,y,g(x))$ ✗ NO (k is fcn not predicate)

Examples:

1) All bicycles are in the garage.
   $\forall x \cdot b(x) \Rightarrow g(x)$
2) Some bikes in garage.
   $\exists x \cdot b(x) \Rightarrow g(x)$!! vacuously true · for non bikes
   weak combo. $\Rightarrow$ should be $\wedge$

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3) Everything is a bicycle or is in the garage.
$\forall x \cdot b(x) \vee g(x)$

4) Something is a bicycle or in garage.
$\exists x \cdot b(x) \vee g(x)$

5) $\forall x \cdot \exists x$ means $\forall x$ is irrelevant.

Scope of quantifiers:

Lowest precedence. only stopped by brackets.

1) $\underbrace{\forall x \cdot \exists y \cdot p(x, y) \wedge q(y)}$

2) $\forall x \cdot (\exists y \cdot p(x, y) \wedge \exists y \cdot q(y)) \wedge r(x, y)$ free

bound to closest quantifier.

Example:

1) All students who like software engineering also like logic.
$\forall x \cdot student(x) \wedge likes(x, SE) \Rightarrow likes(x, logic)$

2) All rich actors collect some valuables Oct 2, 2018
$\forall x \cdot rich(x) \wedge actor(x) \Rightarrow \exists y \cdot valuable(y) \wedge collects(x, y)$

3) Not everyone in Louisiana speaks French, but everyone in L. knows someone in L. who speaks French.
$(\neg[\forall x \cdot in(x, L) \Rightarrow speaks(x, Fr)])$ imp not and
$\wedge (\forall y, \exists z \cdot in(y, L) \Rightarrow knows(y, z) \wedge speaks(z, Fr) \wedge in(y, L)$

can go here

4) Everyone votes for someone

$\forall x \cdot \exists y \cdot votes\_for(x, y)$

5) It is not the case that someone votes for every political party.

$\neg(\exists x \cdot \forall y \cdot political\_party(y) \Rightarrow votes\_for(x, y))$

Now, what does this mean?

$\neg(\forall y \cdot political\_party(y) \Rightarrow \exists x \cdot votes\_for(x, y))$

Not every political party gets a vote.

Multiple interpretations.

Types:

likes(Billy, ice cream) vs likes(Vancouver, icecream)

→ we expect likes(x,y): x should be a person.

→ Non-empty sets of values, like $\mathbb{N}$

Example:

1) $\forall x: \underline{Int} \cdot \exists y: \underline{Int} \cdot x \geq y$

2) $\exists b: \underline{plane} \cdot \underline{flightlevel(b)} > FL290$

Flightlevel: plane → Num

$>$: Num + Num → Bool

Predicates are functions that return bool.

$$\frac{x: T_1, \quad p: Bool}{(\forall x \cdot p): Bool} \quad (\text{and } \exists)$$

Example:

1) Mod 3, pg 51.

2) $\underset{T_1}{\forall x} \cdot \underset{T_3}{\exists y} \cdot \underbrace{q(\underbrace{f(\underset{T_1}{x})}_{T_2}, \underset{T_3}{y})}_{T_2 \times T_3 \to Bool}$

Type Inference:

Get types to make formulas well-typed..

1) $\exists x . p(x) \wedge p(f(x))$

$p : \underline{A} \to Bool$

$f : \underline{A} \to \underline{A}$

$x : \underline{A}$

2) $\forall x, y \cdot q(\underline{f(x,y)}) \Rightarrow q(\underline{y, f(x)})$

Nah,

→ Make some types are disjoint

$\forall x : Q \cdot P(x)$ same as $\forall x \cdot Q(x) \Rightarrow P(x)$

$\exists x : Q \cdot P(x)$ same as $\exists x \cdot Q(x) \wedge P(x)$

Predicate Logic semantics:

$p(g(c), d) \wedge q(c, d)$

Interpretation:

1) Domain = $\{0, 1, 2\}$

2) Mapping:

Syntax	Meaning
c	0
d	1
$g(\cdot)$	$G(x) := (x+1) \bmod 3$
$p(\cdot,\cdot)$	$P(x,y) := x<y$
$q(\cdot,\cdot)$	$Q(x,y) := x=y$

const → elem ∈ domain

(total) fun → domain → domain

predicates → domain → Tr

$\forall x . p(x), D = \{d_1, d_2\}$

$[P(x)]_{x \mapsto d_1}$ AND $[P(x)]_{x \mapsto d_2}$

$[P(\wedge d_1)]$ AND $[P(\wedge d_2)]$

$[p(g(c), d) \wedge q(c, d)]$

$[p(g(c), d)]$ AND $[q(c, d)]$

$P([g(c)], [d])$ AND $Q([c], [d])$

$[g(c)] < [d]$ AND $[c] = [d]$

$G([c]) < [d]$ AND $[c] = [d]$

$([c]+1) \bmod 3 < [d]$ AND $[c] = [d]$

$(0+1) \bmod 3 < 1$ AND $0 = 1$

F AND F → F

Example: Oct 4, 2018

1) $(\exists x.\, g(x) \wedge b(k(x))) \wedge \neg b(c)$ what makes it T?

$D = \{d_1, d_2\}$

$= (G(d_1) \text{ AND } B(K(d_1))) \text{ OR } (G(d_2) \text{ AND } B(K(d_2))) \text{ AND NOT } B(d_1)$

c	d_1
$b(\cdot)$	$B(d_1) = F,\ d_2 = T$
$g(\cdot)$	T, T
$k(\cdot)$	d_1, d_2

results in T

2) $\forall x.\, \exists y.\, p(x) \Leftrightarrow \neg p(y)$

$D = \{d_1, d_2\}$

$p(\cdot)$	$P(d_1) = T$ $P(d_2) = F$

$[\exists y.\, p(d_1) \Leftrightarrow \neg p(y)] \text{ AND } [\exists y.\, p(d_2) \Leftrightarrow \neg y]$

$(([p(d_1) \Leftrightarrow \neg p(d_1)] \text{ OR } [p(d_1) \Leftrightarrow \neg p(d_2)]) \text{ AND }$

$([p(d_2) \Rightarrow \neg p(d_1)] \text{ OR } [p(d_2)] \Leftrightarrow \neg p(d_2)])$

$= T.$

Sat, Taut, Contr:

# of interpretations	Truth Val	Meaning
Some	T	Satisfiable
all	T	tautology
all	F	contradiction

Example:

1) $\forall x.\, p(x) \vee q(x) \not\models ((\forall x.\, p(x)) \vee (\forall x.\, q(x)))$

$p(\cdot)$	$p(d_1) := T$ $p(d_2) := F$
$q(\cdot)$	$q(d_1) := F$ $q(d_2) := T$

Left:

$(P(d_1) \text{ OR } Q(d_1)) \text{ AND } (P(d_2) \text{ OR } Q(d_2))$

Right:

$(P(d_1) \text{ AND } P(d_2)) \text{ OR } (Q(d_1) \text{ AND } Q(d_2))$

Types:
$q(c,d) \wedge \exists z{:}T_1 \cdot \neg(z=c) \wedge q(z,d)$

Domain: $T_1 = \{v_1, v_2\}$ $T_2 = \{w_1\}$

Syntax	Meaning
c	v_1
d	w_1
$q(\cdot,\cdot)$	$Q(v_1, w_1) := T$ $Q(v_2, w_1) := T$

ND for Predicate Logic:
$\forall$ elimination: $\dfrac{\forall x \cdot P}{P[t/x]}$ If formula T $\forall x$, must be T for any x.

Example:
1) 1) $p(c)$ premise
2) $\forall x \cdot p(x) \Rightarrow \neg q(x)$ premise
3) $p(c) \Rightarrow \neg q(c)$ forall_e on 2
4) $\neg q(c)$ imp_e

$p(c), \forall x\, p(x) \Rightarrow \neg q(x) \vdash q(c)$

Substitution:
1) if P is $q(x)$, then $P[y/x]$ is $q(y)$
2) iff P is $\forall x \cdot q(y,x)$ then $P[4/y]$ is $\forall x \cdot q(4,x)$
3) if P is $\forall x \cdot q(x,y) \wedge \forall y \cdot q(x,y)$, then $P[a+b/y]$ is $\forall x \cdot q(x, a+b) \wedge \forall y\, q(x,y)$
4) if P is $\forall x \exists y \cdot q(x,y)$ then $P[f(y)/x]$ is $\forall x \cdot \exists y \cdot q(x,y)$.

- Free var: t is free for z in P
  → in P, no free z's occur in scope of $\forall w$ or $\exists w$ for any free w in t.

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1) If P is $\forall y \cdot a(x) \wedge b(y)$ then $P[g(f(y))/x]$ not allowed
2) If P is $(\forall y \cdot \exists x \cdot q(x,y,z)) \wedge r(y)$ then $P[x+1/y]$ is
$(\forall y \cdot \exists x \cdot q(x,y,z)) \wedge r(x+1)$ and $P[x+1/z]$ is
$(\forall y \cdot \exists x \cdot q(x,y,z)) \wedge r(y)$
- No free vars should be "captured"

$$\frac{P[t/x]}{\exists x \cdot P} \text{ exists\_i}$$ } no bound vars.

Example:
1) $p(c)$ premise
2) $\exists x \cdot p(x)$ by exists_i on 1
3) $p(g(c,y))$ premise
4) $\exists x \cdot p(x)$ by exists_i on 3
5) $p(g(c,y))$ premise
6) $\exists x \cdot p(g(c,x))$ exists_i on 5

$\forall x \cdot a(x) \wedge b(x) \vdash \exists x \cdot a(x)$
1) $\forall x \cdot a(x) \wedge b(x)$ premise
2) $a(c) \wedge b(c)$ forall_e
3) $a(c)$ and_e
4) $\exists x \cdot a(x)$ exists_i

$\forall y \cdot p(g(y), x) \wedge q(y)$
X $\exists z \cdot \forall y \cdot p(z, x) \wedge q(y)$!
↳ can't, g depends on y.

$p(c) \wedge q(x)$
X $\exists x \cdot p(x) \wedge q(x)$!
can't, x becomes bound to $\exists$.

Genuine/Unknown Vars:
Gen: free, any value. ($\forall = T$) x_g
Unknown: free, specific value ($\exists = T$) x_u

Forall_i:

for every x_g {

⋮

$$\frac{P[x_g/x]}{\forall x.P} \text{ forall\_i}$$

}

Oct 11, 2018

for every x_g {

⋮

$$\frac{P(x_g)}{\forall x.P(x)} \text{ forall\_i}$$

}

for every x_g {

⋮

$$\frac{P(f(x_g))}{\forall x.P(f(x))} \text{ by forall\_i}$$

}

Example:

1) $\forall x.p(x) \Rightarrow q(x), \forall x.p(x) \vdash \forall x.q(x)$

1) $\forall x.p(x) \Rightarrow q(x)$ premise
2) $\forall x.p(x)$ premise
3) for every x_g {
   4) $p(x_g) \Rightarrow q(x_g)$ forall_e
   5) $p(x_g)$ forall_e
   6) $q(x_g)$ imp_e
   }
7) $\forall x.q(x)$ by forall_i

2) $\vdash (\forall x.\forall y.p(x,y)) \Rightarrow (\forall y.\forall x.p(x,y))$

1) assume $(\forall x.\forall y.p(x,y))$ {
   2) for every y_g {
      3) for every x_g {
         4) $\forall y.p(x_g, y)$ forall_e on 1
         5) $p(x_g, y_g)$ forall_e on 4
         }
      6) $\forall x.p(x, y_g)$ by forall_i 3-5
      }
   7) $\forall y.\forall x.p(x,y)$ forall_i 2-6
   }
8) done.

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3) $\forall x \cdot p(x,x) \vdash \forall x \cdot \exists y \cdot p(x,y)$

1) $\forall x \cdot p(x,x)$ premise
2) for every x_g {
3) $p(x_g, x_g)$ forall_e on 1
4) $\exists y \cdot p(x_g, y)$ exists_i
}
5) $\forall x \cdot \exists y \cdot p(x,y)$.

Exists Elim:

$\exists x \cdot P$

for some x_u $P[x_u/x]${

$\vdots$

Q

}

Q ← MUST be independent of x_u — exists_e

Example:

1) $\forall x. p(x) \Rightarrow q(x), \exists x \cdot p(x) \vdash \exists x \cdot q(x)$

1) $\forall x \cdot p(x) \Rightarrow q(x)$ premise
2) $\exists x \cdot p(x)$ premise
3) for some x_u $p(x_u)${
4) $p(x_u) \Rightarrow q(x_u)$ forall_e on 1
5) $q(x_u)$ imp_e
6) $\exists x \cdot q(x)$ exists_i
}
7) $\exists x \cdot q(x)$ exists_e on 2, 3-6

2) $\exists x \cdot p(x) \vdash \neg(\forall x \cdot \neg p(x))$

1) $\exists x \cdot p(x)$
2) disprove $\forall x \cdot \neg p(x)${
3) for some x_u $p(x_u)${
4) $\neg p(x_u)$ forall_e on 2
5) false not_e on 3, 4
}
6) false by exists_e
}
7) $\neg(\forall x \cdot \neg p(x))$ by raa.

3) $\neg(\exists x \cdot p(x)) \vdash \forall x \cdot \neg p(x)$

1) $\neg(\exists x \cdot p(x))$ premise
2) for every x_g {
   3) disprove $p(x_g)$ {
      4) $\exists x \cdot p(x)$ exists_i
      5) false not_e
   }
   6) $\neg p(x_g)$
}
7) $\forall x \neg p(x)$ forall_i

4) $\neg(\forall x \cdot p(x)) \vdash \exists x \cdot \neg p(x)$

1) $\neg(\forall x \cdot p(x))$ premise
2) disprove $\neg(\exists x \cdot \neg p(x))$ { ← common patterns
   3) for every x_g {
      4) disprove $\neg p(x_g)$ {
         5) $\exists x \cdot \neg p(x)$ exists_i
         6) false
      }
      7) $p(x_g)$ raa
   }
   8) $\forall x \cdot p(x)$ forall_i
   9) false
}
10) $\exists x \cdot \neg p(x)$

exists_e usually before forall_e.

Prove all forms of DMG as practice.

5) $\forall x \cdot \exists y \cdot \neg w(x,y) \vdash \neg(\exists x \cdot \forall y \cdot w(x,y))$

1) premise
2) disprove $\exists x \cdot \forall y \cdot w(x,y)$ {
   3) for some x_u $\forall y \cdot w(x_u,y)$ {
      4) $\exists y \cdot \neg w(x_u, y)$ forall_e
      5) for some y_u $\neg w(x_u,y_u)$ {
         6) $w(x_u, y_u)$ forall_e on 3
         7) false
      }
      8) false
   }
   9) false
}

Predicate ST:

$\forall x \cdot P(x)$
| forall_nb
$P(t)$

$\exists x \cdot P(x)$
| exists_nb
$P(y_u)$ ← unknown, non-free variable.

Example:

1) $\exists x \cdot p(x) \vdash \neg(\forall x \cdot \neg p(x))$

   1) $\exists x \cdot p(x)$
   2) $\neg\neg(\forall x \cdot \neg p(x))$
   not-not_nb 2
   3) $\forall x \cdot \neg p(x)$
   exists_nb 1
   4) $p(x_u)$
   forall_nb on 3
   5) $\neg p(x_u)$
   closed.

2) $\forall x \cdot p(x, c) \vdash \exists x \cdot p(g(x), x)$

   1) $\forall x \cdot p(x, c)$
   2) $\neg(\exists x \cdot p(g(x), x))$ ← not_exists_nb
   3) $\forall x \cdot \neg p(g(x), x)$
   by forall_nb on 3
   4) $\neg p(g(c), c)$
   by forall_nb on 1
   5) $p(g(c), c)$
   6) closed.

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