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MATH 135 Introduction — Sep 8/2017

Read Chapter 1 on learn, Assignment 0

Chapter 2 Notes:

Definitions:

- Statement: T/F sentence (one or the other) - completely
- Proposition: Statement that needs to be proven
- Theorem: Strong proposition (big) (pythag)
- Lemma: Helper proposition (small) (proven during proof)
- Corollary: Prop. follows prop (theorem) { We know theorem, so ___ . } (need little proof)
- Axiom: Given truth (assumed)

ex: Rem. Theorem → Factor theorem is a corollary

Symbols:

- $\neg A$: Not A (like !A in CS)
- $A \wedge B$: A and B
- $A \vee B$: A or B
- $\equiv$: Logically equivalent (Truth table Left = Right)
- De Morgan's Law:
  - $\neg(A \vee B) \equiv \neg A \wedge \neg B$
  - $\neg(A \wedge B) \equiv \neg A \vee \neg B$
- $A \Rightarrow B$: A (hypothesis) implies B (conclusion) a proposition

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

only situation where false

Example:

1) Negate A w/o negation symbols : $A: (5=2) \vee (3<6<7)$

$\neg A: \neg(5=2) \wedge \neg(3<6<7)$

$(5\neq 2) \wedge \neg[(3<6) \wedge (6<7)]$

$(5 \neq 2) \wedge [(6 \leq 3) \vee (6 \geq 7)]$

Chapter 3 Notes:

Laws:

$A \vee (B \vee C) \equiv (A \vee B) \vee C$ (same for $\wedge$)

$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$

$A \Rightarrow B \equiv (\neg A) \vee B$ } $\neg(A \Rightarrow B) \equiv A \wedge (\neg B)$

Chapter 4 Notes:

* If proving $ax^2+bx+c = 0$, DON'T start w/ eqn
  - ↳ Treat ax^2+bx+c as a standalone expession and try to get 0 (RS).

Proving Implications:

↳ Direct Proofs: Start w/ "Assume A is true", end w/ "∴ B must be true".

Lecture 2

Sep 11/2017

Example:

$x^4 + x^2y + y^2 \geq 5x^2y - 3y^2$ for $x, y \in \mathbb{R}$

1) Discovery (can do anything)

$x^4 - 4x^2y + 4y^2 \geq 0$

$(x^2 - 2y)^2 \geq 0$

2) Prove:

Let $x, y \in \mathbb{R}$

$\therefore (x^2 - 2y)^2 \geq 0$

Expand to get original eqn. QED

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MATH 135 - Chapter 5 & 6 Sep 13/2017

Proving Implications:

Can prove that the counter example (negation of $A \Rightarrow B$) is true $\neg(A \Rightarrow B) \equiv A \wedge (\neg B)$

Definitions:

Statements that are true (no proof required)

↳ ex: even # = $2m$, $m \in \mathbb{Z}$

Divisibility:

$m | n$ $n = km$, $k \in \mathbb{Z}$ } $m/0$ works.

↳ n is a multiple of m

Example:

1) If n is an integer and $3|n$ then $3|n^2$

Let $n \in \mathbb{Z}$ such that $3|n$

$\therefore$, $n = 3k$ and $n^2 = 9k^2$ ✓ need this!

$n^2 = 3(3k^2)$, since $k \in \mathbb{Z}$, $3k^2$ is $\mathbb{Z}$

$\therefore n^2 = 3k$ $3|n^2$ (QED)

Bounds by Divisibility (BBD): $\exists$: such that

Let $a, b \in \mathbb{Z}$ such that $a|b$ and $b \neq 0$. Prove $|a| \leq |b|$

$\therefore b = ka$ when $k \in \mathbb{Z}$

$\because$ (since) $b \neq$, $k \neq 0$

But $k \in \mathbb{Z}$, so $|k| \geq 1$, so

can # these for ref.
$$\left\{\begin{aligned} |b| &= |ka| \\ &= |k||a| \\ &\geq |a| \end{aligned}\right. \quad \text{(QED)}$$

- Since $a|b$ and $b \neq 0$, $|a| \leq |b|$

↳ Then if $|a| > |b|$, $a \nmid b$ or $b = 0$

↳ $|a| \leq |b|$ doesn't tell us anything (can't reverse implications)

Transitivity of Divisibility:

Let $a, b, c \in \mathbb{Z}$. If $a | b$ and $b | c$, then $a | c$.

Let $a, b, c \in \mathbb{Z}$ where $a | b$ and $b | c$:

$\therefore$ there exists $k \in \mathbb{Z} \ni b = ka$ and

" " $m \in \mathbb{Z} \ni c = mb$

$\therefore c = m(ka)$

$= (mk)a$

Since $m, k \in \mathbb{Z}$, $a | c$ QED

$\therefore mk \in \mathbb{Z}$

Divisibility of Integer Combinations: Sep 15, 2017

Let $a, b, c \in \mathbb{Z}$. If $a | b$ and $a | c$, for any $x, y \in \mathbb{Z}$, $a | (bx + cy)$

Let $a, b, c, x, y \in \mathbb{Z}$ where $a | b$ and $a | c$.

$\therefore b = ka$ for some $\mathbb{Z} \ni k$.

$c = na$ for $n \in \mathbb{Z}$.

$\therefore bx + cy$

$= kax + nay$

$= a(kx + ny)$ since $k, x, n, y \in \mathbb{Z}$, $kx + ny = m$ where $m \in \mathbb{Z}$

$= ma$. QED

more *

Sets:

Collection of unique objects (elements)

- No order
- Don't need to have anything in common
- $x \in A$
- $A = \{-2, 5, \{1, 2, 3\}, 9\}$ 4 elements } a trick question
  - $1, 2, 3 \notin A$!
- $\mathbb{N} : \{1 \ldots \infty\}$ Natural #s
- $\mathbb{Z} = \{\pm x : x \in W\}$
- $W : \{0 \ldots \infty\}$ whole #s $\{0\} \cup \mathbb{N}$
- $\mathbb{Q}$
- $\{\}$: Null set $\varnothing$ Empty Set
- $\mathbb{Q} : \{\frac{a}{b}, b \neq 0 : a, b \in \mathbb{Z}\}$ Rationals

Cardinality:
- # of elements.
  - $S = \{1, 2, 3, 4\}$ $\quad |S| = 4$

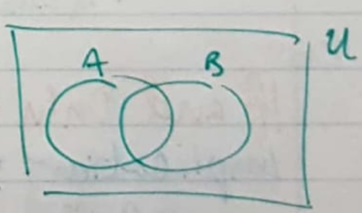
The Universe of Discourse:
$\mathcal{U}$: Contains all items we may need.
- ↳ Eggs? $\mathbb{N}$
- ↳ Time? $\mathbb{R}$
- → Divisibility? $\mathbb{Z}$

} Variable type, kinda.

Set Builder Notation:
$\{x \in \mathcal{U} : P(x)\}$
- ↳ subset of $\mathcal{U}$, elements x satisfy $P(x)$

Example:
1) $\{x \in \mathbb{Z} : (33 | x)\} = \{x \in \mathbb{Z} : (11 | x) \wedge (3 | x)\}$
2) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 144\}$
   - ↳ Set of all points $\bar{w}$ distance 12 from $(0, 0)$
   - ↳ OR: $\{(x, y) : (x \in \mathbb{R}, y \in \mathbb{R}) \wedge (x^2 + y^2 = 144)\}$

Operations:
Unions:
- $A \cup B = \{x \in \mathcal{U} : (x \in A) \vee (x \in B)\}$



Intersections:
- $A \cap B = \{x \in \mathcal{U} : (x \in A) \wedge (x \in B)\}$

Set Difference:
- $A - B = \{x \in \mathcal{U} : (x \in A) \wedge (x \notin B)\}$ A $\bar{w}$/o B

Complement:
- $\overline{A} = \{x : (x \in \mathcal{U}) \wedge (x \notin A)\}$ All $\mathcal{U}$ w/o A $= \mathcal{U} - A$

Cartesian Product:
- $A \times B = \{(x, y) : (x \in A) \wedge (y \in B)\}$ $\quad |A \times B| = |A||B|$ $\quad \mathbb{R} \times \mathbb{R} : \mathbb{R}^2$!

Sep 18, 2017

Subsets:

A is subset of B if $x \in A \Rightarrow x \in B$

↳ B is superset of A

↳ "Proper subset" when elements in B not in A

↳ $A \subseteq B$ subset $A \supseteq B$ superset.

$A \subsetneq B$ proper subset $A \nsubseteq B$ not subset

Example:

1) Prove that $\{n \in \mathbb{N} : 4 \mid (n-3)\} \subseteq \{2k+1 : k \in \mathbb{Z}\}$

Discover:

$4 \mid (n-3)$ then $n-3 = 4a$. If n is odd $= 2k+1$. ← need to get here.

$n = 4a+3$

$n = 4a+2+1$

$n = 2(2a+1)+1$ Let $k \in \mathbb{Z}$

$= 2k+1$.

$\therefore$, if $x \in \{n \in \mathbb{N} : 4 \mid (n-3)\}$ then $x \in \{2k+1 : k \in \mathbb{Z}\}$

$\therefore A \subseteq B$, Q.E.D.

2) Prove that $\{x \in \mathbb{R} : ax^2+bx+c=0\} = \{\frac{-b \pm \sqrt{b^2-4ac}}{2a}\}$

Converse of an Implication:

$B \Rightarrow A$ is converse of $A \Rightarrow B$

↳ $A \Rightarrow B$ doesn't mean $B \Rightarrow A$.

If and Only If:

Implication & Converse are true.

↳ A iff B $= A \longleftrightarrow B$

and { A if B: $B \Rightarrow A$; A only if B: $A \Rightarrow B$ }

if AND only if

A	B	$A \Leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

Proving "iffs":

To prove $A \Leftrightarrow B$, must prove $A \Rightarrow B$ AND $B \Rightarrow A$

Example:

1) Suppose $x, y \geq 0$. Then $x = y$ iff $\frac{x+y}{2} = \sqrt{xy}$.

$\underbrace{x = y}_{A}$ $\quad \underbrace{\frac{x+y}{2} = \sqrt{xy}}_{B}$

$\Rightarrow$ $A \Rightarrow B$ Let $x, y \geq 0$ and $x = y$.

$\therefore \frac{x+y}{2} = \frac{x+x}{2} = x = \sqrt{x^2}$ $(\because x \geq 0)$ NEED THIS ☆

$= \sqrt{xy}$

$\Leftarrow$ $B \Rightarrow A$ Let $x, y \geq 0$ and $\frac{x+y}{2} = \sqrt{xy}$

$x+y = 2\sqrt{xy}$

CAN $\sqrt{\text{both}}$ sides $\rightarrow$ $(x+y)^2 = 4xy$

$(x-y)^2 = 0$

$x = y$.

QED. $\therefore$, $x = y$ iff $\frac{x+y}{2} = \sqrt{xy}$

Set Equality:

$A = B$ if $(A \subseteq B) \wedge (B \subseteq A)$

Example:

Nah.

Quantifiers:

Sep 19, 2017

"Some", "Many", "All"

↳ How many elements in domain satisfy property.

↳ There exists some integer n such that $n^3 + 3 = 30$.

Universal Quantifier:

$\forall$ = "for all" $\rightarrow$ $\forall x \in \mathbb{R}, x^2 \geq 0$

↳ Can talk about null set and be true.

Existential Quantifier:

$\exists$ · At least one is true → $\exists x \in S \ni P(x)$ "there exists" (such that)
↳ Gaurantees that there are elements in S

Quantifier - Variable - Domain (set) - Open sentence

Proving Quantified Statements:

Select Method:

Universal

1) Prove that for every $x \in \mathbb{R}$, $x^2+3x+5>0$
↳ Take a "representative" $x \in S$ to show $p(x)$ is true.
↳ Use properties of S (in this case, $\mathbb{R}$)

Proof: Let $x \in \mathbb{R}$.

$$x^2+3x+5 > x^2+3x+\frac{9}{4}$$
$$= (x+\tfrac{3}{2})^2$$
$$\geq 0$$

$\therefore x^2+3x+5 > 0$ for all $x \in \mathbb{R}$

original exp. is >, so ≥0 means >0.

Construct Method:

Existential

2) There exists a real number $x \in \mathbb{R}$ such that $x^2+3x=5$
↳ Provide an explicit $x \in S$ that makes $P(x)$ true.

Proof: Consider $x = \frac{-3+\sqrt{29}}{2}$.

Then $x^2+3x = \left(\frac{-3+\sqrt{29}}{2}\right)^2 + 3\left(\frac{-3+\sqrt{29}}{2}\right)$
$= 5$ QED

Quantifiers in the Hypothesis:

If $\forall x \in \mathbb{N}, n \mid x$, then $n=1$.

Let $a,b,c \in \mathbb{Z}$. If $\forall x \in \mathbb{Z}, a \mid (bx+c)$, then $a \mid (b+c)$
↳ Substitution method: Assume hypo true, sub any value from domain
↳ Let $a,b,c \in \mathbb{Z}$, and $\forall x \in \mathbb{Z}$, $a \mid (bx+c)$.
Then when $x=1$, $a \mid (b+c)$. QED

This is valid for $\forall$!!!!!

Quantifiers in the Hypothesis:

Let $p \in \mathbb{R}$, $k \in \mathbb{Z}$. If there exists a non-zero $\mathbb{Z}$ q such that $\frac{p}{q} = k$, then p must be $\in \mathbb{Z}$.

If $14 | n$ then $7 | n$.

↳ "There exists" integer k such that ....

Vacuousness

Sep 20, 2017

"There exists a unicorn that eats grass"

Let S be the set of all unicorns

$P(x)$ be "x eats grass"

$\exists x \in S, P(x)$ false, because $S = \emptyset$ Vacuously Fals

"My pug moose has won every math contest he's ever written."

Let S be set of Moose math contests

$P(x)$ be "Moose won x"

$\forall x \in S, P(x)$ true, because $S = \emptyset$ Vacuously True

Vacuously True:

$\forall x \in \emptyset, P(x)$

Vacuously False:

$\exists x \in \emptyset, P(x)$

Negating Quantifiers:

"All real numbers have a positive int. √"

↳ "There exists a real number which doesn't have +int√"

$\neg [\forall x \in S, P(x)] \equiv [\exists x \in S, \neg P(x)]$

$\neg [\exists x \in S, P(x)] \equiv [\forall x \in S, \neg P(x)]$

Proving/Disproving Quantifiers:

1) Single counter example disproves $\forall x \in S, P(x)$
2) Single example doesn't prove $\forall x \in S, P(x)$
3) Single example does prove $\exists x \in S, P(x)$
4) Proving $\exists x \in S, P(x)$ is false? Hard to do.

Nested Quantifiers:

$\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x^3 - y^3 = 1$ is false.

Example:

1) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^3 - y^3 = 1$

Proof: Let $x \in \mathbb{R}$. let $y^3 = x^3 - 1$.

$$x^3 - y^3 = x^3 - (x^3 - 1)$$
$$= 1. \qquad \text{QED}$$

Language Practice

Sep 22, 2017

No multiple of 15 plus a multiple of 6 = 100.

↳ $\forall x, y \in \mathbb{Z}, 15x + 6y \neq 100$

whenever 3 divides both the sum + difference of 2 integers, it also divides each int.

↳ $\forall a, b \in \mathbb{Z}, [(3 | a+b) \wedge (3 | a-b)] \Rightarrow [(3|a) \wedge (3|b)]$

$\forall m \in \mathbb{Z}, ((\exists k \in \mathbb{Z}, m = 2k) \Rightarrow (\exists t \in \mathbb{Z}, 7m^2 + 4 = 2t))$

↳ If m is even then $7m^2+4$ is even (no even $\mathbb{R}$ that's not $\mathbb{Z}$)

$n \in \mathbb{Z} \Rightarrow (\exists m \in \mathbb{Z}, m > n)$

↳ There is no largest integer.

$\forall k, m \in \mathbb{Z}, [(m|k) \Rightarrow ((m^2 = 1) \vee (m^2 = k^2))] \Rightarrow [(|k| \neq 1) \Rightarrow (\sqrt{|k|} \notin \mathbb{N})]$

↳ No prime numbers are perfect □.

Contrapositives:

"If a student earns a grade of 50%+, then the student passed"

"If a student did not pass, they earned a grade below 50%.

$\neg B \Rightarrow \neg A$ is contrapositive of $A \Rightarrow B$

$\equiv$

If $7 \nmid n$ then $14 \nmid n \equiv 14 | n \Rightarrow 7 | n$

If $x^3 + 7x^2 < 9$ then $x < 1.1 \equiv x^3 + 7x^2 \geq 9 \Leftarrow x \geq 1.1$

Implications Cont...

Sep 25, 2017

$(A \wedge B) \Rightarrow C$

$A \Rightarrow (B \wedge C)$ → prove $A \Rightarrow B$ and $A \Rightarrow C$

$(A \vee B) \Rightarrow C$ → prove $A \Rightarrow C$ and $B \Rightarrow C$

$A \Rightarrow (B \vee C)$ "elimination" → prove $(A \wedge \neg B) \Rightarrow C$

Proof by Contradiction:

Prove that the negation of a statement is absurd $(A \wedge \neg A)$ if assumed to be true.

Example:

1) There is no largest integer.

   Assume there is. Add 1 to it. Now the largest integer is both largest and not largest.

When to use Contradiction:

1) Negation of $A \Rightarrow B$ is $A \wedge \neg B$.
   - → If B already has a negation, this is easier.
   - → or if you just like the negation better.

1) Contrapositive is similar.
   - → $\neg B \Rightarrow \neg A$, but contradiction is $A \wedge \neg B$ which leads to $A \wedge \neg B$ which is like $\neg B \Rightarrow \neg A$.

Example:

1) Prove there are ∞ primes.

   Let $n \in \mathbb{N}$. Can be expressed as product of primes "Prime Factorization"

   Assume finite primes. Let $P = \{p_1, p_2 \dots p_k\}$ be this set.

   Let $N = p_1 p_2 p_3 \dots p_k + 1$. We see for any $p \in P$, $p \nmid N$ (breaks PF).

   So N is both $p \mid N$ and $p \nmid N$. Absurd.

   can also say this is also prime.

2) Prove $\sqrt{2}$ is irrational

Assume $\sqrt{2}$ is rational $= \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$.

A and B have no common divisors (lowest fraction)

$b\sqrt{2} = a$

$2b^2 = a^2$ $\quad$ a^2 is even. $\therefore$ a is even.

So $a = 2k$ and $2b^2 = 4k^2$

$b^2 = 2k^2$ $\quad$ Absurd.

$\therefore$ b is even.

3) Prove if $a, b \in \mathbb{Z} \mid a \geq 2$, then $a \nmid b$ or $a \nmid b+1$.

Assume $\underbrace{a, b \in \mathbb{Z}, a \geq 2}_{A}$ $\underbrace{\text{and}}_{\wedge}$ $\underbrace{a \mid b \text{ and } a \mid b+1}_{\neg B}$.

$\therefore \exists k, m \in \mathbb{Z} \mid b = ka \quad b+1 = ma$

$a(m-k) = 1$

$\therefore a \mid 1$

$\underbrace{\therefore a = \pm 1}_{\neg B \text{ led us here.}}$ $\quad$ $\underbrace{\text{but } a \geq 2}_{A}$

Uniqueness:

For each $x \in \mathbb{R}$, $\exists$ a unique $y \in \mathbb{R} \mid (x+1)^2 - x^2 = 2y - 1$.

↳ Only 1 solution exists.

↳ Prove y exists and no other y exists.

1) Isolate y to prove existence.

2) Uniqueness: Assume y is not unique. Let $y_1, y_2 \in \mathbb{R}$, $y_1 \neq y_2$.

① $(x+1)^2 - x^2 = 2y_1 - 1$ and $(x+1)^2 - x^2 = 2y_2 - 1$ ②

$\therefore 2y_1 - 1 = 2y_2 - 1$

$y_1 = y_2$ but $y_1 \neq y_2$, absurd.

$\therefore$ Unique y. QED.

$P(x) \wedge P(y) \Rightarrow x = y$ states uniqueness.

Division Algorithm:

If $a, b \in \mathbb{Z}$, $b > 0 \Rightarrow \exists$ unique q and r where

$a = qb + r \quad 0 \leq r < b$.

dividend — a; divisor — b

Assume $a, b \in \mathbb{Z} > 0$ and q, r not unique.

$\therefore \exists q_1, q_2, r_1, r_2 \in \mathbb{Z} \mid q_1 \neq q_2, r_1 \neq r_2$

$a = q_1 b + r_1 \quad a = q_2 b + r_2 \quad 0 \leq r_1, r_2 < b$

$q_1 b + r_1 = q_2 b + r_2$

$b(q_2 - q_1) = r_1 - r_2 \quad b \mid (r_1 - r_2)$

W/o loss of generality, assume $r_1 > r_2$. b/c one must be greater.

$\therefore r_1 - r_2 > 0$ and $r_1 - r_2 < r_1$

Since $b \mid (r_1 - r_2)$, BBD: $b \leq r_1 - r_2 < r_1$

but $b > r_1 \therefore$ QED.

Injective Functions:

$f: S \to T$ is one to one iff $\forall x_1 \in S\ x_2 \in S$, $(f(x_1) = f(x_2)) \Rightarrow x_1 = x_2$

↳ unique x for every $f(x)$.

Example:

1) $f: [1, \infty) \to (0, 0.5]$ $f(x) = \frac{x}{1+x^2}$ is 1-1.

Surjective Functions: "onto"

$f: S \to T$ iff $\forall y \in T, \exists x \in S \mid f(x) = y$.

↳ Codomain accessible through f.

Example:

1) $f: \mathbb{R} \to (-\infty, 1)$ $f(x) = 1 - e^{-x}$ is onto.

Sigma Notation

Sep 27, 2017

$\sum_{i=m}^{n} x_i$ i is the index variable } Always +1.
m is initial. n is ending.

Pi Notation:

$\prod_{i=m}^{n} x_i$ Exact same as Σ, but with multi.

Example:

1) $\sum_{i=1}^{5} 3i^2 = \underbrace{3(1)^2 + 3(2)^2 + 3(3)^2 + 3(4)^2 + 3(5)^2}_{\Pi \text{ is same but } \times.}$

↳ To increment by 2: $\sum_{i=1}^{5} 3(2i)^2$

Special Cases:

1) $\sum_{i=m}^{m} x_m = x_m$

2) $\sum_{j=5}^{2} x_j = 0$ All sums begin w/ 0

3) $\prod_{i=m}^{m} x_m = x_m$

4) $\prod_{j=5}^{2} x_j = 1$. All multi begin w/ 1.

Mathematical Induction:

Let $P(n)$ be a proposition depending on $n \in \mathbb{N}$.

→ If $P(1)$ is true, and — First dominoe falls

→ $P(k) \Rightarrow P(k+1)$ — Every dominoe makes next fall

→ Then $P(n)$ is true for all $n \in \mathbb{N}$. All fall!

POMI:

1) Base Case:

→ Prove $P(1)$ (doesn't have to be 1).

2) Inductive Hypothesis:

→ Assume $P(k)$ for some $k \in \mathbb{N}$ (should write out $P(k)$)

3) Inductive Conclusion:

→ Prove $P(k) \Rightarrow P(k+1)$ (should write out $P(k+1)$).

Example:

1) $$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}, \underbrace{\forall n, \in \mathbb{N}}$$

Let $P(n)$ be that ↑. $\boxed{\text{WITHOUT } \forall n \in \mathbb{N} \text{ (MISTAKE!)}}$!

Base Case: Show $P(1)$ is true.

$$\sum_{i=1}^{1} i(i+1) = i(i+1) = 1 \times 2 = \frac{1 \times (1+1) \times (1+2)}{3} \quad \therefore P(1) \text{ is true.}$$

Inductive Hypothesis: Assume $P(k)$ is true.

$$\sum_{i=1}^{k} i(i+1) = \frac{k(k+1)(k+2)}{3} \text{ for some } k \in \mathbb{N}$$

Inductive Conclusion: Show $P(k) \Rightarrow P(k+1)$

RTP: $$\sum_{i=1}^{k+1} i(i+1) = \frac{(k+1)((k+1)+1)((k+1)+2)}{3}$$

GOAL → $$= \frac{(k+1)(k+2)(k+3)}{3}$$

Well, $\sum_{i=1}^{k+1} i(i+1) = \sum_{i=1}^{k} i(i+1) + (k+1)[(k+1)+1]$

$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$ (by ind. hypo.)

* need ↑

$= \frac{(k+1)(k+2)(k+3)}{3}$ as desired. QED.

$\therefore$ by POMI, ~~~ $\forall n \in \mathbb{N}$ QED.

2) $n! > 2^n, \forall n \in \mathbb{Z}, n \geq 4$

Let $P(n)$ be the statement $n! > 2^n$.

Base Case: Show $P(4)$ is true.

$4! = 24.$
$2^4 = 16.$ } $24 > 16$. $\therefore P(4)$ is true.

Inductive Hypothesis: ~~Show~~ Assume $P(k)$ is true.

Assume $k! > 2^k$ for some $k \geq 4$.

Inductive Conclusion: Prove $P(k+1) \Leftarrow P(k)$

Prove $(k+1)! > 2^{k+1}$

$(k+1)! > 2^{k+1}$

$= k!(k+1) > 2^{k+1}$

$> (k+1)\, 2^k$ (by IH)

$\geq 5 \cdot 2^k$ since $k \geq 4$

$> 2 \cdot 2^k$

$> 2^{k+1}$

Sep 29, 2017

PomI Practice:

$6 \mid 2n^3 + 3n^2 + n$ $\quad 2n^3 + 3n^2 + n = 6k \quad n \in \mathbb{N}$ ← If $n \in \mathbb{Z}$, you need to show $P(k+1)$ and $P(k-1)$

Base case: Let $n = 1$

$6 = 6k \quad k = 1$

IH: Assume $P(x)$ is true.

$\therefore \exists k \in \mathbb{Z} \mid 2x^3 + 3x^2 + x = 6k$ for some k

IC: Prove $P(x+1)$

$2(x+1)^3 + 3(x+1)^2 + x + 1$

$= 2(x^2+2x+1)(x+1) + 3(x^2+2x+1) + x + 1$

$= 2[x^3 + 2x^2 + x + x^2 + 2x + 1] + 3x^2 + 6x + 3 + x + 1$

$= 2(x^3 + 3x^2 + 3x + 1) + 3x^2 + 7x + 4$

$= 2x^3 + 6x^2 + 6x + 2 + 3x^2 + 7x + 4$

$= (2x^3 + 3x^2 + x) + 6x^2 + 5x + 7x + 6$

$= 6k + 6x^2 + 12x + 6$

$= 6(k + x^2 + 2x + 1) = 6p \quad \therefore 6 \mid P(x+1)$

The sequence $\{x_n\}$, $x_1 = 4$, $x_2 = 68$, $x_m = 2x_{m-1} + 15x_{m-2}$ for $m \geq 3$, $x_n = 2(-3)^n + 10(5)^{n-1}$ for $n \geq 1$ need two base cases 2 trues for another

Let $P(n)$ be "$x_n = 2(-3)^n + 10(5)^{n-1}$" where x_m is ...

needed → $x_1 = 4$, $x_2 = 68$, $x_m = 2x_{m-1} + 15x_{m-2}$ for $m \geq 3$.

Base case: Prove $P(1)$ and $P(2)$

$2(-3)^1 + 10(5)^0 = 4 = x_1 \quad \therefore P(1)$ is true.

$2(-3)^2 + 10(5)^1 = 68 = x_2 \quad \therefore P(2)$ is true.

IH: Assume $P(k)$ and $P(k-1)$ are true.

$$\left.\begin{aligned} x_k &= 2(-3)^k + 10(5)^{k-1} \\ \text{and } x_{k-1} &= 2(-3)^{k-1} + 10(5)^{k-2} \end{aligned}\right\} k \in \mathbb{N} \geq 3.$$

IC: Prove $(P(k) \wedge P(k-1) \Rightarrow P(k+1))$ Want: $x_{k+1} = 2(-3)^{k+1} + 10(5)^k$

$x_{k+1} = 2x_k + 15x_{k-1}$

$= 2[2(-3)^k + 10(5)^{k-1}] + 15[2(-3)^{k-1} + 10(5)^{k-2}]$ by IH

$= 4(-3)^k + 20(5)^{k-1} + 30(-3)^{k-1} + 150(5)^{k-2}$

$= 4(-3)^k + 4(5)^k - 10(-3)^k + (6)(25)(5^{k-2})$

$= 4(-3)^k + 4(5)^k - 10(-3)^k + 6(5)^k$
$= -6(-3)^k + 10(5)^k$
$= 2(-3^{k+1}) + 10(5^k)$ QED.

↑ example where $b=2$.

Principle of Strong Induction:

Let $P(n)$ be a statement $n \in \mathbb{N}$.

1) $P(1), P(2) \dots P(b)$ true for some $b \in \mathbb{Z} > 0$
2) $P(1), P(2) \dots P(k) \Rightarrow P(k+1)$ for all $k \in \mathbb{N}$

Then $P(n)$ is true $\forall n \in \mathbb{N}$. b is "suitably long"

Example:

1) Suppose $x_1 = 3, x_2 = 5$, $x_n = 3x_{n-1} + 2x_{n-2}$ for $n \geq 3$. Then $x_n < 4^n \ \forall n \in \mathbb{N}$.

Base Case: $P(1)$ and $P(2)$ are true.

$3 = x_1 < 4^1$ $5 = x_2 < 4^2$

$(x_k < 4^k) \wedge (x_{k-1} < 4^{k-1})$

$x_{k+1} = 3x_k + 2x_{k-1}$ (hopefully $< 4^{k+1}$)
$< 3(4^k) + 2(4^{k-1})$
$< 3(4^k) + 4(4^{k-1})$
$= 3(4^k) + 4^k$
$= 4^{k+1}$

Fibonacci Sequence

Oct 2, 2017

$f_1 = 1, f_2 = 1$ $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$

$\sum_{r=1}^{n} f_r^2 = f_n f_{n+1}$ for all $n \in \mathbb{N}$

Fibonacci Proofs:

1) $\sum_{r=1}^{n} f_r^2 = f_n f_{n+1} \ \forall n \in \mathbb{N}$. weak

Base Case: $\sum_{r=1}^{1} f_r^2 = 1^2 = 1 \times 1 = f_1 \times f_2$

Inductive Hypothesis: $\sum_{r=1}^{k} f_r^2 = f_k f_{k+1}$

Inductive Conclusion: $\sum_{r=1}^{k+1} f_r^2 = f_{k+1} f_{k+2}$

$$= \underbrace{\left(\sum_{r=1}^{k} f_r^2\right)}_{IH} + f_{k+1}^2$$
$$= f_k f_{k+1} + f_{k+1}^2$$
$$= f_{k+1}(f_k + f_{k+1})$$
$$= f_{k+1} f_{k+2}$$

2) $f_n < \left(\frac{7}{4}\right)^n$ for all $n \in \mathbb{N}$. Strong

Base case: $f_1 = 1 < \left(\frac{7}{4}\right)^1$ ✓ $f_2 = 1 < \left(\frac{7}{4}\right)^2$ ✓

shows base case ... k Now show k+1! → Inductive Hypothesis: $\cancel{\left(f_k < \left(\frac{7}{4}\right)^k\right) \wedge \left(f_{k-1} < \left(\frac{7}{4}\right)^{k-1}\right)}$

$f_1 < \left(\frac{7}{4}\right)$, $f_2 < \left(\frac{7}{4}\right)^2 \ldots f_k < \left(\frac{7}{4}\right)^k$ *need this way

→ "Assume you can go from 1 ... k. Now show how you can get the next one.

Inductive conclusion: $f_{k+1} < \left(\frac{7}{4}\right)^{k+1}$

$$f_{k+1} = f_k + f_{k-1}$$
$$\overset{IH}{<} \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1}$$
$$= \left(\frac{7}{4}\right)^{k-1}\left(\frac{7}{4} + 1\right)$$
$$< \left(\frac{7}{4}\right)^{k-1}\left(\frac{7}{4}\right)^2$$
$$= \left(\frac{7}{4}\right)^{k+1}$$

Discovering Truths:

Closed Form:

Oct 3, 2017

Determine a closed form for $\prod_{r=2}^{n}\left(1-\frac{1}{r^2}\right)$

$= \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)$

$= \left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)$

$= \left(\frac{n^2-1}{n^2}\right)\left(\frac{(n+1)^2-1}{(n+1)^2}\right)$

$= \left(\frac{n^2-1}{n^2}\right)\left(\frac{n^2+2n}{n^2+2n+1}\right)$

$= \frac{(r^2-1)((r+1)^2-1)((r+2)^2-1)\ldots(n^2-1)}{(r^2)(r+1)^2(r+2)^2\ldots(n^2)}$

n	$\prod$
2	$\frac{3}{4}$
3	$\frac{3}{4}\times\frac{8}{9}=\frac{2}{3}=\frac{4}{6}$
4	$\frac{2}{3}\times\frac{15}{16}=\frac{5}{8}$
5	$\frac{5}{8}\times\frac{24}{25}=\frac{3}{5}=\frac{6}{10}$

$\frac{n+1}{2n}$?

Now, proof by induction:

BC: $n=2$ $\quad \prod_{r=2}^{2}\left(1-\frac{1}{r^2}\right)=\frac{3}{4}=\frac{2+1}{2(2)}$ ✓

IH: k: $\quad \prod_{r=2}^{k}\left(1-\frac{1}{r^2}\right)=\frac{k+1}{2k}$

IC: $\prod_{r=2}^{k+1}\left(1-\frac{1}{r^2}\right)$ → want to equal $\left(\frac{k+1+1}{2(k+1)}\right)$

$\hookrightarrow = \left(\underbrace{\prod_{r=2}^{k}\left(1-\frac{1}{r^2}\right)}_{IH}\right)\times\left(1-\frac{1}{(k+1)^2}\right)$

$= \left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right)$ ← could cancel $(k+1)$ but need to state $k\neq-1$.

$= \left(\frac{k+1}{2k}\right)\left(\frac{k^2+2k}{k^2+2k+1}\right)$

= Finish w/ $\frac{k+2}{2(k+1)}$ ✓

Examples:

1) Prove every integer $n > 1$ can be written as a product of primes $(P(n))$

Strong Induction:

BC: $n=2$. 2 is prime ✓

IH: Assume $P(2) \wedge P(3) \wedge P(4) \wedge \dots \wedge P(k)$ ← not this

do this! → or, $P(n)$ true for $2 \le n \le k$ for some $k \ge 2$. *more formal

IC: $k+1$ is prime or composite.

If prime, uh... yeah.

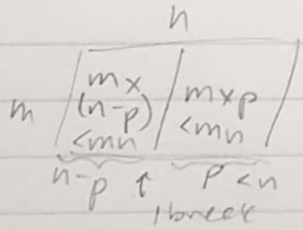
If composite, $\exists\, m, n \in \mathbb{N}$, $k+1 = mn$, $m, n < k+1$

→ Then $P(m)$, $P(n)$ true by IH. So $k+1$ can be written as product of primes (QED).

→ Didn't need 2 base cases, but had to assume string of truth.

2) Exactly $mn-1$ breaks are needed to break an $m \times n$ chocolate bar into unit squares.

Let $t = mn$, which is # of unit squares in choc. bar. $t \in \mathbb{N}$.



k+1 squares $= mn$

each smaller rectangle falls under IH of $P(1) \dots P(k)$,