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* Need textbook for code

MATH 117 - Introduction — Sep 8/2017

Pre-Calculus Review:

Definitions:

Function from set A to set B that assigns each x in A → y in B
↳ each x has only one y (vert. line test)
↳ Doesn't need to be math formula
↳ $f: A \to B$:
- A is the domain (dom(f)), B is codomain ($\mathbb{R}$)
↳ Range of f: (↑ subset of)
- Set of all outputs of f range(f)
- Domain: set of all permissable inputs

Set Notation
↳ $\mathbb{Z}$ - the integers
↳ $\mathbb{Q}$ - the rational numbers
↳ $\mathbb{R}$ - real #'s
↳ $\mathbb{C}$ - $\{a+bi \mid a,b \in \mathbb{R}, i=\sqrt{-1}\}$

Interval Notation → Refer to precalc notes

Inverse Functions

Identity Fcn: $id(x) = x$
↳ g is inv. of f if $g(f(x)) = x$ } And reversed!
↳ "undoes" action of original fcn — dom(f) = rng(f) = $[0,\infty)$
↳ Invertible (inv. fcn) if fcn is one to one of interval

One to one fcn: Pass ——— test "injective" $f(x_1) = f(x_2)$ then $x_1 = x_2$

Intersections: Overlap of sets $A \cap B$ — $[1,5) \cap (2,5] = (2,5)$

Unions: Combining sets $A \cup B$ — $[1,5) \cup (2,5] = [1,5]$
↳ $(0,1) \cup (2,3) = (0,1) \cup (2,3)$
↳ $(0,1) \cap (2,3) = \emptyset$ "null set"
↳ Properties:
↳ $A \cup \emptyset = A$ $A \cap \emptyset = \emptyset$

Subsets: $A \subseteq B$ or $A \subset B$ } A is subset of B

*For inv. fcns: $y = \sqrt{x}$ passes — test but $y = x^2$ not a fcn.
↳ $y = \sqrt{x}$ domain is restricted. dom(g) = rng(f) v.s versa
↳ x^2 is inv, but domain restricted to $[0,\infty)$

Watch

Sep 19, 2017

Composition:

- f, g where $rng(g) \leq dom(f)$
  $\hookrightarrow f \circ g(x) = f(g(x))$
- ① $dom(f \circ g) \subseteq dom(g)$
  ② $rng(f \circ g) \subseteq rng(f)$
  ③ $f \circ g \neq g \circ f$, usually.

Inverses cont...

$dom(g) = rng(f)$
$dom(f) = rng(g)$

Example:

1) $f(x) = 2x - 4 / 3x + 6$
   $f^{-1}(x) = -6y - 4 / 3y - 2$ } find $dom(f)$, $rng(f)$

   $dom(f) = \{x \in \mathbb{R} \mid x \neq -2\} = (-\infty, -2) \cup (-2, \infty)$
   $rng(f) = dom(f^{-1}) = \{x \in \mathbb{R} \mid x \neq 2/3\} = (-\infty, \frac{2}{3}) \cup (\frac{2}{3}, \infty)$

Even & Odd fcns:

Even: $f(x) = f(-x)$
Odd: $f(-x) = -f(x)$

Operations:

$E \times E = E$
$E \times O = O$
$O \times O = E$

} same as if E = +ve int, O: -ve int

Components:

$f_{even}(x) = \frac{1}{2}(f(x) + f(-x))$
$f_{odd}(x) = \frac{1}{2}(f(x) - f(-x))$

} $f(x) = f_e(x) + f_o(x)$

$\therefore$ These only exist when $dom(f)$ not sym. about origin

$(-\infty, -2) \cup (2, \infty)$ ✓
$(-\infty, -5) \cup (2, \infty)$ ✗

Hyperbolic Functions:

$f_o(x) = \frac{1}{2}(e^{x} - e^{-x}) = \sinh(x)$
$f_e(x) = \frac{1}{2}(e^{x} + e^{-x}) = \cosh(x)$

Piecewise Functions: * See 1.5 course notes

Absolute Value: $|x| = \begin{cases} x & \geq 0 \\ -x & < 0 \end{cases}$: $\sqrt{x^2} = (x^2)^{\frac{1}{2}} \neq x$ ← be careful of even denom.

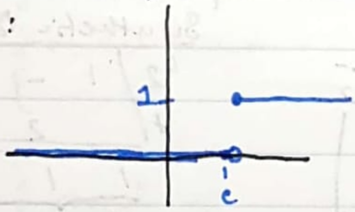
Heavyside fcn (unit step fcn):

$$H(x) = \left.\begin{cases} 0, & x<0 \\ 1, & x \geq 0 \end{cases}\right\}$$ important for multi.

Using $H(x)$:

$$f(x)H(x) = \begin{cases} 0, & x<0 \\ f(x), & x \geq 0 \end{cases}$$

$H(x-c)$:



} Kind of like an on/off switch

Example:

1) $H(x-a) - H(x-b)$ when $a<b$

$$= \begin{cases} 0, & x<a \\ 1, & a \leq x < b \\ 0, & x \geq b \end{cases}$$ a "turns on"

2) Write $f(x)$: $\begin{cases} 5-x, & x<2 \\ \sqrt{x}, & 2 \leq x < 5 \\ 13, & x \geq 5 \end{cases}$ "subtract off" the formula to the left (number line) when a new formula arises.

$$f(x) = \begin{cases} 5-x \\ [\sqrt{x} - (5-x)]H(x-2) \\ [13 - \sqrt{x}]H(x-5) \end{cases} = 5-x + [\sqrt{x} - (5-x)]H(x-2) + [13 - \sqrt{x}]H(x-5)$$

* pg 21 course notes for changing = location

Partial Fraction Decomposition:

Decomposing a rational function:

↳ denom: linear, non reducable quads, and $(\;)^2$.

Example:

1) $\dfrac{5x^2 - 5x + 4}{x^3 - x^2 - x - 2}$

Steps:

1) Factor the denominator
   - likely need root of denom.
   - For deg 3+, use the Rational Roots Theorem
   - Rational Root Theorem:
     - Any rational root of $a_n x^n + \dots a_1 x + a_0$ is the form c/d where:
       ★ $\left\{\begin{array}{l} c \text{ is a factor of } a_0 \\ d \text{ is a factor of } a_n \end{array}\right\}$ Find factor then divide
   - Long Division:

$$\begin{array}{r} x^2 + x + 1 \\ x-2 \overline{) x^3 - x^2 - x - 2} \\ -(x^3 - 2x^2) \\ \hline x^2 - x \\ -(x^2 - 2x) \\ \hline x - 2 \\ -(x-2) \\ \hline 0 \end{array}$$

Synthetic Division:

$$\begin{array}{c|cccc} 2 & 1 & -1 & -1 & -2 \\ + & & 2 & 2 & 2 \\ \hline & 1 & 1 & 1 & 0 \end{array}$$

$\underbrace{1 \quad 1 \quad 1}_{x^2 + x + 1}$

Now @ irreducible roots.

2) Split up the fcn.

$$\frac{5x^2 - 5x + 4}{x^3 - x^2 - x - 2} = \frac{Ax + B}{x^2 + x + 1} + \frac{C}{x - 2}$$

variables must be 1 less $\deg(p(x)) + 1 = \deg(q(x))$

$5x^2 - 5x + 4 = (Ax + B)(x - 2) + C(x^2 + x + 1)$

2 approaches now.

a) Expand RS and compare coefficients to LS.

b) Choose strategic values for x. ★ Much better.
   - Let $x = 2$. cancels first term.
   - Let $x = 0$. cancels A.
   - Let $x = 1$ to get A.

special case when denom is squared

★ $$\frac{7x^5 - 10x^4 - 2x^3 - 45x^2 - x - 13}{(x^2 + x + 1)^2 (x - 2)^2}$$
$$= \frac{Ax + B}{x^2 + x + 1} + \frac{Cx + D}{(x^2 + x + 1)^2} + \frac{E}{(x - 2)} + \frac{F}{(x - 2)^2}$$

Show $A = 2$ $B = -1$ $C = 4$
$D = 1$ $E = 5$ $F = -3$

Independent Reading:

Other Important Piecewise Fcns:

Signum:

$$\text{sgn}(x)\begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$$ { returns sign of #.

Ramp:

$$r(t)\begin{cases} 0 & t<0 \\ ct & t\geq 0 \end{cases}, \quad c\in\mathbb{R}$$ { Heavyside w/ const mult.

Floor:

$\lfloor x \rfloor$ { round down $-1.5 \to -2$, $1.5 \to 1$ } ↓

Ceiling:

$\lceil x \rceil$ { round up $-1.5 \to -1$, $1.5 \to 2$ } ↑

Frac Part:

$\text{frac}(x) : x - \lfloor x \rfloor$ gets decimal.

Partial Fractions Decomp. Cont...   Sep 15, 2017.

1) Factor denominator
2) For each distinct factor $g(x)$ of $q(x)$ w/ exponent $n\geq 1$:
   a) If $g(x)$ is linear, it contributes

$$\frac{A_1}{g(x)} + \frac{A_2}{(g(x))^2} \cdots \frac{A_n}{(g(x))^n}$$

See ex. on back ← b) Irreducible quadratic; it contributes

$$\frac{A_1X+B_1}{g(x)} + \frac{A_2X+B_2}{(g(x))^2} \cdots \frac{A_nX B_n}{(g(x))^n}$$

3) Write over common denom (don't expand numerator)
4) Special Values of x to cancel & solve.
5) Write out final answer.

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Tutorial

Sep 18, 2017

Inequalities & Absolute Values:

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \text{ is } + \\ -f(x) & \text{if } f(x) \text{ is } - \end{cases}$$

Example:

1) $|3x-12| < 9$

1) $3x-12 < 9$

$x < 7$

2) $12-3x < 9$

$x > 1$

2) Describe the set $\{x : |x^2-2| > 1\}$ as a union of finite intervals.

1) $-(x^2-2) > 1$

$(x+1)(x-1) < 0$ DNE OR $-1 < x < 1$ $x \in (-1, 1)$

2) $x^2-2>1$ $\boxed{x \in (\sqrt{3}, \infty) \text{ or } (-\infty, -\sqrt{3})}$ careful w/ [illegible]

$x \in (-\infty, -\sqrt{3}) \cup (-1, 1) \cup (\sqrt{3}, \infty)$

3) $|x-2| < |x+1|$

$(x-2)^2 < (x+1)^2$

$\frac{1}{2} < x$.

Composite Fcns:

1) $f(x) = \frac{1}{x}$ $x \in (0, \infty)$ and $g(x) = 2x+1$ $x \in [-2, 2]$

rng(g(x)) must be in dom(f) and x must be in dom(g(x))

$f(g(x))$ $g(x) > 0$ $x > \frac{-1}{2}$ $\therefore x \in (\frac{-1}{2}, 2]$

2. $g(f(x))$ $-2 \leq f(x) \leq 2$

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Partial Fraction Decomposition Continued:

$$\frac{3x^4+3x^3+2x^2+2x-3}{x^5+2x^4+2x^3+4x^2+x+2}$$

$$= \frac{3x^4+3x^3+2x^2+2x-3}{(x+2)(x^4+2x^2+1)} = \frac{3x^4+3x^3+2x^2+2x-3}{(x+2)(x^2+1)^2} \text{, so...}$$

$$\frac{3x^4+3x^3+2x^2+2x-3}{x^5+2x^4+2x^3+4x^2+x+2} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$= A(x^2+1)^2 + (Bx+C)(x+2)(x^2+1) + (Dx+E)(x+2)$$

Let $x=-2$: $28 = 25A$ $\quad \boxed{\therefore A = 1}$

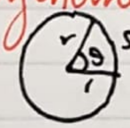
Let $x=0$: $-4 = 2C + 2E$ $\quad \boxed{\therefore E = -2-C}$

Let $x=1$: $7 = 4+6B+6C+3D-6-3C$ $\quad \boxed{\therefore C = 3-2B-D}$

Let $x=-1$: $-8 = -4B-2D$ $\quad \boxed{\therefore B = 2-\frac{D}{2}}$

Let $x=2$: $16 = 8B-8D$ $\quad \boxed{\therefore D = 0}$ $\quad$ Now, go backwards

Trigonometry:



circle subtends arc between sides of the angle.
1 radian is the measure of an angle in a circle that subtends an arc of equal length to the radius.

*rad ≠ unit!

$\sin\theta$: y coord of point on unit circle corresponding to θ
$\cos\theta$: x " " " " " "

Reference Angle: $\bar{\theta}$

Sep 20, 2017

$[0 \leq \theta \leq \frac{\pi}{2}]$ $\qquad |\sin\theta| = \sin\bar{\theta}$ $\qquad |\cos\theta| = \cos\bar{\theta}$

Formulas:

$$\left.\begin{aligned} \sin\theta &= \sin(\theta+2\pi k) \\ \cos\theta &= \cos(\theta+2\pi k) \end{aligned}\right\} k\in\mathbb{Z}$$

$$\left.\begin{aligned} \cos\left(\theta-\frac{\pi}{2}\right) &= \sin\theta \\ \sin\left(\theta-\frac{\pi}{2}\right) &= -\cos\theta \end{aligned}\right\}$$

$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

$\cos^2\theta + \sin^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$\cot^2\theta + 1 = \csc^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$

$\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$

Example:

1) Find all $\theta \in [0, 2\pi)$ that satisfy:

a) $\cos\theta + \sin 2\theta = 0$

b) $\sec^2\theta = 4/3$

a) $\cos\theta + \sin 2\theta = 0$

$\cos\theta + 2\sin\theta\cos\theta = 0$

$\cos\theta(1 + 2\sin\theta) = 0$

$\cos\theta = 0$ and $\sin\theta = \frac{-1}{2}$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

b) $\sec^2\theta = \frac{4}{3}$

$\cos^2\theta = \frac{3}{4}$

$\cos\theta = \pm\frac{\sqrt{3}}{2}$

so

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

2) $\sin\left(\frac{7\pi}{12}\right) = \sin\left(\frac{3\pi}{12} + \frac{4\pi}{12}\right) = \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\cos\frac{\pi}{4}$

$= \frac{\sqrt{2}}{4}(1+\sqrt{3})$

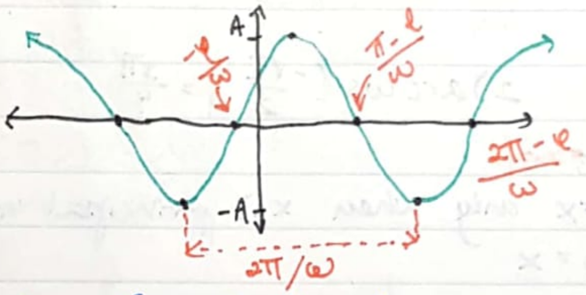
3) $\sin^2\left(\frac{7\pi}{12}\right) = \dfrac{1-\cos\left(\frac{7\pi}{6}\right)}{2} = \dfrac{2+\sqrt{3}}{4}$

General Sine Function:

Sep 22, 2017

$f(x) = A\sin(\omega x + \varphi)$

Period: $\frac{2\pi}{\omega}$ Phase shift: $\frac{-\varphi}{\omega}$ Amplitude: A



Every function $B\sin(wx) + C\cos(wx)$ can be written $A\sin(wx+\varphi)$

Example:

1) $3\sin(2x) + 3\sqrt{3}\cos(2x) = A\sin(2x+\varphi)$

$= A[\sin(2x)\cos(\varphi) + (\sin\varphi)\cos(2x)]$

$= A\cos(\varphi)\sin(2x) + A\sin(\varphi)\cos(2x)$

$\therefore\ 3 = A\cos(\varphi) \quad 3\sqrt{3} = A\sin(\varphi)$ square both sides and add

$A^2\cos^2(\varphi) + A^2\sin^2(\varphi) = 36$

$A = \pm 6$ Take $A = 6$ (-6 also works)

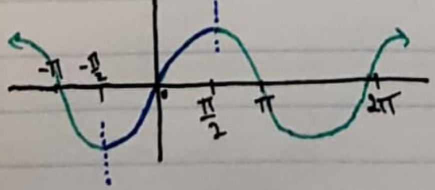
$\therefore \varphi = \frac{\pi}{3}$ $\therefore$ Final: $6\sin(2x + \frac{\pi}{3})$

$$\boxed{\begin{array}{l} A = \pm\sqrt{B^2 + C^2} \\ \left.\begin{array}{l}\cos\varphi = B/A \\ \sin\varphi = C/A\end{array}\right\} \begin{array}{l}\varphi = \arctan(C/B) \\ \text{only when } \varphi \in (\frac{-\pi}{2}, \frac{\pi}{2})\end{array}\end{array}}$$

Inverse Trig:

Sin, cos, tan not inv. but can restrict domain.

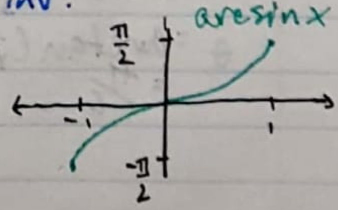
The half-period $[\frac{-\pi}{2}, \frac{\pi}{2}]$ is inv.



$f^{-1}(x) = \arcsin x$

$f: [\frac{-\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$

$f^{-1}: [-1, 1] \to \underbrace{[\frac{-\pi}{2}, \frac{\pi}{2}]}_{\text{Principle values}}$



For cosine: $g: [0,\pi] \to [-1,1]$

$g^{-1}: [-1,1] \to \underbrace{[0,\pi]}_{\text{principle values}}$

$= \arccos(x)$

Examples:

1) $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\pi/4$ 2) $\arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$

3) $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}$ *careful

↳ $\arcsin(\sin x) = x$ only when x is principal value of sine

BUT $\sin(\arcsin x) = x$

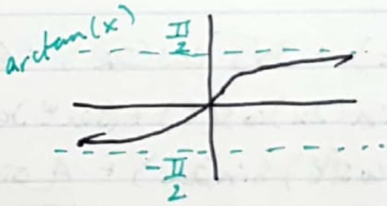
4) $\sin\left(\arcsin\left(\frac{-1}{2}\right)\right) = -1/2$

5) $\sin\left(\arcsin\left(\frac{\pi}{2}\right)\right) = \text{DNE}$ * Watch

For tan: $h: \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$

$h^{-1}: \mathbb{R} \to \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ } P.V.'s

$= \arctan x$



For sec: $(-\infty, -1] \cup [1, \infty) \to \left(-\pi, \frac{-\pi}{2}\right] \cup \left[0, \frac{\pi}{2}\right)$, PVs $= \operatorname{arcsec} x$

For csc: $(-\infty, -1] \cup [1, \infty) \to \left(-\pi, \frac{-\pi}{2}\right) \cup \left(0, \frac{\pi}{2}\right)$, PVs $= \operatorname{arccsc} x$

For cot: $j: (0, \pi) \to \mathbb{R}$ $= \operatorname{arccot} x$

$j^{-1}: \mathbb{R} \to (0, \pi)$ } PVs

1) $\operatorname{arccot}(\sqrt{3}) = \theta$

$\cot\theta = \sqrt{3}$

$\tan\theta = \frac{1}{\sqrt{3}}$

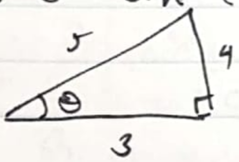
$\theta = \arctan\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{\pi}{6}$

Sometimes Δ trig helps.

1) $\cos(\arcsin(4/5)) = \cos\theta$

Let $\theta = \sin^{-1}(4/5)$



$\therefore \cos\theta = 3/5$.

2) $\cot(\operatorname{arcsec}(13))$

Let $\theta = \operatorname{arcsec}(13)$

$\sec\theta = 13$

$\cos\theta = \frac{1}{13}$ solve for $\frac{1}{\tan\theta}$ using Δ.

3) $\cos(2\arcsin(7/11))$

$= \cos^2(\arcsin(\frac{7}{11})) - \sin^2(\arcsin(\frac{7}{11}))$

$= \cos^2\theta - (\frac{7}{11})^2$

$\theta = \arcsin(\frac{7}{11})$

(Δ) $\sin\theta = \frac{7}{11}$ $\cos\theta = \frac{\sqrt{72}}{11}$ so $\cos^2\theta = \frac{72}{121}$

$= \frac{23}{121}$

4) $\tan(2\arccos(\frac{5}{13}))$

$= \dfrac{2\tan(\arccos\frac{5}{13})}{1 - \tan^2(\arccos\frac{5}{13})}$

$\theta = \arccos\frac{5}{13}$

$\tan\theta = \frac{12}{5}$, so

$= \frac{-120}{119}$

Limits of Sequences:

Let S be an infinite subset of consequtive non-negative integers. A sequence is a function $f: S \Rightarrow \mathbb{R}$

$$f(n) = a_n$$

Usually, $S = \mathbb{N} = \{1, 2, 3 \dots\}$ or $S = \mathbb{N} \cup \{0\}$

To denote sequences

$$\{a_n\}_{n=1}^{\infty}, \quad \{a_n\}_{n=0}^{\infty}, \quad \{a_n\}$$

Example:

1) $\{(-1)^n\}_{n=0}^{\infty} = \{1, -1, 1, -1 \dots\}$
2) $\{3, 9, 15, 21, 27 \dots\} = \{6n+3\}_{n=0}^{\infty} = \{6n-3\}_{n=1}^{\infty}$
$= \{6n-27\}_{n=5}^{\infty}$

Conversions:

To increase starting index by k, substitute n with $(n-k)$

To start $\{6n+3\}_{n=0}^{\infty}$ at $n=5$

$\{6(n-5)+3\}_{n=5}^{\infty}$ simplify.

To start $\{6n-3\}_{n=1}^{\infty}$ at $n=5$

$\{6(n-4)-3\}_{n=5}^{\infty}$ simplify.

Limits of Sequences:

Behavior as $n \to \infty$ $\quad \{\frac{1}{n}\}_{n=1}^{\infty}$

For any +ve number >0 (let ε be this #), we can find an index n where $\frac{1}{n} < \varepsilon$. Consider $1/\varepsilon$, Pick an n where $n > \lceil \frac{1}{\varepsilon} \rceil$

↳ If $n > \lceil \frac{1}{\varepsilon} \rceil$, $n > \frac{1}{\varepsilon}$. Thus $\frac{1}{n} < \varepsilon$

$\therefore$ As $n \to \infty$, $\frac{1}{n} \to 0$ $\quad \lim_{n\to\infty} \frac{1}{n} = 0$

Definition ★★★:

A sequence $\{a_n\}$ converges to the limit L if, for any $\varepsilon > 0$ there exists a natural number $N \in \mathbb{N}$ such that whenever $n > N$, $|a_n - L| < \varepsilon$.

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \ni n > N \to |a_n - L| < \varepsilon$$

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Back to $\{\frac{1}{n}\}_{n=1}^{\infty}$: $a_n = \frac{1}{n}$ $L=0$ $N = \lceil 1/\varepsilon \rceil$

Example:

1) "Use the definition" of sq. limits, prove $\frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$.

$\Leftarrow$ Solution: Required to prove (RTP):

Given any (fixed) $\varepsilon > 0$, we need to find a # N such that $n > N \Rightarrow \left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon$

$\left|\frac{1}{\sqrt{n}} - 0\right| = \frac{1}{\sqrt{n}}$

RTP: $\frac{1}{\sqrt{n}} < \varepsilon$
$n > N$

$\Leftrightarrow$ shows inequality equivalence

$$\frac{1}{\sqrt{n}} < \varepsilon$$
$$\Leftrightarrow \frac{1}{n} < \varepsilon^2$$
$$\Leftrightarrow n > \frac{1}{\varepsilon^2}$$

Let N (must be natural) $= \lceil \frac{1}{\varepsilon^2} \rceil$ if $n > N$, then

$$n > N \Rightarrow n > 1/\varepsilon^2$$
$$\Rightarrow \frac{1}{n} < \varepsilon^2$$
$$\Rightarrow \frac{1}{\sqrt{n}} < \varepsilon \quad \text{QED.}$$

$\therefore \lim_{x \to \infty} \frac{1}{\sqrt{n}} = 0$

2) "Prove" $\frac{n-1}{n} \to 1$ as $n \to \infty$. Sep 27, 2017

Solution: RTP:

$$n > N \Rightarrow \left|\frac{n-1}{n} - 1\right| < \varepsilon$$
$$\Leftrightarrow \left|\frac{-1}{n}\right| < \varepsilon$$
$$\Leftrightarrow \frac{1}{n} < \varepsilon$$
$$\Leftrightarrow n > \frac{1}{\varepsilon}$$

Let $N = \lceil \frac{1}{\varepsilon} \rceil$.

So, when $n > \lceil \frac{1}{\varepsilon} \rceil$, $\left|\frac{n-1}{n} - 1\right| < \varepsilon$ Thus QED.

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Properties of Sequence Limits:

1) $\lim_{n\to\infty} C = C$, $C \in \mathbb{R}$ fixed.

Let $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$

2) $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

Same $\bar{w}$ mult. and division ($B \neq 0$)

3) a) $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for any fixed $p > 0$

b) $\lim_{n\to\infty} r^n = 0$ when $|r| < 1$, $= \infty$ when $|r| > 1$.

4) If f is continuous on open interval containing A:

$\lim_{n\to\infty} f(a_n) = f(A) = f(\lim_{n\to\infty} a_n)$

5) $\lim_{n\to\infty} a_n = \infty$ iff for all $M > 0$, there is an $N \in \mathbb{N}$ such that $n > N \Rightarrow a_n > M$.

Examples:

1) $\left\{ \cos\left(\frac{\pi n}{3n+5}\right) \right\}_{n=1}^{\infty}$ find the limit.

$= \lim_{n\to\infty} \cos\left(\frac{\pi n}{3n+5}\right)$

$= \cos\left(\lim_{n\to\infty} \frac{\pi n}{3n+5}\right)$ } Now, ÷ top/bottom by highest power of n in denom.

Assumptions:
1) lim exists
2) cos is continuous

$= \cos\left(\lim_{n\to\infty} \frac{\pi}{3+\frac{5}{n}}\right) = \cos\left(\lim_{n\to\infty} \frac{\pi}{3}\right) = \frac{1}{2}$

Squeeze Theorem:

Suppose $\{a_n\}, \{b_n\}, \{c_n\}$ where

a) $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$

squished

b) For some $N \in \mathbb{N}$, whenever $n > N$, $a_n \le b_n \le c_n$.

Then $\lim_{n\to\infty} b_n = L$.

Example:

start w/ an inequality.

1) $\lim_{n\to\infty} \frac{\cos(n)}{n}$ We know $-1 \le \cos(\sim) \le 1$

$$-1 \le \cos(n) \le 1$$

$$\frac{-1}{n} \le \frac{\cos(n)}{n} \le \frac{1}{n}$$

$$\therefore \lim_{n\to\infty} \frac{-1}{n} \le \lim_{n\to\infty} \frac{\cos(n)}{n} \le \lim_{n\to\infty} \frac{1}{n}$$

$$0 \le \sim \le 0$$

$\therefore$ The limit is 0.

Indeterminate Forms:

Expressions resulting from direct subst. can't be evaluated.

$$0^0, \frac{0}{0}, \frac{\infty}{\infty}, 1^\infty, \infty - \infty, 0\cdot\infty, \infty^0$$

If f is polynomial of degree $< k$,

$$\lim_{n\to\infty} n^k \pm f(n) = \lim_{n\to\infty} n^k = \infty.$$ } Largest power dominates limit

Example:

1) $\lim_{n\to\infty} \frac{9n^2 - 5n + 3}{5n^2 - 9n + 5}$

$$= \frac{9}{5}$$

2) $\lim_{n\to\infty} \frac{9n+8}{\sqrt{9n^2+8}}$ Divide by highest "effective power" in denom $\sqrt{n^2} = n'$

$$\lim_{n\to\infty} \frac{9 + \cancel{\frac{8}{n}}}{\sqrt{\frac{9n^2}{\cancel{n^2}} + \cancel{\frac{8}{n^2}}}} = 3$$

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3) $\lim_{n\to\infty} \frac{6^n+3^n}{6^{n-1}-4}$ Divide by 6^{n-1} (highest exp of n in denom)

$= \lim_{n\to\infty} \frac{6^n+3^n}{6^{n-1}-4} \cdot \frac{1/6^{n-1}}{1/6^{n-1}}$

$= \lim_{n\to\infty} \frac{6+3\cdot\left(\frac{3^{n-1}}{6^{n-1}}\right)}{1-4/6^{n-1}}$

$= \lim_{n\to\infty} \frac{6+3\cdot\left(\frac{1}{2}\right)^{n-1}}{1-4/6^{n-1}}$

$= \frac{6+0}{1-0} = 6.$

4) $\lim_{n\to\infty} \sqrt{n^2+1} - n \cdot \left(\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}\right)$

$\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} = \frac{1}{\infty} = 0$

$\{a_n\}$ is divergent if $\lim_{n\to\infty} a_n = \pm\infty$ or DNE

Limits of functions:

$\lim_{x\to c} f(x) = L$ iff $\forall \varepsilon > 0,\ \exists \delta > 0 \ni |x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Example:

1) Prove $\lim_{x\to 2} 3x-2 = 4$. *page 59

RTP: Given $\varepsilon > 0$, find $\delta > 0$ where $|x-c| < \delta \Rightarrow |3x-2-4| < \varepsilon$.

$|x-2|$

$|3x-6| < \varepsilon$

$\Leftrightarrow 3|x-2| < \varepsilon$

$\Leftrightarrow |x-2| < \varepsilon/3$ Thus, if $\delta = \varepsilon/3$, is true, so $\lim = 4$.

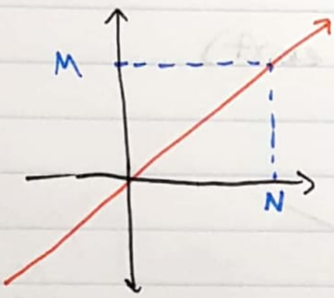
Key Definitions: pg 59

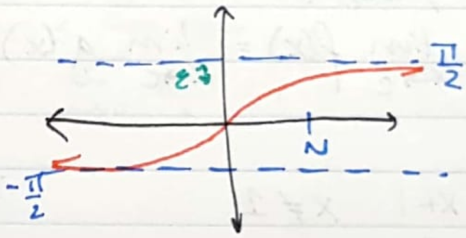
How close you are to a limit

1) $\lim\limits_{x\to c} f(x)=\infty$ $\quad \underbrace{\forall M>0}_{\text{implies } \infty}\ \exists\, \delta>0 \ni \underbrace{|x-c|<\delta}\Rightarrow f(x)>M.$

2) $\lim\limits_{x\to\infty} f(x)=L$ $\quad \forall \varepsilon>0\ \exists N>0 \ni x>N \Rightarrow |f(x)-L|<\varepsilon$

3) $\lim\limits_{x\to\infty} f(x)=\infty$ $\quad \forall M>0\ \exists N>0 \ni x>N \Rightarrow f(x)>M.$





2

Constructing Definitions:

$\boxed{\lim\limits_{x\to c}}$: $\exists\, \delta>0 \ldots$
$\ldots \ni |x-c|<\delta \Rightarrow$

$\boxed{\lim\limits_{x\to\infty}}$: $\exists\, N>0 \ldots$
$\ldots \ni x>N \Rightarrow$

$\boxed{\lim\limits_{x\to-\infty}}$: $\exists\, N>0 \ldots$
$\ldots \ni x<-N \Rightarrow$

$\boxed{\lim f(x)=L}$: $\forall \varepsilon>0\ \exists \ldots \Rightarrow |f(x)-L|<\varepsilon$

$\boxed{\lim f(x)=\infty}$: $\forall M>0\ \exists \ldots \Rightarrow f(x)>M$

$\boxed{\lim f(x)=-\infty}$: $\forall M>0\ \exists \ldots \Rightarrow f(x)<-M.$

One sided limits:

$|x-c|$: 2 sided. $=\begin{cases} x-c, & x>c \quad c^+ \\ c-x, & x<c \quad c^- \end{cases}$

$\lim\limits_{x\to c^-} f(x)=L$: $\forall \varepsilon>0\ \exists\, \delta>0 \ni c-x\ \delta \Rightarrow |f(x)-L|<\varepsilon$

$\lim\limits_{x\to c^+} f(x)=L$: $\forall \varepsilon>0\ \exists\, \delta>0 \ni x-c<\delta \Rightarrow |f(x)-L|<\varepsilon$

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$$\lim_{x \to c^+} f(x) = \infty : \forall M > 0 \ \exists \delta > 0 \ \ni x - c < \delta \Rightarrow f(x) > M.$$

Limits cont... Sep 29, 2017

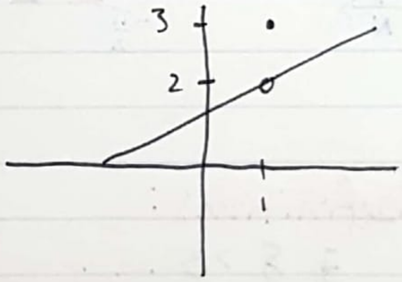
Let $c \in \mathbb{R}$. Open neighbourhood of c is an open interval containing c. (a,b)

↳ If f, g are fcn's and $f(x) = g(x) \ \forall x \in (a,b)$ except $x = c$, then:

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) \quad \text{(if exist)}$$

Example:

1) $f(x) = \begin{cases} x+1 & x \neq 1 \\ 3 & x = 1 \end{cases}$



$$\lim_{x \to 1} f(x) = \lim_{x \to 1} x+1 = 2$$

can use substitution when no proof required.

Same limit theorem applies from sequences

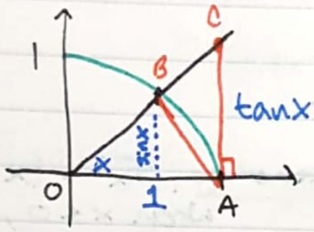
Squeeze Theorem again...

Let f, g, h be functions - $f(x) \leq h(x) \leq g(x)$ on an open neighbourhood of c

If $\lim_{x \to c} f(x) = L = \lim_{x \to c} g(x)$, then $\lim_{x \to c} h(x) = L$.

Special Limit:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$



Area $\Delta OCA \geq$ Area (sector OBA) $\geq$ area (ΔOBA)

$$\frac{1}{2}\tan x \geq \frac{1}{2} x r^2 \geq \frac{1}{2}\sin x$$

$$\tan x \geq x \geq \sin x$$

$$\frac{1}{\cos x} \geq \frac{x}{\sin x} \geq 1.$$

$$\cos x \leq \frac{\sin x}{x} \leq 1 \quad \text{(Now, take limit)}$$

$$\lim_{x\to 0} \cos(x) \leq \lim_{x\to 0} \frac{\sin x}{x} \leq 1$$

$$1 \leq \lim_{x\to 0} \frac{\sin x}{x} \leq 1 \quad \therefore \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Example:

1) $\lim_{x\to 0} \frac{1-\cos x}{x}$

$= \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)}$

$= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{\sin x}{1+\cos x}\right)$

$= (1)\left(\frac{0}{1+1}\right) = 0$

Infinity "Arithmetic":

1) $(\infty)(\infty) = \infty$
2) $(-\infty)(\infty) = -\infty$
3) $(-\infty)(-\infty) = \infty$
4) If $a>1$, $\lim_{x\to\infty} a^x = \infty$, else 0
5) $\frac{c}{\infty} = 0$
6) $\infty + \infty = \infty$
7) When $c \neq 0$:
$(c)(\infty) = \begin{cases} \infty, & c>0 \\ -\infty, & c<0 \end{cases}$

More Limits: Oct 2, 2017.

1) $\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x+1}$ Divide by highest x in denom

$\lim_{x\to\infty} \frac{(\sqrt{x^2+1})}{(x+1)} \left(\frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{x}}\right) = \frac{\sqrt{1+0}}{1+0} = \underline{1}.$ careful

2) $\lim_{x\to -\infty} \frac{\sqrt{x^2+1} \cdot \left(\frac{1}{-\sqrt{x^2}}\right)}{(x+1)\left(\frac{1}{x}\right)}$ because x is negative ∞. $\sqrt{x^2} = |x|$ when $x<0$, $\sqrt{x^2} = -x$!

$\lim_{x\to -\infty} \frac{-\sqrt{1+\frac{1}{x^2}}}{1+\frac{1}{x}} = \underline{-1}.$ two different HAs.

Continuity: * pg 66-67

f is continuous on (a,b) if $\forall x, y \in (a,b)$:
$\forall \varepsilon > 0, \exists \delta > 0 \ni |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

f is continuous at $x=c$ when:
$\lim_{x\to c} f(x) = f(c)$ (and lim is defined + finite).

and... if continuous at $\forall x=c$ in (a,b), continuous on interval.

Usual Steps to Evaluate:
1) $f(c)$ exists.
2) $\lim_{x\to c} f(x)$ exists (+ finite)
3) $\lim_{x\to c} f(x) = f(c)$

Types of Discontinuity:
1) Removable → f not cont. at $x=c$ but $\lim_{x\to c} f(x)$ exists
2) Non removable → $\lim_{x\to c} f(x)$ DNE
↳ Jump $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist but $\neq$
↳ VA (∞) At least one of the → ← lims are finite.

Finding Discontinuities for Piecewise Functions:

1) Check formula, discard those outside x values used
2) Check every transition value.
   ↳ Must do this for every value on both sides → ←

Example:

1) $f(x) = \begin{cases} \frac{x^2-10x+25}{x^2-25} & , x<5 \\ \frac{x-5}{x} & , 5 \le x < 9 \\ \frac{x^2}{8} & , x > 9 \end{cases}$ Be careful

1) Check formulas

i) $\frac{x^2-10x+25}{x^2-25}$ DC at $x=-5, 5$
BUT discard $x=5$ b/c not in interval.

ii) $\frac{x-5}{x}$ DC at $x=0$
BUT discard $x=0$ b/c not in interval.

iii) $\frac{x^2}{8}$ Continuous

From formulas: discontinuous at $\boxed{x=-5}$

2) Transition Values

i) $x=5$

a) $f(5) = \frac{5-5}{5} = 0$ ✓ exists

b) $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} \frac{x^2-10x+25}{x^2-25} = \lim_{x\to 5^-} \frac{x-5}{x+5} = 0.$ ✓

$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} \frac{x-5}{x} = 0$ ✓ $\boxed{\therefore \lim_{x\to 5} f(x) = 0}$

$\therefore f$ is cont. at 5 since $\lim = 0 = f(5)$. need this.

ii) $x=9$

c) $f(9) =$ DNE

$\therefore f$ is disc. at $\boxed{x=9.}$

Now, classify:

At $x=-5$: $\lim_{x\to -5} f(x) = \frac{100}{0}$ → VA/Inf. dc

$x=9$: $\lim_{x\to 9} f(x) =$ DNE since left/right ≠. → Jump dc. } Both non-removable.

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Theorem:

If $\lim_{x \to c} g(x) = L$ and f is continuous at $x = L$ then:

$$\lim_{x \to c} f(g(x)) = f(L) = f\left(\lim_{x \to c} g(x)\right)$$

Example:

1) $\lim_{x \to 7} e^{x^2+5} = e^{54} = e^{\lim_{x \to 7} x^2+5}$

Continuity on Closed Intervals:

f is continuous on $[a,b]$ when:

1) Continuous (a,b)
2) Cont. on the right at $x=a$, left at $x=b$

$[a,b]$

↳ Right continuity:

1) $f(a)$ exists
2) $\lim_{x \to a^+} f(x)$ exists (+ finite)
3) $\lim_{x \to a^+} f(x) = f(a)$

↳ Left continuity:

1) $f(a)$ exists
2) $\lim_{x \to b^-} f(x)$ exists (+ finite)
3) $\lim_{x \to b^-} f(x) = f(a)$

Intermediate Value theorem: Oct 4, 2017.

f continuous on $[a,b]$, function passes every value $[a,b]$

↳ Show $x^5 - 6x^3 - x + 4$ has root on $[0,1]$.

↳ f is continuous on $[0,1]$. $f(0)=4$ $f(1)=-2$.

$\therefore$ By IVT, $(\exists c \in [0,1])$ where $f(c)=0$.

Extreme Value Theorem: pg 69-72

f continuous on $[a,b]$, then f has an abs max/min on $[a,b]$.

Differentiation:

Refer to gr 12 calc notes + EdX Notes.

Proof of power rule: $f'(x) = nx^{n-1}$, $n \in \mathbb{N}$.

$$(x+h)^n = \sum_{i=0}^{n} x^{n-i} h^i \cdot \binom{n}{i}$$

$$= x^n + nhx^{n-1} + h^2 (\text{STUFF...})$$

$$= \lim_{h \to 0} \frac{nhx^{n-1} + h^2(\text{STUFF})}{h}$$

$$= nX^{n-1}$$