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MC 6461
Section 010

MATH 115 - Linear Algebra — Sep 7/2017

Chapter 1 - Euclidean Vector Spaces

Vectors: Magnitude, direction supported on a line $\mathbb{R}^n$
↳ $\mathbb{R}^2$ is set of all vectors in form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ ← real # (components)
↳ $\mathbb{R}^n$ " " $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \vec{x}$ (need arrow), $\mathbb{R}^n$: $\in \mathbb{R}^n$

Duality $\overline{\text{w}}$ pts + $\vec{v}$ in $\mathbb{R}^2$, $\mathbb{R}^n$ (can get pt $\rightleftarrows \vec{v}$)

Refer to gr12 Calc notes for geometric rep, operations, etc...

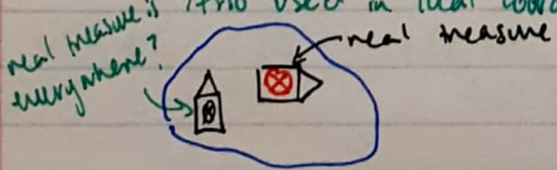
Adding vectors:
↳ Can add vectors in $\mathbb{R}^3$ by using parallelogram method on plane containing vectors.

Zero vector:
$\vec{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$ → initial = terminal pt
↳ supported on ∞ lines, no direction

Theorem:
Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$
↳ $\vec{x} + \vec{y} \in \mathbb{R}^n$ "closure under addition"
↳ $\vec{0} + \vec{x} = \vec{x}$
↳ $t\vec{x} \in \mathbb{R}^n$

Lecture 2 - Sep 12, 2017:

Fixing the origin:
Vectors always anchored on origin for linear algebra.
↳ In physics, often anchored to point where force is applied - "Free vectors"
↳ Also used in local coordinate systems



"Global coordinate system" - origin must be independent of your movement

Lines in $\mathbb{R}^n$:

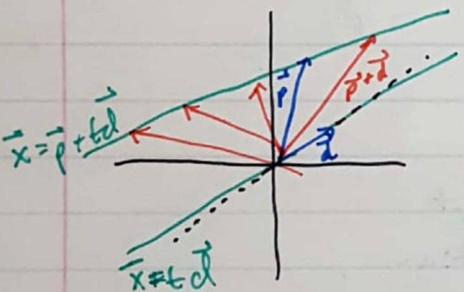
1) Directed line segments

- Let $P, Q \in \mathbb{R}^n$. $\vec{PQ} = \vec{q} - \vec{p}$ (second - first) ← feels not true...
- Origin translated to be at P. ← b/c this (origin location)

Equation of a Line in $\mathbb{R}^n$

- Let $\vec{p}, \vec{d}$ be vectors in $\mathbb{R}^n$, $\vec{d} \neq \vec{0}$, has direction vector $\vec{d}$
- $\{\vec{p} + t\vec{d} \mid t \in \mathbb{R}\} \to \vec{x} = \vec{p} + t\vec{d}, t \in \mathbb{R}$ (just direction vectors)
- $/\!/^{el}$ to line through origin w/ vector eqn $\vec{x} = t\vec{d}$, $t \in \mathbb{R}$

↳ Just a translation by $\vec{p}$.



← all lie on this line

// parallel

★ Old notes: eqns of lines

↳ sym, normal, para

Sym. eqn:

↳ solve for x_1/x_2: $\frac{d_2}{d_1} x_1 + (p_2 - \frac{d_2}{d_1})$ ← $y = mx + b$

↳ If $d_1 = 0$, $d_2 \neq 0$ get a vertical line

$d_2 = 0$, $d_1 \neq 0$ get a horiz. line

★ In $\mathbb{R}^n$, where $x_n = a_{n-1}x_{n-1} + \dots + a_1 x_1$ w/ real coefficients is never the eqn of a line when $n \geq 3$

Two vectors $\vec{u}$ and $\vec{v}$ are $/\!/^{el}$ if one is $\vec{0}$ or sc. multiples.

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Subspaces in $\mathbb{R}^n$:

A non-empty subset S is a subspace of $\mathbb{R}^n$ if for any $\vec{x}, \vec{y} \in S$ and any $t \in \mathbb{R}$, the following holds:

1) $\vec{x} + \vec{y} \in S$
2) $t\vec{x} \in S$

} closure under addition / scalar multi.

Properties:

a) $\vec{0}$ is always in any subspace (let $t=0$)
b) $\{\vec{0}\}$ is subspace of $\mathbb{R}^n$ "trivial subspace"
c) $\mathbb{R}^n$ is subspace of itself
→ Euclidean vector space: set $\mathbb{R}^n$ for $n \geq 1$

Example:

1) Let $S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \;\Big/^{\text{such that}}\; x_1 + x_2 - 8x_3 = 4 \right\}$

Not a subspace of $\mathbb{R}^4$
→ if $\vec{0} \in S$, right eqn $\neq$ true.

2) Let $T = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\Big/\; x_1 x_2 + x_3^2 = 0 \right\}$ ← multi coords and sq. = unlikely to be subspace.

$\vec{0} \in T$, but not subspace of $\mathbb{R}^3$.
→ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in T$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in T$ but sum $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \notin T$ (eqn doesn't work)

3) Let $U = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\Big/\; x_1 + 4x_2 = 0 \right\}$

✓ Subspace of $\mathbb{R}^2$

✓ → $\vec{0} \in U$

✓ → $\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$: $(x_1 + y_1) + 4(x_2 + y_2)$
$= \underbrace{(x_1 + 4x_2)}_{0} + \underbrace{(y_1 + 4y_2)}_{0}$ since $\in U$

✓ → $t\vec{x} = \begin{pmatrix} tx_1 \\ tx_2 \end{pmatrix}$ $\quad t(x_1 + 4x_2) = 0$
$t(0) = 0$ ✓

} 3 conditions

Spanning Sets:

Let $\vec{v}_1 \dots \vec{v}_k$ be k vectors in $\mathbb{R}^n$. Then,

S is all possible L combos of $\vec{v}_{1\dots k}$ $\{ S = \{ t_1\vec{v}_1 \dots t_k\vec{v}_k \mid t_1 \dots t_k \in \mathbb{R} \} \subset \mathbb{R}^n$ *Try to prove this

is a subspace of $\mathbb{R}^n$

↳ Set of all linear combinations is a subspace!

Let $B = \{\vec{v}_1 \dots \vec{v}_k\}$ be set of k vectors in $\mathbb{R}^n$

Let $S = \{t_1\vec{v}_1 \dots t_k\vec{v}_k \mid t_1 \dots t_k \in \mathbb{R}\} \subset \mathbb{R}^n$

↳ The subspace S is "spanned" by B

↳ B "spans" S or is a "spanning set" of S $\} S = \text{span}\{\vec{v}_1 \dots \vec{v}_k\} = \text{span } B$

Example:

1) A line through the origin in $\mathbb{R}^n$, $\vec{x} = t\vec{v}$ $(\vec{v} \neq \vec{0})$ is a subspace of $\mathbb{R}^n$ ($k=1$ from prev. theorem) spanned by any given direction vector $(\vec{v})$.

↳ Because $k=1$, so in $S = \{t_1\vec{v}_1 \mid t_1 \in \mathbb{R}\}$

Theorem:

Let $\vec{v}_1 \dots \vec{v}_k$ be k vectors in $\mathbb{R}^n$. If $\vec{v}_k$ can be expressed as a linear combination of $\vec{v}_1 \dots \vec{v}_{k-1}$ then $\text{span}\{\vec{v}_1 \dots \vec{v}_k\} = \text{span}\{\vec{v}_1 \dots \vec{v}_{k-1}\}$

↳ If you can rep. a $\vec{v}$ in the set as a lin comb. of two other vectors in the set, you can omit it. (same thing (=))

Linear Independance

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set $\{\vec{v}_1 \dots \vec{v}_k \in \mathbb{R}^n\}$ is linearly independent if the only solution to

$$\vec{0} = t_1\vec{v}_1 \dots t_k\vec{v}_k$$

is the trivial solution $t_{1\dots k} = 0$. else $\{$ linearly dependant $\}$

↳ Complicated way of saying no linear combinations within the set.

Theorem:

If the set of vectors $\vec{v}_1 \dots \vec{v}_k$ contains $\vec{0}$, linearly dependent.
↳ Can have any t for the $\vec{0}$ and the rest 0.

Bases:

Subspace $\{\vec{v}_1 \dots \vec{v}_k\}$ in $\mathbb{R}^n$

$\{\vec{v}_1 \dots \vec{v}_k\}$ is a basis for subspace if:
→ linearly independent (no redundant info)
→ spanning set for S.

Example:

1) "The standard basis" in $\mathbb{R}^n$ Just unit vectors
→ $\vec{e}_1 \dots \vec{e}_n$ where $\vec{e}_i \in \mathbb{R}^n$ is the vector whose i^{th} component is 1 and all the others are 0.
→ So..

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Big\}{n-1^{th}} \rightarrow \vec{e}_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ i^{th} \\ \vdots \\ 0 \end{pmatrix}$$ } All component 0, i^{th} component 1.

→ 1) Linearly independent.

$$t_1\vec{e}_1 + \dots + t_n\vec{e}_n = \vec{0}$$

adding gives you $= \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}$ ∴ All $t_1 \dots t_n$ must be 0 for $\vec{0}$.

2) Spanning set for S

$\vec{v} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$ can be written as l. comb of $\vec{e}_1 \dots \vec{e}_n$

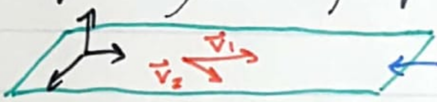
$$\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = v_1\vec{e}_1 + \vec{v}_2\vec{e}_2 \dots \vec{v}_n\vec{e}_n$$ ∴ $\vec{v} \in \text{span}\{\vec{e}_1 .. \vec{e}_n\}$

Surfaces in Higher Dimensions:

Let $\vec{v}_1, \vec{v}_2, \vec{p} \in \mathbb{R}^n$ with $\{\vec{v}_1, \vec{v}_2\}$ linearly independent. then,

$$\vec{x} = \vec{p} + t_1\vec{v}_1 + t_2\vec{v}_2 \quad , t_1, t_2 \in \mathbb{R}$$

is a plane passing through $\vec{p}$. (translated by $\vec{p}$)



← span of the vectors is the plane

Hyperplanes:

Let $\{\vec{v}_1 \dots \vec{v}_{n-1}\}$ be linearly independent. Let $\vec{p} \in \mathbb{R}^n$

$\therefore$, $\vec{x} = \vec{p} + t_1\vec{v}_1 + \dots t_{n-1}\vec{v}_{n-1}$

is a hyperplane passing through $\vec{p}$. (4 dimensions? 3 d. vectors)

Length and Dot Product:

"Dot Product" / Scalar Product / Standard Inner Product

$\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 \dots x_ny_n$ (get a scalar)

Properties:

1) $\vec{x} \cdot \vec{x} \geq 0$. only 0 when $\vec{x} = \vec{0}$
2) $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$
3) $\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$
4) $t\vec{x} \cdot \vec{y} = t(\vec{x} \cdot \vec{y}) = \vec{x} \cdot (t\vec{y})$

Length:

$|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x_1^2 \dots + x_n^2}$ "norm" $\quad \cos\theta = \frac{\vec{x} \cdot \vec{y}}{|\vec{x}||\vec{y}|}$

Properties:

1) $|t \cdot \vec{x}| = |t||\vec{x}|$
2) $|\vec{x} \cdot \vec{y}| \leq |\vec{x}||\vec{y}|$ Cauchy-Schwarz.
3) $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$ Δ ineq.

Sep 15, 2017

Scaler Eqn of A Hyperplane:

Hyperplanes are given by: in $\mathbb{R}^n$...

$$\vec{x} = \vec{p} + t_1\vec{v}_1 \ldots t_{n-1}\vec{v}_{n-1} \quad t \in \mathbb{R}$$

where $\{\vec{v}_1 \ldots {}_{n-1}\}$ is linearly independent.

These hyperplanes have normals. ⇒ $\vec{w}$ n coords $\begin{pmatrix} \vdots \\ n \end{pmatrix}$

- ↳ Non zero
- ↳ Given any $P(x, y, z)$, ignore this
- ↳ Any vector in plane is ⊥ to normal of plane.

Another point x is only in plane iff $\vec{px} \cdot \vec{n} = 0$.

Scalar equation:

$$m_1 x_1 \ldots m_n x_n = d$$

or... $Ax + By + Cz = D$ for $n = 3$.

If...

- $n = 2$? Hyperplane is a line
- $n = 3$? Hyperplane is a plane
- $n > 3$? Hyperplane is a hyperplane.

} One less than n.

Example:

1) $2x_1 + x_2 + 4x_3 = 2$

$2(x_1 - 1) + x_2 + 4x_3 = 0$, or...

$$\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} x_1 - 1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

because $\vec{px}$ is on the plane!

$$\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \cdot \left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] = 0 \quad \therefore \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \underline{\underline{\text{is}}} \text{ on the hyperplane.}$$

Projection and Minimum Distance:

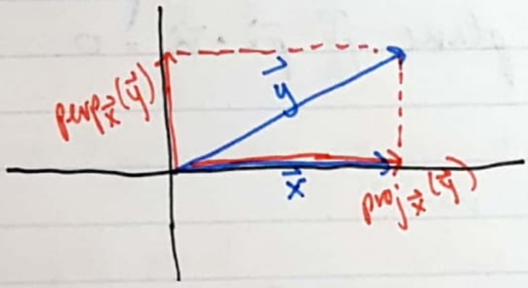
Projections:

$$\mathrm{proj}_{\vec{x}}(\vec{y}) = \left(\frac{\vec{y}\cdot\vec{x}}{\|\vec{x}\|^2}\right)\vec{x}$$

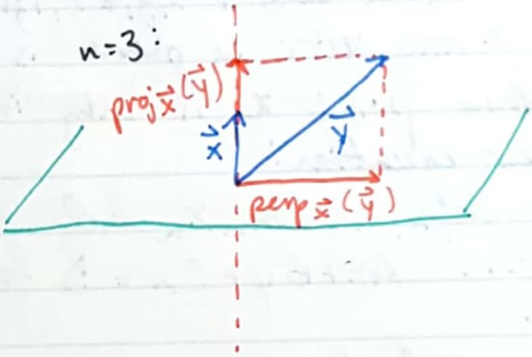
Projection of $\vec{y}$ onto perpendicular of $\vec{x}$:

$$\mathrm{perp}_{\vec{x}}(\vec{y}) = \vec{y} - \mathrm{proj}_{\vec{x}}(\vec{y})$$

when $n=2$:



$n=3$:



Example:

1) $\vec{x} = \begin{pmatrix}1\\1\end{pmatrix}$, $\vec{y} = \begin{pmatrix}3\\2\end{pmatrix}$

$\mathrm{proj}_{\vec{x}}(\vec{y}) = 3\sqrt{2}\begin{pmatrix}1\\1\end{pmatrix}$ $\quad$ $\mathrm{perp}_{\vec{x}}(\vec{y}) = \begin{pmatrix}3\\2\end{pmatrix} - 3\sqrt{2}\begin{pmatrix}1\\1\end{pmatrix}$

Properties of Projections:

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$\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$

1) $\mathrm{proj}_{\vec{x}}(\vec{y}+\vec{z}) = \mathrm{proj}_{\vec{x}}(\vec{y}) + \mathrm{proj}_{\vec{x}}(\vec{z})$
2) $\mathrm{proj}_{\vec{x}}(t\vec{y}) = t\,\mathrm{proj}_{\vec{x}}(\vec{y})$
3) $\mathrm{proj}_{\vec{x}}(\mathrm{proj}_{\vec{x}}(\vec{y})) = \mathrm{proj}_{\vec{x}}(\vec{y})$

Minimum Distance:

Distance btwn point and line/hyperplane is min. distance w/ closest point! ← S is point on line/hyperplane

$\|\mathrm{perp}_{\vec{d}}(\vec{SQ})\|$ → Point Q to line w/ dir. $\vec{v}$: $\vec{d}$

$\|\mathrm{proj}_{\vec{m}}(\vec{SQ})\|$ → Point Q to hyperplane w/ normal $\vec{v}$: $\vec{m}$

Fundamental Example:

Let $Q = (Q_1, Q_2, Q_3) \in \mathbb{R}^3$, $H: m_1x_1 + m_2x_2 + m_3x_3 + d = 0$ (hyperplane)

Distance from H to Q: watch! $+D$ if all on one side

$$\frac{|Ax_1 + Bx_2 + Cx_3 (+D)|}{\sqrt{A^2+B^2+C^2}} \qquad \vec{n} = (A, B, C) \quad P = (x_1, x_2, x_3)$$

Assume $m_1 \neq 0$. Then $S_1 = (\frac{d}{m_1}, 0, 0)$ is on H. Therefore, distance is:

$$\|proj_{\vec{m}}(\vec{SQ})\| = \left\| \frac{(\vec{q} - \vec{s}) \cdot \vec{m}}{\|\vec{m}\|^2} \vec{m} \right\|$$

$$= \frac{|(q_1 - \frac{d}{m_1}) m_1 + q_2 m_2 + q_3 m_3|}{\sqrt{m_1^2 + m_2^2 + m_3^2}}$$

$=$ original formula!

Cross Product:

$\vec{u} \times \vec{v}$ is $\perp$ to both $\vec{u}$ and $\vec{v}$ (dot prod $= 0$)

1) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
2) $\vec{u} \times \vec{u} = \vec{0}$
3) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
4) $t(\vec{u} \times \vec{v}) = (t\vec{u}) \times \vec{v} = \vec{u} \times (t\vec{v})$
5) $(\vec{u} \times \vec{v}) \times \vec{w} \neq \vec{u} \times (\vec{v} \times \vec{w})$ ORDER MATTERS!

Proof of 3): Let $\vec{a} = \vec{v} + \vec{w}$.

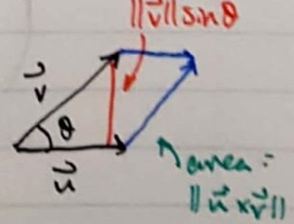
$$\vec{u} \times \vec{a} = u_2(v_3 + w_3) - u_3(v_2 + w_2) \quad \} \text{ first comp. of } \times$$
$$= u_2v_3 + u_2w_3 - u_3v_2 - u_3w_2$$
$$= \underbrace{(u_2v_3 - u_3v_2)}_{\vec{u} \times \vec{v} \text{ comp.}} + \underbrace{(u_2w_3 - u_3w_2)}_{\vec{u} \times \vec{w} \text{ comp.}}$$

And repeat.

Applications:

$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta \quad * \ \theta \in [0, \pi]$

$=$ area of parallelogram spanned by $\vec{u}$ and $\vec{v}$



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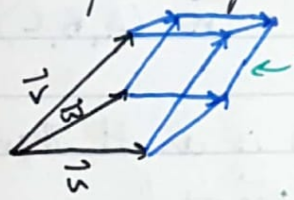
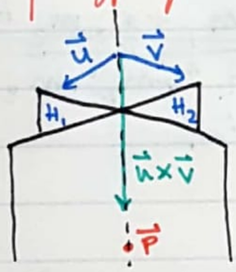
Triple Scalar Product:

$\vec{w} \cdot (\vec{u} \times \vec{v})$

→ Volume of the parallelepiped w/ adj sides $\vec{u}, \vec{w}, \vec{v}$.



Intersection of Hyperplanes:



Let $\vec{u}$ be normal of H_1, $\vec{v}$ for H_2.

The line of int. has dir. vector $\vec{u} \times \vec{v}$.

Example:

1) $H_1: x_1 + x_2 + x_3 = 1$ $H_2: x_1 - x_2 + x_3 = 0$

$\vec{n_1} \times \vec{n_2} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$ $\vec{P} = (\frac{1}{2}, \frac{1}{2}, 0)$ $\vec{x} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, t \in \mathbb{R}$

Chapter 2: Systems of Linear Equations:

Linear Eqns:

A system of m linear equations in n variables $x_1 \dots x_n$ is a set of m eqns:

$$\begin{cases} a_{11} x_1 \cdots a_{1n} x_n = b_1 \\ \vdots \\ a_{m1} x_1 \cdots a_{mn} x_n = b_m \end{cases}$$

The coefficient matrix is...

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

The augmented matrix is...

$$\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_n \end{array}\right) = [A|\vec{b}]$$

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Matrices:

$$\begin{array}{c} \\ i^{th}\text{ row} \\ \\ \end{array}\begin{pmatrix} & \vdots & \\ \cdots & a_{ij} & \cdots \\ & \vdots & \end{pmatrix}$$
j^{th} col

$1 \le i \le m$ equations

$1 \le j \le n$ degree

Row Echelon Form:

Elementary row operations:

1) Multiply by non-zero constant
2) Switch rows
3) Add multiples of rows ★ Can't do $2R_4 - R_5$

Two matrices are row equivalent if one can be obtained by row reducing the other. $A \sim B$

A matrix is in row echelon form if:

1) All 0s at bottom row (don't need row of 0s tho)
2) First # in upper row is ← of first # in lower row

Example:

1) $A: \begin{pmatrix} 1 & 12 & -22 \\ 0 & 2 & 3 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{pmatrix}$ ✓ REF $B: \begin{pmatrix} 1 & 12 & -22 \\ 0 & 2 & 3 \\ 0 & 8 & 4 \end{pmatrix}$ ✗ Not REF

Gaussian Elimination:

$$\begin{pmatrix} 0 & 0 & 1 & 3 \\ 0 & 8 & -1 & 10 \\ 0 & 0 & -3 & 9 \\ 0 & 16 & 4 & -7 \end{pmatrix}$$

1) Identify first non-zero column. Switch rows so that top entry in this column is non zero

first non-zero column

needs to be at top (first non-zero column value)

PIVOT

$$\begin{pmatrix} 0 & 0 & 1 & 3 \\ 0 & 8 & -1 & 10 \\ 0 & 0 & -3 & 9 \\ 0 & 16 & 4 & -7 \end{pmatrix} \Big| R_1 \updownarrow R_2 \sim \begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -3 & 9 \\ 0 & 16 & 4 & -7 \end{pmatrix}$$

2) Use ERO to make all entries below pivot = 0.

$$\begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -3 & 9 \\ 0 & 16 & 4 & -7 \end{pmatrix} \Big| R_4 - 2R_1 \sim \begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -3 & 9 \\ 0 & 0 & 6 & -27 \end{pmatrix}$$

should be 0!

3) Consider all columns right of pivot. Becomes submatrix. Repeat 1+2 for this submatrix w/ all rows below last pivot.

$$\begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -3 & 9 \\ 0 & 0 & 6 & -27 \end{pmatrix} \Big| R_3 + 3R_2 \text{ then } \Big| R_4 - 6R_2 \sim \begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & -45 \end{pmatrix}$$

new pivot

need these 0 now

4) Repeat step 3 until row echelon form.

$$\begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & -45 \end{pmatrix} \Big| R_4 + 5R_3 \sim \begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

new pivot

(would still be REF w/o last row)

Determining Solutions using Matrices: Sep 21, 2017

Consistent - 1+ solutions

Inconsistent - 0 solutions

↳ If last row is $[0 \ldots 0 \mid c]$, $c \neq 0$

If consistent...

↳ # pivots = # variables? Unique Solution. $[0 \ldots c \mid c]$

↳ # pivots < # variables? Infinite solutions. $[0 \ldots 0 \mid 0]$

Example:

1) $\left(\begin{array}{cccc|c} 0 & 8 & -1 & 10 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$

inconsistent

2) $\left(\begin{array}{cccc|c} 0 & 8 & -1 & 10 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

consistent

3 pivots, 4 variables

∴ ∞ solutions

3) $\left(\begin{array}{ccc|c} 8 & -1 & 10 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 9 & 1 \end{array}\right)$

3 pivots, 3 variables

∴ 1 solution

Reduced REF (RREF):

1) REF
2) All leading entries = 1.
3) Leading 1's have all 0s after (in the column)

↳ Any matrix has a equivalent, unique matrix in RREF

$\xleftarrow{\text{REF}}$

$\xleftarrow{\text{RREF}}$

$$\begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -7 \end{pmatrix}$$

Example:

1) $\begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ RREF

$\frac{1}{8}R_1$, then $\frac{1}{9}R_3$

then $R_1 + \frac{1}{8}R_2$

then $R_1 + \left(-\frac{13}{8}\right)R_3$

then $R_2 - 3R_3$

(Do in 1 step tho)

Homogeneity and Rank:

Homogeneous if right side of all eqns $= 0$.

$$\begin{cases} 2x_1 - x_2 = 0 \\ x_1 + 3x_2 - 4x_3 = 0 \end{cases} \rightarrow \vec{x} = \vec{0} \text{ is a trivial solution.}$$

Rank ($\text{rank}(M)$) is # of leading 1's in it's RREF.

Theorem:

1) System is consistent iff rank(coefficient matrix) = rank(augmented)
2) If consistent, parameters needed to express solutions is #variables − rank(coefficient)

Example:

1) $$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 - 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} -x_3 = x_1 + x_2 \\ -3x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_2 = -x_1 \\ x_3 = 0 \end{cases}$$

The set of solutions is $\left\{ \begin{pmatrix} x_1 \\ -x_1 \\ 0 \end{pmatrix} \middle| \, x_1 \in \mathbb{R} \right\}$

Applications:

1) Balancing Chemical Reactions

→ Manganese (Mn)

$$x_1 MnO_4^- + x_2 H_2C_2O_4 + x_3 H^+ \rightarrow x_4 Mn^{2+} + x_5 CO_2 + x_6 H_2O$$

stochiometric coefficients.

C: $2x_2 = x_5$

Mn: $x_1 = x_4$

O: $4x_1 + 4x_2 = 2x_5 + x_6$

H: $2x_2 + x_3 = 2x_6$

Charges: $-x_1 + x_3 = 2x_4$

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{5}{8} & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{5}{4} & 0 \end{array}\right)$$

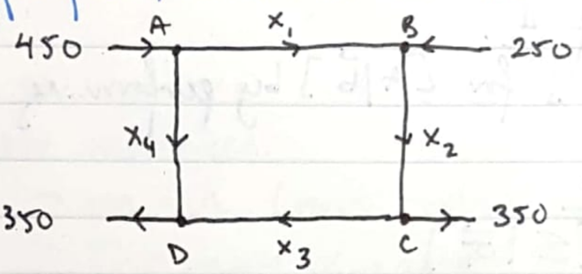
Rank = 5
Vars = 6.

Get $x_1 = \_ x_6$
Let $x_6 = 8$
Get coefficients

Application to Information Theory

Sep 28, 2017.

Traffic flow / communication network.



Each node (ABCD) = antenna.
Each → = one way com. channel
↳ capacity $x_1 x_2 \ldots {}_4$: info/sec
↳ unit: Bits! bit/s

*Applications week 2.

This assumes eq. state (No info gained/lost)

$$\left.\begin{array}{llcl} A: & 450 & = & x_1 + x_4 \\ B: & 250 + x_1 & = & x_2 \\ C: & x_2 & = & x_3 + 350 \\ D: & x_3 + x_4 & = & 350 \end{array}\right\} \begin{array}{l} x_1 + x_4 = 450 \\ -x_1 + x_2 = 250 \\ x_2 - x_3 = 350 \\ x_3 + x_4 = 350. \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 450 \\ -1 & 1 & 0 & 0 & 250 \\ 0 & 1 & -1 & 0 & 350 \\ 0 & 0 & 1 & 1 & 350 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 450 \\ 0 & 1 & 0 & 1 & 700 \\ 0 & 0 & 1 & 1 & 350 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

consistent ∞ soln.

Can be expressed as #vars − rank = 4 − 3 = 1 parameter needed.

Back substitution:

$$\left.\begin{array}{l} x_1 + x_4 = 450 \\ x_2 + x_4 = 700 \\ x_3 + x_4 = 350. \end{array}\right\} \begin{array}{l} x_1 = 450 - x_4 \quad x_4 \in \mathbb{R} \\ x_2 = 700 - x_4 \\ x_3 = 350 - x_4. \end{array}$$

All vars need +! can't have − info

Physical Solutions:

$$\begin{pmatrix} 450 - t \\ 700 - t \\ 350 - t \\ t \end{pmatrix}, 0 \le t \le 350$$

When does a m eq. × n vars matrix have a solution no matter what the constant terms are? $[A|\vec{b}]$, $\vec{b} \in \mathbb{R}^m$

Chapter 3: Matrices

2 matrices are equal when all entries are same.

Special Matrices:

Square matrices:

→ $m = n$ (rows = cols).

→ The main diagonal:

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \quad A_{ii}, 1 \le i \le n$$

→ Upper triangular / Lower triangular

→ All entries below main diagonal are zeroes.

$$\begin{pmatrix} 1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 4 & 5 & 2 \end{pmatrix}$$

→ Diagonal:

→ Both upper and lower triangular

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad diag(d_1, d_2, d_3 \ldots d_n)$$

Vectors:

Sep 27, 2017.

→ $n \times 1$ matrices

Operations on Matrices:

Let A, B be $m \times n$ matrices, s be a scalar.

$(A+B)_{ij} = A_{ij} + B_{ij}$ * both need to be $n \times n$

$(sA)_{ij} = s(A_{ij})$

Theorem:

Let A, B, C be $m \times n$ matrices. $s, t \in \mathbb{R}$.

1) Closure under addition:
   $A+B$ is $m \times n$ matrix.
2) Comm. of addition:
   $A+B = B+A$
3) Assoc. of addition:
   $(A+B)+C = A+(B+C)$
4) Zero Matrix: $\mathbb{O}_{m\times n}$ Only 0 entries.
   $A + \mathbb{O}_{m\times n} = A$.
5) $-A$.
   $A + (-A) = \mathbb{O}_{m\times n}$
6) Closure under scaler multiplication:
   sA is $m\times n$.
7) Assoc. of sc. multi:
   $s(tA) = (st)A$
8) Distributivity:
   $(s+t)A = sA + tA$
9) Dist. 2:
   $s(A+B) = sA + sB$

Transposition:

Let A be $\underline{m\times n}$ [ ]. A^T is a $\underline{n\times m}$ [ ]

$$A_{ij} = A^T_{ji}$$

$$\underset{A}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}} \qquad \underset{A^T}{\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}}$$

Rows become cols, cols become rows

Rows are transposes of vectors.

Properties of Transposition:

Let A, B be $m \times n$ [ ]. Let $s \in \mathbb{R}$.

1) $(A^T)^T = A$. Involution
2) $(A+B)^T = A^T + B^T$
3) $(sA)^T = s(A^T)$

Proof of 1):

$((A^T)^T)_{ij} = (A^T)_{ji} = A_{ij}$

Proof of 2):

$((A+B)^T)_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = A^T + B^T$

Proof of 3):

$((sA)^T)_{ij} = (sA)_{ji} = s(A_{ji}) = s(A^T)$

Multiplication of Matrices:

Let $a_1 \ldots a_n \in \mathbb{R}$. 1) $a_1 + a_2 \ldots + a_n = \sum_{k=1}^{n} a_k$, and...

2) $a_1 \times a_2 \times a_n = \prod_{k=1}^{n} a_k$

Let B be $\boxed{m \times n}$ [ ] $\bar{w}$ rows:

$$(b_{11}, \ldots, b_{1n}) = \vec{b}_1^T$$
$$\vdots \qquad\qquad \vdots$$
$$(b_{m1}, \ldots b_{mn}) = \vec{b}_m^T$$

$(m \times \underline{n} [\ ])(\underline{n} \times p [\ ]) = (m \times p [\ ])$

must be same!

Let A be $\boxed{n \times p}$ [ ] $\bar{w}$ columns

$$\vec{a}_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} \cdots \vec{a}_p = \begin{pmatrix} a_{1p} \\ \vdots \\ a_{np} \end{pmatrix}$$

Then, BA is the $\boxed{m \times p}$ matrix whose ij^{th} entry is:

$$(BA)_{ij} = \vec{b}_i \cdot \vec{a}_j = \sum_{k=1}^{n} (B_{ik})(A_{kj})$$

Scanned by CamScanner

Examples:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & -3 & -4 \end{pmatrix} \quad [2\times 4] \qquad B = \begin{pmatrix} -2 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -2 \end{pmatrix} \quad [4\times 2]$$

$$D = \begin{pmatrix} 0 \\ 1 \end{pmatrix} [2\times 1] \qquad C = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} [2\times 2] \qquad E = (1, 0) \; [1\times 2]$$

1) AB works
2) BA works
3) CD works
4) DC doesn't work
5) EC works
6) CE doesn't work.

Sep 28, 2017

$$AB: \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & -3 & -4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -2 \end{pmatrix}$$

$\boxed{\vec{a}_i \cdot \vec{b}_j}$ for $1 \le i \le m$: & for $1 \le j \le n$:

$$\begin{pmatrix} (1)(-2) + (2)(-1) + (3)(0) + (4)(1) & (1)(1) + (2)(0) + 3(-1) + 4(-2) \\ (-1)(-2) + (-2)(-1) + (-3)0 + (-4)(1) & (-1)(1) + (-2)0 + (-3)(-1) + (-4)(-2) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -10 \\ 0 & 10 \end{pmatrix}$$

Properties: (assume product is defined)

$AB = AC$ doesn't imply $B = C$

$AB \neq BA$ (necessarily)

$A(B+C) = AB + AC$

$t(AB) = (tA)B = A(tB)$

* $(AB)^T = B^T A^T$ Careful. Let A $m\times n$ [ ] and B $n\times p$ [ ], AB defined

↳ A^T — $n\times m$, B^T — $p\times n$ $1 \le i \le p$ and $1 \le j \le m$

$$((AB)^T)_{ij} = AB_{ji} = \sum_{k=1}^{m} (B)_{ki}(A)_{jk} = \sum_{k=1}^{m} (B^T)_{ik}(A^T)_{jk} = (B^T A^T)_{ij}$$

$$\underset{[n\times 1]}{\text{Let } \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} \in \mathbb{R}^n \quad \vec{y} = \underset{[n\times 1]}{\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}} \in \mathbb{R}^n$$

$$\underset{[1\times n]\cdot[n\times 1]}{(\vec{x}^T)\cdot\vec{y}} = (x_1, x_2 \dots x_n)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = x_1y_1 + \dots + x_ny_n = \vec{x}\cdot\vec{y}$$

Definition:

Identity matrix: $n\times n$ $I_n = \text{diag}(1, \dots 1_n) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

- If no ambiguity, $I_n = I$.
- $I_m A = A I_m = A$.

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

Inverse Matrices:

Let A be $n\times n$ $\square$. If there exists an unique $n\times n$ $\square$ B such that $\boxed{AB = BA = I_n}$, then A is invertible with inverse B. (A^{-1})

- Assume not unique. $BA = AB = I = CA = AC = I$.
- $B = BI = B(AC) = (BA)C = IC = C$ $\therefore$ unique.
- If $AB = I_N$, then $BA = I_n$ and $B = A^{-1}$
- Any invertible matrix of size n has max rank n.

Theorem:

Assume A and B are invertible of size n. Let $t \in \mathbb{R}$, $t \neq 0$.

need to state
1) tA is invertible and $(tA)^{-1} = \frac{1}{t}\cdot A^{-1}$
2) AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
3) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Never use A^{-1} before justifying A is invertible.

1) $(tA)\cdot(\frac{1}{t})(A^{-1}) = (t\cdot\frac{1}{t})(A\cdot A^{-1}) = I$
2) $(AB)(B^{-1}A^{-1}) = A[(B)(B^{-1})]A^{-1} = I$
3) $(A^T)\cdot(A^{-1})^T = (A^{-1}A)^T = I^T = I$.

Sep 24, 2017

Invertibility:

If $A\,[\,]$ $B\,[\,]$ such that $AB = I$, then this does NOT imply that A is invertibe w inverse B.

$\rightsquigarrow$ Only holds iff A, B are $\square$. ✱

Example:

1) Consider $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$ $\boxed{\text{BUT}}$ $BA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq I_2$

Proposition:

Let A, B $_{m \times n}$ $[\,]$ where $\forall \vec{x} \in \mathbb{R}^n$:

$A\vec{x} = B\vec{x}$, Then $A = B$

Proof:

Consider $\vec{e}_i$, $1 \leq i \leq n$:

$\vec{e}_i$: $\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ ← i^{th} entry is given. To avoid ambig, $\vec{e}^{\,i} = \vec{e}_i$

$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{1m} & \cdots & a_{mn} \end{bmatrix}$, Then $\underbrace{A\vec{e}^{\,i}}_{[m \times n] \cdot [n \times 1]} \in \mathbb{R}^m$ ← $[m \times 1]$ vector

$= \begin{pmatrix} \vec{a}_1^{\,T} \\ \vdots \\ \vec{a}_m^{\,T} \end{pmatrix}$

the j^{th} component is: $\vec{a}_j \cdot \vec{e}^{\,i} = (a_{j1} \cdots a_{jn}) \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ ← ith

$= a_{j1}(0) + \dots + a_{ji}(1) + \dots + a_{jn}(0)$

$= a_{ji}$ ← This shows:

$A\vec{e}^{\,i} = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ji} \\ \vdots \\ a_{mi} \end{pmatrix} = i^{th}$ column in A. Means all colums for A, B must be equal (i can be any col #).

Finding the Inverse:

Consider n eqns, n vars system.

$$\left.\begin{array}{l} a_{11}x_1 + \dots\ a_{1n}x_n = b_1 \\ \vdots \qquad\qquad\qquad \vdots \\ a_{n1}x_1 + \dots\ a_{nn}x_n = b_n \end{array}\right\}$$ Let A be coef. [ ], $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{R}^n$

Then the system can be rewritten as $A\vec{x} = \vec{b}$.

If A^{-1} exists, $\vec{x} = A^{-1}\vec{b}$

Inverse Matrix Algorithm:

Let A be $\square$

Reduce the matrix $[A|I]$ so A is in RREF.

$*$ 1) If the left block (A) is not I, then A is not invertible.

2) If the form is $[I|B]$, then B is the inverse and A^{-1} exists.

Example:

1) Let $A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$.

$$\left(\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right)_{1R_2 - \frac{2}{3}R_1} \sim \left(\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 0 & \frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right)_{1\frac{1}{3}R_1} \sim \left(\begin{array}{cc|cc} 1 & \frac{4}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right)_{1R_1 - 4R_2} \sim$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 3 & -4 \\ 0 & \frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right)_{1R_2 \cdot 3} \sim \left(\begin{array}{cc|cc} 1 & 0 & 3 & -4 \\ 0 & 1 & -3 & 3 \end{array}\right)$$

$= I \quad \therefore$ A is invertible and is the inverse.

Say "from the I.M.A" A is invertible and $A^{-1} = B$.

Properties of Inverse Matrixes:

Oct 3, 2017

Let A $m \times m$ □. Then, the following statements are equiv.

If one is true, all are true.

i) A is invertible
ii) A has rank m
iii) The RREF of A is I_m
iv) Given any $\vec{b} \in \mathbb{R}^m$, $A\vec{x} = b$ is consistent and has a unique solution
v) Columns of A from linearly independenent vectors.

Proof of iv) $\Rightarrow$ v):

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{m1} & \cdots & a_{mm} \end{pmatrix} = [\vec{a}_1 \ldots \vec{a}_m]$$

where $\vec{a}_i$ is i^{th} col in A.

Let $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{R}^m$

Consider $A\vec{x} = \vec{0}$:

$$\left.\begin{matrix} a_{11} x_1 & \cdots = 0 \\ \vdots & \\ a_{m1} x_1 & \cdots = 0 \end{matrix}\right\} = \sum_{i=1}^{n} \vec{a}_i x_i = \vec{0}$$

came from $\vec{a}_1$

linear independence, opitimized

Example:

1) Let A be a □ where $A^3 = I$. Show A is invertible and determine it's inverse.

$A \times A \times A = I$ Let $D = A^2$

then $AD = I$, where AD are □ (multi is defined).

$\therefore$ The inverse of A is A^2.

2) Let A, B □. Show if A and B are invertible $\Leftrightarrow$ AB is invertible.

$\Rightarrow$ Assume A, B invertible. Then:

$$(AB)(B^{-1}A^{-1}) = (A)(BB^{-1})(A^{-1}) = AIA^{-1} = AA^{-1} = I.$$

Since $B^{-1}A^{-1}$ invertible, $(AB)^{-1}$ is invertible $\bar{w}$ inverse $B^{-1}A^{-1}$!

$\Leftarrow$ Assume AB invertible. Then, $\exists\, \square\, C \ni (AB)C = I,\ C(AB) = I$.
Since $(AB)C = I$, $A(BC) = I$, so A is invertible.
Same w/ B. $A^{-1} = BC$, $B^{-1} = CA$.

Linear Mappings:

Functions are mappings/transformations. If f has dom U, codom V:

$$f : U \longrightarrow V \qquad \Big\{\ f : x \in U \longmapsto f(x) \in V.$$
$$x \longmapsto f(x)$$

maps to

transforms

↳ Range is attainable codomain.

$$f: \begin{array}{c} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^2 \end{array} \qquad g: \begin{array}{c} [-1,1] \to \mathbb{R} \\ x \mapsto x^2 \end{array}$$ Two dif. fcns.

	dom	cod	range
f	$\mathbb{R}$	$\mathbb{R}$	$[0,\infty]$
g	$[-1,1]$	$\mathbb{R}$	$[0,1]$

Matrix Mappings:

Let A be $m \times n$ [ ].

$$f_A : \begin{array}{c} \mathbb{R}^n \to \mathbb{R}^m \\ \underbrace{\underset{[n\times 1]}{\vec{x}} \longmapsto \underset{[m\times n]}{A\vec{x}}}_{\text{need to highlight that } A\vec{x} \text{ exists!}} \end{array}$$

{ Definition of product of 2 matrices shows that matrix mapping f_A is welldefined.

VERY IMPORTANT

Example:

1) Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$. Then $f_A(\vec{x}) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \\ 5x_1 + 6x_2 \end{pmatrix}$ $\mathbb{R}^3$!

Given $m,n \geq 1$, mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear if
1) $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$, $f(\vec{x}+\vec{y}) = f(\vec{x}) + f(\vec{y})$
2) $\forall t \in \mathbb{R}$, $\forall \vec{x} \in \mathbb{R}^n$, $f(t\vec{x}) = t f(\vec{x})$.

A linear operator is a linear mapping whose domain = codomain
↳ $\mathbb{R}^n \to \mathbb{R}^n$ ✱ $[0,1] \longrightarrow \mathbb{R}$ not linear.
↳

Oct 4, 2017

Linearity Continued:

Very similar properties to subspaces.

The following mappings are linear:

a) Given $\vec{v} \in \mathbb{R}^n$, $\text{proj}_{\vec{v}} : \mathbb{R}^n \to \mathbb{R}^n$
$\vec{x} \mapsto \text{proj}_{\vec{v}}(\vec{x})$ ← and perp

→ B/c $\text{proj}_{\vec{v}}(\vec{x}+\vec{y}) = \text{proj}_{\vec{v}}(\vec{x}) + \text{proj}_{\vec{v}}(\vec{y})$
$\text{proj}_{\vec{v}}(t\vec{x}) = t\,\text{proj}_{\vec{v}}(\vec{x})$ } $\vec{v} \neq \vec{0}$

b) Given $\vec{v} \in \mathbb{R}^n$, $S_{\vec{v}} : \mathbb{R}^n \to \mathbb{R}$
$\vec{x} \mapsto \vec{v} \cdot \vec{x}$. ← and cross product $\mathbb{R}^3$!

→ Same properties for dot product.

Not Linear:

a) $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$
$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \sqrt{x_1^2 + \dots x_n^2}$

If linear, $\|\vec{x}+\vec{y}\| = \|\vec{x}\| + \|\vec{y}\|$ ← Δ ineq, tho
→ $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

→ Give counter example w $\vec{e}_1, \vec{e}_2$.

1) Any linear mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ always maps $\vec{O}_n \to \vec{O}_m$ $\binom{t=}{0}$

2) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Let $\vec{x}_1, \dots \vec{x}_k \in \mathbb{R}^n$, $t_1, \dots t_k \in \mathbb{R}$,

→ $f(t_1\vec{x}_1 + \dots t_k\vec{x}_k) = t_1 f(\vec{x}_1) + \dots + t_k f(\vec{x}_k)$

→ $f\left(\sum_{i=1}^{k} t_i \vec{x}_i\right) = \sum_{i=1}^{k} t_i f(\vec{x}_i)$

→ mapping of lin. comb is lin comb. of mapping.

Proof of 2: $k=3$

$f(t_1\vec{x}_1 + t_2\vec{x}_2 + t_3\vec{x}_3) = f(t_1\vec{x}_1 + t_2\vec{x}_2) + f(t_3\vec{x}_3)$
$= f(t_1\vec{x}_1) + f(t_2\vec{x}_2) + f(t_3\vec{x}_3)$
$= t_1 f(\vec{x}_1) + t_2 f(\vec{x}_2) + t_3 f(\vec{x}_3)$

Just example not really proof.

Theorem:

Let $A = m \times n[\,]$, then,

$f_A : \mathbb{R}^n \to \mathbb{R}^m$
$\vec{x} \longmapsto A\vec{x}$ is linear..

Proof: Let $\vec{x}, \vec{y} \in \mathbb{R}^n$, $t \in \mathbb{R}$.

1) $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$
2) $A(t\vec{x}) = t(A\vec{x})$.

Yet Another Theorem:

Let $A = m \times n[\,]$, mapping $f_A : \mathbb{R}^n \to \mathbb{R}^m$.

→ f_A is uniquely determined by the values it takes on the standard basis $(\vec{e}_1, \dots \vec{e}_n) \in \mathbb{R}^n$.

→ $f_A(\vec{x}) = x_1 f_A(\vec{e}_1) + \dots x_n f_A(\vec{e}_n)$, or

$\sum_{k=1}^{n} x_k f_A(\vec{e}_k)$.   components of $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

→ Proof of above:

$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \dots + x_n\vec{e}_n$, so
$f_A(\vec{x}) = x_i f_A(\vec{e}_i)$.

Consider $y = ax$. $\left.\begin{array}{l} f : \mathbb{R} \to \mathbb{R} \\ x \longmapsto ax \end{array}\right\}$ is linear! iff for $\mathbb{R}^1$ That's why it's called linear ö.

→ Conversely, if $g : \mathbb{R} \to \mathbb{R}$ is linear, $\exists\, a \in \mathbb{R}$ (depends on g), where $g(t) = at$, $t \in \mathbb{R}$.

$g(t) = g(t \times 1) = t\,g(1) = g(1)\,t$

Let $a = g(1)$

$f: \mathbb{R} \to \mathbb{R}$ linear $\Leftrightarrow \exists \alpha \in \mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto \alpha x$ $(\alpha = f(1))$ Oct 5, 2017.

So... ↳ $f: \mathbb{R}^n \to \mathbb{R}^m$ linear $\Leftrightarrow \exists A(m \times n)$, $f_A: \mathbb{R}^n \to \mathbb{R}^m$, $\vec{x} \mapsto A\vec{x}$

both ways!

Theorem:
Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then, can be represented as a matrix mapping where $[L]_{m \times n}$:

$$[L] = [L(\vec{e}_1), L(\vec{e}_2), L(\vec{e}_3), \dots, L(\vec{e}_n)]$$

$$L: \mathbb{R}^n \to \mathbb{R}^m$$
$$\vec{e}_i \mapsto L(\vec{e}_i)$$
$$\vec{x} \mapsto L(\vec{x}) = [L]\vec{x}.$$

Proof:
Let $\vec{x} \in \mathbb{R}^n$ be:

$$\vec{x} = x_1 \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \dots + x_n \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} = \sum_{i=0}^{n} x_i \vec{e}_i.$$

By linearity: $L(\vec{x}) = L\left(\sum_{i=1}^{n} x_i \vec{e}_i\right) = \sum_{i=1}^{n} x_i L(\vec{e}_i)$

$\therefore$, $[L]\vec{x} = L(\vec{x})$.

↳ From this we can conclude linearity $\Leftrightarrow$ matrix mapping

Example:

1) Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$, then $C_{\vec{v}}: \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{x} \mapsto \vec{v} \times \vec{x} = [C_{\vec{v}}]\vec{x}$. is linear

Only works for linear mappings.

$$C_{\vec{v}}(\vec{e}_1) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \vec{e}_1 = \begin{pmatrix} 0 \\ v_3 \\ -v_2 \end{pmatrix}$$

$$C_{\vec{v}}(\vec{e}_2) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \vec{e}_2 = \begin{pmatrix} -v_3 \\ 0 \\ v_1 \end{pmatrix}$$

$$C_{\vec{v}}(\vec{e}_3) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \vec{e}_3 = \begin{pmatrix} v_2 \\ -v_1 \\ 0 \end{pmatrix}$$

thus, $[C_{\vec{v}}] = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$

$\therefore$, $\vec{v} \times \vec{x} = [C_{\vec{v}}]\vec{x}$

2) $S_{\vec{v}} : \mathbb{R}^n \to \mathbb{R}$ where $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$

$\vec{x} \longmapsto \vec{v} \cdot \vec{x}$

Let $1 \leq i \leq n$. $S_{\vec{v}}(\vec{e}_i) = \vec{v} \cdot \vec{e}_i = v_i$

$\therefore [S_{\vec{v}}] = [v_1, v_2, \ldots, v_n] = \vec{v}^T$.

$\therefore$ For all $\vec{x} \in \mathbb{R}^n$, $\vec{v} \cdot \vec{x} = [v_1, \ldots, v_n]\vec{x} = \vec{v}^T \vec{x}$.

3) $proj_{\vec{v}}$ $\mathbb{R}^n \to \mathbb{R}^n$ where $\vec{v} \neq \vec{0}$.

$\vec{x} \longmapsto proj_{\vec{v}}(\vec{x})$

$$proj_{\vec{v}}(\vec{e}_i) = \frac{\vec{e}_i \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v} = \frac{1}{\|\vec{v}\|^2} \begin{pmatrix} v_i v_1 \\ \vdots \\ v_i v_n \end{pmatrix}$$

$$\therefore [proj_{\vec{v}}] = \frac{1}{\|\vec{v}\|^2} \begin{pmatrix} v_1 v_1 & \cdots & v_n v_n \\ \vdots & & \vdots \\ v_1 v_n & \cdots & v_n v_n \end{pmatrix}$$

$$[proj_{\vec{v}}]_{ij} = \frac{v_i v_j}{\|\vec{v}\|^2}$$

*Try: prove $[proj_{\vec{v}}] = \frac{1}{\|\vec{v}\|^2} \vec{v}(\vec{v}^T)$

Definitions:

(1) Let $L : \mathbb{R}^n \to \mathbb{R}^m$, $M : \mathbb{R}^n \to \mathbb{R}^m$ be linear mappings.

$L+M : \mathbb{R}^n \to \mathbb{R}^m$ ← sum of each mapping (matrices)

$\vec{x} \longmapsto (L+M)\vec{x} = L(\vec{x}) + M(\vec{x})$

(2) Let $t \in \mathbb{R}$.

$tL : \mathbb{R}^n \to \mathbb{R}^m$

$\vec{x} \longmapsto (tL)(\vec{x}) = tL(\vec{x})$

(3) Let $N : \mathbb{R}^m \to \mathbb{R}^p$ be another linear mapping.

$N \circ L : \mathbb{R}^m \to \mathbb{R}^p$

$\vec{x} \longmapsto (N \circ L)(\vec{x}) = N(L(\vec{x}))$

$\mathbb{R}^m$ must be same w/ L. (like Matrix multi)

*Watch: $\mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \xrightarrow{N} \mathbb{R}^p$

These 3 end up still being linear. *Try: Prove.

Proposition:

1) Matrix sum of lin. maps = sum of indv. matrices.
$[L+M] = [L] + [M]$

2) Matrix of sc. multiple of a lin map = sc. multiple of matrix.
$[tL] = t[L]$

Proof:

1) Let $\vec{x} \in \mathbb{R}^n$: $[L+M]\vec{x} = (L+M)\vec{x} = L(\vec{x}) + M(\vec{x}) = [L]\vec{x} + [M]\vec{x}$

↳ Implies $([L]+[M])\vec{x} = [L+M]\vec{x}$, QED.

↳ Try 2).

THE ANSWER TO ALL YOUR PROBLEMS

Theorem:

Let $L: \mathbb{R}^n \to \mathbb{R}^m$, $N: \mathbb{R}^m \to \mathbb{R}^p$ be two linear mappings.
$[N \circ L] = [N][L]$ ← THIS IS WHY MATRIX MULTI EXISTS!!

↳ Not defined when $[N][L]$ is not defined!!!

Proof:

Let $[L]_{m \times n}$, $[N]_{p \times m}$ where $[N][L]$ is well defined.

Let $\vec{x} \in \mathbb{R}^n$: $[N \circ L](\vec{x}) = (N \circ L)\vec{x} = N(L(x))$
$= N([L]\vec{x}) = [N]([L]\vec{x})$
$= ([N][L])\vec{x}$

Example:

1) Let $L: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \xrightarrow{\mathbb{R}^2 \to \mathbb{R}^2} \begin{pmatrix} 2x_1 + 3x_2 \\ x_1 - x_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$N: \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2 \longmapsto \begin{pmatrix} y_1 + y_2 \\ -3y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

Proposition:

1) Matrix sum of lin. maps = sum of indv. matrices.
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Proof:

1) Let $\vec{x} \in \mathbb{R}^n$: $[L+M]\vec{x} = (L+M)\vec{x} = L(\vec{x}) + M(\vec{x}) = [L]\vec{x} + [M]\vec{x}$ ✓
↳ Implies $([L]+[M])\vec{x} = [L+M]\vec{x}$, QED.
↳ Try 2).

THE ANSWER TO ALL YOUR PROBLEMS

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Let $L: \mathbb{R}^n \to \mathbb{R}^m$, $N: \mathbb{R}^m \to \mathbb{R}^p$ be two linear mappings.
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$= N([L]\vec{x}) = [N]([L]\vec{x})$
$= ([N][L])\vec{x}$

Example:

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$f_A: \mathbb{R}^n \to \mathbb{R}^m$, $\vec{x} \mapsto A\vec{x}$; $f_B: \mathbb{R}^m \to \mathbb{R}^p$, $\vec{y} \mapsto B\vec{y}$ then $\left\{ f_B \circ f_A = f_{BA}: \mathbb{R}^n \to \mathbb{R}^p, \ \vec{x} \mapsto (BA)\vec{x} \right.$

So... → Next page
↳ Must first check that both are linear.

Oct 12, 2017

Let $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. Then...

$\rightarrow N(\vec{a}+\vec{b}) = N(a_1+b_1, a_2+b_2) = \begin{pmatrix} a_1+a_2 \\ -3a_2 \end{pmatrix} + \begin{pmatrix} b_1+b_2 \\ -3b_2 \end{pmatrix}$

$= N(\vec{a}) + N(\vec{b}) \leftarrow$

$\rightarrow N(t\vec{a}) = N(ta_1, ta_2) = t\begin{pmatrix} a_1+a_2 \\ -3a_2 \end{pmatrix}$

$= tN(\vec{a}) \leftarrow$ therefore, N is linear.

Same thing w/ L, both are linear.

[N][L] gives you:

$$\begin{bmatrix} 3 & 2 \\ -3 & 3 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = [N \circ L]\vec{x}$$

Notation:

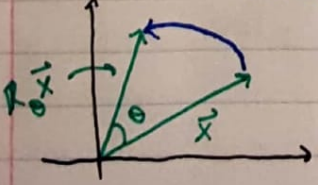
$L(\vec{x}) = L(x_1, x_2, \ldots, x_n)$. (Horizontal for convenience) MIDTERM LINE

Applications:

Week 1: Coordinate System
Week 2: Networks and chemical reactions } Also on midterm
Week 3 & 4: Excercise
Week 4 & 5: X

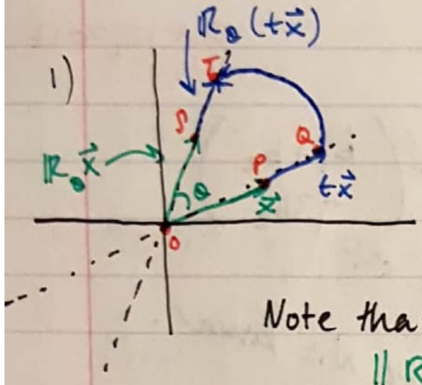
Linear Geometric Transformations: Rotations in the Plane:

Let $\theta \in \mathbb{R}$. The rotation in the plane $R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the mapping which maps $\vec{0} \rightarrow \vec{0}$ and $\vec{x}$ to $R_\theta(\vec{x})$.
↳ Rotates vector $\vec{x}$ counter-c'wise θ.



Is $R_\theta(\vec{x})$ linear?

1)



Since O, P, Q are collinear, so are their images O, S, T.

$\Rightarrow \therefore$, $R_\theta(\vec{x})$ and $R_\theta(t\vec{x})$ are collinear.

$\Rightarrow \exists s \in \mathbb{R} \mid R_\theta(t\vec{x}) = s \cdot R_\theta(\vec{x})$ ← assumes $\neq \vec{0}$.

Note that rotations preserve length.

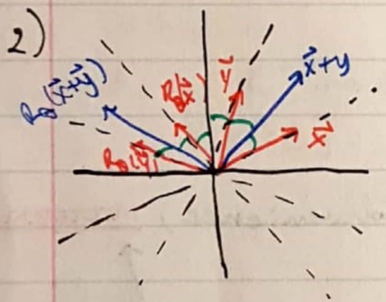
$\| R_\theta(t\vec{x}) \| = \| t\vec{x} \| = |t| \cdot \|\vec{x}\|$

So: $\| s \cdot R_\theta(\vec{x}) \| = |s| \cdot \| R_\theta \vec{x} \| = |s| \cdot \|\vec{x}\|$ } b/c $\vec{x} \neq 0$

$\therefore |s| = |t|$.

Since rotation preserves orientation, $s = t$ QED. $R_\theta(t\vec{x}) = tR_\theta(\vec{x})$

2)



Rotations also preserve angles. ✱

→ Since angles are same, shape of $\|$ogram is conserved. ✱ length

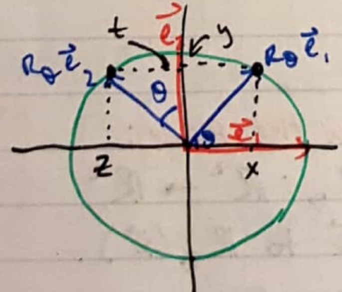
$\therefore \| R_\theta(\vec{x}+\vec{y}) \| = \| R_\theta(\vec{x}) \| + \| R_\theta(\vec{y}) \|$. $\therefore$ linear.

But is θ same? Looong proof, but true.

Std Matrix of R_θ:

Let $[R_\theta] = \begin{pmatrix} x & z \\ y & t \end{pmatrix}$

$R_\theta(\vec{e}_1) = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad$ $R_\theta(\vec{e}_2) = \begin{pmatrix} z \\ t \end{pmatrix}$



$x = \cos\theta$

$y = \sin\theta$

$z = \cos(\theta + \frac{\pi}{2}) = -\sin\theta$

$t = (\sin(\theta + \frac{\pi}{2})) = \cos\theta$

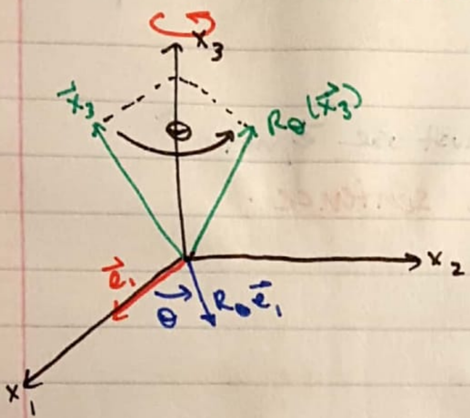
R_θ is linear

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Rotations in $\mathbb{R}^3$:

$R_\theta : \mathbb{R}^3 \to \mathbb{R}^3$

1) $R_\theta(\vec{0}) = \vec{0}$
2) Any vector on x_3 axis is mapped to itself.
3) Any vector not on x_3 axis to $R_\theta(\vec{x})$
   → counterclockwise around x_3 axis



$x_1 - x_2$ plane is invariant (stays in that plane)

Theorem:

$$[R_{\theta_3}] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection in Hyperplane $\mathbb{R}^3$:

Oct 25, 2017

Let H be a hyperplane through origin in $\mathbb{R}^n$.
Let $\vec{m}$ be normal to H. $\vec{m}\cdot\vec{x} = d$ for some $d \in \mathbb{R}$.

→ Since $\vec{0} \in H$, $\vec{m}\cdot\vec{0} = 0$, so $d = 0$.

→ H: $\vec{m}\cdot\vec{x} = 0$.

→ The reflection (or "across") H is the transformation:

$\text{refl}_{\vec{m}} : \mathbb{R}^n \to \mathbb{R}^n$

$\vec{x} \longmapsto \vec{x} - 2\text{proj}_{\vec{m}}(\vec{x}) = Id - 2\text{proj}_{\vec{m}}$

$\vec{x} \mapsto \vec{x} = [I]\vec{x}$

Sum of linear maps, ∴ refl. is linear mapping

Std Matrix of Refl:

$$[\text{refl}_{\vec{m}}] = [\text{Id} - 2\,\text{proj}_{\vec{m}}]$$
$$= [\text{Id}] - 2[\text{proj}_{\vec{m}}]$$
$$= I - 2[\text{proj}_{\vec{m}}]$$

where $[\text{proj}_{\vec{m}}] = \dfrac{(\vec{m})(\vec{m})^T}{\|\vec{m}\|^2}$ (don't need to remember)
→ But need to be able to prove.

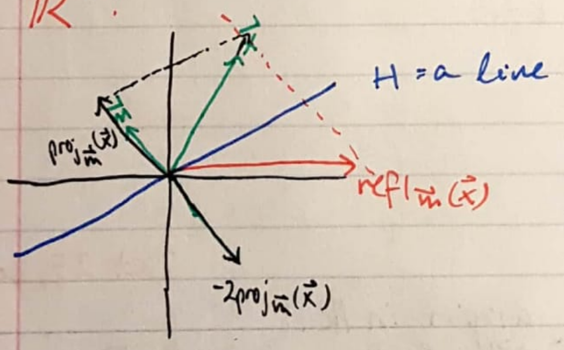
Example:

1) $\text{refl}_{\vec{m}} : \mathbb{R}^3 \to \mathbb{R}^3$ where $\vec{m} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$ Must use $\vec{e}$, not ↑

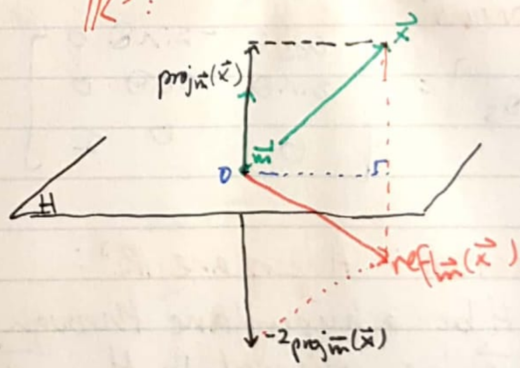
* Let $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \ldots \in \mathbb{R}^3$ Need this sentence.

$\text{refl}_{\vec{e}_1} = \begin{pmatrix} 21/29 \\ 12/29 \\ -16/29 \end{pmatrix}$ etc.

$\mathbb{R}^2$:



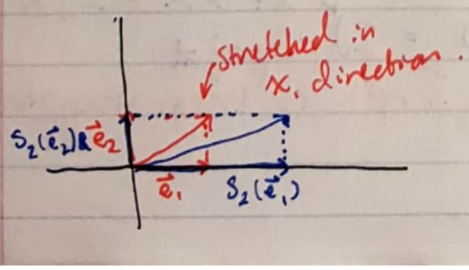
$\mathbb{R}^3$:



Other Transformations:

x_1 dir: Stretch: $t > 0$. $S_t : \mathbb{R}^2 \to \mathbb{R}^2$ $[S_t] = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$. "Shrink" if $t < 1$.

x_2 dir: Stretch: $t > 0$. $S'_t : \mathbb{R}^2 \to \mathbb{R}^2$ $[S'_t] = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$ "Shrink" if $t < 1$.



$t>0$. $T_t : \mathbb{R}^2 \to \mathbb{R}^2$ where $[T_t] = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$

→ Contraction if $t<1$ ⎫
→ Dilation if $t>1$ ⎭ $\vec{x} \mapsto t\vec{x}$

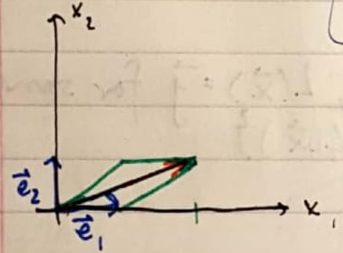
Oct 26, 2017

Shears:

In x_1 dir by s: $\Sigma_s : \mathbb{R}^2 \to \mathbb{R}^2$

$[\Sigma_s] = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ x_2: $\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$

$s=2$: $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$



Inverse Linear Mappings:

L. $\overset{n\to n}{\text{map}}$ invertible if $\exists$ map $\mathbb{R}^n \to \mathbb{R}^n$ where

$L \circ M = M \circ L = Id$

Proof that M is also linear:

$(L \circ M)(\vec{x}) = \vec{x} = (M \circ L)(\vec{x})$

$\vec{x} + \vec{y} = (L \circ M)(\vec{x}) + (L \circ M)(\vec{y})$

$= L\left(M(\vec{x}) + M(\vec{y})\right)$

So, $M(\vec{x}+\vec{y}) = M\left(L(M(\vec{x}) + M(\vec{y}))\right) = (M \circ L)(M(\vec{x}) + M(\vec{y}))$

$= M(\vec{x}) + M(\vec{y})$

And...

$t(\vec{x}) = t\left(L(M(\vec{x}))\right) = L(t(M(\vec{x})))$

So, $M(t(\vec{x})) = (M \circ L)(t M(\vec{x})) = t M(\vec{x})$.

$\therefore$ linear

M is inverse of L iff [L] is inverse of [M]

↳ Means inverse mapping is unique

Example:

1) Consider $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

R_θ is invertible with inverse $R_{-\theta}$.

$$[R_\theta][R_{-\theta}] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Mapping Ranges:

$L : \mathbb{R}^m \to \mathbb{R}^n$. Range(L): All vectors $\vec{y} \in \mathbb{R}^n$, $L(\vec{x}) = \vec{y}$ for some $\vec{x}$.

↳ $\text{Range}(L) = \{\vec{y} \in \mathbb{R}^n : \exists \vec{x} \in \mathbb{R}^m \ni \vec{y} = L(\vec{x})\}$

Null Space:

Null(L): All vectors mapped to $\vec{0}$.

↳ $\text{Null}(L) = \{\vec{x} \in \mathbb{R}^m : L(\vec{x}) = 0\}$ ★never empty ($\vec{0} \to \vec{0}$)

Column Space:

A : $m \times n$ [ ] w/ columns $\vec{a}_1, \dots \vec{a}_n \in \mathbb{R}^m$.

↳ Column space of a is subspace ($\mathbb{R}^m$) spanned by columns of A:

$\text{span}\{\vec{a}_1, \dots \vec{a}_n\} = \{t_1\vec{a}_1 + t_2\vec{a}_2 + \dots + t_n\vec{a}_n : t_{1 \dots n} \in \mathbb{R}\}$.

Linear Mapping Stuff: $L : \mathbb{R}^n \to \mathbb{R}^n$ $[L] = A$. Inv. Matrix/Mapping theorem

i) A is invertible

ii) $\text{Rank}(A) = n$

iii) $RREF = I$

iv) $\forall \vec{b} \in \mathbb{R}^n$, $A\vec{x} = b$ consistent w/ unique

v) Cols in A are lin. independent

vi) Col Space $(A) = \mathbb{R}^n$

vii) L is invertible linear mapping

viii) $\text{Null}(L) = \{\vec{0}\}$ $L(\vec{x}) = 0 \Rightarrow \vec{x} = 0$.

ix): $\text{Range}(L) = \mathbb{R}^n$: $\forall \vec{y} \in \mathbb{R}^n, \exists \vec{x} \in \mathbb{R}^n, \vec{y} = L(\vec{x})$

Invertibility Examples: Oct 27, 2017

1) Prove $proj_{\vec{v}} : \mathbb{R}^n \to \mathbb{R}^n$ is not invertible.

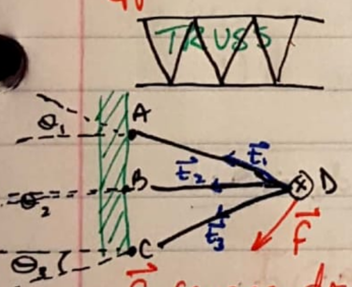
$\because$ More than one vector perp to $\vec{x}$, making Null more than $\{\vec{0}\}$. $\therefore$ not invertible. $\vec{w}\cdot\vec{v}=0$ but $\vec{w}\neq 0$.

2) Stretch by $t>0 : \mathbb{R}^2 \to \mathbb{R}^2$

$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$ invertible w/ inverse $\begin{pmatrix} \frac{1}{t} & 0 \\ 0 & 1 \end{pmatrix}$

3) Refl is inverse of itself.

Applications: Pin Jointed Planar Trusses   Oct 27/31, 2017



Consider: 3 rigid members, 4 pinjoints ABCD.

- Free to rotate at the pin-joints
- Fastened to wall at ABC.

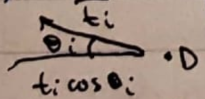
$\vec{F}$ causes displacement $\vec{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ at D. $\vec{d}$ rel. to $\vec{F}$? $\vec{F} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$

A few assumptions:

- Weight is negligible wrt $\vec{F}$
- $\underline{\vec{F}}$ causes compression force (contract/dilate) on each member, t_i = compression where:

$$t_i = \begin{cases} \|\vec{t}_i\| & \text{if contraction} \\ -\|\vec{t}_i\| & \text{if dilation} \end{cases}$$

1) Resolution of forces at D.



Horizontally: $f_1 = t_1\cos\theta_1 + t_2\cos\theta_2 + t_3\cos\theta_3$

Vertically: $f_2 = t_1\sin\theta_1 + t_2\sin\theta_2 + t_3\sin\theta_3$

$$\vec{F} = A\vec{t} : \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 & \cos\theta_2 & \cos\theta_3 \\ \sin\theta_1 & \sin\theta_2 & \sin\theta_3 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

2) Relate t_i to e_i, (extension).

Let e_i (a length) be the compression/extension due to t_i (force).

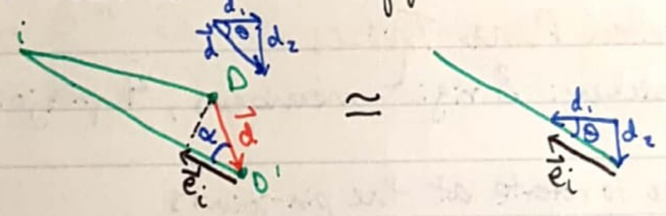
Hooke's Law: $e = kt$ (backwards from physics) b/c ~~k~~ $= k^{-1}$.

k^{-1} is spring constant.

$\vec{e} = k\vec{t}$: $\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$ Assume $k_{123} \neq 0$.

Thus, k is invertible w/ inverse $\begin{pmatrix} k_1^{-1} & 0 & 0 \\ 0 & k_2^{-1} & 0 \\ 0 & 0 & k_3^{-1} \end{pmatrix}$. so $\vec{t} = k^{-1}\vec{e}$.

3) Relate $\vec{e}$ to $\vec{d}$. Assume $\vec{d}$ is small enough so, in first approximation, can be considered to be supported on each member.



$\vec{d} = \vec{e}_i$

Assumes α is very small. $\boxed{e_i = d_1 \cos\theta_i + d_2 \sin\theta_i}$

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ \cos\theta_2 & \sin\theta_2 \\ \cos\theta_3 & \sin\theta_3 \end{pmatrix}}_{A^T} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$ so $\vec{e} = (A^T)\vec{d}$.

Conclusion:

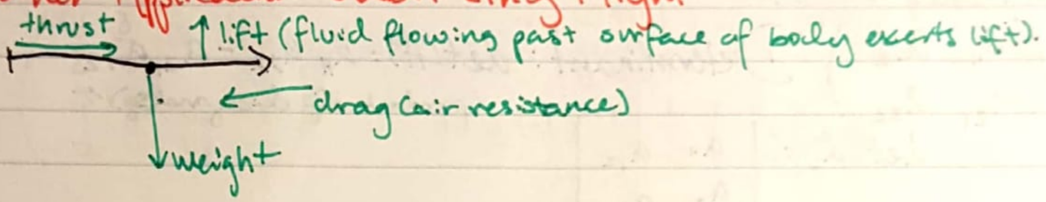
$\vec{f} = A\vec{t} = A(k^{-1}\vec{e}) = Ak^{-1}A^T\vec{d}$ Let $B = Ak^{-1}A^T$

So $\vec{f} = B\vec{d}$ where $B = Ak^{-1}A^T$ (2×2).

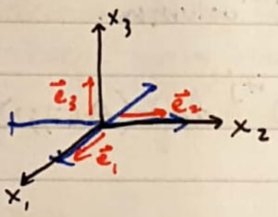
If B is invertible, $\boxed{\vec{d} = B^{-1}\vec{f}}$.

Another Application: Stabilizing Flight:

(sideways plane)



To take off, thrust > drag so lift can take over weight.



- Nose @ tip of $\vec{e}_2$ (initially).

To keep constant direction:

1) A <u>yaw</u> through θ_1. Rotation in x_1, x_2 plane, fixing x_3 axis.

$$Y_{\theta_1} = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2) A <u>pitch</u> through θ_2. Rotation in x_2, x_3 plane, fixing x_1 axis.

$$P_{\theta_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_2 & -\sin\theta_2 \\ 0 & \sin\theta_2 & \cos\theta_2 \end{pmatrix}$$

3) A <u>roll</u> through θ_3. Rotation in x_1, x_3 plane, fixing x_2 axis.

$$R_{\theta_3} = \begin{pmatrix} \cos\theta_3 & 0 & -\sin\theta_3 \\ 0 & 1 & 0 \\ \sin\theta_3 & 0 & \cos\theta_3 \end{pmatrix}$$

The "wrong" nose position $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\vec{x} = (R_{\theta_3} P_{\theta_2} Y_{\theta_1})\vec{e}_2$ Let $T(\theta_1, \theta_2, \theta_3)$ = That thing.

($\vec{e}_2$ is "correct" position)

If it's invertible, autopilot applies the <u>inverse transformation</u> T^{-1}!

↳ Each rotation is invertible, so T is too. Each inverse is the transpose.

$$T(\theta_1, \theta_2, \theta_3)^{-1} = Y_{\theta_1}^T P_{\theta_2}^T R_{\theta_3}^T$$
$$= (R_{\theta_3} P_{\theta_2} Y_{\theta_1})^T$$

Determinants (2x2):

Nov 1, 2017

2x2

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

1) Determinant: $\det(A) = a_{11} a_{22} - a_{21} a_{12}$ (product of diagonals)

$$\det(a) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

3x3

$$a = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Let $A(i,j)$ be 2x2 submatrix obtained by removing i^{th} row, j^{th} column.

The cofactor of a_{ij} is then: $c_{ij} = (-1)^{i+j} \det(A(i,j))$

Example:

1) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$$c_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5 - 8 = -3$$

$$c_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} = 9 - 7 \times 3 = -12$$

Determinants (3x3):

$\det(A) = a_{11} c_{11} + a_{12} c_{12} + a_{13} c_{13}$

Example:

1) $$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \underbrace{1}_{a_{11}} \underbrace{(-1)^{1+1} \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}}_{c_{11}} + \underbrace{2}_{a_{12}} \underbrace{(-1)^{1+2} \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}}_{c_{12}} + \underbrace{3}_{a_{13}} \underbrace{(-1)^{1+3} \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}}_{c_{13}} = 0$$

Generalized Determinants (nxn):

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \qquad c_{ij} = (-1)^{i+j} \det A(i,j)$$

$$\det A = a_{11} c_{11} + a_{12} c_{12} + \dots + a_{1n} c_{1n}$$

$$= \sum_{k=1}^{n} a_{1k} c_{1k}$$

Example:

1) $A = \begin{pmatrix} \underline{1} & \underline{3} & \underline{6} & \underline{10} \\ 2 & 5 & 9 & 13 \\ 4 & 8 & 12 & 15 \\ 7 & 11 & 14 & 16 \end{pmatrix}$ $\det A = \underline{1}(-1)^{1+1}\begin{vmatrix} 5 & 9 & 13 \\ 8 & 12 & 15 \\ 11 & 14 & 16 \end{vmatrix} + \underline{3}(-1)^{1+2}\begin{vmatrix} 2 & 9 & 13 \\ 4 & 12 & 15 \\ 7 & 14 & 16 \end{vmatrix} + \underline{6}(-1)^{1+3}\begin{vmatrix} 2 & 5 & 13 \\ 4 & 8 & 15 \\ 7 & 11 & 16 \end{vmatrix}$

$+ \underline{10}(-1)^{1+4}\begin{vmatrix} 2 & 5 & 9 \\ 4 & 8 & 12 \\ 7 & 11 & 14 \end{vmatrix} = 6$

Things to Note:

1) Determinant of nxn [ ] is a linear combination of [ ] $n-1 \times n-1$.
   → keeps going recursively to show it can be expressed as lin comb. of |2x2| det
2) Cofactor expansion of first row.

Theorem:

Let A be nxn[ ]:

$$\begin{array}{c c} & j^{th} \text{ column} \\ \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} & \begin{array}{l} \\ \\ i^{th} \text{ row} \\ \\ \\ \end{array} \end{array}$$

can use any row or column i, j.

The determinant can be obtained by cofactor expansion by any i, j.

$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots a_{in}C_{in}$
$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots a_{nj}C_{nj}$ } can use any row/col

→ Can choose most $\emptyset$ row/col.

Diagonal/Triangular Matrix Determinants: Nov 2, 2017.

Let A n×n [ ] be diagonal or upper/lower Δ.

Then det A = product of diagonal entries.

$$\begin{vmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{vmatrix} = \begin{vmatrix} a_1 & & \\ & \ddots & \\ 0 & & a_n \end{vmatrix} = \begin{vmatrix} a_1 & & 0 \\ & \ddots & \\ & & a_n \end{vmatrix}$$

$$= \prod_{i=1}^{n} a_i = a_1 a_2 \ldots a_n.$$

Proof: (Start w/ upper Δ)

Base case: $n=2$ $\quad \begin{vmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{vmatrix} = a_{11}a_{22} - 0 \times a_{12} = a_{11}a_{22}$

IH: Assume $n \geq 3$: by cofactor expansion along first (last) row:

$$\Delta: \begin{vmatrix} a_{11} & & & x \\ & \ddots & & \vdots \\ 0 & & & x \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = a_{nn}(-1)^{n+n} \begin{vmatrix} a_{11} & & \\ & \ddots & \\ 0 & & a_{n-1,n-1} \end{vmatrix}$$

By IH: $\begin{vmatrix} a_{11} & & \\ & \ddots & \\ 0 & & a_{n-1,n-1} \end{vmatrix} = a_{11} \cdots a_{n-1,n-1}$

So $\Delta = (a_{nn})(1)(a_{11} \cdots a_{n-1})$. QED.

(Similar for lower Δ and $\swarrow$). Would be last col or row for lower Δ

Properties of the Determinant:

Let A, B be n×n [ ], $r \in \mathbb{N}$. Assume B=A but one row is × by r.

- → All zero column/row ⇒ determinant = 0.
- → $\det A = \det A^T$
- → $\det B = r(\det A)$
- → $\det rA = r^n \det A$ (means det map not linear)

If you swap two rows in A to get B:

- → $\det B = -\det A$

If a has 2 identical rows/columns:

- → $\det A = 0$

If you get B by adding multiples of rows of A:

→ det B = det A

Reducing to Upper Triangular:

1) Swapping rows/colums changes sign
2) Adding rows ($R_1 = R_2 + 2R_2$) doesn't change det.

*Indicate row/column used for cofactor expansion with →

$$\begin{array}{r} \rightarrow \\ \\ \\ \end{array}\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Example:

1)

$$\begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ -1 & 0 & 3 & 1 \\ 3 & 1 & 2 & 0 \end{vmatrix}_{R_3+R_1} = \begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 2 & 4 & 1 \\ 3 & 1 & 2 & 0 \end{vmatrix}_{R_4-3R_1} = \begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & -5 & -1 & 0 \end{vmatrix}_{C_2-5C_3} =$$

$$\begin{vmatrix} 1 & -3 & 1 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & -18 & 4 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix}_{R_3-9R_2} = \begin{vmatrix} 1 & -3 & 1 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -5 & -8 \\ 0 & 0 & -1 & 0 \end{vmatrix}_{\boxed{C_3 \leftrightarrow C_4}} = -\begin{vmatrix} 1 & -3 & 0 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -8 & -5 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 16.$$

Assume A is invertible. $AA^{-1} = I$

$\det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1}) = 1.$

$\therefore, \det A = \frac{1}{\det A^{-1}}$ $\det A \neq 0$ if A is invertible.

↳ check for invertibility.

We know A invertible $\Rightarrow \det A \neq 0$, but $\Leftarrow$? Ya.

Invertibility:

A $n \times n[\,]$:

iff
1) A invertible (+ all those properties that follow)
2) $\det(A) \neq 0$
3) $\operatorname{rank}(A) = n$

Cramer's Rule (Not yet):

→ Cofactor Matrix:

$\operatorname{cof}(A) = n \times n[\,]$ Just take cofactor of every entry in A.

$$A \cdot \operatorname{cof}(A)^T = (\det A) I$$

Sketch of Proof:

Let $A = n \times n[\,]$, $B = A \cdot \operatorname{cof}(A)^T$

$$(B)_{ij} = \sum_{k=1}^{n} (A)_{ik} (\operatorname{cof}(A)^T)_{kj}$$
$$= \sum_{k=1}^{n} (A)_{ik} (\operatorname{cof}(A))_{jk}$$
$$= \sum_{k=1}^{n} a_{ik} C_{jk}$$

If $i = j$, $B_{ii} = \sum_{k=1}^{n} a_{ik} C_{ik} = \det A$. (expanded on i^{th} row)

If $i \neq j$, according to False expansion theorem, $a_{ik} C_{jk} = 0$.

$\therefore B = \operatorname{diag}(\det A, \ldots, \det A)$ QED

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Example:

1) If $n=3$ $i=2$ $j=3$, by False Expansion Theorem,

$$A \cdot cof(A)^T = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$a_{21}C_{31} + a_{22}C_{32} + a_{23}C_{33} = 0$ where a_{ij} from coef. matrix A.

More Invertibility:

Let A $n \times n$ [ ] $\ni$ $\det A \neq 0$.

$\hookrightarrow$ A is invertible $\bar{w}$ inverse:

$$A^{-1} = \frac{cof(A)^T}{\det A}$$ } comes from prev theorem

But... Gaussian >>>>>>>>> except $n=2$.

Gaussian: $O(n^3)$

Cofactor Formula: $O(n!)$

Example:

1) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Assume $\det(A) \neq 0 = ad - bc$

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Systems $\bar{w}$ Invertible Matrices:

$A\vec{x} = \vec{b}$

$\vec{x} = A^{-1}\vec{b}$ } $\vec{x} = \frac{cof(A)^T \vec{b}}{\det A}$

Now Cramer's Rule:

for $1 \le i \le n$,

$$x_i = \frac{\det N_i}{\det A}$$ N_i = A with i^{th} column replaced with $\vec{b}$.

Nov 8, 2017

Example:

1) $\left.\begin{aligned} a_1x+b_1y+c_1z&=d_1\\ a_2x+b_2y+c_2z&=d_2\\ a_3x+b_3y+c_3z&=d_3\end{aligned}\right\}A\vec{t}=\vec{d}$

Assume A is invertible ($\det A\neq 0$)

$\therefore,\quad x=\dfrac{\begin{vmatrix} d_1 & b_1 & c_1\\ d_2 & b_2 & c_2\\ d_3 & b_3 & c_3\end{vmatrix}}{\det A}\quad y=\dfrac{\begin{vmatrix} a_1 & d_1 & c_1\\ a_2 & d_2 & c_2\\ a_3 & d_3 & c_3\end{vmatrix}}{\det A}\quad z=\dfrac{\begin{vmatrix} a_1 & b_1 & d_1\\ a_2 & b_2 & d_2\\ a_3 & b_3 & d_3\end{vmatrix}}{\det A}$

*Only use when $n=2$.

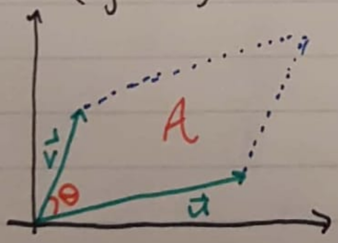
2) $7x_1+2x_2=1$
$4x_1+1x_2=5$

A is invertible since $\det A=-1$.

$x_1=\dfrac{\begin{vmatrix}1&2\\5&1\end{vmatrix}}{-1}=9\qquad x_2=\dfrac{\begin{vmatrix}7&1\\4&5\end{vmatrix}}{-1}=-31$

Geometric Interpretation of the Determinant:

2x2 $A=\begin{pmatrix}x&z\\y&t\end{pmatrix}\quad\vec{u}=\begin{pmatrix}x\\y\end{pmatrix}\quad\vec{v}=\begin{pmatrix}z\\t\end{pmatrix}$ *Assume non-zero, $\vec{u}\neq\vec{v}$



$\text{Area}(A)=\|\vec{u}\|\cdot\|\vec{v}\|\cdot|\sin\theta|$
$=\|\vec{u}\|\cdot\|\vec{v}\|\cdot\sqrt{1-\cos^2\theta}$
$=\|\vec{u}\|\cdot\|\vec{v}\|\cdot\sqrt{1-\left(\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)^2}$ Cauchy!
$=\sqrt{\|\vec{u}\|^2\cdot\|\vec{v}\|^2-(\vec{u}\cdot\vec{v})^2}\quad |\vec{u}\cdot\vec{v}|\leq\|\vec{u}\|\|\vec{v}\|$
$=\sqrt{(x^2+y^2)(z^2+t^2)-(xz+yt)^2}$
$=\sqrt{x^2t^2+y^2z^2-2xzyt}$
$=|xt-zy|$ DETERMINANT

2x2: Thus, $|\det A|$ = Area of $\|^{\text{ogram}}$ spanned by column vectors
↳ $|\det A|=|\det A^T|$. Thus, it works for row vectors too.

$\boxed{3\times 3}$ When $A = 3\times 3\,[\;]$, $|\det A|$ = volume of p'ppiped spanned by column vectors $\vec{u}, \vec{v}, \vec{w}$.

$$|\vec{w}\cdot(\vec{u}\times\vec{v})| = |\det A|$$

Nov 9, 2017

$\boxed{n\times n}$ **General Case:**

Let $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$. Object spanned is an n-dimensional parallelotope.

$$n\text{-Volume}(\vec{v}_1, \ldots, \vec{v}_n) = |\det A|$$

where $A = [\vec{v}_1, \ldots, \vec{v}_n]$

Volume with Linear Mappings:

n-Volume of $L(\vec{v}_1), \ldots, L(\vec{v}_n)$ is $|\det(L(\vec{v}_1), \ldots, L(\vec{v}_n))|$

Let A be the standard mtx of L. $[L] = A$.

$$n\text{-volume}(L(\vec{v}_1), \ldots, L(\vec{v}_n)) = |\det(A\vec{v}_1 \ldots A\vec{v}_n)| = |\det B|$$

Note $B = AC$ where $C = (\vec{v}_1, \ldots, \vec{v}_n)$ * one of the definitions of matrix multi'

$$\{\ |\det B| = |\det AC| = |\det A||\det C|$$

Thus, $n\text{-Volume}(L(\vec{v}_1), \ldots, L(\vec{v}_n)) = |\det A| \cdot n\text{-Volume}(\vec{v}_1, \ldots, \vec{v}_n)$

Other Stuff:

1) When solving $A\vec{x} = \vec{0}$ (A is $n\times n\,[\;]$),
   trivial solution unique $\Leftrightarrow \det A \neq 0 \Leftrightarrow A$ invertible
   $\det A = 0$ means infinite solutions.

2) Sarrus' Rule:

$$\det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{matrix} & & & - & - & - \\ a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \\ & & + & + & + \end{matrix} = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$$

* Only for 3x3 matrices   Faustin's god-like method: Dodgson's C. Method

Change of Basis for a Linear Mapping:

Recall: lin independence, spanning, bases.

Bases Example: $\vec{v}_1 = \binom{4}{3}$ $\vec{v}_2 = \binom{1}{1}$ is basis of $\mathbb{R}^2$.

1) Lin independence:

$$t_1\vec{v}_1 + t_2\vec{v}_2 = 0$$

$$4t_1 + t_2 = 0,\ 3t_1 + t_2 = 0$$

$$M \left\{ \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix} \right. \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

det $A \neq 0$, so only soln is trivial b/c $\vec{t} = \vec{0}M^{-1} = \vec{0}$.

2) Spans $\mathbb{R}^2$:

Show $\exists\, \alpha, \beta \in \mathbb{R} \ni \vec{x} \in \mathbb{R}^2 = \alpha\vec{v}_1 + \beta\vec{v}_2$.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

det $A \neq 0$, so M is invertible

$$\text{so, } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = M^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

$\alpha = x_1 - x_2$

$\beta = -3x_1 + 4x_2$

Notes about bases:

1) If $\{\}$ spans $\mathbb{R}^n$, $|\{\}| \geq n$.
2) If $\{\}$ lin independent in $\mathbb{R}^n$, $|\{\}| \leq n$.

$\therefore$ Bases have n elements. (how dimension is defined)

Changing the Basis:

Nov 10, 2017

1) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$

$$\vec{x} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} -x_1 + 4x_2 \\ -3x_1 + 6x_2 \end{pmatrix} \qquad [L] = \begin{bmatrix} -1 & 4 \\ -3 & 6 \end{bmatrix}$$

basis $\{\vec{e}_1, \vec{e}_2\}$

$$L(\vec{e}_1) = \underline{(-1)}\,\vec{e}_1 + \underline{(-3)}\,\vec{e}_2$$

$$L(\vec{e}_2) = \underline{(4)}\,\vec{e}_1 + \underline{(6)}\,\vec{e}_2$$

Now, consider basis $(\vec{v}_1, \vec{v}_2)$ of $\mathbb{R}^2$ where $\vec{v}_1 = \binom{4}{3}$, $\vec{v}_2 = \binom{1}{1}$

$$L(\vec{v}_1) = L\binom{4}{3} = \binom{8}{6} = \underline{2}\,\vec{v}_1 + \underline{0}\,\vec{v}_2$$

$$L(\vec{v}_2) = L\binom{1}{1} = \binom{3}{3} = \underline{0}\,\vec{v}_1 + \underline{3}\,\vec{v}_2$$

Here, $[L]$ in basis $\{\vec{v}_1, \vec{v}_2\} = B$, $[L]_B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

Summary of Basis Changing:

Let $L: \mathbb{R}^n \to \mathbb{R}^n$ be linear. Let $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis of $\mathbb{R}^n$.

$[L]_B$ = the i^{th} column of $[L]_B$ contains the components of $L(\vec{v}_i)$ decomposed in B.

Assume x is decomposed in B as follows:

$$\vec{x} = t_1\vec{v}_1 + \dots + t_n\vec{v}_n$$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = [L]_B \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} \text{ where}$$

$$L(\vec{x}) = y_1\vec{v}_1 + y_2\vec{v}_2 + \dots + y_n\vec{v}_n$$

Example:

Let $\vec{x} = \vec{v}_1 - \vec{v}_2$

$$= \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3\vec{e}_1 + 2\vec{e}_2$$

$$L(\vec{x}) = [L]\vec{x} = \begin{bmatrix} -1 & 4 \\ -3 & 6 \end{bmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

And $[L]_B \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

$\rightsquigarrow$ So, $L(\vec{x}) = 2\vec{v}_1 - 3\vec{v}_2$.

Relation between these Matrices $[L]$, $[L]_B$:

Consider $P = [\vec{v}_1, \vec{v}_2]$. $\det P = 1$, $P^{-1} = \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix}$

$$P^{-1}[L]P = \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix}\underbrace{\begin{pmatrix} -1 & 4 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}}_{\begin{pmatrix} 8 & 3 \\ 6 & 3 \end{pmatrix}} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = [L]_B$$

Nov 14, 2017

Theorem:

Let $L: \mathbb{R}^n \to \mathbb{R}^n$ be linear. Let $B = \{\vec{v}_1 \dots \vec{v}_n\}$ be basis of $\mathbb{R}^n$.
Let $P = (\vec{v}_1, \dots, \vec{v}_n)$ Then, P is invertible and: from defn of basis/inv.

$$P^{-1}[L]P = [L]_B$$

Proof:

Let $\vec{x} = x_1\vec{e}_1 + \dots + x_n\vec{e}_n = t_1\vec{v}_1 + \dots + t_2\vec{v}_2$ *same things

$L(\vec{x}) = u_1\vec{e}_1 + \dots + u_n\vec{e}_n = y_1\vec{v}_1 + \dots + y_2\vec{v}_2$

$P^{-1}[L]P\begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}$ I ALOTTED TOO MUCH SPACE BUT = $[L]_B\begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}$

$$P^{-1}[L]\vec{x} = [L]_B\begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}$$

$$P^{-1}\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$LS = RS.$$

Similarity:

A, B $n \times n$ [ ] similar if $\exists$ invertible P where $B = P^{-1}AP$ } A, B rep. same lin mapping in 2 bases.

Both represent $L: \vec{x} \longmapsto A\vec{x}$ $(A = [L])(B = [L]_B)$

Example:

1) $A = \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}}_{[L]_B}$, $B = \underbrace{\begin{pmatrix} -1 & 4 \\ -3 & 6 \end{pmatrix}}_{[L]}$ similar since $B = P^{-1}AP$ with $P = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$

Diagonalization:

L is diagonalizable if $\exists$ basis B of $\mathbb{R}^n$ where $[L]_B$ is diagonal.

Example:

1) $L: \vec{x} \to \begin{bmatrix} -1 & 4 \\ -3 & 6 \end{bmatrix}\vec{x}$ is diag$^{\text{ble}}$ in basis $\left\{\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

*note $L(\vec{v}_1) = 2\vec{v}_1$ and $L(\vec{v}_2) = 3\vec{v}_2$ since $[L]_B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

Diagonalizable Matrices:

A [ ] is diag^{le} if A is similar to a diagonal matrix D.
($\exists P$ where $A = P^{-1}DP$)
$\rightarrow$ D is a diagonal form for A.

Example:

1) A is $\begin{pmatrix} -1 & 4 \\ -3 & 6 \end{pmatrix}$. A is diag^{le} since:

$$\begin{pmatrix} -1 & 4 \\ -3 & 6 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$$

Eigenvalues and Eigenvectors:

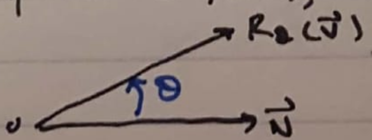
Let $L: \mathbb{R}^n \to \mathbb{R}^n$ be linear. A non-zero $\vec{v} \in \mathbb{R}^n$ is a e'vector to the e'val $\lambda \in \mathbb{R}$ if $L(\vec{v}) = \lambda \vec{v}$. $(\lambda, \vec{v})$ is an eigenvector

Example:

1) Same lin map L. $(2, \vec{v}_1)$ and $(3, \vec{v}_2)$ are eigenpairs.
$(L(\vec{v}_1) = 2\vec{v}_1$ and $L(\vec{v}_2) = 3\vec{v}_2)$

*Note: 0 can be an e'value, but never an e'vector.
↪ If $(\lambda, \vec{v})$ is e'pair for L, $L(\vec{v})$ is collinear to $\vec{v}$.
↪ If $\theta \in (0, 2\pi)$, then R_θ cannot have an e'pair with $\lambda \in \mathbb{R}$.

*$\theta \neq \pi$



} Only eigenvalue is $(-1, \vec{v})$.

Eigen___ $\bar{w}$ Matrices:
$A = n\times n[\,]$. A nonzero $\vec{v} \in \mathbb{R}^n$ is eigenvector with eigenvalue $\lambda \in \mathbb{R}$ if $A\vec{v} = \lambda\vec{v}$. $(\lambda, \vec{v})$ is e'pair for A.

Eigenspaces:
Let A $n\times n[\,]$ with eigenvalue $\lambda \in \mathbb{R}$. The e'space of A for λ is:
$$\mathcal{E}_A(\lambda) = \{\vec{v} \in \mathbb{R}^n \mid A\vec{v} = \lambda\vec{v}\}$$ *CONTAINS ZERO!

$\mathcal{E}_A(\lambda)$ is set of all e'vectors for λ with $\vec{0}$.
↳ E'space is a subspace! Must include $\vec{0}$. (try to prove).

Characteristic Polynomials:
Let A $n\times n[\,]$.
$$C_A(\lambda) = \det(A - \lambda I)$$

Example:
1) $A = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}$ $\quad A - \lambda I = \begin{pmatrix} -1-\lambda & 3 \\ 1 & 1-\lambda \end{pmatrix}$

$$C_A = \det\begin{pmatrix} -1-\lambda & 3 \\ 1 & 1-\lambda \end{pmatrix} = (-1-\lambda)(1-\lambda) - 3 = \lambda^2 - 4 = (\lambda-2)(\lambda+2)$$

E'spaces and Homogeneity.
Let λ be an e'value for $A[\,]_{n\times n}$.
$$\vec{v} \in \mathcal{E}_A(\lambda) \Leftrightarrow A\vec{v} = \lambda\vec{v}$$
$$\Leftrightarrow (A - \lambda I)\vec{v} = \vec{0}$$

*{ So, $\mathcal{E}_A(\lambda)$ is the set of solutions in $\vec{v} \in \mathbb{R}^n$ to:
$$(A - \lambda I)\vec{v} = \vec{0}$$

Characterization of E'values:

Let A $n \times n[\,]$. Then, the set of e'values of A is the set of roots of it's characteristic polynomial.

Example (continued...): Nov 15, 2017

1) If $A = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}$, then $C_A(\lambda) = (\lambda - 2)(\lambda + 2)$.

$\rightsquigarrow$ Thus, A has 2 e'values: $\lambda_1 = 2$, $\lambda_2 = -2$.

$\rightsquigarrow$ So, the e'space $\mathcal{E}_A(\lambda_1)$ = solve $(A - 2I)\vec{v} = \vec{0}$.

ing: $\mathcal{E}_A(2) = \{\vec{v} \in \mathbb{R}^n \mid (A - 2I)\vec{v} = \vec{0}\}$

$A - 2I = \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix} \underset{\mathbb{1}R_1 + 3R_2}{\sim} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ (RREF)

So, $\mathcal{E}_A(2) = \{\vec{v} \in \mathbb{R}^2 \mid v_1 - v_2 = 0\} = \{\vec{v} \in \mathbb{R}^2 \mid \vec{v} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbb{R}^2\}$

$\hookrightarrow = \operatorname{span}\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$

$\mathcal{E}_A(-2) = \{\vec{v} \in \mathbb{R}^n \mid (A + 2I)\vec{v} = \vec{0}\}$

$A + 2I = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$ (RREF)

So, $\mathcal{E}_A(-2) = \{\vec{v} \in \mathbb{R}^2 \mid \vec{v}_1 + 3\vec{v}_2 = 0\} = \{\vec{v} \in \mathbb{R}^2 \mid \vec{v} = \begin{pmatrix} -3v_2 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} \in \mathbb{R}\}$

$\rightarrow = \operatorname{span}\{\begin{pmatrix} -3 \\ 1 \end{pmatrix}\}$

Multiplicity:

Let A $n \times n[\,]$, λ_0 be e'value for A.

a.m.(λ_0) → Algebraic Multiplicity of λ_0 is exponent of $(\lambda - \lambda_0)$ in $C_A($

g.m.(λ_0) → Geometric Multiplicity of λ_0 is #free vars in $(A - \lambda_0 I)\vec{v} = 0$

#vars − rank, so $n - \operatorname{rank}(A - \lambda_0 I)$

Example (continued):

1) $C_A(\lambda) = (\lambda-2)^1(\lambda+2)^1$

$a.m.(2) = 1 = a.m(-2)$.

$g.m(2) = 1$ since $A-2I \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

$g.m(-2) = 1$ since $A+2I \sim \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$

Relation between Multiplicities:

Let A $n\times n$ [ ] $\bar{w}$ e'value λ_0.

$$1 \leq g.m(\lambda_0) \leq a.m.(\lambda_0)$$

Proof:

Let M be $n\times n$ [ ]. Since: M invertible $\Leftrightarrow$ $M\vec{x}=\vec{0}$ unique trivial soln $\Leftrightarrow$ $\overset{\text{rank}}{N}$

Assume λ_0 is e'value.

↪ Means $\exists$ $\boxed{\text{non zero}}$ $\vec{v}$ where $A\vec{v} = \lambda_0\vec{v}$.

↪ $\underbrace{(A-\lambda_0 I)}_{M}\vec{v} = \vec{0}$

Thus, M can't be invertible (there is a non-zero soln).

So, $\text{rank}(\underline{M}) < n \therefore g.m(\lambda_0) = n - \text{rank}(\underline{A-\lambda_0 I}) \geq 1$.

Deficient Eigenvalues:

If λ is e'value of A where:

$$g.m.(\lambda) < a.m.(\lambda)$$

Then λ is a deficient e'value.

Scanned by CamScanner

Why Should we Diagonalize?

Goal: Is A similar to a diag. matrix D?

→ Let A, B similar where $\underline{A} = P^{-1}\underline{B}P$

Then, A and B have:

Easy to check if A diag.
1) Same determinant
2) Same eigenvalues (and c'polynomial)
3) Same rank
4) Same $\underline{\text{trace}}$ = Sum of diagonal elements.

Nov 16, 2017

Proof of 1):

Assume $B = P^{-1}AP$ (P invertible)

$$\det B = \det(P^{-1})\det(A)\det(P) = \det(A)$$

Example:

1) $\begin{pmatrix} -1 & 4 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$

$B \quad = \quad P^{-1} \quad A \quad P$

Consider when A is diagonal. $A = (\mu_1 \ldots \mu_n)$

$$C_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} \mu_1 - \lambda & & 0 \\ & \ddots & \\ 0 & & \mu_n - \lambda \end{vmatrix} = \prod_{i=1}^{n} (\mu_i - \lambda)$$

e'values of diag matrix: diag entries.

a.m.(μ_i) = # of times they appear in diagonal.

Diagonalization Theorem Juan: ★Error

Let $A_{n \times n}$[ ]. A diag^{le} iff
$\exists$ basis $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ of $\mathbb{R}^n$ where:
$\rightarrow$ each $\vec{v}_i$ are e'vectors for A.
If so, $P = [\vec{v}_1, \dots \vec{v}_n]$ diag'izes A into diag form D where:
$D = \text{diag}(\lambda_1, \dots \lambda_n)$, where λ_i is e'value for $\vec{v}_i$.
$A = PDP^{-1}$★ MISTAKE

Diagonalization Theorem Too:

Let A $n \times n$[ ]. A diag^{le} iff:
1) Sum of a.m.'s of e'values $= n$
2) a.m $=$ g.m $\forall$ e'values. $(\alpha_j = n - \text{rank}(A - \mu_j I))$

Diagonalization Theorem Threeeeee:

Let A $n \times n$[ ]. A is diag^{le} if: single $\Rightarrow$
A has n pairwise distinct e'values $(\lambda_1 \dots \lambda_n)$
(unique)

Key Idea Behind Proof:

Assume $(\lambda_i, \vec{v}_i)$ e'pairs for A. (n e'pairs for $n \times n$ [ ]).
Assume also, λ_i's are unique.
$\rightarrow$ Then, $\{\vec{v}_1, \dots, \vec{v}_k\}$ are lin independent (basis)

Position of the Problem:

Let A $n \times n$[ ], diag^{le}. Let $\lambda_1 \dots \lambda_k$ be distinct e'values, each $\overline{w}$ a.m. $m_1 \dots m_k$. Then, A is similar to:

$$D = \text{diag}(\underbrace{\lambda_1, \dots, \lambda_1}_{m_1 \text{ times}}, \dots \underbrace{\lambda_k, \dots \lambda_k}_{m_k \text{ times}})$$

(where $\sum_{i=1}^{k} m_i = n$)

Determining "P":

Since A diag^{le}, am = gm $\forall$ e'value. Let λ_i be given e'value.

↳ Finding vectors in $\mathcal{E}_A(\lambda_i) \Rightarrow$ Solve $(A-\lambda_i I)\vec{v}=\vec{0}$.

↳ Has g.m m_i, so m_i free vars.

$$\vec{v} \in \mathcal{E}_A(\lambda_i) \Leftrightarrow (A-\lambda_i)I\vec{v}=0.$$

$$\Leftrightarrow \vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_{m_i} \\ \vdots \\ v_n \end{pmatrix}$$

} fix

} can be expressed as fcn of above

↳ Implies you can find m_i lin independent vectors in $\mathcal{E}_A(\lambda_i)$ (say, $\vec{v}_i^{(1)} \ldots \vec{v}_i^{(m_i)}$)

Then, $P = [\vec{v}_1^{(1)}, \ldots, \vec{v}_1^{(m_1)}; \vec{v}_2^{(1)}, \ldots, \vec{v}_2^{(m_2)}; \vec{v}_k^{(1)}, \ldots \vec{v}_k^{(m_k)}]$

Example:

Let $\alpha \in \mathbb{R}$. Let

$$A(\alpha) = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ \alpha & 1-\alpha & -1 \end{pmatrix}$$

} What α is $A(\alpha)$ diagonalizable?

1) C' Polynomial:

$$\det(A(\alpha)-\lambda I) = \begin{vmatrix} -\lambda & 1 & -1 \\ -1 & 1-\lambda & 0 \\ \alpha & 1-\alpha & -1-\lambda \end{vmatrix}$$

$$= (-1)(-1)^{1+3}\begin{vmatrix} -1 & 1-\lambda \\ \alpha & 1-\alpha \end{vmatrix} + (-1-\lambda)(-1)^{3+3}\begin{vmatrix} -\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix}$$

$$= -(\lambda^3+\alpha\lambda) = -\lambda(\lambda^2+\alpha)$$

1) If $\alpha < 0$, then $C_\alpha(\lambda) = -\lambda(\lambda-\sqrt{-\alpha})(\lambda+\sqrt{-\alpha})$

↳ D^{le} since C'P has 3 distinct roots (DT3).

2) If $\alpha > 0$, then only root $\lambda=0$ with a.m $=1$.

↳ Not D^{le} since a.m. $\neq n$.

3) If $\alpha = 0$, then $C_0(\lambda) = \lambda^3$

↳ D^{le} since rank$(A(0)-0I) \neq 0$.

$\therefore \alpha < 0$.

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Previous Example Continued... Nov 21, 2017

Let $\alpha = -1$.

$A = P^{-1} D P$ where $D = \mathrm{Diag}(0, 1, -1)$ $\Big\}$ $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ -1 & 2 & -1 \end{bmatrix}$

To find P:

$P = [\vec{v}_1, \vec{v}_2, \vec{v}_3]$

$\vec{v}_1$: A eivector in $E_A(0)$

$\vec{v}_2$: A eivector in $E_A(1)$

$\vec{v}_3$: A eivector in $E_A(-1)$

$\mathcal{E}_A(0) = \{\vec{v} \mid (A - 0I)\vec{v} = 0\}$

$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ -1 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

So, $\mathcal{E}_A(0)$ $\{\vec{v} \mid \begin{pmatrix} 3v_3 \\ v_3 \\ v_3 \end{pmatrix}\}$ $= \mathrm{span}\{\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}\}$

Let $\vec{v}_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$

Repeat for $E_A(1)$, $E_A(-1)$. Take $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 2/3 \\ 1/3 \\ 1 \end{pmatrix}$

Thus, $P = \begin{pmatrix} 3 & 0 & \frac{2}{3} \\ 1 & 1 & \frac{1}{3} \\ 1 & 1 & 1 \end{pmatrix}$

★ If a.m. not all 1?

$A_{3\times3}[\,]$, diag'le. $C_A(\lambda) = -\lambda(\lambda - 1)^2$

$D = \mathrm{Diag}(0, 1, 1)$

$P = [\vec{v}_1, \vec{v}_2, \vec{v}_3]$: $\vec{v}_1$: eivector in $\mathcal{E}_A(0)$

$\vec{v}_2, \vec{v}_3$: 2 eivectors in $E_A(1)$

↳ Linearly independent!

Ex: $E_A(1) = \{\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix} \in \mathbb{R}^3\}$

→ Let $v_1 = 0$, $v_2 = 1$ $\Big\}$ Usually gets 2 lin. independent

$v_2 = 0$, $v_1 = 1$

Markov Chains:

System evolves w̄ time. Let X_n be state after n time intervals.
Markov chain if X_{n+1} only depends on X_n.

Example:

1) Let $N = 110{,}000$ (population)
$n=0$, no one is sick.
At n, the prob of getting sick: 0.2. Recovering: 0.1.
Let $V_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ where x_n is healthy count, y_n sick count.
Let $V_0 = \begin{pmatrix} N \\ 0 \end{pmatrix}$.

$$\left\{\begin{array}{l} x_{n+1} = 0.8x_n + 0.1y_n \\ y_{n+1} = 0.2x_n + 0.9y_n \end{array}\right\} V_{n+1} = \underbrace{\begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}}_{\text{"transition matrix"}} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\boxed{V_n = A^k V_{n-k} = A^n V_0}$$

Assume A is diag^ble. $A = PDP^{-1}$, $D = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$

always ...e

Then: $A^n = (PDP^{-1})^n$
$= PD^nP^{-1}$ where $D^n = \begin{pmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{pmatrix}$

Answer: $P = \frac{2}{3}\begin{pmatrix} 1 & 1 \\ -1 & \frac{1}{2} \end{pmatrix}$ $D = \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix}$

$V_n = (P^{-1}D^nP)V_0$ * Mistake as well (when $P = [\vec{v}_1 \dots \vec{v}_n] \Rightarrow PDP^{-1}$

$$= \begin{pmatrix} \frac{1}{2} & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1^n & 0 \\ 0 & (0.7)^n \end{pmatrix}\begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} N \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \left[\frac{1}{3} + \frac{2}{3}(0.7^n)\right]N \\ \left[\frac{2}{3} - \frac{2}{3}(0.7)^n\right]N \end{pmatrix} \left\{ \text{As } n \to \infty: \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} N \right.$$

Bases of a Subspace:

→ Lin. Independent and Spanning vectors:

→ Vectors are lin. independent if $P = [\vec{v}_1, \vec{v}_2]$ is inv.

→ But, if P is not square ($2\ \vec{v} \in \mathbb{R}^3$):

→ How to determine when $k < n$ vectors lin. ind. $\in \mathbb{R}^n$?

Linear Independence (Revisited):

$\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$. These are lin. independent iff:

$\Leftrightarrow$ Coef. matrix of $t_1\vec{v}_1 + t_2\vec{v}_2 + \ldots + t_k\vec{v}_k = \vec{0}$ has rank k

→ $P = [\vec{v}_1, \ldots \vec{v}_k]$ has rank k.

Example:

1) Are $\vec{v}_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ lin independent?

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix} \sim \left.\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$$ Rank $= 2 = k$.

∴ lin independent.

Spanning (Revisited):

$\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$. These span $\mathbb{R}^n$ iff:

Defn of span:

$\Leftrightarrow$ coef. matrix of $t_1\vec{v}_1 + t_2\vec{v}_2 + \ldots t_k\vec{v}_k = \vec{b}$ has rank n no matter what $\vec{b}$ is.

→ $P = [\vec{v}_1, \ldots, \vec{v}_k]$ has rank n no matter $\vec{b}$.

Example:

1) $\vec{w}_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \vec{w}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \vec{w}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$P = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \left.\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$ Rank $= 3 = n$.

∴ Spanning.

Shortcut:

Let $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$. Then: *with n vectors

1) $\vec{v}_1, \ldots \vec{v}_n$ lin independent

$\Updownarrow$

2) $\vec{v}_1, \ldots, \vec{v}_n$ spans $\mathbb{R}^n$.

Thus, $\{\}$ is a basis of $\mathbb{R}^n$ if either is met. (*n vectors!) (since $k=n$)

→ Thus, prev example is lin independent too.

Example:

1.) Let $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$. Let $S = \text{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$. Then S is a subspace of $\mathbb{R}^n$.

1) $\vec{0} \in S$, let all t's $= 0$
2) $\vec{x} + \vec{y} \in S$, t's are distributed
3) $t\vec{x} \in S$, t is multiplied.

$S = t_1\vec{v}_1 + \ldots t_k\vec{v}_k$

∴, Subspace.

Note:

$\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$. Then, they span $S = \text{span}\{\vec{v}_{1 \ldots k}\}$

→ But, they don't have to be lin. independent.

2) $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \in \mathbb{R}^3$, $S = \text{span}\{\vec{v}_{1,2,3}\}$

Note that $\vec{v}_3 = 2\vec{v}_1 - 3\vec{v}_2$

Independence in Spanning Sets:

Let $\vec{v}_{1 \ldots k} \in \mathbb{R}^n$, $S = \text{span}\{\vec{v}_{1 \ldots k}\}$.

→ Then, any set of linearly independent vectors in S contains at most k elements.

Bases of Subspaces:

Let S be subspace. A basis of S is

→ a set of linearly independent vectors spanning S.

Nov 26, 2017

Dimension for Subspaces:

# of vectors in any basis of S.

Example:

1) Same vectors and S as before.

→ Let $\vec{x} \in S$, $\Rightarrow \exists\ t_1, t_2, t_3$ where

$\vec{x} = t_1\vec{v}_1 + t_2\vec{v}_2 + \underbrace{t_3\vec{v}_3}_{}$ Redundant

$= (t_1+2t_3)\vec{v}_1 + (1-3t_3)\vec{v}_2 \Rightarrow S = \text{Span}\{\vec{v}_1, \vec{v}_2\}$

$\vec{v}_1, \vec{v}_2$ are linear ind. since $[\vec{v}_1, \vec{v}_2]$ has rank 2.

→ Then, since $\vec{v}_1, \vec{v}_2$ lin. independent and span!, it is basis for S (dimension 2).

→ S: vector eqn: $\underbrace{\vec{x} = t_1\vec{v}_1 + t_2\vec{v}_2}_{\text{plane in } \mathbb{R}^3}$

Vector Spaces:

V. space over $\mathbb{R}$ is a set $\mathbb{V}$ w:

vectors $\in \mathbb{V}$ { → Addition operation $(\vec{x}+\vec{y}$, both $\in \mathbb{V})$
→ Scalar multi. operation $(s\vec{x}, \vec{x} \in \mathbb{V}, s \in \mathbb{R})$

→ Properties:

1) $\vec{x}+\vec{y} \in \mathbb{V}$
2) $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$
3) $\vec{0} \in \mathbb{V} \ni \vec{x}+\vec{0} = \vec{x}\ (\in \mathbb{V}) = \vec{0}+\vec{x}$
4) $\exists\ \vec{y} \in \mathbb{V} \ni \vec{x}+\vec{y} = \vec{y}+\vec{x} = \vec{0}$ $(\vec{y} = -\vec{x})$
5) $\vec{x}+\vec{y} = \vec{y}+\vec{x}$
6) $s\vec{x} \in \mathbb{V}$
7) $s(t\vec{x}) = st(\vec{x})$
8) $(s+t)\vec{x} = s\vec{x}+t\vec{x}$
9) $s(\vec{x}+\vec{y}) = s\vec{x}+s\vec{y}$
10) $1\vec{x} = \vec{x}$

(won't use these)

$\boxed{\star\ \mathbb{R}^n \text{ is a } \mathbb{V} \text{ space}}$

Remarks:
1) If $\mathbb{V}$ is not $\mathbb{R}^n$, vectors don't need →
2) Defn of span, lin ind, basis, subspace = same.

Subspaces of $\mathbb{V}$:
Are vector spaces too
Proof:
$\vec{0} \in S$, $\vec{x}+\vec{y} \in S$, $s\vec{x} \in S$.
→ Properties 1, 3, 6 hold. All other properties are true since S is subspace for $\mathbb{V}$.

(will use this) * whenever possible, show $\mathbb{V}$ by showing it's a subspace of a known $\mathbb{V}$ (like $\mathbb{R}^n$)

Polynomial Vector Spaces:
Let $d \geq 1 \in \mathbb{Z}$. Let P_d be set of polynomials $\underline{\deg(\leq d)}$

$$P_d = \sum_{i=0}^{d} a_i x^i$$ where a_i can be 0 ←

These are vector spaces (satisfy properties)
$\therefore P_d,\ d \geq 1$ is a vector space (polynomials are vectors)

Example: Nov 29, 2017

1)
$p_1(x) = x^3 + x^2 + 2x + 1$
$p_2(x) = x^3 + 3x^2 + x + 1$
P_3 $\quad p_3(x) = -3x^3 + 5x^2 + 5x + 3$
$p_4(x) = -2x^2 - x.$

$S = \text{span}\{p_1(x), p_2(x), p_3(x), p_4(x)\}$
↳ Subspace of P_3.
↳ Dimension? (Are they lin. ind.)

Lin ind: $t_1 p_1(x) + t_2 p_2(x) + t_3 p_3(x) + t_4 p_4(x) = 0$

$$x^3(t_1 + t_2 - 3t_3) + x^2(t_1 + 3t_2 + 5t_3 - 2t_4) + x(2t_1 + t_2 + 5t_3 - t_4) + (t_1 + t_2 + 3t_3) = 0$$

Each coefficient must $= 0$.

$$\left.\begin{array}{l} t_1 + t_2 - 3t_3 = 0 \\ t_1 + 3t_2 + 5t_3 - 2t_4 = 0 \\ 2t_1 + t_2 + 5t_3 - t_4 = 0 \\ t_1 + t_2 + 3t_3 = 0 \end{array}\right\} \underbrace{\begin{bmatrix} 1 & 1 & -3 & 0 \\ 1 & 3 & 5 & -2 \\ 2 & 1 & 5 & -1 \\ 1 & 1 & 3 & 0 \end{bmatrix}}_{A} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = 0$$

Row reduce until you cry.

$$A \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\therefore t_1 = t_2 = t_3 = t_4 = 0$.

$\therefore$ They are linearly independent ($\dim(S) = 4$)

$\therefore$ Basis of S.

2) Let $q(x) = 1 + 5x - 5x^2 + x^3$. Does it belong to S?

$\exists\, t_{1 \ldots 4} \ni t_i p_i(x) = q(x)$?

$$A \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -5 \\ 1 \end{pmatrix} \qquad [A|\vec{b}] \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \text{RANDOM} \\ 0 & 0 & 1 & 0 & \text{GARBAGE} \\ 0 & 0 & 0 & 1 & \end{array}\right)$$

RREF = Identity. $\therefore \vec{b}$ doesn't matter. So, yes, $q(x) \in S$.

Polynomial Dimensions:

P_d has dimension $d+1$.

Proof:

$P_d = \text{Span}\{x^0, x^1, \ldots, x^{d-1}, x^d\}$. $d+1$ elements in Basis of P_d.

Vector Spaces of Matrices:

Nov 30, 2017

$m, n \in \mathbb{Z} \geq 1$. Let:

$M(m,n) = \{$all $m \times n$ matrices $\bar{w}$ $\mathbb{R}$ entries$\}$

→ Vector space over $\mathbb{R}$. (Check 10 properties)

→ Dimension $= mn$.

→ A basis of $M(m,n)$ is the set of "elementary matrices". For $1 \leq i \leq m$ $1 \leq j \leq n$

↳ $(i,j)^{th}$ elem. matrix E_{ij} : All zero with 1 at i,j

Examples:

1) Let $n \geq 1$. Let $S(n \times n)$ be the set of $n \times n$ sym. matrices.

$$S(n \times n) = \{A \in M(n,n) \mid A^T = A\}.$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \therefore \begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} a & c \\ c & d \end{pmatrix}$$

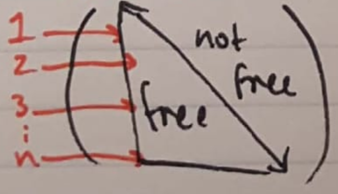
→ Prove 3 properties to show that $S(n \times n)$ is a vector space

1) $x + y =$ sym $(A+B)^T = A^T + B^T = A + B$ $(A, B \in S(n \times n))$

2) $sx =$ sym $(sA)^T = sA^T = sA$ $(A \in S(n \times n))$

3) $0 =$ sym. $\emptyset_{n \times n} = \emptyset_{n \times n}^T$.

#Free: ↳ Dimension? # of free vars.



$$\text{Dim} = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

2) $\Sigma(n \times n) = \{A \in M(n,n) \mid \det(A) = 0\}$. Show $\neg$ subspace of $M(n \times n)$

$x + y \neq \det 0$. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \det 1$.

3) Try: Set of matrices $\bar{w}$ 0 trace is v space.

Curve Fitting:

Given $n+1$ points, $M_i = (x_i, y_i)$ $(0 \le i \le n)$, to find a polynomial deg $\le n$ passing through these points.

Take $n=3$ and $: M\{(-1,-7), (0,-1), (1,-1), (2,5)\}$

$$\begin{array}{l} A \\ \text{coef.} \\ \text{matrix} \end{array} \underbrace{\begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{bmatrix}}_{[x^3,\ x^2,\ x^1,\ x^0]} = \begin{bmatrix} -7 \\ -1 \\ -1 \\ 5 \end{bmatrix}$$

* Backwards. A should be that big matrix, $\vec{a}$ should be $\begin{pmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix}$. $\vec{a} = A^{-1}\vec{b}$.

Answer: $A^{-1}\vec{b} = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

Theorem:

$n+1$ DISTINCT POINTS $\Rightarrow$ A invertible $\Rightarrow$ unique polynomial of deg n.