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MC 6461
Section 010

MATH 115 - Linear Algebra Sep 7/2017

Chapter 1 - Euclidean Vector Spaces

Vectors: Magnitude, direction supported on a line $\mathbb{R}^n$
↳ $\mathbb{R}^2$ is set of all vectors in form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ ← real # (components)
↳ $\mathbb{R}^n$ " " " $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \vec{x}$ (need arrow), $\mathbb{R}^n$: $\in \mathbb{R}^n$

Duality $\bar{\text{w}}$ pts + $\vec{v}$ in $\mathbb{R}^2$, $\mathbb{R}^n$ (can get pt $\rightleftarrows \vec{v}$)

Refer to gr 12 Calc notes for geometric rep, operations, etc...

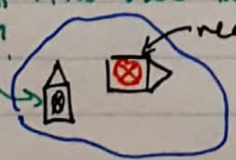
Adding vectors:
↳ Can add vectors in $\mathbb{R}^3$ by using parallelogram method on plane containing vectors.

Zero vector:
$\vec{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$ → initial = terminal pt
↳ supported on ∞ lines, no direction

Theorem:
Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$
↳ $\vec{x} + \vec{y} \in \mathbb{R}^n$ "closure under addition"
↳ $\vec{0} + \vec{x} = \vec{x}$
↳ $t\vec{x} \in \mathbb{R}^n$

Lecture 2 - Sep 12, 2017:

Fixing the origin:
Vectors always anchored on origin for linear algebra.
↳ In physics, often anchored to point where force is applied - "Free vectors"
↳ Also used in local coordinate systems



"Global coordinate system" - origin must be independent of your movement

Lines in $\mathbb{R}^n$:

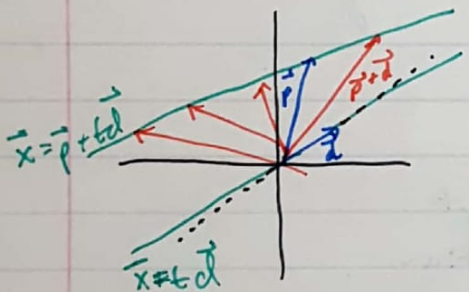
1) Directed line segments

- Let $P, Q \in \mathbb{R}^n$. $\overrightarrow{PQ} = \vec{q} - \vec{p}$ (second - first) ← tech not true...
- Origin translated to be at P. ↳ b/c this (origin location)

Equation of a Line in $\mathbb{R}^n$

- Let $\vec{p}, \vec{d}$ be vectors in $\mathbb{R}^n$, $\vec{d} \neq \vec{0}$, has direction vector $\vec{d}$
- $\{\vec{p} + t\vec{d} \mid t \in \mathbb{R}\} \rightarrow \vec{x} = \vec{p} + t\vec{d}, t \in \mathbb{R}$ (just direction vectors)
- $/\!/^{\text{ll}}$ to line through origin w/ vector eqn $\vec{x} = t\vec{d}$, $t \in \mathbb{R}$

↳ Just a translation by $\vec{p}$.



← all lie on this line

// parallel

★ Old notes: eqns of lines

↳ sym, normal, para

Sym. eqn:

↳ solve for x_1 / x_2: $\frac{d_2}{d_1} x_1 + (p_2 - \frac{d_2}{d_1})$ $y = mx + b$

↳ If $d_1 = 0$, $d_2 \neq 0$ — get a vertical line

$d_2 = 0$, $d_1 \neq 0$ — get a horiz. line

★ In $\mathbb{R}^n$, where $x_n = a_{n-1}x_{n-1} + \dots + a_1 x_1$ w/ real coefficients is never the eqn of a line when $n \geq 3$

Two vectors $\vec{u}$ and $\vec{v}$ are $/\!/^{\text{ll}}$ if one is $\vec{0}$ or sc. multiples.

Sep 13, 2017

Subspaces in $\mathbb{R}^n$:

A non-empty subset S is a subspace of $\mathbb{R}^n$ if for any $\vec{x}, \vec{y} \in S$ and any $t \in \mathbb{R}$, the following holds:

1) $\vec{x}+\vec{y} \in S$
2) $t\vec{x} \in S$

$\left.\right\}$ closure under addition/scalar mult.

Properties:

a) $\vec{0}$ is always in any subspace (let $t=0$)

b) $\{\vec{0}\}$ is subspace of $\mathbb{R}^n$ "trivial subspace"

c) $\mathbb{R}^n$ is subspace of itself

→ Euclidean vector space: set $\mathbb{R}^n$ for $n \geq 1$

Example:

1) Let $S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle/\; x_1 + x_2 - 8x_3 = 4 \right\}$ (such that)

Not a subspace of $\mathbb{R}^4$

→ if $\vec{0} \in S$, right eqn $\neq$ true.

2) Let $T = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle/\; x_1 x_2 + x_3^2 = 0 \right\}$ ← multi coords and sq. = unlikely to be subspace.

$\vec{0} \in T$, but not subspace of $\mathbb{R}^3$.

→ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in T$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in T$ but sum $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \notin T$ (eqn doesn't work)

3) Let $U = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle/\; x_1 + 4x_2 = 0 \right\}$

✓ Subspace of $\mathbb{R}^2$

✓ → $\vec{0} \in U$

✓ → $\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$: $(x_1 + y_1) + 4(x_2 + y_2)$

$= \underbrace{(x_1 + 4x_2)}_{0} + \underbrace{(y_1 + 4y_2)}_{0}$ since $\in U$

✓ → $t\vec{x} = \begin{pmatrix} tx_1 \\ tx_2 \end{pmatrix}$ $\quad t(x_1 + 4x_2) = 0$

$t(0) = 0$ ✓

3 conditions

Spanning Sets:

Let $\vec{v}_1 \ldots \vec{v}_k$ be k vectors in $\mathbb{R}^n$. Then,

S is all possible lin combos of $\vec{v}_{1 \ldots k}$ $S = \{t_1\vec{v}_1 \ldots t_k\vec{v}_k \mid t_1 \ldots t_k \in \mathbb{R}\} \subset \mathbb{R}^n$ *Try to prove this

is a subspace of $\mathbb{R}^n$

↳ set of all linear combinations is a subspace!

Let $B = \{\vec{v}_1 \ldots \vec{v}_k\}$ be set of k vectors in $\mathbb{R}^n$

Let $S = \{t_1\vec{v}_1 \ldots t_k\vec{v}_k \mid t_1 \ldots t_k \in \mathbb{R}\} \subset \mathbb{R}^n$

↳ The subspace S is "spanned" by B

↳ B "spans" S or is a "spanning set" of S

$\Big\}$ $S = \text{span}\{\vec{v}_1 \ldots \vec{v}_k\} = \text{span } B$

Example:

1) A line through the origin in $\mathbb{R}^n$, $\vec{x} = t\vec{v}$ $(\vec{v} \neq \vec{0})$ is a subspace of $\mathbb{R}^n$ ($k=1$ from prev. theorem) spanned by any given direction vector $(\vec{v})$.

↳ Because $k=1$, so in $S = \{t_1\vec{v}_1 \mid t_1 \in \mathbb{R}\}$

Theorem:

Let $\vec{v}_1 \ldots \vec{v}_k$ be k vectors in $\mathbb{R}^n$. If $\vec{v}_k$ can be expressed as a linear combination of $\vec{v}_1 \ldots \vec{v}_{k-1}$ then $\text{span}\{\vec{v}_1 \ldots \vec{v}_k\} = \text{span}\{\vec{v}_1 \ldots \vec{v}_{k-1}\}$

↳ If you can rep. a $\vec{v}$ in the set as a lin comb. of two other vectors in the set, you can omit it. (same thing (=))

Linear Independance

Sep 14, 2017

set $\{\vec{v}_1 \ldots \vec{v}_k \in \mathbb{R}^n\}$ is linearly independent if the only solution to

$$\vec{0} = t_1\vec{v}_1 \ldots t_k\vec{v}_k$$

is the trivial solution $t_{1 \ldots k} = 0$. else $\{$ linearly dependant $\}$

↳ Complicated way of saying no linear combinations within the set.

Theorem:

If the set of vectors $\vec{v}_1 \dots \vec{v}_k$ contains $\vec{0}$, linearly dependent.

↳ Can have any t for the $\vec{0}$ and the rest 0.

Bases:

Subspace $\{\vec{v}_1 \dots \vec{v}_k\}$ in $\mathbb{R}^n$

$\{\vec{v}_1 \dots \vec{v}_k\}$ is a basis for subspace if:

→ linearly independent (no redundant info)

→ Spanning set for S.

Example:

1) "The standard basis" in $\mathbb{R}^n$ Just unit vectors

→ $\vec{e}_1 \dots \vec{e}_n$ where $\vec{e}_i \in \mathbb{R}^n$ is the vector whose i^{th} component is 1 and all the others are 0.

→ So..

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Big\} n-1^{th} \rightarrow \vec{e}_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ i^{th} \\ \vdots \\ 0 \end{pmatrix}$$ } All component 0, i^{th} component 1.

→ 1) Linearly independent.

$$t_1\vec{e}_1 + \dots + t_n\vec{e}_n = \vec{0}$$

adding gives you $= \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}$ ∴ All $t_1 \dots t_n$ must be 0 for $\vec{0}$.

2) Spanning set for S

$\vec{v}\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$ can be written as l. comb of $\vec{e}_1 \dots \vec{e}_n$

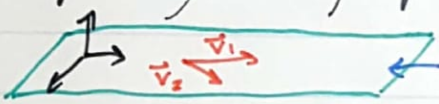
$$\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = v_1\vec{e}_1 + \vec{v}_2\vec{e}_2 \dots \vec{v}_n\vec{e}_n$$ ∴ $\vec{v} \in \operatorname{span}\{\vec{e}_1 \dots \vec{e}_n\}$

Surfaces in Higher Dimensions:

Let $\vec{v}_1, \vec{v}_2, \vec{p} \in \mathbb{R}^n$ with $\{\vec{v}_1, \vec{v}_2\}$ linearly independent. then,

$$\vec{x} = \vec{p} + t_1\vec{v}_1 + t_2\vec{v}_2 \quad , t_1, t_2 \in \mathbb{R}$$

is a plane passing through $\vec{p}$. (translated by $\vec{p}$)



← span of the vectors is the plane

Hyperplanes:

Let $\{\vec{v}_1 \dots \vec{v}_{n-1}\}$ be linearly independent. Let $\vec{p} \in \mathbb{R}^n$

$\therefore$, $\vec{x} = \vec{p} + t_1\vec{v}_1 + \dots t_{n-1}\vec{v}_{n-1}$

is a hyperplane passing through $\vec{p}$. (4 dimensions? 3 d. vectors)

Length and Dot Product:

"Dot Product" / Scalar Product / Standard Inner Product

$\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 \dots x_ny_n$ (get a scalar)

Properties:

1) $\vec{x} \cdot \vec{x} \geq 0$. only 0 when $\vec{x} = \vec{0}$
2) $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$
3) $\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$
4) $t\vec{x} \cdot \vec{y} = t(\vec{x} \cdot \vec{y}) = \vec{x} \cdot (t\vec{y})$

Length:

$|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x_1^2 \dots + x_n^2}$ "norm" $\quad \cos\theta = \dfrac{\vec{x} \cdot \vec{y}}{|\vec{x}||\vec{y}|}$

Properties:

1) $|t \cdot \vec{x}| = |t||\vec{x}|$
2) $|\vec{x} \cdot \vec{y}| \leq |\vec{x}||\vec{y}|$ Cauchy-Schwarz.
3) $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$ Δ ineq.

Sep 15, 2017

Scaler Eqn of A Hyperplane:

Hyperplanes are given by: in $\mathbb{R}^n$ . . .

$$\vec{x} = \vec{p} + t_1\vec{v}_1 \ldots t_{n-1}\vec{v}_{n-1} \quad t \in \mathbb{R}$$

where $\{\vec{v}_1 \ldots {}_{n-1}\}$ is linearly independent.

These hyperplanes have normals. $\Rightarrow \vec{w}$ n coords $\begin{pmatrix} \vdots \\ n \end{pmatrix}$

- Non zero
- Given any $P(x,y,z)$, ignore this
- Any vector in plane is $\perp$ to normal of plane.

Another point x is only in plane iff $\vec{px} \cdot \vec{n} = 0$.

Scalar equation:

$$m_1 x_1 \ldots m_n x_n = d$$

or . . . $Ax + By + Cz = D$ for $n = 3$.

If . . .

- $n = 2$? Hyperplane is a line
- $n = 3$? Hyperplane is a plane
- $n > 3$? Hyperplane is a hyperplane.

} One less than n.

Example:

1) $2x_1 + x_2 + 4x_3 = 2$

$2(x_1 - 1) + x_2 + 4x_3 = 0$, or . . .

$$\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} x_1 - 1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

because $\vec{px}$ is on the plane!

$$\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \cdot \left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] = 0 \quad \therefore \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ \underline{is} on the hyperplane.}$$

Projection and Minimum Distance:

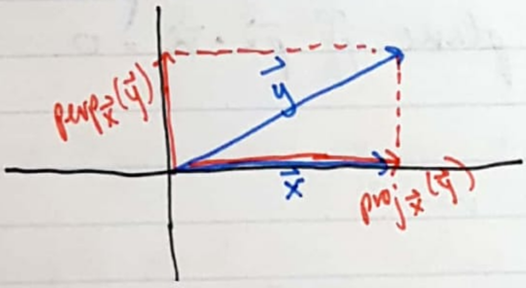
Projections:

$$\text{proj}_{\vec{x}}(\vec{y}) = \left(\frac{\vec{y}\cdot\vec{x}}{\|\vec{x}\|^2}\right)\vec{x}$$

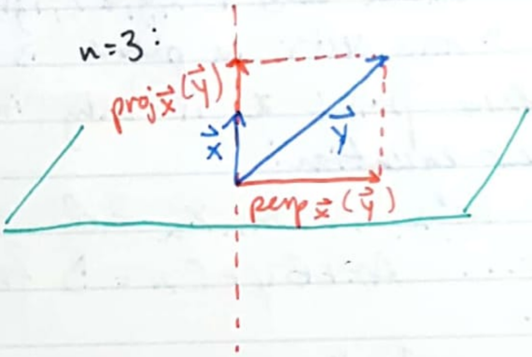
Projection of $\vec{y}$ onto perpendicular of $\vec{x}$:

$$\text{perp}_{\vec{x}}(\vec{y}) = \vec{y} - \text{proj}_{\vec{x}}(\vec{y})$$

when $n=2$:



$n=3$:



Example:

1) $\vec{x} = \begin{pmatrix}1\\1\end{pmatrix}$, $\vec{y} = \begin{pmatrix}3\\2\end{pmatrix}$

$$\text{proj}_{\vec{x}}(\vec{y}) - 3\sqrt{2}\begin{pmatrix}1\\1\end{pmatrix} \qquad \text{perp}_{\vec{x}}(\vec{y}) : \begin{pmatrix}3\\2\end{pmatrix} - 3\sqrt{2}\begin{pmatrix}1\\1\end{pmatrix}$$

Properties of Projections:

Sep 19, 2017

$\vec{x},\vec{y},\vec{z} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$

1) $\text{proj}_{\vec{x}}(\vec{y}+\vec{z}) = \text{proj}_{\vec{x}}(\vec{y}) + \text{proj}_{\vec{x}}(\vec{z})$
2) $\text{proj}_{\vec{x}}(t\vec{y}) = t\,\text{proj}_{\vec{x}}(\vec{y})$
3) $\text{proj}_{\vec{x}}(\text{proj}_{\vec{x}}(\vec{y})) = \text{proj}_{\vec{x}}(\vec{y})$

Minimum Distance:

Distance btwn point and line/hyperplane is min. distance $\bar{w}$ closest point! ← S is point on line/hyperplane

$\|\text{perp}_{\vec{d}}(\vec{SQ})\|$ → Point Q to line $\bar{w}$ <u>dir.</u> $\vec{v}$: $\vec{d}$

$\|\text{proj}_{\vec{m}}(\vec{SQ})\|$ → Point Q to hyperplane $\bar{w}$ <u>normal</u> $\vec{v}$: $\vec{m}$

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Fundamental Example:

Let $Q = (Q_1, Q_2, Q_3) \in \mathbb{R}^3$, $H: m_1x_1 + m_2x_2 + m_3x_3 + d = 0$ (hyperplane)

Distance from H to Q:

$$\frac{|Ax_1 + Bx_2 + Cx_3 (+D)|}{\sqrt{A^2+B^2+C^2}}$$

watch! +D if all on one side

$\vec{n} = (A, B, C)$ $P = (x_1, x_2, x_3)$

Assume $m_1 \neq 0$. Then $S_1 = (\frac{d}{m_1}, 0, 0)$ is on H. Therefore, distance is:

$$\|proj_{\vec{m}}(\vec{SQ})\| = \left\| \frac{(\vec{q} - \vec{s}) \cdot \vec{m}}{\|\vec{m}\|^2} \vec{m} \right\|$$

$$= \frac{|(q_1 - \frac{d}{m_1})m_1 + q_2m_2 + q_3m_3|}{\sqrt{m_1^2 + m_2^2 + m_3^2}}$$

= original formula!

Cross Product:

$\vec{u} \times \vec{v}$ is $\perp$ to both $\vec{u}$ and $\vec{v}$ (dot prod = 0)

1) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
2) $\vec{u} \times \vec{u} = \vec{0}$
3) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
4) $t(\vec{u} \times \vec{v}) = (t\vec{u}) \times \vec{v} = \vec{u} \times (t\vec{v})$
5) $(\vec{u} \times \vec{v}) \times \vec{w} \neq \vec{u} \times (\vec{v} \times \vec{w})$ ORDER MATTERS!

Proof of 3): Let $\vec{a} = \vec{v} + \vec{w}$.

$\vec{u} \times \vec{a} = u_2(v_3 + w_3) - u_3(v_2 + w_2)$ } first comp. of $\times$

$= u_2v_3 + u_2w_3 - u_3v_2 - u_3w_2$

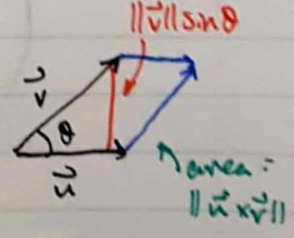
$= \underbrace{(u_2v_3 - u_3v_2)}_{\vec{u} \times \vec{v} \text{ comp.}} + \underbrace{(u_2w_3 - u_3w_2)}_{\vec{u} \times \vec{w} \text{ comp.}}$

And repeat.

Applications:

$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta$ * $\theta \in [0, \pi]$

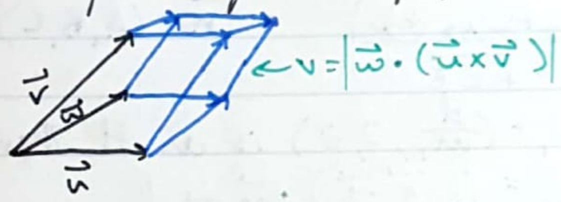
= area of parallelogram spanned by $\vec{u}$ and $\vec{v}$



Triple Scalar Product:

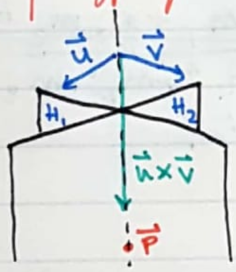
$\vec{w} \cdot (\vec{u} \times \vec{v})$

→ Volume of the parallelepiped w adj sides $\vec{u}, \vec{w}, \vec{v}$.



Intersection of Hyperplanes:

Let $\vec{u}$ be normal of H_1, $\vec{v}$ for H_2.
The line of int. has dir. vector $\vec{u} \times \vec{v}$.



Example:

1) $H_1: x_1 + x_2 + x_3 = 1 \quad H_2: x_1 - x_2 + x_3 = 0$

$\vec{n_1} \times \vec{n_2} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \quad \vec{P} = \left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right) \quad \vec{x} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, t \in \mathbb{R}$

Chapter 2: Systems of Linear Equations:

Linear Eqns:

A system of m linear equations in n variables $x_1 \ldots x_n$ is a set of m eqns:

$$\begin{cases} a_{11} x_1 \cdots a_{1n} x_n = b_1 \\ \vdots \\ a_{m1} x_1 \cdots a_{mn} x_n = b_m \end{cases}$$

The coefficient matrix is...

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

The augmented matrix is...

$$\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right) = [A|\vec{b}]$$

Sep 20, 2017

Matrices:

$$\begin{array}{cc} & \\ i^{th}\text{ row} & \begin{pmatrix} & \vdots & \\ \cdots & a_{ij} & \cdots \\ & \vdots & \end{pmatrix} \\ & \quad j^{th}\text{ col} \end{array}$$

$1 \le i \le m$ equations

$1 \le j \le n$ degree

Row Echelon Form:

Elementary row operations:

1) Multiply by non-zero constant
2) Switch rows
3) Add multiples of rows * Can't do $2R_4 - R_5$

Two matrices are row equivalent if one can be obtained by row reducing the other. $A \sim B$

A matrix is in row echelon form if:

1) All 0s at bottom row (don't need row of 0s tho)
2) First # in upper row is ← of first # in lower row

Example:

1) $A: \begin{pmatrix} 1 & 12 & -22 \\ 0 & 2 & 3 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{pmatrix}$ ✓ REF $\quad B: \begin{pmatrix} 1 & 12 & -22 \\ 0 & 2 & 3 \\ 0 & 8 & 4 \end{pmatrix}$ ✗ Not REF

Gaussian Elimination:

$$\begin{pmatrix} 0 & 0 & 1 & 3 \\ 0 & 8 & -1 & 10 \\ 0 & 0 & -3 & 9 \\ 0 & 16 & 4 & -7 \end{pmatrix}$$

1) Identify first non-zero column. Switch rows so that top entry in this column is non zero

first non-zero column

needs to be at top (first non-zero column value)

PIVOT

$$\begin{pmatrix} 0 & 0 & 1 & 3 \\ 0 & 8 & -1 & 10 \\ 0 & 0 & -3 & 9 \\ 0 & 16 & 4 & -7 \end{pmatrix} \Big| R_1 \updownarrow R_2 \sim \begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -3 & 9 \\ 0 & 16 & 4 & -7 \end{pmatrix}$$

2) Use ERO to make all entries below pivot = 0.

$$\begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -3 & 9 \\ 0 & 16 & 4 & -7 \end{pmatrix} \Big| R_4 - 2R_1 \sim \begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -3 & 9 \\ 0 & 0 & 6 & -27 \end{pmatrix}$$

should be 0!

3) Consider all columns right of pivot. Becomes submatrix. Repeat 1+2 for this submatrix w/ all rows below last pivot.

$$\begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -3 & 9 \\ 0 & 0 & 6 & -27 \end{pmatrix} \Big| R_3 + 3R_2 \text{ then } \Big| R_4 - 6R_2 \sim \begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & -45 \end{pmatrix}$$

new pivot

need these 0 now

4) Repeat step 3 until row echelon form.

$$\begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & -45 \end{pmatrix} \Big| R_4 + 5R_3 \sim \begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

new pivot

(would still be REF w/o last row)

Determining Solutions using Matrices: Sep 21, 2017

Consistent - 1+ solutions

Inconsistent - 0 solutions
- If last row is $[0 \ldots 0 \mid c]$, $c \neq 0$

If consistent...
- # pivots = # variables? Unique Solution. $[0 \ldots c \mid c]$
- # pivots < # variables? Infinite Solutions. $[0 \ldots 0 \mid 0]$

Example:

1) $\left(\begin{array}{cccc|c} 0 & 8 & -1 & 10 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$

inconsistent

2) $\left(\begin{array}{cccc|c} 0 & 8 & -1 & 10 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

consistent

3 pivots, 4 variables

∴ ∞ solutions

3) $\left(\begin{array}{ccc|c} 8 & -1 & 10 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 9 & 1 \end{array}\right)$

3 pivots, 3 variables

∴ 1 solution

Reduced REF (RREF):

1) REF
2) All leading entries = 1.
3) Leading 1's have all 0s after (in the column)
- Any matrix has a equivalent, unique matrix in RREF

REF

RREF

$\begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -7 \end{pmatrix}$

Example:

1) $\begin{pmatrix} 0 & 8 & -1 & 10 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ RREF

$\frac{1}{8}R_1$, then $\frac{1}{9}R_3$

then $R_1 + \frac{1}{8}R_2$

then $R_1 + \left(\frac{-13}{8}\right)R_3$

then $R_2 - 3R_3$

(Do in 1 step tho)

Homogeneity and Rank:

Homogeneous if right side of all eqns $= 0$.

$$\begin{cases} 2x_1 - x_2 = 0 \\ x_1 + 3x_2 - 4x_3 = 0 \end{cases} \rightarrow \vec{x} = \vec{0} \text{ is a trivial solution.}$$

Rank ($\text{rank}(M)$) is # of leading 1's in it's RREF.

Theorem:

1) System is consistent iff rank(coefficient matrix) = rank(augmented)
2) If consistent, parameters needed to express solutions is #variables − rank(coefficient)

Example:

1) $$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 - 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} -x_3 = x_1 + x_2 \\ -3x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_2 = -x_1 \\ x_3 = 0 \end{cases}$$

The set of solutions is $\left\{ \begin{pmatrix} x_1 \\ -x_1 \\ 0 \end{pmatrix} \middle| x_1 \in \mathbb{R} \right\}$

Applications:

1) Balancing Chemical Reactions

→ Manganese (Mn)

$$x_1 MnO_4^- + x_2 H_2C_2O_4 + x_3 H^+ \rightarrow x_4 Mn^{2+} + x_5 CO_2 + x_6 H_2O$$

stoichiometric coefficients

$$\begin{aligned} C:&\ 2x_2 &= x_5 \\ Mn:&\ x_1 &= x_4 \\ O:&\ 4x_1 + 4x_2 &= 2x_5 + x_6 \\ H:&\ 2x_2 + x_3 &= 2x_6 \\ \text{Charges}:&\ -x_1 + x_3 &= 2x_4 \end{aligned}$$

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{5}{8} & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{5}{4} & 0 \end{array}\right)$$

Rank = 5
Vars = 6.

Get $x_1 = \_x_6$
Let $x_6 = 8$
Get coefficients

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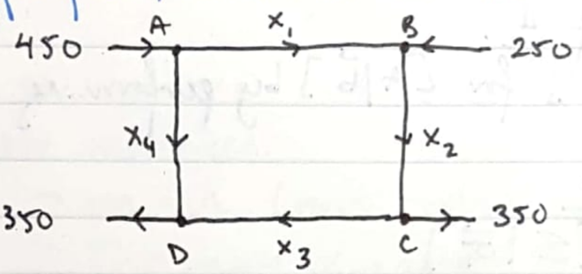
Application to Information Theory

Sep 28, 2017.

Traffic flow / communication network.



Each node (ABCD) = antenna.

Each → = one way com. channel

↳ Capacity $x_1 x_2 \ldots _4$: info/sec

↳ Unit: Bits! bit/s

*Applications week 2.

This assumes eq. state (No info gained/lost)

$$\left.\begin{array}{llll} A: & 450 & = & x_1 + x_4 \\ B: & 250 + x_1 & = & x_2 \\ C: & x_2 & = & x_3 + 350 \\ D: & x_3 + x_4 & = & 350 \end{array}\right\} \begin{array}{l} x_1 + x_4 = 450 \\ -x_1 + x_2 = 250 \\ x_2 - x_3 = 350 \\ x_3 + x_4 = 350. \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 450 \\ -1 & 1 & 0 & 0 & 250 \\ 0 & 1 & -1 & 0 & 350 \\ 0 & 0 & 1 & 1 & 350 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 450 \\ 0 & 1 & 0 & 1 & 700 \\ 0 & 0 & 1 & 1 & 350 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

consistent ∞ soln.

Can be expressed as #vars − rank = 4 − 3 = 1 parameter needed.

Back substitution:

$$\left.\begin{array}{l} x_1 + x_4 = 450 \\ x_2 + x_4 = 700 \\ x_3 + x_4 = 350. \end{array}\right\} \begin{array}{l} x_1 = 450 - x_4 \quad x_4 \in \mathbb{R} \\ x_2 = 700 - x_4 \\ x_3 = 350 - x_4. \end{array}$$

All vars need +! can't have − info

Physical Solutions:

$$\begin{pmatrix} 450 - t \\ 700 - t \\ 350 - t \\ t \end{pmatrix}, \quad 0 \le t \le 350$$

{ When does a m eq. × n vars matrix have a solution no matter what the constant terms are? $[A|\vec{b}]$, $\vec{b} \in \mathbb{R}^m$

Chapter 3: Matrices

2 matrices are equal when all entries are same.

Special Matrices:

Square matrices:

→ $m = n$ (rows = cols).

→ The main diagonal:

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \quad A_{ii}, 1 \le i \le n$$

→ Upper triangular / Lower triangular

→ All entries below main diagonal are zeroes.

$$\begin{pmatrix} 1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 4 & 5 & 2 \end{pmatrix}$$

→ Diagonal:

→ Both upper and lower triangular

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \operatorname{diag}(d_1, d_2, d_3 \ldots d_n)$$

Vectors:

Sep 27, 2017.

→ $n \times 1$ matrices

Operations on Matrices:

Let A, B be $m \times n$ matrices, s be a scalar.

$(A+B)_{ij} = A_{ij} + B_{ij}$ *both need to be $n \times n$

$(sA)_{ij} = s(A_{ij})$

Theorem:

Let A, B, C be $m \times n$ matrices. $s, t \in \mathbb{R}$.

1) Closure under addition:
   $A + B$ is $m \times n$ matrix.
2) Comm. of addition:
   $A + B = B + A$
3) Assoc. of addition:
   $(A+B) + C = A + (B+C)$
4) Zero Matrix: $\mathbb{O}_{m \times n}$ Only 0 entries.
   $A + \mathbb{O}_{m \times n} = A$.
5) $-A$.
   $A + (-A) = \mathbb{O}_{m \times n}$
6) Closure under scaler multiplication:
   sA is $m \times n$.
7) Assoc. of sc. multi:
   $s(tA) = (st)A$
8) Distributivity:
   $(s+t)A = sA + tA$
9) Dist. 2:
   $s(A+B) = sA + sB$

Transposition:

Let A be $\underline{m \times n}$ [ ]. A^T is a $\underline{n \times m}$ [ ]

$$A_{ij} = A^T_{ji}$$

$$\underset{A}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}} \qquad \underset{A^T}{\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}}$$

Rows become cols, cols become rows

Rows are transposes of vectors.

Properties of Transposition:

Let A, B be $m \times n$ [ ]. Let $s \in \mathbb{R}$.

1) $(A^T)^T = A$. Involution
2) $(A+B)^T = A^T + B^T$
3) $(sA)^T = s(A^T)$

Proof of 1):

$((A^T)^T)_{ij} = (A^T)_{ji} = A_{ij}$

Proof of 2):

$((A+B)^T)_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = A^T + B^T$

Proof of 3):

$((sA)^T)_{ij} = (sA)_{ji} = s(A_{ji}) = s(A^T)$

Multiplication of Matrices:

Let $a_1 \ldots a_n \in \mathbb{R}$. 1) $a_1 + a_2 \ldots + a_n = \sum_{k=1}^{n} a_k$, and...

2) $a_1 \times a_2 \times a_n = \prod_{k=1}^{n} a_k$

Let B be $\boxed{m \times n}$ [ ] $\bar{w}$ rows:

$(b_{11}, \ldots, b_{1n}) = \vec{b}_1^T$

$\vdots$

$(b_{m1}, \ldots b_{mn}) = \vec{b}_m^T$

$(m \times \underline{n}\ [\ ])(\underline{n} \times p[\ ]) = (m \times p[\ ])$

must be same!

Let A be $\boxed{n \times p}$ [ ] $\bar{w}$ columns

$\vec{a}_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} \ldots \vec{a}_p = \begin{pmatrix} a_{1p} \\ \vdots \\ a_{np} \end{pmatrix}$

Then, BA is the $\boxed{m \times p}$ matrix whose ij^{th} entry is:

$(BA)_{ij} = \vec{b}_i \cdot \vec{a}_j = \sum_{k=1}^{n} (B_{ik})(A_{kj})$

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Examples:

$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & -3 & -4 \end{pmatrix}$ [2x4] $B = \begin{pmatrix} -2 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -2 \end{pmatrix}$ [4x2]

$D = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ [2x1] $C = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$ [2x2] $E = (1, 0)$ [1x2]

1) AB works
2) BA works
3) CD works
4) DC doesn't work
5) EC works
6) CE doesn't work.

Sep 28, 2017

$AB: \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & -3 & -4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -2 \end{pmatrix}$

$\boxed{\vec{a}_i \cdot \vec{b}_j}$: $\begin{pmatrix} (1)(-2) + (2)(-1) + (3)(0) + (4)(1) & (1)(1) + (2)(0) + 3(-1) + 4(-2) \\ (-1)(-2) + (-2)(-1) + (-3)0 + (-4)(1) & (-1)(1) + (-2)0 + (-3)(-1) + (-4)(-2) \end{pmatrix}$

for $1 \le i \le m$:
↳ for $1 \le j \le n$:

$= \begin{pmatrix} 0 & -10 \\ 0 & 10 \end{pmatrix}$

Properties: (assumes product is defined)

$AB = AC$ doesn't imply $B = C$

$AB \neq BA$ (necessarily)

$A(B+C) = AB + AC$

$t(AB) = (tA)B = A(tB)$

* $(AB)^T = B^T A^T$ Careful. Let A $m \times n$[ ] and B $n \times p$[ ], AB defined

↳ A^T — $n \times m$, B^T — $p \times n$ $1 \le i \le p$ and $1 \le j \le m$

$((AB)^T)_{ij} = AB_{ji} = \sum_{k=1}^{m} (B)_{ki} (A)_{jk} = \sum_{k=1}^{m} (B^T)_{ik} (A^T)_{jk} = (B^T A^T)_{ij}$

Let $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ $[n \times 1]$ $\quad \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$ $[n \times 1]$

$$\underset{[1 \times n] \cdot [n \times 1]}{(\vec{x}^T) \cdot \vec{y}} = (x_1, x_2 \ldots x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = x_1 y_1 + \ldots + x_n y_n = \vec{x} \cdot \vec{y}$$

Definition:

Identity matrix: $n \times n$ $I_n = \text{diag}(1_1 \ldots 1_n) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

- If no ambiguity, $I_n = I$.
- $I_m A = A I_m = A$.

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

Inverse Matrices:

Let A be $n \times n$ □. If there exists an unique $n \times n$ □ B such that $\boxed{AB = BA = I_n}$, then A is invertible with inverse B. (A^{-1})

- Assume not unique. $BA = AB = I = CA = AC = I$.
- $B = BI = B(AC) = (BA)C = IC = C$ $\therefore$ unique.
- If $AB = I_N$, then $BA = I_n$ and $B = A^{-1}$
- Any invertible matrix of size n has max rank n.

Theorem:

Assume A and B are invertible of size n. Let $t \in \mathbb{R}$, $t \neq 0$.

need to state

1) tA is invertible and $(tA)^{-1} = \frac{1}{t} \cdot A^{-1}$
2) AB is invertible and $(AB)^{-1} = B^{-1} A^{-1}$
3) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

1) $(tA) \cdot (\frac{1}{t})(A^{-1}) = (t \cdot \frac{1}{t})(A \cdot A^{-1}) = I$
2) $(AB)(B^{-1}A^{-1}) = A[(B)(B^{-1})]A^{-1} = I$
3) $(A^T) \cdot (A^{-1})^T = (A^{-1}A)^T = I^T = I$.

Never use A^{-1} before justifying A is invertible.

Invertibility:

Sep 24, 2017

If A [ ] B [ ] such that $AB = I$, then this does NOT imply that A is invertible w inverse B.

→ Only holds iff A, B are □. ✶

Example:

1) Consider $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$ $\boxed{\text{BUT}}$ $BA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq I_2$

Proposition:

Let A, B $m \times n$ [ ] where $\forall \vec{x} \in \mathbb{R}^n$:

$A\vec{x} = B\vec{x}$, Then $A = B$

Proof:

Consider $\vec{e}_i$, $1 \leq i \leq n$:

$\vec{e}$: $\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ ← i^{th} entry is given. To avoid ambig, $\vec{e}^i = \vec{e}_i$

$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{1m} & \cdots & a_{mn} \end{bmatrix}$, Then $\underbrace{A\vec{e}^i}_{[m \times n] \cdot [n \times 1]} \in \mathbb{R}^m$ ← [m×1] vector

$= \begin{pmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_m^T \end{pmatrix}$ ↓ the j^{th} component is: $\vec{a}_j \cdot \vec{e}^i = (a_{j1} \cdots a_{jn}) \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ ← ith

$= a_{j1}(0) \ldots + \ldots a_{ji}(1) \ldots + \ldots a_{jn}(0)$

$= a_{ji}$ ← This shows:

$A\vec{e}^i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ji} \\ \vdots \\ a_{mi} \end{pmatrix}$ = i^{th} column in A. Means all colums for A, B must be equal (i can be any col #).

Finding the Inverse:

Consider n equns, n vars system.

$$a_{11}x_1 + \dots a_{1n}x_n = b_1$$
$$\vdots \qquad\qquad \vdots$$
$$a_{n1}x_1 + \dots a_{nn}x_n = b_n$$

Let A be coef. [ ], $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{R}^n$

Then the system can be rewritten as $A\vec{x} = \vec{b}$.

If A^{-1} exists, $\vec{x} = A^{-1}\vec{b}$

Inverse Matrix Algorithm:

Let A be $\boxed{}$

Reduce the matrix $[A|I]$ so A is in RREF.

$*$ 1) If the left block (A) is not I, then A is not invertible.
2) If the form is $[I|B]$, then B is the inverse and A^{-1} exists.

Example:

1) Let $A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$.

$$\left(\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right)_{1R_2 - \frac{2}{3}R_1} \sim \left(\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 0 & \frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right)_{1\frac{1}{3}R_1} \sim \left(\begin{array}{cc|cc} 1 & \frac{4}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right)_{1R_1 - 4R_2} \sim$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 3 & -4 \\ 0 & \frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right)_{1R_2 \cdot 3} \sim \left(\begin{array}{cc|cc} 1 & 0 & 3 & -4 \\ 0 & 1 & -3 & 3 \end{array}\right)$$

$= I \quad \therefore$ A is invertible and is the inverse.

Say "from the I.M.A" A is invertible and $A^{-1} = B$.

Properties of Inverse Matrixes:

Oct 3, 2017

Let A $m \times m$ □. Then, the following statements are equiv.

If one is true, all are true.
- i) A is invertible
- ii) A has rank m
- iii) The RREF of A is I_m
- iv) Given any $\vec{b} \in \mathbb{R}^m$, $A\vec{x} = b$ is consistent and has a unique solution
- v) Columns of A from linearly independent vectors.

Proof of iv) $\Rightarrow$ v):

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{m1} & \cdots & a_{mm} \end{pmatrix} = [\vec{a}_1 \dots \vec{a}_m]$$

where $\vec{a}_i$ is i^{th} col in A.

Let $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{R}^m$

Consider $A\vec{x} = \vec{0}$:

$$\left.\begin{matrix} a_{11} x_1 & \dots = 0 \\ \vdots & \\ a_{m1} x_1 & \dots = 0 \end{matrix}\right\} = \sum_{i=1}^{n} \vec{a}_i x_i = \vec{0}$$

linear independence, opitimized

came from $\vec{a}_1$

Example:

1) Let A be a □ where $A^3 = I$. Show A is invertible and determine it's inverse.

$A \times A \times A = I$ Let $D = A^2$

then $AD = I$, where AD are □ (multi is defined).

$\therefore$ The inverse of A is A^2.

2) Let A, B □. Show if A and B are invertible $\Leftrightarrow$ AB is invertible.

$\Rightarrow$ Assume A, B invertible. Then:

$$(AB)(B^{-1}A^{-1}) = (A)(BB^{-1})(A^{-1}) = AIA^{-1} = AA^{-1} = I.$$

Since $B^{-1}A^{-1}$ invertible, $(AB)^{-1}$ is invertible $\bar{w}$ inverse $B^{-1}A^{-1}$!

$\Leftarrow$ Assume AB invertible. Then, $\exists$ □ $C \ni (AB)C = I$, $C(AB) = I$.
Since $(AB)C = I$, $A(BC) = I$, so A is invertible.
same $\bar{w}$ B. $A^{-1} = BC$, $B^{-1} = CA$.

Linear Mappings:

Functions are mappings/transformations. If f has dom U, codom V:

maps to
$$f : U \longrightarrow V$$
transforms
$$x \longmapsto f(x)$$
$\{ f : x \in U \longmapsto f(x) \in V$.

↳ Range is attainable codomain.

$f: \dfrac{\mathbb{R} \to \mathbb{R}}{x \mapsto x^2}$ $\qquad$ $g: \dfrac{[-1,1] \to \mathbb{R}}{x \mapsto x^2}$ Two dif. fcns.

	dom	cod	range
f	$\mathbb{R}$	$\mathbb{R}$	$[0, \infty]$
g	$[-1,1]$	$\mathbb{R}$	$[0,1]$

Matrix Mappings:

Let A be $m \times n$ [ ].

$$f_A : \mathbb{R}^n \to \mathbb{R}^m$$
$$\underbrace{\underset{[n \times 1]}{\vec{x}} \longmapsto \underset{[m \times n]}{A\vec{x}}}_{\text{need to highlight that } A\vec{x} \text{ exists!}}$$

{ Definition of product of 2 matrices shows that matrix mapping f_A is welldefined.
VERY IMPORTANT

Example:

1) Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$. Then $f_A(\vec{x}) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \\ 5x_1 + 6x_2 \end{pmatrix}$ $\mathbb{R}^3$!

Given $m, n \geq 1$ mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear if
1) $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$, $f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y})$
2) $\forall t \in \mathbb{R}$, $\forall \vec{x} \in \mathbb{R}^n$, $f(t\vec{x}) = t f(\vec{x})$.

A linear operator is a linear mapping whose domain = codomain
↳ $\mathbb{R}^n \to \mathbb{R}^n$ $\quad$ ✱ $[0,1] \longrightarrow \mathbb{R}$ not linear.

Oct 4, 2017

Linearity Continued:

The following mappings are linear:

a) Given $\vec{v} \in \mathbb{R}^n$, $proj_{\vec{v}}: \mathbb{R}^n \to \mathbb{R}^n$
$\vec{x} \mapsto proj_{\vec{v}}(\vec{x})$ ← and perp

→ B/C $proj_{\vec{v}}(\vec{x}+\vec{y}) = proj_{\vec{v}}(\vec{x}) + proj_{\vec{v}}(\vec{y})$ $\}\ \vec{v} \neq \vec{0}$
$proj_{\vec{v}}(t\vec{x}) = t\,proj_{\vec{v}}(\vec{x})$

b) Given $\vec{v} \in \mathbb{R}^n$, $S_{\vec{v}}: \mathbb{R}^n \to \mathbb{R}$
$\vec{x} \mapsto \vec{v} \cdot \vec{x}$. ← and cross product $\mathbb{R}^3$ ← !

→ Same properties for dot product.

Very similar properties to subspaces.

Not Linear:

a) $\|\cdot\|: \mathbb{R}^n \to \mathbb{R}$
$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \sqrt{x_1^2 + \dots x_n^2}$

If linear, $\|\vec{x}+\vec{y}\| = \|\vec{x}\| + \|\vec{y}\|$ ← Δ ineq, tho
→ $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

→ Give counter example w $\vec{e}_1, \vec{e}_2$.

1) Any linear mapping $f: \mathbb{R}^n \to \mathbb{R}^m$ always maps $\vec{0}_n \to \vec{0}_m$ $(t=0)$

2) Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be linear. Let $\vec{x}_1, \dots \vec{x}_k \in \mathbb{R}^n$, $t_1, \dots t_k \in \mathbb{R}$,

→ $f(t_1\vec{x}_1 + \dots t_k\vec{x}_k) = t_1 f(\vec{x}_1) + \dots + t_k f(\vec{x}_k)$

→ $f\left(\sum_{i=1}^{k} t_i \vec{x}_i\right) = \sum_{i=1}^{k} t_i f(\vec{x}_i)$

→ mapping of lin. comb is lin comb. of mapping.

Proof of 2: $k=3$

$f(t_1\vec{x}_1 + t_2\vec{x}_2 + t_3\vec{x}_3) = f(t_1\vec{x}_1 + t_2\vec{x}_2) + f(t_3\vec{x}_3)$

Just example not really proof. $\{$ $= f(t_1\vec{x}_1) + f(t_2\vec{x}_2) + f(t_3\vec{x}_3)$
$= t_1 f(\vec{x}_1) + t_2 f(\vec{x}_2) + t_3 f(\vec{x}_3)$

Theorem:

Let $A = m \times n[\ ]$, then,

$f_A : \mathbb{R}^n \to \mathbb{R}^m$
$\vec{x} \longmapsto A\vec{x}$ is linear..

Proof: Let $\vec{x}, \vec{y} \in \mathbb{R}^n$, $t \in \mathbb{R}$.
1) $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$
2) $A(t\vec{x}) = t(A\vec{x})$!

Yet Another Theorem:

Let $A = m \times n[\ ]$, mapping $f_A : \mathbb{R}^n \to \mathbb{R}^m$.

→ f_A is uniquely determined by the values it takes on the standard basis $(\vec{e}_1, \ldots \vec{e}_n) \in \mathbb{R}^n$.

→ $f_A(\vec{x}) = x_1 f_A(\vec{e}_1) + \ldots x_n f_A(\vec{e}_n)$, or

$\sum_{k=1}^{n} x_k f_A(\vec{e}_k)$. components of $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

→ Proof of above:
$\vec{x} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \ldots + x_n \vec{e}_n$, so
$f_A(\vec{x}) = x_i f_A(\vec{e}_i)$.

Consider $y = ax$. $\left.\begin{array}{l} f : \mathbb{R} \to \mathbb{R} \\ x \longmapsto ax \end{array}\right\}$ is linear! That's why it's called linear ö.

iff for $\mathbb{R}^1$

→ Conversely, if $g : \mathbb{R} \to \mathbb{R}$ is linear, $\exists\ a \in \mathbb{R}$ (depends on g), where $g(t) = at$, $t \in \mathbb{R}$.

$g(t) = g(t \times 1) = t\,g(1) = g(1)\,t$

Let $a = g(1)$

$f: \mathbb{R} \to \mathbb{R}$ linear $\Leftrightarrow \exists \alpha \in \mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto \alpha x$ $(\alpha = f(1))$ Oct 5, 2017.

so... $\hookrightarrow$ $f: \mathbb{R}^n \to \mathbb{R}^m$ linear $\Leftrightarrow \exists A(m \times n)$, $f_A: \mathbb{R}^n \to \mathbb{R}^m$, $\vec{x} \mapsto A\vec{x}$

both ways!

Theorem:

Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then, can be represented as a matrix mapping where $[L]_{m \times n}$:

$$[L] = [L(\vec{e}_1), L(\vec{e}_2), L(\vec{e}_3), \ldots, L(\vec{e}_n)]$$

$$L: \mathbb{R}^n \to \mathbb{R}^m$$
$$\vec{e}_i \mapsto L(\vec{e}_i)$$
$$\vec{x} \mapsto L(\vec{x}) = [L]\vec{x}.$$

Proof:

Let $\vec{x} \in \mathbb{R}^n$ be:

$$\vec{x} = x_1 \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \ldots + x_n \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} = \sum_{i=0}^{n} x_i \vec{e}_i.$$

By linearity: $L(\vec{x}) = L\left(\sum_{i=1}^{n} x_i \vec{e}_i\right) = \sum_{i=1}^{n} x_i L(\vec{e}_i)$

$\therefore$, $[L]\vec{x} = L(\vec{x})$.

$\hookrightarrow$ From this we can conclude linearity $\Leftrightarrow$ matrix mapping

Example:

1) Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$, then $\left.\begin{array}{l} C_{\vec{v}}: \mathbb{R}^3 \to \mathbb{R}^3 \\ \vec{x} \mapsto \vec{v} \times \vec{x} = [C_{\vec{v}}]\vec{x}. \end{array}\right\}$ is linear

Only works for linear mappings.

$$\left\{\begin{array}{l} C_{\vec{v}}(\vec{e}_1) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \vec{e}_1 = \begin{pmatrix} 0 \\ v_3 \\ -v_2 \end{pmatrix} \\ C_{\vec{v}}(\vec{e}_2) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \vec{e}_2 = \begin{pmatrix} -v_3 \\ 0 \\ v_1 \end{pmatrix} \\ C_{\vec{v}}(\vec{e}_3) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \vec{e}_3 = \begin{pmatrix} v_2 \\ -v_1 \\ 0 \end{pmatrix} \end{array}\right\}$$

thus, $[C_{\vec{v}}] = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$

$\therefore$, $\vec{v} \times \vec{x} = [C_{\vec{v}}]\vec{x}$

2) $S_{\vec{v}} : \mathbb{R}^n \to \mathbb{R}$ where $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$

$\vec{x} \longmapsto \vec{v} \cdot \vec{x}$

Let $1 \leq i \leq n$. $S_{\vec{v}}(\vec{e}_i) = \vec{v} \cdot \vec{e}_i = v_i$

$\Rightarrow \therefore [S_{\vec{v}}] = [v_1, v_2 \ldots v_n] = \vec{v}^T$.

$\Rightarrow \therefore$ For all $\vec{x} \in \mathbb{R}^n$, $\vec{v} \cdot \vec{x} = [v_1, \ldots, v_n]\vec{x} = \vec{v}^T\vec{x}$.

3) $proj_{\vec{v}} \; \mathbb{R}^n \to \mathbb{R}^n$ where $\vec{v} \neq \vec{0}$.

$\vec{x} \longmapsto proj_{\vec{v}}(\vec{x})$

$$proj_{\vec{v}}(\vec{e}_i) = \frac{\vec{e}_i \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{1}{\|\vec{v}\|^2}\begin{pmatrix} v_i v_1 \\ \vdots \\ v_i v_n \end{pmatrix}$$

$$\therefore [proj_{\vec{v}}] = \frac{1}{\|\vec{v}\|^2}\begin{pmatrix} v_1 v_1 & \cdots & v_n v_n \\ \vdots & & \vdots \\ v_1 v_n & \cdots & v_n v_n \end{pmatrix}$$

$[proj_{\vec{v}}]_{ij} = \frac{v_i v_j}{\|\vec{v}\|^2}$ *Try: prove $[proj_{\vec{v}}] = \frac{1}{\|\vec{v}\|^2}\vec{v}(\vec{v}^T)$

Definitions:

(1) Let $L : \mathbb{R}^n \to \mathbb{R}^m$, $M : \mathbb{R}^n \to \mathbb{R}^m$ be linear mappings.

$L+M : \mathbb{R}^n \to \mathbb{R}^m$ ← sum of each mapping (matrices)

$\vec{x} \longmapsto (L+M)\vec{x} = L(\vec{x}) + M(\vec{x})$

(2) Let $t \in \mathbb{R}$.

$tL : \mathbb{R}^n \to \mathbb{R}^m$

$\vec{x} \longmapsto (tL)(\vec{x}) = tL(\vec{x})$

(3) Let $N : \mathbb{R}^m \to \mathbb{R}^p$ be another linear mapping.

$N \circ L : \mathbb{R}^m \to \mathbb{R}^p$

$\vec{x} \longmapsto (N \circ L)(\vec{x}) = N(L(\vec{x}))$

} $\mathbb{R}^m$ must be same w/ L. (like Matrix multi)

*Watch: $\mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \xrightarrow{N} \mathbb{R}^p$

These 3 end up still being linear. *Try: Prove.

Proposition:

1) Matrix sum of lin. maps = sum of indv. matrices.
$[L+M] = [L] + [M]$

2) Matrix of sc. multiple of a lin map = sc. multiple of matrix.
$[tL] = t[L]$

Proof:

1) Let $\vec{x} \in \mathbb{R}^n$: $[L+M]\vec{x} = (L+M)\vec{x} = L(\vec{x}) + M(\vec{x}) = [L]\vec{x} + [M]\vec{x}$

↳ Implies $([L]+[M])\vec{x} = [L+M]\vec{x}$, QED.

↳ Try 2).

THE ANSWER TO ALL YOUR PROBLEMS

Theorem:

Let $L: \mathbb{R}^n \to \mathbb{R}^m$, $N: \mathbb{R}^m \to \mathbb{R}^p$ be two linear mappings.
$[N \circ L] = [N][L]$ ← THIS IS WHY MATRIX MULTI EXISTS!!
↳ Not defined when $[N][L]$ is not defined!!!

Proof:

Let $[L]_{m \times n}$, $[N]_{p \times m}$ where $[N][L]$ is well defined.
Let $\vec{x} \in \mathbb{R}^n$: $[N \circ L](\vec{x}) = (N \circ L)\vec{x} = N(L(x))$
$= N([L]\vec{x}) = [N]([L]\vec{x})$
$= ([N][L])\vec{x}$

Example:

1) Let $L: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \xrightarrow{\mathbb{R}^2 \to \mathbb{R}^2} \begin{pmatrix} 2x_1 + 3x_2 \\ x_1 - x_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$N: \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2 \longmapsto \begin{pmatrix} y_1 + y_2 \\ -3y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

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