MATH 119

Approximation: Jan 3, 2018

1) Numerical: with computer

2) Analytical: with calc

Linear Approximation:

Find a line $L(x)$ similar to fcn.

$\rightarrow \sin(x^2) \approx \sin(x)$

Example:

1) Find $L(x)$ at $(a, f(a))$

1) $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

2) $y - f(a) = f'(a)(x-a)$

3) $L(x) = y = f(a) + f'(a)(x-a)$

2) $\sin(0.1)$? $\sin(0) = 0$, so...

$$f'(0) = \lim_{x \to 0} \frac{\sin(x) - 0}{x - 0} \overset{L'H}{=} \lim_{x \to 0} \cos(x) = 1$$

$L(x) = 0 + 1(x - 0) = x$. so $\sin(0.1) \approx 0.1$.

3) $e^{0.1}$? $e^0 = 1$, so...

$$f'(0) = \lim_{x \to 0} \frac{e^x - 1}{x} = e^0 = 1.$$

$L(x) = 1 + 1(x-0) = x+1$ so $e^{0.1} \approx 0.1 + 1 \approx 1.1$

3) $m = \frac{m_0}{\sqrt{1 - (\frac{v}{c})^2}}$. Find approx for small v.

$f(x) = \frac{1}{\sqrt{1-x^2}}$ (constants can come back at end)

$g(u) = \frac{1}{\sqrt{1-u}}$ $v = 0 \Rightarrow u = 0$, so approx near $u = 0$.

$g'(0) = \frac{1}{2}$. so,

$L(u) = 1 + \frac{1}{2}(u - 0)$

$= \frac{1}{2}u + 1$.

$\therefore f(x) \approx \frac{1}{2}x^2 + 1$

$m \approx m_0(1 + \frac{1}{2}(\frac{v}{c}

Root Finding:

Jan 5, 2018

$x - e^{-x} = 0$

1) Bisection:

$f(0) = -1$ and $f(1) \approx 0.63$ — $(0, 1)$

$f(1/2) \approx -0.11$ — $(\frac{1}{2}, 1)$

$f(3/4) \approx 0.28$ — $(\frac{1}{2}, 3/4)$

$f(5/8) \approx 0.09$ — $(\frac{1}{2}, 5/8)$

$\therefore$ root @ 0.6.

Always converges, but slowly.

2) Newton's:

Replace $f(x)$ w/ $L(x)$. start w/ bisection to get close (0.6).

$(0.06, 0.05)$ $\quad L_{x_0}(x) = 0.05 + 1.55(x - 0.6) = 0$

$x \approx 0.5677$

repeat w/ this as starting pt.

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

* → Doesn't work w/o defined deriv.

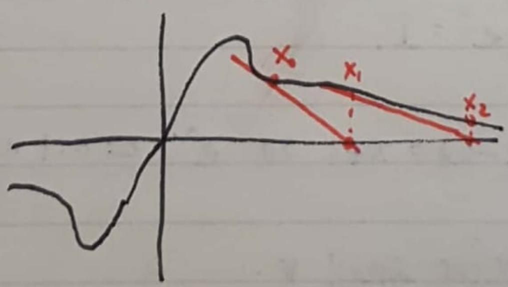
→ and deriv $\neq 0$ at root

If $f'(x) = 0$ at root: $f(x) = x^n$

$$x_k = x_{k-1}\left(1 - \frac{1}{n}\right)$$
$$= \left(1 - \frac{1}{n}\right)^k$$

no faster than bisection (slow)

→ Doesn't work sometimes w/ bad starting pt.



gets further from root (or dif. root)
(use bisection first)

Polynomial Interpolation:

Jan 8, 2018

$(0,5),(1,3),(2,6),(3,4)$ $y = a + bx + cx^2 + dx^3$

$y_0: 5 = a$

$y_1: 3 = a + b + c + d$

$y_2: 6 = a + 2b + 4c + 8d$

$y_3: 4 = a + 3b + 9c + 27d$

$\Delta y_0 = y_1 - y_0 = b + c + d$

$\Delta y_1 = y_2 - y_1 = b + 3c + 7d$

$\Delta y_2 = y_3 - y_2 = b + 5c + 19d$

repeat: $\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = 2c + 6d$

$\Delta^2 y_1 = \Delta y_2 - \Delta y_1 = 2c + 12d$

And, once more,

$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = 6d.$

$\boxed{a = y_0}$ $\boxed{d = \frac{\Delta^3 y_0}{6}}$ Now, back sub.

$c = \dfrac{\Delta^2 y_0 - 6d}{2} = \dfrac{\Delta^2 y_0 - \Delta^3 y_0}{2}$

$b = \Delta y_0 - c - d = \Delta y_0 - \dfrac{\Delta^2 y_0}{2} + \dfrac{\Delta^3 y_0}{3}$

y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
5			
	-2		
3		5	
	3		-10
6		-5	
	-2		
4			

Sub to get $a = 5, d = \frac{-5}{3}, c = \frac{15}{2}, b = \frac{-47}{6}$

General form: $y = y_0 + x\Delta y_0 + x(x-1)\dfrac{\Delta^2 y_0}{2} + x(x-1)(x-2)\dfrac{\Delta^3 y_0}{6}$

n points: $y = y_0 + x\Delta y_0 + x(x-1)\dfrac{\Delta^2 y_0}{2} + \dots + x(x-1) \times \dots \times (x-n+1)\dfrac{\Delta^n y_0}{n!}$

↑ mult;.

Assumptions:

1) x coord... 0,1,2,3...

2) Points are equidistant ↑ removes this assumption

Newton Forward Distance Formula:

n+1 equidistand nodes, $x_n = x_0 + nh$ (h = distance).

$y = y_0 + \dfrac{(x-x_0)}{h}\Delta y_0 + \dfrac{(x-x_0)(x-x_1)}{2!\,h^2}\Delta^2 y_0 + \dots + \dfrac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{n!\,h^n}\Delta^n y_0$

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Example:

1) Estimate $f(2.45)$ if $f(x)$ passes through $(2,4), (2.2,5), (2.4,4), (2.6,2)$

$x_0 = 2$

$h = 0.2$

$n = 4$

y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
4	1	-2	1
5	-1	-1	
4	-2		
2			

$$y = 4 + \frac{(x-2)}{0.2}\cdot(1) + \frac{(x-2)(x-2.2)}{2\cdot(0.2)^2}(-2) + \frac{(x-2)(x-2.2)(x-2.4)}{6(0.2)^3}\cdot(1)$$

$f(2.45) \approx 3.56$ (no clue on how accurate)

Linear Interpolation:

Assume straight line btwn each pair.

Asymptotes can't be approx by polynomials.

Given $(x_0,y_0)(x_1,y_1)$ $\Big\}$ $y = y_0 + \frac{(x-x_0)}{(x_1-x_0)}(y_1-y_0)$

$$f(x) = \frac{(x_1-x)}{(x_1-x_0)}y_0 + \frac{(x-x_0)}{(x_1-x_0)}y_1$$

$f(2.45)$: $(2.4,4), (2.6,2)$

$$f(2.45) \approx \frac{0.15}{0.2}(4) + \frac{0.05}{0.2}(2) \approx 3.5$$

Taylor Polynomials:

$(x_0, f(x_0)), (x_0+\Delta x, f(x_0+\Delta x)), (x_0+2\Delta x, f(x_0+2\Delta x))$

NFDF: $y = y_0 + (x-x_0)\frac{\Delta y_0}{\Delta x} + \frac{1}{2}(x-x_0)(x-x_1)\frac{\Delta^2 y_0}{(\Delta x)^2}$

Let $\Delta x \to 0$ (merge points)

$$y \to \underbrace{y_0 + (x-x_0)f'(x_0)}_{\text{Linear approx}} + \frac{1}{2}(x-x_0)^2 f''(x_0)$$

whole thing quadratic approx. (more accurate)

$$P_{n,x_0}(x) = f(x) + f'(x_0)(x-x_0) + \frac{1}{2}f(x_0)(x-x_0)^2 + \dots + \frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n$$

$$= \sum_{K=0}^{n} \frac{f^{(K)}(x_0)}{K!}(x-x_0)^K \quad \text{where} \quad \frac{\Delta^k y_0}{(\Delta x)^k} \to f^{(k)}(x_0)$$

Example:

1) $f(x) = e^x$ near $x=0$

$x_0 = 0$ Lin app: $f(x) \approx 1 + x$

$f(x_0) = 1$ Quad app: $f(x) \approx 1 + x + \frac{1}{2}x^2$

$f'(x_0) = 1 = f''(x_0)$ ← matches concavity

2) $\sqrt{4.5} \approx \sqrt{4} = 2$

$f'(x) = \frac{1}{2\sqrt{x}} \Big|_{x=4} = \frac{1}{8}$

$f''(x) = \frac{-1}{4\sqrt{x^3}} = \frac{-1}{32}$

$y \approx 2 + \frac{1}{8} - \frac{1}{32}\left(\frac{1}{2}\right)^3$

≈ 2.12109

3) $\int_0^{0.5} \tan^{-1}(x^2)\,dx \approx \int_0^{0.5} P_{6,0}(x) \cdot dx$

$f(x) = \tan^{-1}(x^2)$ $f(0) = 0$ $P_{6,0}(x) = \frac{1}{2} \cdot 2 \cdot (x)^2 + \frac{1}{3}x^6$

$f'(x) = \frac{2x}{1+x^4}$ $f'(0) = 0$ $= x^2 + \frac{1}{3}x^6$

$f''(0) = 2$, and so on ... (then integrate).

Maclaurin's Approach:

Want $P_{n,x_0}(x)$ and it's deriv. to have same $f(x_0), f'(x_0), f''(x_0)$ etc.

$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots + a_n(x - x_0)^n$

$p(x_0) = a_0 = f(x_0)$ Derive to find that:

$p'(x_0) = a_1 = f'(x_0)$

$p''(x_0) = 2a_2$, so $a_2 = \frac{f''(x_0)}{2}$ } keep going to get original T. poly. eqn.

→ implies unique approximation.

Taylor Polynomial centered at $x_0 = 0$: Maclaurin Poly.

$$P_{n,0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

Shortcut:

Linear: $f(x) \approx f(0) + f'(0)x$ Let $x = t^2$

$f(t^2) \approx f(0) + f'(0)t^2$ ← Quadratic

$h(t) = f(t^2)$ $Q(t) = f(0) + f'(0)t^2$

→ $Q(t)$ is $TP_2(\cdot)$ for $h(t)$ ⇒ same $f(x_0)$, $f'(x_0)$

① $\begin{cases} h(0) = f(0) \\ Q(0) = f(0) \end{cases}$ ② $\left.\begin{cases} h'(t) = 2t f'(t^2) \\ Q'(t) = 2f'(0)t \end{cases}\right\}$ $\begin{matrix} h'(0) = 0 \\ Q'(0) = 0 \end{matrix}$

③ $\left.\begin{cases} h''(t) = 2f'(t^2) + 2tf''(t^2)(2t) \\ Q''(t) = 2f'(0) \end{cases}\right\}$ $\begin{matrix} h''(0) = 2f'(0) \\ Q''(0) = 2f'(0) \end{matrix}$

1, 2, 3 means $Q(t)$ is 2nd order T.P. for $h(t)$.

General Case:

Finding mnth M.Poly for $f(kt^m)$ can be done by letting $x = kt^m$ ($k \in \mathbb{R}, m > 0$) and finding nth order poly.

→ $f^{(mn)}(0)$ must exist.

Example:

1) $f(x) = e^{x^2}$ near $x = 0$. $f(t) = e^t$ where $t = x^2$.

$g(t) \approx 1 + t + \frac{1}{2}t^2$

so $g(x^2) \approx 1 + x^2 + \frac{1}{2}x^4$

Accuracy of Taylor Polynomials: Jan 12, 2018

$|a+b| \leq |a| + |b|$ – Scalars. Integrals?

$$\left|\int_a^b f(x)dx\right| = \left|\sum_{i=1}^{n} f(x_i)\Delta x_i\right| \quad (\text{as } n \to \infty)$$

$$\leq \sum_{i=1}^{n} |f(x_i)\Delta x_i| \quad b \geq a, \text{ so } \Delta x_i \text{ is } +.$$

$$= \sum_{i=1}^{n} |f(x_i)| \Delta x_i$$

$$= \int_a^b |f(x)| \cdot dx$$

$a = x_0$, $b = x$ (variable).

$$\int_{x_0}^{x} f'(t)\cdot dt = f(x) - f(x_0)$$

$$f(x) = \underbrace{f(x_0)}_{P_{0,x_0}(x)} + \int_{x_0}^{x} f'(t)\,dt$$

Now, IBP: $\left\{ \begin{array}{ll} u = f'(t) & dv = dt \\ du = f''(t)dt & v = t \end{array} \right.$

$$= f(x_0) + \Big[t f'(t)\Big]_{x_0}^{x} - \int_{x_0}^{x} t f''(t)\cdot dt$$

$$= f(x_0) + x f'(x) - x_0 f'(x_0) - \int_{x_0}^{x} t f''(t)\,dt$$

$$= \underbrace{f(x_0) + (x - x_0) f'(x_0)}_{P_{1,x_0}(x)} \underbrace{- x f'(x_0) + x f'(x) - \int_{x_0}^{x} t f''(t)\,dt}_{\text{remainder}}$$

$$= P_{1,x_0}(x) + x\left(f'(x) - f'(x_0)\right) - \int_{x_0}^{x} t f''(t)\cdot dt$$

$$= P_{1,x_0}(x) + x\int_{x_0}^{x} f''(t)\cdot dt - \int_{x_0}^{x} t f''(t)\,dt$$

$$= P_{1,x_0}(x) + \int_{x_0}^{x} (x - t) f''(t)\cdot dt$$

IBP again $\left\{ \begin{array}{ll} u = f''(t) & dv = (x-t)dt \\ du = f'''(t) & v = -\frac{1}{2}(x-t)^2 \end{array} \right.$

$$= P_{1,x_0}(x) + \left[\frac{-1}{2}(x-t)^2 f''(t)\right]_{x_0}^{x} - \int_{x_0}^{x} \frac{-1}{2}(x-t)^2 f'''(t)\cdot dt$$

$$= \underbrace{P_{1,x_0}(x) + \frac{1}{2}(x - x_0)^2 f''(x_0)}_{P_{2,x_0}(x)} + \frac{1}{2}\int_{x_0}^{x} (x-t)^2 f'''(t)\,dt$$

$$= P_{2,x_0}(x) + \underbrace{\frac{1}{2}\int_{x_0}^{x} (x-t)^2 f'''(t)\cdot dt}_{}$$ Remainder

Taylor Theorem $\bar{w}$ Integral Remainder:

Suppose f has $n+1$ deriv. at x_0. Then,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_n(x) = P_{n,x_0}(x) + R_n(x)$$

$$R_n(x) = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\cdot dt$$

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Suppose $|f^{(n+1)}(t)| \le K$ for $t \in [x_0, x]$ (some constant K)

↳ just bound derives to same line (no VAs, so)

$$|R_n(x)| = \left| \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t) \cdot dt \right|$$

If $x > x_0$, then Δ ineq. applies.

$$\le \int_{x_0}^{x} \left| \frac{(x-t)^n}{n!} f^{(n+1)}(t) \cdot dt \right|$$

$t \in [x_0, x]$, so $(x-t)^n \ge 0$.

$$= \int_{x_0}^{x} \frac{(x-t)^n}{n!} |f^{(n+1)}(t)| dt$$

$$\le \int_{x_0}^{x} \frac{(x-t)^n}{n!} K \cdot dt$$

$$= \frac{-K}{(n+1)!} \left[(x-t)^{n+1} \right]_{x_0}^{x}$$

$$= \frac{K}{(n+1)!} (x-x_0)^{n+1}$$

Else if $x < x_0$, then:

Just swap $x \leftrightarrow x_0$, negative gets abs valued.

now, $|R_n(x)| = \left| \int_{x}^{x_0} \frac{(x-t)^n}{n!} f^{(n+1)}(t) dt \right|$

If n is even, you get $-$ve version. Odd you get $+$.

Taylor's Inequality:

$$|R_n(x)| \le K \frac{|x-x_0|^{n+1}}{(n+1)!}$$ where $|f^{(n+1)}(z)| \le K \ \forall z \in [x_0, x]$

Finding K: Jan 15, 2018

1) Approx e w/ 7^{th} Mc. Poly. Given $e = 2.7???\ldots$

$f(x) = e^x,\ x_0 = 0$

$f^{(m)}(x_0) = 1$

$P_{7,0}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}$

$P(1) = 2.7182\ldots$

$f^{(8)}(x) = e^x$, so

$|e^x| \le K$ for some $x \in [0,1]$.

* $e^1 > e^0$ (increasing) on interval. ∴ endpoint (e^1) is max.

$e^1 < 3$, so let $K = 3$ closer K = less error.

$|R_n(x)| \le 3 \frac{|x|^{n+1}}{(n+1)!}$

$|R_7(1)| \le 3 \cdot \frac{1}{8!}$

$= \frac{1}{13440}$ (less than 10^{-4}, 0.0001, away).

∴ accurate to 3 dec places for sure (except rounding, maybe)

$P(1) \approx 2.71825$

Change by + or − 1 still rounds to:

≈ 2.718 ∴ On $[0,1]$,

$P_{7,0}(x) - \frac{x^8}{13440} \le e^x \le P_{7,0}(x) + \frac{x^8}{13440}$ } applies to whole interval

2) Suppose $f^{(n+1)}(x) = (x^2+2)e^{-x}$, interval $[0,2]$. Find K.

(x^2+2): Increasing

↳ max = 6 ($x = 2$)

e^{-x} : Decreasing

↳ max = 1 ($x = 0$)

∴ Entire function bounded by 6.

* Both fcns positive on interval

3) Suppose $f^{(n+1)}(x) = (x^2-1)x$, interval $[0,1]$.

$0 \le x \le 1$

$-1 \le x^2 - 1 \le 0$.

need magnitudes.

$|(x^2-1)x| = |(x^2-1)||x|$

$\le (1)(1)$

$= 1$

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4) $f^{(n+1)}(x) = x^3 - 2x^2 - 5x + 30$ on $[-3, 0]$ — Jan 17, 2018

Not monotonic.

$$f^{(n+1)}(x) \le |x^3| + |-2x^2| + |-5x| + |30|$$
$$= |x|^3 + 2|x|^2 + 5|x| + 30$$

Now, monotonic (decreasing).

$g(0) = 30$, $g(-3) = 90$. $\therefore \le 90$.

5) $f^{(n+1)}(x) = \sin x - \cos x + \ln x - e^{-x}$ on $[\pi, 2\pi]$

$$\le |\sin x| + |-\cos x| + |\ln x| + |e^{-x}|$$
$$\le 1 + 1 + \ln(2\pi) + 1$$
$$\le 3 + \ln(e^3)$$
$$\le 6.$$

Approximation for Integrals:

1) $f(x) = \int_0^x e^{t^2} dt \qquad x \in [\frac{-1}{2}, \frac{1}{2}]$

Let $g(u) = e^u$.
$\approx P_{2,0}(u)$
$= 1 + u + \frac{1}{2}u^2$

$e^{t^2} \approx 1 + t^2 + \frac{1}{2}t^4 \quad (P_{4,0}(t))$

$$\therefore f(x) \approx \int_0^x (1 + t^2 + \tfrac{1}{2}t^4)\,dt$$
$$= x + \tfrac{1}{3}x^3 + \tfrac{1}{10}x^5 \qquad f(\tfrac{1}{2}) \approx 0.544791\overline{66}$$

Accuracy: $e^u = P_{2,0}(u) + R_2(u)$

$|R_2(u)| \le K \frac{|u|^3}{3!}$

$x \in [\frac{-1}{2}, \frac{1}{2}]$, $t \in [0, x]$, $\in [-\frac{1}{2}, \frac{1}{2}]$, $u \in [0, \frac{1}{4}]$

$|g^{(3)}(z)|$ on $z \in [0, \frac{1}{4}]$:
$\le e^{\frac{1}{4}}$
< 2 (or 3)

$\therefore |R_2(u)| \le \frac{|u|^3}{2}$
$= \frac{u^3}{2}$ (interval +)

$\therefore e^{t^2} \approx 1 + t^2 + \frac{1}{2}t^4 + R_2(t^2)$

$|R_2(t^2)| \le \frac{t^6}{2}$

→

$$\int_0^x e^{t^2}\,dt = \int_0^x P_{2,0}(t^2)\,dt + \int_0^x R_2(t^2)\,dt$$

$x>0$: where $\left|\int_0^x R_{2,0}(t^2)\,dt\right| \leq \int_0^x |R_2(t^2)|\,dt$ for $x \in [0, \frac{1}{2}]$

$$\leq \frac{x^7}{14}$$

If $x<0$, then:

$$\left|\int_0^x R_{2,0}(t^2)\,dt\right| = \left|\int_x^0 R_{2,0}(t^2)\,dt\right|$$

$$\leq \int_x^0 |R_{2,0}(t^2)|\,dt$$

$$\leq \frac{-x^7}{14}$$

$\therefore \left|\int_0^x R_2(t^2)\,dt\right| \leq \frac{|x|^7}{14}$

$\therefore \int_0^x e^{t^2}\,dt = x + \frac{1}{3}x^3 + \frac{1}{10}x^5 \left(\pm \frac{x^7}{14}\right)$

$\int_0^{\frac{1}{2}} e^{t^2}\,dt = 0.5447916\overline{6} \pm 0.00056.$

2) $f(x) = \int_0^x \cos(t)\,dt \quad x \in [0, \frac{1}{2}]$

Jan 18, 2018

Same thing: Can integrate first, then approx $\sin(x)$ $\bar{w}$ $P_{5,0}(x)$. OR:

Approx $\cos(x)$ first, then integrate.

$$f(x) \approx \int_0^x 1 - \frac{t^2}{2!} + \frac{t^4}{4!}\,dt$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Accuracy: $x \in [0, \frac{1}{2}] \Rightarrow t \in [0, \frac{1}{2}]$

$|R_5(t)| \leq 1 \cdot \frac{|t^6|}{6!}$ since $g^{(6)}(t) = -\cos(t)$ (bounded by 1).

$\left|\int_0^x R_5(t)\,dt\right| \leq \int_0^x \frac{t^6}{6!} = \frac{x^7}{7!}$ $\quad \therefore f(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} \left(\pm \frac{x^7}{7!}\right)$

Bound is next term

Jan 22, 2018

Infinite Series:

$f(x) = \sin(x)$

$$P_{2n+1,0}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \Big\} x = 3?$$

0, 1, 0, −1, etc

↳ converges, P_{11} and $P_{13,0}(x) \cdot 0.141$.

↳ no matter what x is.

$f(x) = \frac{1}{1+x}$. $P_{n,0}(x) = (-1)^n x^n$ $f(2) = \frac{1}{3}$

↳ Diverges, 1, −1, 3, −5 . . . .

what's the diff?

$$\sin(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \underbrace{R_n(x)}$$

want $\lim_{n\to\infty} R_n(x) = 0$ for convergence

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \Big\} \text{For } \underline{\text{any } n}, \ K=1.$$

$(n+1)!$ grows faster than $|x|^{n+1}$, so $\lim_{n\to\infty} = 0$.

$$g(x) = \lim_{n\to\infty} \sum_{k=0}^{n} (-1)^k x^k + \lim_{n\to\infty} R_n(x)$$

$|R_n(x)| \leq K \frac{|x|^{n+1}}{(n+1)!}$ on $[0,2]$, $K = 1, 1$, then 2, 6 . . . .

$K = (n+1)!$ is the best bound. ↑exact bounds

If $|x| > 1$: $\lim_{n\to\infty} R_n(x) = \infty$, so diverges. } within $[-1,1]$
else, $\lim_{n\to\infty} R_n(x) = 0$, so converges. } is "good enough"

Taylor Series:

$$\sum_{k=0}^{\infty} \frac{f^k(x_0)(x-x_0)^k}{k!} \Big\} \text{ if } \lim_{n\to\infty} R_n(x) = 0, \text{ convergence.}$$

Infinite series:

Jan 24, 2018

$\sum_{k=0}^{\infty} a_k$ for constants a_k.

Sequence $\{a_k\}$, list of #s
Series is partial-sum (or ∞)

Reindexing:

$$\sum_{k=q}^{\infty} a_k = \sum_{j=0}^{\infty} a_{j+q}$$

→ if started at 0 vs 1 for $\frac{1}{2^k}$, they will both converge (but to different values).

Geometric Series:

$\sum_{k=0}^{\infty} ar^k$ r: ratio. $\sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}$

↳ $= \lim_{n\to\infty} \frac{a(1-r^{n+1})}{1-r}$

If $|r|<1$: $\lim = \frac{a}{1-r}$ (converges)
If $|r|>1$: Diverges
If $r=1$: Diverges
If $r=-1$: Diverges $[(-1)^{n+1}$ as $n\to\infty]$ 0, 1, 0, 1 ....

Examples:

1) $\sum_{k=0}^{\infty} \frac{(-4)^{3k}}{5^{k+1}} = \sum_{k=0}^{\infty} \left(\frac{-64}{5}\right)^k \cdot \frac{1}{5}$ Diverges $|r|>1$

2) $\sum_{k=1}^{\infty} 4\left(\frac{2}{5}\right)^k = \sum_{k=0}^{\infty} 4\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)^k$ Converges to $\frac{8}{3}$

Divergence Test:

If $\lim_{k\to\infty} a_k \neq 0$, then $\sum a_k$ diverges. } number you add by $\neq 0$.

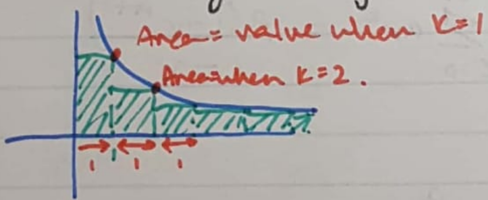
If $\lim_{k\to\infty} a_k = 0$, means nothing. (necessary, not sufficient)

Examples:

1) $\sum_{k=1}^{\infty} \frac{k^2+2}{4k^2-k}$ ← $\lim_{k\to\infty} \frac{k^2+2}{4k^2-k} = \frac{1}{4}$ ∴ diverges.

2) $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges. (why?)

Upper bound converges



box area ≤ curve area

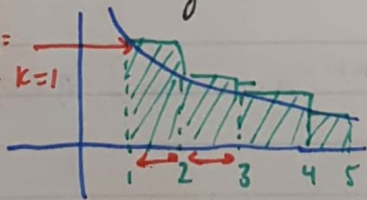
$$\sum_{k=1}^{\infty} \frac{1}{k^2} \leq \int_1^{\infty} \frac{1}{k^2}\,dk + 1 = 2$$

∴ converges to ≤2.

3) $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges * But lim = 0

Lower bound diverges

Area = value at k=1



box area ≥ curve area

$$\sum_{k=1}^{\infty} \frac{1}{k} \geq \int_1^{\infty} \frac{1}{k}\,dk = [\ln k]_1^{\infty}$$

Diverges, ∴ fcn diverges.

Integral Test:

$\sum_{k=k_0}^{\infty} a_k$ converges $\Longleftrightarrow$ $\int_{k_0}^{\infty} f(x)\,dx$ converges. $f(k)=a_k$

↳ Works for:

- $a_k > 0$. (or <0 and multiply converge by -1)
- f continuous ($\int$ exists)
- $f \to 0$ as $x \to \infty$ (Divergence test above)

Jan 26, 2018

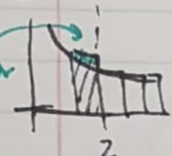
Example:

1) $\sum_{k=2}^{\infty} \frac{1}{k(\ln(k))^2} \rightarrow \lim_{n\to\infty} \int_2^n \frac{1}{x(\ln(x))^2} \cdot dx$

u-sub $u = \ln x$, $du = \frac{1}{x} dx$.

$\lim_{n\to\infty} \int_{\ln 2}^{\ln(n)} \frac{1}{u^2} du = \frac{1}{\ln(2)}$ $\therefore$ converges.

2) Lower bound:



Term for early values

$\sum_{k=1}^{\infty} \frac{1}{k^2} \geq \int_1^{\infty} \frac{1}{x^2} dx \iff 1 + \int_2^{\infty} \frac{1}{x^2} dx$ → more accurate if we take actual value.

$= 1 \qquad = \frac{3}{2}$

$\therefore \frac{3}{2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \leq 2$

To increase accuracy, evaluate more initial terms.

$\sum_{k=1}^{n} \frac{1}{k^2} + \int_{n+1}^{\infty} \frac{1}{k^2} dx \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \leq \sum_{k=1}^{n} \frac{1}{k^2} + \int_n^{\infty} \frac{1}{k^2} dx$ $\uparrow n = \uparrow$ accurate.

General Bounds:

$S_n = \sum_{k=1}^{n} a_k \qquad S_n + \int_{n+1}^{\infty} f(x) \cdot dx \leq \sum_{k=1}^{\infty} a_k \leq S_n + \int_n^{\infty} f(x) \cdot dx$

P-Series Test:

$\sum \frac{1}{k^p}$ converges $\iff p > 1$. (b/c integral test)

Comparision Test:

$\sum a_k$, $a_k > 0$. Identify $\sum b_k$ where $a_k \leq b_k \ \forall k$, and $\sum b_k$ converges. Then a_k converges.

↳ Same w/ diverging $\leq b_k$.

↳ Pick geometric or p-series.

Example: Jan 29, 2018

1) $\sum_{k=1}^{\infty} \frac{1}{2^k+\sqrt{k}}$; $a_k > 0$ and $\lim_{k\to\infty} a_k = 0$.

Not easy to integrate. lim = 0, so no help.

Not geometric/p series.

$\frac{1}{2^k+\sqrt{k}} \le \frac{1}{2^k}$ } Geometric $(\frac{1}{2})^k$, converges $|r|<1$.

Limit Comparision Test:

$\sum \frac{1}{k^2+2} \le \sum \frac{1}{k^2}$ (comp. Test)

$\sum \frac{1}{k^2-2}$; If $\lim_{k\to\infty} \frac{a_k}{b_k} = L$ (constant), then

$\uparrow$ non-zero.

$\sum a_k$ and $\sum b_k$ both converge/diverge.

$\lim_{k\to\infty} \frac{k^2}{k^2-2} = 1$. } ∴ same behaviour, both converge.

Alternating Series Test:

$\sum_{k=0}^{\infty} (-1)^k a_k$ $(a_k > 0)$

If $\lim_{k\to\infty} a_k = 0$ and $\{a_k\}$ eventually decreases, then converges.

Example:

$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ $\lim_{k\to\infty} \frac{1}{k} = 0$, and $\frac{1}{k}$ decreases. ∴ convergence.

Alternating Series Estimation Theorem:

If we use n^{th} term as estimate.

$$|S - S_n| \le a_{n+1}$$

"Absolute Convergence":

$\Sigma |a_k|$ converges. $\quad \Sigma \frac{(-1)^k}{k^2}$

"Conditional Convergence"

Σa_k converges, $\Sigma |a_k|$ doesn't. $\quad \Sigma \frac{(-1)^k}{k}$

The Ratio Test:

If $\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = L \ldots$

$L < 1$? Σa_k is abs. convergent.

$L > 1$? Σa_k is divergent.

$L = 1$? IDK.

Example:

1) $\underbrace{\Sigma (-1)^k \frac{2^k}{k!}}_{a_k \ (-1)^k \text{ gets } 1 \cdot 1}$ $\quad \lim_{k \to \infty} \left| \frac{\frac{2^{k+1}}{(k+1)!}}{\frac{2^k}{k!}} \right| = 0. < 1 \therefore$ abs conv.

2) $\Sigma \frac{3^k}{k^2 2^k}$ $\quad \lim_{k \to \infty} \left| \frac{\frac{3^{k+1}}{(k+1)^2 2^{k+1}}}{\frac{3^k}{k^2 2^k}} \right| = \frac{3}{2} > 1 \therefore$ divergent

The Root Test:

$L = \lim_{k \to \infty} \sqrt[k]{|a_k|}$ } Same cond. as ratio test.

Ratio/Root Test check if series behaves like geometric.

$$\lim_{k \to \infty} \left| \frac{ar^{k+1}}{ar^k} \right| = r$$

Jan 31, 2018

Power Series:

$$\sum_{k=0}^{\infty} c_k (x-x_0)^k$$ } Call T. series if obtained from f. deriv, etc.

convergence? Ratio test:

$$L = \lim_{k\to\infty} \left|\frac{c_{k+1}}{c_k}\right| |x-x_0|$$

For convergence, $L < 1$

$$|x-x_0| < \underline{\lim_{k\to\infty} \left|\frac{c_k}{c_{k+1}}\right|}$$ "close enough" R (radius of convergence)

↳ $R = 0$? convergence at $x = x_0$

↳ $R = \infty$? convergence $\forall x$

↳ R not? ? convergence for $x \in \underline{(x_0 - R, x_0 + R)}$ check endpoints

Example:

1) $f(x) = \sin x$ $\quad P_{\infty,0}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

$$L = \lim_{k\to\infty} \left|\frac{x^2}{(2k+2)(2k+3)}\right| < 1$$

$$x^2 < \lim_{k\to\infty} (2k+2)(2k+3) = R$$

$$\therefore x^2 < \infty$$

2) $g(x) = \frac{1}{1+x}$ $\quad P_{\infty,0}(x) = \sum_{k=0}^{\infty} (-1)^k x^k$

$$L = \lim_{k\to\infty} \left|\frac{x^{k+1}}{x^k}\right| = |x| < 1. \quad (R=1)$$

3) $\sum_{k=1}^{\infty} \frac{(x-3)^k}{k4^k}$ $\quad L = \lim_{k\to\infty} \left|\frac{(x-3)^{k+1} 4^k}{(k+1)4^{k+1}(x-3)^k}\right|$

$$= \lim_{k\to\infty} \left|\frac{k(x-3)}{4(k+1)}\right| \qquad \therefore |x-3| < 4$$

$$\left|\frac{x-3}{4}\right| < \lim_{k\to\infty} \left|\frac{k+1}{k}\right| = 1$$

→

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$x \in (-1, 7)$ convergence.

↳ Check endpoints

If $x = -1$, $x - 3 = -4$

$$\sum_{k=1}^{\infty} \frac{(-4)^k}{k4^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \Big\} \text{ Conditional Convergence}$$

If $x = 7$, $x - 3 = 4$

$$\sum_{k=1}^{\infty} \frac{4^k}{k4^k} = \sum_{k=1}^{\infty} \frac{1}{k} \Big\} \text{ Divergence}$$

$\therefore$ $x \in [-1, 7)$ for convergence.

Manipulation of Power Series:

If $\sum c_k (x - x_0)^k$ has r.o.c. R,

↳ Differentiate terms

↳ Integrate

↳ $\times$ by Constant

↳ Add to another series of radius $\geq R$. ($\times$ and $\div$ works too)

All resulting series will have r.o.c. R.

Example:

1) $f(x) = \frac{1}{1-x} : \sum_{k=0}^{\infty} x^k$ $\quad L = \lim_{k \to \infty} |x| \Rightarrow R = 1$

$f'(x) = \frac{+1}{(1-x)^2}$ $\quad \frac{d}{dx} \sum_{\boxed{k=0}}^{\infty} x^k = \sum_{\boxed{k=1}}^{\infty} k x^{k-1}$. $\quad$ R still 1.

$F(x) = -\ln(1-x)$ $\quad \int \sum_{k=0}^{\infty} x^k \, dx = \sum_{\boxed{k=0}}^{\infty} \int x^k \, dx = \sum_{\boxed{k=0}}^{\infty} \frac{x^{k+1}}{k+1} + C$

Find C: Let $x = 0$

$$\ln(1) = \sum_{k=0}^{\infty} \frac{0}{k+1} + C \quad \therefore C = 0$$

Feb 2, 2018

2) $-\ln(1-x) = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} \Rightarrow \int \ln(1-x) = -\int \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$ R still 1

3) $\frac{1}{(1-x)^2} + \ln(1-x) = \sum_{k=1}^{\infty} k x^{k-1} - \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$

$= \sum_{k=0}^{\infty} (k+1)x^k - \sum_{k=1}^{\infty} \frac{1}{k} x^k$

$= 1 + \sum_{k=1}^{\infty} \left(k+1-\frac{1}{k}\right)x^k \Big\}$ T. series with R = 1.

*Note: R is same, but interval could be diff.
↳ check endpoints.

4) $\frac{x}{3+2x} = x\left(\frac{1}{3+2x}\right) = \frac{x}{3}\left(\frac{1}{1-\left(\frac{-2x}{3}\right)}\right)$

$= \frac{x}{3}\left[\sum_{k=0}^{\infty} \left(\frac{-2}{3}x\right)^k\right]$ $\quad |\frac{2}{3}x| < 1 \Rightarrow |x| < \frac{3}{2}$

$= \sum_{k=0}^{\infty} (-1)^k \frac{2^k x^{k+1}}{3^{k+1}}$

↳ x by $\frac{1}{3} \Rightarrow$ R still $\frac{3}{2}$
↳ "x" is power series (lol), so we multiplied by series.

Simple Functions to Reduce To:

$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad |x| < 1$

$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \qquad$ All x (∞)

$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \qquad$ All x (∞)

$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad$ All x (∞)

$(1+x)^m = 1 + mx + m(m-1)\frac{x^2}{2!} + \ldots + m(m-1) * \ldots * (m-n+1)\frac{x^n}{n!}$

Big O:
f is of order g as $x \to x_0$ if $\exists A > 0 \ni$
$|f(x)| \leq A|g(x)| \Rightarrow f(x) = O(g(x))$

Example:

1) If $P_{4,0}(x) = P_{5,0}(x) = P_{6,0}(x)$
$f(x) \approx a + bx + cx^2 + dx^3 + ex^4$ } Doesn't convey what happens after
$\hookrightarrow f(x) = a + bx + cx^2 + dx^3 + ex^4 + O(x^7)$
$\hookrightarrow$ Conveys next valid $P(x)$.

2) $|x^3| \leq 1|x^2|$ as $x \to 0$, so $x^3 = O(x^2)$
Around $x = 0$, $x^a = O(x^b)$ for $a \geq b$.
That's why we can replace rest of series w/ $O(x^n)$.

3) $|\sin(x)| \leq |x|$ $\forall x$ (look @ t-series).
$\therefore \sin(x) = O(x)$
$\sin(x) = x - \frac{x^3}{3!} = x + O(x^3)$

Rules: Feb 3, 2018

$k\,O(x^n) = O(x^n)$
$O(x^m) + O(x^n) = O(x^{\min\{m,n\}})$
$O(x^m) \cdot O(x^n) = O(x^{mn})$
$\frac{O(x^m)}{x^n} = O(x^{m-n})$

Example:

1) $|R_n(x)| \leq \frac{K}{(n+1)!}(x - x_0)^{n+1} = O((x-x_0)^{n+1})$
$f(x) = P_{n,x_0}(x) + O((x - x_0)^{n+1})$

2) $f(x) = \sqrt{1+x} + \sin(x)$ around $x_0 = 0$.

$\sqrt{1+x} = 1 + \frac{1}{2}x + O(x^2)$

$\sin(x) = x + O(x^3)$

$f(x) = 1 + \frac{3}{2}x + O(x^2)$ as $x \to 0$.

3) $f(x) = e^{x^2}$ Let $t = x^2$

$$\left.\begin{aligned} f(t) &= e^t \\ &= 1 + t + \frac{t^2}{2} + O(t^3) \\ f(x) &= 1 + x^2 + \frac{x^4}{2} + O(x^6) \end{aligned}\right\}$$

Subbing works the same.

4) $f(x) = e^x \sin(x)$

$= (1 + x + \frac{x^2}{2} + O(x^3))(x + O(x^3))$

$= x + x^2 + \frac{x^3}{2} + O(x^4) + O(x^3) + O(x^4) + \frac{O(x^5)}{2} + O(x^6)$

$= x + x^2 + O(x^3)$

5) $f(x) = \cos(x)$. Find Approx w/o T. series for cos.

$\sin(x) = x - \frac{x^3}{6} + O(x^5)$

$\rightarrow \cos(x) = 1 - \frac{x^2}{2} + O(x^4)$

Limits:

1) $\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{x + O(x^3)}{x} = \lim_{x \to 0} 1 + O(x^2) = 1$

2) $\lim_{x \to 0} \frac{\sin(x)\ln(x+1)}{x^4} = \lim_{x \to 0} \frac{(x + O(x^3))(x + O(x^2))}{x^4} = \infty$

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Multivariate Calculus: Feb 7, 2018

Functions of 2+ variables:

Graphing:

For n variables:

$$f: \mathbb{R}^n \mapsto \mathbb{R}$$

Level curve: pick a value for one variable (usually z)

$$z = k = f(x, y)$$

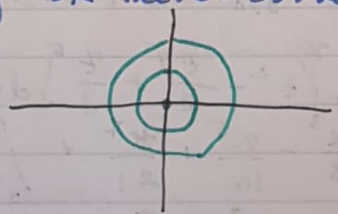
Contour Plot:

Series of level curves for dif. k's.

Example:

1) $z = x^2 + y^2$.

$x^2 + y^2 = k$ (level curve)



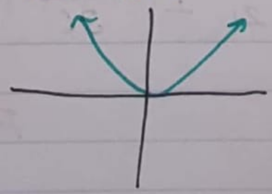
$k=0$ $(0,0)$

$k=1$ circle of radius 1.

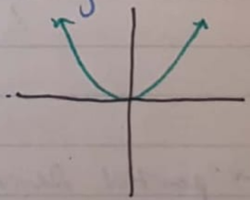
⋮

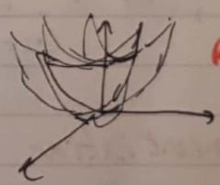
circle of radius $\sqrt{k}$

Set $x=0$



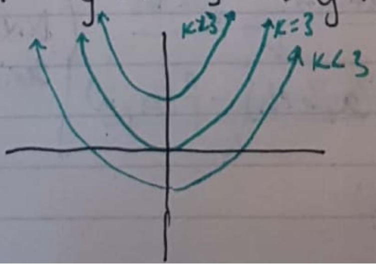
Set $y=0$



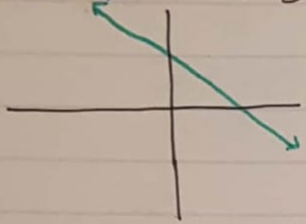


Result

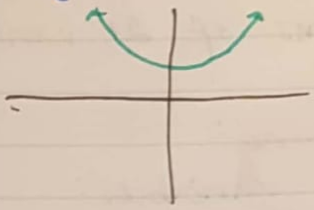
2) $x^2 - y = k - 3 \Rightarrow y = x^2 + 3 - k$



Set $x=0$ $z=3-y$



Set $y=0$ $z=x^2+3$



More variables:

$w = x^2+y^2+z^2 = K$. Level surfaces: spheres.

↳ Can't visualize.

Midterm Review

1) 7^{th} order approx + error $\int_0^x \ln(t^2+1)\,dt$ $x \in [0, \frac{1}{2}]$

$f(u) = \ln(u+1)$

$f'(u) = \frac{1}{u+1}$ $P_{3,0}(x) = u - \frac{u^2}{2} + \frac{u^3}{3}$

$f(u) = P_{3,0}(u) + O(u^4)$

$f''(u) = \frac{-1}{(u+1)^2}$ $= \int_0^x \left(t^2 - \frac{t^4}{2} + \frac{t^6}{3}\right) dt + \int_0^x R_3(u)\cdot dt$ sub t^2

$f'''(u) = \frac{2}{(u+1)^3}$ $\leq \frac{x^3}{3} - \frac{x^5}{10} + \frac{x^7}{21} + \int_0^x \frac{6}{4!} |t|^8 \cdot dt$

$f^{(4)}(u) = \frac{-6}{(u+1)^4}$ $\Rightarrow K = 6$ $= \frac{x^3}{3} - \frac{x^5}{10} + \frac{x^7}{21} \pm \frac{x^9}{36}$

Differentiation Feb 9, 2018

One var at a time → "partial derivative"

↳ $f_x(a,b)$ (wrt x) at (a,b):

$$\left.\begin{aligned} f_x(a,b) &= \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} \\ f_y(a,b) &= \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h} \end{aligned}\right\} \text{Directional Derivative}$$

$= \frac{\partial}{\partial x} f(x,y) = f_x(x,y)$

Example:

1) $f(x,y) = x^2 y + \frac{x}{y}$

$$f_x(x,y) = \lim_{h\to 0} \frac{(x+h)^2 y + \frac{x+h}{y} - x^2 y - \frac{x}{y}}{h}$$

$$= 2xy + \frac{1}{y}$$ * normal differentiation (y constant).

$$f_y(x,y) = x^2 + \frac{-x}{y^2}$$

2) Need limit defn: $f(x,y) = (x^3+y^3)^{\frac{1}{3}}$. $f_y(0,0)$?

→ get $\frac{0}{0}$ w/ deriv. rules.

$$\lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = 1$$

Higher Derivatives:

$f(x,y) = 2\sin(xy) + y^3$

$f_x(x,y) = 2y\cos(xy)$ → $\frac{d^2f}{dx^2} = 2y(-y\sin(xy))$

$f_y(x,y) = 2x\cos(xy) + 3y^2$ ⇒ $\frac{d^2f}{dxdy} = 2\cos(xy) - 2xy\sin(xy)$

$= \frac{d^2f}{dydx} = 2\cos(xy) - 2xy\sin(xy)$

Clairaut's Theorem:

If f_x, f_y, f_{xy} exist near (a,b) and f_{xy} cont. ~~at~~ (a,b):

Then f_{yx} exists and $f_{xy} = f_{yx}$

→ order doesn't matter, sometimes one way is easier

Example:

1) $f(x,y) = \ln|y| - xye^{y^2}$. Find $f_{xxyyy}(x,y)$.

→ 3 y is messy. Do 2 x first. Get 0.

Multi-Var T. Series: Feb 26, 2018

→ 2 vars at (x_0, y_0)

1) wrt x:

$$f(x,y) \approx f(x_0,y) + \frac{f_x(x_0,y)}{1}(x-x_0) + \frac{f_{xx}(x_0,y)}{2!}(x-x_0)^2 + \dots$$

2) wrt y:

Subbing x_0 → $$f(x_0,y) \approx f(x_0,y_0) + f_y(x_0,y_0)(y-y_0) + \frac{f_{yy}(x_0,y_0)}{2!}(y-y_0)^2 + \dots$$

$$f_x(x_0,y) \approx f_x(x_0,y_0) + f_{xy}(x_0,y_0)(y-y_0) + \frac{f_{xyy}(x_0,y_0)}{2!}(y-y_0)^2 + \dots$$

→ In General:

$$f(x,y) \approx f(x_0,y_0) \quad \text{const}$$
$$+ f_y(x_0,y_0)(y-y_0) + f_x(x_0,y_0)(x-x_0) \quad \text{linear}$$
$$+ \frac{f_{yy}(x_0,y_0)}{2!}(y-y_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{f_{xx}(x_0,y_0)}{2!}(x-x_0)^2 \quad \text{quad}$$

$P_0 = (x_0, y_0)$, $h = x - x_0$, $k = y - y_0$

$$f(x,y) \approx f(P_0) + f_x(P_0)h + f_y(P_0)k$$
$$+ \frac{1}{2!}\left(f_{xx}(P_0)h^2 + 2f_{xy}(P_0)hk + f_{yy}(P_0)k^2\right)$$
$$+ \frac{1}{3!}\left(f_{xxx}(P_0)h^3 + 3f_{xxy}(P_0)h^2k + 3f_{xyy}(P_0)hk^2 + f_{yyy}(P_0)k^3\right)$$

Pascal's Δ.

Example:

1) $f(x,y) = \sin(x)\ln(y)$ at $(0,1)$

$f(P_0) = 0$

$f_x = \cos(x)\ln(y)$ $\quad f_x(P_0) = 0$ $\quad f_{xxx}(P_0) = 0$

$f_y = \frac{\sin(x)}{y}$ $\quad f_y(P_0) = 0$ $\quad f_{yyy}(P_0) = 0$

$\vdots$ $\quad f_{xx}(P_0) = 0$ $\quad f_{xxy}(P_0) = 0$

$\vdots$ $\quad f_{yy}(P_0) = 0$ $\quad f_{xyy}(P_0) = -1$

$f_{xy} = \frac{\cos(x)}{y}$ $\quad f_{xy}(P_0) = 1$

$$\therefore f(x,y) \approx hk - \frac{1}{2!}hk^2$$
$$= x(y-1) - \frac{1}{2!}x(y-1)^2$$

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Another method: * only when $f(x,y) = g(x)h(y)$ for some funcs.

$f(x,y) = \sin(x)\ln(y)$

$$\approx \left(x - \frac{x^3}{3!} + O(x^5)\right)\left((y-1) - \frac{(y-1)^2}{2!} + \frac{(y-1)^3}{3!} + O(y^4)\right)$$

$$= x(y-1) - \frac{1}{2!}x(y-1)^2 \quad (O(x^3y) = O(x^4))$$

Change in f:

$f(x,y) - f(x_0,y_0) \approx f_x(P_0)h + f_y(P_0)k$

$\Delta f = f_x(P_0)\Delta x + f_y(P_0)\Delta y$

$\Delta x, \Delta y \to 0 \Rightarrow$ error $\to 0$.

$\boxed{df = f_x dx + f_y dy}$ Linear approx for $(x,y) \to (x_0,y_0)$

Example:

1) $f(x,y) = e^{x+2y}$ $(0.1, 0.02)$.

$\to f(0,0) = 1$, so...

$f(x,y) - f(0,0) = \Delta f \approx df = f_x dx + f_y dy$

$= e^{x+2y}(0,0)dx + 2e^{x+2y}(0,0)dy$

$f(x,y) - f(0,0) = dx + 2dy = 0.1 + 2(0.02) = 0.14$

so $f(x,y) \approx 1.14$

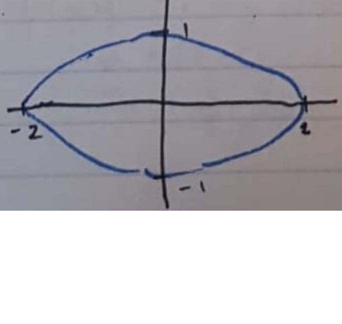
Parametric Equations — Feb 2

ex: $x^2 + y^2 = 1 \Rightarrow x = \cos\theta,\ y = \sin\theta,\ \theta \in [0, 2\pi]$

Example:

1) $x^2 + 4y^2 = 4$ — $x = 2\cos\theta,\ y = \sin\theta$.

$(2\cos\theta)^2 + 4\sin^2\theta = 4$



$$\vec{r}(\theta) = \begin{bmatrix} 2\cos\theta \\ \sin\theta \end{bmatrix}$$

2) $y = x^2$, $x \in [-2, 2]$

$x = t$, $y = t^2$ $\vec{r}(t) \begin{bmatrix} t \\ t^2 \end{bmatrix}$, $t \in [-2, 2]$. (Trivial)

3) line btwn $(2,3), (4,7)$

$\vec{r}(t) = \vec{m}t + (2,3)$

$= \binom{2}{4} t + \binom{2}{3}$ $t \in [0, 1]$

Eqn of a Line:

Given (x_1, y_1) (x_2, y_2)

$\vec{r}(t) = \underbrace{(x_1, y_1)}_{P_0} + t \underbrace{(x_2 - x_1, y_2 - y_1)}_{\vec{m}}$ $t \in [0, 1]$ for segment.

$\mathbb{R}^3$:

$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$

$x^2 + y^2 + z^2 = 1$

$x = \cos\theta \sin\varphi$

$y = \sin\theta \sin\varphi$

$z = \cos\varphi$

Chain Rule on Drugs:

$f(x,y) = x^2 e^y$, $x = t^2 - 1$, $y = \sin(t)$

↳ Can differentiate in one var, or:

$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

$= (2xe^y)(2t) + (x^2 e^y)(\cos(t))$, sub x, y

$= 4t(t^2 - 1)e^{\sin(t)} + (t^2 - 1)^2 \cos(t) e^{\sin(t)}$

Gradient Vector

March 2, 2018

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ "grad f"

Lin Approx $\bar{w}$ Grad f:

$f(\vec{x}) \approx f(\vec{a}) + f_x(\vec{a})(x-a) + f_y(\vec{a})(y-b)$.

$\vec{a} = (a, b)$
$\vec{x} = (x, y)$

$= f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$

Chain Rule $\bar{w}$ Grad f:

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt}$$

$$= \nabla f \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

Directional Derivative:

At $\vec{a}(a,b)$ in direction of unit vector $\vec{u}(u_1, u_2)$

$$D_{\vec{u}} f(a,b) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h} \qquad g(h) = f(\vec{a} + h\vec{u})$$

$$= \lim_{h \to 0} \frac{g(h) - g(0)}{h} = g'(0)$$

$x = a + u_1 h$
$y = b + u_2 h$

$$= \left[\frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh}\right]_{h=0}$$

$$= (\nabla f \cdot \vec{u})_{h=0} = \boxed{\nabla f(\vec{a}) \cdot \vec{u}}$$

Example:

$f(x,y) = x^2 + y^2$. Slope at $(1,-1)$ in dir $(3,4)$.

$\vec{a} = (1,-1)$ $\qquad \nabla f = (2x, 2y)$

$\vec{u} = \left(\frac{3}{5}, \frac{4}{5}\right)$ $\qquad D_{\vec{u}} f(1,-1) = \frac{6}{5} - \frac{8}{5} = \frac{-2}{5}$.

Representation of ∇f:

$D_{\vec{u}}(a,b) = \nabla f(a,b) \cdot \vec{u} = \|\nabla f(a,b)\| (1) \cos\theta$

→ max 0, 2π (when $\vec{u}$ is p^{ll} to ∇f)

→ ∇f is the direction with max slope.

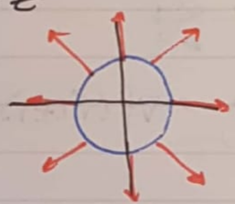
→ To find slope:

$\vec{u} = \frac{\nabla f}{\|\nabla f\|} \Rightarrow D_{\vec{u}} f(a,b) = \frac{\nabla f(a,b)}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2}} \cdot \nabla f(a,b) = \|\nabla f\|$

→ $\therefore$ Max slope is $\|\nabla f\|$

→ $\perp$ to contour plot.

$x^2 + y^2 = z$



$\nabla f = (2x, 2y)$
"Vector field"

Example:

1) $f(x,y,z) = 3x^2 - 5xy + xyz$. Direction + mag of steepest slope from $(2,2,7)$?

$\nabla f = (6x - 5y + yz, -5x + xz, xy)$

At $(2,2,7)$, $\nabla f = (16, 4, 4)$ $\therefore$ Direction $(16,4,4)$, slope $\|\nabla f\|$

Chain Rule Forms:

1) Change of coordinate system

$z = f(x,y) \quad x = g(s,t) \quad y = h(s,t) \qquad (x,y) \Rightarrow (s,t)$

$$\frac{dz}{ds} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

Example:

1) $z = x^2 + y^2$. $x = r\cos\theta$, $y = r\sin\theta$ (p. coord. system)

$z = r^2(\cos^2\theta + \sin^2\theta) = r^2$

$\frac{dz}{dr} = 2r$. $\frac{dz}{d\theta} = 0$

Chain Rule: $\frac{dz}{dr} = (2x)\cos\theta + (2y)\sin\theta$

$= 2r(\cos^2\theta + \sin^2\theta) = 2r$.

2) One var relation

$w = f(x, y, z)$

$x = s^2 t$, $y = s$, $z = e^t$.

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \cancel{\frac{\partial f}{\partial z}\frac{\partial z}{\partial s}} = 0$$

$= 3s^2 t$

Chain Rule Exception: Mar 7, 2018

$$\left.\begin{aligned} W &= x^3 + 2y \\ y &= x\sin(z) \end{aligned}\right\} \quad \frac{\partial w}{\partial x} = \frac{\partial w}{\partial x}\cancel{\frac{dx}{dx}}^{1} + \frac{\partial w}{\partial y}\frac{dy}{dx}$$

* $\frac{\partial w}{\partial x}$ appears twice. It's ok, one is with y subbed in.

$$\frac{\partial w}{\partial x} = 3x^2 + 2\sin(z)$$

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial x}\cancel{\frac{dx}{dz}}^{0} + \frac{\partial w}{\partial y}\frac{dy}{dz} = 2x\cos(z)$$

Optimization:

Horizontal tangent plane.

$\nabla f = \vec{0}$ $\left(\frac{dz}{dx}, \frac{dz}{dy} \text{ both } 0\right)$

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Example:

$z = x^2 + y^2$.

$\nabla f = (2x, 2y)$

$\left.\begin{array}{l} 2x = 0 \Rightarrow x = 0 \\ 2y = 0 \Rightarrow y = 0 \end{array}\right\}$ crit. point $(0,0)$

Types of Critical Points:

CP at (x_0, y_0)

L. Max: $f(x,y) \leq f(x_0, y_0)$ around CP.

L. Min: $f(x,y) \geq f(x_0, y_0)$ around CP.

Else: saddle point. (Dif. directions).

Concavity:

Mar 9, 2018

$$f(x,y) \approx f(x_0,y_0) + \cancel{f_x(x_0,y_0)}^{0}(x-x_0) + \cancel{f_y(x_0,y_0)}^{0}(y-y_0) +$$
$$+ \frac{1}{2}\left[f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)(y-y_0)^2\right]$$

$$f(x,y) - f(x_0,y_0) \approx \frac{1}{2}\left[f_{xx}(P_0)h^2 + 2f_{xy}(P_0)hk + f_{yy}(P_0)k^2\right]$$

Does sign change?

↳ Yes: saddle (zero exists)

↳ No: max/min

Divide by k^2 (always +, doesn't change sign. $m = \frac{h}{k}$.

$f_{xx}(P_0)m^2 + 2f_{xy}(P_0)m + f_{yy}(P_0)$ (quadratic)

↳ If $b^2 - 4ac < 0 \Rightarrow$ max/min

↳ " $> 0 \Rightarrow$ saddle

↳ " $= 0$ = Not necessarily.

$b^2 - 4ac = 4(f_{xy}(P_0))^2 - 4f_{xx}(P_0)f_{yy}(P_0)$

Let $D = f_{xx}(P_0)f_{yy}(P_0) - (f_{xy}(P_0))^2$.

$D > 0 \Rightarrow$ extreme (max/min) → $f_{xx}(P_0) > 0 \Rightarrow$ min; → $f_{xx}(P_0) < 0 \Rightarrow$ max; $= 0$: try another D.

$< 0 \Rightarrow$ saddle

$= 0 \Rightarrow$ No info.

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Example:

1) $z = x^2+y^2$. CP $(0,0)$.

$f_{xx} = 2$ $\quad f_{xy} = 0$. $\quad D = 4 > 0$ (extreme)

$f_{yy} = 2$. $\quad f_{xx} > 0$ ∴ minimum.

2) $f(x,y) = x^4+y^4-4xy$

$\nabla f = \vec{0} \Rightarrow$ CPs at $(0,0)$, $(1,1)$, $(-1,-1)$

$f_{xx} = 12x^2$ $\quad f_{xy} = -4$

$f_{yy} = 12y^2$

$(0,0)$:

$D = -16 < 0$ ∴ saddle.

$(1,1)$:

$D > 0$ and $f_{xx} > 0$ ∴ minimum

$(-1,-1)$:

$D > 0$ and $f_{xx} > 0$ ∴ minimum

3) $f = 2x^4 - 3x^2y + y^2$

$\nabla f = \vec{0} \Rightarrow$ CP at $(0,0)$

$D(0,0) = 0$ ∴ inconclusive.

Factor to get $f = (2x^2-y)(x^2-y)$

↳ Check if + or − around $(0,0)$.

↳ $f > 0 \Rightarrow$

1) $2x^2 - y > 0$, $x^2 - y > 0$

2) $2x^2 - y < 0$, $x^2 - y < 0$

↳ $f < 0 \Rightarrow$

1) $2x^2 - y > 0$, $x^2 - y < 0$

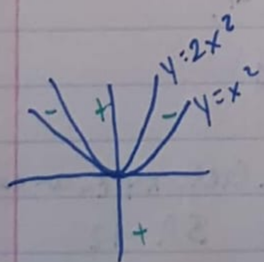
2) $2x^2 - y < 0$, $x^2 - y > 0$ } impossible

$y < 2x^2$, $y > x^2$.

Let $x = t$, $y = 1.5t^2 \Rightarrow$ line through $(0,0)$ w neg. f except at

→ Decrease heading away $(0,0)$ on this line.

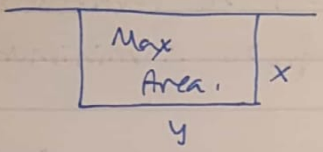
→ Direction where slope is negative ∴ f not minimum



Both also possible ($y = \frac{1}{2}x^2$, $100x^2$) ∴ saddle point.

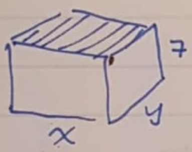
Constrained Optimization: Mar

400m fencing.



Get eqn $\bar{w}$ one var.

Rectangular box open at top. $V = 4m^2$. Minimize SA.



$SA = 2yz + 2xz + xy$

$xyz = 4$

- Optimal value when constraint curve + level curve are tangential.
- $\perp$ lines must be $\parallel$ too
  - $\rightarrow \nabla f$ is orthogonal to level curves.
  - $\rightarrow \nabla g$ is orthogonal to constraint
- Solve $\nabla f = \lambda \nabla g$ (λ constant)

Method of Lagrange Multipliers: Mar 14, 2018

1) Maximize $A = xy$ where $2x + y = 400$.

$f(x,y) = xy$, $g(x,y) = 2x + y$

$\nabla f = (y, x)$ $\nabla g = (2, 1)$

$y = 2\lambda$ $x = \lambda$ $2x + y = 400$.

$\therefore x = 100, y = 200$.

2) Min SA, $V = 4m^2$. $SA = 2yz + 2xz + xy$. $xyz = 4$.

$\nabla f = (y + 2z, x + 2z, 2x + 2y)$, $\nabla g = (yz, xz, xy)$

$y + 2z = \lambda(yz)$ $xyz = 4$

$x + 2z = \lambda(xz)$ 4 eqn, 4 unknown. Get xyz on each eqn.

$2x + 2y = \lambda(xy)$ $(x, y, z) = (\frac{1}{2}, \frac{1}{2}, 1)$ $SA = 12$

3. $f(x,y) = 4x^2 + 10y^2$ subject to $g(x,y) = x^2+y^2 \underline{\leq 4}$. Find optimal points.

Can use L.M. for $x^2+y^2=4$.

→ If a CP has $x^2+y^2<4$, then it'll be a local extrema.

1) Find CPs

$\nabla f = (8x, 20y) = \vec{0} \Rightarrow$ CP $(0,0)$. } inside constraint.

2) Check boundary (constraint)

Yes, $0<4$.

3) L.M. for edge.

$$\left.\begin{aligned} 8x &= 2x\lambda \\ 20y &= 2y\lambda \end{aligned}\right\} x^2+y^2=4. \quad x=0/y=0/\lambda=4/\lambda=10.$$

$x=0 \Rightarrow y = \pm 2$ $\quad$ $\lambda=4 \Rightarrow (x,y) = (\pm 2, 0)$ } same

$y=0 \Rightarrow x = \pm 2.$ $\quad$ $\lambda=10 \Rightarrow (x,y) = (0, \pm 2)$ } cases

$\therefore (0,2), (0,-2), (2,0), (-2,0), (0,0)$.

4) Sub to find optima

$(0,0)$ min. $(0,2), (0,-2)$

Integration:

$\int x^2 + y^2 \, dy = x^2 y + \frac{1}{3}y^3 + \underline{C(x)}$ } Volume btwn surface + xy plane.

Need $C(x)$, not C, since $\frac{d}{dy} C(x) = 0$.

Example:

1) $\nabla f = (6xy^2 + e^y, 6x^2y + xe^y + \sin(y))$. Find y.

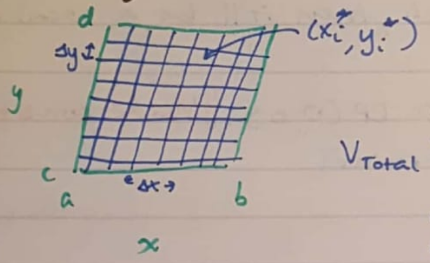
$\int f_x \, dx = \int 6xy^2 + e^y \, dx = 3y^2x^2 + e^y x + g(y)$

$\int f_y \, dy = \int 6x^2y + xe^y + \sin(y) \, dy = 3x^2y^2 + xe^y - \cos(y) + h(x)$

$\therefore h(x) = C$ and $g(y) = -\cos(y)$. $f = 3x^2y^2 + xe^y - \cos(y) + C$

Integrating on 2 vars: Mar 16, 2018

Rect. region $a \le x \le b$, $c \le y \le d$.



$$V_{\text{one square}} \approx f(x_i^*, y_i^*)\,\Delta x \Delta y$$

$$V_{Total} \approx \sum_i \sum_j f(x_i^*, y_j^*)\,\Delta x \Delta y$$

$$= \lim_{\Delta x, \Delta y \to 0} \sum_i \sum_j f(x_i^*, y_j^*)\,\Delta x \Delta y$$

$$= \iint_R f(x,y)\cdot \underbrace{dx\,dy}_{dA}$$

To calculate:

→ V of a slice: $\left(\int_c^d f(x_i^*, y_j^*)\,dy\right)\Delta x$

↳ sum over all intervals, take limit.

↳ $V_{Total} = \underbrace{\int_a^b \left(\int_c^d f(x,y)\cdot dy\right) dx}$.

Rect. region order doesn't matter (pick easier)

Example:

1) $0 \le x \le \ln(2)$ $0 \le y \le 1$. Find V between $f = xe^{xy}$ and xy plane.

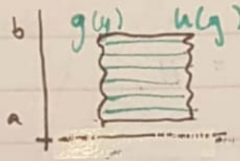
$$V = \int_0^{\ln(2)} \left(\int_0^1 x e^{xy}\cdot dy\right) dx$$

$$= \int_0^{\ln(2)} \left[\frac{x}{x} e^{xy}\right]_0^1 dx$$

$$= \int_0^{\ln(2)} (e^x - 1)\,dx = 2 - \ln(2) - 1 = 1 - \ln(2)$$

Mar 19, 2018

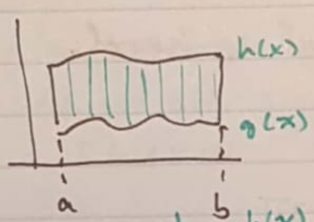
If R is not rectangular:

1) 

$a \leq y \leq b$

$g(y) \leq x \leq h(y)$.

$$\int_a^b \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy$$ } can't switch sides here.



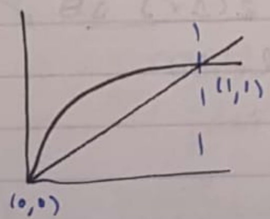
$g(x) \leq y \leq h(x)$

$a \leq x \leq b$

$$\int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$

Example:

1) V between $z = xy$ + the xy plane between $y = x$, $y = \sqrt{x}$.



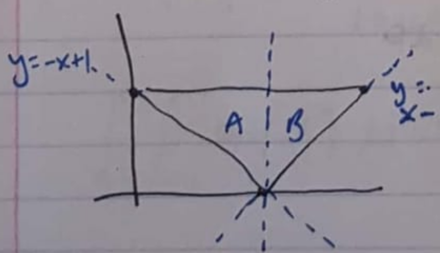
$0 \leq x \leq 1$

$x \leq y \leq \sqrt{x}$ } OR: $y \leq x \leq y^2$, $0 \leq y \leq 1$.

$$\int_0^1 \int_x^{\sqrt{x}} xy\,dy\,dx = \frac{1}{24}.$$ OR:

$$\int_0^1 \int_{y^2}^{y} xy\,dx\,dy = \frac{1}{24}$$

2) V Between $f = x+y$ + xy plane btwn $(0,1), (1,0), (2,1)$



$0 \leq y \leq 1$

$1-y \leq x \leq 1+y$

1 integral.

} OR: A: $0 \leq x \leq 1$, $-x+1 \leq y \leq 1$ B: $1 \leq x \leq 2$, $x-1 \leq y \leq 1$

2 integrals.

More Non-Rectangular Regions: Mar 21, 2018

1) $f(x,y)=1$ and xy plane btwn $x^2+y^2=1$.

Expect V: height × circle area $= 1 \times \pi r^2 = \pi$.

$-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$
$-1 \le y \le 1$

$$V = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 1 \cdot dx \cdot dy$$

$$= \int_{-1}^{1} 2\sqrt{1-y^2} \cdot dy \quad \text{Trig sub } y = \sin\theta$$

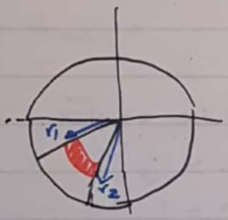
If $f(x,y)$ is more complex, this method is hard.

∴ Polar coordinates

$x = r\cos\theta$, $y = r\sin\theta$

$0 \le r \le 1$
$0 \le \theta \le 2\pi$.

$$V = \int_0^{2\pi} \int_0^1 1 \, dA \quad \text{where } dA = dx\,dy = r\,dr\,d\theta \quad \text{(below)}$$



$$A = \pi(r_2^2 - r_1^2)\frac{\Delta\theta}{2\pi} = \frac{1}{2}(r_2^2 - r_1^2)\Delta\theta$$

$$= \frac{1}{2}((r_1 + \Delta r)^2 - r_1^2)\Delta\theta$$

$$= r_1 \Delta r \Delta\theta + \underline{\frac{1}{2}(\Delta r)^2 \Delta\theta} \quad \text{Approaches 0 faster}$$

$$\lim_{\Delta r, \Delta\theta \to 0} = r\Delta r\Delta\theta = r\,dr\,d\theta$$

So, $V = \int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^1 d\theta = \pi$ ✓.

2) $\iint_R x^3 + xy^2 \, dA$ where R is the first quad. of unit circle.

$x = r\cos\theta$
$y = r\sin\theta$

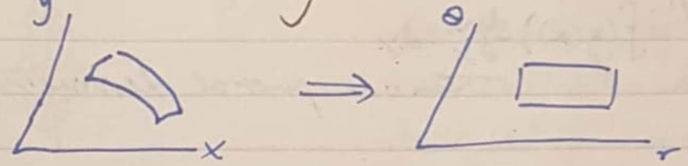
$0 \le r \le 1$
$0 \le \theta \le \frac{\pi}{2}$

$$V \int_0^{\frac{\pi}{2}} \int_0^1 r(r^3\cos^3\theta + r\cos\theta\, r^2\sin^2\theta)\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{1}{5}\cos\theta\, d\theta = \frac{1}{5}$$

Change of Variables:

What does using P.C. do?



rectangular region.

Example:

1) Region between $y = \frac{5}{x}, \frac{10}{x}, x, 2x$.

Rewrite: $xy = 5, 10$ $\quad \frac{y}{x} = 1, 2$ $\quad$ Let $u = xy$, $v = \frac{y}{x}$

$5 \le u \le 10$ $\quad 1 \le v \le 2$. ← Rectangle.

What is dA?

If $x = x(u,v)$, $y = y(u,v)$ then $\iint_{R_{xy}} f(x,y)\,dx\,dy$

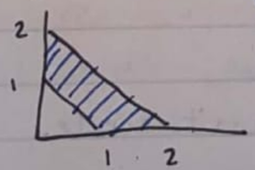
$$= \iint_{R_{uv}} f[x(u,v), y(u,v)] \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

where $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\right|$ (Jacobian)

If $x = r\cos\theta$, $y = r\sin\theta$

So, $J = \left|\det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}\right| = r$

2) $\iint_D \cos\left(\frac{y-x}{y+x}\right) dA$



$y = 1-x$ $\quad x = 0$ $\quad x + y = 1$

$y = 2-x$ $\quad y = 0$. $\quad x + y = 2$

Let $u = x+y$, $v = y-x$

$1 \le u \le 2$. $\quad$ If $x = 0 \Rightarrow u = v$, $y = 0 \Rightarrow -u = v$

$-u \le v \le u$

$$\int_1^2 \int_{-u}^{u} \cos\left(\frac{v}{u}\right) \cdot dA \quad \Big\} \; dA = \frac{1}{2}\,du\,dv$$

One var. Jacobian: Mar 23, 2018

$f(u)$, $u = g(x)$

$$\int_{g(a)}^{g(b)} f(u)\,du = \int_a^b f(g(x))\,\frac{du}{dx}\cdot dx$$

$\frac{du}{dx}$ → one partial derivative.

Interpretation of Integrals:

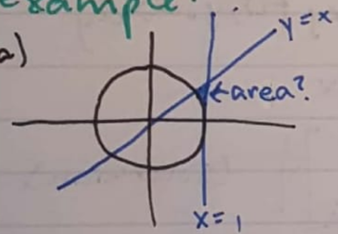
1) $\iint_R f(x,y)\,dx\,dy$ - Volume between f + xy plane within R.

2) $\iiint_R f(x,y,z)\,dx\,dy\,dz$ - Volume in 4^{th} dimension

3) $\int_0^{2\pi}\int_0^1 r\,dr\,d\theta$ - Area of circle ($\iint_R dx\,dy \Rightarrow$ area of R)

Example:

a)



$x = r\cos\theta$, $y = r\cos\theta$

$1 \leq r \leq \sec\theta$ $\quad x=1 \Rightarrow r = \sec\theta$

$0 \leq \theta \leq \frac{\pi}{4}$

$$\int_0^{\frac{\pi}{4}}\int_1^{\sec\theta} 1\,(r\,dr\,d\theta) = \int_0^{4} \frac{\sec^2\theta - 1}{2}\,d\theta = \frac{1}{2} - \frac{\pi}{8}$$

4) $f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\text{Area}(R)}\iint_R f(x,y)\,dA$

Another Example:

b) Avg distance from origin of all points in unit circle.

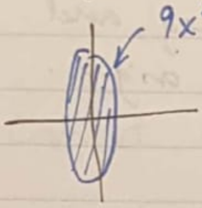
$f(x,y) = \sqrt{x^2+y^2}$

$$f_{avg} = \frac{1}{\iint_R dx\,dy}\iint_R f(x,y)\,dA = \frac{1}{\pi(1)^2}\int_0^{2\pi}\int_0^1 \left[\left(\sqrt{x^2+y^2}\right)(r\,dr\,d\theta)\right]$$

$$= \frac{1}{\pi}\int_0^{2\pi}\int_0^1 r^2\,dr\,d\theta$$

$$= \frac{2}{3}$$

c) $\iint_R x^2\, dA$



Polar coords doesn't quite work.

Let $x=2u,\ y=3v$
$\Rightarrow$ now $u^2+v^2=1$ } could've done $x=2r\cos\theta$, $y=3r\sin\theta$

now let $u=r\cos\theta,\ v=r\sin\theta$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = 6. \quad \therefore dA = 6\,du\,dv = 6r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^1 (4r^2\cos^2\theta)(6r\,dr\,d\theta)$$ } Trig sub to get ans

Triple Integrals: Mar 26, 2018

$D=[a,b]\times[c,d]\times[e,f]$ $\quad a\le x\le b \quad c\le y\le d \quad e\le z\le f.$

– Approximate rect. prism.

$$\int_e^f\int_c^d\int_a^b f(x,y,z)\,\underbrace{dx\,dy\,dz}_{\to dV}$$

Example:

1) $$\int_0^1\int_1^2\int_2^3 8xyz\cdot dV$$
$$= \int_0^1\int_1^2 20yz\cdot dy\cdot dz = 15$$

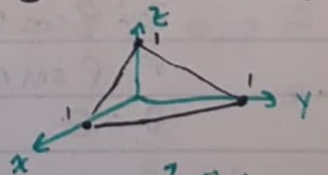
Non-Rectangular Regions:

$\int_D 2x\cdot dV$ where D = region under $x+y+z=1$ in first octant. $(x,y,z\ge 0)$

$0\le x\le 1$

$0\le y\le 1-x$

$0\le z\le 1-x-y$



$$\int_0^1\int_0^{1-x}\int_0^{1-x-y} 2x\cdot dz\cdot dy\cdot dx$$ } This order b/c bounds.

$$= \frac{1}{12}$$

Example:

1) $\int_D xz \cdot dV$ $\quad D$ = region in first octant below $z=y$ and inside cylinder $r=1$ center @ origin.

Polar coordinates x, y. Leave z alone.

↳ "cylindrical coordinates"

$$\left.\begin{array}{l} 0 \le r \le 1 \\ 0 \le \theta \le \frac{\pi}{2} \\ 0 \le z \le r\sin\theta \end{array}\right\} \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{r\sin\theta} xz.$$

Find dV:

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = \det\begin{vmatrix} \cos\theta & r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \leftarrow \text{cofactor expansion}$$

$$= \det\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$\therefore$ $dV = r \cdot dr \cdot d\theta \cdot dz$

$$\int_0^{\pi/2} \int_0^1 \int_0^{r\sin\theta} r^2 \cos\theta\, z \cdot d\theta \cdot dr.$$

2) Vsphere radius m?

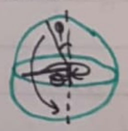
Mar 28, 2018

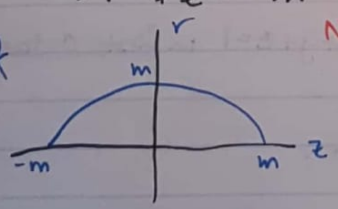
Look at x, y; convert to "cylindrical" coords.

$x = r\cos\theta \quad y = r\sin\theta \quad z = z \quad 0 \le r \le m \quad 0 \le \theta \le 2\pi$

Sphere: $x^2 + y^2 + z^2 = m^2$

↳ $r^2 + z^2 = m^2$ (circle)





top half only.

Now use ρ coords:

$z = \rho\cos\varphi \qquad 0 \le \rho \le m$

$r = \rho\sin\varphi \qquad 0 \le \varphi \le \pi \leftarrow \frac{\pi}{2}$ for $\frac{1}{2}$ sphere

$\theta = \theta$

"spherical coords" can jump to this

$\therefore$, $x = \rho\sin\varphi\cos\theta$

$y = \rho\sin\varphi\sin\theta$

$z = \rho\cos\varphi$

$dV = ?$

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| = \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\theta \end{vmatrix}$$

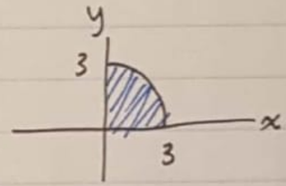
$dV = \rho^2 \sin\varphi$

Scanned by CamScanner

$$\therefore V = \int_0^{\pi}\int_0^{2\pi}\int_0^{m} \rho^2 \sin\varphi \cdot d\rho \cdot d\theta \cdot d\varphi = \frac{4}{3}\pi m^3$$

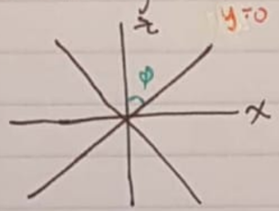
3) Convert $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\, dz\, dx\, dy$ to sph. coords.

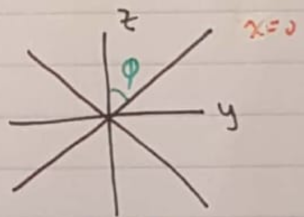
$$\left.\begin{aligned} &0 \le y \le 3 \\ &0 \le x \le \sqrt{9-y^2} \\ &\sqrt{x^2+y^2} \le z \le \sqrt{18-x^2-y^2} \end{aligned}\right\}$$ xy plane:



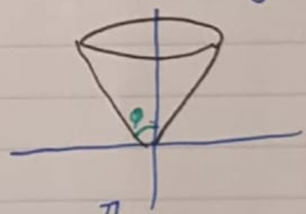
z: Bounded by sphere w/ radius $\sqrt{18}$ and:

Similar to: $z = x^2+y^2$





But $z^2 = x^2+y^2$:



$0 \le \rho \le 3\sqrt{2}$ ← radius

$0 \le \theta \le \frac{\pi}{2}$ ← angle from +x axis (just x and y)

$0 \le \varphi \le \frac{\pi}{4}$ ← angle from +z axis

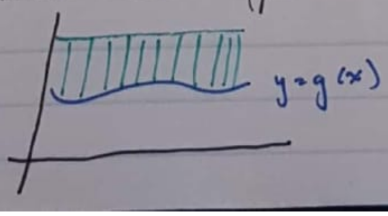
$$\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{2}}\int_0^{3\sqrt{2}} \rho^2(\rho^2 \sin\varphi)\, d\rho\, d\theta\, d\varphi$$

Steps for Spherical Coords: Apr 2, 2018

1) Figure out what the region looks like.
2) Get conversions for ρ, θ, φ w/ bounds.

Path Integrals:

- Area under surface along curve



Parameterize $y = g(x) = r(t)$
↳ intervals of width Δt.