MATH 119

Approximation: Jan 3, 2018

1) Numerical: with computer
2) Analytical: with calc

Linear Approximation:

Find a line $L(x)$ similar to fcn.

$\rightarrow \sin(x^2) \approx \sin(x)$

Example:

1) Find $L(x)$ at $(a, f(a))$

1) $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

2) $y - f(a) = f'(a)(x - a)$

3) $L(x) = y = f(a) + f'(a)(x - a)$

2) $\sin(0.1)$? $\sin(0) = 0$, so...

$$f'(0) = \lim_{x \to 0} \frac{\sin(x) - 0}{x - 0} \overset{L'H}{=} \lim_{x \to 0} \cos(x) = 1$$

$L(x) = 0 + 1(x - 0) = x$. so $\sin(0.1) \approx 0.1$.

3) $e^{0.1}$? $e^0 = 1$, so...

$$f'(0) = \lim_{x \to 0} \frac{e^x - 1}{x} = e^0 = 1.$$

$L(x) = 1 + 1(x - 0) = x + 1$ so $e^{0.1} \approx 0.1 + 1 \approx 1.1$

3) $m = \frac{m_0}{\sqrt{1 - (\frac{v}{c})^2}}$. Find approx for small v.

$f(x) = \frac{1}{\sqrt{1 - x^2}}$ (constants can come back at end)

$g(u) = \frac{1}{\sqrt{1 - u}}$ $v = 0 \Rightarrow u = 0$, so approx near $u = 0$.

$g'(0) = \frac{1}{2}$. so,

$L(u) = 1 + \frac{1}{2}(u - 0)$

$= \frac{1}{2}u + 1$.

$\therefore f(x) \approx \frac{1}{2}x^2 + 1$

$m \approx m_0 (1 + \frac{1}{2}(\frac{v}{c}$

Root Finding: Jan 5, 2018

$x - e^{-x} = 0$

1) Bisection:

$f(0) = -1$ and $f(1) \approx 0.63$ $(0,1)$

$f(1/2) \approx -0.11$ $(\frac{1}{2}, 1)$

$f(3/4) \approx 0.28$ $(\frac{1}{2}, 3/4)$

$f(5/8) \approx 0.09$ $(\frac{1}{2}, 5/8)$

$\therefore$ root @ 0.6.

Always converges, but slowly.

2) Newton's:

Replace $f(x)$ w/ $L(x)$. start w/ bisection to get close (0.6).

$(0.06, 0.05)$ $L_{x_0}(x) = 0.05 + 1.55(x - 0.6) = 0$

$\rightarrow x \approx 0.5677$

repeat w/ this as starting pt.

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

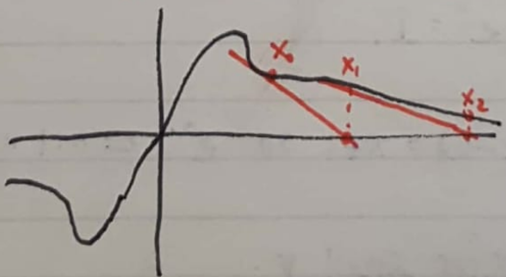
* { → Doesn't work w/o defined deriv.
→ and deriv $\neq 0$ at root

If $f'(x) = 0$ at root: $f(x) = x^n$

$$x_k = x_{k-1}\left(1 - \frac{1}{n}\right) = \left(1 - \frac{1}{n}\right)^k$$

} no faster than bisection (slow)

→ Doesn't work sometimes w/ bad starting pt.



gets further from root (or dif. root)
(use bisection first)

Polynomial Interpolation:

Jan 8, 2018

$(0,5),(1,3),(2,6),(3,4)$ $\qquad y = a+bx+cx^2+dx^3$

$y_0: 5 = a$

$y_1: 3 = a+b+c+d$

$y_2: 6 = a+2b+4c+8d$

$y_3: 4 = a+3b+9c+27d$

$\left.\begin{array}{l} \Delta y_0 = y_1 - y_0 = b+c+d \\ \Delta y_1 = y_2 - y_1 = b+3c+7d \\ \Delta y_2 = y_3 - y_2 = b+5c+19d \end{array}\right\}$ repeat: $\begin{array}{l} \Delta^2 y_0 = \Delta y_1 - \Delta y_0 = 2c+6d \\ \Delta^2 y_1 = \Delta y_2 - \Delta y_1 = 2c+12d \end{array}$

And, once more,

$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = 6d.$

$\boxed{a = y_0}$ $\boxed{d = \frac{\Delta^3 y_0}{6}}$ Now, back sub.

$c = \frac{\Delta^2 y_0 - 6d}{2} = \frac{\Delta^2 y_0 - \Delta^3 y_0}{2}$

$b = \Delta y_0 - c - d = \Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3}$

y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
5			
	-2		
3		5	
	3		-10
6		-5	
	-2		
4			

Sub to get $a=5$, $d = \frac{-5}{3}$, $c = \frac{15}{2}$, $b = \frac{-47}{6}$

General form: $y = y_0 + x\Delta y_0 + x(x-1)\frac{\Delta^2 y_0}{2} + x(x-1)(x-2)\frac{\Delta^3 y_0}{6}$

n points: $y = y_0 + x\Delta y_0 + x(x-1)\frac{\Delta^2 y_0}{2} + \dots + x(x-1) \times \dots \times (x-n+1)\frac{\Delta^n y_0}{n!}$

mult.

Assumptions:

1) x coord... 0,1,2,3...

2) Points are equidistant ↑ removes this assumption

Newton Forward Distance Formula:

n+1 equidistand nodes, $x_n = x_0 + nh$ (h: distance).

$y = y_0 + \frac{(x-x_0)}{h}\Delta y_0 + \frac{(x-x_0)(x-x_1)}{2!\,h^2}\Delta^2 y_0 + \dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{n!\,h^n}\Delta^n y_0$

Example:

1) Estimate $f(2.45)$ if $f(x)$ passes through $(2,4),(2.2,5),(2.4,4),(2.6,2)$

$x_0 = 2$

$h = 0.2$

$n = 4$

$$\begin{array}{ccccccc} y_0 & & \Delta y_0 & & \Delta^2 y_0 & & \Delta^3 y_0 \\ 4 & & & & & & \\ & & 1 & & & & \\ 5 & & & & -2 & & \\ & & -1 & & & & 1 \\ 4 & & & & -1 & & \\ & & -2 & & & & \\ 2 & & & & & & \end{array}$$

$$y = 4 + \frac{(x-2)}{0.2}\cdot(1) + \frac{(x-2)(x-2.2)}{2\cdot(0.2)^2}(-2) + \frac{(x-2)(x-2.2)(x-2.4)}{6(0.2)^3}\cdot(1)$$

$f(2.45) \approx 3.56$ (no clue on how accurate)

Linear Interpolation:

Assume straight line btwn each pair.

* Asymptotes can't be approx by polynomials.

Given $(x_0,y_0)(x_1,y_1)$ $\left\{ y = y_0 + \frac{(x-x_0)}{(x_1-x_0)}(y_1-y_0) \right.$

$$f(x) = \frac{(x_1-x)}{(x_1-x_0)}y_0 + \frac{(x-x_0)}{(x_1-x_0)}y_1$$

$f(2.45)$: $(2.4,4),(2.6,2)$

$$f(2.45) \approx \frac{0.15}{0.2}(4) + \frac{0.05}{0.2}(2) \approx 3.5$$

Taylor Polynomials:

$(x_0, f(x_0)),\ (x_0+\Delta x, f(x_0+\Delta x)),\ (x_0+2\Delta x, f(x_0+2\Delta x))$

NFDF: $y = y_0 + (x-x_0)\frac{\Delta y_0}{\Delta x} + \frac{1}{2}(x-x_0)(x-x_1)\frac{\Delta^2 y_0}{(\Delta x)^2}$

Let $\Delta x \to 0$ (merge points)

$$y \to \underbrace{y_0 + (x-x_0)f'(x_0)}_{\text{Linear approx}} + \frac{1}{2}(x-x_0)^2 f''(x_0)$$

whole thing quadratic approx. (more accurate)

$$P_{n,x_0}(x) = f(x) + f'(x_0)(x-x_0) + \frac{1}{2}f(x_0)(x-x_0)^2 + \dots + \frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \quad \text{where} \quad \frac{\Delta^k y_0}{(\Delta x)^k} \to f^{(k)}(x_0)$$

Example:

1) $f(x) = e^x$ near $x=0$

$x_0 = 0$

$f(x_0) = 1$

$f'(x_0) = 1 = f''(x_0)$

Lin app: $f(x) \approx 1 + x$

Quad app: $f(x) \approx 1 + x + \frac{1}{2}x^2$ ← matches concavity

2) $\sqrt{4.5} \approx \sqrt{4} = 2$

$f'(x) = \frac{1}{2\sqrt{x}} \Big|_{x=4} = \frac{1}{8}$

$f''(x) = \frac{-1}{4\sqrt{x^3}} = \frac{-1}{32}$

$y \approx 2 + \frac{1}{8} - \frac{1}{32}\left(\frac{1}{2}\right)^3$

≈ 2.12109

3) $\int_0^{0.5} \tan^{-1}(x^2)\,dx \approx \int_0^{0.5} P_{6,0}(x) \cdot dx$

$f(x) = \tan^{-1}(x^2)$ $f(0) = 0$

$f'(x) = \frac{2x}{1+x^4}$ $f'(0) = 0$

$f''(0) = 2$, and so on ...

$P_{6,0}(x) = \frac{1}{2} \cdot 2 \cdot (x)^2 + \frac{1}{3}x^6$

$= x^2 + \frac{1}{3}x^6$ (then integrate).

Maclaurin's Approach:

Want $P_{n,x_0}(x)$ and it's deriv. to have same $f(x_0), f'(x_0), f''(x_0)$ etc.

$p(x) = a_0 + a_1(x - x_0) + a_2(x-x_0)^2 + \ldots + a_n(x-x_0)^n$

$p(x_0) = a_0 = f(x_0)$ Derive to find that:

$p'(x_0) = a_1 = f'(x_0)$

$p''(x_0) = 2a_2$, so $a_2 = \frac{f''(x_0)}{2}$

} keep going to get original T. poly. eqn.

→ implies unique approximation.

Taylor Polynomial centered at $x_0 = 0$: Maclaurin Poly.

$$P_{n,0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

Shortcut:

Linear: $f(x) \approx f(0) + f'(0)x$ Let $x = t^2$

$f(t^2) \approx f(0) + f'(0)t^2$ ← Quadratic

$h(t) = f(t^2)$ $Q(t) = f(0) + f'(0)t^2$

→ $Q(t)$ is $TP_2(\cdot)$ for $h(t)$ ⇒ same $f(x_0)$, $f'(x_0)$

① $\begin{cases} h(0) = f(0) \\ Q(0) = f(0) \end{cases}$ ② $\left.\begin{cases} h'(t) = 2t f'(t^2) \\ Q'(t) = 2f'(0)t \end{cases}\right\}$ $\begin{matrix} h'(0) = 0 \\ Q'(0) = 0 \end{matrix}$

③ $\left.\begin{cases} h''(t) = 2f'(t^2) + 2tf''(t^2)(2t) \\ Q''(t) = 2f'(0) \end{cases}\right\}$ $\begin{matrix} h''(0) = 2f'(0) \\ Q''(0) = 2f'(0) \end{matrix}$

1, 2, 3 means $Q(t)$ is 2^{nd} order T.P. for $h(t)$.

General Case:

Finding mn^{th} M.Poly for $f(kt^m)$ can be done by letting $x = kt^m$ ($k \in \mathbb{R}, m > 0$) and finding n^{th} order poly.

→ $f^{(mn)}(0)$ must exist.

Example:

1) $f(x) = e^{x^2}$ near $x = 0$. $f(t) = e^t$ where $t = x^2$.

$g(t) \approx 1 + t + \frac{1}{2}t^2$

so $g(x^2) \approx 1 + x^2 + \frac{1}{2}x^4$

Accuracy of Taylor Polynomials: Jan 12, 2018

$|a+b| \le |a| + |b|$ – Scalars. Integrals?

$$\left| \int_a^b f(x)dx \right| = \left| \sum_{i=1}^{n} f(x_i)\Delta x_i \right| \quad (\text{as } n \to \infty)$$

$$\le \sum_{i=1}^{n} |f(x_i)\Delta x_i| \quad b \ge a, \text{ so } \Delta x_i \text{ is } +.$$

$$= \sum_{i=1}^{n} |f(x_i)| \Delta x_i$$

$$= \int_a^b |f(x)| \cdot dx$$

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$a = x_0$, $b = x$ (variable).

$$\int_{x_0}^{x} f'(t) \cdot dt = f(x) - f(x_0)$$

$$f(x) = \underbrace{f(x_0)}_{P_{0,x_0}(x)} + \int_{x_0}^{x} f'(t)\,dt$$

Now, IBP: $\begin{cases} u = f'(t) & dv = dt \\ du = f''(t)dt & v = t \end{cases}$

$$= f(x_0) + \Big[t f'(t)\Big]_{x_0}^{x} - \int_{x_0}^{x} t f''(t) \cdot dt$$

$$= f(x_0) + x f'(x) - x_0 f'(x_0) - \int_{x_0}^{x} t f''(t)\,dt$$

$$= \underbrace{f(x_0) + (x - x_0) f'(x_0)}_{P_{1,x_0}(x)} \underbrace{- x f'(x_0) + x f'(x) - \int_{x_0}^{x} t f''(t)\,dt}_{\text{remainder}}$$

$$= P_{1,x_0}(x) + x\big(f'(x) - f'(x_0)\big) - \int_{x_0}^{x} t f''(t) \cdot dt$$

$$= P_{1,x_0}(x) + x \int_{x_0}^{x} f''(t) \cdot dt - \int_{x_0}^{x} t f''(t)\,dt$$

$$= P_{1,x_0}(x) + \int_{x_0}^{x} (x - t) f''(t) \cdot dt$$

IBP again $\begin{cases} u = f''(t) & dv = (x-t)dt \\ du = f'''(t) & v = -\frac{1}{2}(x-t)^2 \end{cases}$

$$= P_{1,x_0}(x) + \left[\frac{-1}{2}(x-t)^2 f''(t)\right]_{x_0}^{x} - \int_{x_0}^{x} \frac{-1}{2}(x-t)^2 f'''(t) \cdot dt$$

$$= \underbrace{P_{1,x_0}(x) + \frac{1}{2}(x - x_0)^2 f''(x_0)}_{P_{2,x_0}(x)} + \frac{1}{2}\int_{x_0}^{x} (x-t)^2 f'''(t)\,dt$$

$$= P_{2,x_0}(x) + \underbrace{\frac{1}{2}\int_{x_0}^{x} (x-t)^2 f'''(t) \cdot dt}_{\text{Remainder}}$$

Taylor Theorem w Integral Remainder:

Suppose f has $n+1$ deriv. at x_0. Then,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_n(x) = P_{n,x_0}(x) + R_n(x)$$

$$R_n(x) = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t) \cdot dt$$

Suppose $|f^{(n+1)}(t)| \leq K$ for $t \in [x_0, x]$ (some constant K)

↳ just bound derivs to same line (no VAs, so)

$$|R_n(x)| = \left| \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t) \cdot dt \right|$$

If $x > x_0$, then Δ ineq. applies.

$$\leq \int_{x_0}^{x} \left| \frac{(x-t)^n}{n!} f^{(n+1)}(t) \cdot dt \right|$$

$t \in [x_0, x]$, so $(x-t)^n \geq 0$.

$$= \int_{x_0}^{x} \frac{(x-t)^n}{n!} \left| f^{(n+1)}(t) \right| dt$$

$$\leq \int_{x_0}^{x} \frac{(x-t)^n}{n!} K \cdot dt$$

$$= \frac{-K}{(n+1)!} \left[(x-t)^{n+1} \right]_{x_0}^{x}$$

$$= \frac{K}{(n+1)!} (x-x_0)^{n+1}$$

Else if $x < x_0$, then:

Just swap $x \leftrightarrow x_0$, negative gets abs valued.

now, $|R_n(x)| = \left| \int_{x}^{x_0} \frac{(x-t)^n}{n!} f^{(n+1)}(t) dt \right|$

If n is even, you get $-$ve version. Odd you get $+$.

Taylor's Inequality:

$$|R_n(x)| \leq K \frac{|x-x_0|^{n+1}}{(n+1)!} \quad \text{where } |f^{(n+1)}(z)| \leq K \ \forall z \in [x_0, x]$$

Finding K: Jan 15, 2018

1) Approx e w/ 7^{th} Mc. Poly. Given $e = 2.7???...$

$f(x)=e^x,\ x_0=0$

$f^{(m)}(x_0)=1$

$P_{7,0}(x) = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\frac{x^7}{7!}$

$P(1) = 2.7182...$

$f^{(8)}(x)=e^x$, so

$|e^x| \le K$ for some $x \in [0,1]$.

★ $e^1 > e^0$ (increasing) on interval. ∴ endpoint (e^1) is max.

$e^1 < 3$, so let $K=3$ closer K = less error.

$|R_n(x)| \le 3\frac{|x|^{n+1}}{(n+1)!}$

$|R_7(1)| \le 3 \cdot \frac{1}{8!}$

$= \frac{1}{13440}$ (less than 10^{-4}, 0.0001, away).

∴ accurate to 3 dec places for sure (except rounding, maybe)

$P(1) \approx 2.71825$

→ Change by + or − 1 still rounds to:

≈ 2.718 ∴ On $[0,1]$,

$P_{7,0}(x) - \frac{x^8}{13440} \le e^x \le P_{7,0}(x) + \frac{x^8}{13440}$ } applies to whole interval

2) Suppose $f^{(n+1)}(x) = (x^2+2)e^{-x}$, interval $[0,2]$. Find K.

(x^2+2): Increasing

↳ max = 6 ($x=2$)

e^{-x} : Decreasing

↳ max = 1 ($x=0$)

∴ Entire function bounded by 6.

* Both fcns positive on interval

3) Suppose $f^{(n+1)}(x) = (x^2-1)x$, interval $[0,1]$.

$0 \le x \le 1$

$-1 \le x^2-1 \le 0$.

need magnitudes.

$|(x^2-1)x| = |(x^2-1)||x|$

$\le (1)(1)$

$= 1$

4) $f^{(n+1)}(x) = x^3 - 2x^2 - 5x + 30$ on $[-3, 0]$ Jan 17, 2018

Not monotonic.

$$f^{(n+1)}(x) \le |x^3| + |-2x^2| + |-5x| + |30|$$
$$= |x|^3 + 2|x|^2 + 5|x| + 30$$

Now, monotonic (decreasing).

$g(0) = 30$, $g(-3) = 90$. $\therefore \le 90$.

5) $f^{(n+1)}(x) = \sin x - \cos x + \ln x - e^{-x}$ on $[\pi, 2\pi]$

$$\le |\sin x| + |-\cos x| + |\ln x| + |e^{-x}|$$
$$\le 1 + 1 + \ln(2\pi) + 2$$
$$\le 3 + \ln(e^3)$$
$$\le 6.$$

Approximation for Integrals:

1) $f(x) = \int_0^x e^{t^2} dt$ $x \in [-\frac{1}{2}, \frac{1}{2}]$

Let $g(u) = e^u$.
$$\approx P_{2,0}(u)$$
$$= 1 + u + \frac{1}{2}u^2$$

$\left.\right\} e^{t^2} \approx 1 + t^2 + \frac{1}{2}t^4$ $(P_{4,0}(t))$

$$\therefore f(x) \approx \int_0^x (1 + t^2 + \frac{1}{2}t^4)\,dt$$
$$= x + \frac{1}{3}x^3 + \frac{1}{10}x^5$$

$f(\frac{1}{2}) \approx 0.5447916\overline{6}$

Accuracy: $e^u = P_{2,0}(u) + R_2(u)$

$|R_2(u)| \le K \frac{|u|^3}{3!}$

$x \in [-\frac{1}{2}, \frac{1}{2}]$, $t \in [0, x]$ $\in [-\frac{1}{2}, \frac{1}{2}]$ $\big\} u \in [0, \frac{1}{4}]$

$|g^{(3)}(z)|$ on $z \in [0, \frac{1}{4}]$:
$$\le e^{\frac{1}{4}}$$
$$< 2 \text{ (or 3)}$$
$$\therefore |R_2(u)| \le \frac{|u|^3}{2}$$
$$= \frac{u^3}{2} \text{ (interval +)}$$

$\therefore e^{t^2} = 1 + t^2 + \frac{1}{2}t^4 + R_2(t^2)$

$|R_2(t^2)| \le \frac{t^6}{2}$

$\rightarrow$

$$\int_0^x e^{t^2}\cdot dt = \int_0^x P_{2,0}(t^2)\cdot dt + \int_0^x R_2(t^2)\cdot dt$$

$x>0$: where $\left|\int_0^x R_{2,0}(t^2)\,dt\right| \le \int_0^x |R_2(t^2)|\,dt$ for $x \in [0,\frac{1}{2}]$

$$\le \frac{x^7}{14}$$

If $x<0$, then:

$$\left|\int_0^x R_{2,0}(t^2)\cdot dt\right| = \left|\int_x^0 R_{2,0}(t^2)\cdot dt\right|$$

$$\le \int_x^0 |R_{2,0}(t^2)|\cdot dt$$

$$\le \frac{-x^7}{14}$$

$$\therefore \left|\int_0^x R_2(t^2)dt\right| \le \frac{|x|^7}{14}$$

$$\therefore \int_0^x e^{t^2}\cdot dt = x + \frac{1}{3}x^3 + \frac{1}{10}x^5 \left(\pm \frac{x^7}{14}\right)$$

$$\int_0^{\frac{1}{2}} e^{t^2}\cdot dt = 0.5447916\overline{6} \pm 0.00056.$$

2) $f(x) = \int_0^x \cos(t)\cdot dt \quad x \in [0,\frac{1}{2}]$ Jan 18, 2018

Same thing: Can integrate first, then approx $\sin(x)$ $\bar{w}$ $P_{5,0}(x)$. OR: Approx $\cos(x)$ first, then integrate.

$$f(x) \approx \int_0^x 1 - \frac{t^2}{2!} + \frac{t^4}{4!}\cdot dt$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Accuracy: $x \in [0,\frac{1}{2}] \Rightarrow t \in [0,\frac{1}{2}]$

$|R_5(t)| \le 1\cdot \frac{|t^6|}{6!}$ since $g^{(6)}(t) = -\cos(t)$ (bounded by 1).

$$\left|\int_0^x R_5(t)\cdot dt\right| \le \int_0^x \frac{t^6}{6!} = \frac{x^7}{7!} \quad \therefore f(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}\left(\pm \frac{x^7}{7!}\right)$$

Bound is next term

Jan 22, 2018

Infinite Series:

$f(x) = \sin(x)$

$P_{2n+1,0}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$ } $x = 3$?

0, 1, 0, −1, etc

↳ converges, P_{11} and $P_{13,0}(x) \cdot 0.141$.

↳ no matter what x is.

$f(x) = \frac{1}{1+x}$. $P_{n,0}(x) = (-1)^n x^n$ $f(2) = \frac{1}{3}$

↳ Diverges, 1, −1, 3, −5 . . . .

what's the diff?

$\sin(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_n(x)$

want $\lim_{n\to\infty} R_n(x) = 0$ for convergence

$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}$ } For any n, $K = 1$.

$(n+1)!$ grows faster than $|x|^{n+1}$, so $\lim_{n\to\infty} = 0$.

$g(x) = \lim_{n\to\infty} \sum_{k=0}^{n} (-1)^k x^k + \lim_{n\to\infty} R_n(x)$

$|R_n(x)| \le K \frac{|x|^{n+1}}{(n+1)!}$ on $[0, 2]$, $K = 1, 1$, then 2, 6 . . . .

$K = (n+1)!$ is the best bound. ↑exact bounds

If $|x| > 1$: $\lim_{n\to\infty} R_n(x) = \infty$, so diverges.
else, $\lim_{n\to\infty} R_n(x) = 0$, so converges.
} within $[-1, 1]$ is "good enough"

Taylor Series:

$\sum_{k=0}^{\infty} \frac{f^k(x_0)(x-x_0)^k}{k!}$ } if $\lim_{n\to\infty} R_n(x) = 0$, convergence.

Infinite Series:

Jan 24, 2018

$\sum_{k=0}^{\infty} a_k$ for constants a_k.

Sequence $\{a_k\}$, list of #s

Series is partial-sum (or ∞)

Reindexing:

$$\sum_{k=q}^{\infty} a_k = \sum_{j=0}^{\infty} a_{j+q}$$

→ if started at 0 vs 1 for $\frac{1}{2^k}$, they will both converge (but to different values).

Geometric Series:

$\sum_{k=0}^{\infty} ar^k$ r: ratio. $\sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}$

$\hookrightarrow = \lim_{n\to\infty} \frac{a(1-r^{n+1})}{1-r}$

If $|r|<1$: $\lim = \frac{a}{1-r}$ (converges)

If $|r|>1$: Diverges

If $r=1$: Diverges

If $r=-1$: Diverges $[(-1)^{n+1}$ as $n\to\infty]$ 0, 1, 0, 1 ....

Examples:

1) $\sum_{k=0}^{\infty} \frac{(-4)^{3k}}{5^{k+1}} = \sum_{k=0}^{\infty} \left(\frac{-64}{5}\right)^k \cdot \frac{1}{5}$ Diverges $|r|>1$

2) $\sum_{k=1}^{\infty} 4\left(\frac{2}{5}\right)^k = \sum_{k=0}^{\infty} 4\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)^k$ Converges to $\frac{8}{3}$

Divergence Test:

If $\lim_{k\to\infty} a_k \neq 0$, then $\sum a_k$ diverges. } number you add by $\neq 0$.

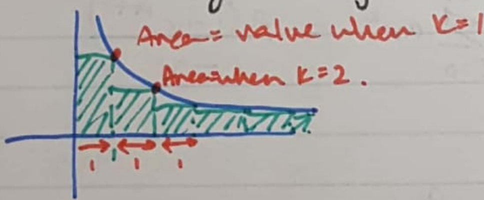
If $\lim_{k\to\infty} a_k = 0$, means nothing. (necessary, not sufficient)

Examples:

1) $\sum_{k=1}^{\infty} \frac{k^2+2}{4k^2-k}$ ← $\lim_{k\to\infty} \frac{k^2+2}{4k^2-k} = \frac{1}{4}$ ∴ diverges.

2) $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges. (why?)

Upper band converges



box area ≤ curve area

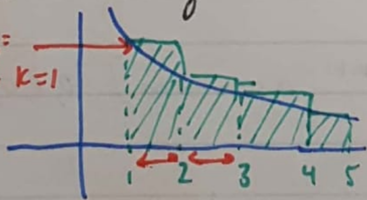
$$\sum_{k=1}^{\infty} \frac{1}{k^2} \leq \int_1^{\infty} \frac{1}{k^2}\, dk + 1 = 2$$

∴ converges to ≤2.

3) $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges * But lim = 0

Lower band diverges

Area = value at k=1



box area ≥ curve area

$$\sum_{k=1}^{\infty} \frac{1}{k} \geq \int_1^{\infty} \frac{1}{k}\, dk = [\ln k]_1^{\infty}$$

Diverges, ∴ fcn diverges.

Integral Test:

$\sum_{k=k_0}^{\infty} a_k$ converges $\Leftrightarrow$ $\int_{k_0}^{\infty} f(x)\,dx$ converges. $f(k) = a_k$

↳ Works for:

- $a_k > 0$. (or < 0 and multiply converge by -1)
- f continuous ($\int$ exists)
- $f \to 0$ as $x \to \infty$ (Divergence test above)

Jan 26, 2018

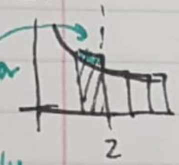
Example:

1) $\sum_{k=2}^{\infty} \frac{1}{k(\ln(k))^2} \rightarrow \lim_{n\to\infty} \int_2^n \frac{1}{x(\ln(x))^2} \cdot dx$

u-sub $u = \ln x$, $du = \frac{1}{x} dx$.

$\lim_{n\to\infty} \int_{\ln 2}^{\ln(n)} \frac{1}{u^2} du = \frac{1}{\ln(2)}$ $\therefore$ converges.

2) Lower bound:



Term for early values

$\sum_{k=1}^{\infty} \frac{1}{k^2} \geq \int_1^{\infty} \frac{1}{x^2} dx \;(=)\; \underline{1} + \int_2^{\infty} \frac{1}{x^2} dx$ → more accurate if we take actual value.

$= 1 \qquad = \frac{3}{2}$

$\therefore \frac{3}{2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \leq 2$

To increase accuracy, evaluate more initial terms.

$\sum_{k=1}^{n} \frac{1}{k^2} + \int_{n+1}^{\infty} \frac{1}{k^2} dx \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \leq \sum_{k=1}^{n} \frac{1}{k^2} + \int_n^{\infty} \frac{1}{k^2} dx$ ↑ n = ↑ accurate.

General Bounds:

$S_n = \sum_{k=1}^{n} a_k \qquad S_n + \int_{n+1}^{\infty} f(x) \cdot dx \leq \sum_{k=1}^{\infty} a_k \leq S_n + \int_n^{\infty} f(x) \cdot dx$

P-Series Test:

$\sum \frac{1}{k^p}$ converges $\Leftrightarrow p > 1$. (b/c integral test)

Comparision Test:

$\sum a_k$, $a_k > 0$. Identify $\sum b_k$ where $a_k \leq b_k \; \forall k$, and $\sum b_k$ converges. Then a_k converges.

↳ Same $\bar{w}$ diverging $\leq b_k$.

↳ Pick geometric or p-series.

Example: Jan 29, 2018

1) $\sum_{k=1}^{\infty} \frac{1}{2^k+\sqrt{k}}$; $a_k > 0$ and $\lim_{k\to\infty} a_k = 0$.

Not easy to integrate. lim = 0, so no help.
Not geometric/p series.

$\frac{1}{2^k+\sqrt{k}} \leq \frac{1}{2^k}$ } Geometric $(\frac{1}{2})^k$, converges $|r|<1$.

Limit Comparision Test:

$\sum \frac{1}{k^2+2} \leq \sum \frac{1}{k^2}$ (comp. Test)

$\sum \frac{1}{k^2-2}$; If $\lim_{k\to\infty} \frac{a_k}{b_k} = L$ (constant), then non-zero.
$\sum a_k$ and $\sum b_k$ both converge/diverge.

$\lim_{k\to\infty} \frac{k^2}{k^2-2} = 1$. } $\therefore$ same behaviour, both converge.

Alternating Series Test:

$\sum_{k=0}^{\infty} (-1)^k a_k$ $(a_k > 0)$

If $\lim_{k\to\infty} a_k = 0$ and $\{a_k\}$ eventually decreases, then converges.

Example:

$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ $\lim_{k\to\infty} \frac{1}{k} = 0$, and $\frac{1}{k}$ decreases. $\therefore$ convergence.

Alternating Series Estimation Theorem:

If we use n^{th} term as estimate.

$$|S - S_n| \leq a_{n+1}$$

"Absolute Convergence":

$\sum |a_k|$ converges. $\qquad \sum \frac{(-1)^k}{k^2}$

"Conditional Convergence"

$\sum a_k$ converges, $\sum |a_k|$ doesn't. $\qquad \sum \frac{(-1)^k}{k}$

The Ratio Test:

If $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L \ldots$

$L < 1$? $\sum a_k$ is abs. convergent.

$L > 1$? $\sum a_k$ is divergent.

$L = 1$? IDK.

Example:

1) $\sum \underbrace{(-1)^k \frac{2^k}{k!}}_{a_k}$ $\qquad \lim_{k\to\infty} \left|\frac{\frac{2^{k+1}}{(k+1)!}}{\frac{2^k}{k!}}\right| = 0. < 1 \therefore$ abs conv.

$(-1)^k$ gets 1·1

2) $\sum \frac{3^k}{k^2 2^k}$ $\qquad \lim_{k\to\infty} \left|\frac{\frac{3^{k+1}}{(k+1)^2 2^{k+1}}}{\frac{3^k}{k^2 2^k}}\right| = \frac{3}{2} > 1 \therefore$ divergent

The Root Test:

$L = \lim_{k\to\infty} \sqrt[k]{|a_k|}$ } same cond. as ratio test.

Ratio/Root Test check if series behaves like geometric.

$$\lim_{k\to\infty} \left|\frac{ar^{k+1}}{ar^k}\right| = r$$

Power Series: Jan 31, 2018

$$\sum_{k=0}^{\infty} c_k (x-x_0)^k$$ } Call T. Series if obtained from f. deriv, etc.

convergence? Ratio test:

$$L = \lim_{k\to\infty} \left|\frac{c_{k+1}}{c_k}\right| |x-x_0|$$

For convergence, $L < 1$

$$|x-x_0| < \underline{\lim_{k\to\infty} \left|\frac{c_k}{c_{k+1}}\right|}$$ "close enough" R (radius of convergence)

↳ $R = 0$? convergence at $x = x_0$

↳ $R = \infty$? convergence $\forall x$

↳ R not ?? convergence for $x \in \underline{(x_0 - R, x_0 + R)}$ check endpoints

Example:

1) $f(x) = \sin x$ $\quad P_{\infty,0}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

$$L = \lim_{k\to\infty} \left|\frac{x^2}{(2k+2)(2k+3)}\right| < 1$$

$$x^2 < \lim_{k\to\infty} (2k+2)(2k+3) = R$$

$$\therefore x^2 < \infty$$

2) $g(x) = \frac{1}{1+x}$ $\quad P_{\infty,0}(x) = \sum_{k=0}^{\infty} (-1)^k x^k$

$$L = \lim_{k\to\infty} \left|\frac{x^{k+1}}{x^k}\right| = |x| < 1. \quad (R = 1)$$

3) $\sum_{k=1}^{\infty} \frac{(x-3)^k}{k4^k}$ $\quad L = \lim_{k\to\infty} \left|\frac{(x-3)^{k+1} 4^k}{(k+1) 4^{k+1} (x-3)^k}\right|$

$$= \lim_{k\to\infty} \left|\frac{k(x-3)}{4(k+1)}\right| \qquad \therefore |x-3| < 4$$

$$\left|\frac{x-3}{4}\right| < \lim_{k\to\infty} \left|\frac{k+1}{k}\right| = 1$$

→

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$x \in (-1, 7)$ convergence.
↳ Check endpoints

If $x = -1$, $x - 3 = -4$

$$\sum_{k=1}^{\infty} \frac{(-4)^k}{k 4^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \Big\} \text{ Conditional convergence}$$

If $x = 7$, $x - 3 = 4$

$$\sum_{k=1}^{\infty} \frac{4^k}{k 4^k} = \sum_{k=1}^{\infty} \frac{1}{k} \Big\} \text{ Divergence}$$

$\therefore$ $x \in [-1, 7)$ for convergence.

Manipulation of Power Series:

If $\sum c_k (x - x_0)^k$ has r.o.c. R,
↳ Differentiate terms
↳ Integrate
↳ $\times$ by constant
↳ Add to another series of radius $\geq R$. ($\times$ and $\div$ works too)

All resulting series will have r.o.c. R.

Example:

1) $f(x) = \frac{1}{1-x}$: $\sum_{k=0}^{\infty} x^k$ $\quad L = \lim_{k \to \infty} |x| \Rightarrow R = 1$

$f'(x) = \frac{+1}{(1-x)^2}$ $\quad \frac{d}{dx} \sum_{\boxed{k=0}}^{\infty} x^k = \sum_{\boxed{k=1}}^{\infty} k x^{k-1}$. $\quad$ R still 1.

$F(x) = -\ln(1-x)$ $\quad \int \sum_{k=0}^{\infty} x^k \, dx = \sum_{\boxed{k=0}}^{\infty} \int x^k \, dx = \sum_{\boxed{k=0}}^{\infty} \frac{x^{k+1}}{k+1} + C$

Find C: Let $x = 0$

$$\ln(1) = \sum_{k=0}^{\infty} \frac{0}{k+1} + C \quad \therefore C = 0$$

Feb 2, 2018

2) $-\ln(1-x) = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} \Rightarrow \int \ln(1-x) = -\int \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$ R still 1

3) $\frac{1}{(1-x)^2} + \ln(1-x) = \sum_{k=1}^{\infty} k x^{k-1} - \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$

$= \sum_{k=0}^{\infty} (k+1)x^k - \sum_{k=1}^{\infty} \frac{1}{k} x^k$

$= 1 + \sum_{k=1}^{\infty} (k+1-\frac{1}{k})x^k$ } T. series with R = 1.

* **Note**: R is same, but interval could be diff.
↳ check endpoints.

4) $\frac{x}{3+2x} = x\left(\frac{1}{3+2x}\right) = \frac{x}{3}\left(\frac{1}{1-(\frac{-2x}{3})}\right)$

$= \frac{x}{3}\left[\sum_{k=0}^{\infty} \left(\frac{-2}{3}x\right)^k\right]$

$= \sum_{k=0}^{\infty} (-1)^k \frac{2^k x^{k+1}}{3^{k+1}}$

$|\frac{2}{3}x| < 1 \Rightarrow |x| < \frac{3}{2}$
↳ x by $\frac{1}{3} \Rightarrow$ R still $\frac{3}{2}$
↳ "x" is power series (lol), so we multiplied by series.

Simple Functions to Reduce To:

$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ $|x| < 1$

$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$ All x (∞)

$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ All x (∞)

$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ All x (∞)

$(1+x)^m = 1 + mx + m(m-1)\frac{x^2}{2!} + \ldots + m(m-1) * \ldots * (m-n+1)\frac{x^n}{n!}$

Big O:

f is of order g as $x \to x_0$ if $\exists A > 0 \in$
$|f(x)| \le A|g(x)| \Rightarrow f(x) = O(g(x))$

Example:

1) If $P_{4,0}(x) = P_{5,0}(x) = P_{6,0}(x)$
$f(x) \approx a + bx + cx^2 + dx^3 + ex^4$ } Doesn't convey what happens after
↳ $f(x) = a + bx + cx^2 + dx^3 + ex^4 + O(x^7)$
↳ Conveys next valid $P(x)$.

2) $|x^3| \le 1|x^2|$ as $x \to 0$, so $x^3 = O(x^2)$
Around $x = 0$, $x^a = O(x^b)$ for $a \ge b$.
That's why we can replace rest of series w/ $O(x^n)$.

3) $|\sin(x)| \le |x|$ $\forall x$ (look @ t-series).
$\therefore \sin(x) = O(x)$
$\sin(x) = x - \frac{x^3}{3!} = x + O(x^3)$

Rules:

Feb 3, 2018

$kO(x^n) = O(x^n)$
$O(x^m) + O(x^n) = O(x^{\min\{m,n\}})$
$O(x^m) \cdot O(x^n) = O(x^{mn})$
$\frac{O(x^m)}{x^n} = O(x^{m-n})$

Example:

1) $|R_n(x)| \le \frac{k}{(n+1)!}(x - x_0)^{n+1} = O((x-x_0)^{n+1})$
$f(x) = P_{n,x_0}(x) + O((x - x_0)^{n+1})$

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2) $f(x) = \sqrt{1+x} + \sin(x)$ around $x_0 = 0$.

$$\sqrt{1+x} = 1 + \frac{1}{2}x + o(x^2)$$

$$\sin(x) = x + o(x^3)$$

$$f(x) = 1 + \frac{3}{2}x + o(x^2) \text{ as } x \to 0.$$

3) $f(x) = e^{x^2}$ Let $t = x^2$

$$f(t) = e^t$$

$$= 1 + t + \frac{t^2}{2} + o(t^3)$$

$$f(x) = 1 + x^2 + \frac{x^4}{2} + o(x^6)$$

Subbing works the same.

4) $f(x) = e^x \sin(x)$

$$= \left(1 + x + \frac{x^2}{2} + o(x^3)\right)\left(x + o(x^3)\right)$$

$$= x + x^2 + \frac{x^3}{2} + o(x^4) + o(x^3) + o(x^4) + \frac{o(x^5)}{2} + o(x^6)$$

$$= x + x^2 + o(x^3)$$

5) $f(x) = \cos(x)$. Find Approx w/o T. series for cos.

$$\sin(x) = x - \frac{x^3}{6} + o(x^5)$$

$$\rightsquigarrow \cos(x) = 1 - \frac{x^2}{2} + o(x^4)$$

Limits:

1) $\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{x + o(x^3)}{x} = \lim_{x \to 0} 1 + o(x^2) = 1$

2) $\lim_{x \to 0} \frac{\sin(x)\ln(x+1)}{x^4} = \lim_{x \to 0} \frac{(x + o(x^3))(x + o(x^2))}{x^4} = \infty$