Dr. Graeme Turner
MC 6513
g3turner@uwaterloo.ca
* Need textbook for code

MATH 117 - Introduction

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Pre-Calculus Review:

Definitions:

Function from set A to set B that assigns each x in $A \to y$ in B
- each x has only one y (vert. line test)
- Doesn't need to be math formula
- $f: A \to B$:
  - A is the domain (dom(f)), B is codomain ($\mathbb{R}$) ↑ subset of
- Range of f:
  - Set of all outputs of f range(f)
- Domain: set of all permissable inputs

Set Notation
- $\mathbb{Z}$ - the integers
- $\mathbb{Q}$ - the rational numbers
- $\mathbb{R}$ - real #'s
- $\mathbb{C}$ - $\{a + bi \mid a, b \in \mathbb{R}, i = \sqrt{-1}\}$

Interval Notation → Refer to precalc notes

Inverse Functions

Identity Fcn: $id(x) = x$
- g is inv. of f if $g(f(x)) = x$ } And reversed!
- "undoes" action of original fcn dom(f) = rng(f) = $[0, \infty)$
- Invertible (inv. fcn) if fcn is one to one of interval

One to one fcn: Pass — test "injective" $f(x_1) = f(x_2)$ then $x_1 = x_2$

Intersections: Overlap of sets $A \cap B$ $[1,5) \cap (2,5] = (2,5)$

Unions: Combining sets $A \cup B$ $[1,5) \cup (2,5] = [1,5]$
- $(0,1) \cup (2,3) = (0,1) \cup (2,3)$
- $(0,1) \cap (2,3) = \emptyset$ "null set"
- Properties:
  - $A \cup \emptyset = A$ $A \cap \emptyset = \emptyset$

Subsets: $A \subseteq B$ or $A \subset B$ } A is subset of B

Watch

*For inv. fcns: $y = \sqrt{x}$ passes — test but $y = x^2$ not a fcn.
- $y = \sqrt{x}$ domain is restricted. dom(g) = rng(f) v.s versa
- x^2 is inv, but domain restricted to $[0, \infty)$

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Composition:

- f, g where $rng(g) \subseteq dom(f)$
  $\hookrightarrow f \circ g(x) = f(g(x))$
- ① $dom(f \circ g) \subseteq dom(g)$
  ② $rng(f \circ g) \subseteq rng(f)$
  ③ $f \circ g \neq g \circ f$, usually.

Inverses cont...

$dom(g) = rng(f)$
$dom(f) = rng(g)$

Example:

1) $f(x) = 2x-4/3x+6$
$f^{-1}(x) = -6y-4/3y-2$ } find $dom(f)$, $rng(f)$

$dom(f) = \{x \in \mathbb{R} \mid x \neq -2\} = (-\infty, -2) \cup (-2, \infty)$

$rng(f) = dom(f^{-1}) = \{x \in \mathbb{R} \mid x \neq 2/3\} = (-\infty, \frac{2}{3}) \cup (\frac{2}{3}, \infty)$

Even & Odd fcns:

Even: $f(x) = f(-x)$
Odd: $f(-x) = -f(x)$

Operations:

$E \times E = E$
$E \times O = O$
$O \times O = E$

} same as if E = +ve int, O = -ve int

Components:

$f_{even}(x) = \frac{1}{2}(f(x) + f(-x))$
$f_{odd}(x) = \frac{1}{2}(f(x) - f(-x))$
} $f(x) = f_e(x) + f_o(x)$

$\therefore$ These only exist when $dom(f)$ not sym. about origin

$(-\infty, -2) \cup (2, \infty)$ ✓
$(-\infty, -5) \cup (2, \infty)$ ✗

Hyperbolic Functions:

$f_o(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh(x)$
$f_e(x) = \frac{1}{2}(e^x + e^{-x}) = \cosh(x)$

Piecewise Functions: * See 1.5 course notes

Absolute Value: $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$: $\sqrt{x^2} = (x^2)^{\frac{1}{2}} \neq x$ be careful of even denom.

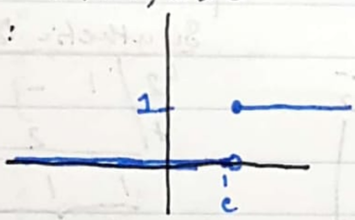
Heavyside fcn (unit step fcn):

$$H(x) = \left.\begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}\right\} \text{important for multi.}$$

Using $H(x)$:

$$f(x)H(x) = \begin{cases} 0, & x < 0 \\ f(x), & x \geq 0 \end{cases}$$

$H(x-c)$:



} kind of like an on/off switch

Example:

1) $H(x-a) - H(x-b)$ when $a < b$

$$= \begin{cases} 0, & x < a \\ 1, & a \leq x < b \\ 0, & x \geq b \end{cases}$$ a "turns on"

2) Write $f(x)$: $\begin{cases} 5-x, & x < 2 \\ \sqrt{x}, & 2 \leq x < 5 \\ 13, & x \geq 5 \end{cases}$ "subtract off" the formula to the left (number line) when a new formula arises.

$$f(x) = \begin{cases} 5-x \\ [\sqrt{x} - (5-x)]H(x-2) \\ [13 - \sqrt{x}]H(x-5) \end{cases} = 5-x + [\sqrt{x} - (5-x)]H(x-2) + [13 - \sqrt{x}]H(x-5)$$

* pg 21 course notes for changing = location

Partial Fraction Decomposition:

Decomposing a rational function:

↳ denom: linear, non reducable quads, and $(\ )^2$.

Example:

1) $\dfrac{5x^2 - 5x + 4}{x^3 - x^2 - x - 2}$

Steps:

1) Factor the denominator
- ↳ likely need root of denom.
- ↳ For deg 3+, use the Rational Roots Theorem
- ↳ Rational Root Theorem:
  - ↳ Any rational root of $a_n x^n + \ldots a_1 x + a_0$ is the form c/d where:
  - ★ $\left\{\begin{array}{l} c \text{ is a factor of } a_0 \\ d \text{ is a factor of } a_n \end{array}\right\}$ Find factor then divide
- ↳ Long Division:

$$\begin{array}{r} x^2+x+1 \\ x-2 \overline{) x^3-x^2-x-2} \\ -(x^3-2x^2) \\ \hline x^2-x \\ -(x^2-2x) \\ \hline x-2 \\ -(x-2) \\ \hline 0 \checkmark \end{array}$$

Synthetic Division:

$$\begin{array}{c|cccc} 2 & 1 & -1 & -1 & -2 \\ + & & 2 & 2 & 2 \\ \hline & 1 & 1 & 1 & 0 \checkmark \end{array}$$

$\underbrace{1 \quad 1 \quad 1}_{x^2+x+1}$

Now @ irreducible roots.

2) Split up the fcn.

$$\frac{5x^2-5x+4}{x^3-x^2-x-2} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-2}$$

variables must be 1 less $\deg(p(x))+1 = \deg(q(x))$

$5x^2-5x+4 = (Ax+B)(x-2) + C(x^2+x+1)$

2 approaches now.

a) Expand RS and compare coefficients to LS.

b) Choose strategic values for x. ★ Much better.
- ↳ Let $x=2$. cancels first term.
- ↳ Let $x=0$. cancels A.
- ↳ Let $x=1$ to get A.

special case when denom is squared

★ $$\frac{7x^5-10x^4-2x^3-45x^2-x-13}{(x^2+x+1)^2(x-2)^2}$$

$$= \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2} + \frac{E}{(x-2)} + \frac{F}{(x-2)^2}$$

Show A=2 B=-1 C=4
D=1 E=5 F=-3

Independent Reading:
Other Important Piecewise Fcns:

Signum:

$$sgn(x) \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$$ } returns sign of #.

Ramp:

$$r(t) \begin{cases} 0 & t<0 \\ ct & t\geq 0 \end{cases}, \quad c \in \mathbb{R}$$ } Heavyside w const mult.

Floor:

$\lfloor x \rfloor$ { round down

$-1.5 \to -2$
$1.5 \to 1$ } ↓

Ceiling:

$\lceil x \rceil$ { round up

$-1.5 \to -1$
$1.5 \to 2$ } ↑

Frac Part:

$frac(x): x - \lfloor x \rfloor$ gets decimal.

Partial Fractions Decomp. Cont... Sep 15, 2017.

1) Factor denominator

2) For each distinct factor $g(x)$ of $q(x)$ w exponent $n \geq 1$:

a) If $g(x)$ is linear, it contributes

$$\frac{A_1}{g(x)} + \frac{A_2}{(g(x))^2} \cdots \frac{A_n}{(g(x))^n}$$

See ex. on back ← { b) Irreducible quadratic, it contributes

$$\frac{A_1X+B_1}{g(x)} + \frac{A_2X+B_2}{(g(x))^2} \cdots \frac{A_nX B_n}{(g(x))^n}$$

3) Write over common denom (don't expand numerator)

4) Special Values of x to cancel & solve.

5) Write out final answer.

Tutorial

Sep 18, 2017

Inequalities & Absolute Values:

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \text{ is } + \\ -f(x) & \text{if } f(x) \text{ is } - \end{cases}$$

Example:

1) $|3x-12| < 9$

1) $3x - 12 < 9$

$x < 7$

2) $12 - 3x < 9$

$x > 1$

2) Describe the set $\{x : |x^2-2| > 1\}$ as a union of finite intervals.

1) $-(x^2-2) > 1$

$(x+1)(x-1) < 0$ DNE OR $-1 < x < 1$ $x \in (-1, 1)$

2) $x^2 - 2 > 1$ $\boxed{x \in (\sqrt{3}, \infty) \text{ or } (-\infty, -\sqrt{3})}$ careful in Stewart

$x \in (-\infty, -\sqrt{3}) \cup (-1, 1) \cup (\sqrt{3}, \infty)$

3) $|x-2| < |x+1|$

$(x-2)^2 < (x+1)^2$

$\frac{1}{2} < x$.

Composite Fcns:

1) $f(x) = \frac{1}{x}$ $x \in (0, \infty)$ and $g(x) = 2x+1$ $x \in [-2, 2]$

rng(g(x)) must be in dom(f) and x must be in dom(g(x))

$f(g(x))$ $g(x) > 0$ $x > \frac{-1}{2}$ $\therefore x \in (\frac{-1}{2}, 2]$

2. $g(f(x))$ $-2 \leq f(x) \leq 2$

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Partial Fraction Decomposition Continued:

$$\frac{3x^4+3x^3+2x^2+2x-3}{x^5+2x^4+2x^3+4x^2+x+2}$$

$$=\frac{3x^4+3x^3+2x^2+2x-3}{(x+2)(x^4+2x^2+1)}=\frac{3x^4+3x^3+2x^2+2x-3}{(x+2)(x^2+1)^2}\text{, so...}$$

$$\frac{3x^4+3x^3+2x^2+2x-3}{x^5+2x^4+2x^3+4x^2+x+2}=\frac{A}{x+2}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$$

$$=A(x^2+1)^2+(Bx+C)(x+2)(x^2+1)+(Dx+E)(x+2)$$

Let $x=-2$: $28=25A$ $\boxed{\therefore A=1}$

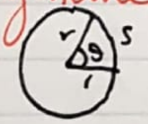
Let $x=0$: $-4=2C+2E$ $\boxed{\therefore E=-2-C}$

Let $x=1$: $7=4+6B+6C+3D-6-3C$ $\boxed{\therefore C=3-2B-D}$

Let $x=-1$: $-8=-4B-2D$ $\boxed{\therefore B=2-\frac{D}{2}}$

Let $x=2$: $16=8B-8D$ $\boxed{\therefore D=0}$ Now, go backwards

Trigonometry:



circle subtends arc between sides of the angle.
1 radian is the measure of an angle in a circle that subtends an arc of equal length to the radius.

*rad ≠ unit!

$\sin\theta$: y coord of point on unit circle corresponding to θ
$\cos\theta$: x " " " " " "

Reference Angle: $\bar{\theta}$

Sep 20, 2017

$[0 \le \theta \le \frac{\pi}{2}]$ $|\sin\theta| = \sin\bar{\theta}$ $|\cos\theta| = \cos\bar{\theta}$

Formulas:

$\left.\begin{aligned}\sin\theta &= \sin(\theta+2\pi k)\\ \cos\theta &= \cos(\theta+2\pi k)\end{aligned}\right\} k\in\mathbb{Z}$

$\cos\left(\theta-\frac{\pi}{2}\right)=\sin\theta$

$\sin\left(\theta-\frac{\pi}{2}\right)=-\cos\theta$

$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

$\cos^2\theta + \sin^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$\cot^2\theta + 1 = \csc^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$

$\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$

Example:

1) Find all $\theta \in [0, 2\pi)$ that satisfy:

a) $\cos\theta + \sin 2\theta = 0$

b) $\sec^2\theta = 4/3$

a) $\cos\theta + \sin 2\theta = 0$

$\cos\theta + 2\sin\theta\cos\theta = 0$

$\cos\theta(1 + 2\sin\theta) = 0$

$\cos\theta = 0$ and $\sin\theta = \frac{-1}{2}$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

b) $\sec^2\theta = \frac{4}{3}$

$\cos^2\theta = \frac{3}{4}$

$\cos\theta = \pm\frac{\sqrt{3}}{2}$

so

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

2) $\sin\left(\frac{7\pi}{12}\right) = \sin\left(\frac{3\pi}{12} + \frac{4\pi}{12}\right) = \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\cos\frac{\pi}{4}$

$= \frac{\sqrt{2}}{4}(1 + \sqrt{3})$

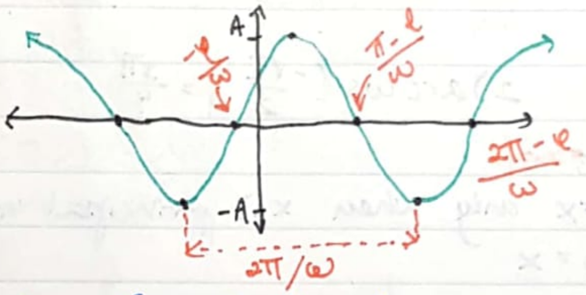
3) $\sin^2\left(\frac{7\pi}{12}\right) = \dfrac{1-\cos\left(\frac{7\pi}{6}\right)}{2} = \dfrac{2+\sqrt{3}}{4}$

General Sine Function:

Sep 22, 2017

$f(x) = A\sin(\omega x + \varphi)$

Period: $\frac{2\pi}{\omega}$ Phase shift: $\frac{-\varphi}{\omega}$ Amplitude: A



Every function $B\sin(\omega x) + C\cos(\omega x)$ can be written $A\sin(\omega x + \varphi)$

Example:

1) $3\sin(2x) + 3\sqrt{3}\cos(2x) = A\sin(2x + \varphi)$

$= A[\sin(2x)\cos(\varphi) + (\sin\varphi)\cos(2x)]$

$= A\cos(\varphi)\sin(2x) + A\sin(\varphi)\cos(2x)$

$\therefore\ 3 = A\cos(\varphi) \quad 3\sqrt{3} = A\sin(\varphi)$ square both sides and add

$A^2\cos^2(\varphi) + A^2\sin^2(\varphi) = 36$

$A = \pm 6$ Take $A = 6$ (-6 also works)

$\therefore\ \varphi = \frac{\pi}{3}$ $\therefore$ Final: $6\sin(2x + \frac{\pi}{3})$

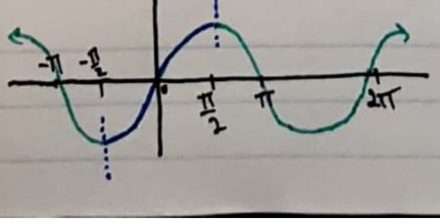
$A = \pm\sqrt{B^2 + C^2}$

$\left.\begin{array}{l}\cos\varphi = B/A \\ \sin\varphi = C/A\end{array}\right\}$ $\varphi = \arctan(C/B)$ only when $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$

Inverse Trig:

Sin, cos, tan not inv. <u>but</u> can restrict domain.

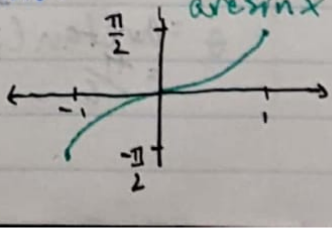
The half-period $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is inv.



$f^{-1}(x) = \arcsin x$

$f: [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$

$f^{-1}: [-1, 1] \to \underbrace{[-\frac{\pi}{2}, \frac{\pi}{2}]}_{\text{Principle values}}$



For cosine: $g: [0,\pi] \to [-1,1]$

$g^{-1}: [-1,1] \to \underbrace{[0,\pi]}$ principle values

$= \arccos(x)$

Examples:

1) $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\pi/4$ 2) $\arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$

3) $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}$ *careful

$\hookrightarrow \arcsin(\sin x) = x$ only when x is principal value of sine

BUT $\sin(\arcsin x) = x$

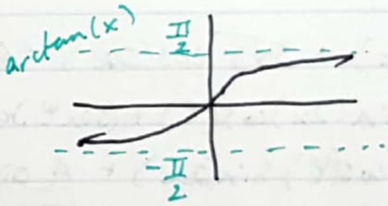
4) $\sin\left(\arcsin\left(\frac{-1}{2}\right)\right) = -1/2$

5) $\sin\left(\arcsin\left(\frac{\pi}{2}\right)\right) = \text{DNE}$ * Watch

For tan: $h: \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$

$h^{-1}: \mathbb{R} \to \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ } P.V.'s

$= \arctan x$



For sec: $(-\infty,-1] \cup [1,\infty) \to \underline{\left(-\pi, \frac{-\pi}{2}\right] \cup \left[0, \frac{\pi}{2}\right)}$, PVs $= \operatorname{arcsec} x$

For csc: $(-\infty,-1] \cup [1,\infty) \to \underline{\left(-\pi, \frac{-\pi}{2}\right) \cup \left(0, \frac{\pi}{2}\right)}$, PVs $= \operatorname{arccsc} x$

For cot: $j: (0,\pi) \to \mathbb{R}$ $= \operatorname{arccot} x$

$j^{-1}: \mathbb{R} \to (0,\pi)$ } PVs

1) $\operatorname{arccot}(\sqrt{3}) = \theta$

$\cot\theta = \sqrt{3}$

$\tan\theta = \frac{1}{\sqrt{3}}$

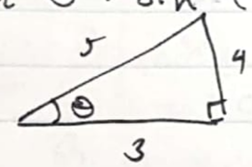
$\theta = \arctan\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{\pi}{6}$

Sometimes Δ trig helps.

1) $\cos(\arcsin(4/5)) = \cos\theta$

Let $\theta = \sin^{-1}(4/5)$



$\therefore \cos\theta = 3/5$.

2) $\cot(\operatorname{arcsec}(13))$

Let $\theta = \operatorname{arcsec}(13)$

$\sec\theta = 13$

$\cos\theta = \frac{1}{13}$ solve for $\frac{1}{\tan\theta}$ using Δ.

3) $\cos(2\arcsin(7/11))$

$= \cos^2(\arcsin(\frac{7}{11})) - \sin^2(\arcsin(\frac{7}{11}))$

$= \cos^2\theta - (\frac{7}{11})^2$

$\theta = \arcsin(\frac{7}{11})$

(Δ) $\sin\theta = \frac{7}{11}$ $\cos\theta = \frac{\sqrt{72}}{11}$ so $\cos^2\theta = \frac{72}{121}$

$= \frac{23}{121}$

4) $\tan(2\arccos(\frac{5}{13}))$

$= \dfrac{2\tan(\arccos\frac{5}{13})}{1-\tan^2(\arccos\frac{5}{13})}$

$\theta = \arccos\frac{5}{13}$

$\tan\theta = \frac{12}{5}$, so

$= \frac{-120}{119}$

Limits of Sequences:

Let S be an infinite subset of consecutive non-negative integers. A sequence is a function $f: S \Rightarrow \mathbb{R}$

$f(n) = a_n$

Usually, $S = \mathbb{N} = \{1, 2, 3 \dots\}$ or $S = \mathbb{N} \cup \{0\}$

To denote sequences

$\{a_n\}_{n=1}^{\infty}$, $\{a_n\}_{n=0}^{\infty}$, $\{a_n\}$

Example:

1) $\{(-1)^n\}_{n=0}^{\infty} = \{1, -1, 1, -1 \dots\}$
2) $\{3, 9, 15, 21, 27 \dots\} = \{6n+3\}_{n=0}^{\infty} = \{6n-3\}_{n=1}^{\infty}$
$= \{6n-27\}_{n=5}^{\infty}$

Conversions:

To increase starting index by k, substitute n with $(n-k)$

To start $\{6n+3\}_{n=0}^{\infty}$ at $n=5$

$\{6(n-5)+3\}_{n=5}^{\infty}$ simplify.

To start $\{6n-3\}_{n=1}^{\infty}$ at $n=5$

$\{6(n-4)-3\}_{n=5}^{\infty}$ simplify.

Limits of Sequences:

Behavior as $n \to \infty$ $\quad \{\frac{1}{n}\}_{n=1}^{\infty}$

For any +ve number >0 (let ε be this #), we can find an index n where $\frac{1}{n} < \varepsilon$. Consider $1/\varepsilon$, Pick an n where $n > \lceil \frac{1}{\varepsilon} \rceil$

$\hookrightarrow$ If $n > \lceil \frac{1}{\varepsilon} \rceil$, $n > \frac{1}{\varepsilon}$. Thus $\frac{1}{n} < \varepsilon$

$\therefore$ As $n \to \infty$, $\frac{1}{n} \to 0$ $\quad \lim_{n\to\infty} \frac{1}{n} = 0$

Definition ★★★:

A sequence $\{a_n\}$ converges to the limit L if, for any $\varepsilon > 0$ there exists a natural number $N \in \mathbb{N}$ such that whenever $n > N$, $|a_n - L| < \varepsilon$.

$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \ni n > N \to |a_n - L| < \varepsilon$

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Back to $\{\frac{1}{n}\}_{n=1}^{\infty}$: $a_n = \frac{1}{n}$ $L = 0$ $N = \lceil 1/\varepsilon \rceil$

Example:

1) "Use the definition" of sq. limits, prove $\frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$.

$\Leftarrow$ Solution: Required to prove (RTP):

Given any (fixed) $\varepsilon > 0$, we need to find a # N such that $n > N \Rightarrow \left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon$

$\left|\frac{1}{\sqrt{n}} - 0\right| = \frac{1}{\sqrt{n}}$

RTP: $\frac{1}{\sqrt{n}} < \varepsilon$
$n > N$

$\Leftrightarrow$ shows inequality equivalence

$\frac{1}{\sqrt{n}} < \varepsilon$

$\Leftrightarrow \frac{1}{n} < \varepsilon^2$

$\Leftrightarrow n > \frac{1}{\varepsilon^2}$ Let N (must be natural) $= \lceil \frac{1}{\varepsilon^2} \rceil$ if $n > N$, then

$n > N \Rightarrow n > 1/\varepsilon^2$

$\Rightarrow \frac{1}{n} < \varepsilon^2$

$\Rightarrow \frac{1}{\sqrt{n}} < \varepsilon$ QED. $\therefore \lim_{x \to \infty} \frac{1}{\sqrt{n}} = 0$

2) "Prove" $\frac{n-1}{n} \to 1$ as $n \to \infty$.   Sep 27, 2017

Solution: RTP:

$n > N \Rightarrow \left|\frac{n-1}{n} - 1\right| < \varepsilon$

$\Leftrightarrow \left|\frac{-1}{n}\right| < \varepsilon$

$\Leftrightarrow \frac{1}{n} < \varepsilon$

$\Leftrightarrow n > \frac{1}{\varepsilon}$ Let $N = \lceil \frac{1}{\varepsilon} \rceil$.

So, when $n > \lceil \frac{1}{\varepsilon} \rceil$, $\left|\frac{n-1}{n} - 1\right| < \varepsilon$ Thus QED.

Properties of Sequence Limits:

1) $\lim_{n\to\infty} C = C$, $C \in \mathbb{R}$ fixed.

Let $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$

2) $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

Same $\bar{w}$ multi: and division $(B \neq 0)$

3) a) $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for any fixed $p>0$

b) $\lim_{n\to\infty} r^n = 0$ when $|r|<1$, $= \infty$ when $|r|>1$.

4) If f is continuous on open interval containing A:
$\lim_{n\to\infty} f(a_n) = f(A) = f\left(\lim_{n\to\infty} a_n\right)$

5) $\lim_{n\to\infty} a_n = \infty$ iff for all $M>0$, there is an $N \in \mathbb{N}$ such that $n>N \Rightarrow a_n > M$.

Examples:

1) $\left\{ \cos\left(\frac{\pi n}{3n+5}\right) \right\}_{n=1}^{\infty}$ find the limit.

$= \lim_{n\to\infty} \cos\left(\frac{\pi n}{3n+5}\right)$

$= \cos\left(\lim_{n\to\infty} \frac{\pi n}{3n+5}\right)$ } Now, $\div$ top/bottom by highest power of n in denom.

Assumptions:
1) lim exists
2) cos is continuous

$= \cos\left(\lim_{n\to\infty} \frac{\pi}{3+\frac{5}{n}}\right) = \cos\left(\lim_{n\to\infty} \frac{\pi}{3}\right) = \frac{1}{2}$

Squeeze Theorem:

Suppose $\{a_n\}, \{b_n\}, \{c_n\}$ where

a) $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$

b) For some $N \in \mathbb{N}$, whenever $n > N$, $a_n \leq \underset{\text{squished}}{b_n} \leq c_n$.

Then $\lim_{n\to\infty} b_n = L$.

Example:

1) $\lim_{n\to\infty} \frac{\cos(n)}{n}$ We know $-1 \leq \cos(\cancel{n}) \leq 1$ ← start w/ an inequality.

$-1 \leq \cos(n) \leq 1$

$\frac{-1}{n} \leq \frac{\cos(n)}{n} \leq \frac{1}{n}$

$\therefore \lim_{n\to\infty} \frac{-1}{n} \leq \lim_{n\to\infty} \frac{\cos(n)}{n} \leq \lim_{n\to\infty} \frac{1}{n}$

$0 \leq \cancel{\quad} \leq 0$ $\therefore$ The limit is 0.

Indeterminate Forms:

Expressions resulting from direct subst. can't be evaluated.

$0^0, \frac{0}{0}, \frac{\infty}{\infty}, 1^\infty, \infty - \infty, 0 \cdot \infty, \infty^0$

If f is polynomial of degree $< k$,

$\lim_{n\to\infty} n^k \pm f(n) = \lim_{n\to\infty} n^k = \infty$. } Largest power dominates limit

Example:

1) $\lim_{n\to\infty} \frac{9n^2 - 5n + 3}{5n^2 - 9n + 5}$

$= \frac{9}{5}$

2) $\lim_{n\to\infty} \frac{9n+8}{\sqrt{9n^2+8}}$ Divide by highest "effective power" in denom $\sqrt{n^2} = n'$

$\lim_{n\to\infty} \frac{9 + \cancel{\frac{8}{n}}}{\sqrt{\frac{9n^2}{\cancel{n^2}} + \cancel{\frac{8}{n^2}}}} = 3$

3) $\lim_{n\to\infty} \frac{6^n+3^n}{6^{n-1}-4}$ Divide by 6^{n-1} (highest exp of n in denom)

$= \lim_{n\to\infty} \frac{6^n+3^n}{6^{n-1}-4} \cdot \frac{1/6^{n-1}}{1/6^{n-1}}$

$= \lim_{n\to\infty} \frac{6+3\cdot\left(\frac{3^{n-1}}{6^{n-1}}\right)}{1-4/6^{n-1}}$

$= \lim_{n\to\infty} \frac{6+3\cdot\left(\frac{1}{2}\right)^{n-1}}{1-4/6^{n-1}}$

$= \frac{6+0}{1-0} = 6.$

4) $\lim_{n\to\infty} \sqrt{n^2+1} - n \cdot \left(\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}\right)$

$\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} = \frac{1}{\infty} = 0$

$\{a_n\}$ is divergent if $\lim_{n\to\infty} a_n = \pm\infty$ or DNE

Limits of functions:

$\lim_{x\to c} f(x) = L$ iff $\forall \varepsilon > 0, \exists \delta > 0 \ni |x-c| < \delta \Rightarrow |f(x)-L| < \varepsilon$

Example:

1) Prove $\lim_{x\to 2} 3x-2 = 4.$ *page 59

RTP: Given $\varepsilon > 0$, find $\delta > 0$ where $\underset{|x-2|}{|x-c|} < \delta \Rightarrow |3x-2-4| < \varepsilon.$

$|3x-6| < \varepsilon$

$\Leftrightarrow 3|x-2| < \varepsilon$

$\Leftrightarrow |x-2| < \varepsilon/3$ Thus if $\delta = \varepsilon/3$, is true, so $\lim = 4$.

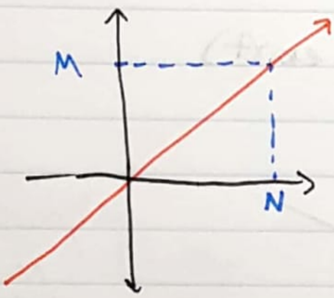
Key Definitions: pg 59

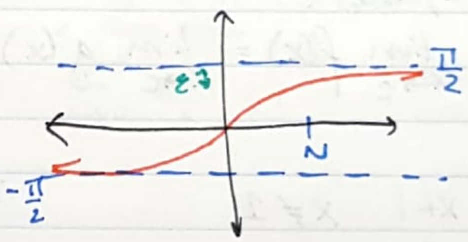
How close you are to a limit

1) $\lim_{x \to c} f(x) = \infty$ $\quad \underbrace{\forall M > 0}_{\text{implies } \infty} \; \exists \delta > 0 \ni \underbrace{|x-c| < \delta}_{} \Rightarrow f(x) > M.$

2) $\lim_{x \to \infty} f(x) = L$ $\quad \forall \varepsilon > 0 \; \exists N > 0 \ni x > N \Rightarrow |f(x) - L| < \varepsilon$

3) $\lim_{x \to \infty} f(x) = \infty$ $\quad \forall M > 0 \; \exists N > 0 \ni x > N \Rightarrow f(x) > M.$





2

Constructing Definitions:

$\boxed{\lim_{x \to c}}$: $\exists \delta > 0 \ldots$
$\ldots \ni |x-c| < \delta \Rightarrow$

$\boxed{\lim_{x \to \infty}}$: $\exists N > 0 \ldots$
$\ldots \ni x > N \Rightarrow$

$\boxed{\lim_{x \to -\infty}}$: $\exists N > 0 \ldots$
$\ldots \ni x < -N \Rightarrow$

$\boxed{\lim f(x) = L}$: $\forall \varepsilon > 0 \; \exists \ldots \Rightarrow |f(x) - L| < \varepsilon$

$\boxed{\lim f(x) = \infty}$: $\forall M > 0 \; \exists \ldots \Rightarrow f(x) > M$

$\boxed{\lim f(x) = -\infty}$: $\forall M > 0 \; \exists \ldots \Rightarrow f(x) < -M.$

One sided limits:

$|x - c|$: 2 sided. $= \begin{cases} x - c, & x > c \quad c^+ \\ c - x, & x < c \quad c^- \end{cases}$

$\lim_{x \to c^-} f(x) = L$: $\forall \varepsilon > 0 \; \exists \delta > 0 \ni c - x \; \delta \Rightarrow |f(x) - L| < \varepsilon$

$\lim_{x \to c^+} f(x) = L$: $\forall \varepsilon > 0 \; \exists \delta > 0 \ni x - c < \delta \Rightarrow |f(x) - L| < \varepsilon$

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$\lim_{x \to c^+} f(x) = \infty : \forall M > 0 \; \exists \delta > 0 \ni x - c < \delta \Rightarrow f(x) > M.$

Limits cont... Sep 29, 2017

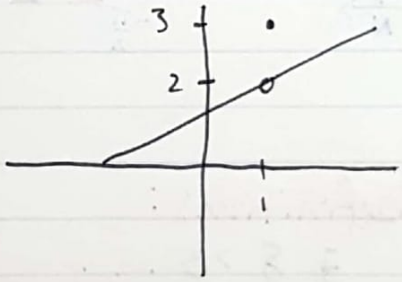
Let $c \in \mathbb{R}$. Open neighbourhood of c is an open interval containing c. (a,b)

↳ If f, g are fcn's and $f(x) = g(x)$ $\forall x \in (a,b)$ except $x = c$, then:

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) \quad \text{(if exist)}$$

Example:

1) $f(x) = \begin{cases} x+1 & x \neq 1 \\ 3 & x = 1 \end{cases}$



$\lim_{x \to 1} f(x) = \lim_{x \to 1} x+1$

$= 2$

can use substitution when no proof required.

Same limit theorem applies from sequences

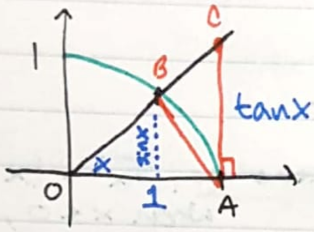
Squeeze Theorem again...

Let f, g, h be functions - $f(x) \leq h(x) \leq g(x)$ on an open neighbourhood of c

If $\lim_{x \to c} f(x) = L = \lim_{x \to c} g(x)$, then $\lim_{x \to c} h(x) = L$.

Special Limit:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$



Area $\Delta OCA \geq$ Area (sector OBA) $\geq$ area (ΔOBA)

$$\frac{1}{2}\tan x \geq \frac{1}{2} x r^2 \geq \frac{1}{2}\sin x$$

$$\tan x \geq x \geq \sin x$$

$$\frac{1}{\cos x} \geq \frac{x}{\sin x} \geq 1.$$

$$\cos x \leq \frac{\sin x}{x} \leq 1 \quad \text{(Now, take limit)}$$

$$\lim_{x\to 0} \cos(x) \leq \lim_{x\to 0} \frac{\sin x}{x} \leq 1$$

$$1 \leq \lim_{x\to 0} \frac{\sin x}{x} \leq 1 \quad \therefore \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Example:

1) $$\lim_{x\to 0} \frac{1-\cos x}{x}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)}$$

$$= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{\sin x}{1+\cos x}\right)$$

$$= (1)\left(\frac{0}{1+1}\right) = 0$$

Infinity "Arithmetic":

1) $(\infty)(\infty) = \infty$
2) $(-\infty)(\infty) = -\infty$
3) $(-\infty)(-\infty) = \infty$
4) If $a>1$, $\lim_{x\to\infty} a^x = \infty$, else 0
5) $\frac{c}{\infty} = 0$
6) $\infty + \infty = \infty$
7) When $c \neq 0$:
$$(c)(\infty) = \begin{cases} \infty, & c>0 \\ -\infty, & c<0 \end{cases}$$

More Limits:

Oct 2, 2017.

1) $\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x+1}$ Divide by highest x in denom

careful

$$\lim_{x\to\infty} \frac{(\sqrt{x^2+1})}{(x+1)} \left(\frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{x}}\right) = \frac{\sqrt{1+0}}{1+0} = \underline{1}.$$

2) $\lim_{x\to -\infty} \frac{\sqrt{x^2+1}\cdot\left(\frac{1}{-\sqrt{x^2}}\right)}{(x+1)\left(\frac{1}{x}\right)}$ because x is negative ∞. $\sqrt{x^2} = |x|$
when $x<0$, $\sqrt{x^2} = -x$!

$$\lim_{x\to-\infty} \frac{-\sqrt{1+\frac{1}{x^2}}}{1+\frac{1}{x}} = \underline{-1}.$$ two different HAs.

Continuity: * pg 66-67

f is continuous on (a,b) if $\forall x, y \in (a,b)$:
$\forall \varepsilon>0, \exists \delta>0 \ni |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$

f is continuous at $x=c$ when:
$\lim_{x\to c} f(x) = f(c)$ (and lim is defined + finite).

and... if continuous at $\forall x=c$ in (a,b), continuous on interval.

Usual Steps to Evaluate:

1) $f(c)$ exists.
2) $\lim_{x\to c} f(x)$ exists (+ finite)
3) $\lim_{x\to c} f(x) = f(c)$

Types of Discontinuity:

1) Removable → f not cont. at $x=c$ but $\lim_{x\to c} f(x)$ exists
2) Non removable → $\lim_{x\to c} f(x)$ DNE
   ↳ Jump $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist but $\neq$
   ↳ VA (∞) At least one of the → ← lims are finite.

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Finding Discontinuities for Piecewise Functions:

1) Check formula, discard those outside x values used
2) Check every transition value.
   ↳ **Must** do this for every value on both sides → ←

Example:

1) $f(x) = \begin{cases} \frac{x^2-10x+25}{x^2-25}, & x<5 \\ \frac{x-5}{x}, & 5 \le x < 9 \\ \frac{x^2}{8}, & x > 9 \end{cases}$ Be careful

1) Check formulas

i) $\frac{x^2-10x+25}{x^2-25}$ DC at $x=-5, 5$
BUT discard $x=5$ b/c not in interval.

ii) $\frac{x-5}{x}$ DC at $x=0$
BUT discard $x=0$ b/c not in interval.

iii) $\frac{x^2}{8}$ Continuous

From formulas: discontinuous at $\boxed{x=-5}$

2) Transition Values

i) $x=5$

a) $f(5) = \frac{5-5}{5} = 0$ ✓ exists

b) $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} \frac{x^2-10x+25}{x^2-25} = \lim_{x\to 5^-} \frac{x-5}{x+5} = 0.$ ✓

$\lim_{x\to 5^+} f(x) = \lim_{x \to 5^+} \frac{x-5}{x} = 0$ ✓ $\boxed{\therefore \lim_{x\to 5} f(x) = 0}$

$\therefore f$ is cont. at 5 since $\lim = 0 = f(5)$. need this.

ii) $x=9$

c) $f(9) = \text{DNE}$

$\therefore f$ is disc. at $\boxed{x=9}$.

Now, classify:

At $x=-5$: $\lim_{x\to -5} f(x) = \frac{100}{0}$ → VA/Inf. dc

$x=9$: $\lim_{x\to 9} f(x) = \text{DNE}$ since left/right ≠. → Jump dc.

Both non-removable.

Theorem:

If $\lim_{x \to c} g(x) = L$ and f is continuous at $x = L$ then:

$$\lim_{x \to c} f(g(x)) = f(L) = f(\lim_{x \to c} g(x))$$

Example:

1) $\lim_{x \to 7} e^{x^2+5} = e^{54} = e^{\lim_{x \to 7} x^2+5}$

Continuity on Closed Intervals:

f is continuous on $[a,b]$ when:

1) Continuous (a,b)
2) Cont. on the right at $x=a$, left at $x=b$

$[a,b]$

→ Right continuity:

1) $f(a)$ exists
2) $\lim_{x \to a^+} f(x)$ exists (+ finite)
3) $\lim_{x \to a^+} f(x) = f(a)$

→ Left continuity:

1) $f(a)$ exists
2) $\lim_{x \to b^-} f(x)$ exists (+ finite)
3) $\lim_{x \to b^-} f(x) = f(a)$

Intermediate Value theorem: Oct 4, 2017.

f continuous on $[a,b]$, function passes every value $[a,b]$

→ Show $x^5 - 6x^3 - x + 4$ has root on $[0,1]$.

→ f is continuous on $[0,1]$. $f(0) = 4$ $f(1) = -2$.

$\therefore$ By IVT, $(\exists c \in [0,1])$ where $f(c) = 0$.

Extreme Value Theorem: pg 69-72

f continuous on $[a,b]$, then f has an abs max/min on $[a,b]$.

Differentiation:

Refer to gr 12 calc notes + EdX Notes.

Proof of power rule: $f'(x) = nx^{n-1}$, $n \in \mathbb{N}$.

$$(x+h)^n = \sum_{i=0}^{n} x^{n-i} h^i \cdot \binom{n}{i}$$

$$= x^n + nhx^{n-1} + h^2 (\text{STUFF}\ldots)$$

$$= \lim_{h \to 0} \frac{nhx^{n-1} + h^2(\text{STUFF})}{h}$$

$$= nx^{n-1}$$

Extreme Value Theorem: pg 69-72

f continuous on $[a,b]$, then f has an abs max/min on $[a,b]$.

Differentiation:

Refer to gr 12 calc notes + EdX Notes.

Proof of power rule: $f'(x) = nx^{n-1}$ $n \in \mathbb{N}$.

$$(x+h)^n = \sum_{i=0}^{n} x^{n-i} h^i \cdot \binom{n}{i}$$

proof in gr 12 physics notes

$$= x^n + nhx^{n-1} + h^2(\text{STUFF}\ldots)$$

$$= \lim_{h\to 0} \frac{nhx^{n-1} + h^2(\text{STUFF})}{h}$$

$$= nx^{n-1}$$

Rules: (Refer to gr 12 and EdX calc notes).

Proof of product rule:

$$p'(x) = \lim_{h\to 0} \frac{p(x+h) - p(x)}{h}$$

$$= \lim_{h\to 0} \frac{[f(x+h)][g(x+h)] - f(x)g(x)}{h}$$

equivalent to zero

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) + \underbrace{[-f(x)g(x+h) + f(x)g(x+h)]}_{\text{equivalent to zero}} - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{g(x+h)[f(x+h) - f(x)]}{h} + \lim_{h\to 0} f(x)\frac{[g(x+h) - g(x)]}{h}$$

$$= g(x)f'(x) + f(x)g'(x)$$

Examples:

1) $p(x) = (x^2+1)\sin(x)$ → product rule

$p'(x) = 2x\sin x + (x^2+1)\cos x$

2) $g(x) = \dfrac{e^x + x^3}{x^2 \sin x}$ → quotient rule.

$f(x) = e^x + x^3$ $\quad g(x) = (x^2+1)\sin x$ } etc...

$f'(x) = e^x + 3x^2$ $\quad g'(x) = 2x\sin x$

Special Derivatives:

$\frac{d}{dx}\tan(x) = \sec^2 x$

$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$

$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$

$\frac{d}{dx}\cot(x) = -\csc^2(x)$

$\frac{d}{dx}\sqrt{f(x)} = \frac{1}{2\sqrt{f(x)}} \cdot f'(x)$

$\frac{d}{dx}\sinh(x) = \cosh(x)$

$\frac{d}{dx}\cosh(x) = \sinh(x)$

$\frac{d}{dx}\tanh(x) = \text{sech}^2(x)$

$\frac{d}{dx}\ln(x) = \frac{1}{x}$

$\frac{d}{dx}a^x = a^x \ln(a)$ } $a \neq 0, 1; a > 0$.

$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln a}$

$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$ } Watch sign ($\frac{d}{dx}\cos x = -\sin x$)

$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$

$\frac{d}{dx}\text{arcsec}(x) = \frac{1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}\text{arccsc}(x) = \frac{-1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}\text{arccot}(x) = \frac{-1}{1+x^2}$

Oct 13, 2017

Differentiability:

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \text{ exists}$$

$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Continuity $\not\Rightarrow$ Diff. $|x|$

Diff $\Rightarrow$ Continuity : Not continuous $\Rightarrow$ Not diff.

Proof: RTP $\lim_{x \to c} f(x) = f(c)$. Given $f'(c)$ exists.

$$\left(\lim_{x \to c} f(x)\right) - f(c)$$

$$= \lim_{x \to c} (f(x) - f(c)) \cdot \frac{x-c}{x-c}$$

we know this exists

$$= \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c}\right) \cdot \lim_{x \to c} (x - c)$$

$$= f'(c) \cdot 0 = 0.$$

Oct 25, 2017

Implicit Differentiation:

Unit Circle Example:

$x^2 + y^2 = 1$

$|y| = \sqrt{1-x^2}$ $\quad y: \begin{cases} \sqrt{1-x^2} & y \geq 0 \\ -\sqrt{1-x^2} & y < 0 \end{cases}$

when $y \geq 0$:

$$\frac{dy}{dx}\left(y = \sqrt{1-x^2}\right) = y: \frac{1}{2\sqrt{1-x^2}}(-2x) = \frac{-x}{\sqrt{1-x^2}} = \frac{-x}{y}.$$

when $y < 0$:

$$\frac{dy}{dx}\left(y = -\sqrt{1-x^2}\right) = \frac{-1}{2\sqrt{1-x^2}}(-2x) = -\frac{x}{y}$$

$\leftarrow$ b/c y is $-$ve

$\therefore \frac{dy}{dx} = \frac{-x}{y}.$

ID method of last example:

1) $x^2+y^2=1$

$$\frac{d(x^2+y^2)}{dx} = \frac{d}{dx}(1)$$

$$2x + 2y \cdot y' = 0$$

$$y' = \frac{-x}{y}.$$

2) $3x^2y^3 + 13x = 12 + 2y$

$$\frac{d}{dx}(3x^2y^3 + 13x) = \frac{d}{dx}(12 + 2y)$$

$$6xy^3 + 9x^2y^2y' + 13 = 2y'$$

$$y' = \frac{6xy^3+13}{2-9x^2y^2}$$

Derivatives of the Inverse:

Suppose a fcn is diff + invertible. we want deriv. f^{-1}. $x = f(y)$.

$$\frac{d(f^{-1}(x))}{dx} \qquad 1 = \frac{d\,f(y)}{dx}$$

$$1 = f'(y) \cdot \frac{dy}{dx}$$

$$\frac{dx}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

$$\boxed{\therefore \frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}}$$

Example:

1) $y = \ln x$ $\quad f^{-1}(x) = \ln x,\ f(x) = e^x,\ f'(x) = e^x$

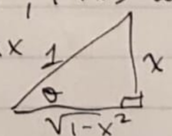
$$\frac{d}{dx}\ln x = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

2) $y = \arcsin(x)$ $\quad f^{-1}(x) = \arcsin(x),\ f(x) = \sin x,\ f'(x) = \cos x$

+ root (look @ arcsin PVs)

$$\frac{d}{dx}\arcsin x = \frac{1}{\cos(\arcsin x)}$$

Let $\theta = \arcsin x$



so $\cos\theta = \sqrt{1-x^2}$

$$\frac{1}{\sqrt{1-x^2}}$$

3) $y = a^x \quad f^{-1}(x) = a^x,\ f(x) = \log_a(x),\ f'(x) = \frac{1}{x\ln a}$

$$\frac{d}{dx}(a^x) = \frac{1}{\left(\frac{1}{a^x \ln a}\right)} = a^x \ln a.$$

4) $\frac{d}{dx}\arctan(x^3)$? $\quad f'(x) = \frac{1}{1+(x^3)^2} \cdot \frac{d}{dx}(x^3) = \frac{3x^2}{1+x^6}.$

5) $\frac{dy}{dx}$ of $\arcsin(2^y) = 5x^2y^3$ *use $\frac{dy}{dx}$ > y' for mistakes.

$$\frac{1}{\sqrt{1-(2^y)^2}} \cdot \frac{d}{dx}(2^y) = 5\left(2xy^3 + 3x^2y^2\frac{dy}{dx}\right)$$

$$\frac{1}{\sqrt{1-2^{2y}}} \cdot (2^y)(\ln(2))\left(\frac{dy}{dx}\right) = 10xy^3 + 15x^2y^2\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(\frac{2^y \ln 2}{\sqrt{1-2^{2y}}} - 15x^2y^2\right) = 10xy^3$$

Logarithmic Differentiation:

$f(x) = (\cos(x))^{x^2+1}$

Take ln of both sides:

$$\ln(f(x)) = \ln\left[(\cos(x))^{x^2+1}\right]$$

$$\ln(f(x)) = (x^2+1)\ln(\cos x)$$

Now Derive:

$$\frac{d}{dx}\ln(f(x)) = \frac{d}{dx}(x^2+1)\ln(\cos x)$$

$$\frac{1}{f(x)} \cdot f'(x) = 2x\ln\cos x + (x^2+1)\left(-\frac{\sin x}{\cos x}\right)$$

don't leave in there

$$f'(x) = \left[2x\ln\cos x + (x^2+1)(-\tan x)\right]\left[\cos(x)^{x^2+1}\right]$$

Example:

1) $f(x) = x^x,\ x > 0$

$\ln(f(x)) = x\ln x$

$\frac{f'(x)}{f(x)} = \ln(x) + \cancel{x}\cdot\frac{1}{\cancel{x}} = f(x)(\ln(x)+1)$

$= x^x(\ln(x)+1)$

$\operatorname{dom}(\ln x) = (0, \infty)$

$\operatorname{dom}(\ln|x|) = \mathbb{R}\setminus\{0\}$

$\frac{d}{dx}\ln|x| = \left(\frac{1}{|x|}\right)\frac{d}{dx}|x| \quad = \frac{|x|}{x}$

$= \frac{1}{x}$

So, $\frac{d}{dx}\ln(|f(x)|) = \frac{f'(x)}{f(x)}$

2) $f(x) = \frac{(x-2)^3(x^2-2x+5)^4}{(2x+1)^2(x^2-7)^3}$ Take $\ln|f(x)|$ Oct 27, 2017

$\ln|f(x)| = \ln|\text{that}| = 3\ln|x-2| + 4\ln|x^2-2x+5| - 2\ln|2x+1| - 3$

Differentiating: *ONLY VALID WHEN $f(x) \neq 0$ or DNE

$\frac{f'(x)}{f(x)} = \frac{3}{x-2} + \frac{8x-8}{x^2-2x+5} - \frac{4}{2x+1} - \frac{6x}{x^2-7}$

$f'(x) = f(x)[\text{that}]$ where $x \neq 2$, $x^2-2x+5 \neq 0$, $2x+1 \neq 0$, $x^2-7 \neq 0$.

Use $|f(x)|$ when $\ln(f(x))$ not defined (function is negative)

Extrema and Monotonicity:

Let f defined at $x=c$.

1) Global max at $x=c$ if: $f(c) \geq f(x)$ $\forall x \in dom(f)$ open neighborhood

2) Local max at $x=c$ if: $f(c) \geq f(x)$ $\forall x \in I$. I is int: containing c

→ Mins too. c can't be an endpoint

Critical Numbers:

$c \in dom(f)$

a) $f'(c) = 0$

b) $f'(c) = DNE$

$\Big\}$ $(c, f(c))$ critical point

→ Cusp when $f(c)$ exists but nada $f'(c)$

$\vee$ $\wedge$ $\curlyvee$ $\curlywedge$ $\int$

Fermat's Theorem: (Not Little or Last)

Rel min/max $\Rightarrow$ critical point / number

Increasing: $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ $(f'(x) > 0)$ Oct 30, 2017

→ Monotonic if either increasing/decreasing on interval.

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Increasing / Decreasing:

$f'(x)$ sign can only change at crit #'s and discontinuities.

Testing for increasing / decreasing: Let f be dif. on (a,b).

1) Increasing $f'(x)>0 \ \forall x \in (a,b)$
2) Decreasing $f'(x)<0 \ \forall x \in (a,b)$
3) Neither $f(x)=0 \ \forall x \in (a,b)$ *not monotonic

First Derivative Test for Extrema:

Intervals where $f'(x)=0$ or DNE. $x=c$ crit #.

	$x<c$	$x>c$	Type
$f'(x)$ sign	−	+	local min
	+	−	local max

{ Make sure it's not VA before classifying.

Example:

1) $f(x)=x^3-3x^2-24x$

1) $f'(x)=0$ or DNE → Crit #s

$f'(x)=3(x-4)(x+2)$ $x=4,-2$ CN's

2) Table:

	$(-\infty,-2)$	$(-2,4)$	$(4,\infty)$
$f'(x)$	+	−	+
$f(x)$	↗	↘	↗

Increasing on $(-\infty,2]\cup[4,\infty)$

square brackets → since defined for $f(x)$. If VA? Na bro. ( ).

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Concavity / Inflection:

Nov 1, 2017

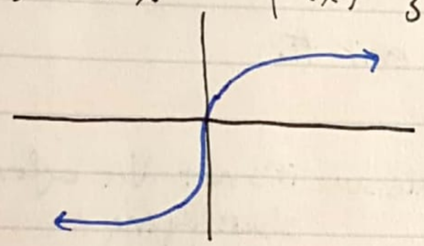
1) Concave up on interval when $f''(x) > 0$. ∪
2) Concave down on interval when $f''(x) < 0$. ∩

When concavity changes, point of inflection.

→ Hypercritical numbers: $f''(c) = 0$ or DNE.

Example:

1) $f(x) = \sqrt[3]{x}$ $\quad f'(x) = \frac{1}{3}x^{-2/3}$ $\quad f''(x) = \frac{-2}{3}x^{-\frac{5}{3}}$



$f'(0)$ and $f''(0)$ DNE.

No extrema at $x=0$, but POI, yeah.

Check out $\sqrt[3]{x^2}$.

↳ $g'(0)$ global max

↳ $g''(0)$ No POI.

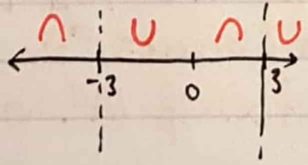
Note: Concavity can only change at hcrit #s.

2) $g(x) = \frac{x}{x^2-9}$ $\quad g'(x) = \frac{-(x^2+9)}{(x^2-9)^2}$ $\quad g''(x) = \frac{2x(x^2+27)}{(x^2-9)^3}$

$g''(x)$ DNE at $x = \pm 3$.

$g''(x)$ 0 when $2x(x^2+27) = 0$, so $x = 0$.

↳ VAs at $x = \pm 3$. Sub hc #s to $g''(x)$.



POI at $x=0$. $(0,0)$.

Second Derivative Test:

Let $f'(c) = 0$. Assume f'' is cont. on open neighbourhood $\bar{\omega}$ c.

1) $f''(c) > 0$? CCU, rel min ∪
2) $f''(c) < 0$? CCD, rel max ∩
3) $f''(c) = 0$? Doesn't tell u anything. (can be any, $x^3, x^4, -x^4$)

Curve Sketching:

AGr 12 calc notes.

$f(x) = x^3$ (example curve)

	increasing $f'(x)>0$	decreasing $f'(x)<0$
CCU $f''(x)>0$		
CCD $f''(x)<0$		

} accelerating/decellerating forwards + backwards.

1) Find x-ints, y-ints.
2) Find discontinuities
    ↳ Including RD's and respective y-value of RD.
    ↳ VAs - don't need one sided limits b/c inc/dec table
3) Horizontal Asymptotes
    ↳ $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ (must do both)
4) Find critical numbers w/ $f'(x)$
    ↳ Intervals of ↗ ↘. Classify extrema.
5) Find hypercritical numbers w/ $f''(x)$.
    ↳ Intervals of ∩ ∪. Classify POIs.

Optimization

Nov 3, 2017

To find extrema on $[a,b]$ (refer to EVT):

1) Find CP's
2) Points of $x=a$, $x=b$.

} Largest y = max, smallest y = min

Example:

1) Find extrema of $f(x) = x^2 - 4x + 4$ on $[-1, 4]$.

$f'(x) = 2x - 4$

$f'(x) = 0 \Rightarrow x = 2$.

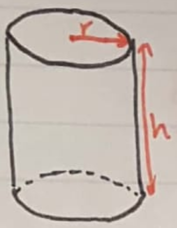
$f(-1) = 9$ max$(-1, 9)$

$f(2) = 0$ min$(2, 0)$

$f(4) = 4$

Example: Nov 6 2017

1) A can manufacturer wishes to minimize amount of tin used to make 2L can. Dimensions?



$V = \pi r^2 h = 2$ "secondary"

$SA = 2\pi r^2 + 2\pi r h$ "primary"

$2 = \pi r^2 h$
$h = \frac{2}{\pi r^2}$

$\to SA(r) = 2\pi r^2 + 2\pi r\left(\frac{2}{\pi r^2}\right)$
$= 2\pi r^2 + \frac{4}{r}$

$SA'(r) = 4\pi r - \frac{4}{r^2}$ DNE when $r = 0$

$SA'(r) = 0$ when $r = \pi^{-\frac{1}{3}}$

2nd Derivative Test:

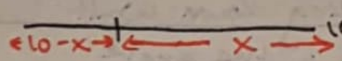
$S''(\pi^{-\frac{1}{3}}) > 0$ $\therefore$, min. Only crit #, so absolute min.

SA continuous on $(0, \infty)$. Now solve for height.

$\therefore r = \pi^{-\frac{1}{3}}$ dm, $h = 2\pi^{-\frac{1}{3}}$ dm

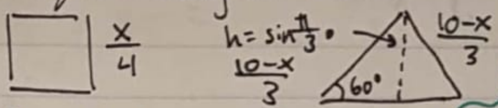
$\to$ Volume in L, $L = dm^3$, so units must be dm^3

2) A 10cm wire is cut into max two pieces. One bent to □, other bent to eq. Δ. How to cut to max/min area?



$x \in [0, 10]$

Let square length be x.



$$Area(x) = \frac{x^2}{16} + \frac{1}{2}\underbrace{\left(\frac{10-x}{3}\right)}_{b}\underbrace{\left(\frac{\sqrt{3}}{2}\right)\left(\frac{10-x}{3}\right)}_{h}$$

$$A(x) = \frac{x^2}{16} + \frac{\sqrt{3}(10-x)^2}{36}$$

$$A'(x) = \frac{x}{8} - \frac{\sqrt{3}}{18}(10-x)$$

$A'(x) = 0$ when $x = \frac{40\sqrt{3}}{9+4\sqrt{3}}$ } Closed I, so must check endpoints.

$A(0) \doteq 4.81$
$A\left(\frac{40\sqrt{3}}{9+4\sqrt{3}}\right) \doteq 2.72$
$A(10) = 6.25$

} ∴ max area when $x = 10$ (no Δ) cm
min area when $x = \frac{40\sqrt{3}}{9+4\sqrt{3}}$ cm

Generalized Optimization Problems:

1) Draw pic
2) Get ur saucy 2 equations
   - Primary: what you wanna optimize *extra spicy
   - secondary: relates var to constant
3) Isolate a variable in secondary eqn
4) Find crit points using $f'(x)$. Classify extrema with:
   - Test for closed interval when makes sense
   - First/Second derivative test (must be open)

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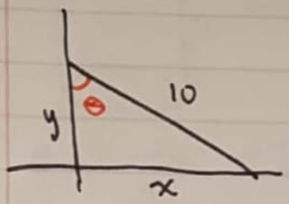
Related Rates:

1) A 10 m ladder rests against wall. Base of ladder slides away at 1 m/s. How fast is top of ladder sliding down wall when base is 8m from wall, and at what rate is the angle btwn top of ladder and wall changing at the same moment?



$x^2+y^2=100$

x and y depend on t. Dif wrt t.

a) $x=8$ $\quad \frac{dx}{dt}=1$

6 (pythag) $y=?$ $\quad \frac{dy}{dt}=$ WTF.

$$\frac{d}{dt}(x^2+y^2)=\frac{d}{dt}(100)$$

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

$$\frac{dy}{dt}=\frac{-x}{y}\left(\frac{dx}{dt}\right)=\frac{-4}{3}\text{m/s}.$$ $\boxed{\therefore \text{sliding down at } 4/3 \text{ m/s}}$

b) $x=8$ $y=6$ $\theta=?$

$\frac{dx}{dt}=1$ $\quad \frac{dy}{dt}=6$

$\frac{d\theta}{dt}=$ WTF.

$$\sin\theta=\frac{x}{10}$$

$$\cos\theta\frac{d\theta}{dt}=\frac{1}{10}\left(\frac{dx}{dt}\right)=\frac{1}{10}$$

$$\frac{d\theta}{dt}=\frac{\sec\theta}{10}=\frac{1}{y}\boxed{=\frac{1}{6}\text{ rad/s}}$$

More Related Rates:

Nov 8, 2017

1) A spherical balloon is being inflated w helium at $9\pi^2 cm^3/s$. Radius when $V = 36\pi cm^3$? (rate of change of radius)

$V_{sphere} = \frac{4}{3}\pi r^3 = 36\pi$

3 from

$\frac{dV}{dt} = 9\pi^2$

$r = ?$

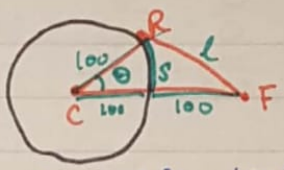
$\frac{dr}{dt} = ?$

differentiated

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{9\pi^2}{4\pi(3)^2} = \frac{\pi}{4}$ cm/s.

2) A runner is running along a circular track. 7 m/s, $r = 100m$. Friend standing 200m from center of track. After runner passes friend, what rate is distance btwn them changing when 200m apart?



$l = 200$ $\frac{dl}{dt} = ?$

$S = 100\theta$ $\frac{ds}{dt} = 7$

$\theta :$ $\frac{d\theta}{dt}$

Cosine law: $l^2 = 100^2 + 200^2 - 2(100)(200)\cos\theta$

$l^2 = 50000 - 40000\cos\theta$

☆ $2l\frac{dl}{dt} = 40000\sin\theta\frac{d\theta}{dt}$ Need $\sin\theta$ and $\frac{d\theta}{dt}$

To find $\sin\theta$...

$l^2 = 50000 - 40000\cos\theta$ (sub $l = 200$)

$\cos\theta = \frac{1}{4}$. $\sin\theta$ (also +) $= \sqrt{1-\cos^2\theta} = \frac{\sqrt{15}}{4}$.

To find $\frac{d\theta}{dt}$...

$\theta = \frac{s}{r} = \frac{s}{100}$

$\frac{d\theta}{dt} = \frac{ds}{dt}\left(\frac{1}{100}\right) = \frac{7}{100}$ Now go back to ☆

$\frac{dl}{dt} = \frac{20000}{l}\sin\theta\frac{d\theta}{dt} = \frac{7\sqrt{15}}{4}$ m/s. (increasing)

L'Hôpital's Rule:

Let (a,b) be open neighbourhood of c. $I=(a,c)\cup(c,b)$ (No c).

Suppose f and g are on I where:

① f and g differentiable on I (not nec. at c)

② $g'(x) \neq 0$ on I.

③ $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ $\quad \lim_{x\to c} \frac{f(x)}{g(x)} = \frac{0}{0}$

or: $\lim_{x\to c} f(x) = \pm\infty$ and $\lim_{x\to c} g(x) = \pm\infty$ $\quad \lim_{x\to c} \frac{f(x)}{g(x)} = \pm\frac{\infty}{\infty}$

④ $\lim_{x\to c} \frac{f(x)}{g(x)} = L$ where $L \in \mathbb{R}$ or $L = \pm\infty$

Then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$

If $c = \pm\infty$,

change I to $(-\infty, a) : c \Rightarrow \infty$

$(a, \infty) : c \Rightarrow -\infty$

Examples:

1) $\lim_{x\to 2} \frac{x-2}{x^2-4} \overset{DS}{=} \frac{0}{0}$. *Must show indeterminate form every time you use L'Hôpital's Rule!!

$\overset{LH}{=} \lim_{x\to 2} \frac{1}{2x} = \frac{1}{4}$

2) $\lim_{x\to\infty} \frac{\ln(x)}{x} \overset{L'H}{=} \frac{1}{x} = 0.$

3) $\lim_{x\to\infty} \frac{(\ln(x))^3}{x} \overset{D.S}{=} \frac{\infty}{\infty}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{3(\ln x)^2 \cdot \frac{1}{x}}{1} = \lim_{x\to\infty} \frac{3(\ln x)^2}{x} = \frac{\infty}{\infty}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{6\ln x}{x} \overset{DS}{=} \frac{\infty}{\infty}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{6}{x} = 0.$

4) $\lim_{x\to\infty} \frac{e^x}{x} \overset{DS}{=} \frac{\infty}{\infty}$ even if x was raised to any power, or!

$\overset{L'H}{=} \lim_{x\to\infty} e^x = \infty$

★ 5) $\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x+1} \overset{D.S.}{=} \frac{\infty}{-\infty}$

$\overset{L'H}{=} \lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} \overset{DS}{=} \frac{-\infty}{\infty}$

$\overset{L'H}{=} \lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x}$ Infinite loop... can't L'H. (divide by $\frac{1}{-\sqrt{x^2}} / \frac{1}{x}$)

$0 \cdot \infty$ Forms:

$0 \cdot \infty = \frac{0}{1/\infty} = \frac{0}{0}$

$\hookrightarrow = \frac{\infty}{1/0} = \frac{\infty}{\pm\infty}$ Need to know 0^+ or 0^-

Examples:

1) $\lim_{x\to 0^+} x^2(\ln x)^2 = \lim_{x\to 0^+} \frac{(\ln x)^2}{1/x^2} \overset{DS}{=} \frac{-\infty}{\infty}$ Leave logs in numerator..

$\overset{L'H}{=} \lim_{x\to 0^+} \frac{2\ln(x)/x}{-2/x^3} = \lim_{x\to 0^+} -x^2\ln(x) \overset{DS}{=} 0\cdot-\infty = \lim_{x\to 0^+} \frac{\ln x}{-1/x^2}$

$\overset{L'H}{=} \lim_{x\to 0^+} \frac{-\ln x}{1/x^2} \overset{DS}{=} \frac{\infty}{\infty}$

$\overset{L'H}{=} \lim_{x\to 0^+} \frac{-1/x}{-2/x^3} = 0.$

$\infty - \infty$ Forms:

Common denominator or conjugate.

Try $\lim\limits_{x \to 0^+} \csc(x) - \cot(x) = 0$

Exponential Forms: $0^0, \infty^0, 1^\infty$

Basically log differentiation but for limits.

$\lim\limits_{x \to 0^+} x^x \overset{D.S}{=} 0^0$. Let $L = \lim\limits_{x \to 0^+} x^x$

$\ln(L) = \lim\limits_{x \to 0^+} \ln(x^x)$

DO NOT FORGET

$= \lim\limits_{x \to 0^+} x \ln x = 0$. $\boxed{\ln L = 0, \text{ so } L = 1}$

Examples:

1) $\lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^x \overset{D.S.}{=} 1^\infty$ Let L = that

$\ln L = \lim\limits_{x \to \infty} \left(x \ln\left(1 + \frac{1}{x}\right)\right) \overset{D.S}{=} \infty \cdot 0$

$= \lim\limits_{x \to \infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{1/x} \overset{D.S}{=} \frac{0}{0}$

$\overset{L'H}{=} \lim\limits_{x \to \infty} \frac{\frac{1}{1 + 1/x} \cdot \frac{-1}{x^2}}{\frac{-1}{x^2}} = 1$. $\ln L = 1$, so $L = e$

one of the definitions of e

2) $\lim\limits_{x \to 0^+} |\ln(x)|^x \overset{DS}{=} \infty^0 = L$

$\ln L = \lim\limits_{x \to 0^+} (x \ln|\ln x|) \overset{D.S}{=} 0 \cdot \infty$

$= \lim\limits_{x \to 0^+} \frac{\ln|\ln x|}{1/x} \overset{D.S.}{=} \frac{\infty}{\infty}$

$\overset{L'H}{=} \lim\limits_{x \to 0^+} \frac{\frac{1}{\ln x} \cdot \frac{1}{x}}{\frac{-1}{x^2}} \overset{D.S}{=} \frac{0}{-\infty} = 0$ $\ln L = 0$, so $L = e^0 = 1$.

Integration - Anti-Derivatives:

Antiderivatives: (an) Antiderivative of $f'(x)$ is $f(x)$.

→ 2 antiderivatives of a function only differ by constant C.

Proof:

Let $F'(x) = f(x)$, $G'(x) = f(x)$.

$$H(x) = F(x) - G(x)$$
$$H'(x) = F'(x) - G'(x)$$
$$= f(x) - f(x) = 0$$ means $H(x)$ is constant.

General antiderivative of f is indefinite integral of f.

$$\int f(x)\,dx = F(x) + C$$

Thus, $\int 2x\,dx = x^2 + C$.

Properties of Indefinite Integral:

1) $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$.
2) $\int (k f(x))\,dx = k \int f(x)\,dx$

List of Anti-Derivatives:

$$\int \sin x\,dx = -\cos x + C$$
$$\int \cos x\,dx = \sin x + C$$
$$\int \sec^2 x\,dx = \tan x + C$$
$$\int (\sec x \tan x)\,dx = \sec(x) + C$$
$$\int (\csc x \cot x)\,dx = -\csc(x) + C$$
$$\int \csc^2(x)\,dx = -\cot(x) + C$$
$$\int \left(\frac{1}{1+x^2}\right) dx = \arctan(x) + C$$
$$\int \left(\frac{1}{\sqrt{1-x^2}}\right) dx = \arcsin(x) + C$$
$$\int \left(\frac{1}{x\sqrt{x^2-1}}\right) dx = \operatorname{arcsec}(x) + C$$

$$\int e^x\,dx = e^x + C$$
$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$
$$\int \frac{1}{x}\,dx = \ln|x| + C$$
$$\int \cosh(x)\,dx = \sinh(x) + C$$
$$\int \sinh(x)\,dx = \cosh(x) + C$$
$$\int x^n\,d(x) = \frac{x^{n+1}}{n+1} + C \quad n \neq -1$$
$$\int k\,dx = kx + C$$
$$\int 0\,dx = C$$
$$\int \sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}} + C$$

Example:

1) $\int \sqrt{x}\, dx$

$= \int (x^{\frac{1}{2}})\, dx = \frac{2}{3} x^{\frac{3}{2}} + C$

2) $\int (\sqrt{x^3} + \sqrt[3]{x^2})\, dx$

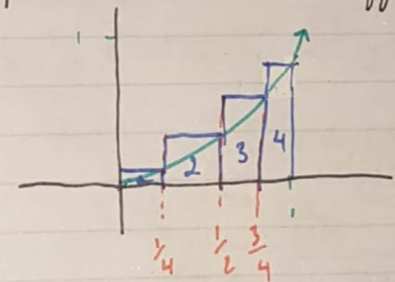
$= \frac{1}{\frac{3}{2}+1} x^{\frac{3}{2}+1} + C_1 + \frac{1}{\frac{2}{3}+1} x^{\frac{2}{3}+1} + C_2$

$= \frac{2}{5} x^{\frac{5}{2}} + \frac{3}{5} x^{\frac{5}{3}} + C_1 + C_2$ call this "C"

Riemann Sums/Definite Integrals:

Area under the curve:

1) $f(x) = x^2$ on $[0,1]$. Approx. area under curve?



$[0,1] \to 4$ subintervals of equal length

Area $= \sim \frac{15}{32}$ (bad...)

$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$ Riemann Sum

$\hookrightarrow \lim_{n\to\infty} R_n$ is area!

Let $\Delta x = \frac{1}{n}$ (taking n subsections)

$x_0 = 0, x_1 = \frac{1}{n}, x_2 = \frac{2}{n}$ $\quad x_i = i\Delta x = \frac{i}{n}$

$\left.\begin{array}{l} f(x_i) = x_i^2 = i^2/n^2 \\ f(x_i)\Delta x = i^2/n^3 \end{array}\right\} \sum_{i=1}^{n} f(x_i)\Delta x = \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{1}{n^3}\left(\frac{2n^3+3n^2+n}{6}\right)$

$\lim_{n\to\infty} \frac{2n^3+3n^2+n}{6n^3} = \frac{1}{3}$

Can use any $x_i^* \in [x_{i-1}, x_i]$ for rectangle.

If rectangle height taken from left x_i...

$$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x$$

Right point...

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

Midpoint...

$$M_n = \sum_{i=1}^{n} f(\overline{x}_i)\Delta x \qquad \overline{x}_i = \frac{x_{i-1}+x_i}{2} \qquad \overline{x}_i = a + \left(\frac{2i+1}{2}\right)\Delta x$$

Summary:

Let f be continuous on $[a,b]$, $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$, $x_i^* \in [x_{i-1}, x_i]$

→ Definite integral of f from $a \to b$:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Algorithm:

1) Compute $\Delta x = \frac{b-a}{n}$
2) Compute $x_i = a + i\Delta x$ (or $\overline{x}_i = a + \left(\frac{2i+1}{2}\right)\Delta x$ if M_n)
3) Compute $f(x_i)$
4) Compute $f(x_i)\Delta x$
5) Simplify:

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x \Big\} \text{Using summation formulas}$$

6) Area $= \lim_{n\to\infty} R_n$

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Summation Formulas:

1) $\sum_{i=1}^{n} c = n \cdot c$

2) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}$

4) $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

5) $(a+b)^3 = a^3+3a^2b+3ab^2+b^2$

Example:

1) Find $\int_0^2 (x^3+x)\,dx$ (Use R_n when given choice)

1) $\Delta x = \frac{2-0}{n} = \frac{2}{n}$

2) $x_i = a + i\Delta x$ * Must Know this method

$= 0 + i \cdot \frac{2}{n}$

$= \frac{2}{n} i$

3) $f(x_i) = x_i^3 + x_i$

$= \frac{8}{n^3} i^3 + \frac{2}{n} i$

4) $f(x_i)\Delta x = \left(\frac{8}{n^3} i^3 + \frac{2}{n} i\right)\left(\frac{2}{n}\right)$

$= \frac{16}{n^4} i^3 + \frac{4}{n^2} i$

5) $R_n = \sum_{i=1}^{n} \left[\frac{16}{n^4} i^3 + \frac{4}{n^2} i\right]$

$= \sum_{i=1}^{n} \frac{16}{n^4} i^3 + \sum_{i=1}^{n} \frac{4}{n^2} i$

$= \frac{16}{n^4}\left(\frac{n^4+2n^3+n^2}{4}\right) + \frac{4}{n^2}\left(\frac{n^2+n}{2}\right)$

$= \frac{4n^4+8n^3+4n^2}{n^4} + \frac{2n^2+2n}{n^2}$

6) $\lim_{n\to\infty} \frac{4n^4+8n^3+4n^2}{n^4} + \lim_{n\to\infty} \frac{2n^2+2n}{n^2} = 6$

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Properties of Definite Integral:

1) $\int_a^b k\,dx = k(b-a)$

2) $\int_a^a f(x)\,dx = 0$

3) $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

4) $\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$

5) $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

6) if $c \in (a,b)$:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$ *Piecewise

7) If $f(x) \le g(x)$ $\forall x \in [a,b]$ then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

8) For f continuous on $[a,b]$, EVT gaurantees max/min on interval.

$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$ M max, m min

Fundamental Theorem of Calculus:

Nov 13, 2017

FTC1: f continuous on $[a,b]$, then: (proof in book)

$$g(x) = \int_a^x f(t)\,dt$$

is differentiable on (a,b) and $g'(x) = f(x)$. (g is anti-derivative of f)

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Example:

1) $\frac{d}{dx}\int_{5}^{x}\frac{\cos^3(t)}{t^2+t+1}\,dt = \frac{\cos^3(x)}{x^2+x+1}$

doesn't matter what this is.

2) $\frac{d}{dx}\int_{3}^{\sqrt{x}}\sin(t)\,dt$

Let $g(x)=\int_3^x \sin(t)\,dt$. $h(x)=g(\sqrt{x})=$ that

$h'(x) = g'(\sqrt{x})\cdot\frac{d}{dx}(\sqrt{x})$

Since $g'(x)=\sin(x)$, $h'(x)=\sin(\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right)$

In general:

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x)$$

3) $\frac{d}{dx}\int_{x^2}^{\sin(x)} e^t\,dt$ Note that: $\int_{x^2}^{\sin(x)} e^t dt = \int_{x^2}^{a} e^t dt + \int_a^{\sin x} e^t dt$ $a \in dom(e^t)$ (usually 0)

$\int_{x^2}^{\sin x} e^t dt = -\int_a^{x^2} e^t dt + \int_a^{\sin(x)} e^t dt$

So, $\frac{d}{dx}\int_{x^2}^{\sin x} e^t dt = -e^{x^2}\cdot 2x + e^{\sin x}\cos x$

In general;

$$\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x)$$

FCT2: f continuous on $[a,b]$, F be any antiderivative of f.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$ } Much better than R_n.

Example:

1) $\int_0^1 x^2\,dx$

$F(x) = \int x^2\,dx = \frac{1}{3}x^3 + C$

So: $\int_0^1 x^2\,dx = \left(\frac{1}{3}(1)^3 + C\right) - \left(\frac{1}{3}(0) + C\right) = \frac{1}{3}$. (c disappears e'rytime)

Looks like: $\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$

2) $\int_0^2 (x^3 + x)\,dx$

$= \left(\frac{1}{4}x^4 + \frac{1}{2}x^2\right)\Big|_0^2 = 6$

3) $\int_0^{\pi} \sin(x)\,dx = -\cos(x)\Big|_0^{\pi} = 2$ ö *Note: $-\cos x\Big|_0^{\pi} = \cos x\Big|_{\pi}^{0}$

U-Substitution:

Since $h'(x) = f'(g(x)) \cdot g'(x)$, u-sub undoes.

Given $\int f(g(x)) \cdot \underline{g'(x)\,dx}$ Let $u = g(x)$. So, $\frac{du}{dx} = g'(x)$. $du = g'(x)\,dx$

$= \int f(u)\,du$ } Find function and sc. multi of derivative.

↳ Attached to dx by multi.

Example:

1) $\int 2x\sqrt{x^2+1}\,dx = \int \sqrt{x^2+1}\,2x\,dx$

Let $u = x^2 + 1$. $du = 2x\,dx$.

$= \int \sqrt{u}\,dx = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(x^2+1)^{\frac{3}{2}} + C$

must sub

2) $\int x\sqrt{x^2+1}\,dx$

Let $u = x^2+1$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$= \int \sqrt{u}\cdot\frac{1}{2}du = \frac{1}{2}\left(\frac{2}{3}u^{\frac{3}{2}}+C\right) = \frac{1}{3}(x^2+1)^{3/2}+C$

3) $\int 2x^3\sqrt{x^2+1}\,dx$

Let $u = x^2+1$
$du = 2x\,dx$
$x^2 = u-1$

$= \int (u-1)\sqrt{u}\,du = \int (u\sqrt{u} - \sqrt{u})\,du$
$= \int (u^{\frac{3}{2}} - u^{\frac{1}{2}})\,du = \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C$

and sub x^2+1.

Good to know

4) $\int \tan(x)\,dx = \int \frac{\sin x}{\cos x}dx$

Let $u = \cos x$
$du = -\sin x\,dx$
$-du = \sin x\,dx$

$= -\int \frac{1}{u}du = -\ln|u|+C = -\ln|\cos(x)|+C$
$= \ln|\cos(x)|^{-1}+C = \ln|\sec(x)|+C$

5) $\int \frac{\sin(\sqrt{x})}{\sqrt{x}}dx$

Let $u = \sqrt{x}$
$2du = \frac{1}{\sqrt{x}}dx$

$= 2\int \sin(u)\,du = -2\cos(\sqrt{x})+C$

6) $\int \frac{\cos(\ln(x))}{x}dx$

Let $u = \ln x$
$du = \frac{1}{x}dx$

$= \int \cos(u)\,du = +\sin(\ln(x))+C$

7) $\int_1^4 \frac{x}{4x^2+1}dx$

Nov 15, 2017

Let $u = 4x^2+1$
$x\,dx = \frac{1}{8}du$

Ⓐ Find antiderivative + apply FCT2
Ⓑ Change limits in terms of u + apply FCT2

A: $\to \frac{1}{8}\ln|4x^2+1|\Big|_1^4 = \frac{1}{8}\ln 13$

B: $\to u|_{x=1} = 5$
$u|_{x=4} = 65$

$\int_5^{65}\frac{1}{8u}du = \frac{1}{8}\ln 13 = \frac{1}{8}\ln|u|\Big|_5^{65}$

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8) $\int \frac{1}{1+e^x}\,dx$ No u here... must make it

$= \int \frac{1}{1+e^x}\left(\frac{e^{-x}}{e^{-x}}\right)$

$= \int \frac{e^{-x}}{e^{-x}+1}$ Let $u = e^{-x}+1$

$du = -e^{-x}dx$

$= -\int \frac{1}{u}\,du = -\ln|u| + C = -\ln|e^{-x}+1| + C$

$= \ln\left|\frac{1}{e^{-x}+1}\right| + C = \ln\left|\frac{e^x}{1+e^x}\right| + C$

9) $\int \cos^4(x)\sin^3(x)\,dx$ → break it up.

$= \int \cos^4(x)(1-\cos^2 x)\sin x\,dx$ } Let $u = \cos x$, $-du = \sin x\,dx$

$= -\int u^4(1-u^2)\,du$

$= \frac{1}{7}u^7 - \frac{1}{5}u^5 + C$ (and sub).

Integration by Parts:

Undoes product rule.

Prod rule: $\frac{d}{dx}uv = \frac{du}{dx}v + u\frac{dv}{dx}$

Now, Integrate both sides:

$$\int \frac{d}{dx}uv\,dx = \int \frac{du}{dx}v\,dx + \int u\frac{dv}{dx}\,dx$$

$$uv = \int v\,du + \int u\,dv$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Example:

1) $\int x\cos(x)\,dx$

$u = x$ $dv = \cos x\,dx$

$du = dx$ $v = \sin x$ } $\int u\,dv = uv - \int v\,du$

$\int x\cos x = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$.

2) $\int x^2 \cos(x)\,dx$

$u = x^2$ $\quad dv = \cos x\,dx$

$du = 2x\,dx$ $\quad v = \sin x$

$\int x^2 \cos x\,dx = x^2 \sin x - 2\underbrace{\int x\sin(x)\,dx}_{\text{must do again}}$

$u_1 = x \quad dv_1 = \sin x$

$du_1 = dx \quad v_1 = -\cos x$

$\int x^2 \cos x\,dx = x^2 \cdot \sin x + 2(u_1 v_1 - \int v_1 du_1)$

$= x^2 \sin x + 2x\cos x - \int \cos x\,dx$

$= x^2 \sin x + 2x\cos x - \sin x + C$ $\quad$ *Mistake $\cdot 2$

$= (x^2 - 1)\sin x + 2x\cos(x) + C.$

3) $\int e^x \cos(x)\,dx$

$u = e^x \quad dv = \cos x\,dx$

$du = e^x dx \quad v = \sin x$

$\int e^x \cos x\,dx = e^x \sin x - \underbrace{\int e^x \sin x\,dx}_{\text{again...}}$

$u = e^x \quad dv = \sin x\,dx$

$du = e^x dx \quad v = -\cos x$

$\int e^x \cos x\,dx = e^x \sin x - (-e^x \cos x + \int e^x \cos x\,dx)$

$\int e^x \cos x\,dx = e^x \sin x + e^x \cos x - \int e^x \cos x\,dx$ should be -ve.

$\int e^x \cos x\,dx = \dfrac{e^x \sin x + e^x \cos x}{2} + C. \quad = \dfrac{e^x}{2}(\sin x + \cos x) + C.$

4) $\int \ln(x)\,dx$

$u = \ln x \quad dv = dx$

$du = \frac{1}{x}dx \quad v = x$

$\int \ln x\,dx = x\ln x - \int \cancel{x} \cdot \frac{1}{\cancel{x}}\,dx$ can cancel b/c $x > 0$ ($\ln$)

$\boxed{= x\ln(x) - x + C}$. Should know

5) $\int x^3 \ln x\,dx$

Not as nice: $u = x^3 \quad dv = \ln x\,dx$

$du = 3x^2 dx \quad v = x\ln x - x$

$\int x^3 \ln x\,dx = x^4 \ln(x) - x^4 - 3\int (x^3 \ln x - x^3)\,dx$

Thus ~~juan~~ $u = \ln x \quad dv = x^3\,dx$

$du = \frac{1}{x}dx \quad v = \frac{1}{4}x^4$

$\int x^3 \ln x\,dx = \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 dx$

$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$

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6) $\int x^2 e^{x^2} dx$ $\int e^{x^2} dx$ is impossible.

Let $u = x^2$
$\frac{1}{2} du = x dx$
$\frac{1}{2} u du = x^3 dx$

$\frac{1}{2}\int u e^u du$ Now IBP

Let $w = u$ $dv = e^u du$
$dw = du$ $v = e^u$

$wv - \int v dw = ue^u - \int e^u du$
$= ue^u - e^u + C$

so $I = \frac{e^{x^2}(x^2-1)}{2} + C$

ILATE: Order for IBP choices: Nov 17, 2017.

I nverse Trig
L ogs
A lgebraic (polynomials)
T rig
E xponentials

Let $u =$ thing highest on scale.

More on U-Subs:

1) $\int f(ax) dx = \frac{1}{a} F(ax) + C$

2) $\int \sec(x) dx$
$= \int \sec(x)\left(\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\right) dx$

Let $u = \sec(x) + \tan(x)$
$du = [(\sec x)(\tan x) + \sec^2 x] dx$

$= \int \frac{1}{u} du = \ln|u| + C = \ln|\sec x + \tan x| + C$

3) $\int \csc(x) dx = -\ln|\csc(x) + \cot(x)| + C.$

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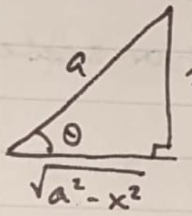
Trigonometric Substitution: *pg 130-131 Mean Values. (Need it)

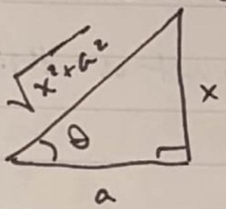
For integrals involving:

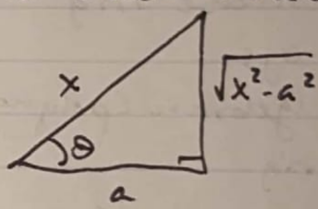
- $\sqrt{a^2-x^2}$
- $\sqrt{x^2+a^2}$
- $\sqrt{x^2-a^2}$

} U-Sub and IBP are messy

$\sqrt{a^2-x^2}$	$\sqrt{x^2+a^2}$	$\sqrt{x^2-a^2}$
$x=a\cdot\sin\theta$	$x=a\tan\theta$	$x=a\sec\theta$
$dx=a\cos\theta\,d\theta$	$dx=a\sec^2\theta\,d\theta$	$dx=a\sec\theta\tan\theta\,d\theta$



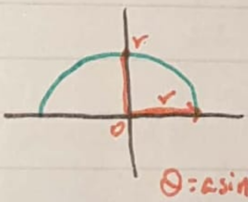




Example:

1) Find area of ○ w radius r.

*Recall: $\boxed{\cos^2\theta=\frac{1+\cos 2\theta}{2} \quad \sin^2\theta=\frac{1-\cos 2\theta}{2}}$



$f(x)=\sqrt{r^2-x^2}$ $\quad A=\int_{-r}^{r}\sqrt{r^2-x^2}\,dx$

Let $x=r\cdot\sin\theta$ $\quad \theta=\arcsin(\frac{x}{r})$

$dx=r\cos\theta\,d\theta$

$$\int\sqrt{r^2-x^2}\,dx=\int\sqrt{r^2-r^2\sin^2\theta}\cdot r\cos\theta\,d\theta$$
$$=\int\sqrt{r^2\cos^2\theta}\cdot r\cos\theta\,d\theta$$
$$=\int r^2\cos^2\theta\,d\theta$$
$$=r^2\int\cos^2\theta\,d\theta$$

$$=\frac{r^2}{2}\int(1+\cos 2\theta)\,d\theta$$
$$=\frac{r^2}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)+C$$
$$=\frac{r^2}{2}\left(\arcsin(\tfrac{x}{r})+\sin\theta\cos\theta\right)+C$$
$$=\frac{r^2}{2}\left(\arcsin(\tfrac{x}{r})+\left(\tfrac{x}{r}\right)\left(\tfrac{\sqrt{r^2-x^2}}{r}\right)\right)+C$$

So, $A=\frac{r^2}{2}\left(\arcsin(\frac{x}{r})+\left(\frac{x}{r}\right)\left(\frac{\sqrt{r^2-x^2}}{r}\right)\right)\Big]_{-r}^{r}$
$=\frac{\pi r^2}{2}$

So $A_{circle}=\pi r^2$

2) $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

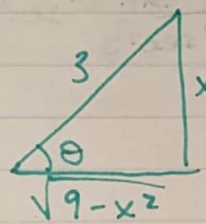
Let $x = 3\sin\theta$
$dx = 3\cos\theta\, d\theta$

$$= \int \frac{\sqrt{9-9\cos^2\theta}}{9\sin^2\theta} \cdot 3\cos\theta\, d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= \int \left(\frac{1}{\sin^2\theta} - 1\right) d\theta$$

$$= \int (\csc^2\theta - 1)\, d\theta = -\cot\theta - \theta + C$$



$x = 3\sin\theta$
$\arcsin\left(\frac{x}{3}\right) = \theta$

$$= \frac{-\sqrt{9-x^2}}{x} - \arcsin\left(\frac{x}{3}\right) + C$$

pg 136, c) 3) $\int \frac{x^3}{\sqrt{4-x^2}}\,dx$ Trig sub: Let $x = 2\sin\theta$.

Or... can use U-Sub:
Let $u = 4 - x^2$
$du = -2x\,dx$
$-\frac{1}{2} du = x\,dx$

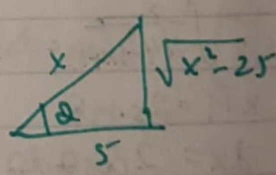
$$-\frac{1}{2}\int \frac{4-u}{\sqrt{u}}\,du = -\frac{1}{2}\int\left(\frac{4}{\sqrt{u}} - \frac{u}{\sqrt{u}}\right)du$$

$$= -\frac{1}{2}\left(8\sqrt{u} - \frac{2}{3}u^{\frac{3}{2}}\right) + c$$

$$= -4\sqrt{4-x^2} + \frac{1}{3}(4-x^2)^{3/2} + c$$

4) $\int \frac{1}{\sqrt{x^2-25}}\,dx$

$x = 5\sec\theta$
$dx = 5\sec\theta\tan\theta\, d\theta$



$$= \int \frac{5\sec\theta\tan\theta}{\sqrt{25\sec^2\theta - 25}}\, d\theta$$

$$= \int \frac{\sec\theta\tan\theta}{\tan\theta}\, d\theta = \ln|\sec\theta + \tan\theta| + c$$

$$= \ln\left|\frac{x}{5} + \frac{\sqrt{x^2-25}}{5}\right| + c$$

$$= \ln\left|x + \sqrt{x^2-25}\right| - \ln 5 + C$$

$$= \ln\left|x + \sqrt{x^2-25}\right| + C$$

5) $\int \frac{x^3}{(4x^2+9)^{3/2}}\,dx$ Can u-Sub $4x^2+9$ → better...

Nov 20, 2017

Let $2x = 3\tan\theta$, $dx = \frac{3}{2}\sec^2\theta\,d\theta$ $\Big\}$ $4x^2+9 = 4\left(\frac{9}{4}\tan^2\theta + 9\right)$ $= 9\sec^2\theta$

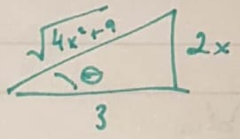
$= \frac{3}{2}\int \frac{\frac{27}{8}\tan^3\theta}{(9\sec^2\theta)^{3/2}} \cdot \sec^2\theta\,d\theta$ (he made mistake: $\frac{3}{2}\int \frac{27/8\,\tan^3\theta}{9\sec^2\theta}\sec^2\theta\,d\theta$)

$= \frac{81}{16}\int \frac{\tan^3\theta}{27\sec^3\theta} \cdot \sec^2\theta\,d\theta$

$= \frac{3}{16}\int \frac{\tan^3\theta}{\sec\theta}\,d\theta$

$= \frac{3}{16}\int \frac{\sin^3\theta}{\cos^2\theta}\,d\theta = \frac{3}{16}\int \frac{\sin^2\theta\sin\theta}{\cos^2\theta}\,d\theta = \frac{3}{16}\int \frac{(1-\cos^2\theta)\sin\theta}{\cos^2\theta}\,d\theta$

$u = \cos\theta$, $du = \sin\theta\,d\theta$ $\Big\}$ $= \frac{-3}{16}\int \frac{1-u^2}{u^2}\,du = \frac{-3}{16}\int \frac{1}{u^2}\,du + \frac{3}{16}\int du$



$\sec\theta = \frac{\sqrt{4x^2+9}}{3}$ $= \frac{3}{16}\left(\frac{1}{u} + u\right) + C$

$\cos\theta = \frac{3}{\sqrt{4x^2+9}}$ $= \frac{3}{16}\left(\frac{\sqrt{4x^2+9}}{3} + \frac{3}{\sqrt{4x^2+9}}\right) + C$

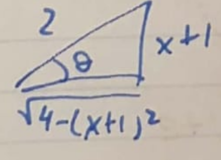
6) $\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$ Complete the square: $-x^2-2x+3$

$-(x^2+2x)+3$

$-(x+1)^2+3+1 = -(x+1)^2+4$ ✓

Now, $\int \frac{x}{\sqrt{4-(x+1)^2}}\,dx$

Let $x+1 = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$ $\Big\}$ $\int \frac{2\sin\theta - 1}{2\cos\theta}\,2\cos\theta\,d\theta$



$= \int (2\sin\theta - 1)\,d\theta = -2\cos\theta - \theta + C$

$= \frac{-\cancel{2}\sqrt{4-(x+1)^2}}{\cancel{2}} - \arcsin\left(\frac{x+1}{2}\right) + C$

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Area Between Curves:

Given f, g & $[a,b]$ with $f(x) \geq g(x)$ on interval, A between curves:

$$\int_a^b (f(x)-g(x))\,dx$$ } The curves can intersect many times.

Example:

1) $y=2x+3$ and $y=x^2$. Area of enclosed area?

1) Find intersection values of x

$x = 3, -1$

2) Test for "ceiling"

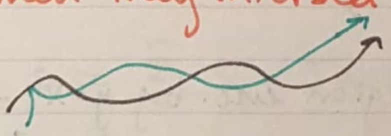
Test value & $[-1, 3]$

At $x=0$, $2x+3 > x^2$

3) Integrate

$$\int_{-1}^{3} (2x+3-x^2)\,dx = x^2+3x-\frac{1}{3}x^3 \Big|_{-1}^{3} = \frac{32}{3}$$

When they intersect more than once:



} Do above for each interval

Example:

1) A btwn $y=x$ and $y=x^3$

1) $x = x^3$

$x = 0, -1, 1$

2): $[-1, 0]$: $x^3 > x$

$[0, 1]$: $x > x^3$

3) $$\int_{-1}^{1} |x-x^3|\,dx = \int_{-1}^{0}(x^3-x)\,dx + \int_0^1 (x-x^3)\,dx$$

$$= \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_{-1}^{0} + \left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1$$

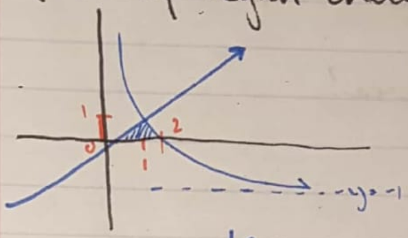
$$= \frac{1}{2}$$

Changing Variable: Nov 22, 2017

- when cieling/floor changes

Example:

1) Area of region enclosed by: $y=x$, $y=\frac{2}{x}-1$, and + x axis



$\int_0^1 x\,dx + \int_1^2 \left(\frac{2}{x}-1\right) dx$ (normal way)

or, change variable in terms of y

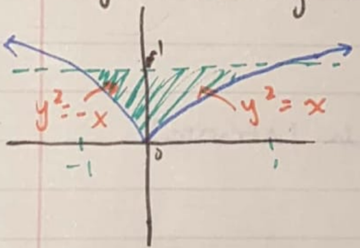
$y=x$ $\quad$ $y=\frac{2}{x}-1$

$x=y$ $\quad$ $x=\frac{2}{y+1}$ } Floor: left, Cieling: Right

$A: \int_0^1 \left(\left(\frac{2}{y+1}\right)-y\right) dy$ } Range: y values.

$= 2\ln|y+1| - \frac{1}{2}y^2 \Big|_0^1 = 2\ln 2 - \frac{1}{2}$

2) $y=1$ and $y=\sqrt{|x|}$



$\int_0^1 (y^2-(-y^2))\,dy = \frac{2}{3}$

Basically area of region enc. by $y=x^2, -x^2$ on $[0,1]$

PFD Again:

Useful formula: $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$

Will see: $\int \frac{1}{(ax+b)}\,dx = \frac{1}{a}\ln|ax+b|+C$ $\quad$ $\int \frac{1}{ax^2+bx+d}\,dx$

$n>1$ $\int \frac{1}{(ax+b)^n}\,dx = \frac{1}{a}\cdot\frac{(ax+b)^{-n+1}}{-n+1}+C$ $\quad$ $n>1$ $\int \frac{1}{(ax^2+bx+d)^n}\,dx$

$\int \frac{hx+k}{ax^2+bx+d}\,dx$ $\quad$ $n>1$ $\int \frac{hx+k}{(ax^2+bx+d)^n}\,dx$

(If $b=0$, $\int \frac{1}{ax^2+d}\,dx$ (atan))

Example:

1) $\int \frac{5x^2-5x+4}{x^3-x^2-x-2}\,dx = \int \frac{3x-1}{x^2+x+1}\,dx + \int \frac{2}{x-2}\,dx$ $= 2\ln|x-2|+C$

① $\int \frac{3x-1}{x^2+x+1}\,dx$ $\quad u = x^2+x+1$, $du = (2x+1)\,dx$ $\quad \frac{3}{2}\,du = (3x+\frac{3}{2})\,dx$

$= \int \frac{3x+3/2}{x^2+x+1}\,dx - \int \frac{5/2}{x^2+x+1}\,dx$ {complete the sq.

$= \frac{3}{2}\int \frac{1}{u}\,du - \frac{5}{2}\int \frac{1}{(x+\frac{1}{2})^2+3/4}\,dx$ arctan formula. $a = \frac{\sqrt{3}}{2}$

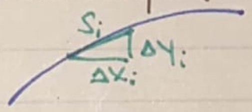
*course notes

$= \frac{3}{2}\ln|x^2+x+1| - \frac{5}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+c$

$\therefore \int = \frac{3}{2}\ln|x^2+x+1| - \frac{5}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + 2\ln|x-2| + c$

Arclength of a Curve:

Divide $[a,b]$ into n intervals:



$S_i = \sqrt{\Delta x^2+\Delta y^2}$

$= \sqrt{\Delta x^2\left(1+\frac{\Delta y^2}{\Delta x^2}\right)}$

$= \Delta x\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}$

$S \simeq \sum_{i=1}^{n} S_i$ Take $n\to\infty$

$S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

For F on $[a,b]$: $S = \int_a^b \sqrt{1+(f'(x))^2}\,dx$

Good to know: $\int f(ax+b)+c = \frac{1}{a}F(ax+b)+c$

Example:

1) Arclen $x^{3/2}$ on $[1,4]$?

$$s = \int_1^4 \sqrt{1+\left(\tfrac{3}{2}\sqrt{x}\right)^2}\,dx \qquad \left(\text{apply } \tfrac{1}{a}F(ax+b)+c\right)$$

$$= \frac{4}{9}\cdot\frac{2}{3}\left(1+\frac{9}{4}x\right)^{3/2}\Big|_1^4$$

2) Arclen semi circle of rad. r?

$f(x)=\sqrt{r^2-x^2}$ $\qquad f'(x)=\frac{-x}{\sqrt{r^2-x^2}}$

$$s = \int_{-r}^{r}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$$

$$= \int_{-r}^{r}\sqrt{\frac{r^2}{r^2-x^2}}\,dx = r\int_{-r}^{r}\frac{1}{\sqrt{r^2-x^2}}\,dx \Big\{ \text{arcsin sub}$$

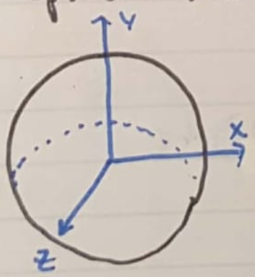
$$\left.\begin{aligned} x &= r\sin\theta \\ dx &= r\cos\theta\, d\theta \end{aligned}\right\} = r\int_{x=-r}^{x=r} 1\,d\theta = r\arcsin\left(\frac{x}{r}\right)\Big|_{-r}^{r} = \pi r$$

Solids of Revolution/Volumes:

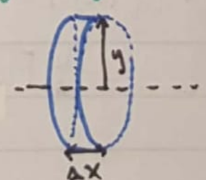
Rn for volumes (one variable)

Example:

1) Vol sphere radius r.



Height: $y=\sqrt{r^2-x^2}$ → Cut a disk height y at x. (Δx width)



$A=\pi y^2 = \pi(r^2-x^2)$

$V_i = V(x_i) = \pi(r^2-x_i^2)\Delta x$

$$V \approx \sum_{i=1}^{n} A(x_i)\Delta x = \int_{-r}^{r}\pi(r^2-x^2)\,dx = \pi\left(r^2x-\frac{x^3}{3}\right)\Big|_{-r}^{r}$$

$$= \frac{4}{3}\pi r^3$$

Idea for Solids of Rev:

→ Divide vol. into pieces

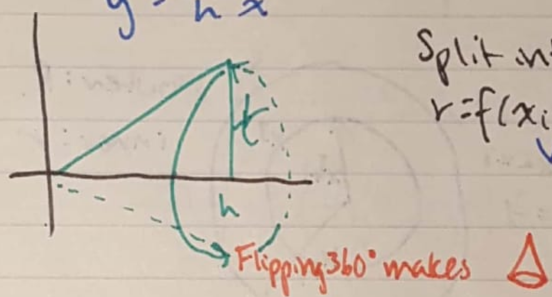
→ Get $A(x)\Delta x$ (Volume of slice) just i.t.o. x.

Example:

Disc/Washer Method:

1) V_{cone} $r = t$, height h?

$y = \frac{t}{h} x$



Split into n discs.

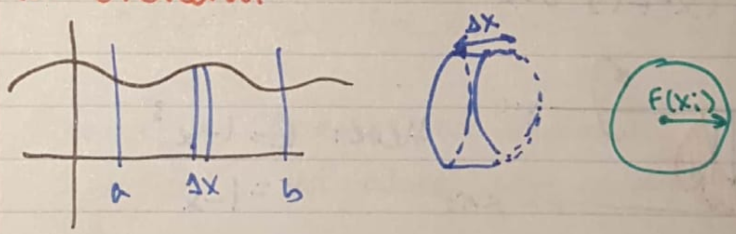
$r = f(x_i)$ width $= \Delta x$

$V_i = \pi (f(x_i))^2 \Delta x$ (πr^2 width)

$n \to \infty$: $V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi \left(\frac{t^2}{h^2} x_i^2\right) \Delta x$

$= \int_0^h \pi \left(\frac{t^2}{h^2} x^2\right) dx = \frac{1}{3}\pi t^2 h$

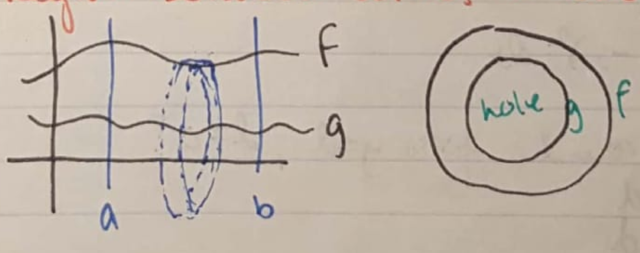
In General:



Disc Method:

$V = \pi \int_a^b (f(x))^2 dx$

Region Between Curves w/ Solids:



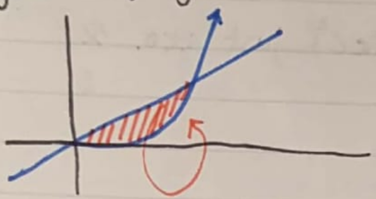
washer!

Area: $\pi(f(x)^2 - g(x)^2)$

$V = \pi \int_a^b \left[f(x)^2 - g(x)^2\right] dx$

Example:

1) $y = x$ & $y = x^3$ on $[0,1]$ about x axis



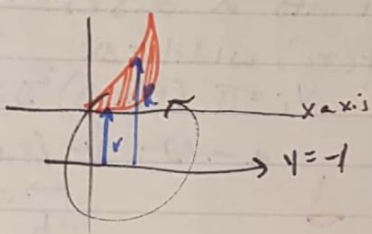
Area of cross section $= \pi(x^2 - x^6)$

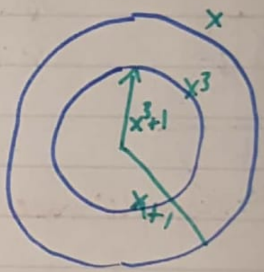
$$V = \pi \int_0^1 (x^2 - x^6)\,dx$$

$$= \frac{4\pi}{21}.$$

2) Same ↑ but around:

a) $y = -1$





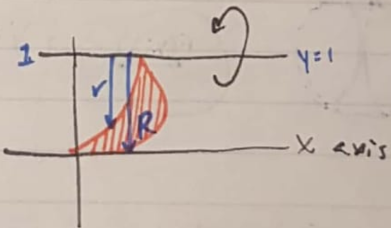
outer: R

inner: r

$$\text{Area} = \pi\left((x+1)^2 - (x^3+1)^2\right)$$

$$V = \pi \int_0^1 (x+1)^2 - (x^3+1)^2\,dx$$

★ b) $y = 1$.



Here: $R = 1 - x^3$

$r = 1 - x$

$$V = \pi \int_0^1 (1-x^3)^2 - (1-x)^2\,dx$$

★ revolving around axis $y = d$, $d \neq 0$?

$R \to R - d$

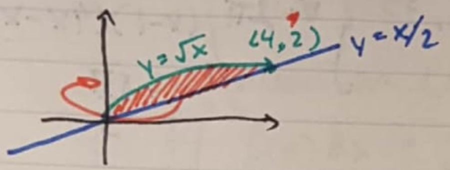
$r \to r - d$

Scanned by CamScanner

Nov 24, 2017

Revolving about y-axis:

V when $y = \frac{x}{2}$ and $y = \sqrt{x}$ is rev. about $x = 0$?



Disc wrt y (not x).
→ width of Δy.

Outer: $f(y) = 2y$
Inner: $g(y) = y^2$

$$V = \pi \int_0^2 \left((2y)^2 - (y^2)^2\right) dy$$

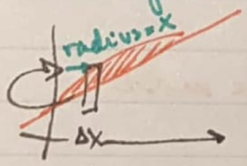
$$= \pi \int_0^2 (4y^2 - y^4)\, dy = \frac{64}{15}\pi$$

Shell Method:

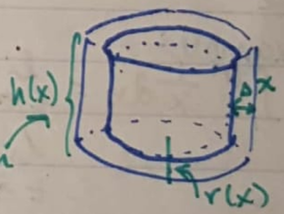
Disc/washer method integrate $\parallel$ to axis of rev.
→ Shell: $\perp$ to axis.

Example: (same as above)

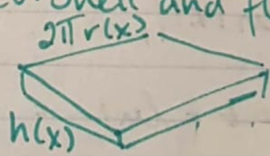
1)



Cut along perp. axis.



Cut shell and flatten



$r(x) = x$
$h(x) = \sqrt{x} - \frac{x}{2}$

$h(x_i) = f(x_i) - g(x_i)$

$a = 0, b = 4$. So, volume of shell: $V_i = 2\pi r(x_i) h(x_i) \Delta x$

$$V \approx \sum_{i=1}^{n} 2\pi r(x_i) h(x_i) \Delta x$$

$$V = 2\pi \int_a^b r(x_i) h(x_i)\, dx$$

For y-axis: $V = 2\pi \int_a^b x\, h(x)\, dx$

2) V : $y = 4x - x^2$, $y = 3$ about the line $x = 1$.
$r(x) \to r(x) - 1$

Intersections: $(1,3)$, $(3,3)$
$h(x) = 4x - x^2 - 3$
$r(x) = x - 1$

$V = 2\pi \int_1^3 (x-1)(4x-x^2-3)\,dx = \frac{8\pi}{3}$

* 3) V : $y = 3 - x$ and $x = 4 - (y-1)^2$ about $y = 0$.
Easier in terms of y (Shell Method)!

$x = 3 - y$
$x = 4 - (y-1)^2$
Intersect at $y = 0$, $y = 3$.
ceil: $4-(y-1)^2$ floor: $3-y$

$V = 2\pi \int_0^3 y\left(4 - (y-1)^2 - (3-y)\right) dy = \frac{27\pi}{2}$.

Improper Integrals:
∞, asymptotes, discontinuities.
1) Integrating to ∞:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

f cont. on $[a,\infty)$
finite? converge. Else: diverge.

Example:
1) $\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$ *This is a mark –

$= \lim_{t\to\infty} \ln|x| \Big|_1^t$

$= \lim_{t\to\infty} \ln t - \ln 1 = \infty$.

2) $\int_1^\infty \frac{1}{x^p}\,dx$, $p > 1$ $= \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx$

$= \lim_{t\to\infty} \frac{1}{-p+1} \cdot x^{-p+1} \Big|_1^t = \lim_{t\to\infty} \frac{1}{-p+1}\left(\frac{1}{t^{p-1}} - 1\right)$

If $p - 1 > 0$, $\frac{1}{t^{p-1}} \to 0$

$= \frac{1}{p-1}$.

else, diverge.

2) Integrating on Asymptotes:

Nov 27, 2017

f continuous on $[a,b)$ but D.C. at $x=b$, then:

RS $\left\{ \int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx \right.$

f continuous on $(a,b]$ but D.C. at $x=a$, then:

LS $\left\{ \int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx \right.$

Example:

1) $\int_0^1 \frac{1}{x^p} dx \quad (p>1) = \lim_{t \to 0^+} \int_t^1 \frac{1}{x^p} dx$

$= \lim_{t \to 0^+} \left(\frac{1}{-p+1} \cdot \frac{1}{x^{p-1}} \right) \Big|_t^1$

$= \frac{1}{-p+1} (1-\infty) = +\infty.$

3) Integrate over Discontinuities:

f cont. on $[a,b]$ but D.C at $x=c$, then:

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

$= \lim_{t \to c^-} \int_a^t f(x)dx + \lim_{t \to c^+} \int_t^b f(x)dx$

Example: * watch for discontinuities!

$\boxed{\neq \frac{-1}{x-2} \Big|_1^3}$

1) $\int_1^3 \frac{1}{(x-2)^2} dx = \lim_{t \to 2^-} \int_1^t \frac{1}{(x-2)^2} dx + \lim_{t \to 2^+} \int_t^3 \frac{1}{(x-2)^2} dx$

$= \lim_{t \to 2^-} \left(\frac{-1}{x-2} \right)\Big|_1^t + \lim_{t \to 2^+} \left(\frac{-1}{x-2} \right)\Big|_t^3$

$= \lim_{t \to 2^-} \left(\frac{-1}{t-2} \right) - 1 + \left(-1 - \lim_{t \to 2^+} \frac{1}{t-2} \right)$

$= \infty - 1 + (0) + [-1 + \infty]$

$= \infty$

4) More Infinities:

$$\int_{-\infty}^{\infty} f(x)\,dx \ \Big\} \text{ Pick an easy } c \in \mathbb{R} \text{ where:}$$

$$= \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

5) Losing track of titles:

f continuous on (a,b):

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{t\to a^+} \int_t^c f(x)\,dx + \lim_{t\to b^-} \int_c^t f(x)\,dx.$$

Extra Stuff:

$$\int_{-a}^{a} (\text{odd fcn})\,dx = 0.$$