

MATH 117 - Introduction

Sep 8/2012

Pre-Calculus Review:

Definitions:

Function from set A to set B that assigns each $x \in A \rightarrow y \in B$
 ↳ each x has only one y (vert. line test)

↳ Doesn't need to be math formula

↳ $f: A \rightarrow B$:
 - A is the domain, B is codomain (IR)

↳ Range of f :
 - Set of all outputs of range (f)
 ↳ Domain: set of all permissible inputs

Set Notation

↳ \mathbb{Z} - the integers

↳ \mathbb{Q} - the rational numbers

↳ \mathbb{R} - real #'s

↳ $\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}, i = \sqrt{-1}\}$

Interval Notation

Inverse Functions

Identity fun: $\{ id(x) = x \}$ ↳ g is inv. of f if $g(f(x)) = x$ { And reversed!
 ↳ "undoes" action of original fun } $\text{dom}(f) = \text{range}(f) = [0, \infty)$
 ↳ Invertible (inv-fun) if fun is one-to-one of interval

One to one fun: Pass $\xrightarrow{\text{one-to-one}}$ test "injective" $f(x_1) = f(x_2)$ then $x_1 = x_2$

Intersections: Overlap of sets $A \cap B$ $\{[1, 5] \cap [2, 5] = [2, 5]\}$

Unions: Combining sets $A \cup B$ $\{[1, 5] \cup [2, 5] = [1, 5]\}$

↳ $(0, 1) \cup (2, 3) = (0, 1) \cup (2, 3)$

↳ $(0, 1) \cap (2, 3) = \emptyset$ "null set"

↳ Properties:

↳ $A \cup \emptyset = A$ $A \cap \emptyset = \emptyset$

Subsets: $A \subseteq B$ or $A \subset B$ } A is subset of B

For inv. funs: $y = \sqrt{x}$ passes $\xrightarrow{\text{one-to-one}}$ test but $y = x^2$ not a fun.

↳ $y = \sqrt{x}$ domain is restricted. $\text{dom}(g) = \text{range}(f)$ v.s. versa

↳ x^2 is inv., but domain restricted to $[0, \infty)$

Composition:

- f, g where $\text{rng}(g) \subseteq \text{dom}(f)$
 $f \circ g(x) = f(g(x))$
- ① $\text{dom}(f \circ g) \subseteq \text{dom}(g)$
- ② $\text{rng}(f \circ g) \subseteq \text{rng}(f)$
- ③ $f \circ g \neq g \circ f$, usually.

Inverses cont...

$$\text{dom}(g) = \text{rng}(f)$$

$$\text{dom}(f) = \text{rng}(g)$$

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Example:

1) $f(x) = \frac{2x-4}{3x+6}$ { find $\text{dom}(f), \text{rng}(f)$
 $f^{-1}(x) = -\frac{6y-4}{3y-2}$
 $\text{dom}(f) = \{x \in \mathbb{R} | x \neq -2\} = (-\infty, -2) \cup (-2, \infty)$
 $\text{rng}(f) = \text{dom}(f^{-1}) = \{x \in \mathbb{R} | x \neq \frac{2}{3}\} = (-\infty, \frac{2}{3}) \cup (\frac{2}{3}, \infty)$

Even & Odd func:

$$\text{Even: } f(x) = f(-x)$$

$$\text{Odd: } f(-x) = -f(x)$$

Operations:

$$E \times E = E$$

$$E \times O = O$$

$$O \times O = E$$

{ same as if
E = +ve int
O = -ve int

Components:

$$\text{even}(x) = \frac{1}{2}(f(x) + f(-x)) \quad \left\{ \begin{array}{l} f(x) = f_e(x) + f_o(x) \\ f_o(x) = \frac{1}{2}(f(x) - f(-x)) \end{array} \right.$$

These only exist when $\text{dom}(f)$ not sym.
about origin

$$(-\infty, -2) \cup (2, \infty) \checkmark$$

$$(-\infty, -5) \cup (2, \infty) \times$$

Hyperbolic Functions:

$$f_o(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh(x)$$

$$f_e(x) = \frac{1}{2}(e^x + e^{-x}) = \cosh(x)$$

Piecewise Functions: + Sec 1.5 course notes
 Absolute Value: $|x| = \begin{cases} x & : x \geq 0 \\ -x & : x < 0 \end{cases}$ be careful of even denom.

Heaviside Function (unit step fun):

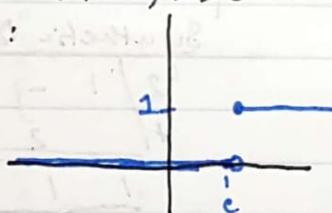
$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Important for multi.

Using $H(x)$:

$$f(x)H(x) = \begin{cases} 0, & x < 0 \\ f(x), & x \geq 0 \end{cases}$$

$H(x-c)$:



Kind of like an on/off switch

Example:

$$1) f/(x-a) - f/(x-b) \text{ when } a < b$$

$$= \begin{cases} 0, & x < a \\ 1, & a \leq x < b \\ 0, & x \geq b \end{cases}$$

a "turns on"

$$2) \text{ Write } f(x): \begin{cases} 5-x, & x < 2 \\ \sqrt{x}, & 2 \leq x < 5 \\ 13, & x \geq 5 \end{cases}$$

"subtract off" the formula to the left (number line) when a new formula arises.

$$f(x) = \begin{cases} 5-x \\ [\sqrt{x} - (5-x)]H(x-2) \\ [13 - \sqrt{x}]H(x-5) \end{cases} = 5-x + [\sqrt{x} - (5-x)]H(x-2) + [13 - \sqrt{x}]H(x-5)$$

* pg 21 course notes for changing location

Partial Fraction Decomposition:

Decomposing a rational function:

↳ denom: linear, non reducible quads, and $(\)^2$.

Example:

$$1) \frac{5x^2 - 5x + 4}{x^3 - x^2 - x - 2}$$

Steps:

1) Factor the denominator

↳ likely need root of denom.

↳ For deg 3+, use the Rational Roots Theorem

↳ Rational Root Theorem:

↳ Any rational root of $a_nx^n + \dots + a_0$ is the form $\frac{c}{d}$ where:

* { c is a factor of a_0 } Find factors then divide
 d is a factor of a_n

↳ Long Division:

$$\begin{array}{r} x^2 + x + 1 \\ x-2 \overline{)x^3 - x^2 - x - 2} \\ - (x^3 - 2x^2) \\ \hline x^2 - x \\ - (x^2 - 2x) \\ \hline x - 2 \\ - (x-2) \\ \hline 0 \end{array}$$

Synthetic Division:

$$\begin{array}{r} 2 | 1 & -1 & -1 & -2 \\ & 2 & 2 & 2 \\ \hline & 1 & 1 & 1 & 0 \end{array}$$

$x^2 + x + 1$

Now @ irreducible roots.

2) Split up the fun.

$$\frac{5x^2 - 5x + 4}{x^3 - x^2 - x - 2} = \frac{Ax + B}{x^2 + x + 1} + \frac{C}{x-2} \quad \begin{array}{l} \text{variables must be} \\ 1 \text{ less deg}(p(x)) + 1 = \\ \deg(q(x)) \end{array}$$

$$5x^2 - 5x + 4 = (Ax + B)(x-2) + C(x^2 + x + 1)$$

2 approaches now.

a) Expand RS and compare coefficients to LS.

b) Choose strategic values for x . * Mock letter.

↳ Let $x=2$, cancels first term.

↳ Let $x=0$, cancels A.

↳ Let $x=1$ to get A.

when denom is squared
special case

* $\frac{x^5 - 10x^4 - 2x^3 - 45x^2 - x - 13}{(x^2 + x + 1)^2 (x-2)^2}$

$$= \frac{Ax + B}{x^2 + x + 1} + \frac{Cx + D}{(x^2 + x + 1)^2} + \frac{E}{(x-2)} + \frac{F}{(x-2)^2}$$

$$\text{Show } A=2 \ B=-1 \ C=4$$

$$D=1 \ E=5 \ F=-3$$

Independent Reading:

Other Important Piecewise Func:

Signum:

$$\text{sgn}(x) \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

{ returns sign of #.

Ramp:

$$r(t) \begin{cases} 0 & t < 0 \\ ct & t \geq 0 \end{cases}, c \in \mathbb{R}$$

{ Heaviside unit const mult.

Floor:

$$Lx \downarrow \begin{cases} \text{round down} & -1.5 \rightarrow -2 \\ & 1.5 \rightarrow 1 \end{cases}$$

Ceiling:

$$\lceil x \rceil \begin{cases} \text{round up} & -1.5 \rightarrow -1 \\ & 1.5 \rightarrow 2 \end{cases}$$

Frac Part:

$$\text{frac}(x) = x - Lx \downarrow \text{get decimal.}$$

Partial Fractions Decomp. Cont...

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1) Factor denominator

2) For each distinct factor $g(x)$ of $g(x)$ w/ exponent $n \geq 1$:

a) If $g(x)$ is linear, it contributes

$$\frac{A_1}{g(x)} + \frac{A_2}{(g(x))^2} + \dots + \frac{A_n}{(g(x))^n}$$

b) Irreducible quadratic, it contributes

$$\frac{A_1x+B_1}{g(x)} + \frac{A_2x+B_2}{(g(x))^2} + \dots + \frac{A_nx+B_n}{(g(x))^n}$$

*Set $x = 0$ {
on back}*

3) Write over common denominator (don't expand numerator)

4) Special values of x to cancel? solve.

5) Write out final answer.

Tutorial

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Inequalities w/ Absolute Values:

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

Example:

1) $|3x-12| < 9$

1) $3x-12 < 9$
 $x < 7$

2) $12-3x < 9$
 $x > 1$

3) $|x-2| < |x+1|$

$(x-2)^2 < (x+1)^2$
 $\frac{1}{2} < x$

2) Describe the set $\{x : |x^2-2| > 1\}$ as a union of finite intervals.

1) $-(x^2-2) > 1$
 $(x+1)(x-1) < 0$ DNE OR $-1 < x < 1 \quad x \in (-1, 1)$

2) $x^2-2 > 1 \quad |x \in (\sqrt{3}, \infty) \text{ or } (-\infty, -\sqrt{3})|$ careful in trace
 $x \in (-\infty, -\sqrt{3}) \cup (-1, 1) \cup (\sqrt{3}, \infty)$

Composite Func:

1) $f(x) = \frac{1}{x} \quad x \in (0, \infty)$ and $g(x) = 2x+1 \quad x \in [-2, 2]$

$\text{rng}(g(x))$ must be in $\text{dom}(f)$ and x must be in $\text{dom}(g(x))$

$f(g(x)) \quad g(x) > 0 \quad x > -\frac{1}{2} \quad \therefore x \in (-\frac{1}{2}, 2]$

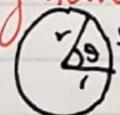
2) $g(f(x)) \quad -2 \leq f(x) \leq 2$

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Partial Fraction Decomposition Continued:

$$\begin{aligned}
 & \frac{3x^4 + 3x^3 + 2x^2 + 2x - 3}{x^5 + 2x^4 + 2x^3 + 4x^2 + x + 2} \\
 = \frac{3x^4 + 3x^3 + 2x^2 + 2x - 3}{(x+2)(x^4 + 2x^2 + 1)} & = \frac{3x^4 + 3x^3 + 2x^2 + 2x - 3}{(x+2)(x^2+1)^2}, \text{ so...} \\
 \frac{3x^4 + 3x^3 + 2x^2 + 2x - 3}{x^5 + 2x^4 + 2x^3 + 4x^2 + x + 2} & = \frac{A}{x+2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} \\
 & = A(x^2+1)^2 + (Bx+C)(x+2)(x^2+1) + (Dx+E)(x+2) \\
 \text{Let } x = -2: \quad 28 & = 25A \quad \therefore A = 1 \\
 \text{Let } x = 0: \quad -4 & = 2C + 2E \quad \therefore E = -2 - C \\
 \text{Let } x = 1: \quad 7 & = 4 + 6B + 6C + 3D - 3C \quad \therefore C = 3 - 2B - D \\
 \text{Let } x = -1: \quad -8 & = -4B - 2D \quad \therefore B = 2 - \frac{D}{2} \\
 \text{Let } x = 2: \quad 16 & = 8B - 8D \quad \therefore D = 0 \quad \text{Now, go backwards}
 \end{aligned}$$

Trigonometry:



circle subtends arc between sides of the angle.

1 radian is the measure of an angle in a circle that subtends an arc of equal length to the radius.

$\sin \theta$: y coordinate of point on unit circle corresponding to θ

$\cos \theta$: x " " " " "

Reference Angle: $\bar{\theta}$

$$[0 \leq \theta \leq \pi] \quad |\sin \theta| = \sin \bar{\theta} \quad |\cos \theta| = \cos \bar{\theta}$$

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Formulas:

$$\begin{aligned}
 \sin \theta &= \sin(\theta + 2\pi k) \quad \left. \begin{array}{l} \\ \end{array} \right\} k \in \mathbb{Z} \\
 \cos \theta &= \cos(\theta + 2\pi k) \quad \left. \begin{array}{l} \\ \end{array} \right\} \\
 \cos\left(\theta - \frac{\pi}{2}\right) &= \sin \theta \\
 \sin\left(\theta - \frac{\pi}{2}\right) &= -\cos \theta
 \end{aligned}$$

$$\begin{aligned}\sin(x \pm y) &= \sin x \cos y \pm \sin y \cos x \\ \cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y \\ \tan(x \pm y) &= \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}\end{aligned}\quad \left\{ \begin{array}{l} \text{ } \\ \text{ } \\ \text{ } \end{array} \right.$$

$$\begin{aligned}\cos^2 \theta + \sin^2 \theta &= 1 \\ 1 + \tan^2 \theta &= \sec^2 \theta\end{aligned}\quad \left\{ \begin{array}{l} \text{ } \\ \text{ } \end{array} \right.$$

$$\cot^2 \theta + 1 = \csc^2 \theta$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

$$\sin x \sin y = \frac{1}{2} [\cos(x-y) - \cos(x+y)]$$

$$\sin x \cos y = \frac{1}{2} [\sin(x-y) + \sin(x+y)]$$

Example:

1) Find all $\theta \in [0, 2\pi)$ that satisfy:

$$a) \cos \theta + \sin 2\theta = 0$$

$$b) \sec^2 \theta = 4/3$$

$$a) \cos \theta + \sin 2\theta = 0$$

$$\cos \theta + 2 \sin \theta \cos \theta = 0$$

$$\cos \theta (1 + 2 \sin \theta) = 0$$

$$\cos \theta = 0 \quad \text{and} \quad \sin \theta = \frac{-1}{2}$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \quad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$b) \sec^2 \theta = \frac{4}{3}$$

$$\cos^2 \theta = \frac{3}{4}$$

$$\cos \theta = \pm \frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$2) \sin\left(\frac{7\pi}{12}\right) = \sin\left(\frac{3\pi}{12} + \frac{4\pi}{12}\right) = \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \sin\frac{\pi}{4} \cos\frac{\pi}{3} + \sin\frac{\pi}{3} \cos\frac{\pi}{4}$$

$$= \frac{\sqrt{2}}{2} (1 + \sqrt{3})$$

$$3) \sin^2\left(\frac{7\pi}{12}\right) = \frac{1 - \cos\left(\frac{7\pi}{6}\right)}{2} = \frac{2 + \sqrt{3}}{4}$$

General Sine Function:

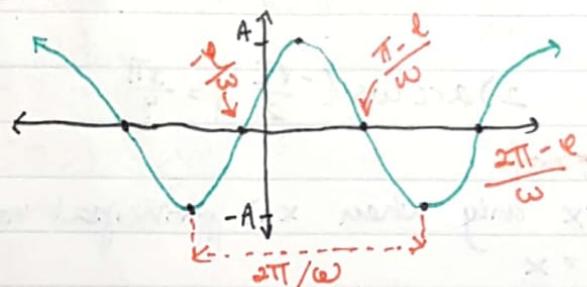
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$$f(x) = A \sin(\omega x + \varphi)$$

Period: $\frac{2\pi}{\omega}$

Phase shift: $-\frac{\varphi}{\omega}$

Amplitude: A



Every function $B \sin(\omega x) + C \cos(\omega x)$ can be written $A \sin(\omega x + \varphi)$

Example:

$$1) 3 \sin(2x) + 3\sqrt{3} \cos(2x) = A \sin(2x + \varphi)$$

$$= A [\sin(2x)\cos(\varphi) + (\sin\varphi)\cos(2x)]$$

$$= A \cos(\varphi) \sin(2x) + A \sin(\varphi) \cos(2x)$$

$$\therefore 3 = A \cos(\varphi) \quad 3\sqrt{3} = A \sin(\varphi) \quad \text{square both sides and add}$$

$$A^2 \cos^2(\varphi) + A^2 \sin^2(\varphi) = 36$$

$$A = \pm 6 \quad \text{Take } A = 6 \quad (-6 \text{ also works})$$

$$\therefore \varphi = \frac{\pi}{3} \quad \therefore \text{Final: } 6 \sin(2x + \frac{\pi}{3})$$

$$A = \pm \sqrt{B^2 + C^2}$$

$$\cos \varphi = B/A \quad \left\{ \begin{array}{l} \varphi = \arctan(C/B) \\ \text{only when } \varphi \in (-\frac{\pi}{2}, \frac{\pi}{2}) \end{array} \right.$$

$$\sin \varphi = C/A$$

Inverse Trig:

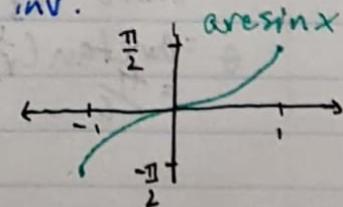
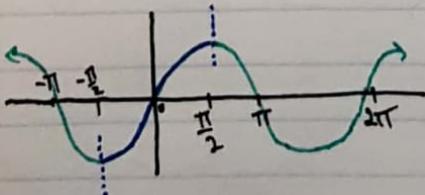
\sin, \cos, \tan not inv. but can restrict domain.

The half-period $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is inv.

$$f'(x) = \arcsin x$$

$$f: [-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow [-1, 1]$$

Principal values

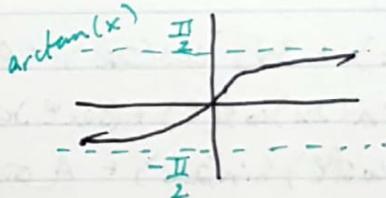


For cosine: $g: [0, \pi] \rightarrow [-1, 1]$
 $g^{-1}: [-1, 1] \rightarrow [0, \pi]$
 $= \arccos(x)$ principal values

Example:

- 1) $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$
- 2) $\arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$
- 3) $\arcsin(\sin(\frac{\pi}{4})) = \frac{\pi}{4}$ *correct
 $\hookrightarrow \arcsin(\sin x) = x$ only when x is principal value of sine
 BUT $\sin(\arcsin x) = x$
- 4) $\sin(\arcsin(\frac{-1}{2})) = -\frac{1}{2}$
- 5) $\sin(\arcsin(\frac{\pi}{2})) = \text{DNE}$ *watch

For tan: $h: (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}$
 $h^{-1}: \mathbb{R} \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$ P.V.
 $= \arctan x$



For sec: $(-\infty, -1] \cup [1, \infty) \rightarrow (-\pi, -\frac{\pi}{2}] \cup [0, \frac{\pi}{2})$ PVs = arcsec x

For csc: $(-\infty, -1] \cup [1, \infty) \rightarrow (-\pi, -\frac{\pi}{2}) \cup (0, \frac{\pi}{2})$ PVs = arccsc x

For cot: $j: (0, \pi) \rightarrow \mathbb{R}$ = arccot x
 $j^{-1}: \mathbb{R} \rightarrow (0, \pi)$ PVs

1) $\arccot(\sqrt{3}) = \theta$

$$\cot \theta = \sqrt{3}$$

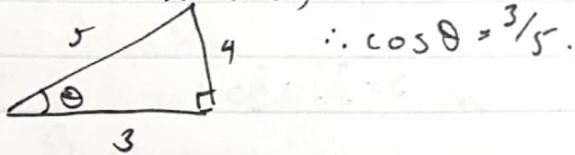
$$\tan \theta = \frac{1}{\sqrt{3}}$$

$$\theta = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

Sometimes trig helps.

$$1) \cos(\arcsin(4/5)) = \cos \theta$$

$$\text{Let } \theta = \sin^{-1}(4/5)$$



$$\therefore \cos \theta = 3/5.$$

$$2) \cot(\operatorname{arcsec}(13))$$

$$\text{Let } \theta = \operatorname{arcsec}(13)$$

$$\sec \theta = 13$$

$$\cos \theta = \frac{1}{13} \text{ solve for } \tan \theta \text{ using } \Delta.$$

$$3) \cos(2\arcsin(\frac{7}{11}))$$

$$= \cos^2(\arcsin(\frac{7}{11})) - \sin^2(\arcsin(\frac{7}{11}))$$

$$= \cos^2 \theta - (\frac{7}{11})^2$$

$$\theta = \arcsin(\frac{7}{11})$$

$$(\Delta) \quad \sin \theta = \frac{7}{11} \quad \cos \theta = \frac{\sqrt{12}}{11} \quad \text{so } \cos^2 \theta = \frac{72}{121}$$

$$4) \tan(2\arccos(\frac{5}{13}))$$

$$= \frac{2 \tan(\arccos \frac{5}{13})}{1 - \tan^2(\arccos \frac{5}{13})}$$

$$\theta = \arccos \frac{5}{13}$$

$$\tan \theta = \frac{12}{5}, \text{ so}$$

$$= \frac{-120}{125}$$

Limits of Sequences:

Let S be an infinite subset of consecutive non-negative integers. A sequence is a function $f: S \rightarrow \mathbb{R}$

$$f(n) = a_n$$

Usually, $S = \mathbb{N} = \{1, 2, 3, \dots\}$ or $S = \mathbb{N} \cup \{0\}$

To denote sequences

$$\{a_n\}_{n=1}^{\infty}, \{a_n\}_{n=0}^{\infty}, \{a_n\}$$

Example:

$$1) \{(-1)^n\}_{n=0}^{\infty} = \{1, -1, 1, -1, \dots\}$$

$$2) \{3, 9, 15, 21, 27, \dots\} = \{6n+3\}_{n=0}^{\infty} = \{6n-3\}_{n=1}^{\infty}$$

Conversions:

To increase starting index by k , substitute n with $(n-k)$

To start $\{6n+3\}_{n=0}^{\infty}$ at $n=5$

$$\{6(n-5)+3\}_{n=5}^{\infty} \text{ simplify.}$$

To start $\{6n-3\}_{n=1}^{\infty}$ at $n=5$

$$\{6(n-4)-3\}_{n=5}^{\infty} \text{ simplify.}$$

Limits of Sequences:

Behavior as $n \rightarrow \infty$ $\{\frac{1}{n}\}_{n=1}^{\infty}$

For any $\epsilon > 0$ (let ϵ be this #), we can find an index n where $\frac{1}{n} < \epsilon$. Consider $\frac{1}{\epsilon}$, pick an n where $n > \lceil \frac{1}{\epsilon} \rceil$

$\Rightarrow n > \lceil \frac{1}{\epsilon} \rceil, n > \frac{1}{\epsilon}$. Thus $\frac{1}{n} < \epsilon$

\therefore As $n \rightarrow \infty, \frac{1}{n} \rightarrow 0$

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Definition ***:

A sequence $\{a_n\}$ converges to the limit L if, for any $\epsilon > 0$ there exists a natural number $N \in \mathbb{N}$ such that whenever $n > N$, $|a_n - L| < \epsilon$.

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n > N \rightarrow |a_n - L| < \epsilon$$

Back to $\sum_{n=1}^{\infty} a_n$: $a_n = \frac{1}{n}$ $L=0$ $N = \lceil \frac{1}{\varepsilon} \rceil$

Example:

1) Use the definition of sq. limits, prove $\frac{1}{\sqrt{n}} \rightarrow 0$ as $n \rightarrow \infty$.

\Leftarrow Solution: Required to prove (RTP):

Given any (fixed) $\varepsilon > 0$, we need to find a $\# N$ such that $n > N \Rightarrow |\frac{1}{\sqrt{n}} - 0| < \varepsilon$

$$|\frac{1}{\sqrt{n}} - 0| = \frac{1}{\sqrt{n}}$$

$$\text{RTP: } \frac{1}{\sqrt{n}} < \varepsilon$$

$$n > N$$

\Leftrightarrow shows inequality equivalence

$$\frac{1}{\sqrt{n}} < \varepsilon$$

$$\Leftrightarrow \frac{1}{n} < \varepsilon^2$$

$\Leftrightarrow n > \frac{1}{\varepsilon^2}$ Let N (must be natural) = $\lceil \frac{1}{\varepsilon^2} \rceil$ if $n > N$, then

$$n > N \Rightarrow n > \frac{1}{\varepsilon^2}$$

$$\Rightarrow \frac{1}{n} < \varepsilon^2$$

$$\Rightarrow \frac{1}{\sqrt{n}} < \varepsilon \quad \text{QED.} \quad \therefore \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$$

2) "Prove" $\frac{n-1}{n} \rightarrow 1$ as $n \rightarrow \infty$.

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Solution: RTP:

$$n > N \Rightarrow \left| \frac{n-1}{n} - 1 \right| < \varepsilon$$

$$\Leftrightarrow \left| \frac{-1}{n} \right| < \varepsilon$$

$$\Leftrightarrow \frac{1}{n} < \varepsilon$$

$$\Leftrightarrow n > \frac{1}{\varepsilon}$$
 Let $N = \lceil \frac{1}{\varepsilon} \rceil$.

So, when $n > \lceil \frac{1}{\varepsilon} \rceil$, $\left| \frac{n-1}{n} - 1 \right| < \varepsilon$ Thus QED.

Properties of Sequence Limits:

$$1) \lim_{n \rightarrow \infty} c = c, \quad c \in \mathbb{R} \text{ fixed.}$$

Let $\lim_{n \rightarrow \infty} a_n = A, \lim_{n \rightarrow \infty} b_n = B$

$$2) \lim_{n \rightarrow \infty} (a_n \pm b_n) = A \pm B = \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n$$

Same for multi. and division ($B \neq 0$)

$$3) a) \lim_{n \rightarrow \infty} \frac{1}{n^p} = 0 \text{ for any fixed } p > 0$$

$$b) \lim_{n \rightarrow \infty} r^n = 0 \text{ when } |r| < 1, = \infty \text{ when } |r| > 1.$$

4) If f is continuous on open interval containing A :

$$\lim_{n \rightarrow \infty} f(a_n) = f(A) = f\left(\lim_{n \rightarrow \infty} a_n\right)$$

$$5) \lim_{n \rightarrow \infty} a_n = \infty \text{ iff for all } M > 0, \text{ there is an}$$

$N \in \mathbb{N}$ such that $n > N \Rightarrow a_n > M$.

Examples:

$$1) \left\{ \cos\left(\frac{\pi n}{3n+5}\right) \right\}_{n=1}^{\infty} \text{ find the limit.}$$

$$= \lim_{n \rightarrow \infty} \cos\left(\frac{\pi n}{3n+5}\right)$$

$$= \cos\left(\lim_{n \rightarrow \infty} \frac{\pi n}{3n+5}\right) \quad \begin{array}{l} \text{Now, } \div \text{ top/bottom by highest power} \\ \text{of } n \text{ in denom.} \end{array}$$

Assumption:
1) \lim exists
2) \cos is continuous

$$= \cos\left(\lim_{n \rightarrow \infty} \frac{\pi}{3 + \frac{5}{n}}\right) = \cos\left(\lim_{n \rightarrow \infty} \frac{\pi}{3}\right) = \frac{1}{2}$$

Squeeze Theorem:

Suppose $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ where

a) $\lim_{n \rightarrow \infty} a_n = L = \lim_{n \rightarrow \infty} c_n$

squeezed
↙

b) For some $N \in \mathbb{N}$, whenever $n > N$, $a_n \leq b_n \leq c_n$.
Then $\lim_{n \rightarrow \infty} b_n = L$.

Example:

1) $\lim_{n \rightarrow \infty} \frac{\cos(n)}{n}$ We know $-1 \leq \cos(n) \leq 1$

$$-1 \leq \cos(n) \leq 1$$

$$\frac{-1}{n} \leq \frac{\cos(n)}{n} \leq \frac{1}{n}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{-1}{n} \leq \lim_{n \rightarrow \infty} \frac{\cos(n)}{n} \leq \lim_{n \rightarrow \infty} \frac{1}{n}$$

$$0 \leq \frac{\cos(n)}{n} \leq 0 \quad \therefore \text{The limit is } 0.$$

Indeterminate Forms:

Expressions resulting from direct subst. can't be evaluated.

$$0^0, \frac{0}{0}, \frac{\infty}{\infty}, 1^\infty, \infty - \infty, 0 \cdot \infty, \infty^\infty$$

If f is polynomial of degree $< k$,

$$\lim_{n \rightarrow \infty} n^k + f(n) = \lim_{n \rightarrow \infty} n^k = \infty. \quad \begin{cases} \text{Largest power} \\ \text{dominates limit} \end{cases}$$

Example:

1) $\lim_{n \rightarrow \infty} \frac{4n^2 - 5n + 3}{5n^2 - 4n + 5} = \frac{4}{5}$

2) $\lim_{n \rightarrow \infty} \frac{9n+8}{\sqrt{9n^2+8}}$ Divide by highest "effective power" in denom $\sqrt{n^2} = n$

$$\lim_{n \rightarrow \infty} \frac{9 + \frac{8}{n^2}}{\sqrt{\frac{9n^2+8}{n^2}}} = 3$$

3) $\lim_{n \rightarrow \infty} \frac{6^n + 3^n}{6^{n-1} - 4}$ Divide by 6^{n-1} (highest exp of n in denom)

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \frac{6^n + 3^n}{6^{n-1} - 4} \cdot \frac{1/6^{n-1}}{1/6^{n-1}} \\
 &= \lim_{n \rightarrow \infty} \frac{6 + 3 \cdot \left(\frac{3^{n-1}}{6^{n-1}}\right)}{1 - 4/6^{n-1}} \\
 &= \lim_{n \rightarrow \infty} \frac{6 + 3 \cdot \left(\frac{1}{2}\right)^{n-1}}{1 - 4/6^{n-1}} \\
 &= \frac{6 + 0}{1 - 0} = 6.
 \end{aligned}$$

4) $\lim_{n \rightarrow \infty} \sqrt{n^2 + 1} - n \cdot \left(\frac{\sqrt{n^2 + 1} + n}{\sqrt{n^2 + 1} + n} \right)$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n^2 + 1}} = \frac{1}{\infty} = 0$$

$\{a_n\}$ is divergent if $\lim_{n \rightarrow \infty} a_n = \pm \infty$ or DNE

Limits of functions:

$$\lim_{x \rightarrow c} f(x) = L \text{ iff } \forall \varepsilon > 0, \exists \delta > 0 \ni |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Example:

$$1) \text{ Prove } \lim_{x \rightarrow 2} 3x - 2 = 4.$$

*page 59

$$|x - 2|$$

RTP: Given $\varepsilon > 0$, find $\delta > 0$ where $|x - c| < \delta \Rightarrow |3x - 2 - 4| < \varepsilon$.

$$|3x - 6| < \varepsilon$$

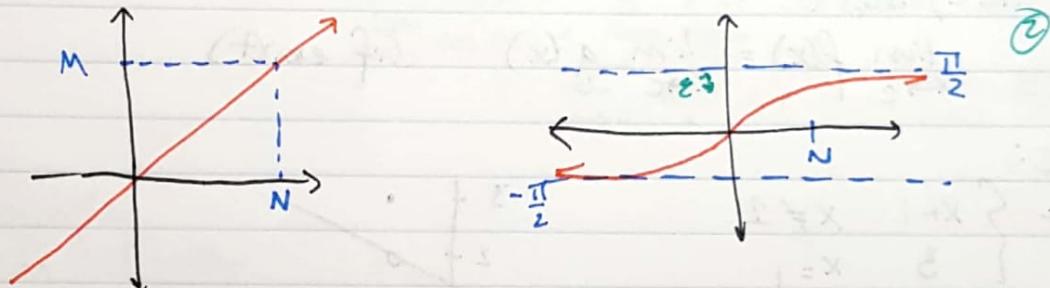
$$\Leftrightarrow 3|x - 2| < \varepsilon$$

$$\Leftrightarrow |x - 2| < \frac{\varepsilon}{3}$$

Thus, if $\delta = \frac{\varepsilon}{3}$, true, so $\lim = 4$.

Key Definitions: pg 59

- 1) $\lim_{x \rightarrow c} f(x) = \infty \quad \forall M > 0 \exists \delta > 0 \ni |x - c| < \delta \Rightarrow f(x) > M.$ How close you are to a limit implies ∞
- 2) $\lim_{x \rightarrow \infty} f(x) = L \quad \forall \varepsilon > 0 \exists N > 0 \ni x > N \Rightarrow |f(x) - L| < \varepsilon$
- 3) $\lim_{x \rightarrow \infty} f(x) = \infty \quad \forall M > 0 \exists N > 0 \ni x > N \Rightarrow f(x) > M.$



Constructing Definitions:

- $\boxed{\lim_{x \rightarrow c}}$: $\exists \delta > 0 \dots \exists |x - c| < \delta \Rightarrow \dots$
- $\boxed{\lim_{x \rightarrow \infty}}$: $\exists N > 0 \dots \exists x > N \Rightarrow \dots$
- $\boxed{\lim_{x \rightarrow -\infty}}$: $\exists N > 0 \dots \exists x < -N \Rightarrow \dots$
- $\boxed{\lim f(x) = L}$: $\forall \varepsilon > 0 \exists \dots \Rightarrow |f(x) - L| < \varepsilon$
- $\boxed{\lim f(x) = \infty}$: $\forall M > 0 \exists \dots \Rightarrow f(x) > M$
- $\boxed{\lim f(x) = -\infty}$: $\forall M > 0 \exists \dots \Rightarrow f(x) < -M$.

One sided limits:

$$|x - c| : 2 \text{ sided.} = \begin{cases} x - c, & x > c \\ c - x, & x < c \end{cases}$$

$$\lim_{x \rightarrow c^-} f(x) = L: \forall \varepsilon > 0 \exists \delta > 0 \ni c - x < \delta \Rightarrow |f(x) - L| < \varepsilon$$

$$\lim_{x \rightarrow c^+} f(x) = L: \forall \varepsilon > 0 \exists \delta > 0 \ni x - c < \delta \Rightarrow |f(x) - L| < \varepsilon$$

$$\lim_{x \rightarrow c^+} f(x) = \infty : \forall M > 0 \exists \delta > 0 \ni x - c < \delta \Rightarrow f(x) > M.$$

Limits Cont...

Sep 29, 2017

Let $c \in \mathbb{R}$. Open neighbourhood of c is an open interval containing c . (a, b)

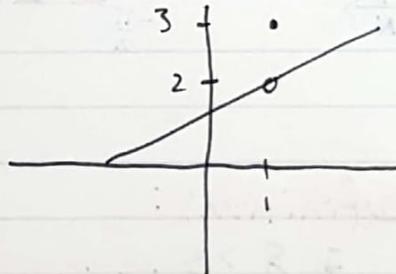
→ If f, g are fun's and $f(x) = g(x) \quad \forall x \in (a, b)$ except $x = c$, then:

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) \quad (\text{if exist})$$

Example:

$$1) f(x) = \begin{cases} x+1 & x \neq 1 \\ 3 & x = 1 \end{cases}$$

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} x+1 = 2$$



can use substitution
when no proof required.

Same limit theorem applies from sequences

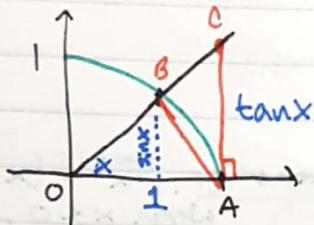
Squeeze Theorem again...

Let f, g, h be functions - $f(x) \leq h(x) \leq g(x)$ on an open neighbourhood of c

If $\lim_{x \rightarrow c} f(x) = L = \lim_{x \rightarrow c} g(x)$, then $\lim_{x \rightarrow c} h(x) = L$.

Special Limit:

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1.$$



Area $\Delta OCA \geq$ Area (sector OBA) \geq area (ΔOBA)

$$\frac{1}{2} \tan x \geq \frac{1}{2} \times r^2 \geq \frac{1}{2} \sin x$$

$$\tan x \geq \frac{x}{r} \geq \sin x$$

$$\frac{1}{\cos x} \geq \frac{\sin x}{x} \geq 1$$

$$\cos x \leq \frac{\sin x}{x} \leq 1 \quad (\text{Now, take limit})$$

$$\lim_{x \rightarrow 0} \cos(x) \leq \lim_{x \rightarrow 0} \frac{\sin x}{x} \leq 1$$

$$1 \leq \lim_{x \rightarrow 0} \frac{\sin x}{x} \leq 1 \quad \therefore \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

Example:

$$1) \lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$$

$$= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x(1 + \cos x)}$$

$$= \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right) \left(\lim_{x \rightarrow 0} \frac{\sin x}{1 + \cos x} \right)$$

$$= (1)(\frac{0}{1+1}) = 0$$

Infinity "Arithmetic":

$$1) (\infty)(\infty) = \infty$$

$$2) (-\infty)(\infty) = -\infty$$

$$3) (-\infty)(-\infty) = \infty$$

$$4) \text{ If } a > 1, \lim_{x \rightarrow \infty} a^x = \infty, \text{ else } 0 \quad 7) \text{ when } c \neq 0:$$

$$5) \frac{c}{\infty} = 0$$

$$6) \infty + \infty = \infty$$

$$(c)(\infty) = \begin{cases} \infty, & c > 0 \\ -\infty, & c < 0 \end{cases}$$

More limits:

Oct 2, 2017.

$$1) \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{x + 1} \quad \text{Divide by highest } x \text{ in denom}$$

$$\lim_{x \rightarrow \infty} \left(\frac{\sqrt{x^2 + 1}}{(x+1)} \right) \left(\frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{x}} \right) = \frac{\sqrt{1+0}}{1+0} = \underline{1}.$$

$$2) \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 1} \cdot \left(\frac{1}{-\sqrt{x^2}} \right)}{(x+1) \left(\frac{1}{x} \right)}$$

because x is negative ∞ . $\sqrt{x^2} = |x|$
when $x < 0$, $\sqrt{x^2} = -x$!

$$\lim_{x \rightarrow -\infty} \frac{-\sqrt{1 + \frac{1}{x^2}}}{1 + \frac{1}{x}} = \underline{-1}. \quad \text{two different HTs.}$$

Continuity: * pg 66-67

f is continuous on (a, b) if $\forall x, y \in (a, b)$:
 $\forall \epsilon > 0, \exists \delta > 0 \Rightarrow |x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$

f is continuous at $x=c$ when:

$$\lim_{x \rightarrow c} f(x) = f(c) \quad (\text{and limit is defined + finite}).$$

and... if continuous at $\forall x=c$ in (a, b) , continuous on interval.

Usual Steps to Evaluate:

1) $f(c)$ exists.

2) $\lim_{x \rightarrow c} f(x)$ exists (+finite)

3) $\lim_{x \rightarrow c} f(x) = f(c)$

Types of Discontinuity:

1) Removable $\rightarrow f$ not cont. at $x=c$ but $\lim_{x \rightarrow c} f(x)$ exists

2) Non removable $\rightarrow \lim_{x \rightarrow c} f(x) \text{ DNE}$

↳ Jump $\lim_{x \rightarrow c^-} f(x)$ and $\lim_{x \rightarrow c^+} f(x)$ exist but \neq

↳ V.A (∞) At least one of the \rightarrow \leftarrow limits are finite.

Finding Discontinuities for Piecewise Functions:

- 1) Check formula; discard those outside x values used
- 2) Check every transition value.

↳ Must do this for every value on both sides → ←

Example:

$$f(x) = \begin{cases} \frac{x^2 - 10x + 25}{x^2 - 25}, & x < 5 \\ \frac{x-5}{x}, & 5 \leq x < 9 \\ \frac{x^2}{8}, & x > 9 \end{cases}$$

Be careful

- 1) Check formulas

i) $\frac{x^2 - 10x + 25}{x^2 - 25}$

DC at $x = -5, 5$
BUT discard $x = 5$ b/c not in interval.

ii) $\frac{x-5}{x}$

DC at $x = 0$

iii) $\frac{x^2}{8}$

Continuous

From formulas: discontinuous at $x = -5$

- 2) Transition Values

i) $x = 5$

a) $f(5) = \frac{5-5}{5} = 0$ ✓ exists

b) $\lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5^-} \frac{x^2 - 10x + 25}{x^2 - 25} = \lim_{x \rightarrow 5^-} \frac{x-5}{x+5} = 0$

$\lim_{x \rightarrow 5^+} f(x) = \lim_{x \rightarrow 5^+} \frac{x-5}{x} = 0$ — | ∵ $\lim_{x \rightarrow 5} f(x) = 0$ |
 $\therefore f$ is cont. at 5 since $\lim = f(5)$. need this.

ii) $x = 9$

c) $f(9) = \text{DNE}$

∴ f is disc. at $x = 9$.

Now, classify:

At $x = -5$: $\lim_{x \rightarrow -5} f(x) = \frac{100}{0}$ → VA/Inf. dc

$x = 9$: $\lim_{x \rightarrow 9} f(x) = \text{DNE}$ since left/right ≠. → Jump dc. [Both non-removable]

Theorem:

If $\lim_{x \rightarrow c} g(x) = L$ and f is continuous at $x=L$ then:

$$\lim_{x \rightarrow c} f(g(x)) = f(L) = f(\lim_{x \rightarrow c} g(x))$$

Example:

$$1) \lim_{x \rightarrow 7} e^{x^2+5} = e^{54} = e^{\lim_{x \rightarrow 7} x^2+5}$$

Continuity on Closed Intervals:

f is continuous on $[a, b]$ when:

1) continuous (a, b)

2) cont. on the right at $x=a$, left at $x=b$

$$[a, b] \quad \xrightarrow{a} \xrightarrow{b}$$

→ Right continuity:

1) $f(a)$ exists

2) $\lim_{x \rightarrow a^+} f(x)$ exists (+finite)

3) $\lim_{x \rightarrow a^+} f(x) = f(a)$

→ Left continuity:

1) $f(a)$ exists

2) $\lim_{x \rightarrow b^-} f(x)$ exists (+finite)

3) $\lim_{x \rightarrow b^-} f(x) = f(a)$

Intermediate Value theorem:

Oct 4, 2017.

f continuous on $[a, b]$, function passes every value $[a, b]$

→ Show $x^5 - 6x^3 - x + 4$ has root on $[0, 1]$.

→ f is continuous on $[0, 1]$. $f(0) = 4$ $f(1) = -2$.

∴ By I.V.T., $(\exists c \in [0, 1])$ where $f(c) = 0$.

Extreme Value Theorem: pg 69-72

f continuous on $[a, b]$, then f has an abs max/min on $[a, b]$.

Differentiation:

Refer to gr 12 calc notes + EdX Notes.

Proof of power rule: $f'(x) = nx^{n-1}$, $n \in \mathbb{N}$.

$$\begin{aligned}(x+h)^n &= \sum_{i=0}^n x^{n-i} h^i \cdot \binom{n}{i} \\&= x^n + nhx^{n-1} + h^2 \text{ (STUFF...)} \\&\stackrel{h \rightarrow 0}{\rightarrow} \lim \frac{nhx^{n-1} + h^2 \text{ (STUFF)}}{h} \\&= nx^{n-1}\end{aligned}$$

Extreme Value Theorem: pg 69-72

f continuous on $[a, b]$, then f has an abs max/min on $[a, b]$.

Differentiation:

Refer to gr 12 calc notes + EdX Notes.

Proof of power rule: $f'(x) = nx^{n-1}$, $n \in \mathbb{N}$.

$$\begin{aligned}(x+h)^n &= \sum_{i=0}^n x^{n-i} h^i \cdot \binom{n}{i} && \text{proof in gr 12 physics notes} \\ &= x^n + nhx^{n-1} + h^2 (\text{STUFF...}) \\ &\approx \lim_{h \rightarrow 0} \frac{nhx^{n-1} + h^2 (\text{STUFF})}{h} \\ &= nx^{n-1}\end{aligned}$$

Rules: (Refer to gr 12 and EdX calc notes)

Proof of product rule:

$$\begin{aligned}p'(x) &= \lim_{h \rightarrow 0} \frac{p(x+h) - p(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[f(x+h)][g(x+h)] - f(x)g(x)}{h} && \text{equivalent to zero} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) + [-f(x)g(x+h) + f(x)g(x+h)] - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{g(x+h)[f(x+h) - f(x)]}{h} + \lim_{h \rightarrow 0} f(x) \frac{[g(x+h) - g(x)]}{h} \\ &= g(x)f'(x) + f(x)g'(x)\end{aligned}$$

Examples:

1) $p(x) = (x^2 + 1) \sin(x)$ \rightarrow product rule
 $p'(x) = 2x \sin x + (x^2 + 1) \cos x$

2) $g(x) = \frac{e^x + x^3}{x^2 \sin x}$ \rightarrow quotient rule.

$$\begin{aligned} f(x) &= e^x + x^3 \\ f'(x) &= e^x + 3x^2 \end{aligned} \quad \begin{aligned} g(x) &= (x^2 + 1) \sin x \\ g'(x) &= 2x \sin x \end{aligned} \quad \left. \begin{aligned} &\text{etc...} \\ &g'(x) = \dots \end{aligned} \right\}$$

Special Derivatives:

$$\frac{d}{dx} \tan(x) = \sec^2 x$$

$$\frac{d}{dx} \sec(x) = \sec(x) \tan(x)$$

$$\frac{d}{dx} \csc(x) = -\csc(x) \cot(x)$$

$$\frac{d}{dx} \cot(x) = -\csc^2(x)$$

$$\frac{d}{dx} \sqrt{f(x)} = \frac{1}{2\sqrt{f(x)}} \cdot f'(x)$$

$$\frac{d}{dx} \sinh(x) = \cosh(x)$$

$$\frac{d}{dx} \cosh(x) = \sinh(x)$$

$$\frac{d}{dx} \tanh(x) = \operatorname{sech}^2(x)$$

$$\frac{d}{dx} \ln(x) = \frac{1}{x}$$

$$\frac{d}{dx} a^x = a^x \ln(a) \quad \left. \begin{aligned} &a \neq 0, 1; a > 0 \\ &\log_a(x) = \frac{1}{x} \ln(a) \end{aligned} \right\}$$

$$\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}} \quad \left. \begin{aligned} &\text{watch sign } (\frac{d}{dx} \cos x = -\sin x) \\ &\arccos(x) = \frac{-1}{\sqrt{1-x^2}} \end{aligned} \right\}$$

$$\frac{d}{dx} \arccos(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx} \operatorname{arcsec}(x) = \frac{1}{x \sqrt{x^2-1}}$$

$$\frac{d}{dx} \operatorname{arccsc}(x) = \frac{-1}{x \sqrt{x^2-1}}$$

$$\frac{d}{dx} \operatorname{arccot}(x) = \frac{-1}{1+x^2}$$

Oct 13, 2017

Differentiability:

$$f'(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} \text{ exists}$$

$$= \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

Continuity \Rightarrow Diff. $|x|$

Diff \Rightarrow Continuity : Not continuous \Rightarrow Not diff.

Proof: RTE $\lim_{x \rightarrow c} f(x) = f(c)$. Given $f'(c)$ exists.

$$\left(\lim_{x \rightarrow c} f(x) \right) = f(c)$$

$$= \lim_{x \rightarrow c} (f(x) - f(c)) \cdot \frac{x - c}{x - c}$$

we know this exists

$$= \lim_{x \rightarrow c} \left(\frac{f(x) - f(c)}{x - c} \right) \cdot \lim_{x \rightarrow c} (x - c)$$

$$= f'(c) \cdot 0 = 0.$$

Oct 25, 2017

Implicit Differentiation:

Unit Circle Example:

$$x^2 + y^2 = 1$$

$$|y| = \sqrt{1-x^2} \quad y = \begin{cases} \sqrt{1-x^2} & y \geq 0 \\ -\sqrt{1-x^2} & y < 0 \end{cases}$$

when $y \geq 0$:

$$\frac{dy}{dx} (y = \sqrt{1-x^2}) = y = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}} = \frac{-x}{y}.$$

when $y < 0$:

$$\frac{dy}{dx} (y = -\sqrt{1-x^2}) = \frac{-1}{2\sqrt{1-x^2}} \cdot (-2x) = \frac{x}{y} \quad \therefore \frac{dy}{dx} = \frac{-x}{y}.$$

\leftarrow b/c y is -ve

1D method of last example:

$$1) \begin{aligned} x^2 + y^2 &= 1 \\ \frac{d(x^2 + y^2)}{dx} &= \frac{d}{dx}(1) \\ 2x + 2y \cdot y' &= 0 \\ y' &= -\frac{x}{y} \end{aligned}$$

$$2) \begin{aligned} 3x^2y^3 + 13x &= 12 + 2y \\ \frac{d}{dx}(3x^2y^3 + 13x) &= \frac{d}{dx}(12 + 2y) \\ 6xy^3 + 9x^2y^2y' + 13 &= 2y' \\ y' &= \frac{6xy^3 + 13}{2 - 9x^2y^2} \end{aligned}$$

Derivatives of the Inverse:

Suppose a fun is diff + invertible. we want deriv. f^{-1} : $x = f(y)$.

$$\frac{d(f^{-1}(x))}{dx} : 1 = \frac{dF(y)}{dx}$$

$$\begin{aligned} 1 &= f'(y) \cdot \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))} \end{aligned}$$

$$\boxed{\therefore \frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}}$$

Example:

$$1) y = \ln x \quad f^{-1}(x) = \ln x, \quad f(x) = e^x, \quad f'(x) = e^x$$
$$\frac{d \ln x}{dx} = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

$$2) y = \arcsin(x) \quad f^{-1}(x) = \arcsin(x), \quad f(x) = \sin x, \quad f'(x) = \cos x$$
$$\frac{d \arcsin x}{dx} = \frac{1}{\cos(\arcsin x)}$$

*+ root
(look @ arcsin
PVS)*

Let $\theta = \arcsin x$

So $\cos \theta = \sqrt{1-x^2}$

$$\frac{1}{\sqrt{1-x^2}}$$

$$3) y = a^x \quad f^{-1}(x) = a^x, \quad f(x) = \log_a(x), \quad f'(x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx} (a^x) = \frac{1}{(\ln a)} = a^x \ln a.$$

$$4) \frac{d}{dx} \arctan(x^3)? \quad f'(x) = \frac{1}{1+(x^3)^2} \cdot \frac{d}{dx}(x^3) = \frac{3x^2}{1+x^6}.$$

$$5) \frac{dy}{dx} \text{ of } \arcsin(2^y) = 5x^2y^3 \quad \text{use } \frac{dy}{dx} > y' \text{ for mistakes.}$$

$$\frac{1}{\sqrt{1-(2^y)^2}} \cdot \frac{d}{dx}(2^y) = 5(2xy^3 + 3x^2y^2 \frac{dy}{dx})$$

$$\frac{1}{\sqrt{1-2^{2y}}} \cdot (2^y)(\ln(2))(\frac{dy}{dx}) = 10xy^3 + 15x^2y^2 \frac{dy}{dx}$$

$$\frac{dy}{dx} \left(\frac{2^y \ln 2}{\sqrt{1-2^{2y}}} - 15x^2y^2 \right) = 10xy^3$$

Logarithmic Differentiation:

$$f(x) = (\cos(x))^{x^2+1}$$

Take \ln of both sides:

$$\ln(f(x)) = \ln[(\cos(x))^{x^2+1}]$$

$$\ln(f(x)) = (x^2+1) \ln(\cos x)$$

Now Define:

$$\frac{d}{dx} \ln(f(x)) = \frac{d}{dx} (x^2+1) \ln(\cos x)$$

$$\frac{1}{f(x)} \cdot f'(x) = 2x \ln \cos x + (x^2+1) \left(-\frac{\sin x}{\cos x} \right)$$

$$f'(x) = [2x \ln \cos x + (x^2+1)(-\tan x)] \underline{(\cos x)^{x^2+1}}$$

don't leave in more

Example:

$$1) f(x) = x^x, \quad x > 0$$

$$\ln(f(x)) = x \ln x$$

$$\frac{f'(x)}{f(x)} = \ln(x) + \cancel{x} = f(x)(\ln(x)+1)$$

$$= x^x(\ln(x)+1)$$

$$\begin{cases} \text{dom}(\ln x) = (0, \infty) \\ \text{dom}(\ln|x|) = \mathbb{R} \setminus \{0\} \end{cases}$$

$$\frac{d}{dx} \ln|x| = \frac{1}{|x|} \frac{d}{dx} |x| \quad \text{So, } \frac{d}{dx} \ln(|f(x)|)$$

$$= \frac{1}{x} \frac{d}{dx} |f(x)| = \frac{f'(x)}{f(x)}$$

$$2) f(x) = \frac{(x-2)^3 (x^2-2x+5)^4}{(2x+1)^2 (x^2-7)^3} \quad \text{Take } \ln|f(x)|$$

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$$\ln|f(x)| = \ln \left| \text{that} \right| = 3\ln|x-2| + 4\ln|x^2-2x+5| - 2\ln|2x+1| - 3$$

Differentiating:

* ONLY VALID WHEN $f(x) \neq 0$ or DNE

$$\frac{f'(x)}{f(x)} = \frac{3}{x-2} + \frac{8x-8}{x^2-2x+5} - \frac{4}{2x+1} - \frac{6x}{x^2-7}$$

$$f'(x) = f(x) \left[\text{that} \right] \quad \text{where } x \neq 2, x^2-2x+5 \neq 0, 2x+1 \neq 0, x^2-7 \neq 0.$$

Use $|f(x)|$ when $\ln(f(x))$ not defined (function is negative)

Extrema and Monotonicity:

Let f defined at $x=c$.

- (1) Global max at $x=c$ if: $f(c) \geq f(x) \quad \forall x \in \text{dom}(f)$ open neighborhood.
 - (2) Local max at $x=c$ if: $f(c) \geq f(x) \quad \forall x \in I$. I is int. containing c
- Mins too. c can't be an endpoint

Critical Numbers:

$c \in \text{dom}(f)$

a) $f'(c) = 0$

b) $f'(c) = \text{DNE}$

→ Cusp when $f'(c)$ exists but $\lim_{x \to c} f'(x) \neq f'(c)$

$\vee \quad \wedge \quad \nearrow \quad \searrow \quad \{$

Fermat's Theorem: (Not Little or Least)

Rel min/max \Rightarrow critical point / number

Increasing: $x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \quad (f'(x) > 0)$

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Monotone if either increasing/decreasing on interval.

Increasing/Decreasing:

$f'(x)$ sign can only change at crit #'s and discontinuities.

Testing for increasing/decreasing: Let f be dif. on (a, b) .

1) Increasing $f'(x) > 0 \quad \forall x \in (a, b)$

2) Decreasing $f'(x) < 0 \quad \forall x \in (a, b)$

3) Neither $f'(x) = 0 \quad \forall x \in (a, b)$ *not monotonic

First Derivative Test for Extrema:

Intervals where $f'(x) = 0$ or DNE. $x=c$ crit #.

	$x < c$	$x > c$	Type
$f'(x)$ sign	-	+	local min
	+	-	local max

{ Make sure it's not VA before classifying. }

Example:

1) $f(x) = x^3 - 3x^2 - 24x$

1) $f'(x) = 0$ or DNE \rightarrow Crit #'s

$$f'(x) = 3(x-4)(x+2) \quad x=4, -2 \quad \text{CN}'s$$

2) Table:

	$(-\infty, -2)$	$(-2, 4)$	$(4, \infty)$
$f'(x)$	+	-	+
$f(x)$	\nearrow	\searrow	\nearrow

Increasing on $(-\infty, 2] \cup [4, \infty)$

Square brackets \rightarrow since defined for $f(x)$. If VA? No bro. (.)

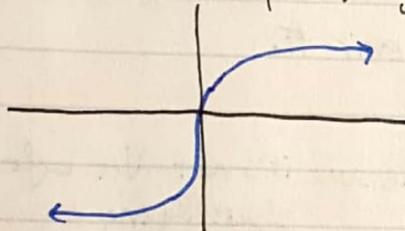
Concavity/Inflection:

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- 1) Concave up an interval when $f''(x) > 0$. \cup
 - 2) Concave down an interval when $f''(x) < 0$. \cap
- When concavity changes, point of inflection.
→ **Hypercritical numbers**: $f''(c) = 0$ or DNE.

Example:

1) $f(x) = \sqrt[3]{x}$ $f'(x) = \frac{1}{3}x^{-2/3}$ $f''(x) = -\frac{2}{3}x^{-5/3}$



$f'(0)$ and $f''(0)$ DNE.

No extrema at $x=0$, but POI, yeah.
Check out $\sqrt[3]{x^2}$.

↪ $g'(0)$ global max

↪ $g''(0)$ No POI.

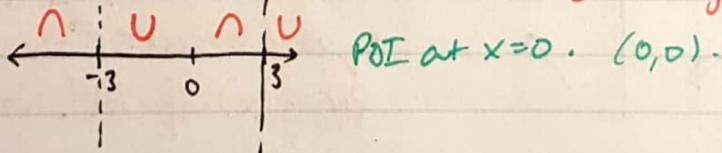
Note: Concavity can only change at limit #s.

2) $g(x) = \frac{x}{x^2-9}$ $g'(x) = \frac{-(x^2+9)}{(x^2-9)^2}$ $g''(x) = \frac{2x(x^2+27)}{(x^2-9)^3}$

$g''(x)$ DNE at $x = \pm 3$.

$g''(x) 0$ when $2x(x^2+27) = 0$, so $x=0$.

↪ VAs at $x = \pm 3$. Sub hc#s to $g''(x)$.



Second Derivative Test:

Let $f'(c) = 0$. Assume f'' is cont. on open neighbourhood \bar{c} .

1) $f''(c) > 0$? CCU, rel min \cup

2) $f''(c) < 0$? CCD, rel max \cap

3) $f''(c) = 0$? Doesn't tell u anything. (can be any, $x^3, x^4, -x^4$)

Curve Sketching:

ΔGr 12 calc notes.

$$f(x) = x^3 \text{ (example curve)}$$

	increasing	decreasing
$f'(x) > 0$	\	/
$f''(x) > 0$	\	/
$f'(x) < 0$	/	\

} accelerating / decelerating
forwards + backwards.

1) Find $x_{\text{ints}}, y_{\text{ints}}$.

2) Find discontinuities

↳ including RD's and respective y -value of RD.

↳ VAs - don't need one-sided limits b/c inc/dec table

3) Horizontal Asymptotes

↳ $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$ (must do both)

4) Find critical numbers \bar{x} in $f'(x)$

↳ intervals of \nearrow \searrow . Classify extrema.

5) Find hypercritical numbers \bar{x} in $f''(x)$.

↳ intervals of \cap \cup . Classify POIs.

Optimization

Nov 3, 2017

To find extrema on $[a, b]$ (refer to Evt):

1) Find CP's

2) Points of $x=a, x=b$. } Largest $y = \max$, smallest $y = \min$

Example:

1) Find extrema of $f(x) = x^2 - 4x + 4$ on $[-1, 4]$.

$$f'(x) = 2x - 4$$

$$f'(x) = 0 \Rightarrow x = 2$$

$$f(-1) = 9 \quad \max(1, 9)$$

$$f(2) = 0 \quad \min(2, 0)$$

$$f(4) = 4$$