Applied Probability for Computer Science

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Poisson Processes

Preliminary Notation and Definitions

- Since a discrete-space process, at each fixed t takes a countable number of possible values, $\{a_1, a_2, a_3, \dots\}$, it is a common convention to simply use the values $\{1, 2, 3, \dots\}$ to refer to the states of the process. Therefore, in what follows, we assume that $\mathbb{N}$ is the support of $X(t)$ ➔ **Warning:** If the natural numbers are only used as labels to represent the possible states of the process, quantities as the mean and covariance functions will become meaningless.

Counting process

A stochastic process $X(t)$ is called a **counting process** if

1. $X(t) \in \mathbb{N}$ for all $t \in T \subset \mathbb{R}$, the **time domain**
2. $X(s) \leq X(t)$ for all $s \leq t$.

- Counts are nonnegative integers, $X(t) \in \{0, 1, 2, 3, \dots\}$ ➔ All counting processes are discrete-state processes

Preliminary Notation and Definitions

- As time passes, one can count additional items ➔ sample paths of a counting process are always non-decreasing

☞ Unless otherwise stated, we will assume that all Markov processes are **time homogeneous**, i.e., for any time increment $h > 0$, the **transition probability** $\mathbb{P}\left[X_{t+h} = j | X_t = i\right]$ depends on h but not on t, so that we may write

$$p_{ij}(h) = \mathbb{P}\left[X_{t+h} = j | X_t = i\right]$$

- $p_i j(h)$ is the conditional probability that the state of the process h time units into the future will be j given that the process is presently in state i, regardless of the present time t

Poisson Process: 3 Definitions

Poisson process: Definition 1

A **Poisson process** with intensity λ is a continuous-time counting process $N = \{N(t) : t \geq 0\}$ taking values in $S = \{0, 1, 2, \ldots\}$ such that:

(a) $N(0) = 0$,

(b) The increments are independent and stationary (the distribution of $N(t+h) - N(t)$ depends only on h).

(c) $\mathbb{P}[N(h) = 1] = \lambda h + o(h)$ and $\mathbb{P}[N(h) \geq 2] = o(h)$

Poisson Process: 3 Definitions

Poisson process: Definition 2

A **Poisson process** with intensity λ is a continuous-time counting process $N = \{N(t) : t \geq 0\}$ that satisfies the following properties:

(a) $N(0) = 0$,

(b) for any pair of disjoint intervals, $(s_1, s_1 + t_1]$, $(s_2, s_2 + t_2]$, $t_1, t_2, s_1, s_2 \geq 0$ the increments $N(s_1 + t_1) - N(s_1)$ and $N(s_2 + t_2) - N(s_2)$ are independent,

(c) for any $t, s \geq 0$, $N(t + s) - N(s) \sim \text{Po}\,(\lambda t)$.

Poisson Process: 3 Definitions

Poisson process: definition 3

A **Poisson process** with intensity λ is a continuous-time counting process $N = \{N(t) : t \geq 0\}$ that satisfies the following properties:

(a) $N(0) = 0$,

(b) let $T_0 = 0$ and $T_n = \inf\{t : N(t) = n\}$ be the n-th arrival time, i.e. the time at which the process jumps from state $n-1$ to state, for n, $i = 1, 2, \ldots$. Then the interarrival times $X_n = T_n - T_{n-1}$ are i.i.d. exponential random variables with rate λ.

➔ **Notation:** If $N = \{N(t) : t \geq 0\}$ is a Poisson Process (i.e. it satisfies the conditions of Definitions 1, 2 or 3), we will write

$$N \sim \mathsf{PP}(\lambda)$$

What does $o(h)$ mean?

Recall: Our initial definition of the Poisson process (definition 6.14 of the Baron textbook and definitions 1 or 1* in the Poisson Process notes) includes the expression $o(h)$ but what does this mean, exactly?

Landau symbols

Considera continuous variable x tending to some limit $x_0 \in \mathbb{R} \cup \{-\infty, \infty\}$ and two functions, $f(x)$ and $g(x) > 0$. The **Landau symbols**, commonly known as **big-O** and **little-o** are defined as follows:

- $f(x) = O(g(x)) \Leftrightarrow |f(x)| < Mg(x)$ for some constant M and all values of x
- $f(x) = o(g(x)) \Leftrightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$

What does $o(h)$ mean?

We are interested in the case $x_0 = 0$ and $g(x) = x$, so, $f(x) = o(x)$ can be informally interpreted as f *goes to zero faster than* x, since

- $f(x) = o(x) \Rightarrow \lim_{x\to 0} \frac{f(x)}{x} = 0$
- This also implies $\lim_{x\to 0} f(x) = 0$

☞ Notice that $f_1(x) = o(x), f_2(x) = o(x) \Rightarrow f_1(x) + f_2(x) = o(x)$, so we informally write $o(x) + o(x) = o(x)$, which also holds for countable sums.

➔ We now return to the 3 definitions and properties of the **Poisson Process** in the corresponding file available on Moodle

Definition 1 $\Rightarrow$ Definition 2

Poisson marginal distributions

If $N \sim$ PP(λ) according to Definition 1, then $N(t) \sim$ Po (λt) for all $t > 0$, as specified in Definition 2. , i.e.

$$\mathbb{P}\left[N(t) = j\right] = \frac{(\lambda t)^j}{j!} e^{-\lambda t}, \quad j = 0, 1, 2, \ldots$$

Indeed, we can condition use the law of total probability to obtain the distribution of $N(t+h)$ by conditioning on $N(t)$:

$$\begin{aligned} \mathbb{P}\left[N(t+h) = j\right] &= \sum_{i=0}^{\infty} \mathbb{P}\left[N(t+h) = j | N(t) = i\right] \mathbb{P}\left[N(t) = i\right] \\ &= \sum_{i=0}^{\infty} \mathbb{P}\left[N(t) = i\right] \mathbb{P}\left[(j-i) \text{ arrivals in } (t, t+h]\right] \end{aligned}$$

Definition 1 $\Rightarrow$ Definition 2

We now split the sum by considering the cases in Definition 1.c:

$$\begin{aligned}
\mathbb{P}\left[N(t+h)=j\right] =& \mathbb{P}\left[N(t)=j\right]\mathbb{P}\left[\text{no arrivals in } (t,t+h]\right] \\
& + \mathbb{P}\left[N(t)=j-1\right]\mathbb{P}\left[\text{one arrival in } (t,t+h]\right] \\
& + \sum_{i=2}^{j} \mathbb{P}\left[N(t)=j-i\right]\mathbb{P}\left[\text{i arrivals in } (t,t+h]\right] \\
& = \mathbb{P}\left[N(t)=j\right]\left[1-\lambda h - o(h)\right] \\
& + \mathbb{P}\left[N(t)=j-1\right]\left[\lambda h + o(h)\right] \\
& + \sum_{i=2}^{j} \mathbb{P}\left[N(t)=j-i\right] o(h)
\end{aligned}$$

Letting, $p_j(t) = \mathbb{P}\left[N(t)=j\right]$, for $j = 0, 1, \ldots$, we have

$$\begin{aligned}
p_j(t+h) &= \lambda h p_{j-1}(t) + (1-\lambda h)p_j(t) + o(h), \quad j > 0, \\
p_0(t+h) &= (1-\lambda h)p_0(t) + o(h)
\end{aligned}$$

Definition 1 $\Rightarrow$ Definition 2

Subtracting $p_j(t)$ and dividing by h on both sides of the equalities and letting h go to 0, we obtain a system of differential equations:

$$\begin{aligned} p_j'(t) &= \lambda p_{j-1}(t) - \lambda p_j(t), \quad j \neq 0, \\ p_0'(t) &= -\lambda p_0(t), \end{aligned}$$

with boundary condition (given by condition **(1.a)**),

$$p_j(0) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

☞ We will encountere similar differential equations in the context holding or sojourn times of general CTMCs

Definition 1 $\Rightarrow$ Definition 2

- It is (relatively) easy to see, for $j = 0$, that $p_0(t) = e^{-\lambda t}$. Substituting in the equation for $j = 1$, we obtain

$$p_1'(t) = \lambda e^{-\lambda t} - \lambda p_1(t).$$

- Once again, this differential equation can be solved by integration and a change of variable, leading to

$$p_1(t) = \lambda t e^{-\lambda t}.$$

  Substituting in the equation for $j = 2$, we obtain

$$p_2'(t) = \lambda t e^{-\lambda t} - \lambda p_2(t).$$

Definition 1 $\Rightarrow$ Definition 2

- This time, the solution requires more work, and integration by parts, but it can be shown to be:

$$p_2(t) = \frac{(\lambda t)^2}{2} e^{-\lambda t}.$$

$$\vdots$$

- Iterating, we find the general solution:

$$p_j(t) = \frac{(\lambda t)^j}{j!} e^{-\lambda t}.$$

This corresponds to the probability mass function of a Poisson random variable with rate parameter λt.

Definition 1 $\Rightarrow$ Definition 2

Poisson distribution of the increments

If $N \sim \mathsf{PP}(\lambda)$ then the increments $N(t+h) - N(h) \sim \mathsf{Po}\,(\lambda t)$ for all $t > 0$ and $h \geq 0$.

Indeed, by conditions **(1.a)** and **(1.b)**,

$$\mathbb{P}\left[N(t+h) - N(h) = j\right] = \mathbb{P}\left[N(t) - N(0) = j\right] = \mathbb{P}\left[N(t) = j\right] = p_j(t)$$

☞ If $N \sim \mathsf{Po}\,(\lambda)$, the unit-time increment $N(h+1) - N(h)$ has a Poisson distribution with rate λ. Therefore, the **intensity parameter** Poisson process is often also called the **rate parameter**.

Definition 2 $\Rightarrow$ Definition 3

We now consider the sequence of random variables $T_0, T_1, \ldots$ given by

$$T_0 = 0, \quad T_n = \inf\{t : N(t) = n\}$$

- This is called the sequence of **arrival times**, since T_n represents the time of the n-th arrival.

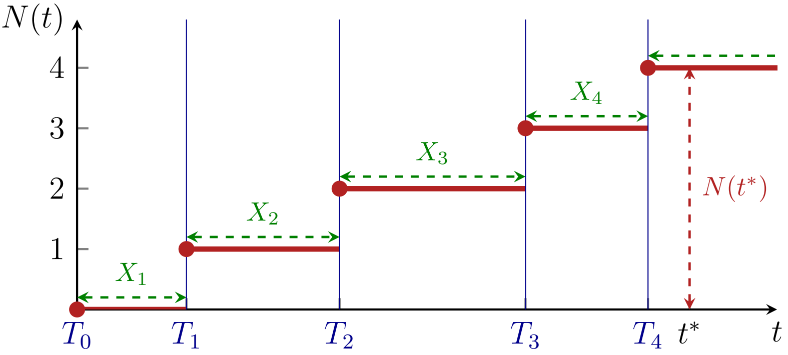
And the sequence of random variables $X_1, X_2, \ldots$ given by

$$X_n = T_n - T_{n-1}$$

- This is called the sequence of **interarrival times**, since X_n represents the time between the n-th and the $(n-1)$-th arrivals.

Definition 2 $\Rightarrow$ Definition 3

Example: A path of a Poisson process in the interval $[0, t]$



Definition 2 $\Rightarrow$ Definition 3

Exponential interarrival times

If $N \sim$PP(λ), the interarrival times $X_1, X_2, \ldots$ are independent and identically distributed exponential random variables with parameter λ

Remember the inevitability of the exponential distribution, discussed previously? ➔ This will be generalized to the distribution of the holding times when we study general homogeneous continuous time Markov chains (condition **(2.b)** implies the Markov property)

☞ Clearly, knowledge of N allows us to calculate the interarrival times, so Definition 2 implies Definition 3.

Definition 3 $\Rightarrow$ Definition 1

- Definition 3 tells us that the converse is also true, i.e. if $X_1, X_2, \ldots$ is a sequence of i.i.d. Exp(λ) random variables, the process N defined by,
$$N(t) = \max\{n : T_n \leq t\}, \quad \text{for } T_n = \sum_{i=1}^{n} X_i,$$
is a Poisson process with parameter λ.

Indeed, noting that
$$N(t) \geq n \quad \text{if and only if} \quad T_n \leq t,$$
we can find
$$\begin{aligned}
\mathbb{P}\left[N(t) = 1\right] &= \mathbb{P}\left[T_1 \leq t \leq T_2\right] = \mathbb{P}\left[X_1 \leq t \leq X_1 + X_2\right] \\
&= \int_0^\infty \mathbb{P}\left[X_1 \leq t \leq X_1 + X_2 | X_1 = s\right] \lambda e^{-\lambda s} ds \\
&= \lambda \int_0^t \mathbb{P}\left[X_2 > t - s\right] e^{-\lambda s} ds = \lambda \int_0^t e^{-\lambda(t-s)} e^{-\lambda s} ds \\
&= \lambda t e^{-\lambda t} = \lambda t + \lambda t(e^{-\lambda t} - 1).
\end{aligned}$$

Definition 3 $\Rightarrow$ Definition 1

But

$$\lambda t(e^{-\lambda t} - 1) \to 0 \quad \text{and} \quad \frac{\lambda t(e^{-\lambda t} - 1)}{t} \to 0 \quad \text{as} \quad t \downarrow 0,$$

so $\lambda t(e^{-\lambda t} - 1) = o(t)$ and $\mathbb{P}\left[N(t) = 1\right] = \lambda t + o(t)$.

Similarly,

$$\begin{aligned}
\mathbb{P}\left[N(t) \geq 2\right] &= \mathbb{P}\left[T_2 \leq t\right] = \mathbb{P}\left[X_1 + X_2 \leq t\right] \\
&= \int_0^\infty \mathbb{P}\left[X_1 + X_2 \leq t | X_1 = s\right] \lambda e^{-\lambda s} ds \\
&= \lambda \int_0^t \mathbb{P}\left[X_2 \leq t - s\right] e^{-\lambda s} ds = \lambda \int_0^t \left(1 - e^{-\lambda(t-s)}\right) e^{-\lambda s} ds \\
&= \lambda \int_0^t \left(e^{-\lambda s} - e^{-\lambda t}\right) ds = 1 - e^{-\lambda t} - \lambda t e^{-\lambda t}.
\end{aligned}$$

Definition 3 $\Rightarrow$ Definition 1

But

$$1 - e^{-\lambda t} - \lambda t e^{-\lambda t} \to 0 \quad \text{and} \quad \frac{1 - e^{-\lambda t} - \lambda t e^{-\lambda t}}{t} \to 0 \quad \text{as} \quad t \downarrow 0,$$

so $1 - e^{-\lambda t} - \lambda t e^{-\lambda t} = \mathbb{P}[N(t) \geq 2] = o(t)$.
Finally, $\mathbb{P}[N(t) = 0] = 1 - \mathbb{P}[N(t) = 1] + \mathbb{P}[N(t) \geq 2] = 1 - \lambda t + o(t)$, so Definition 3 implies Definition 1.

☞ We have shown that

$$\mathbf{(1)} \Rightarrow \mathbf{(2)} \Rightarrow \mathbf{(3)} \Rightarrow \mathbf{(1)},$$

so the three definitions are equivalent! Together, they provide different ways to view the Poisson Process and study its features.

Properties of the Poisson process

- Later, we will see that the Poisson process is a particular case of a homogeneous CTMC characterized by transition probabilities

$$p_{m,n+m}(h) = \mathbb{P}\left[N(t+h) = n+m | N(t) = m\right]$$

satisfying

$$p_{m,n+m}(h) = \begin{cases} 1 - \lambda h + o(h) & \text{if } n = 0, \\ \lambda h + o(h) & \text{if } n = 1, \\ \lambda o(h) & \text{if } n > 1. \end{cases}$$

- We have shown that

$$\begin{aligned} p_{m,n+m}(h) &= \mathbb{P}\left[N(t+h) = n+m | N(t) = m\right] \\ &= \mathbb{P}\left[N(h) = n | N(0) = 0\right] = \mathbb{P}\left[N(h) = n\right] \end{aligned}$$

and we know $N(t) \sim \text{Po}\left(\lambda t\right)$

Properties of the Poisson process

- The process can, alternatively, be characterized by the holding times, also called interarrival times, $X_i \overset{i.i.d.}{\sim} \mathrm{Exp}\left(\lambda\right)$

- This results in arrival times

$$T_n = \sum_{i=1}^{n} X_i \sim \mathrm{Gamma}\left(n, \lambda\right).$$

Superposition

Superposition of Poisson processes

Let $N_1 = \{N_1(t) : t \geq 0\}$ and $N_2 = \{N_2(t) : t \geq 0\}$ be two independent Poisson processes with intensities λ_1 and λ_2, respectively. Then, the process $N = \{N(t) : t \geq 0\}$ given by $N(t) = N_1(t) + N_2(t)$ for al $t \geq 0$, is also a Poisson process with intensity $\lambda = \lambda_1 + \lambda 2$.

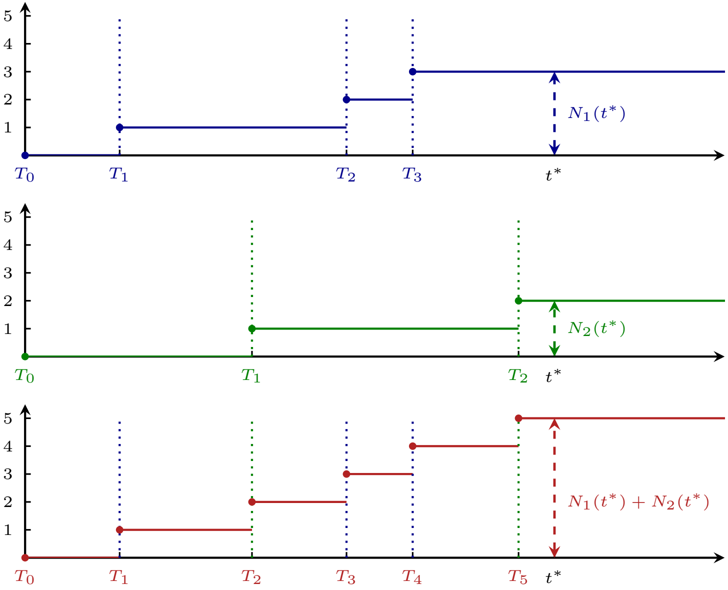
- The proof is left as an exercise!

- The result can be extended to the sum of a countable number of poisson process:

$$N_i \overset{ind}{\sim} \mathsf{PP}(\lambda_i) \quad \Rightarrow \quad N = \sum_{i=1}^{k} N_i \sim \mathsf{PP}\left(\sum_{i=1}^{k} \lambda_i\right)$$

- It can also be extended to an uncountable sum of independent Poisson processes, provided the sum of the rates is finite.

Superposition



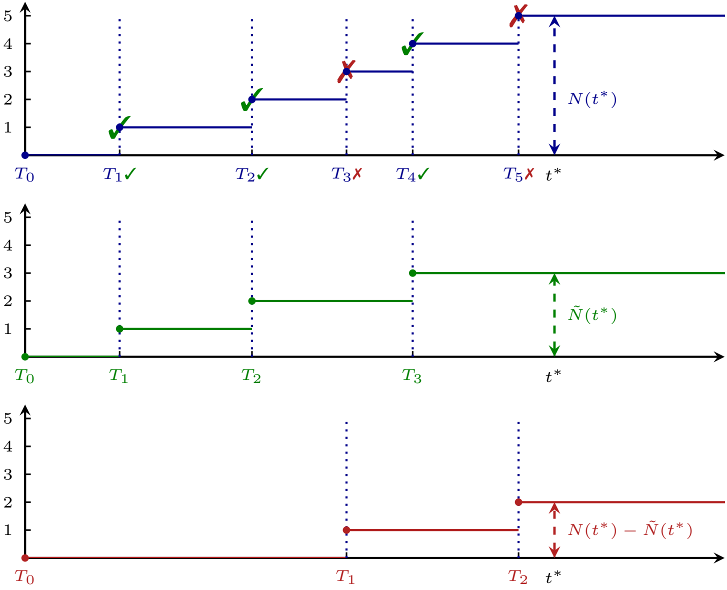
Thinning or Splitting

Thinning or splitting a Poisson process

Let $N = \{N(t) : t \geq 0\}$ be a Poisson processes with intensity λ and let $\tilde{N} = \{\tilde{N}(t) : t \geq 0\}$ be the process obtained by retaining each arrival event with probability p, and discarding it with probability $1 - p$. Then, $\tilde{N} \sim \mathrm{PP}(\lambda p)$. Furthermore, the process $M = \{M(t) : t \geq 0\}$ formed by the discarded arrivals, i.e. $M(t) = N(t) + \tilde{N}(t)$ for al $t \geq 0$, is also a Poisson process with intensity $\lambda(1 - p)$ and independent of $\tilde{N}$.

☞ The proof is left as an exercise!

Thinning or Splitting



Thinning or Splitting

- The result can be extended to a countable number of thinned processes, i.e.

$$\tilde{N}_i \overset{ind}{\sim} \mathsf{PP}(\lambda p_i), \quad \text{for } p_i > 0, \sum_{i=1}^{k} p_i = 1,$$

provided each arrival of the original process N is assigned, idependently, to a single process $\tilde{N}_i$ with probability p_i.

Mean and Covariance functions

We have seen that the marginal distributions of the Poisson process are Poisson, i.e.

$$N \sim \mathsf{PP}(\lambda) \quad \Rightarrow \quad N(t) \sim \mathsf{Po}\,(\lambda t)$$

So, clearly, the Poisson process is NOT stationary:

- $\mu_N(t) = \mathbb{E}\,[N(t)] = \lambda t$ depends on t
- $\sigma^2_N(t) = \mathsf{Var}\,[N(t)] = \lambda t$ depends on t

We can also calculate the covariance function of the process. For $s > t$

$$\begin{aligned}\sigma_N(t,s) &= \mathrm{Cov}[N(t), N(s)] = \mathrm{Cov}[N(t), N(t) + N(s) - N(t)] \\ &= \mathrm{Cov}[N(t), N(t)] + \mathrm{Cov}[N(t), N(s) - N(t)] = \\ &\mathsf{Var}\,[N(t)] + 0 = \lambda t\end{aligned} \tag{1}$$

By a similar argument, for $s < t$, $\sigma_N(t,s) = \lambda s$. So, in general

$$\sigma_N(t,s) = \lambda \min\{t, s\}$$

Examples

☞ **CD Example 7.25** Database queries to a certain data warehouse occur randomly throughout the day. On average, 0.8 queries arrive per second during regular business hours. Assume a Poisson process model is applicable here.

- What is the probability of exactly 1 query in the first second and exactly 2 queries in the four seconds thereafter

➔ Let $N(t)$ denote the number of queries in the first t seconds. Then, $N \sim \text{PP}(0.8)$ and $N(1), N(5) - N(1)$ are independent Poisson random variables with means $0.8 \cdot 1 = 0.8$ and $0.8 \cdot 4 = 3.2$, respectively. Hence,

$$\mathbb{P}\left[N(1) = 1, N(5) - N(1) = 2\right] = \frac{e^{-0.8} 0.8^1}{1!} \frac{e^{-3.2} 3.2^2}{2!} = 0.075$$

Examples

- What is the probability of exactly 1 query in the first second and exactly 2 queries in the first 5 seconds?

➔ This time, the two time intervals of interest are not disjoint, so we need to condition in order to find the required probability

$$
\begin{aligned}
\mathbb{P}\left[N(1)=1, N(5)=2\right] &= \mathbb{P}\left[N(1)=1\right]\mathbb{P}\left[N(5)=2|N(1)=1\right] \\
&= \mathbb{P}\left[N(1)=1\right]\mathbb{P}\left[N(5)-N(1)=1\right] \\
&= \frac{e^{-0.8}0.8^1}{1!}\frac{e^{-3.2}3.2^1}{1!} = 0.0469
\end{aligned}
$$

☞ Because the intervals $[0,1]$ and $[0,5]$ are overlapping, the variables $N(1)$ and $N(5)$ are correlated, with $\sigma_N(1,5) = 0.8 \min 1, 5 = 0.8$ As one might expect, the covariance is positive, since an increase in $N(1)$ is clearly related to an increase in $N(5)$

Examples

☞ **CD Example 7.27** Two roads feed into the northbound lanes on the Anderson Street Bridge. During rush hour, the number of vehicles arriving from the first road can be modeled by a Poisson process with a rate parameter of 10 per minute, while arrivals from the second road form an independent Poisson process with rate 8 cars per minute.

- What is the probability that a total of more than 100 vehicles will arrive via the two feeder roads in the first 5 min of rush hour?

➔ Let $N(t)$ denote the total number of cars entering the northbound lanes. Then, by the superposition property, $N \sim \text{PP}(10 + 8 = 18)$ and

$$\mathbb{P}\left[N(5) > 100\right] = 1 - \mathbb{P}\left[N(5) \leq 100\right] = 1 - \text{ppois}(100, 5 * 18) = 0.1349$$

Examples

CD Example 7.28 At a certain large hospital, patients enter the emergency room at a mean rate of 15 per hour. Suppose 20% of patients arrive in critical condition, i.e., they require immediate treatment. Assume patient arrivals meet the conditions of a Poisson process.

- What is the probability that more than 50 patients arrive in the next 4 hours?

➔ Let $N(t)$ denote the total number of patient arrivals (regardless of condition). Then $N(4) \sim \mathsf{Po}\,(15 * 4 = 60)$, so

$$\mathbb{P}\left[N(4) > 50\right] = 1 - \mathbb{P}\left[N(4) \leq 50\right] = 1 - \mathrm{ppois}(50, 60) = 0.8923$$

- What is the probability that more than 10 critical patients arrive in the next 4 hours?

Examples

- Let $N_c(t)$ denote the number of critical patients arriving within t hours. Then $N_c \sim \text{PP}(0.2 \cdot 15 = 3)$, thus

$$\mathbb{P}\left[N_c(4) > 10\right] = 1 - \mathbb{P}\left[N_c(4) \leq 10\right] = 1 - \text{ppois}(10, 3 * 4) = 0.6528$$

- What is the probability that more than 10 critical patients and more than 40 noncritical patients arrive in the next 4 hours?

- Let $N_n(t)$ denote the number of noncritical patients arriving within t hours. Then $N_n \sim \text{PP}(0.8 \cdot 15 = 12)$. Moreover, $N_n(4)$ and $N_c(4)$ are independent, hence

$$\begin{aligned}\mathbb{P}\left[N_c(4) > 10, N_n(4) > 40\right] &= \mathbb{P}\left[N_c(4) > 10\right]\mathbb{P}\left[N_n(4) > 40\right] \\ &= 0.6528 \cdot 0.8617 = 0.5625\end{aligned}$$

Examples

CD Example 7.26 Consider again the database queries described in Example 7.25 (from lesson 10), i.e. $N \sim \text{PP}(0.8)$, where $N(t)$ denotes the number of queries in the first t seconds

- What is the expected waiting time, from the beginning of regular business hours, until the arrival of the 50th query?

➔ Let T_{50} denote the time to the 50th query from the beginning of regular business hours. Then, $T_{50} \sim \text{Gamma}\,(50, 0.8)$, so $\mathbb{E}\left[Y_{50}\right] = 50(1/0.8) = 62.5$

- If 50 or more queries arrive within the first minute, system users will experience a significant backlog in subsequent minutes because of processing time. What is the probability this happens?

➔ A backlog occurs if $Y_{50} \leq 60$ seconds. The probability of this event is

$$\mathbb{P}\left[Y_{50} \leq 60\right] = \text{pgamma}(60, 50, 0.8) = 0.4054$$

Examples

- Alternatively, we may notice that a backlog occurs if the number of queries in the first 60 seconds, is 50 or more, so the same probability can be calculated as

$$\mathbb{P}\left[N(60) \geq 50\right] = 1 - \mathrm{ppois}(49, 60 * 0.8) = 0.4054$$

Poisson Process Simulation

👉 To better understand the Poisson process, an interesting question is: knowing the rate λ, how can we simulate one or more paths (using **R**)?

👉 File **L20-PoissonProcessSimulation.R** available on Moodle

➜ What did we observe?

1. The paths of the process are step functions: the process moves in jumps
2. The times at which the jumps occur are random, but the jumps themselves are deterministic: conditional on the when, the where is not random!
3. The waiting time until the next jump is independent on the current state of the process

➜ The first condition is a consequence of the discrete (countable) state space of the process, but the other two could be "relaxed" to obtain more general continuous time, discrete space processes: **Continous Time Markov Chains (CTMC)**