Applied Probability for Computer Science

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Continuous Time Markov Chains (CTMC)

Introduction

- We will study the main concepts from the textbook **CD Section 7.7**...
- ... plus some additional technical details!

Continuous-time Markov Chain (CTMC)

A family $X = \{X(t) : t \geq 0\}$ of random variables taking values in a countable state space S and indexed by the half-line $[0, \infty)$ is called a **continous-time Markov chain** if it satisfies the **Markov property**:

$$\mathbb{P}\left[X(t_n) = x_j | X(t_1) = x_{i_1}, \ldots, X(t_{n-1}) = x_{i_{n-1}}\right] = \mathbb{P}\left[X(t_n) = x_j | X(t_{n-1}) = x_{i_{n-1}}\right]$$

for all $j, i_1, \ldots, i_{n-1} \in S$ and any sequence $t_1 < t_2 < \cdots < t_n$ of times.

Introduction

- In what follows, we will assume that S is a subset of the integers (or that it can be labeled by such), and we will write $\mathbb{P}\left[X(t_n) = j | X(t_{n-1}) = i_{n-1}\right]$ to simplify notation.
  - In the context of Poisson processes, all transition probabilities were characterized in Definition 1, relying on infinitesimal calculus (limits, derivatives and the "little o" notation)
  - More generally, for Continuous Time Markov Chains, we will define a matrix $\mathbf{G}$, called the *generator* of the chain, which will completely describe the behaviors of the process
  - We will see that there is an alternative way to study continuous-time processes by considering, separately, the random times at which the process *jumps* (transitions from one state to the other) and the discrete-time (*embeded*) random process obtained by listing the transitions of the original process (where it jumps to), without considering the times (when).

The transition semigroup

Homogeneous CTMC

Let $p_{ij}(s,t) = \mathbb{P}\left[X(t) = j | X(s) = i\right]$, for $i, j \in S$ and $s \leq t$ be the **transition probabilities** of a CTMC. The chain is called **homogeneous** if, for each pair (i,j) the transition probabilities depend on time only through the difference $t - s$. In other words:

$$p_{ij}(s,t) = p_{ij}(0, t-s) \text{ for all } i, j \in S, s, t \geq 0.$$

➜ From now on, unless explicitly stated, we will assume that X is a **homogeneous** CTMC (HCTMC) and the $\#S \times \#S$ matrix with entries $p_{ij}(t) := p_{ij}(0,t)$ will be denoted by $\mathbf{P}_t$.

The transition semigroup

Transition semigroup

The family $\{\mathbf{P}_t : t \geq 0\}$ is the **transition semigroup** of the process and it satisfies:

1. $\mathbf{P}_0 = \mathbf{I}$, the identity matrix
2. For every $t > 0$, $\mathbf{P}_t$ is a **stochastic matrix**, i.e. it has non-negative entries and the sum of the elements of each row is 1.
3. The **Chapman-Kolmogorov equations** $\mathbf{P}_{s+t} = \mathbf{P}_s \mathbf{P}_t$ for all $s, t \geq 0$.

- The stochastic semigroup $\{\mathbf{P}_t\}$, together with the distribution of the initial point $X(0)$ determine the behavior of the process $X(t)$

The transition semigroup

$$\mathbf{P}_t = \begin{bmatrix} p_{11}(t) & p_{12}(t) & p_{13}(t) & p_{14}(t) & \dots \\ p_{21}(t) & p_{22}(t) & p_{23}(t) & p_{24}(t) & \dots \\ p_{31}(t) & p_{32}(t) & p_{33}(t) & p_{34}(t) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$p_{ij}(t) = \mathbb{P}\left[X(s+t) = j | X(s) = i\right] = \mathbb{P}\left[X(t) = j | X(0) = i\right]$$

- Each row of the matrix represents the current state (where the chain is moving from) ➔ This is the conditioning state, that is why each row adds up to 1
- Each column represents the future state (where the chain is moving to)

The transition semigroup

Example: Let N is a Poisson process with intensity λ. Then, for all $i = 0, 1, \ldots$

- $p_{ij} = 0$ for all $j < i$
- $p_{ii} = 1 - \lambda t + o(t)$
- $p_{i,i+1} = \lambda t + o(t)$
- $p_{ij} = o(t)$ for all $j \geq i + 2$

So

$$\mathbf{P}_t = \begin{bmatrix} 1 - \lambda t + o(t) & \lambda t + o(t) & o(t) & o(t) & \ldots \\ 0 & 1 - \lambda t + o(t) & \lambda t + o(t) & o(t) & \ldots \\ 0 & 0 & 1 - \lambda t + o(t) & \lambda t + o(t) & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The transition semigroup

Note: We said, in the context of CTMCs, we consider the state space to be $\mathbb{N} = \{1, 2, 3, \ldots\}$ or a subset. Depending on the context, it is sometimes more convenient to consider $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

Furthermore, we have seen that $N_{s+t} - N_s \sim \text{Po}(\lambda t)$, so we can rewrite

$$\mathbf{P}_t = \begin{bmatrix} e^{-\lambda t} & \lambda t e^{-\lambda t} & \frac{(\lambda t)^2}{2} e^{-\lambda t} & \frac{(\lambda t)^3}{6} e^{-\lambda t} & \cdots \\ 0 & 1e^{-\lambda t} & \lambda t e^{-\lambda t} & \frac{(\lambda t)^2}{2} e^{-\lambda t} & \cdots \\ 0 & 0 & e^{-\lambda t} & \lambda t e^{-\lambda t} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Notice that the condition

$$p_{ij}(t) = \mathbb{P}[X(s+t) = j | X(s) = i] = \mathbb{P}[X(t) = j | X(0) = i]$$

is still satisfied, even if for the Poisson process, the events $X(0) = i$ have probability zero for all $i > 0$

The transition semigroup

Standard semigroup

The semigroup $\{\mathbf{P}_t\}$ is called **standard** if $\mathbf{P}_t \to \mathbf{I}$ as $t \downarrow 0$. In other words, as $t \downarrow 0$, $p_{ii}(t) \to 1$ and, for $i \neq j$, $p_{ij}(t) \to 0$.

Note that the semigroup is standard if and only if its elements, $p_{ij}(t)$ are continuous functions of t.

➜ From now on, unless explicitly stated, we will assume that $\{\mathbf{P}_t\}$ is a **standard** semigroup.

The generator

Suppose that, at time $t \geq 0$, the chain X is at state i. Let $h > 0$ be small and consider what could happen in the small time interval $(t, t+h)$:

- no change in the state of the process, with probability $p_{ii}(h) + o(h)$, where $o(h)$ is an error term taking into account the possibility that the chain moves out of i and back, within the time interval.
- the chain arrives at state j with probability $p_{ij}(h) + o(h)$ where $o(h)$ is an error term taking into account the possibility that the chain moves into j, then out and back in, within the time interval.

  👉 In mathematics, the expression $o(h)$ is used to indicate a quantity such that $o(h) \to 0$ and $o(h)/h \to 0$ as $t \to 0$.

The generator

Generator

There exist constants $\{g_{ij} : i, j \in S\}$ such that, for sufficiently small h,

$$p_{ij}(h) \approx g_{ij}h \text{ if } i \neq j, \quad p_{ii} \approx 1 + g_{ii}h.$$

The matrix with entries g_{ij} is denoted by $\mathbf{G}$ and called the **generator** of the HCTMC.

➜ The **generator** $\mathbf{G}$ takes over the role of the transition matrix $\mathbf{P}$ for discrete-time chains, allowing to characterize the behavior of the process X.

The generator

The existence of the $\{g_{ij} : i, j \in S\}$ can be proved and it implies the differentiability of $p_{ij}(t)$ at $t = 0$. In matrix form, we can write:

$$\lim_{h \downarrow 0} \frac{\mathbf{P}_h - \mathbf{I}}{h} = \mathbf{G}$$

and, in particular

$$\lim_{h \downarrow 0} \frac{p_{ij}(h)}{h} = g_{ij} \quad \text{for all } i \neq j.$$

The $\{g_{ij}\}$ are usually called the **instantaneous transition rates**.

Assume $X(t) = i$, then, for sufficiently small h

$$X(t+h) = \begin{cases} i & \text{with probability } 1 + g_{ii}\, h + o(h) \\ j \neq i & \text{with probability } g_{ij}\, h + o(h) \end{cases}$$

The generator

Assuming that h is small enough for the error terms to be negligible, we have

$$1 = \sum_{j \in S} p_{ij}(h) \approx 1 + g_{ii}h + \sum_{j \neq i} g_{ij}h = 1 + h \sum_{j \in S} g_{ij},$$

which leads to the equation

$$\sum_{j \in S} g_{ij} = 0$$

or, in matrix form $\mathbf{G1} = \mathbf{0}$, where $\mathbf{1}$ and $\mathbf{0}$ are column vectors of ones and zeros, respectively.
In other words, the rows of $\mathbf{G}$ add up to 0 and we can write

$$g_{ii} = - \sum_{j \in S : j \neq i} g_{ij}.$$

The $\{g_{ii}\}$ are sometimes called the **instantaneous exit rates**.

The generator

The generator of a process is, therefore, a square matrix with:

- The instantaneous exit rates on the diagonal
- The instantaneous transition rates outside the diagonal

👉 **Note:** When $\#S$ is infinite some issues may arise if $-g_{ii}$ is infinite for one or more $i \in S$, so an additional condition to make things work is that $\sup_{i \in S}\{-g_{ii}\} < \infty$. In this course, we will focus on cases for which this condition is satisfied.

👉 In some context, the generator is denoted by $\mathbf{Q}$ and called the **Q-matrix** of the process.

Example: Birth process

Birth process

A **birth process** with intensities $\lambda_0, \lambda_1, \lambda_2, \ldots$ is a stochastic process $N = \{N(t) : t \geq 0\}$ taking values in $S = \{0, 1, 2, \ldots\}$ such that:

1. It is positive and non-decreasing: $N(0) \geq 0$ and $N(s) \leq N(t)$ for $s < t$
2. $$\mathbb{P}\left[N(t+h) = n + m | N(t) = n\right] = \begin{cases} \lambda_n h + o(h) & \text{if } m = 1 \\ o(h) & \text{if } m > 1 \\ 1 - \lambda_n h + o(h) & \text{if } m = 0 \end{cases}$$
3. Given $N(s)$, the increment $N(t) - N(s)$ is independent of all arrivals prior to s, for all $s < t$.

By the second and third conditions, N is an HCTMC.

Example: Birth process

The generator of a birth process, is:

$$\mathbf{G} = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \dots \\ 0 & -\lambda_1 & \lambda_1 & 0 & 0 & \dots \\ 0 & 0 & -\lambda_2 & \lambda_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

☞ The **Poisson process** is a particular case, with $\lambda_n = \lambda$ for all n

Forward and backward equations

We have seen that $\mathbf{G}$ can be obtained from the $\{\mathbf{P}_t\}$. The converse is also usually true:

Suppose that $X(0) = i$ for some $i \in S$. Then, by the Chapman-Kolmogorov equations, for sufficiently small h, considering the transition probabilities for $X(t+h)$ given $X(t)$ yields

$$\begin{aligned} p_{ij}(t+h) &= \sum_{k \in S} p_{ik}(t) p_{kj}(h) \\ &\approx p_{ij}(t)(1 + g_{jj}h) + \sum_{k \in S: k \neq j} p_{ik}(t) g_{kj} h \\ &= p_{ij}(t) + h \sum_{k \in S} p_{ik}(t) g_{kj} \end{aligned}$$

Therefore

$$\frac{p_{ij}(t+h) - p_{ij}(t)}{h} = \sum_{k \in S} p_{ik}(t) g_{kj}$$

Forward and backward equations

Letting $h \downarrow 0$, the left hand side of the equality becomes $p'_{ij}(t)$, the (right) derivative of $p_{ij}(t)$ with respect to t and the right hand side is unaffected.

Forward equations

For all $i, j \in S$ and $t \geq 0$ we have that

$$p'_{ij}(t) = \sum_{k \in S} p_{ik}(t) g_{kj}$$

Or, in matrix notation, $\mathbf{P}'_t = \mathbf{P}_t \mathbf{G}$

A similar calculation can be made considering the transition probabilities for $X(t+h)$ given $X(h)$:

Forward and backward equations

$$\begin{aligned} p_{ij}(t+h) &= \sum_{k\in S} p_{ik}(h)p_{kj}(t) \\ &\approx (1+g_{ii}h)p_{ij}(t) + \sum_{k\in S:k\neq j} g_{ik}hp_{kj}(t) \\ &= p_{ij}(t) + h\sum_{k\in S} g_{ik}p_{kj}(t) \end{aligned}$$

Backward equations

For all $i, j \in S$ and $t \geq 0$ we have that

$$p'_{ij}(t) = \sum_{k\in S} g_{ik}p_{kj}(t)$$

Or, in matrix notation, $\mathbf{P}'_t = \mathbf{G}\mathbf{P}_t$

Example: A two-state process

Consider a two-state chain X with $S = \{1, 2\}$ and generator

$$\mathbf{G} = \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix}$$

The forward equations give

$$\begin{aligned} p'_{11}(t) &= p_{11}(t)g_{11} + p_{12}(t)g_{21} \\ &= -\mu p_{11} + \lambda p_{12} \end{aligned} \qquad \begin{aligned} p'_{12}(t) &= p_{11}(t)g_{12} + p_{12}(t)g_{22} \\ &= \mu p_{11} - \lambda p_{12} \end{aligned}$$

$$\begin{aligned} p'_{22}(t) &= p_{21}(t)g_{12} + p_{22}(t)g_{22} \\ &= \mu p_{21} - \lambda p_{22} \end{aligned} \qquad \begin{aligned} p'_{21}(t) &= p_{21}(t)g_{11} + p_{22}(t)g_{12} \\ &= -\mu p_{21} + \lambda p_{22} \end{aligned}$$

With the initial condition given by the semigroup property $\mathbf{P}_0 = \mathbf{I}$:

$$p_{11}(0) = p_{22}(0) = 1, \quad p_{12}(0) = p_{21}(0) = 0$$

Example: A two-state process

- **Solving the forward equations directly.** First, we notice that by the semigroup properties of $\{\mathbf{P}_t\}$,

$$p_{11}(t) + p_{12}(t) = 1, \quad p_{21}(t) + p_{22}(t) = 1,$$

so it is sufficient to solve two differential equations.

We consider the system:

$$\begin{aligned} & p'_{11}(t) = -\mu p_{11} + \lambda p_{12} = \lambda - (\mu + \lambda) p_{11}(t), \quad p_{11}(0) = 1 \\ \Leftrightarrow \quad & 1 = \frac{p'_{11}(t)}{\lambda - (\mu + \lambda) p_{11}(t)}, \quad p_{11}(0) = 1 \\ \Leftrightarrow \quad & \int_0^t ds = \int_0^t \frac{p'_{11}(s) ds}{\lambda - (\mu + \lambda) p_{11}(s)} \end{aligned}$$

Example: A two-state process

The integral on the left-hand side is easily solved and equal to t. To solve the integral on the right-hand side, we can use a change of variable:

$$u = \lambda - (\mu + \lambda)p_{11}(s) \quad \Rightarrow \quad du = -(\mu + \lambda)p'_{11}(s)$$

Therefore,

$$\begin{aligned}\int \frac{p'_{11}(s)ds}{\lambda - (\mu + \lambda)p_{11}(s)} &= \frac{-1}{\mu + \lambda}\int \frac{du}{u} = \frac{-1}{\mu + \lambda}\log u \\ &= \frac{-1}{\mu + \lambda}\log\left[\lambda - (\mu + \lambda)p_{11}(s)\right]\end{aligned}$$

And,

$$\begin{aligned}t &= \frac{-1}{\mu + \lambda}\left(\log\left[\lambda - (\mu + \lambda)p_{11}(t)\right] - \log\left[\lambda - (\mu + \lambda)p_{11}(0)\right]\right) \\ &= \frac{-1}{\mu + \lambda}\left(\log\left[\lambda - (\mu + \lambda)p_{11}(t)\right] - \log\left[\lambda - (\mu + \lambda)1\right]\right)\end{aligned}$$

Example: A two-state process

By the properties of the natural logarithm, this gives:

$$t = \frac{-1}{\mu + \lambda} \log \left[\frac{\lambda - (\mu + \lambda) p_{11}(t)}{-\mu} \right]$$

Finally, solving for $p_{11}(t)$, we obtain

$$p_{11}(t) = \frac{\lambda + \mu e^{-(\mu+\lambda)t}}{\mu + \lambda}$$

$$p_{12}(t) = 1 - p_{11}(t) = \frac{\mu + \mu e^{-(\mu+\lambda)t}}{\mu + \lambda}$$

Exercise: Show that an analogous procedure leads to

$$p_{22}(t) = \frac{\mu + \lambda e^{-(\mu+\lambda)t}}{\mu + \lambda}; \quad p_{21}(t) = \frac{\lambda + \lambda e^{-(\mu+\lambda)t}}{\mu + \lambda}$$

And show that the backward equations lead to the same solution.

Matrix Exponential

Matrix Exponential

Let $\mathbf{A}$ be a square matrix. The **matrix exponential** $e^{\mathbf{A}}$ is the square matrix of the same size as $\mathbf{A}$ given by

$$e^{\mathbf{A}} = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{A}^n = \mathbf{I} + \mathbf{A} + \frac{1}{2}\mathbf{A}^2 + \frac{1}{6}\mathbf{A}^3 + \dots$$

- The matrix exponential is the matrix version of the exponential function and reduces to the ordinary exponential function e^a when $\mathbf{A} = a$ is a 1×1 matrix.
- The matrix $e^{\mathbf{A}}$ is well-defined, as its defining series converges for any square matrix $\mathbf{A}$.

Matrix Exponential

The matrix exponential satisfies many of the properties of the exponential function:

1. $e^{\mathbf{0}} = \mathbf{I}$ where $\mathbf{0}$ is a square matrix with all entries equal to zero.
2. $e^{\mathbf{A}} e^{-\mathbf{A}} = \mathbf{I}$
3. $e^{s\mathbf{A}} e^{t\mathbf{A}} = e^{(s+t)\mathbf{A}} = e^{(t+s)\mathbf{A}} = e^{t\mathbf{A}} e^{s\mathbf{A}}$
4. If $\mathbf{AB} = \mathbf{BA}$, then $e^{\mathbf{AB}} = e^{\mathbf{A}} e^{\mathbf{B}} = e^{\mathbf{B}} e^{\mathbf{A}}$
5. $\frac{d}{dt} e^{t\mathbf{A}} = \mathbf{A} e^{t\mathbf{A}} = e^{t\mathbf{A}} \mathbf{A}$

Forward and backward equations: general solution

Both the forward and backward equations are systems of differential equations.

Forward and backward equations: general solution

Subject to the (boundary) condition $\mathbf{P}_0 = \mathbf{I}$, satisfied by the transition semigroup, the forward and backward equations, $\mathbf{P}_t' = \mathbf{P}_t\mathbf{G}$ and $\mathbf{P}_t' = \mathbf{G}\mathbf{P}_t$ (under some technical conditions which we will assume) have a unique solution:

$$\mathbf{P}_t = e^{t\mathbf{G}}$$

Indeed, letting $\mathbf{P}_t = e^{t\mathbf{G}}$ gives

$$\begin{aligned}\mathbf{P}_t' = \frac{d}{dt}e^{t\mathbf{G}} &= \mathbf{G}e^{t\mathbf{G}} = e^{t\mathbf{G}}\mathbf{G} \\ &= \mathbf{P}_t\mathbf{G} = \mathbf{G}\mathbf{P}_t.\end{aligned}$$

Example: Two-state process revisited

In **R**, *advanced* matrix operations, such as the matrix powers and matrix exponential can be obtained using the **expm** package. You will have to install it the first time you use it:

```
> install.packages("expm")
```

➜ It works for numerical matrices only. For example, letting $\lambda = 1$ and $\mu = 2$, we can define $\mathbf{P}_t = e^{t\mathbf{G}}$ as a function of t:

```
> library("expm")
> lambda <- 1
> mu <- 2
```

- The generator:

```
> G = matrix(c(-mu, mu, lambda, -lambda),nrow=2,byrow=T)
```

Example: Two-state process revisited

```
> G
     [,1] [,2]
[1,]   -2    2
[2,]    1   -1
```

- The transition semigroup, as a function of t:

```
> P_t <- function(t) expm(t*G)
```

- Verifying that the initial condition is satisfied:

```
> P_t(0)
     [,1] [,2]
[1,]    1    0
[2,]    0    1
```

- Calculating the transition probabilities for a fixed value of t, in this case $t = 3$:

```
> P_t(3)
```

Example: Two-state process revisited

```
           [,1]      [,2]
[1,] 0.3334156 0.6665844
[2,] 0.3332922 0.6667078
```

- We can verify that the matrix exponential yields the same result obtained as the solution of the differential equations for fixed $t(=3)$:

```
> P_3 = matrix(c(lambda+mu*exp(-(mu+lambda)*3),
+                mu-mu*exp(-(mu+lambda)*3),
+                lambda-lambda*exp(-(mu+lambda)*3),
+                mu+lambda*exp(-(mu+lambda)*3)),
+                nrow=2,byrow=T)/(lambda+mu)
> P_3
           [,1]      [,2]
[1,] 0.3334156 0.6665844
[2,] 0.3332922 0.6667078
```

Example: Two-state process revisited

Symbolic software systems, such as **Mathematica** and **Maple**, work in exact integer arithmetic, and can be used to find the matrix exponential when the generator matrix contains symbolic parameters.

➜ **Wolfram Alpha**, (https://www.wolframalpha.com/) which is freely available on the web, has the command **Matrix Exp** for computing the matrix exponential. Try visiting the website and typing the following command on the search bar:

```
Matrix Exp[t*{{ -mu, mu}, {lambda, -lambda}}]
```

👉 Note: The space between the words "Matrix" and "Exp" is important. If the space is missing, the exponential will be applied component by component, which is not what we want. Other formats for the command, such as "matrix exponential" will produce the desired result, as long as the space is present.

Matrix Exponential and diagonalization

Diagonalization

A square matrix $\mathbf{A}$ is **diagonalizable** if it can be written as $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$, where

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_k \end{bmatrix}$$

$\{d_1, \dots, d_k\}$ are the eigenvalues of $\mathbf{A}$ and $\mathbf{V}$ is an invertible matrix whose columns are the corresponding eigenvectors.

Diagonalization can be used to efficiently compute the powers of a matrix $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$:

Matrix Exponential and diagonalization

$$\begin{aligned}\mathbf{A}^n &= \left(\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\right)^n = \left(\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\right)\left(\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\right)\cdots\left(\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\right)\\ &= \mathbf{V}\mathbf{D}\left(\mathbf{V}^{-1}\mathbf{V}\right)\mathbf{D}\left(\mathbf{V}^{-1}\mathbf{V}\right)\cdots\left(\mathbf{V}^{-1}\mathbf{V}\right)\mathbf{D}\mathbf{V}^{-1} = \mathbf{V}\mathbf{D}^n\mathbf{V}^{-1},\end{aligned}$$

since $\mathbf{D}^n$ is simply the diagonal matrix with diagonal elements d_i^n.

The matrix exponential becomes:

$$\begin{aligned}e^{t\mathbf{A}} &= \sum_{n=0}^{\infty}\frac{1}{n!}\mathbf{A}^n = \sum_{n=1}^{\infty}\frac{1}{n!}(\mathbf{V}\mathbf{D}^n\mathbf{V}^{-1})^n\\ &= \mathbf{V}\left(\sum_{n=1}^{\infty}\frac{1}{n!}\mathbf{D}^n\right)\mathbf{V}^{-1} = \mathbf{V}e^{\mathbf{D}}\mathbf{V}^{-1},\end{aligned}$$

where $e^{\mathbf{D}}$ is simply the diagonal matrix with diagonal elements e^{d_i}.

Example: back to the two-state process

The generator

$$\mathbf{G} = \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix}$$

of the two-state HCTMC is diagonalizable with eigenvalues $d_1 = 0$ and $d_2 = -(\lambda + \mu)$ and corresponding eigenvectors $\mathbf{v}_1 = (1, 1)^\intercal$ and $\mathbf{v}_2 = (-\mu, \lambda)^\intercal$.

Therefore, we can write:

$$\mathbf{G} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} = \begin{bmatrix} 1 & -\mu \\ 1 & \lambda \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -(\lambda + \mu) \end{bmatrix} \begin{bmatrix} \lambda/(\lambda + \mu) & \mu/(\lambda + \mu) \\ -1/(\lambda + \mu) & 1/(\lambda + \mu) \end{bmatrix}$$

Example: back to the two-state process

And the transition function for the process is:

$$
\begin{aligned}
\mathbf{P}_t &= e^{t\mathbf{G}} = \mathbf{V} e^{t\mathbf{D}} \mathbf{V}^{-1} \\
&= \begin{bmatrix} 1 & -\mu \\ 1 & \lambda \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-t(\lambda+\mu)} \end{bmatrix} \begin{bmatrix} \lambda/(\lambda+\mu) & \mu/(\lambda+\mu) \\ -1/(\lambda+\mu) & 1/(\lambda+\mu) \end{bmatrix} \\
&= \frac{1}{\lambda+\mu} \begin{bmatrix} 1 & -\mu \\ 1 & \lambda \end{bmatrix} \begin{bmatrix} \lambda & \mu \\ -e^{-t(\lambda+\mu)} & e^{-t(\lambda+\mu)} \end{bmatrix} \\
&= \frac{1}{\lambda+\mu} \begin{bmatrix} \lambda + \mu e^{-t(\lambda+\mu)} & \mu - \mu e^{-t(\lambda+\mu)} \\ \lambda - \lambda e^{-t(\lambda+\mu)} & \mu + \lambda e^{-t(\lambda+\mu)} \end{bmatrix}
\end{aligned}
$$

Example: back to the two-state process

➔ Eigenvalues, eigenvectors and inverses for symbolic matrices can be found via **Wolfram Alpha**, (https://www.wolframalpha.com/). Try the following commands:

```
eigenvalues{{ -mu, mu}, {lambda, -lambda}}
eigenvectors{{ -mu, mu}, {lambda, -lambda}}
diagonalize{{ -mu, mu}, {lambda, -lambda}}
inverse{{ 1, -mu}, {1, lambda}}
```

☞ You may need to recall that, if $\mathbf{v}_i$ is an eigenvector for a matrix $\mathbf{A}$ corresponding to an eigenvalue d_i, then $c\mathbf{v}_i$ is also an eigenvector corresponding to d_i for any $c \in \mathbb{R}$.

Example: back to the two-state process

➜ Eigenvalues, eigenvectors and inverses for numerical matrices can be calculating using **R**:
For example, letting $\lambda = 1$ and $\mu = 2$,

```
> lambda <- 1
> mu <- 2
> G = matrix(c(-mu, mu, lambda, -lambda),nrow=2,byrow=T)
```

- the function **eigen** is used to find the eigenvalues and eigenvectors of $\mathbf{G}$, defining the matrices $\mathbf{V}$ and $\mathbf{D}$:

```
> ev <- eigen(G)
> D <- diag(ev$values)
> D
     [,1] [,2]
[1,]   -3    0
[2,]    0    0
```

Example: back to the two-state process

```
> V <- ev$vectors
> V
           [,1]       [,2]
[1,] -0.8944272 -0.7071068
[2,]  0.4472136 -0.7071068
```

The eigenvalues are always returned in decreasing order according to their size, and each column of the entry vectors corresponds to the elements in the entry values.

We can verify that $\mathbf{VDV}^{-1} = G$, recalling that solve is used to find the inverse of a matrix and %*% indicates the matrix multiplication.

```
> V %*% D %*% solve(V)
     [,1] [,2]
[1,]   -2    2
[2,]    1   -1
```

Example: back to the two-state process

Finally, we can recover the transition matrix for fixed t, say $t = 3$:

```
> P_3 <- V %*% diag(exp(3*ev$values)) %*% solve(V)
> P_3
          [,1]      [,2]
[1,] 0.3334156 0.6665844
[2,] 0.3332922 0.6667078
```

Or define the transition function, depending on t

```
> P_t <- function(t)
+   {V %*% diag(exp(t*ev$values)) %*% solve(V)}
> P_t(3)
          [,1]      [,2]
[1,] 0.3334156 0.6665844
[2,] 0.3332922 0.6667078
```

Holding time

Holding time

Suppose that $X(s) = i$ and let $U_i = \inf\{t \geq 0 : X(s+t) \neq i\}$ be the further time until the chain changes its state. U_i is called a **holding time** or **sojourn time** and it is exponentially distributed with parameter $g_i := -g_{ii}$.

Indeed, by the Markov property and the homogeneity of the chain, we can see that U_i has the lack of memory property. Let $u, t \geq 0$,

$$\begin{aligned}\mathbb{P}\left[U_i > t+u | U > t\right] &= \mathbb{P}\left[U_i > t+u | X(s+t) = i\right] \\ &= \mathbb{P}\left[X(s+t+u) | X(s+t) = i\right] \\ &= \mathbb{P}\left[X(u) | X(0) = i\right] = \mathbb{P}\left[U_i > u\right].\end{aligned}$$

It follows that U_i must follow an exponential distribution, as it is the only continuous distribution with the memoryless property.

Holding time

When $X(s) = i$, the process leaves state i to enter state $j \neq i$ with a rate g_{ij}. Therefore, the total exit rate from state i is

$$g_i := \sum_{j \neq i} g_{ij} = -g_{ii}.$$

In other words, the process *leaves* state i, on average, g_i times per time unit.

- The time between consecutive *exits* from state i is $U_i \sim \text{Exp}\,(g_i)$ (recall the relation between the exponential and the poisson distributions!)

(Conditional) transition probabilities

Suppose that $u < U_i < u + h$ and suppose that the chain jumps only once in the time interval $(u, u + h]$. Then, informally

$$\mathbb{P}[\text{the chain jumps from } i \text{ to } j | \text{the chain jumps}]$$
$$\approx \frac{p_{ij}(h)}{1 - p_{ii}(h)} \to -\frac{g_{ij}}{g_{ii}} \quad \text{as } h \downarrow \infty$$

Transition probability

The (conditional) **transition probability** that a transition to state j occurs (at a given time t) given that a transition out of state i occurs (exactly at time t) is

$$\tilde{p}_{ij} = -\frac{g_{ij}}{g_{ii}} = \frac{g_{ij}}{g_i}$$

We use $\tilde{\mathbf{P}}$ to denote the transition probability matrix.

Irreducible HCTMC

Irreducible HCTMC

A HCTMC X is **irreducible** if for any pair of states, $i, j \in S$, we have that $p_{ij}(t) > 0$.

It can be shown that for a continuous markov chain, either $p_{ij}(t) = 0$ for all $t > 0$, or $p_{ij}(t) > 0$ for all $t > 0$. So a HCTMC is irreducible if the probability of reaching state j from state i in a time t is positive for any $i, j \in S$ and $t > 0$.

- ➜ It is possible to verify the irreducibility for a HCTMC in terms of the generator $\mathbf{G}$

Irreducible HCTMC

Irreducibility condition

Let X be a continuous-time Markov chain with generator $\mathbf{G} = [g_{ij}]$ and transition semigroup $\mathbf{P}_t = \exp\{t\mathbf{G}\}$. X is irreducible if and only if for any pair i, j of states, there exists a sequence $k_1, k_2, \ldots, k_n$ of states such that

$$g_{i,k_1} g_{k_1,k_2} \cdots g_{k_n,j} \neq 0$$

Example: The **birth process** is not irreducible, since it is non-decreasing.

Example: The **two-states process** is irreducible for any $\mu > 0$, $\lambda > 0$.

Example: Birth-death process

Birth-death process

Suppose that the number $X(t)$ of individuals alive in some population at time t evolves in the following way:

1. X is a Markov chain taking values in $\{0, 1, 2, \ldots\}$,
2. the infinitesimal transition probabilities are given by

$$\mathbb{P}\left[X(t+h) = n+m | X(t) = n\right] = \begin{cases} \lambda_n h + o(h) & \text{if } m = 1, \\ \mu_n h + o(h) & \text{if } m = -1, \\ o(h) & \text{if } |m| > 1, \end{cases}$$

3. the **birth rates** $\lambda_0, \lambda_1, \ldots$ and the **death rates** $\mu_0, \mu_1, \ldots$ satisfy $\lambda_i \geq 0$, $\mu_i \geq 0$, $\mu_0 = 0$.

Then X is called a **birth-death process**.

Example: Birth-death process

The generator $\mathbf{G} = [g_{ij} : i, j \geq 0]$ is given by

$$\mathbf{G} = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \dots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- The birth-death process is irreducible if $\lambda_i > 0$ for all $i \geq 0$ and $\mu_i > 0$ for all $i \geq 1$.
- In many applications, it is interesting to consider the case when $\lambda_0 = 0$. In this case, $0 \in S$ is an absorbing state representing the extinction of the population, and the chain is not irreducible.

Example

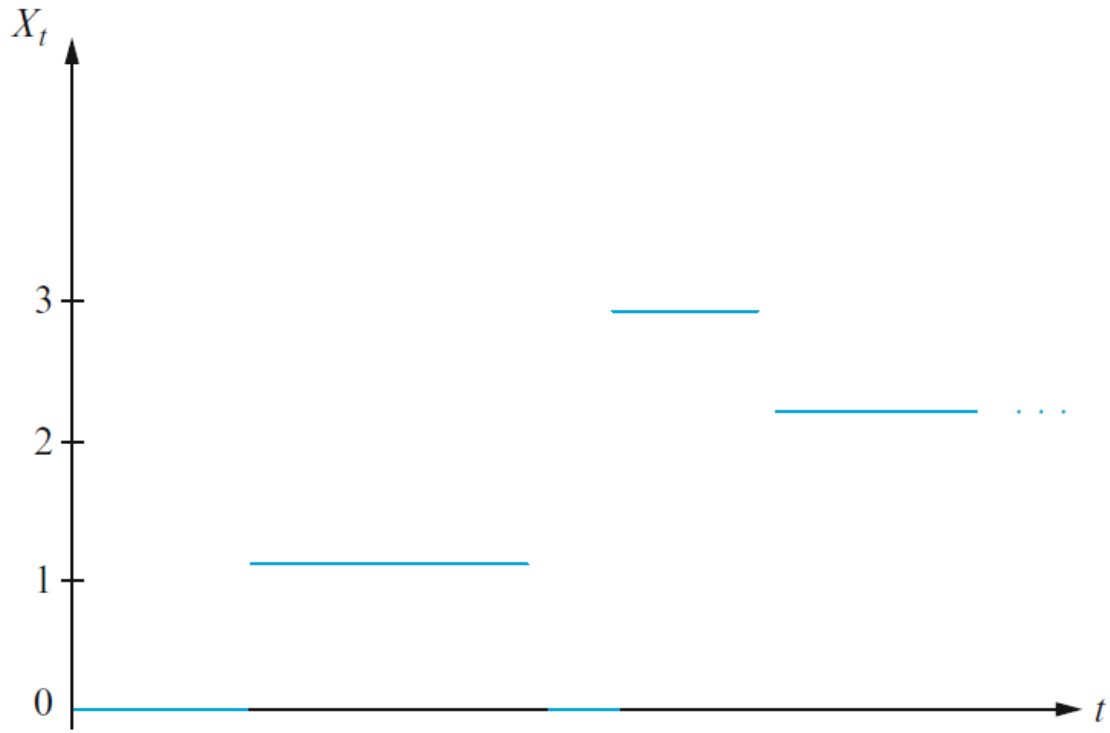
(CD) Example 7.33 A nurse checks in on patients in three hospital rooms and also spends time at the nurses' station. Identify these "states" as 0=nurses' station and 1, 2, 3=the three patient rooms. Let X_t denote the nurse's location t hours into her shift. The nurse begins her shift at the nurse's station $(X_0 = 0)$, spends some time there, and then moves periodically from room to room. Assume that the process X_t can be modelled as a CTMC with generator

$$\mathbf{G} = \begin{bmatrix} -6.5 & 4 & 0.5 & 2 \\ 3 & -8 & 4 & 1 \\ 3 & 0.5 & -7.5 & 4 \\ 3 & 0 & 1 & -4 \end{bmatrix}$$

- We know, for instance, that the nurse walks from the nurses' station to room 3 an average of twice per hour, since the rate at which she transitions from state 0 to state 3 is $g_{0,3} = 2$.

Example

- The following figure represents a possible path of the process



The nurse spends a while at her station (state 0), then some time in patient room 1, then back at her station, over into room 3, and finally into room 2

Example

- If we wanted to simulate a path like this, we would begin by simulating a realization, u_0 from the holding time $U_0 \sim \text{Exp}\,(6.5)$, to determine the initial amount of time spent by the nurse at the station before beginning her rounds.

➔ Then we would need to decide which room she visits when she leaves the station at time u_0. Should it be room 1, 2 or 3? What is the probability that she visits room 1, or room 2 or room 3? In other words, what is the conditional distribution of $X(u_0)$ given that there is a transition at time $t = u_0$?

➔ In the long run, how much of her time does the nurse spend at the station? And in each of the rooms?

Transition probabilities and long term behaviour

- Once a sojourn in state i has ended, the process transitions (instantaneously) to a particular state $j \neq i$. An interesting quantity to study is the probability that the process transitions (instantaneously) to state j at time t given that it transitions from state i (at time t)

$$P(\text{Transition to } j | \text{Transition out of } i) = \lim_{h \downarrow 0} \mathbb{P}\left[X_{t+h} = j | X_t = i, X_{t+h} \neq i\right]$$

  For a time homogeneous process, these probabilities should not depend on t

- We may also be interested in the average amount of time that the chain spends in state i **in the long run**

Stationary distribution and long term behaviour

If $X(0)$ has marginal distribution $\boldsymbol{\mu}_0 = [\mu_1(0), \mu_2(0), \mu_3(0) \ldots]^\mathsf{T}$, where $\mu_i(0) = \mathbb{P}[X(0) = i]$, by the law of total probability, the marginal distribution of $X(t)$, $\boldsymbol{\mu}_t = [\mu_1(t), \mu_2(t), \mu_3(t) \ldots]^\mathsf{T}$ satisfies

$$\begin{aligned}\mu_j(t) = \mathbb{P}[X(t) = j] &= \sum_{i \in S} \mathbb{P}[X(0) = i]\, \mathbb{P}[X(t) = j | X(0) = i] \\ &= \sum_{i \in S} \mu_i(0) p_{ij}(t)\end{aligned}$$

Or, in matrix form, $\boldsymbol{\mu}_t^\mathsf{T} = \boldsymbol{\mu}_0^\mathsf{T} \mathbf{P}_t$, i.e.

$$[\mu_1(t), \mu_2(t), \mu_3(t) \ldots] = [\mu_1(0), \mu_2(0), \mu_3(0) \ldots] \begin{bmatrix} p_{1,1}(t) & p_{1,2}(t) & p_{1,3}(t) & \cdots \\ p_{2,1}(t) & p_{2,2}(t) & p_{2,3}(t) & \cdots \\ p_{3,1}(t) & p_{3,2}(t) & p_{3,3}(t) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Stationary distribution and long term behaviour

Definition: Stationary distribution

The vector $\boldsymbol{\pi} = [\pi_1, \pi_2, \pi_3, \ldots]^\mathsf{T}$ is a **stationary distribution** of the chain X if $\pi_i \geq 0$ for all $i \in S$, $\sum_{i \in S} \pi_i = 1$ and

$$\boldsymbol{\pi}^\mathsf{T} = \boldsymbol{\pi}^\mathsf{T} \mathbf{P}_t \quad \text{for all } t \geq 0.$$

The stationary distribution $\boldsymbol{\pi}$ is also called the **steady state distribution** because if the process starts at the stationary distribution, then it stays in the stationary distribution:

$$X(0) \sim \boldsymbol{\mu}_0 = \boldsymbol{\pi} \quad \Rightarrow \quad X(t) \sim \boldsymbol{\mu}_t = \boldsymbol{\pi},$$

since $\boldsymbol{\mu}_0^\mathsf{T} \mathbf{P}_t = \boldsymbol{\pi}^\mathsf{T} \mathbf{P}_t = \boldsymbol{\pi}^\mathsf{T}$.
In this case, it is said that the chain is in the **steady state** and $\pi_i = \mathbb{P}[X(t) = i]$ can be interpreted as the average amount of time that the chain spends in state i *in the long run*.

Stationary distribution and long term behaviour

Long term behaviour

Let X be an irreducible HCTMC with standard semigroup $\{\mathbf{P}_t\}$ of transition probabilities. If there exists a stationary distribution $\boldsymbol{\pi}$, then it is unique and for all $i, j \in S$,

$$p_{ij}(t) = \mathbb{P}\left[X(t) = j | X(0) = i\right] \to \pi_j \text{ as } t \to \infty.$$

So, if the stationary distribution $\boldsymbol{\pi}$ exists, in the long run, the marginal distribution of $X(t)$ will approach $\boldsymbol{\pi}$, regardless of the initial distribution

➔ The chain will eventually reach the steady state (at least approximately)

☞ If S is infinite, the stationary distribution may not exist. In this case $p_{ij}(t) \to 0$ as $t \to \infty$ for all $i, j \in S$ and we say that the process *explodes* or *drifts off* to infinity.

Back to the example

(CD) Example 7.34 - Recall: A nurse checks in on patients in three hospital rooms and also spends time at the nurses' station. Identify these "states" as 0=nurses' station and 1, 2, 3=the three patient rooms. Let X_t denote the nurse's location t hours into her shift. The nurse begins her shift at the nurse's station $(X_0 = 0)$, spends some time there, and then moves periodically from room to room. Assume that the process X_t can be modelled as a CTMC with generator

$$\mathbf{Q} = \begin{bmatrix} -6.5 & 4 & 0.5 & 2 \\ 3 & -8 & 4 & 1 \\ 3 & 0.5 & -7.5 & 4 \\ 3 & 0 & 1 & -4 \end{bmatrix}$$

- The time intervals spent by the nurse at the nurses' station are exponentially distributed with parameter $\lambda = q_0 = 6.5$. Hence, the average length of time she spends there is $1/\lambda = 1/6.5$ h ≈ 9.23 min

Back to the example

- Similarly, the average sojourn time in patient room 3 is $1/q_3 = 1/4$ h ≈ 15 min
- When the nurse leaves her station, the likelihoods that she next visits rooms 1, 2, and 3 are, respectively,

$$\tilde{p}_{01} = \frac{4}{6.5} = \frac{8}{13}; \quad \tilde{p}_{02} = \frac{0.5}{6.5} = \frac{1}{13}; \quad \tilde{p}_{03} = \frac{2}{6.5} = \frac{4}{13}$$

- Similarly, when the nurse leaves patient room 1, there is a $3/8$ chance she'll return to the nurses' station, a $4/8$ probability of moving on to room 2, and a $1/8$ chance of checking the patients in room 3

Notice: In general, the transition probabilities exiting a sojourn spent in state i are proportional to the instantaneous transition rates out of state i

In the long run...

➜ In the long run, how much of her time does the nurse spend at the station? And in each of the rooms?

We have seen that the proportion of time that the nurse spends at the station is given by the π_0, the corresponding element of the **stationary distribution**. Similarly, the nurse spends a proportion π_i of her time in rom i.

👉 But how do we find π? ➜ The stationary distribution of a HCTMC can be found by solving the **Global Balance Equations**.

Global balance equations

Finding the stationary distribution

A distribution $\boldsymbol{\pi}$ is the stationary distribution of a HCTMC with transition semigroup $\mathbf{P}_t$ if and only if

$$\boldsymbol{\pi}^{\mathsf{T}}\mathbf{G} = \mathbf{0}.$$

Indeed, remembering that $\mathbf{G}^0 = \mathbf{I}$, we have

$$\begin{aligned}
\boldsymbol{\pi}^{\mathsf{T}}\mathbf{G} = \mathbf{0} \Leftrightarrow & \quad \boldsymbol{\pi}^{\mathsf{T}}\mathbf{G}\frac{\mathbf{G}^{n-1}}{n!} = \mathbf{0} \text{ for all } n \geq 1 \\
\Leftrightarrow & \quad \boldsymbol{\pi}^{\mathsf{T}} + \sum_{n=1}^{\infty} \boldsymbol{\pi}^{\mathsf{T}}\frac{(t\mathbf{G})^n}{n!} = \boldsymbol{\pi}^{\mathsf{T}} + \mathbf{0} \text{ for all } t \geq 0 \\
\Leftrightarrow & \quad \boldsymbol{\pi}^{\mathsf{T}}\sum_{n=0}^{\infty}\frac{(t\mathbf{G})^n}{n!} = \boldsymbol{\pi}^{\mathsf{T}} \text{ for all } t \geq 0 \\
\Leftrightarrow & \quad \boldsymbol{\pi}^{\mathsf{T}}\mathbf{P}_t = \boldsymbol{\pi}^{\mathsf{T}} \text{ for all } t \geq 0
\end{aligned}$$

Global balance equations

The previous result can be interpreted in terms of the flux in and out of a given state $i \in S$, by developing the matrix product $\boldsymbol{\pi}^{\mathsf{T}}\mathbf{G} = \mathbf{0}$ and recalling that

$$g_{ii} = -\sum_{j:j\neq i} g_{ij}.$$

Global balance equations

A distribution $\boldsymbol{\pi}$ is the stationary distribution of a HCTMC with transition semigroup $\mathbf{P}_t$ if and only if it satisfies the **global balance equations**

$$\pi_i \sum_{j:j\neq i} g_{ij} = \sum_{j:j\neq i} \pi_j g_{ji}$$

Global balance equations

- The left hand side of the balance equations can be interpreted as the flux out of state i: the mean proportion of time that the process spends in state i multiplied by the total rate of transitions out of state i.
- The right hand side of the balance equations can be interpreted as the flux into state i: the sum over all other states $j \neq i$ of the mean proportion of time that the process spends in state j multiplied by the rate of transitions from j into i.

In other words, at the steady state, the time that the process spends in a state i multiplied by the rate at which it leaves must equal, on average, to the sum of times that the process spends in other states multiplied by the rates at which it returns to i.

Global balance equations

Example: Two-state process Recall the chain X with $S = \{1, 2\}$ and generator

$$\mathbf{G} = \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix}$$

Developing the matrix product $\boldsymbol{\pi}^{\mathsf{T}}\mathbf{G} = \mathbf{0}$ we obtain

$$\begin{aligned} -\mu\pi_1 \quad & +\lambda\pi_2 \quad = 0 \\ \mu\pi_1 \quad & -\lambda\pi_2 \quad = 0. \end{aligned}$$

In order to obtain a unique solution, we must consider the condition

$$\pi_1 + \pi_2 = 1; \quad \pi_1, \pi_2 \in [0, 1].$$

Global balance equations

Therefore, the stationary distribution for this process is

$$\boldsymbol{\pi}^{\mathsf{T}} = \left[\frac{\lambda}{\mu+\lambda}, \frac{\mu}{\mu+\lambda}\right]^{\mathsf{T}}$$

☞ **Exercize:** Find the stationary distribution by calculating the limit of $\mathbf{P}_t$ as $t \to \infty$.

In general, the global balance equations define an irreducible system of equations, so in order to obtain a unique solution, we must consider the normalization condition

$$\sum_{i \in S} \pi_i = 1.$$

In matrix form, this can be done by substituting one of the columns of the matrix $\mathbf{G}$ with a row of ones, and placing a one in the corresponding place of the row vector $\mathbf{0}$.

Global balance equations

When $\#S = N < \infty$, we will substitute the N-th column of $\mathbf{G}$.

- Most software systems used to compute solutions to systems of linear equations consider systems of the type $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ and $\mathbf{b}$ are a matrix and vector, respectively, of known constants, and x is the vector for which we wish to solve.

  It is therefore convenient to define $\mathbf{A} = \tilde{\mathbf{G}}^{\mathsf{T}}$, where $\tilde{\mathbf{G}}$ is obtained by substituting the last column of $\mathbf{G}$ with a column of ones, and $\mathbf{b} = \mathbf{e}_N \in \mathbb{R}^N$ is the N-th canonical vector (the last element is 1 and all others are 0).

Global balance equations

Example: For the two-state process, we can find the stationary distribution $\boldsymbol{\pi}$ by solving the system $\mathbf{A}\boldsymbol{\pi} = \mathbf{b}$ given by

$$\begin{bmatrix} -\mu & \lambda \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Assuming $\mu = 2$ and $\lambda = 1$, we can do this in **R**:

- The number of states is $\#S = N = 2$

```
> N=2
```

- Define the generator

```
> mu <- 2
> lambda <- 1
> G <- matrix(c(-mu, mu, lambda, -lambda),
+             ncol=N, byrow=T)
> G
```

Global balance equations

```
     [,1] [,2]
[1,]   -2    2
[2,]    1   -1
```

- Define the matrix A by substituting a column from $\mathbf{G}$ and then transposing:

```
> A <- G
> A[,N] <- rep(1,N)
> A <- t(A)
> A
```

```
     [,1] [,2]
[1,]   -2    1
[2,]    1    1
```

- Define the vector $\mathbf{b} = \mathbf{e}_2$

```
> b <- c(rep(0,N-1),1)
> b
```

Global balance equations

```
[1] 0 1
```

- Find the stationary distribution by solving the system $\mathbf{A}\boldsymbol{\pi} = \mathbf{b}$:

```
> p <- solve(A, b)
> p
[1] 0.3333333 0.6666667
```

☞ We can verify this is the same distribution obtained before:

```
> c(lambda, mu)/(lambda+mu)
[1] 0.3333333 0.6666667
```

Global balance equations

Example: Consider a birth-death process with generator $\mathbf{G} = [g_{ij} : i, j \geq 0]$ given by

$$\mathbf{G} = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \dots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

If $\lambda_0 > 0$, the process is irreducible, so the stationary distribution $\boldsymbol{\pi}$, if it exists, can be found as the unique solution to the global balance equations $\boldsymbol{\pi}^{\mathsf{T}}\mathbf{G} = \mathbf{0}$. This is a system with an infinite number of equations:

$$\begin{aligned} -\lambda_0\pi_0 + \mu_1\pi_1 &= 0, \\ \lambda_{n-1}\pi_{n-1} - (\lambda_n + \mu_n)\pi_n + \mu_{n+1}\pi_{n+1} &= 0 \quad \text{if } n \geq 1. \end{aligned}$$

Global balance equations

We can solve this system by induction. Informally,

- $-\lambda_0\pi_0 + \mu_1\pi_1 = 0$

$$\Rightarrow \quad \pi_1 = \frac{\lambda_0}{\mu_1}\pi_0$$

- $\lambda_0\pi_0 - (\lambda_1 + \mu_1)\pi_1 + \mu_2\pi_2 = 0$

$$\begin{aligned}
\Rightarrow \quad & \lambda_0\pi_0 - (\lambda_1 + \mu_1)\frac{\lambda_0}{\mu_1}\pi_0 + \mu_2\pi_2 = 0 \\
\Rightarrow \quad & -\frac{\lambda_0\lambda_1}{\mu_1}\pi_0 + \mu_2\pi_2 = 0 \\
\Rightarrow \quad & \pi_2 = \frac{\lambda_0\lambda_1}{\mu_1\mu_2}\,\pi_0
\end{aligned}$$

- In general,

$$\pi_n = \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}\,\pi_0, \quad n \geq 1.$$

Global balance equations

The vector $\boldsymbol{\pi}$ is a stationary distribution if and only if $\sum_n \pi_n = 1$, which may happen if and only if

$$1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} < \infty.$$

If this holds, then

$$\pi_0 = \left(1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu) n} \right)^{-1}$$

and all other probabilities π_n, $n \geq 1$ can be obtained recursively.

- This means that the process settles into equilibrium (with stationary distribution $\boldsymbol{\pi}$) if and only if this sum is finite, a condition requiring that the birth rates are not too large relative to the death rates.

Simulating a HCTMC

We can simulate a path from a HCTMC X with finite support S and generator $\mathbf{G}$, in the time interval $[0, T]$ in the following way:

1. Set $n = 0$, $t = t_0 = 0$ and simulate the initial value, $x_0 = i_0$ from the initial marginal distribution $\boldsymbol{\mu}_0$, i.e. $\mathbb{P}[X(0) = i] = \mu_i(0)$
   - In many cases a fixed value i_0 is chosen, so that $\mu_{i_0}(0) = 1$ and $\mu_i(0) = 0$ for all $i \neq i_0$.
2. Simulate the holding time $U_{i_0} \sim \text{Exp}(g_{i_0})$ and set $t = t_1 = u_{i_0}$.
3. If $t \leq T$ set $t_n = T$, $n = n - 1$ and stop; otherwise set $i = x_n$, simulate X_{n+1}, a discrete random variable with probability distribution given by the i-th row of $\tilde{\mathbf{P}}$, i.e. $\mathbb{P}[X_{n+1} = j \neq i] = g_{ij}/g_i$ then set $n = n + 1$, $i = x_n$, then generate the holding time $u_i \sim \text{Exp}(g_i)$ and set $t = t + u_i$, $t_n = t$.
4. Return to step 3.

Simulating a HCTMC

The output of the algorithm is the number n of jumps of the process up to time T, the vector $[t_1, t_2, \ldots, t_n]$ of times at which the process jumps, and the vector $[x_0, x_1, \ldots, x_n]$ of states visited. The full path of the process, up to time T is obtained by noticing that $x(t) = x_m$ for all $t \in [t_m, t_{m+1})$, for $m = 0, \ldots, n$, with $t_{n+1} = T$.

Example: Simulating a single path from a 3-state HCTMC with generator

$$\mathbf{G} = \begin{bmatrix} -4 & 2 & 2 \\ 1 & -2 & 1 \\ 0.5 & 1 & -1.5 \end{bmatrix}$$

up to time $T = 10$ (the interval $[0, T]$ is sometimes called the time frame).

Simulating a HCTMC

👉 In **R**:

- Set the generator and time frame:

```
> T <- 10
> G <- matrix(c(-4, 2, 2, 1, -2, 1, 0.5, 1, -1.5),
+             ncol=3, byrow=T)
> G
     [,1] [,2] [,3]
[1,] -4.0    2  2.0
[2,]  1.0   -2  1.0
[3,]  0.5    1 -1.5
```

- Calculate matrix $\tilde{\mathbf{P}}$ of transition probabilities given the jump:

```
> P_tilde <- -G / diag(G)
> diag(P_tilde) <- 0
> P_tilde
```

Simulating a HCTMC

```
           [,1]      [,2] [,3]
[1,] 0.0000000 0.5000000  0.5
[2,] 0.5000000 0.0000000  0.5
[3,] 0.3333333 0.6666667  0.0
```

- Fix the seed of the random number generator and initialize the chain:

```
> set.seed(9878)
> n <- 0
> t <- 0
> times <- 0
> x_t <- 1
> x <- x_t
```

- Simulate the first holding time:

```
> t <- rexp(1, -G[x_t, x_t])
```

Simulating a HCTMC

- Simulate sequentially until sopping rule:

```
> while (t <= T){
+   n <- n+1
+   times <- c(times,t)
+   x_t <- sample(1:3, 1, prob = P_tilde[x_t,])
+   x <- c(x, x_t)
+   t <- t + rexp(1, -G[x_t, x_t])
+ }
```

- Plot the path:

```
> plot(c(times,T), c(x,x[n+1]), type="s",
+       col="blue", yaxt="n")
> axis(side=2, at=1:3)
```

Simulating a HCTMC