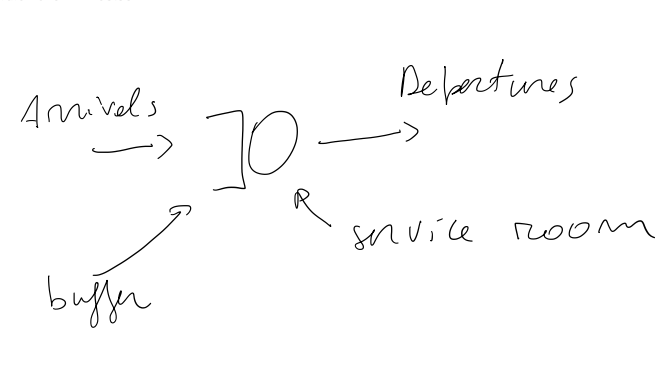
Inspection paradox

lunedì 27 febbraio 2023 08:55

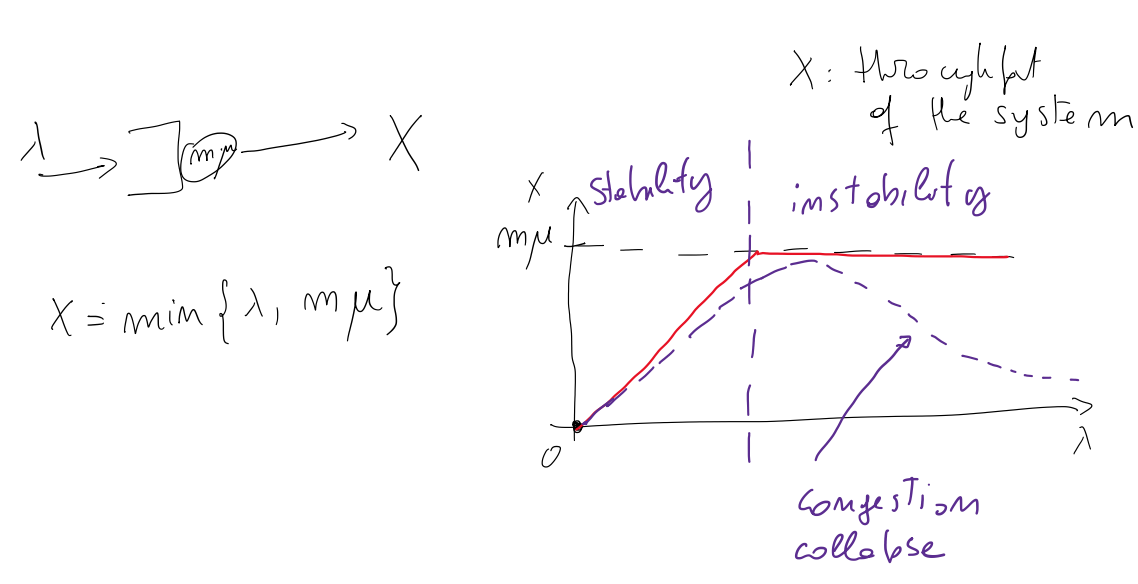


λ job/s is the arrival stream

μ job/s is the service rate of each server

m servers

$\boxed{\lambda < m\mu}$ stability condition for work conserving queues



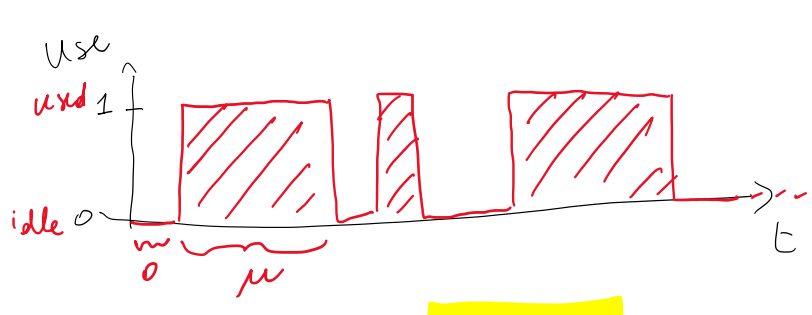
$X = \min\{\lambda, m\mu\}$

suppose m = 1 (single server)

λ<μ stability

→ X = λ

utilization = ? $U = \frac{\lambda}{\mu}$



$\mu \cdot U = \lambda$ ⇒ $\boxed{U = \frac{\lambda}{\mu}}$ UTILIZATION LAW FOR SINGLE SERVER

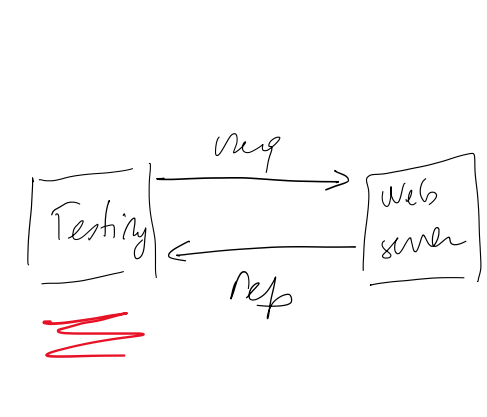
LITTLE'S LAW

$\bar N$: expected n. of jobs in the system

$\bar R$: expected response time of the system

X: throughput

$\boxed{\bar N = X \cdot \bar R}$



Example

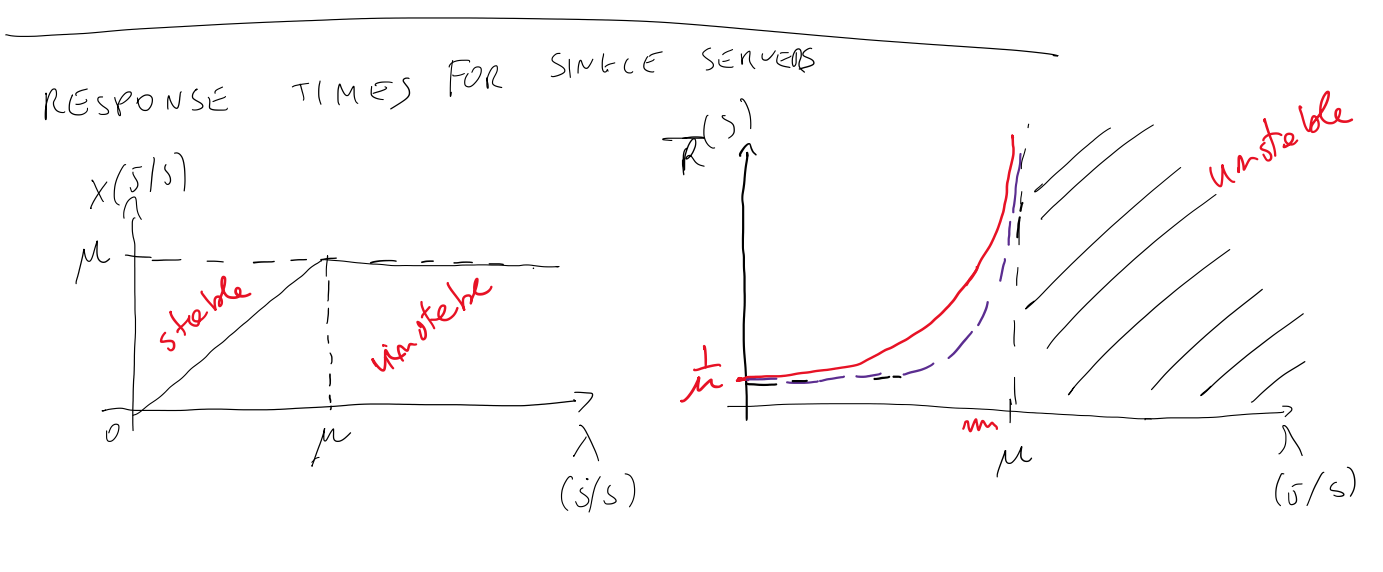
$\bar R$ = 20 ms

X = 15 req/s   Stable systems

$\bar N = 20 \cdot 10^{-3} \cdot 15 = 0.3$ jobs

$\boxed{\bar N = \lambda \bar R}$

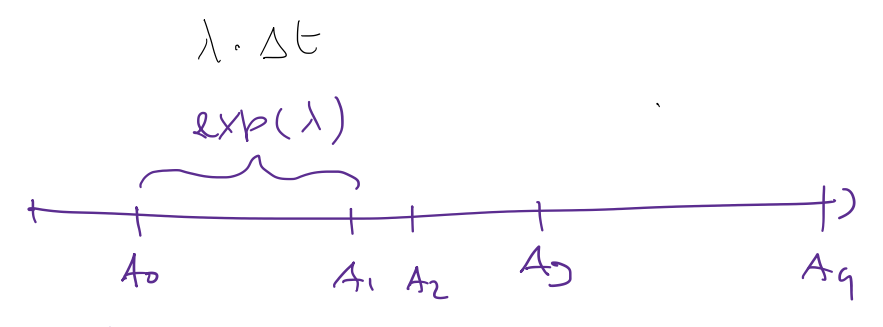
RESPONSE TIMES FOR SINGLE SERVER



λ > μ

POISSON ARRIVAL PROCESS

Δt   N: # of arrivals in this time interval is distributed according to a Poisson r.v. with parameter λ·Δt



Example

λ = 4 j/s

Av. distance between two consecutive jobs is 0.25 s

if Δt = 5 s the parameter of a Poisson R.V. is Δt·λ = 20 j

Δt

LITTLE'S LAW

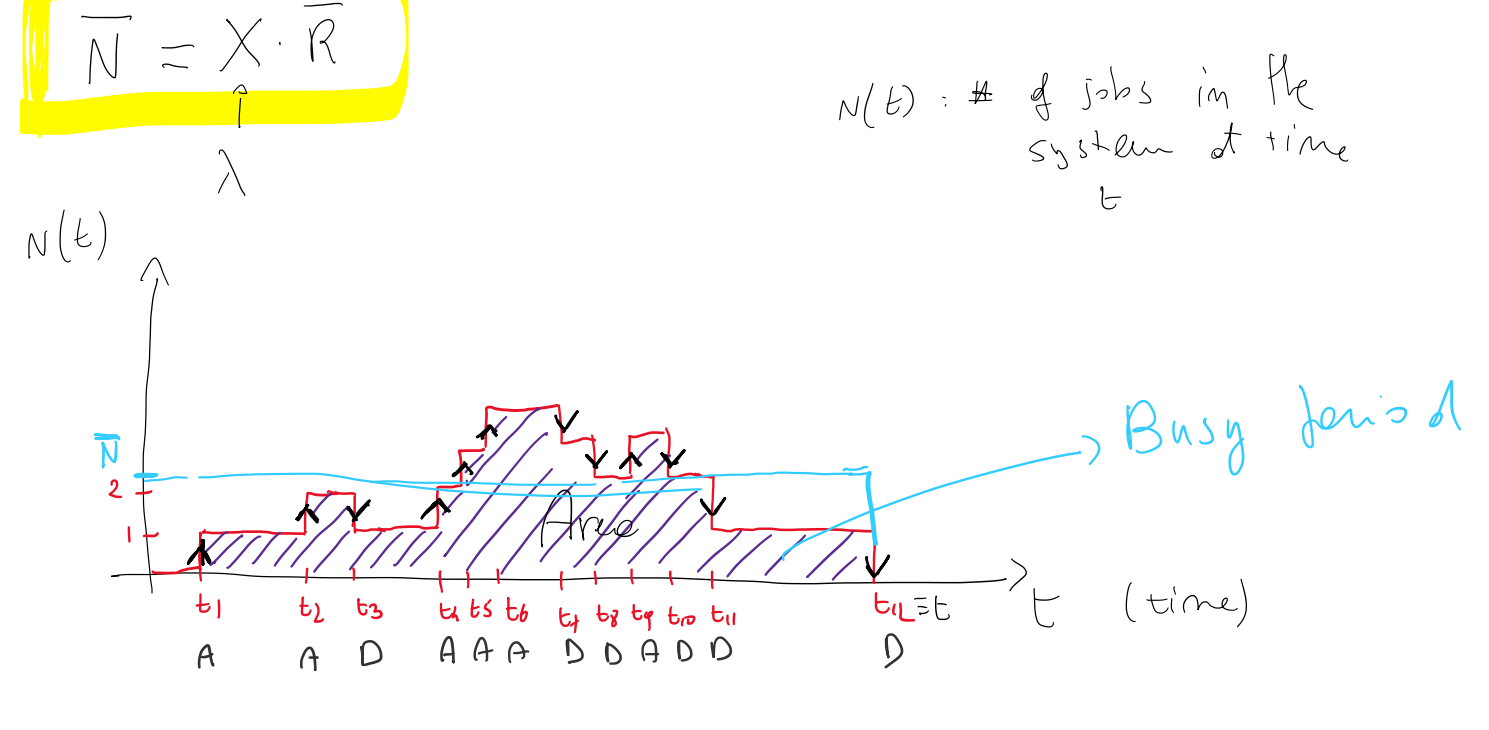
1) No work wasted on jobs (no creation/destruction of jobs)

2) System stable $\boxed{X = \lambda}$ ↳ throughput = arrival rate

$\boxed{\bar N = X \cdot \bar R}$

λ

N(t): # of jobs in the system at time t



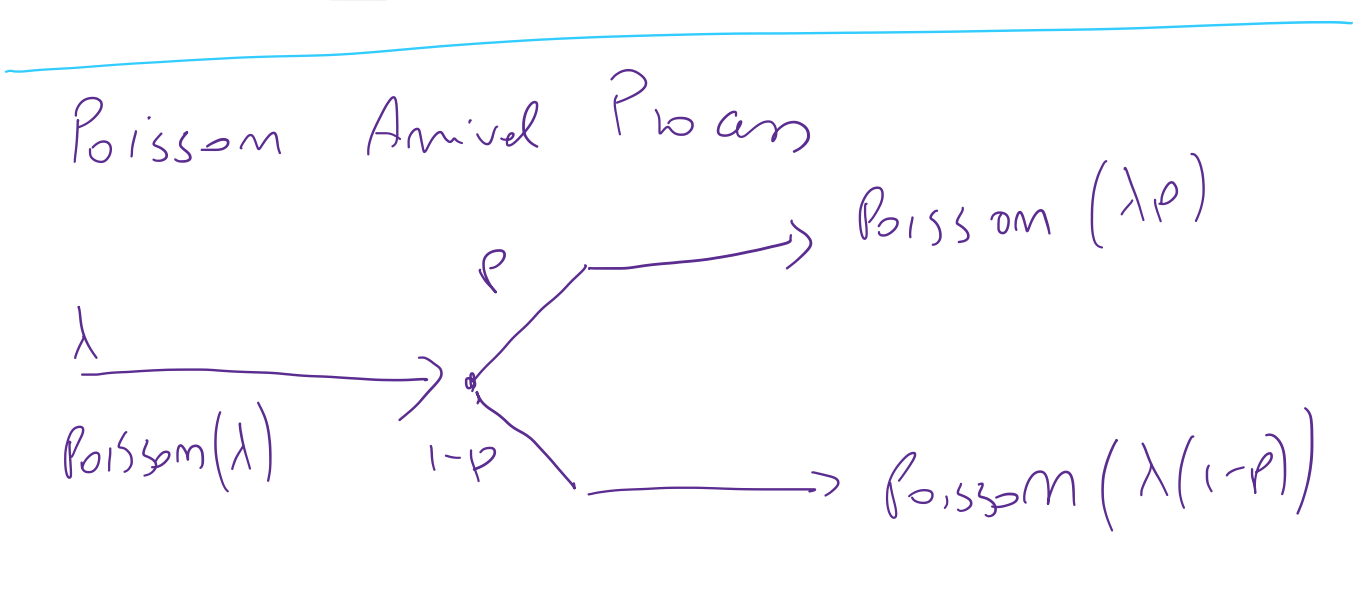
$\bar N = \frac{Area}{t}$ $\bar R = \frac{Area}{\#Arrivals}$

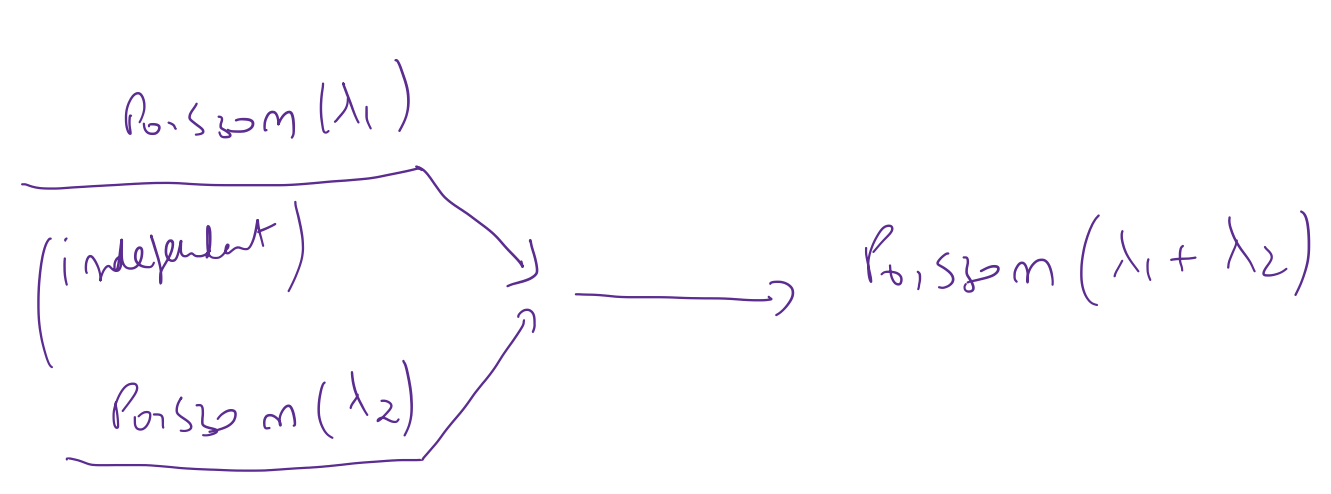
$Area = \bar N \cdot t$ $Area = \bar R \cdot \#Arrivals$

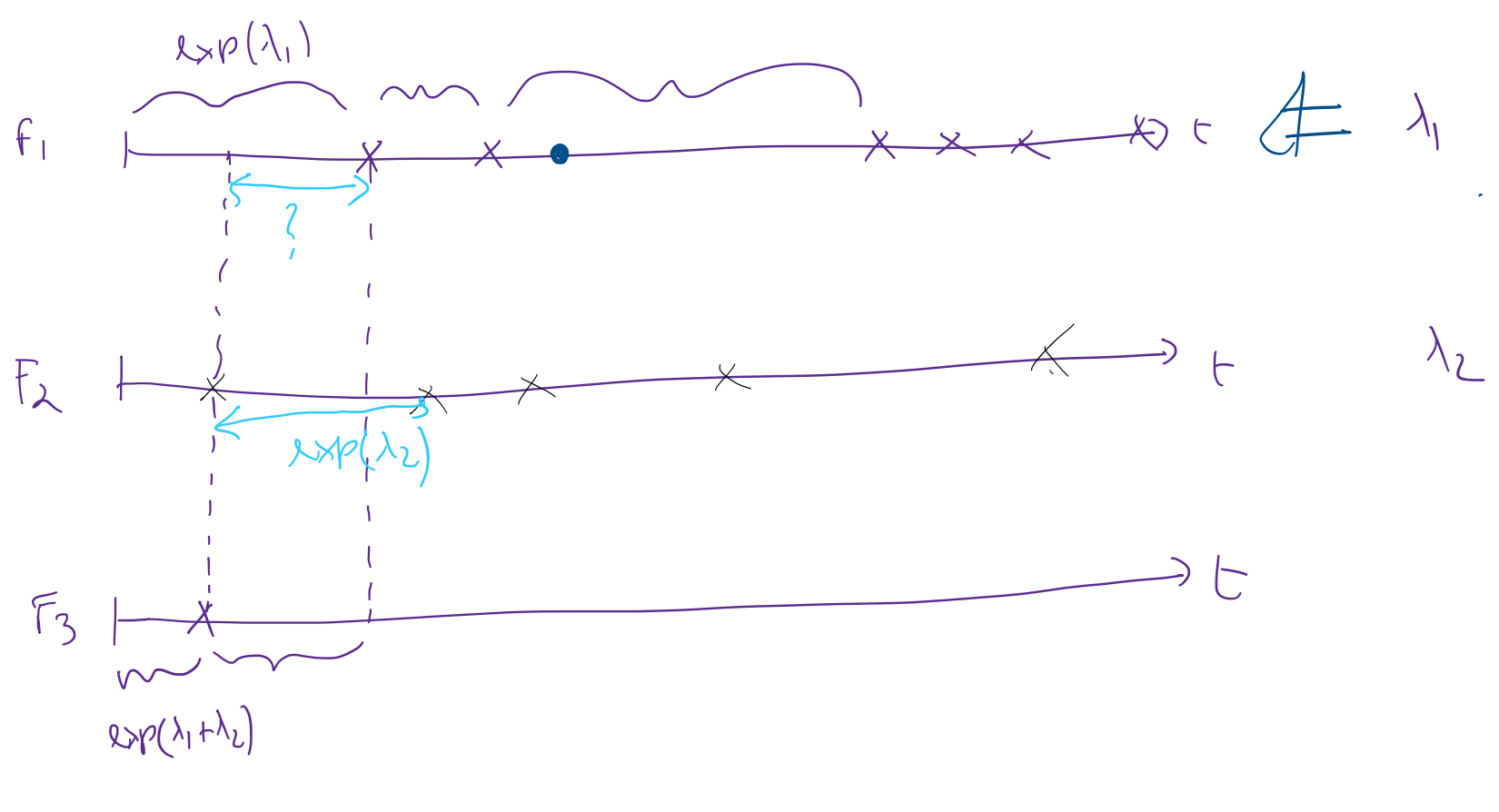
$\bar N \cdot \cancel{t} = \bar R \cdot \frac{\#Arrivals}{t}$

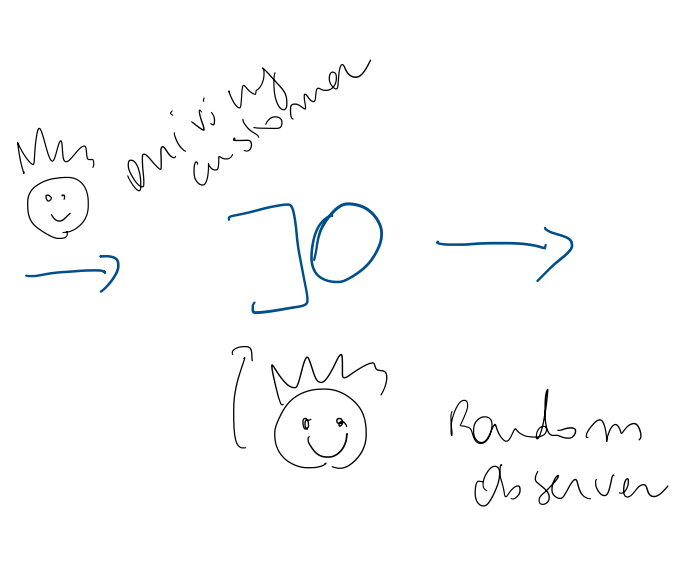
$\boxed{\bar N = \bar R \cdot \lambda}$ $\lambda = X$ $\boxed{\bar N = \bar R \cdot X}$

Poisson Arrival Process

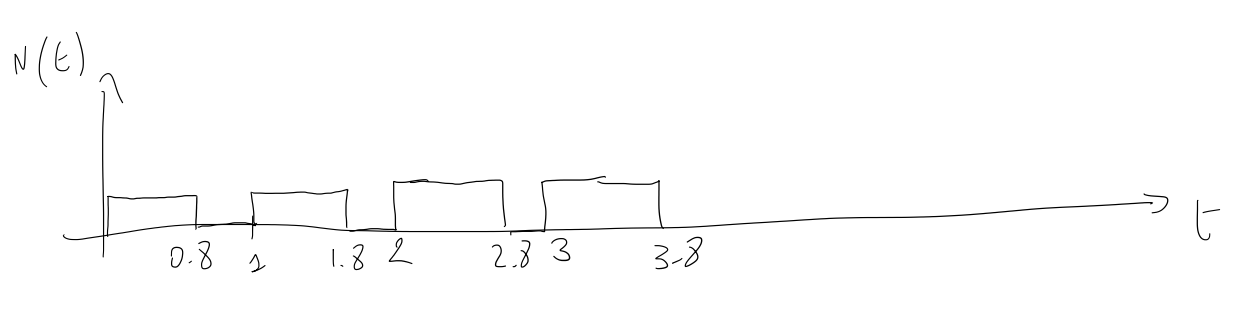








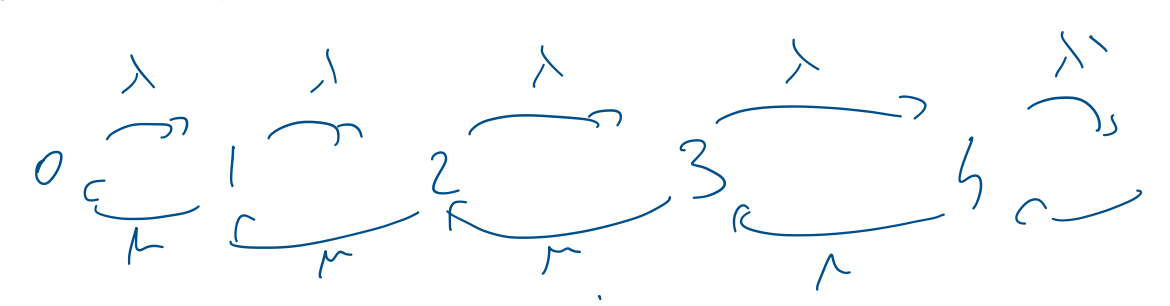
- Arrival is deterministic with inter-arrival time of 1 s
- Service is deterministic with service time of 0.8 s



- Random observer sees utilization of 80%
- Arriving customer sees utilization of 0%

$\boxed{PASTA:}$ Poisson Arrivals Sees Time Averages

The random observer point of view is the same of the point of view of the arriving customer.



$\pi(i) = \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^i$

$E[N] = \frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}}$ M/M/1

$\bar R = \frac{1}{\mu} \cdot \bar N + \frac{1}{\mu} = \frac{1}{\mu}(\bar N + 1)$

✓1) Stable
✓2) Poisson Arrivals
✓3) Exponential Service

$\bar N = \lambda \cdot \bar R \rightarrow \boxed{\bar R = \frac{\bar N}{\lambda}}$

$\frac{\bar N}{\lambda} = \frac{1}{\mu}(\bar N + 1)$ $\frac{\bar N}{\lambda} - \frac{\bar N}{\mu} = \frac{1}{\mu}$

$\bar N\left(\frac{1}{\lambda} - \frac{1}{\mu}\right) = \frac{1}{\mu}$ $\bar N\left(\frac{\mu-\lambda}{\lambda\mu}\right) = \frac{1}{\mu}$

$\bar N = \frac{\lambda\cancel{\mu}}{\mu-\lambda} \cdot \frac{1}{\cancel{\mu}} = \frac{\lambda}{\mu-\lambda} = \frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}}$

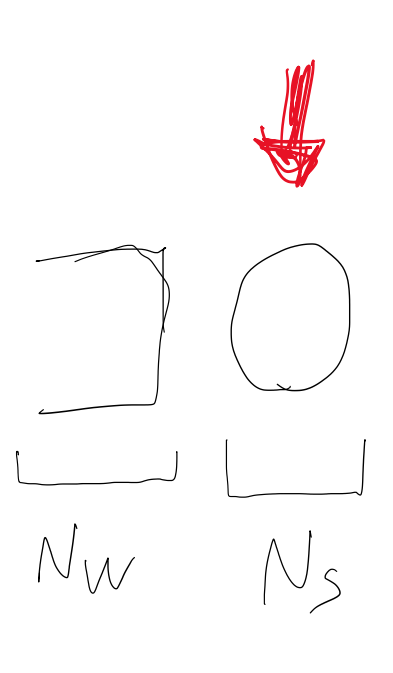
$\bar R = \frac{1}{\mu-\lambda}$

M/M/1   λ<μ $\frac{1}{\mu}$ → Expected service time

1) $U = \frac{\lambda}{\mu}$ $\boxed{\rho \triangleq \frac{\lambda}{\mu}}$

2) $\bar N = \frac{\rho}{1-\rho}$

3) $\bar R = \frac{1}{\mu-\lambda}$



$X = \lambda$

$\bar N_W = \frac{\rho}{1-\rho} - \rho = \frac{\cancel{\rho} - \cancel{\rho} + \rho^2}{1-\rho} = \frac{\rho^2}{1-\rho}$

$N = N_W + N_S$

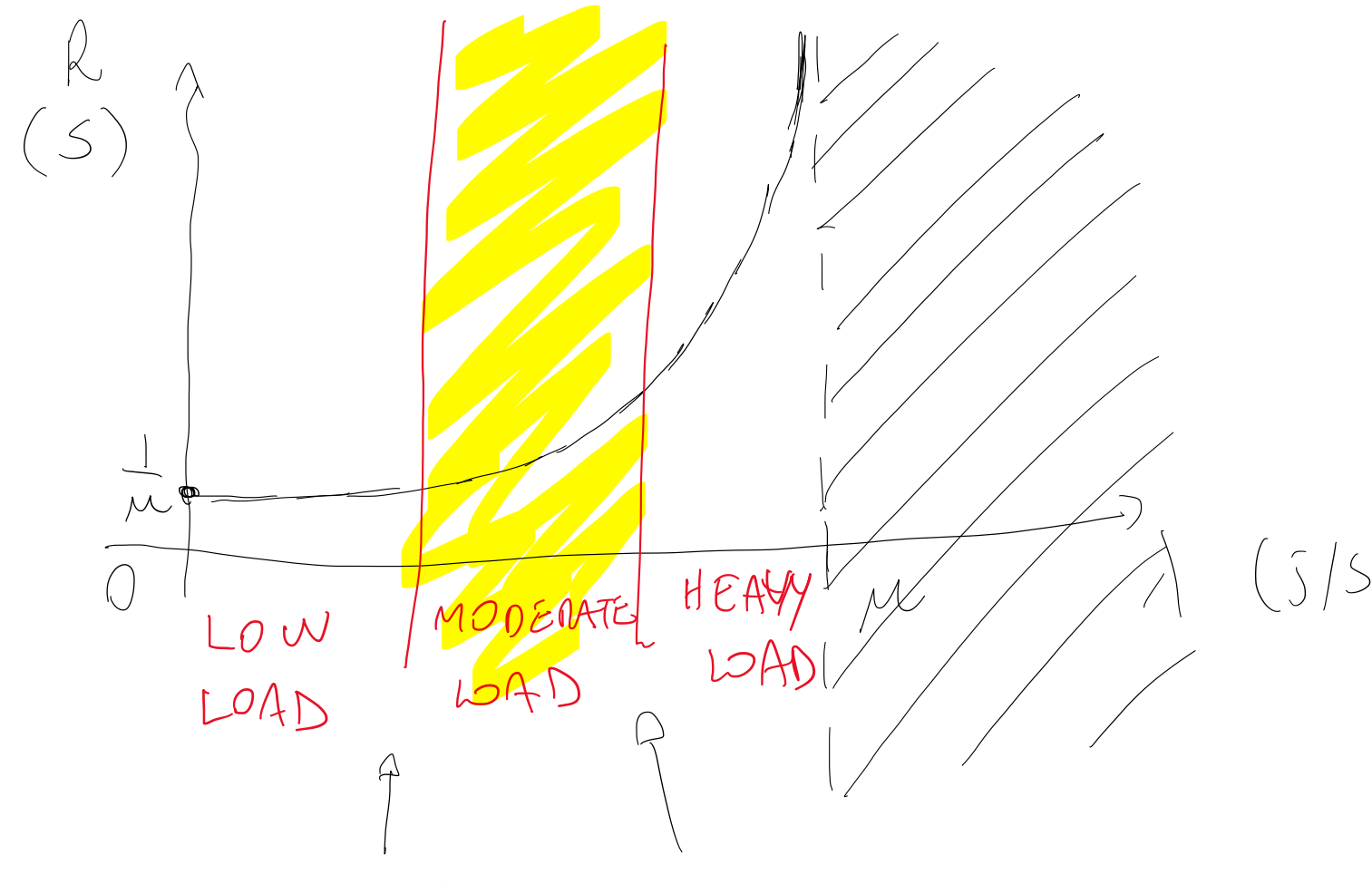
$\bar N = \boxed{\bar N_W} + \bar N_S$

$\bar W$ waiting time

$\bar N_S = \underbrace{(1-U)}_{\text{prob. of empty syst.}} \cdot 0 + \underbrace{U}_{\text{prob. of busy system}} \cdot 1 = U$

$\bar R = \bar W + \frac{1}{\mu}$

$\bar W = \bar R - \frac{1}{\mu} = \frac{1}{\mu-\lambda} - \frac{1}{\mu} = \frac{\mu-\mu+\lambda}{\mu(\mu-\lambda)} = \rho \cdot \frac{1}{\mu-\lambda}$

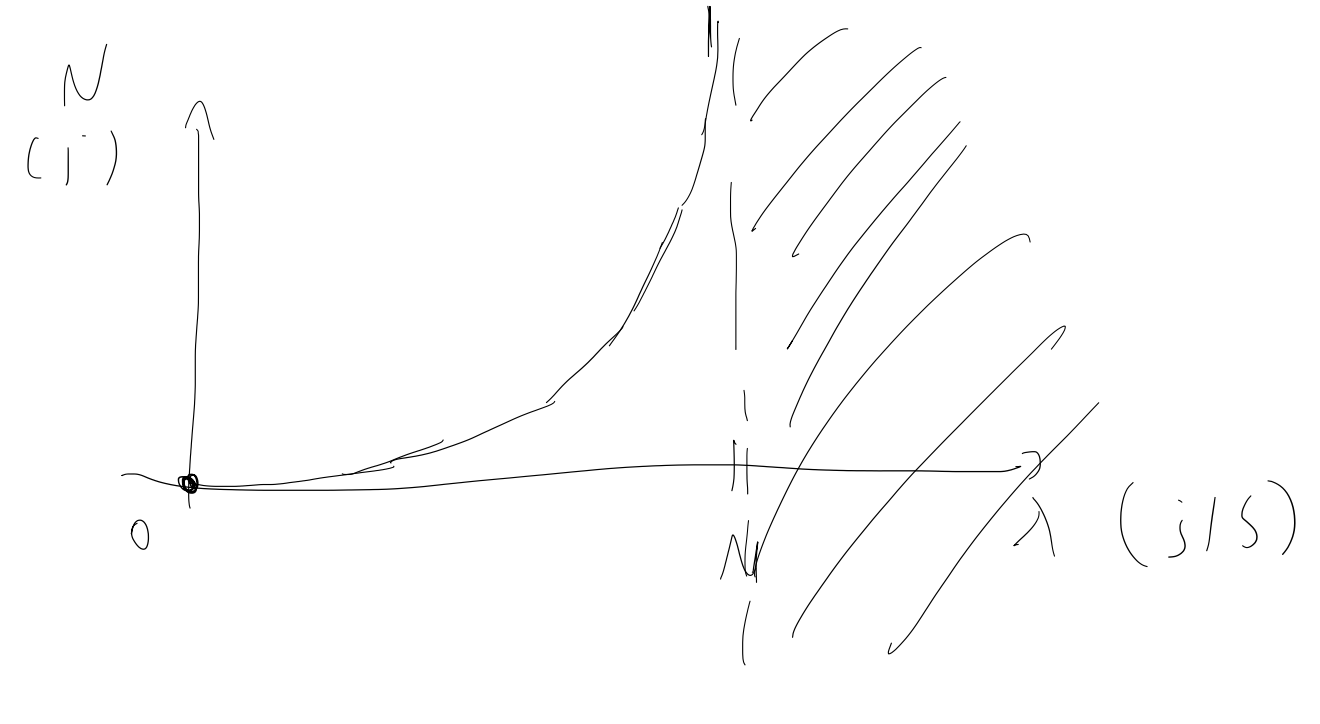


$\boxed{\bar R = \frac{1}{\mu-\lambda}}$

λ<μ

$\lim_{\lambda \to \mu^-} \frac{1}{\mu-\lambda} = +\infty$

ρ≃0.4   ρ≃0.7



$\bar N = \frac{\rho}{1-\rho}$ ⇒ $\rho \triangleq \frac{\lambda}{\mu}$

λ = 15 j/s arrival rate (traffic intensity)

μ = 20 j/s service rate

expected service time is $\frac{1}{20}$ = 0.05 s = 50 ms

$\bar R = \frac{1}{\mu-\lambda} = \frac{1}{20-15} = \frac{1}{5}$ = 0.2 s → 200 ms

$\lambda_1 = \lambda + 10\%$ = 16.5 j/s

$\bar R_1 = \frac{1}{\mu-\lambda} = \frac{1}{20-16.5} = \frac{1}{3.5}$ = 290 ms

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λ = 10 j/s
μ = 20 j/s $\bar R = \frac{1}{10}$ = 100 ms

λ_1 = 11 j/s $\bar R_1 = \frac{1}{9}$ = 110 ms

INSPECTION PARADOX

X: r.v. model first card

Y: r.v. model second card

$\Pr\{Y=Q\} = \frac{1}{3}$ ✓

$\Pr\{Y=Q \mid X=Q\} = 0$

$\Pr\{Y=Q \mid X\neq Q\} = \frac{1}{2}$

$\Pr\{Y=Q\} = \overset{0}{\cancel{\Pr\{Y=Q \mid X=Q\}}}\ \cancel{\Pr\{X=Q\}} + \Pr\{Y=Q \mid X\neq Q\}\ \Pr\{X\neq Q\} = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$ ✓

X continuous $F_X(x) = \Pr\{X \leq x\}$

Y discrete $\boxed{f_X(x) = \frac{\partial F_X(x)}{\partial x}}$

↳ is not the prob. that X assumes value x

$E[X] = \int_{\mathbb{R}} x\, f_X(x)\, dx$

$E[Z] = \sum_{j\in\mathbb{N}} j \cdot P_j$

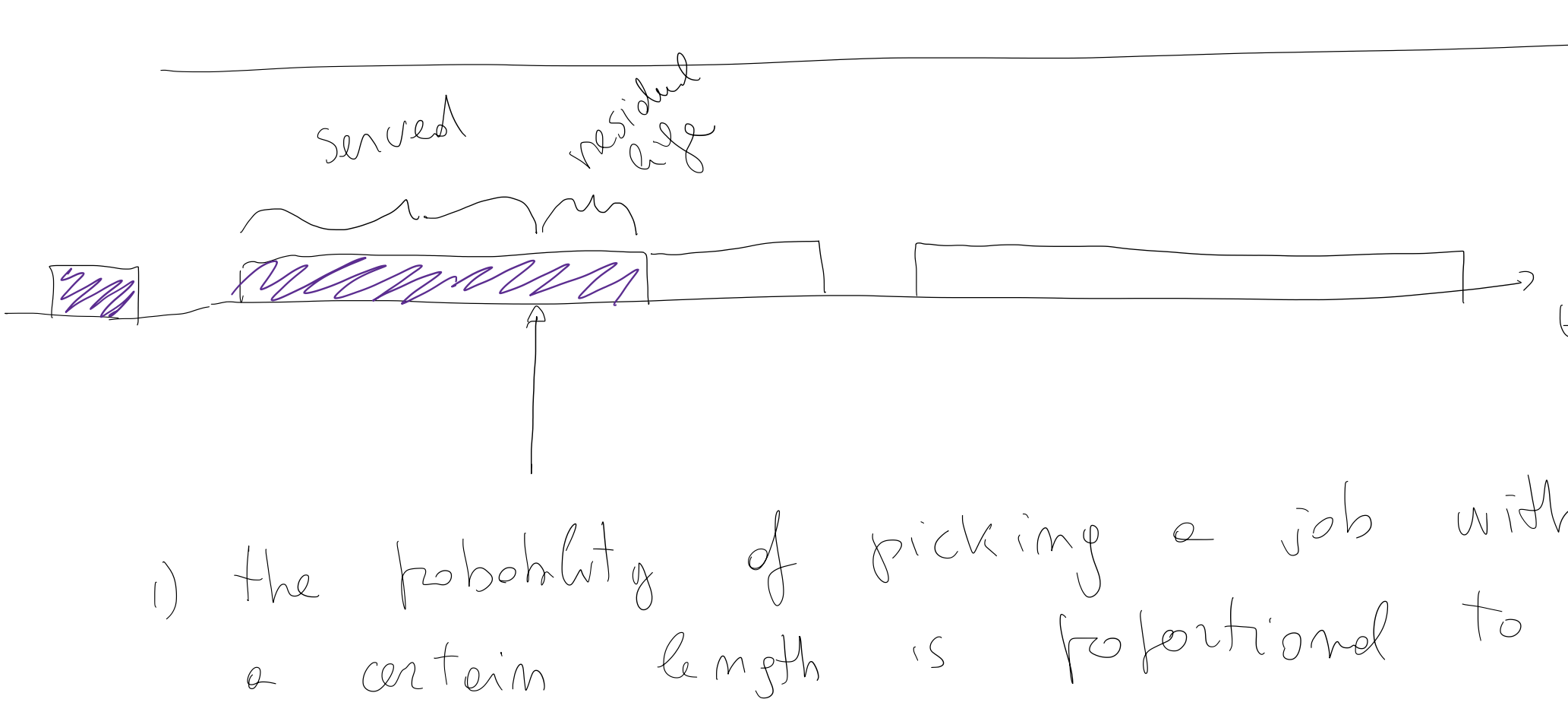
discrete

$f_X(x) \cdot dx$ → is the prob. that X takes a value in proximity of x

$X \mid Y = y_1$ $\Pr\{Y=y_1\} + \Pr\{Y=y_2\} = 1$

$X \mid Y = y_2$ y_1 and y_2 are disjoint

$f_X(x) = f_X(x \mid Y=y_1) \cdot \Pr\{Y=y_1\} + f_X(x \mid Y=y_2) \cdot \Pr\{Y=y_2\}$



1) the probability of picking a job with a certain length is proportional to its length

2) suppose that you know that the job that you are observing has a duration x then the expected residual life is $\frac{x}{2}$

X duration of a job $f_X(x)$ pdf $F_X(x) = \Pr\{X \leq x\}$ CDF

Y duration of the job that is observed at random time $f_Y(x)$ pdf $F_Y(x)$ cdf

① $f_Y(x)\, dx = K \cdot f_X(x) \cdot x \cdot dx$

② $\int_0^\infty f_Y(x)\, dx = 1$ condition for $f_Y(x)$ to be a pdf

$K \int_0^\infty f_X(x) \cdot x \cdot dx = 1$

$K\, E[X] = 1$ ⇒ $K = \frac{1}{E[X]}$

③ $\boxed{f_Y(x) = \frac{1}{E[X]} f_X(x) \cdot x}$

④ $E[Y] = \int_0^\infty x \cdot f_Y(x)\, dx = \int_0^\infty \frac{1}{E[X]} \cdot f_X(x) \cdot x^2\, dx$

$= \frac{1}{E[X]} \int_0^\infty x^2 f_X(x)\, dx = \frac{E[X^2]}{E[X]} = \frac{Var[X] + E^2[X]}{E[X]}$

$Var[X] = E[X^2] - E^2[X]$ ⇒ $E[X^2] = Var[X] + E^2[X]$

⑤ $\boxed{\text{Expected residual life} = \frac{E[X^2]}{2E[X]} = \frac{1}{2}\left(\frac{Var[X]}{E[X]} + E[X]\right)}$

suppose $X \sim exp(\mu)$ $E[X] = \frac{1}{\mu}$ $Var[X] = \frac{1}{\mu^2}$

Expected residual life $\triangleq E[\xi] = \frac{1}{2}\left(\frac{\frac{1}{\mu^2}}{\frac{1}{\mu}} + \frac{1}{\mu}\right)$

$= \frac{1}{2}\left(\frac{1}{\mu^2} \cdot \mu + \frac{1}{\mu}\right) = \frac{1}{2}\left(\frac{2}{\mu}\right) = \frac{1}{\mu} = E[X]$ ✓