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The Kerr-CFT correspondence

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Abstract

While a full theory of quantum gravity remains elusive, it was hypothesized in [1] that quantum gravity in the near horizon region of an extremal Kerr black hole is holographically dual to a two dimensional conformal field theory, living on the boundary of the near horizon geometry. This is called the *Kerr-CFT correspondence* conjecture. In this thesis, we review this conjecture and briefly mention some of its applications.

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Introduction

The unification of quantum mechanics and general relativity into one complete theory is the holy grail of modern day theoretical physics. In a lot of systems that we study in theoretical physics, only one of these theories is important while the effects of the other one may be safely ignored. Black holes, however, are one system where the effects of both these theories are important. To see this heuristically, one should recall that the Bekenstein–Hawking formula for black hole entropy has factors of G and \hbar appearing with the same power. An understanding of the physics of black holes therefore promises to offer insight into the theory of quantum gravity.

The Kerr-CFT correspondence conjecture states that the physics of the near horizon region of an extremal Kerr black hole, that is, a black hole whose angular momentum and mass satisfy the relation $J = GM_{ADM}^2$, is governed by a two dimensional conformal field theory living on the boundary of the near horizon geometry. Since a full description of the near horizon geometry would necessarily involve a theory of quantum gravity, this conjecture says that quantum gravity in the near horizon geometry of an extremal Kerr black hole is holographically dual to a conformal field theory. Apart from insights into quantum gravity, this conjecture also allows us to analytically study a lot of astrophysical phenomena that occur near the Kerr black holes that exist in our universe. Previously, a lot of these phenomena could only be analyzed numerically. The aim of this thesis is to review in detail and re-derive the important results that lead up to the Kerr-CFT correspondence conjecture.

Chapter 2 contains a detailed discussion of Kerr black holes, their near horizon geometry and a brief reference to some of their remarkable properties. Chapter 3 describes how generators of gauge transformations can be derived using the *Covariant Phase Space Formalism*. We discuss how in manifolds with boundaries, one can obtain conserved charges associated to a certain class of gauge transformations which form the *Asymptotic Symmetry Group*. In Chapter 4, we use the results of Chapter 3 to explore the *Near Horizon Extremal Kerr* geometry. The *Asymptotic symmetry algebra* in this case turns out to be one copy of

the Virasoro algebra with a central charge that exactly reproduces the expression for the entropy of the Kerr black hole using Cardy's formula. The applications of the Kerr-CFT correspondence conjecture are briefly discussed in Chapter 5.

Conventions and notation:

- Metric signature: $(-, +, +, +)$
- Levi-Civita tensor: $\epsilon = \frac{1}{d!} \sqrt{-g} \tilde{\epsilon}_{\mu_1, \mu_2, \dots, \mu_d} dx^{\mu_1} \wedge dx^{\mu_2} \cdots \wedge dx^{\mu_d}$
- Convention for the Levi-Civita tensor density $\tilde{\epsilon}_{\tau r \theta \varphi} = 1$
- Speed of light, $c = 1$.
- Planck's constant \hbar and Newton's constant G are kept explicit.
- d : Exterior derivative on the spacetime manifold.
- Interior derivative between two forms θ, Ω is denoted by $\theta \cdot \Omega$
- \star : Hodge dual.
- \wedge : Wedge product.
- \approx : Weak or on-shell equivalence.
- \equiv : Definition.
- NHEK: Near Horizon Extremal Kerr.
- ASG: Asymptotic Symmetry Group.
- PB: Poisson Bracket.
- DB: Dirac Bracket.
- LB: Lie Bracket.
- CFT: Conformal Field Theory.

Kerr black holes

The aim of this chapter is to discuss the basic properties, thermodynamics and the near horizon geometry of Kerr black holes.

1 The Kerr ‘miracle’

Chandrasekhar famously remarked [3]: “*Kerr’s solution has also surpassing theoretical interest: it has many properties that have the aura of the miraculous about them.*”

Indeed, given its complicated nature and that it took 50 years after Schwarzschild’s solution to discover it, just the existence of the solution in closed form may be one of those miracles. The metric is given as follows [1]:¹

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a \sin^2 \theta d\phi)^2 + \frac{\sin^2 \theta}{\rho^2}[(\hat{r}^2 + a^2)d\phi - adt]^2 + \frac{\rho^2}{\Delta}d\hat{r}^2 + \rho^2 d\theta^2, \quad (2.1)$$

where:

$$\begin{aligned} \Delta &\equiv \hat{r}^2 - 2M\hat{r} + a^2, & \rho^2 &\equiv \hat{r}^2 + a^2 \cos^2 \theta, \\ a &\equiv \frac{GJ}{M}, & M &\equiv GM_{\text{ADM}}. \end{aligned} \quad (2.2)$$

Here, $t \in (-\infty, \infty)$, $\hat{r} \in [0, \infty)$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. Notice that the metric has two manifest Killing vectors given by

$$K \equiv \frac{\partial}{\partial t} \quad \text{and} \quad m \equiv \frac{\partial}{\partial \phi}. \quad (2.3)$$

It is possible to show that these are the only Killing vectors admitted by the Kerr geometry. The corresponding Komar integrals give the black hole mass M_{ADM} and the angular momen-

¹The radial coordinate is denoted as \hat{r} to distinguish it from another coordinate r which we will define later on.

tum J , respectively. Hence, the Kerr solution corresponds to a stationary, axisymmetric, rotating black hole with angular momentum J . Moreover, since a is a constant, the mass parameter M is proportional to the angular momentum J . This result just follows from explicitly evaluating the Komar integral corresponding to the $\frac{\partial}{\partial\phi}$ Killing vector. In the asymptotic limit, as $\hat{r} \rightarrow \infty$, $\frac{\rho^2}{\Delta} \rightarrow 1$ and the metric becomes:

$$ds^2 = -dt^2 + \hat{r}^2 \sin^2 \theta d\phi^2 + d\hat{r}^2 + \hat{r}^2 d\theta^2,$$

which is the usual flat metric in spherical polar coordinates. This means that the ADM mass is a well defined quantity for this geometry and justifies expressions (2.2). The solution has two horizons, an inner and an outer one, which correspond to the two solutions of $\Delta = 0$.

$$\hat{r}_\pm = M \pm \sqrt{M^2 - a^2}. \quad (2.4)$$

The range of allowed values for J is fixed by the condition that a black hole horizon has to enclose the (ring) singularity at $\hat{r} = 0$:

$$\frac{-M^2}{G} \leq J \leq \frac{M^2}{G}. \quad (2.5)$$

It is easy to see that if J lies outside this range, there is no event horizon and there would be a naked singularity. Quantum mechanically, of course, $J = \hbar j$. Notice that at $a = 0$ which corresponds to zero angular momentum, there is just one event horizon given by $\hat{r} = 2M$ and we recover the Schwarzschild solution. In the subsequent discussion, we will mainly be concerned with the *extremal Kerr black hole* for which the bound on J saturates,

$$J = \frac{M^2}{G} \quad \Leftrightarrow \quad a = M, \quad (2.6)$$

and the two horizons coincide. We now proceed to deriving two important quantities corresponding to the Kerr geometry, namely the Hawking temperature and the angular velocity of the outer event horizon.

1.1 Angular velocity of the Kerr horizon

Let us consider an observer moving along a geodesic in Kerr geometry. The two conserved quantities corresponding to the two Killing vectors are energy and angular momentum, given by, $E = -u_\mu K^\mu$ and $L = m^\mu u_\mu$, where u^μ is the observer's 4 velocity.² In particular,

²The minus sign has been put in by hand to define the energy as a positive quantity.

let us consider a zero angular momentum observer (ZAMO) for whom $L = u_\phi = 0$. The *horizon angular velocity* of a Kerr black hole is defined as the angular velocity of such an observer evaluated on \hat{r}_+ [6]. Since $L = u_\phi = 0 = g_{\phi\phi}u^\phi + g_{\phi t}u^t$, we have:

$$\Omega_+ = \frac{d\phi}{dt}\Big|_{\hat{r}_+} = \frac{\frac{d\phi}{d\tau}}{\frac{dt}{d\tau}}\Big|_{\hat{r}_+} = \frac{u^\phi}{u^t}\Big|_{\hat{r}_+} = -\frac{g_{\phi t}}{g_{\phi\phi}}\Big|_{\hat{r}_+} = \frac{a}{2M\hat{r}_+}, \quad (2.7)$$

1.2 Hawking temperature and entropy

The area of the event horizon corresponding to $\hat{r} = \hat{r}_+$ is given by:

$$A_+ = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sqrt{\det g}|_{\hat{r}_+} = 2M\hat{r}_+ \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta = 8\pi M\hat{r}_+. \quad (2.8)$$

Using now Bekenstein’s result that the black hole entropy equals a quarter of the horizon area measured in Planck units [5], we recover:

$$S_{BH} = \frac{A_+}{4\hbar G} = \frac{2\pi M\hat{r}_+}{\hbar G}. \quad (2.9)$$

To obtain an expression for the Hawking temperature T_H , we impose the first law of black hole thermodynamics,

$$dM_{ADM} = T_H dS_{BH} + \Omega_+ dJ. \quad (2.10)$$

So we differentiate both sides of equation (2.9) with respect to M while holding the angular momentum constant:

$$\frac{\partial S_{BH}}{\partial M}\Big|_J = \frac{2\pi}{\hbar G} \frac{\partial}{\partial M} [(M(M + \sqrt{M^2 - (GJ/M)^2}))]\Big|_J.$$

Performing the algebra we get:

$$\left(\frac{\partial M}{\partial S_{BH}}\right)^{-1}\Big|_J = \frac{\hbar G}{2\pi} \left[\frac{\sqrt{M^2 - (GJ/M)^2}}{2M\sqrt{M^2 - (GJ/M)^2} + 2M^2} \right] = \frac{\hbar G}{4\pi} \left[\frac{\hat{r}_+ - M}{M\hat{r}_+} \right].$$

Since $M = GM_{ADM}$, we finally recover:

$$T_H = \left(\frac{\partial M_{ADM}}{\partial S}\right)^{-1}\Big|_J = \frac{\hbar}{4\pi} \left[\frac{\hat{r}_+ - M}{M\hat{r}_+} \right]. \quad (2.11)$$

Notice that for an extreme Kerr black hole $\hat{r}_+ = M$ and the Hawking temperature vanishes, $T_H = 0$. However, the entropy is non zero and is given by:

$$S_{BH} = \frac{2\pi M^2}{\hbar G} = \frac{2\pi J}{\hbar}. \quad (2.12)$$

1.3 Other ‘miracles’ of Kerr

Here we briefly mention some of the other remarkable properties of Kerr black holes, which are discussed in detail in [2], [4], [8] and [19].

- **Kerr-Schild form:** The Kerr black hole metric can be written in the following manner:

$$g_{\mu\nu} = \eta_{\mu\nu} + 2Hk_\mu k_\nu, \quad (2.13)$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric, k is a null vector, and $H \propto M$ is a scalar function. This is called the *Kerr-Schild form*. Since the only dependence on M is through H , one can think of the Kerr metric as a (linear in M) fluctuation about flat spacetime, quite a remarkable property for an exact solution of the non-linear Einstein equations.

- **Carter’s constant:** As mentioned in Sec. 1, the Kerr metric has two Killing vectors, ∂_t and ∂_ϕ . These provide two constants of motion for a geodesic in the Kerr geometry, namely, the energy and angular momentum. The normalization of the 4-velocity provides another conserved quantity. However, for geodesic equations to be integrable in 4 spacetime dimensions, we need 4 integrals of motion. Remarkably, there is another ‘hidden’ constant of motion, originally derived through a method of separation of variables in the Hamilton–Jacobi equation [7]. The existence of this additional constant of motion is a consequence of the fact that the Kerr geometry admits a *Killing–Yano tensor* $f_{\mu\nu}$ obeying: $f_{\mu\nu} = -f_{\mu\nu}, \nabla_{(\lambda} f_{\mu)\nu} = 0$, and given by [2]:

$$f = a \cos \theta dr \wedge (dt - a \sin^2 \theta d\phi) - r \sin \theta d\theta \wedge [adt + (r^2 + a^2)d\phi]. \quad (2.14)$$

This additional constant of motion, called Carter’s constant, is a quantity quadratic in velocities, given by:

$$Q_{\text{Carter}} = u^\mu u^\nu K_{\mu\nu}, \quad (2.15)$$

where $K_{\mu\nu} = f_\mu^\nu f_{\lambda\nu}, \nabla_{(\mu} K_{\nu\rho)} = 0$. This fourth constant of motion makes the geodesic equations in Kerr geometry completely integrable. Despite the fact that we know the explicit expression for the Killing–Yano tensor and use it to construct a conserved quantity, its geometric origin is still not fully understood [2]. The fact that something whose origin we do not fully understand gives us just the right amount of information to make geodesic equations in Kerr integrable is again something that gives the Kerr geometry an aura of the ‘miraculous.’

- **Frame dragging:** Recall that the location of the outer event horizon of the Kerr black hole corresponds to the largest root of Δ . However, it is easy to see from (2.1)

that g_{tt} changes sign at $\Delta = a^2 \sin^2 \theta$ and hence ∂_t becomes spacelike outside the event horizon. The surface corresponding to the zero of g_{tt} is called the *ergosphere*. The region in between the event horizon and the *ergosphere* is called the *ergoregion*. Notice that the metric (2.1) only involves cross terms between $d\phi$ and dt . Moreover, the term $g_{\phi t}$ is negative in that region. As a result, for trajectories to have tangent vectors that are timelike in this region, the tangent vector must acquire an additional ∂_ϕ piece. Consequently, any observer in that region must co-rotate with the black hole. Since observers cannot stay stationary in the *ergoregion* and must co-rotate with the black hole, the black hole is said to “drag” local frames around it. This is an example of a phenomenon called *frame dragging*.

- **Penrose process and superradiance:** Since the Killing vector ∂_t becomes spacelike in the *ergoregion*, $E = -u^\mu K_\mu$ becomes negative. If the Kerr black hole absorbs such a particle with negative energy, (positive) energy can be extracted from the black hole. This is called the *Penrose process*. A more famous analogue of this process is called *superradiance*. Namely, waves sent towards the black hole with certain frequencies return to the exterior region with an increased amplitude. In both cases, rotational energy of the black hole is extracted and the black hole loses part of its angular momentum.

2 The near horizon geometry

Let us now focus on the extremal Kerr black hole. By re-defining the coordinates in a certain way, one can zoom in to the near horizon region and think of it as a spacetime on its own. This is called the *Near Horizon Extremal Kerr (NHEK)* geometry [9].³ In this section, we re-derive the corresponding limit and study some properties of the NHEK geometry.

Following the approach given in [1], [9] and [14], new dimensionless, co-rotating coordinates are defined as follows:

$$t' = \frac{\lambda t}{2M}, \quad y = \frac{\lambda M}{\hat{r} - M}, \quad \phi' = \phi - \frac{t}{2M}, \quad \theta = \theta, \quad (2.16)$$

where $t' \in (-\infty, \infty)$, $y \in [0, \infty)$, $\theta \in [0, \pi]$ and $\phi' \sim \phi' + 2\pi$. Taking the near horizon limit corresponds to taking the limit $\lambda \rightarrow 0$. In these coordinates, points that are close to the horizon get blown up while everything that is far from it approaches $y \rightarrow 0$. In particular,

³It was shown in [9] that for global coordinates (given below), the near horizon region is a geodesically complete manifold and can be treated as an independent spacetime.

the event horizon is now at $y = \infty$. The metric becomes:

$$ds^2 = 2GJ\Omega^2 \left[\frac{-dt'^2 + dy^2}{y^2} + d\theta^2 + \Lambda^2(d\phi' + \frac{dt'}{y})^2 \right], \quad (2.17)$$

$$\Omega^2 \equiv \frac{1 + \cos^2 \theta}{2}, \quad \Lambda \equiv \frac{2 \sin \theta}{1 + \cos^2 \theta}. \quad (2.18)$$

Notice that if we keep t' and θ constant and then define $y \equiv e^{-\sigma}$, the induced metric is:

$$ds'^2 = 2GJ\Omega^2(d\sigma^2 + \Lambda^2d\phi'^2). \quad (2.19)$$

The location of the horizon, $y = \infty$ corresponds to $\sigma = -\infty$. In particular, the geometry at fixed t' and θ looks like an infinite cylinder or an infinite throat as it is usually referred to in the literature. The event horizon in these coordinates corresponds to the bottom of this infinite throat. Notice that the t', y part of (2.17) bears a resemblance to the metric of AdS_2 . To see this explicitly, recall that AdS_2 embedded in \mathbb{R}^3 is a hyperboloid given by $-X^2 - Y^2 + Z^2 = -1$. In our choice of coordinates, X, Y, Z can be parametrized as follows [14]:

$$X + Z = \frac{1}{y}, \quad X - Z = y - \frac{t'}{y^2}, \quad Y = \frac{t'}{y}.$$

The coordinates t', y are analogs of Poincare coordinates on AdS_2 and they cover only part of the NHEK geometry. One way to see that is to notice that in the parametrization given above, $X + Z$ is always positive because $y \in [0, \infty)$. To cover the whole throat geometry, we make the following parametrization using global coordinates τ, r , as given in [14]:

$$X = \sqrt{1 + r^2} \cos \tau, \quad Y = \sqrt{1 + r^2} \sin \tau, \quad Z = r.$$

By comparing these two expressions, we obtain:

$$y = (\sqrt{1 + r^2} \cos \tau + r)^{-1}, \quad t' = y \sqrt{1 + r^2} \sin \tau. \quad (2.20)$$

As a result:

$$-\frac{dt'^2}{y^2} + \frac{dy^2}{y^2} = -(1 + r^2)d\tau^2 + \frac{dr^2}{1 + r^2}, \quad \frac{dt'}{y} = rd\tau + d\gamma,$$

where $\gamma = \ln \left| \frac{\cos \tau + r \sin \tau}{1 + \sin \tau \sqrt{1 + r^2}} \right|$. Defining finally:

$$\phi' \equiv \varphi + \ln \left| \frac{\cos \tau + r \sin \tau}{1 + \sqrt{1 + r^2} \sin \tau} \right|, \quad \theta = \theta, \quad (2.21)$$

the metric (2.17) simplifies to:

$$ds^2 = 2GJ\Omega^2 \left[-(1+r^2)d\tau^2 + \frac{dr^2}{1+r^2} + d\theta^2 + \Lambda^2(d\varphi + rd\tau)^2 \right], \quad (2.22)$$

with $\tau \in (-\infty, \infty)$, $\varphi \sim \varphi + 2\pi$, $\theta \in [0, \pi]$, and $r \in (-\infty, \infty)$. This is the form of NHEK metric that we will use from now on. Note that τ is not identified here to avoid closed timelike curves. Moreover, in these coordinates, there is a boundary at $r = \infty$ ⁴ which corresponds to the entrance of the throat which was at $y = 0$ in Poincare coordinates. This is where the full, asymptotically flat Kerr geometry (outside the black hole horizon) joins the near horizon geometry. Notice also that in these coordinates, as $r \rightarrow \infty$ we no longer get the Minkowski metric and so the near horizon extremal Kerr geometry is no longer asymptotically flat.

Enhanced symmetry of NHEK

From (2.22), one can see that $\xi_0 = \partial_\varphi$ is a Killing vector of the NHEK geometry. In addition, the time-like Killing vector becomes part of an extended set of isometries generated by [1]:

$$J_1 = \frac{2r}{\sqrt{1+r^2}} \sin \tau \partial_\tau - 2\sqrt{1+r^2} \cos \tau \partial_r + \frac{2 \sin \tau}{1+r^2} \partial_\varphi, \quad (2.23)$$

$$J_{-1} = -\frac{2r}{\sqrt{1+r^2}} \cos \tau \partial_\tau - 2\sqrt{1+r^2} \sin \tau \partial_r - \frac{2 \cos \tau}{1+r^2} \partial_\varphi, \quad (2.24)$$

$$J_0 = 2\partial_\tau. \quad (2.25)$$

Notice that none of these Killing vectors has a ∂_θ piece. They, therefore, all act within a fixed θ slice of the four-dimensional geometry. Moreover, they satisfy the following Lie bracket relations:

$$[J_1, J_{-1}]_{\text{LB}} = 2J_0, \quad [J_1, J_0]_{\text{LB}} = 2J_{-1}, \quad [J_{-1}, J_0]_{\text{LB}} = -2J_1, \quad [\xi_0, J_i]_{\text{LB}} = 0.$$

The generators given above thus form an $SL(2, \mathcal{R}) \times U(1)$ algebra. The $U(1)$ corresponds to the Killing vector ξ_0 . Recall that $SL(2, \mathcal{R})$ is the isometry group of AdS_2 . The algebra thus reflects the fact that in a fixed θ slice, the NHEK geometry is locally $AdS_2 \times U(1)$. Globally, it is a quotient ($\varphi \sim \varphi + 2\pi$) of "warped" AdS_3 in the sense that the τ, r plane describes AdS_2 while the φ part describes a Hopf fibration of S^1 on AdS_2 . Near the equator,

⁴There is also a boundary at $r = -\infty$ but we will not be concerned with that here. We can impose sufficiently strong boundary conditions to ensure that there is no interesting physics at that boundary.

the radius of the Hopf-fibred S^1 is maximum and it goes to zero at the poles. Hence, near the equator, the AdS_3 quotient is stretched and near the poles it is squashed.

There is one last important observation to make before concluding this chapter. Consider the coordinate relations given in equation (2.20)–(2.21). Using the chain rule for partial differentiation and these relations, it follows that in the limit $r \rightarrow \infty$ and $\tau \rightarrow 0$, $\partial_\tau \propto \frac{2M}{G}\partial_t + \partial_\phi$. This means that the variation of the “asymptotic (Komar) charge” corresponding to this Killing vector obeys: $\delta Q_\tau \propto \delta(\frac{M^2}{G} - J)$. In the extremal Kerr case, we know that $\frac{M^2}{G} = J$. Since we want to keep the condition of extremality in place throughout, we will enforce that δQ_τ vanishes. We will come back to this point in Chapter 4.

This concludes our discussion of the geometry of Kerr black holes. In the next chapter, we will describe a general procedure for constructing generators and conserved charges in gauge theories defined on manifolds with boundaries.

Gauge symmetries and conserved charges

The aim of this chapter is to discuss the construction of generators and conserved charges corresponding to the gauge symmetries that underlie gauge invariant theories. After explaining the general setup, we will concentrate on two important examples: Maxwell's theory and Einstein gravity. Based on our understanding of the latter, we will study conserved charges corresponding to a particular type of diffeomorphisms in the NHEK geometry in Chapter 4. The formalism that is used in this chapter is called the *Covariant Phase Space Formalism* for gauge theories, and was originally developed in [12], [13] and [15]. Most of the ideas and derivations presented in this chapter are based on [10] and [11].

1 Structure, setup and notation

Our starting point is to define the notion of a *phase space*. We start by considering \mathcal{F} , the space of all possible field configurations denoted by $\Phi(x)$, which satisfy a particular set of boundary conditions. These include field configurations which do not satisfy the field equations in our theory. In particular, the set of on-shell field configurations form a subspace which is denoted by $\bar{\mathcal{F}}$. The fields take values on the spacetime manifold, which we take to have spacetime dimension d . A particular configuration of the field on the spacetime manifold is represented by a point in this space. Next, we define *presymplectic structure* on this space. This structure satisfies the same properties as *symplectic structure* (see Appendix D), except for the fact that it has degeneracies. A reduction over these degeneracies is equivalent to taking a *symplectic quotient* (see, for example, [15]). The resulting space Γ has symplectic structure on it, it has the structure of a manifold, and is called the *phase space*.

Let δ denote a field variation that lies in the tangent space of the phase space manifold.

In what follows we consider bosonic fields which satisfy $\delta_1\delta_2\Phi - \delta_2\delta_1\Phi = 0$. A *field theory* is defined by the Lagrangian $L[\Phi]$, which is a *top form* (a form of degree d), written as $L(\Phi) \equiv \epsilon\mathcal{L}$ where ϵ is the volume form and $\mathcal{L} = \mathcal{L}(\Phi)$ is the Lagrangian density; Φ in the argument denotes all the dynamical fields in the system. The variation of the Lagrangian with respect to Φ is:

$$\delta L[\Phi] = E[\Phi]\delta\Phi + d\Theta[\delta\Phi, \Phi], \quad (3.1)$$

where $E[\Phi] = \frac{\delta L}{\delta\Phi}$ is the Euler–Lagrange equation and $\Theta[\delta\Phi, \Phi]$ is a $(d-1)$ -form, called the *presymplectic potential*. A summation over fields Φ is assumed here. Moreover, δ , the variation of fields on the phase space manifold, and d , the exterior derivative on the spacetime manifold, commute by construction. The solution to (3.1) has the form:

$$\Theta[\delta\Phi, \Phi] = \Theta^{ref}[\delta\Phi, \Phi] + dY[\delta\Phi, \Phi]. \quad (3.2)$$

Θ^{ref} collectively refers to the terms that are left behind after one does integration by parts on the variation of the Lagrangian and collects the terms which are multiplied by $\delta\Phi$, and Y is some arbitrary $(d-2)$ -form. The fact that the potential is defined up an “arbitrary term” is a recurring theme which will appear in other *presymplectic* quantities as well. We will not write these extra terms explicitly from here on because for the purposes of computing the variation in the conserved charge, which is essentially all we will be interested in here, these terms will not matter. These extra terms are indicative of the fact that there are degeneracies in our definitions. In fact, as mentioned earlier, to obtain a *symplectic manifold*, one should really fix all of these ambiguities and mod out the degeneracies. The quantities obtained after that will be *symplectic*. However, for our purposes, these extra terms just define a reference value or “zero” for the charge. We will therefore assume that the degeneracies have been fixed and will just talk about *symplectic quantities* from now on.

Let us define the *symplectic current* as:

$$\omega[\delta_1\Phi, \delta_2\Phi, \Phi] \equiv \delta_1\Theta[\delta_2\Phi, \Phi] - \delta_2\Theta[\delta_1\Phi, \Phi]. \quad (3.3)$$

This quantity has the following important property. Let Φ be a solution of the equations of motion and $\delta_1\Phi, \delta_2\Phi$ be solutions to the linearized field equations around Φ , that is, $\delta_1E[\Phi] \equiv \left.\frac{\delta E[\Phi]}{\delta\Phi}\right|_{\Phi} [\delta_1\Phi]^A = 0$ and $\delta_2E[\Phi] \equiv \left.\frac{\delta E[\Phi]}{\delta\Phi}\right|_{\Phi} [\delta_2\Phi]^B = 0$, then¹:

$$d\omega[\delta_1\Phi, \delta_2\Phi, \Phi] \approx 0. \quad (3.4)$$

¹Indices A, B denote the components of the variations for some choice of coordinates on the phase space manifold. The summation convention is understood.

This can be seen as follows:

$$\begin{aligned} d\omega[\delta_1\Phi, \delta_2\Phi, \Phi] &= \delta_1 d\Theta[\delta_2\Phi, \Phi] - \delta_2 d\Theta[\delta_1\Phi, \Phi] \\ &= \delta_1(\delta_2 L[\Phi] - E[\Phi]\delta_2\Phi) - \delta_2(\delta_1 L[\Phi] - E[\Phi]\delta_1\Phi) \\ &= -\delta_1 E[\Phi]\delta_2\Phi + \delta_2 E[\Phi]\delta_1\Phi \approx 0, \end{aligned}$$

where in the first step, we have used (3.3) and the commutativity of δ and d and in the second step, we have used (3.1). Since Φ satisfies the equations of motion and $\delta_1\Phi$ and $\delta_2\Phi$ satisfy the linearized equations of motion by assumption, the two terms in the last expression vanish individually on shell.

Next, we define the *symplectic two form* Ω_{AB} :²

$$\Omega_{AB}[\delta_2\Phi]^A[\delta_1\Phi]^B = \int_{\Sigma} \omega[\delta_1\Phi, \delta_2\Phi, \Phi], \quad (3.5)$$

where Σ is some spacelike surface on the spacetime manifold. In this formalism, the integral of the symplectic current over the boundary is always zero by construction. As a result, if one continuously deforms the spacelike surface being integrated over, using Stokes' theorem, one sees that the difference between the integrals over the two surfaces is proportional to $d\omega$ integrated over the spacetime region between the two surfaces. As a result of (3.4), we see that this vanishes on shell. Therefore, the on shell definition of the symplectic form, is independent of the spacelike surface that is integrated over. This freedom in the choice of spacelike hypersurface is essentially why this formalism is covariant.

2 Generators of gauge symmetries

Having defined the symplectic two-form Ω_{AB} , the Poisson bracket for any two functions H and F of the fields Φ is defined as:

$$\{H, F\}_{PB} \equiv \Omega^{AB} \frac{\delta H}{[\delta\Phi]^A} \frac{\delta F}{[\delta\Phi]^B}, \quad (3.6)$$

where $\Omega^{AB}\Omega_{BC} = \delta_C^A$.

We are interested in calculating the generators of gauge transformations that cause the flow from one field configuration to another on the phase space manifold. The generator of a

²As a result of (3.3) Ω_{AB} is antisymmetric in the two indices. By defining an exterior derivative on the phase space, as done in [10], one can also show that it is closed.

gauge transformation H_ζ satisfies the following relation by definition:

$$\{H_\zeta, \Phi^A\}_{PB} = [\delta_\zeta \Phi]^A = [\mathcal{L}_\zeta \Phi]^A, \quad (3.7)$$

where ζ is the vector field along which the flow occurs. Obviously, from this definition we have:

$$[\delta_\zeta \Phi]^A = \Omega^{BC} \frac{\delta H_\zeta}{[\delta \Phi]^B} \frac{\delta \Phi^A}{[\delta \Phi]^C} = \Omega^{BA} \frac{\delta H_\zeta}{[\delta \Phi]^B}. \quad (3.8)$$

Multiplying both sides by Ω_{AC} we get that:

$$\frac{\delta H_\zeta}{[\delta \Phi]^C} = \Omega_{AC} [\delta_\zeta \Phi]^A. \quad (3.9)$$

So we have shown:

$$\delta H_\zeta = [\delta \Phi]^C \frac{\delta H_\zeta}{[\delta \Phi]^C} = \Omega_{AC} [\delta_\zeta \Phi]^A [\delta \Phi]^C = \int_{\Sigma} \omega [\delta \Phi, \delta_\zeta \Phi, \Phi]. \quad (3.10)$$

The charge corresponding to a gauge transformation is defined to be the on shell value of the generator. In general, the charge, labeled by \mathcal{Q}_ζ in this case, is given by an integration in the phase space over an arbitrary curve which connects a reference field configuration Φ_0 and Φ ³.

$$\mathcal{Q}_\zeta[\Phi] \equiv \int_{\Phi_0}^{\Phi} \delta \mathcal{Q}_\zeta + \mathcal{Q}_\zeta[\Phi_0], \quad (3.11)$$

For infinitesimal field transformation, which is what we will be interested in in this thesis, we define:

$$\mathcal{Q}_\zeta[\Phi] \equiv \mathcal{Q}_\zeta[\Phi] - \mathcal{Q}_\zeta[\Phi_0] = \delta \mathcal{Q}_\zeta. \quad (3.12)$$

where we have essentially set $\mathcal{Q}_\zeta[\Phi_0]$ to be our reference value or “zero” for the charge.

3 Generators and charges in Maxwell’s theory

Let us now look at a concrete example of the formalism developed in the previous section. We consider Maxwell theory’s:

$$S_{\text{Maxwell}} = \int \left[\frac{-1}{4} F_{\mu\nu} F^{\mu\nu} \right] \epsilon \quad (3.13)$$

³This assumes that the charges are integrable, that is, the definition of charge does not depend on the path of integration over the phase space. The proof of this is omitted but may be found in [10] and [11].

embedded in flat spacetime with a boundary. The variation of the action gives:

$$\delta S_{\text{Maxwell}} = \int \epsilon [F_{\mu\nu} \partial^\mu \delta A^\nu] = \int \epsilon [(\partial^\mu (F_{\mu\nu} \delta A^\nu) - (\partial^\mu F_{\mu\nu}) \delta A^\nu)]. \quad (3.14)$$

Comparing this with equation (3.1) we get: $(\star\Theta)^\mu [\delta A, A] = F^{\mu\nu} \delta A_\nu$. We are interested in calculating the generator associated with the gauge transformation: $\delta_\alpha A_\mu = \partial_\mu \alpha(x)$. The corresponding symplectic form is then, using (3.2), given as:

$$\omega[\delta A, \delta_\alpha A, A] = d\Sigma_\mu (\delta F^{\mu\nu} \delta_\alpha A_\nu - \delta_\alpha F^{\mu\nu} \delta A_\nu), \quad (3.15)$$

where δA represents some other variation in A which does not correspond to a gauge transformation, that is, it is not of the form $\delta_\alpha A_\mu = \partial_\mu \alpha(x)$, but is some other solution of the linearized equations of motion. Since $F^{\mu\nu}$ is gauge invariant, the second term in (3.15) vanishes. Since δH_ζ denotes $\int_\Sigma \omega[\delta\Phi, \delta_\zeta\Phi, \Phi]$, we see that:

$$\begin{aligned} \delta H_{\alpha(x)} &= \int_\Sigma d\Sigma_\mu \delta F^{\mu\nu} \partial_\nu \alpha(x) \\ &= \int_\Sigma d\Sigma_\mu \delta [\partial_\nu (F^{\mu\nu} \alpha(x)) - \partial_\nu (F^{\mu\nu}) \alpha(x)]. \end{aligned} \quad (3.16)$$

Here, we have used the fact that δ does not act on $\alpha(x)$. From here, one can just read off the generator:

$$H_{\alpha(x)} = \int_\Sigma d\Sigma_\mu ((\partial_\nu (F^{\mu\nu} \alpha(x)) - \partial_\nu (F^{\mu\nu}) \alpha(x)). \quad (3.17)$$

The second term vanishes on shell and the first term becomes an integral over the boundary using Stokes' theorem. Consequently, we get:

$$\mathcal{Q}_\alpha = \int_{\partial\Sigma} d\Sigma_{\mu\nu} [F^{\mu\nu} \alpha(x)], \quad (3.18)$$

where \mathcal{Q}_α is the on shell value of the generator or the corresponding charge. As a useful check, we see that as $\alpha \rightarrow 1$ near the boundary, we get the electric charge as the on shell value the generator which is what we know it should be for the global U(1) transformation. We will now apply this formalism to Einstein gravity.

4 Generators and charges in Einstein gravity

In this section, we explicitly construct the generators corresponding to diffeomorphisms in Einstein gravity. The Einstein–Hilbert Lagrangian is invariant under local coordinate transformations up to total derivative terms. Let us denote the vector field that generates

these coordinate transformations by ζ . Then we have

$$\delta_\zeta L = \mathcal{L}_\zeta L = \zeta \cdot dL + d(\zeta \cdot L) = d(\zeta \cdot L), \quad (3.19)$$

where we have used Cartan's identity $\mathcal{L}_X \Omega = X \cdot d\Omega + d(X \cdot \Omega)$ in addition to the fact that dL is zero since L is defined to be a top degree form. The following quantity is then called the *first Noether current*:

$$J_\zeta = \Theta[\delta_\zeta \Phi, \Phi] - M_\zeta[\Phi], \quad M_\zeta = \zeta \cdot L. \quad (3.20)$$

Taking the exterior derivative of the above equation, we get:

$$dJ_\zeta = d\Theta[\delta_\zeta \Phi, \Phi] - dM_\zeta[\Phi] = \delta_\zeta L - E[\Phi]\delta_\zeta \Phi - \delta_\zeta L = -E[\Phi]\delta_\zeta \Phi,$$

where we have used equations (3.1) and (3.19). When the equations of motion hold, $E[\Phi] = 0$ and hence $dJ \approx 0$. This corresponds to Noether's first theorem. In a similar fashion, we see the following: after moving the covariant derivatives in the $\delta_\zeta \Phi$ part of $E[\Phi]\delta_\zeta \Phi$ onto $E[\Phi]$ by using the product rule for covariant differentiation,⁴ we get:

$$E[\Phi]\delta_\zeta \Phi = \zeta \cdot N(E[\Phi], \Phi) + dS_\zeta(E[\Phi], \Phi), \quad (3.21)$$

Here, S_ζ corresponds to the extra term that one gets in addition to Noether's first current, in the expression for *Noether's second current*. Recall that this term vanishes on shell. Moreover, the first term on the right hand side of (3.21) vanishes off shell because of Noether's second theorem and $N(E[\Phi], \Phi)$ corresponds to the *Bianchi identity*. As a result, we get the following off shell relation:

$$dJ_\zeta + dS_\zeta(E[\Phi], \Phi) = 0 \Rightarrow J_\zeta + S_\zeta(E[\Phi]) = dQ_\zeta,$$

where in the last line, we have invoked Poincare's algebraic lemma. The Q_ζ ⁵ is a $(d-2)$ -form and it is the Noether charge density associated with ζ . Now we vary equation (3.20) with respect to the fields. We assume here that ζ is independent of the fields.⁶

$$\delta J_\zeta = \delta\Theta[\delta_\zeta \Phi, \Phi] - \zeta \cdot \delta L, \quad (3.22)$$

$$\Rightarrow \zeta \cdot \delta L = \zeta \cdot d\Theta[\delta\Phi, \Phi] + \zeta \cdot (E[\Phi]\delta\Phi) = \mathcal{L}_\zeta\Theta[\delta\Phi, \Phi] - d(\zeta \cdot \Theta[\delta\Phi, \Phi]) + \zeta \cdot (E[\Phi]\delta\Phi),$$

⁴Remember that identifying Φ with the metric g we get $\delta_\zeta g_{\mu\nu} = \mathcal{L}_\zeta g_{\mu\nu} = \nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu$.

⁵The Noether charge is represented by Q whereas the charge corresponding to gauge transformations is represented by \mathcal{Q} .

⁶A more general derivation of this result applicable to the case when it does depend on the fields may be found in [10].

where in the first equality, equation (3.1) and in the second, Cartan's identity has been used. Now, recall that δ, d commute. Hence, we have that: $\delta J_\zeta = -\delta S_\zeta + d(\delta Q_\zeta)$. Using this, equation (3.22) and the expression below it, we see that:

$$-\delta S_\zeta + d(\delta Q_\zeta) = (\delta\Theta[\delta_\zeta\Phi, \Phi] - \mathcal{L}_\zeta(\Theta[\delta\Phi, \Phi])) + d(\zeta \cdot \Theta[\delta\Phi, \Phi]) - \zeta \cdot (E[\Phi]\delta\Phi). \quad (3.23)$$

Now, if we compare this expression with (3.3), we see that the first two terms on the r.h.s. become $\omega(\delta\Phi, \delta_\zeta\Phi, \Phi)$. Hence,

$$\omega(\delta\Phi, \delta_\zeta\Phi, \Phi) = (-\delta S_\zeta + \zeta \cdot (E[\Phi]\delta\Phi)) + d(\delta Q_\zeta - \zeta \cdot \Theta[\delta\Phi, \Phi]). \quad (3.24)$$

On shell, the first two terms on the r.h.s. vanish and we are left with the following expression:

$$\delta H_\zeta = \int_{\Sigma} \omega[\delta\Phi, \delta_\zeta\Phi, \Phi] \approx \int_{\partial\Sigma} k_\zeta[\delta\Phi, \Phi], \quad k_\zeta[\delta\Phi, \Phi] \equiv \delta Q_\zeta - \zeta \cdot \Theta[\delta\Phi, \Phi]. \quad (3.25)$$

The result above shows that the infinitesimal variation of the generator has a term on the boundary which is in general non-vanishing even on shell. Recall that the generators of gauge transformations correspond to constraints in manifolds without boundaries. In this scenario, they behave as constraints in the bulk of the manifold. We see from (3.25) that the variation of these generators acquires a non-zero boundary term. This means that the Poisson bracket of two such constraints does not vanish on shell. These are therefore called *second class constraints*. The natural way to deal with a system with second class constraints is to replace all Poisson brackets with Dirac brackets (see Appendix C). As a result, in the next chapter, we will compute infinitesimal variations of generators and charges using Dirac brackets.

Conserved charges for Einstein gravity

Our next task is to explicitly construct generators for diffeomorphisms in Einstein's gravity. We will work in 4 spacetime dimensions. The Einstein–Hilbert action is given by⁷:

$$S_{EH} = \frac{1}{16\pi G} \int \epsilon R. \quad (3.26)$$

⁷The symplectic current and the symplectic two-form are insensitive to the addition of a Gibbons–Hawking term (See, for example, [11]) and so we will work with just this piece of the action.

We want to calculate the quantity $k_\zeta[\delta\Phi, \Phi]$, (3.25). A variation of the action with respect to the metric gives us the following:

$$\delta S_{EH} = \frac{1}{16\pi G} \int \epsilon \left[\delta g^{\mu\nu} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) - \nabla_\mu \nabla^\mu \delta g_\alpha^\alpha + \nabla_\mu \nabla_\nu \delta g^{\mu\nu} \right]. \quad (3.27)$$

Comparing this expression with (3.1), we can read off that:

$$\begin{aligned} \Phi(x^\alpha) &\rightarrow g_{\mu\nu}(x^\alpha), & E[\Phi]\delta\Phi &= \frac{d^d x \sqrt{-g}}{16\pi G} G_{\mu\nu} \delta g^{\mu\nu}, \\ (\star\Theta)^\mu[\delta\Phi, \Phi] &= \frac{1}{16\pi G} (\nabla_\nu \delta g^{\mu\nu} - \nabla^\mu \delta g_\alpha^\alpha). \end{aligned}$$

Moreover, the corresponding expression for the Noether charge is well-known:⁸

$$(\star Q^{\mu\nu})_\zeta \approx \frac{1}{16\pi G} (\nabla^\nu \zeta^\mu - \nabla^\mu \zeta^\nu). \quad (3.28)$$

Taking the Hodge dual, this reads

$$Q_\zeta \approx \frac{-1}{16\pi G} \frac{\sqrt{-g}}{2!2!} \tilde{\epsilon}_{\mu\nu\alpha_1\alpha_2} (\nabla^\mu \zeta^\nu - \nabla^\nu \zeta^\mu) dx^{\alpha_1} \wedge dx^{\alpha_2}, \quad (3.29)$$

and results in

$$\delta Q_\zeta \approx \frac{-\tilde{\epsilon}_{\mu\nu\alpha_1\alpha_2}}{32\pi G} \left[(\delta \sqrt{-g}) g^{\mu\beta} (\nabla_\beta \zeta^\nu) - \sqrt{-g} \delta g^{\mu\beta} (\nabla_\beta \zeta^\nu) + \sqrt{-g} g^{\mu\beta} \delta (\nabla_\beta \zeta^\nu) \right] dx^{\alpha_1} \wedge dx^{\alpha_2}, \quad (3.30)$$

Similarly we have

$$\begin{aligned} \zeta \cdot \Theta &= \zeta \cdot \left(\frac{-1}{16\pi G} \frac{\sqrt{-g}}{3!} \tilde{\epsilon}_{\mu\alpha_1\dots\alpha_3} (\nabla_\alpha \delta g^{\alpha\mu} - \nabla^\mu \delta g_\alpha^\alpha) dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_3} \right), \\ &= \frac{-1}{16\pi G} \frac{\sqrt{-g}}{2!} \tilde{\epsilon}_{\mu\nu\alpha_2\alpha_3} (\nabla_\alpha \delta g^{\alpha\mu} - \nabla^\mu \delta g_\alpha^\alpha) \zeta^\nu dx^{\alpha_2} \wedge dx^{\alpha_3}, \end{aligned} \quad (3.31)$$

where we have made the following definition: $\delta g^{\mu\nu} \equiv g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta} = -\delta(g^{\mu\nu})$. Now using;

$$\delta \sqrt{-g} = \frac{1}{2\sqrt{-g}} \delta g_\alpha^\alpha, \quad \delta \Gamma_{\mu\nu}^\lambda = \frac{1}{2} [g^{\lambda\sigma} (\nabla_\mu \delta g_{\sigma\nu} + \nabla_\nu \delta g_{\sigma\mu} - \nabla_\sigma \delta g_{\mu\nu})],$$

we get:

$$\delta Q_\zeta \approx \frac{-1}{16\pi G} \frac{\sqrt{-g}}{2!} \tilde{\epsilon}_{\mu\nu\alpha_1\alpha_2} \left[\frac{\delta g_\alpha^\alpha}{2} (\nabla^\mu \zeta^\nu) - \delta g^{\mu\beta} \nabla_\beta \zeta^\nu + \zeta^\alpha \nabla^\mu \delta g_\alpha^\nu \right] dx^{\alpha_1} \wedge dx^{\alpha_2}. \quad (3.32)$$

⁸Notice that for a global Killing vector ζ , this is the well known expression for the Komar charge. Instead of deriving (3.28), we just used the fact that on-shell, the expression for Q_ζ must coincide with the expression for the charge corresponding to a global Killing vector ζ , that is, the Komar charge. A more detailed derivation, however, is given in [11].

Putting all of this together:

$$k_{\zeta}^{EH}[\delta g, g] \approx \frac{-1}{16\pi G} \frac{\sqrt{-g}}{2!} \tilde{\epsilon}_{\mu\nu\alpha_1\alpha_2} \left[\frac{\delta g_{\alpha}^{\alpha}}{2} (\nabla^{\mu}\zeta^{\nu}) - \delta g^{\mu\beta} \nabla_{\beta}\zeta^{\nu} + \zeta^{\alpha} \nabla^{\mu} \delta g_{\alpha}^{\nu} + (\nabla_{\alpha} \delta g^{\alpha\mu} - \nabla^{\mu} \delta g_{\alpha}^{\alpha}) \zeta^{\nu} \right] dx^{\alpha_1} \wedge dx^{\alpha_2}. \quad (3.33)$$

Rearranging these terms and denoting $\delta g^{\mu\nu} \rightarrow h^{\mu\nu}$ this is equivalent to:

$$k_{\zeta}^{EH}[h, g] \approx \frac{\sqrt{-g}}{32\pi G} \tilde{\epsilon}_{\alpha\beta\mu\nu} [\zeta^{\nu} \nabla^{\mu} h - \zeta^{\nu} \nabla_{\sigma} h^{\mu\sigma} + \zeta_{\sigma} \nabla^{\nu} h^{\mu\sigma} + \frac{1}{2} h \nabla^{\nu} \zeta^{\mu} - h^{\nu\sigma} \nabla_{\sigma} \zeta^{\mu}] dx^{\alpha} \wedge dx^{\beta}. \quad (3.34)$$

This expression is perhaps the most important result of this chapter and it will be used in the next chapter to calculate charges in NHEK.⁹

Conservation and finiteness of charges

As described at the end of Sec. 2, the integral $\int_{\Sigma} \omega[\delta\Phi, \delta_{\zeta}\Phi, \Phi]$ does not depend on the choice of the spacelike hypersurface Σ given that the symplectic flux through the boundary is zero. We can choose this hypersurface to be a surface of constant time. The statement that the charges are conserved then follows from the fact that the definition of charge does not depend on the choice of the spacelike hypersurface. Moreover, we require the conserved charges to be finite. Since the integrals are to be computed on the boundary of NHEK which lies at $r = \infty$, this requires imposing a suitable set of boundary conditions which will be discussed in the next chapter.

⁹This expression differs from the one given in [1] by two terms. This difference is because of a different choice of the (pre)symplectic current. In the literature, the expression we used is called the *Iyer-Wald symplectic current* and the one used in [1] is called the *invariant symplectic current*. See, for example, [16] for details. This difference is a consequence of the degeneracy in the definition of the presymplectic current, as we mentioned earlier. However, the additional terms do not contribute to the charges we compute for NHEK, as can be checked by explicit calculation.

The Kerr-CFT correspondence conjecture

In the preceding chapter, we described a procedure for evaluating conserved charges corresponding to diffeomorphisms on manifolds with boundaries. In this chapter, we will apply that procedure to derive charges corresponding to the *asymptotic symmetry group* of the NHEK geometry (2.22). We then compute their Dirac brackets and canonically quantize the theory by taking these to commutators. The discussion in this chapter closely follows [1] and [2].

1 Asymptotic symmetry group

To construct well defined charges corresponding to diffeomorphisms, we need to impose appropriate boundary conditions on the metric fluctuations. Before defining what these boundary conditions should be, we need to introduce the concepts of *allowed* and *trivial diffeomorphisms*.

We will consider metric perturbations near the boundary of the form $g_{\mu\nu} = g_{\mu\nu}^0 + h_{\mu\nu}$ where $g_{\mu\nu}^0$ is the background metric and $h_{\mu\nu}$ describes the fluctuations about it. Choosing the asymptotic fall off for $h_{\mu\nu}$ corresponds to specifying our boundary conditions. Only fluctuations that have the right dependence on the radial coordinate r are allowed. *Allowed diffeomorphisms* are then defined as diffeomorphisms which take $g_{\mu\nu}$ to $g'_{\mu\nu}$ such that $g'_{\mu\nu} = g_{\mu\nu}^0 + h'_{\mu\nu}$ where $h'_{\mu\nu}$ belongs to the set of allowed fluctuations. It is illuminating to consider an example of diffeomorphisms that are not allowed. Recall that in asymptotically Minkowski space [2], the boundary conditions correspond to $1/r$ fluctuations about the background Minkowski metric. In this case, diffeomorphisms which go like $\mathcal{O}(1)$ or $\mathcal{O}(r)$, for example, will not be allowed.

Diffeomorphisms allowed by the boundary conditions but which yield vanishing charges are

called *trivial diffeomorphisms*. We will point to a concrete example of these in the following section. The *asymptotic symmetry group* (ASG) is the group of diffeomorphisms that are allowed by the boundary conditions and yield finite, non-zero charges:

$$ASG = \frac{\text{Allowed Diffeomorphisms}}{\text{Trivial Diffeomorphisms}}.$$

The corresponding algebra describes the generators of these transformations. We will later see that the ASG corresponds to the symmetries of a quantum theory on the boundary of our manifold, at $r = \infty$. The states in the quantum theory transform in a representation of the ASG and are annihilated by the trivial diffeomorphisms. One could be led to the impression, at this stage, that a wide range of boundary conditions can be imposed. However, it turns out that in general there is a narrow window of boundary conditions which give well defined, non trivial charges [2]. For instance, if instead of the usual $\mathcal{O}(1/r)$ fall-off, we impose a $\mathcal{O}(1/\sqrt{r})$ fall-off in 4d Minkowski spacetime, one can show that the generators blow up and the charges are therefore ill-defined. This would be an example of boundary conditions that are too weak. If, on the other hand, we impose boundary conditions that are too strong, for example an $\mathcal{O}(r^{-17})$ fall-off, the corresponding charges all vanish and the corresponding boundary theory is trivial.

As examples of asymptotic symmetry groups [2], we have the Poincare group at the spatial infinity of asymptotically Minkowski spacetime and $SO(3, 2)$ for AdS_4 . The asymptotic symmetry group, however, need not be the isometry group of the background metric. For example, it is well known that the asymptotic symmetry group of Minkowski spacetime at null infinity is the infinite dimensional BMS group. Similarly, as we will soon see, the asymptotic symmetry group corresponding to the near horizon Kerr geometry is not the isometry group of the NHEK as one might naively expect.

2 Boundary conditions

A consistent set of boundary conditions for the NHEK geometry was given in [1]. In general, one can define any set of boundary conditions such that they give rise to well-defined, non-trivial charges on the boundary and respect the constraints on the system¹. The physics at the boundary could in general depend on the choice of boundary conditions. The approach in [1] differs from the one in [18] where the authors acted on the background metric with the group of AdS_3 transformations in an attempt to concretely define the notion of *asymptotically anti de Sitter space*. In [1], the authors go, in some sense, the other way. They define

¹For example, for NHEK, the boundary conditions would have to respect the extremality constraint as discussed later.

a set of boundary conditions, check that they define a consistent boundary theory and then analyze that theory. The boundary conditions given in [1] in terms of the fluctuations $h_{\mu\nu}$ about the background metric are:

$$\begin{pmatrix} h_{\tau\tau} = \mathcal{O}(r^2) & h_{\tau r} = \mathcal{O}(r^{-2}) & h_{\tau\theta} = \mathcal{O}(r^{-1}) & h_{\tau\varphi} = \mathcal{O}(1) \\ h_{rr} = h_{\tau r} & h_{rr} = \mathcal{O}(r^{-3}) & h_{r\theta} = \mathcal{O}(r^{-2}) & h_{r\varphi} = \mathcal{O}(r^{-1}) \\ h_{\theta\tau} = h_{\tau\theta} & h_{\theta r} = \mathcal{O}(r^{-2}) & h_{\theta\theta} = \mathcal{O}(r^{-1}) & h_{\theta\varphi} = h_{\varphi\theta} \\ h_{\varphi\tau} = h_{\tau\varphi} & h_{\varphi r} = \mathcal{O}(r^{-1}) & h_{\varphi\theta} = \mathcal{O}(r^{-1}) & h_{\varphi\varphi} = \mathcal{O}(1). \end{pmatrix}$$

The expression above only describes the r dependence of the fluctuations. In general, we can have any dependence on τ, θ, φ in these terms. One should add here that $\mathcal{O}(r^{-n})$ means that only terms that go like r^{-p} where $p \in \mathbb{Z}$ and $p \geq n$ are allowed. For example, $\mathcal{O}(1)$ means that terms with r dependence given by $r^0, r^{-1}, r^{-2}, \dots$ are allowed. Instead of trying to derive these boundary conditions from some first principle, we will show that they lead to well-defined, non-trivial charges. Note that in recent work, for example in [10], some potential problems with the boundary conditions used in [1] have been pointed out. In this thesis, we have contained ourselves to the original derivation of the Kerr-CFT conjecture given in [1] and so have ignored these problems. The next task is to construct the most general vector field such that the corresponding diffeomorphism preserves these boundary conditions. One can write an ansatz as a power series in r but it turns out that restricting to these boundary conditions results in the following most general vector field [1]:²

$$\zeta = [C + \mathcal{O}(r^{-3})]\partial_\tau + [-r\epsilon'(\varphi) + \mathcal{O}(1)]\partial_r + [\mathcal{O}(r^{-1})]\partial_\theta + [\epsilon(\varphi) + \mathcal{O}(r^{-2})]\partial_\varphi, \quad (4.1)$$

where $\epsilon(\varphi)$ is an arbitrary smooth function and C is some arbitrary constant. The sub-leading terms result in vanishing charges and therefore generate trivial diffeomorphisms [1]. Moreover, as we will soon see, the charge corresponding to ∂_τ vanishes. As a result, we see that the ASG is generated by:

$$\zeta_\epsilon = \epsilon(\varphi)\partial_\varphi - r\epsilon'(\varphi)\partial_r, \quad (4.2)$$

Since φ is periodic, $\epsilon(\varphi)$ can be expanded in Fourier modes. It is therefore convenient to define $\epsilon_n(\varphi) \equiv -e^{-in\varphi}$. It then follows that:

$$\zeta_n \equiv \zeta(\epsilon_n) = \zeta(-e^{-in\varphi}) = -e^{-in\varphi}\partial_\varphi - ine^{-in\varphi}r\partial_r. \quad (4.3)$$

²We checked that this is the most general form by observing that if we try and tamper with the r dependence in these terms, the boundary conditions are violated.

The generators of the ASG defined in this way satisfy the relation:

$$i[\zeta_m, \zeta_n]_{\text{LB}} = (m - n)\zeta_{m+n}. \quad (4.4)$$

The diffeomorphism generated by ζ_ϵ does not keep the metric invariant. In fact, by computing in the usual way $\delta g_{\mu\nu} = \nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu$ ³, we get:

$$\delta_\epsilon g = 4JG\Omega^2 \left[r^2(1 - \Lambda^2)\epsilon'(\varphi)d\tau^2 - \frac{r\epsilon''(\varphi)}{1+r^2}d\varphi dr + \Lambda^2\epsilon'(\varphi)d\varphi^2 - \frac{\epsilon'(\varphi)}{(1+r^2)^2}dr^2 \right]. \quad (4.5)$$

3 Generator Algebra

Our next task is to construct the charges corresponding to these diffeomorphisms. We will use the formalism developed in the previous chapter. Recall that since we are dealing with a system with second class constraints, all Poisson brackets are replaced with Dirac brackets, as discussed in Appendix C. The expression for the charge corresponding to a diffeomorphism such that $g_{\mu\nu} \rightarrow g_{\mu\nu} + h_{\mu\nu}$ was derived in Chapter 3 and reads (taking out a factor of $\frac{1}{8\pi G}$):

$$\delta\mathcal{Q}_\zeta = \mathcal{Q}_\zeta[g] = \frac{1}{8\pi G} \int_{\partial\Sigma} k_\zeta[h, g], \quad (4.6)$$

$$k_\zeta[h, g] \approx \frac{\sqrt{-g}}{4} \tilde{\epsilon}_{\alpha\beta\mu\nu} \left[\zeta^\nu \nabla^\mu h - \zeta^\nu \nabla_\sigma h^{\mu\sigma} + \zeta_\sigma \nabla^\nu h^{\mu\sigma} + \frac{1}{2} h \nabla^\nu \zeta^\mu - h^{\nu\sigma} \nabla_\sigma \zeta^\mu \right] dx^\alpha \wedge dx^\beta, \quad (4.7)$$

where $h \equiv g^{\mu\nu}h_{\mu\nu}$ and in the first equality, we have used (3.12). From (4.5), it is easy to check that h vanishes here.⁴ Computing (4.7) for the generators of the ASG of the NHEK geometry and ignoring terms that vanish upon integration over $\partial\Sigma$, we got, upon using the Mathematica package given in [17], the following expressions:

$$\begin{aligned} k_{\partial_\tau} &= -\left(\frac{1}{4\Lambda}r[(\Lambda^4 + \Lambda^2 - 2)h_{\varphi\varphi} + \frac{\Lambda^4}{r^2}h_{\tau\tau}]\right) - \frac{1}{4\Lambda}[r^3\Lambda^4h_{rr} + 2r^2\Lambda\partial_\theta(\Lambda h_{r\theta}) \\ &\quad + 2\Lambda^2r\partial_\tau h_{r\varphi} + 2(\Lambda^2 - 1)r^2\partial_r h_{\varphi\varphi} + 2\Lambda^4h_{\tau\varphi} - \Lambda^2r(\Lambda^2 - 2 + 2r\partial_r h_{\theta\theta})]d\theta \wedge d\varphi, \\ k_{\zeta_\epsilon} &= \frac{1}{4\Lambda}\left[2\Lambda^2\epsilon'r h_{r\varphi} - \epsilon\Lambda^2(\Lambda^2\frac{h_{\tau\tau}}{r^2} + (\Lambda^2 + 1)h_{\varphi\varphi} + 2r\partial_\varphi h_{r\varphi})\right]d\theta \wedge d\varphi, \end{aligned} \quad (4.8)$$

³The fluctuations $h_{\mu\nu}$ are by assumption infinitesimal. Moreover, throughout this thesis, we will only be interested in first order fluctuations to the metric. Therefore, the raising and lowering of indices and the Christoffel symbols will always be defined using the background metric g given in (2.22).

⁴More generally, h vanishes because of the constraints arising from the Bianchi identity: $\nabla_\mu G^{\mu 0} = 0$. The proof of this is omitted but may be found in [1].

where Σ was chosen to be a constant time (spacelike) hypersurface and the boundary lies at $r = \infty$. Since we are interested in looking at extremal Kerr black holes, we want to preserve the following equality:

$$\frac{M^2}{G} - J = 0. \quad (4.9)$$

We saw in Chapter 2 that the variation of the conserved charge corresponding to ∂_τ gives us $\delta(\frac{M^2}{G} - J)$. Since we want to fix this variation to be zero, we require that:

$$\delta \mathcal{Q}_{\partial_\tau} = \mathcal{Q}_{\partial_\tau} = 0. \quad (4.10)$$

This is indeed borne out by calculation: we see that the diffeomorphism generated by ∂_τ leaves the NHEK metric invariant. Consequently, all terms in the expression for $k_{\partial_\tau} = 0$ above are zero and the expression identically vanishes. This serves as a check that the boundary conditions given in Sec. 2 preserve extremality. In a system with second class constraints, the analog of (3.6) is defined using Dirac brackets in the following manner:

$$\{\mathcal{Q}_\zeta, F\}_{\text{DB}} \equiv \mathcal{L}_\zeta F, \quad (4.11)$$

where F denotes some function of fields on the phase space. From (D.5), we have:

$$\{\mathcal{Q}_\zeta, \mathcal{Q}_\xi\}_{\text{DB}} = \mathcal{Q}_{[\zeta, \xi]_{\text{LB}}} + C_{\zeta\xi}, \quad (4.12)$$

Notice that from (4.11), we have that:

$$\{\mathcal{Q}_\zeta, \mathcal{Q}_\xi\}_{\text{DB}} = \mathcal{L}_\zeta \mathcal{Q}_\xi. \quad (4.13)$$

The variation of the charge on right hand side of (4.13) was calculated in [16] and it was shown that:

$$\{\mathcal{Q}_\zeta, \mathcal{Q}_\xi\}_{\text{DB}} = \mathcal{Q}_{[\zeta, \xi]_{\text{LB}}} + \frac{1}{8\pi G} \int_{\partial\Sigma} k_\zeta [\mathcal{L}_\xi g, g]. \quad (4.14)$$

Hence, we see that:

$$C_{\zeta\xi} = \frac{1}{8\pi G} \int_{\partial\Sigma} k_\zeta [\mathcal{L}_\xi g, g], \quad (4.15)$$

In terms of Fourier modes, we have that:

$$C_{\zeta_m \zeta_n} = \frac{1}{8\pi G} \int_{\partial\Sigma} k_{\zeta_m} [\mathcal{L}_{\zeta_n} g, g], \quad (4.16)$$

where g represents (2.22). One can now read off the following expressions from (4.5):

$$\mathcal{L}_{\zeta_n} g_{\tau\tau} = 4GJ\Omega^2(1 - \Lambda^2)r^2ine^{-in\varphi}, \quad \mathcal{L}_{\zeta_n} g_{r\varphi} = \frac{-2GJ\Omega^2r}{1+r^2}n^2e^{-in\varphi}, \quad (4.17)$$

$$\mathcal{L}_{\zeta_n} g_{\varphi\varphi} = 4GJ\Omega^2\Lambda^2ine^{-in\varphi}, \quad \mathcal{L}_{\zeta_n} g_{rr} = -\frac{4GJ\Omega^2}{(1+r^2)^2}ine^{-in\varphi}. \quad (4.18)$$

4 Central charge and the statement of the conjecture

We now plug (4.17) and (4.18) into (4.8) to calculate (4.16):

$$\begin{aligned} C_{\zeta_m \zeta_n} &= \frac{1}{8\pi G} \int_{\partial\Sigma} k_{\zeta_m} (\mathcal{L}_{\zeta_n} g, g), \\ &= \frac{1}{8\pi G} \int d\theta d\varphi \left[(-imn^2r^2)(e^{-i(m+n)\varphi})\left(\frac{GJ\Omega^2\Lambda}{1+r^2}\right) + (in)(e^{-i(m+n)\varphi})(GJ\Omega^2(\Lambda^3 - \Lambda^5)) \right. \\ &\quad \left. + (in)(e^{-i(m+n)\varphi})(\Lambda^5 + \Lambda^3)(GJ\Omega^2) + (in^3r^2)(e^{-i(m+n)\varphi})\left(\frac{GJ\Omega^2\Lambda}{1+r^2}\right) \right]. \end{aligned} \quad (4.19)$$

Doing the integration over φ , taking the limit $r \rightarrow \infty$, and finally integrating over θ we get:

$$\begin{aligned} C_{\zeta_m \zeta_n} &= \frac{1}{8\pi G} \int d\theta (4\pi in(GJ\Lambda^3\Omega^2) + 4\pi in^3r^2\left(\frac{GJ\Lambda\Omega^2}{1+r^2}\right))\delta_{n+m,0}, \\ &= \frac{1}{8\pi G} \int d\theta (4\pi inGJ\Lambda^3\Omega^2 + 4\pi in^3GJ\Lambda\Omega^2)\delta_{n+m,0}, \\ &= i(n^3 + 2n)J\delta_{m+n,0} = -i(m^3 + 2m)J\delta_{m+n,0}. \end{aligned} \quad (4.20)$$

We can now plug this into the expression for the Dirac bracket of conserved charges. Making the following definition:

$$\hbar L_n \equiv \mathcal{Q}_{\zeta_n} + \frac{3J}{2}\delta_{n,0}, \quad (4.21)$$

we see that

$$\mathcal{Q}_{[\zeta_m, \zeta_n]} = \mathcal{Q}_{-i(m-n)\zeta_{n+m}} = -i(m-n)\hbar L_{m+n} + 3imJ\delta_{n+m,0}. \quad (4.22)$$

Putting this in the full expression for the Dirac bracket and then performing the Dirac quantization step $\{\quad\}_{\text{DB}} \rightarrow \frac{-i}{\hbar} [\quad, \quad]$,

$$\{\mathcal{Q}_{\zeta_m}, \mathcal{Q}_{\zeta_n}\}_{\text{DB}} = \mathcal{Q}_{[\zeta_m, \zeta_n]_{\text{LB}}} + C_{\zeta_m \zeta_n}, \quad (4.23)$$

becomes:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{Jm}{\hbar}(m^2 - 1)\delta_{m+n,0}. \quad (4.24)$$

The reason for making the above redefinition for \mathcal{Q} in terms of L was to allow us to compare the subsequent expression to the well known expression for the *Virasoro algebra* of generators of a chiral CFT which is given as:⁵

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}, \quad (4.25)$$

where c is the *central charge* of the CFT. Comparing the two expressions, we see that the algebra of generators of the asymptotic symmetry group of NHEK corresponds to the algebra of a conformal field theory with central charge:

$$c = \frac{12J}{\hbar}. \quad (4.26)$$

Using the expression for Cardy's formula reviewed in Appendix A, we can calculate the entropy of extremal Kerr black hole. Using the formula for the effective temperature of the Kerr black hole derived in Appendix B, we get :

$$S = \frac{\pi^2 T c}{3} = \frac{12J\pi^2}{6\pi\hbar} = \frac{2\pi J}{\hbar}, \quad (4.27)$$

which matches the expression derived in Chapter 2, equation (2.12). Note, however, that it is not obvious, apriori, that one can use Cardy's formula here. In our derivation of Cardy's formula, given in Appendix A, we assumed that the temperature is large compared to the central charge. This condition is not satisfied here because typically, the central charge, $\frac{12J}{\hbar}$ is very large for astrophysical near extremal Kerr black holes. For example GRS 1915+105 has $c = (2 \pm 1) \times 10^{79}$ [1]. The authors of [9] take the view that the fact that Cardy's formula gives the right entropy is one evidence for its applicability. Moreover, another sufficient condition for the applicability of Cardy's formula [9] is that the temperature be large as compared to the smallest excitation in the CFT. This essentially means that a large number of degrees of freedom are excited in the theory. It can be shown [2] that this condition is satisfied here.

Let us summarize the main result of this chapter. We computed the Dirac brackets of asymptotic charges in NHEK. Following the Dirac quantization procedure, we then promoted these to commutators and recovered one half of a Virasoro algebra. The corresponding central charge gave the right value for the entropy of the black hole. This then leads to the conjecture that the theory of gravity in the bulk of NHEK, when quantized, is dual

⁵This is actually just one half of the Virasoro algebra. Typically, for a non-chiral CFT, there would be another similar expression for the conjugate Virasoro modes. This term has just the “left moving” central charge. In a non-chiral CFT, there would also be a “right moving” central charge appearing in the commutator of the conjugate Virasoro modes. See, for example, [28] for an introduction to CFTs.

to a chiral CFT living on the boundary of the NHEK geometry. Alternatively stated, this conjecture says that extremal Kerr black holes are holographically dual to a chiral CFT with central charge $c = \frac{12J}{\hbar}$. This is called the *Kerr-CFT correspondence conjecture*, originally put forth in [1].

Having provided a detailed discussion of the Kerr-CFT correspondence conjecture, we will allude to some of its applications in the next chapter.

Applications and conclusions

We conclude this essay by mentioning some applications of the Kerr-CFT correspondence. A detailed discussion is beyond the scope of this thesis and the interested reader is referred to [19, 23, 24, 25]. Before describing the applications, it is worth pointing out that the Kerr-CFT conjecture is particularly important because it means that we can apply all the machinery of conformal field theories to study various phenomena that occur near the horizon of extremal Kerr black holes. It potentially offers analytic solutions to problems that were previously solvable only numerically at best. One particularly relevant example is that of the GRS 1915 + 105 stellar black hole in the Milky Way galaxy which is almost extremal ($JG/M^2 = 0.98$ [2]) and can therefore be studied using this correspondence. Other examples include the Cygnus X-1 [26] and the supermassive black hole at the center of Seyfert-1.2 galaxy MCG-6-30-15 [27]. In the context of applications of this conjecture, it is also worth pointing out that in [14], an extension of the Kerr-CFT correspondence to dimensions higher than 4 has also been made. We now briefly mention some problems to which the Kerr-CFT correspondence has been successfully applied.

- *Force-Free Electrodynamics around extremal Kerr black holes:* The energy that powers the emission of collimated jets of electromagnetic radiation from quasars is thought to be derived from Kerr black holes that are surrounded by a magnetosphere containing a plasma [23]. These processes can be described by the highly non-linear equations of Force-Free Electrodynamics, characterized by $F^{\mu\nu}J_\nu = 0$. While most analyses of these processes in the past were numerical, the authors of [23] were able to exactly solve the corresponding equations by making use of the conformal symmetry of the NHEK region.
- *Black Hole Superradiance:* Scalar fields incident on extremal Kerr black holes whose frequency ω and angular momentum m satisfy the following relation $0 \leq \omega \leq m\Omega$ where $\Omega = \frac{1}{2M}$ is the angular velocity of the extremal Kerr horizon, are reflected back with a negative absorption probability. The amplitude of the final wave is thus

larger than that of the initial wave. This was shown in the seventies by Starobinsky, Churilov, Press and Teukolsky (see [24] for a detailed list of references). The authors of [24] were able to re-derive the corresponding scattering cross sections using the Kerr-CFT correspondence where the black hole corresponds to a particular thermal state in the CFT. The scalar field has a dual operator in the CFT and the absorption probability is related to the 2-point function of this operator.

- *Gravitational waves:* The gravitational radiation produced by a massive object orbiting in the NHEK geometry was studied in [25]. Adding this massive object corresponds to deforming the dual CFT. The authors showed that the result for the graviton number flux through the horizon, as computed in the gravity picture, is exactly reproduced in the dual CFT as the transition rate from the CFT vacuum.

From the Kerr-Schild form to superradiance, some of the most remarkable concepts in theoretical physics are associated with Kerr black holes. In addition, the Kerr-CFT correspondence conjecture is an important step towards a full understanding of quantum gravity. It offers insight into what properties a theory of quantum gravity in the NHEK geometry should have by proposing that this theory is dual to a chiral conformal field theory living on the NHEK boundary. Moreover, the fact that entropy calculation in the CFT gives the right value of entropy for the black hole shows that the entropy of a black hole is related to degrees of freedom that live near the horizon. This reinforces the idea that black hole microstates live near the horizon, rather than in the causally inaccessible interior. This is important in the context of a resolution of the information paradox. Although recent work has shown compelling evidence for the Kerr-CFT conjecture, a complete understanding of the dual CFT is still to be achieved. For instance, obtaining consistent right moving and left moving Virasoro Algebras away from extremality would be an important step towards a more complete understanding of the dual CFT. To the best of our knowledge, this has not been achieved so far. It remains to be seen what the future holds in store.

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Appendices

Cardy's formula

A derivation of Cardy's formula for a two dimensional, unitary conformal field theory on a torus is given in [28].¹ We include it here for the sake of completeness. The partition function in the low temperature limit is given by:

$$Z(\beta) = \sum_{states} e^{-\beta E} \approx e^{-\beta E_{vac}} (\beta \rightarrow \infty). \quad (\text{A.1})$$

Using the fact that $E_{vac} = \frac{-c}{12}$, where c is the central charge,² we get the following low temperature behavior:

$$Z(\beta) \approx e^{\frac{\beta c}{12}} (\beta \rightarrow \infty). \quad (\text{A.2})$$

¹This derivation assumes that theory is non chiral and the left moving central charge and the right moving central charge are the same: $c = c_R = c_L$.

²This is a well known result for the ground state energy of a free scalar field described by a conformal field theory ($c_R = c_L$) on a torus. Generically, $E_{vac} = \frac{-c_R - c_L}{24}$. See, for example, [28] for a derivation.

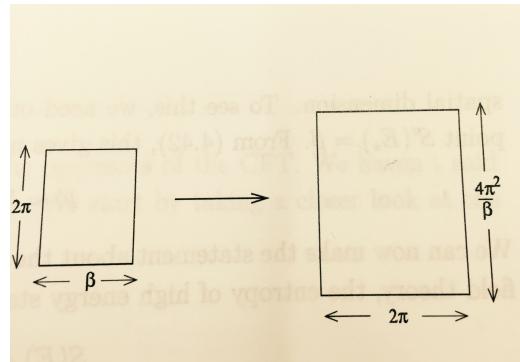


Figure A.1: Modular invariance.

We define coordinates on the torus as τ and σ with periods β and 2π respectively. Since both directions on the torus are equivalent, we can just as well switch their labels, $\tau \leftrightarrow \sigma$, as depicted in Figure A.1. To compare to the original partition function, we need to rescale our coordinate variables so we make the following change of variables so that σ' still runs from 0 to 2π :

$$\sigma' \equiv \frac{2\pi}{\beta}\tau, \quad \tau' \equiv \frac{2\pi}{\beta}\sigma, \quad (\text{A.2})$$

τ' now has period $\frac{4\pi^2}{\beta}$. Since we just switched labels and did coordinate redefinitions, our partition function should remain unchanged. Hence, we obtain the following important relationship between the high and low temperature behaviors of the partition function.

$$Z[\beta' \equiv \frac{4\pi^2}{\beta}] = Z[\beta]. \quad (\text{A.3})$$

This is an example of modular invariance and it allows us to study the high temperature behaviour of the conformal field theory using its low temperature behaviour. In particular, we get the following relation:

$$Z(\beta') \approx e^{\frac{\pi^2 c}{3\beta'}} (\beta' \rightarrow 0), \quad (\text{A.4})$$

which allows us to derive the following expressions, in the limit of high temperature, for the entropy and free energy of the conformal field theory in terms of the central charge.

$$S(\beta') = (1 - \beta' \partial_{\beta'}) \log Z = \frac{2\pi^2 c}{3\beta'}, \quad (\text{A.5})$$

$$F(\beta') = -\partial_{\beta'} \log Z = \frac{\pi^2 c}{3\beta'^2}. \quad (\text{A.6})$$

These expressions hold for a non chiral conformal field theory as mentioned above. Repeating these calculations for a chiral CFT by setting c_R in the expression for energy zero, we get:

$$S(\beta') = \frac{\pi^2 c_L}{3\beta'}, \quad (\text{A.7})$$

$$F(\beta') = \frac{\pi^2 c_L}{6\beta'^2}. \quad (\text{A.8})$$

Effective temperature of extremal Kerr

Despite the fact that the Hawking temperature of an extremal Kerr black hole is zero, quantum fields near the black hole are in a thermal state [2]. Here, we derive the effective temperature seen by these fields. The first law of black hole thermodynamics and the extremality constraint together imply the following relation [19]:

$$0 = T_H \delta S_{ext} = \delta M_{ADM}^{ext} - \Omega_H^{ext} \delta J, \quad (\text{B.1})$$

where Ω_H is the angular velocity of the black hole horizon. This implies that at extremality:

$$\delta M_{ADM}^{ext} = \Omega_H^{ext} \delta J. \quad (\text{B.2})$$

Additionally, we also know that entropy $S = S(J)$ can, at extremality where $T_H = 0$, be written as:

$$\delta S_{ext} = \frac{1}{T_\phi} \delta J, \quad (\text{B.3})$$

where:

$$\frac{1}{T_\phi} = \frac{\delta S_{ext}}{\delta J}.$$

Now, we write down the first law for a non-extremal Kerr black hole as:

$$\delta S = \frac{1}{T_H} (\delta M_{ADM} - \Omega_H \delta J), \quad (\text{B.4})$$

and carefully taking the extremality limit. We first take $\delta M_{ADM} = \delta M_{ADM}^{ext}$, plug this expression into (B.4) and use (B.2). We get:

$$T_\phi = \lim_{T_H \rightarrow 0} \frac{T_H}{\Omega_H^{ext} - \Omega_H} = \left. \frac{-\partial T_H}{\frac{\partial \Omega_H}{\partial \hat{r}_+}} \right|_{\hat{r}_+ = \hat{r}_{ext}} = \frac{\hbar}{2\pi}, \quad (\text{B.5})$$

We have used the fact that the approach to extremality is controlled by \hat{r} . Since in our derivation of Cardy's formula Appendix A, we defined temperature as a dimensionless quantity, we absorb the \hbar in J to redefine T_ϕ as a dimensionless quantity. Consequently, we have:

$$T_\phi = \frac{1}{2\pi}. \quad (\text{B.6})$$

Hence we see that extremal Kerr black holes have an effective temperature, even though their Hawking temperature is identically zero.

Constraints and Dirac brackets

In this appendix, we include a discussion of constraints and Dirac brackets which is relevant to the calculation of generators of gauge symmetries in Chapter 3.

Types of constraints

First class constraints: Constraints whose Poisson brackets with all other constraints vanish on shell are called *first class constraints*.

Second class constraints: Constraints for which there exists at least one other constraint with which their Poisson bracket is non-vanishing even on shell are called *second class constraints*. This other constraint must then be a second class constraint itself because there will be at least one other constraint with which its Poisson bracket is non zero.

The natural way to deal with second class constraints is to replace Poisson brackets with *Dirac brackets* which is what we proceed to show next. We follow the derivation given in [10].

Dirac brackets

Assume that we have two second class constraints whose Poisson bracket is given as follows:

$$\Delta_{mn} = \{\chi_m, \chi_n\}_{\text{PB}}, \quad (\text{C.1})$$

for some non zero Δ_{mn} . If Δ is a degenerate matrix then there exists a non-zero vector λ^n such that $\Delta_{mn}\lambda^n = 0$. This implies that:

$$\{\chi_m, \chi_n\}_{\text{PB}}\lambda^n = \{\chi_m, \lambda^n \chi_n\}_{\text{PB}} = 0, \quad (\text{C.2})$$

However, this would imply that there exists $\chi = \lambda^n \chi_n$ whose Poisson bracket with some other second class constraint χ_m is zero. Therefore, we have a contradiction because this

would imply that χ_m is not second class. We therefore we conclude that Δ as defined above must be non-degenerate and therefore invertible. Denoting its inverse with raised indices, the following bracket, called the *Dirac bracket*, is then defined:

$$\{f, g\}_{\text{DB}} \equiv \{f, g\}_{\text{PB}} - \{f, \chi_m\}_{\text{PB}} \Delta^{mn} \{\chi_n, g\}_{\text{PB}}. \quad (\text{C.3})$$

One can check that this satisfies all the properties of a bracket, namely bilinearity, antisymmetry and the Jacobi identity. It is clear from the definition above that the Dirac bracket of a second class constraint with any other constraint vanishes. Moreover, the Dirac bracket of any two first class constraints reduces to their Poisson bracket because the second term vanishes. As a result, by introducing Dirac brackets, the second class constraints are always obeyed. Therefore, it is natural to replace Poisson brackets with Dirac brackets in a system with second class constraints. The canonical way of quantizing such a system is to take:

$$\{\cdot, \cdot\}_{\text{DB}} \rightarrow \frac{-i}{\hbar} [\cdot, \cdot]. \quad (\text{C.4})$$

Symplectic mechanics

We discuss some important aspects of symplectic mechanics which are useful for the covariant phase space formalism discussed in Chapter 3. The discussion here is based on [10].

A *symplectic manifold* is a manifold that is equipped with a form Ω which satisfies the following properties:

- Ω is a two form: $\Omega_{ab} = -\Omega_{ba}$.
- Ω is closed: $d\Omega = 0$.
- Ω is non-degenerate: $\Omega_{ab}X^b = 0 \Rightarrow X^a = 0$, i.e, $\det \Omega_{ab} \neq 0$.

Such a form is called a *symplectic* form. We further define Ω^{ab} such that $\Omega^{ab}\Omega_{bc} = \delta_c^a$. A vector field X over the symplectic manifold, or equivalently the phase space, is called a *symplectic symmetry* if the Lie derivative of the symplectic two form along X vanishes. Therefore:

$$0 = \mathcal{L}_X\Omega = X \cdot d\Omega + d(X \cdot \Omega) = d(X \cdot \Omega) = 0, \quad (\text{D.1})$$

where the Cartan identity and the closure of the symplectic two from have been invoked. It follows from Poincare's lemma that at least locally,

$$X \cdot \Omega = dH_X \quad (\text{D.2})$$

for some function H_X . This implies that $X^a\Omega_{ab} = \partial_b H_X$. Multiplying this equation with Ω^{bc} , we get $X^c = \Omega^{bc}\partial_b H_X$. Consider now the Poisson bracket of H_X with some other function f :

$$\{H_X, f\}_{\text{PB}} \equiv \Omega^{ab}\partial_b f \partial_a H_X = (\Omega^{ab}\partial_a H_X)\partial_b f = X^b\partial_b f = \mathcal{L}_X f, \quad (\text{D.3})$$

where the definition of the Poisson bracket was used in the first equality. This equation shows that H_X is the generator of evolution along the vector field X . The on-shell value of this generator is the corresponding charge.

Symplectic symmetries form an algebra under the Lie bracket: given two symplectic symmetries, their Lie bracket is also a symplectic symmetry. The way to see that is as follows. If X and Y are symplectic symmetries then:

$$\mathcal{L}_{[X,Y]}\Omega = (\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X)\Omega = 0. \quad (\text{D.4})$$

¹which proves our claim. Now, let us consider the following:

$$\begin{aligned} \partial_a H_{[X,Y]} &= \Omega_{ba}[X, Y]^b = \Omega_{ba}\mathcal{L}_X Y^b = \mathcal{L}_X(\Omega_{ba}Y^b) = \mathcal{L}_X(dH_Y)_a = (d(X \cdot dH_Y))_a \\ &= \partial_a(\Omega^{bc}(\partial_b H_X)(\partial_c H_Y)) = \partial_a(\{H_X, H_Y\}_{\text{PB}}), \end{aligned}$$

where in the first, fourth and sixth equality, (D.2) was used, the second equality follows from the definition of the Lie derivative, the third equality is true because X is a symplectic symmetry, in the fifth equality, Cartan's identity was used and the last equality follows from the definition of the Poisson bracket. Hence, we see that:

$$\{H_X, H_Y\}_{\text{PB}} = H_{[X,Y]} + C. \quad (\text{D.5})$$

One can see that since the derivation above relates derivatives on both sides, it admits an additional term C such that $dC = 0$. This is the extra term given in (D.5). Since this term is a constant, its Poisson bracket with all other symmetry generators is zero. It represents the central extension of the algebra. In a system with second class constraints, the result (D.5) holds for Dirac brackets.

¹Here, we have used the well known result $\mathcal{L}_{[X,Y]} = \mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X$ for Lie derivatives.

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