Trotterized Heisenberg

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We consider the Heisenberg model on a 1-D lattice with nearest neighbor interactions. We construct a circuit that allows you to do a Trotterized time-evolution of this Hamiltonian using single qubit rotations and CNOT gates and implement this in the Qiskit SDK.

I will consider the isotropic, XXX Heisenberg spin chain with n spins and coupling constant J, without periodic boundary conditions. Generalizations to anisotropic spin chains are straightforward and so, for simplicity, I will restrict myself to this case here. The hamiltonian of this system is given by

$$H_{\text{Heis.}} = \sum_{<ij>}^{n} J\left[{\sigma_x}^{(i)}{\sigma_x}^{(j)} + {\sigma_y}^{(i)}{\sigma_y}^{(j)} + {\sigma_z}^{(i)}{\sigma_z}^{(j)}\right], \tag{1}$$

where σ_x, σ_y and σ_z represent the Pauli matrices. To compute the time evolution operator, we use Trotter's formula

$$e^{-i\sum_k H_k t} = (\prod_k e^{-iH_k \frac{t}{r}})^r + O(\frac{t^2}{r}), \tag{2}$$

where k denotes each pairwise interaction and r denotes the number of Trotter steps. For example, for $n = 3$ spins and 2 Trotter steps, this becomes

$$\begin{aligned}
&e^{-it[J({\sigma_x}^{(1)}{\sigma_x}^{(2)}+{\sigma_y}^{(1)}{\sigma_y}^{(2)}+{\sigma_z}^{(1)}{\sigma_z}^{(2)}+{\sigma_x}^{(2)}{\sigma_x}^{(3)}+{\sigma_y}^{(2)}{\sigma_y}^{(3)}+{\sigma_z}^{(2)}{\sigma_z}^{(3)})]} \\
&= \left[e^{-i\frac{t}{2}({\sigma_x}^{(1)}{\sigma_x}^{(2)}+{\sigma_y}^{(1)}{\sigma_y}^{(2)}+{\sigma_z}^{(1)}{\sigma_z}^{(2)})}e^{-i\frac{t}{2}({\sigma_x}^{(2)}{\sigma_x}^{(3)}+{\sigma_y}^{(2)}{\sigma_y}^{(3)}+{\sigma_z}^{(2)}{\sigma_z}^{(3)})}\right]^2 + O(\frac{t^2}{2}) \\
&\approx [e^{-i\frac{t}{2}{\sigma_x}^{(1)}{\sigma_x}^{(2)}}e^{-i\frac{t}{2}{\sigma_y}^{(1)}{\sigma_y}^{(2)}}e^{-i\frac{t}{2}{\sigma_z}^{(1)}{\sigma_z}^{(2)}}e^{-i\frac{t}{2}{\sigma_x}^{(2)}{\sigma_x}^{(3)}}e^{-i\frac{t}{2}{\sigma_y}^{(2)}{\sigma_y}^{(3)}}e^{-i\frac{t}{2}{\sigma_z}^{(2)}{\sigma_z}^{(3)}}]^2,
\end{aligned} \tag{3}$$

where we have set $J = 1$ for simplicity and, in the last step, used the fact that ${\sigma_x}^{(i)}{\sigma_x}^{(j)}, {\sigma_y}^{(i)}{\sigma_y}^{(j)}$, and ${\sigma_z}^{(i)}{\sigma_z}^{(j)}$ all mutually commute. The next step is to write our evolution operator in terms of CNOT and single qubit rotation gates. That is all done in the qiskit notebook accompanying this file.