DSP1 – Practice Homework

Exercise 1. Review of complex numbers.

(a) Let $s[n] = \frac{1}{2^n} + j\frac{1}{3^n}$. Compute $\sum_{n=1}^{\infty} s[n]$.

(b) Do the same with $s[n] = \left(\frac{j}{3}\right)^n$.

(c) Characterize the set of complex numbers satisfying $z^* = z^{-1}$.

(d) Find 3 distinct complex numbers $\{z_0, z_1, z_2\}$ for which $z_i^3 = 1$.

(e) Compute the infinite product $\prod_{n=1}^{\infty} e^{j\pi/2^n}$

Exercise 2. Sampling music

A music song recorded in a studio is stored as a digital sequence on a CD. The analog signal representing the music is 2 minutes long and is sampled at a frequency $f_s = 44100\ s^{-1}$. How many samples should be stored on the CD?

Exercise 3. Moving average

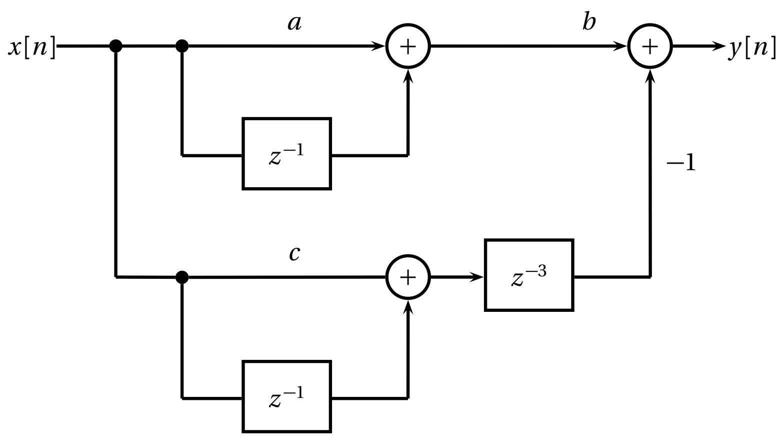
Consider the following signal,

$$x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]. \tag{1}$$

Compute its moving average $y[n] = \frac{x[n]+x[n-1]}{2}$, where we call $x[n]$ the input and $y[n]$ the output.

Exercise 4. SP with Lego

Given the following filter:



What is the input-output relationship?

Exercise 5. Bases
Let $\{\mathbf{x}^{(k)}\}_{k=0,\dots,N-1}$ be a basis for a subspace S. Prove that any vector $\mathbf{z} \in S$ is *uniquely* represented in this basis. *Hint: remember that the vectors in a basis are linearly independent and use this to prove the thesis by contradiction.*

Exercise 6. Vector Spaces.
Consider the four diagonals of a three-dimensional unit cube as vectors in $\mathbb{R}^3$. Are they mutually orthogonal?

Exercise 7. DFT Formula.
Derive a simple expression for the DFT of the time-reversed signal

$$\mathbf{x}_r = [x[N-1]\; x[N-2] \dots x[1]\; x[0]]^T$$

in terms of the DFT $\mathbf{X}$ of the signal $\mathbf{x}$. Hint: you may find it useful to remark that $W_N^k = W_N^{-(N-k)}$.

Exercise 8. DFT Manipulation.
Consider a length-N signal $\mathbf{x} = [x[0]\; x[1]\; \dots\; x[N-1]]^T$ and the corresponding vector of DFT coefficients $\mathbf{X} = [X[0]\; X[1] \dots X[N-1]]^T$.

Consider now the length-$2N$ signal obtained by interleaving the values of $\mathbf{x}$ with zeros

$$\mathbf{x}_2 = [x[0]\;\; 0\;\; x[1]\;\; 0\;\; x[2]\;\; 0\;\; \dots\;\; x[N-1]\;\; 0]^T$$

.
Express $\mathbf{X}_2$ (the $2N$-point DFT of $\mathbf{x}_2$) in terms of $\mathbf{X}$.

Exercise 9. The Structure of DFT Formulas.
The DFT and IDFT formulas are similar, but not identical. Consider a length-N signal $\mathbf{x}$ with $x[n], N = 0,\dots,N-1$. What is the length-N signal $y[n]$ obtained as

$$\mathbf{y} = \text{DFT}\{\text{DFT}\{\mathbf{x}\}\}?$$

In other words, what are the effects of applying twice the DFT transform?

Exercise 10. DFT of the Autocorrelation.
Consider a sequence $\mathbf{x}$ of finite length N. Let $\mathbf{X}$ denote the N point DFT of $\mathbf{x}$ and define the circular autocorrelation sequence $\mathbf{r}_\mathbf{x}$ as

$$r_x[m] = \sum_{n=0}^{N-1} x[n]x^*[(n-m) \mod N].$$

Express $\mathbf{r}_\mathbf{x}$ in terms of $\mathbf{X}$. *[Hint: build a signal $S[n] = X[n]X^*[n]$, compute its inverse DFT and work backwards.]*

Exercise 11. Subsampling the DFT
Consider $\mathbf{x} \in \mathbb{C}^N$ with N even and its N-point DFT $\mathbf{X}$. Define an $(N/2)$-length vector $\mathbf{Y}$ as $Y[k] = X[2k], k = 0,\dots,N/2-1$. Compute the inverse DFT of $\mathbf{Y}$.

Exercise 12. Plancherel-Parseval Equality.
Let $x[n]$ and $y[n]$ be two complex-valued sequences and $X(e^{j\omega})$ and $Y(e^{j\omega})$ their corresponding DTFTs.

(a) Show that

$$\langle x[n], y[n]\rangle = \frac{1}{2\pi}\langle X(e^{j\omega}), Y(e^{j\omega})\rangle,$$

where we use the inner products for $l_2(\mathbb{Z})$ and $L_2([-\pi,\pi])$ respectively.

(b) What is the physical meaning of the above formula when $x[n] = y[n]$?

Exercise 13. DTFT properties.
Derive the time-reverse and time-shift properties of the DTFT.
