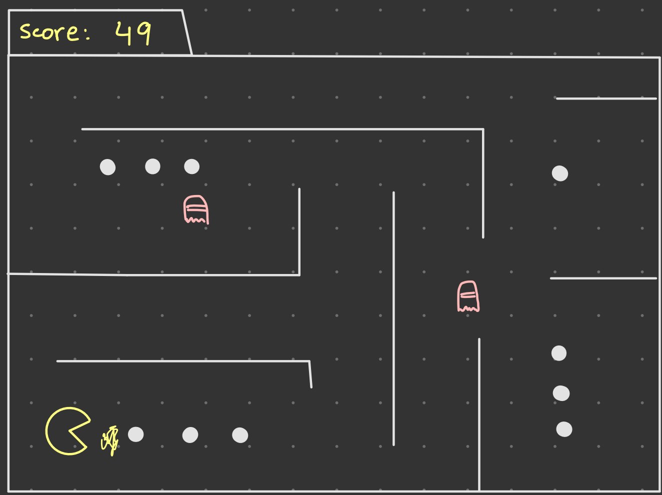
Solutions to Linear Equations

$$A x = y$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & \ddots & \vdots \\ \vdots & & & \vdots \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

But what is this anyway??

- Set up a scenario
- Decide on features
- Conduct an experiment
- Find "values" for ~~features~~ weights



- Tell us "how our results are affected by each feature."

* This is essentially all of modern machine learning!!

Oh!

Solving Linear Equations

1. Gaussian Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a'_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y'_2 \\ y'_3 \end{bmatrix}$$

2. Gauss-Jordan Elimination

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y'_1 \\ y'_2 \\ y'_3 \end{bmatrix}$$

3. LU decomposition

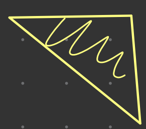
Given the matrix **A**, "decompose" it into LU.

$$Ax = y \rightsquigarrow L\underbrace{Ux}_{m} = y$$

$Ux = m$

$Lm = y$

$L =$ 

$U =$ 

$$L\underbrace{Ux}_{m} = y$$

$$Lm = y$$

ℓ_{11}

ℓ_{21}

ℓ_{22}

$$\begin{bmatrix} \sim & 0 & 0 \\ \sim & \sim & 0 \\ \sim & \sim & \sim \end{bmatrix}\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Since L is lower triangular, it is easy to find m

Once we have 'm' ...

$$Ux = m$$

Again ... trivial.

$$\begin{bmatrix} \sim & \sim & \sim \\ 0 & \sim & \sim \\ 0 & 0 & \sim \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}$$

But, how to find U and L ?

Three methods (all very similar)

"factorization"

- 1s on diagonal of L ⇝ Dolittle
- 1s on diagonal of U ⇝ Crout
- form is LL^T ⇝ Cholesky

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 5 \\ -1 & 1 & -5 & 3 \\ 3 & 1 & 7 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 31 \\ -2 \\ 18 \end{bmatrix}$$

Step 1: Write the decomposition

$$\underset{L}{\begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} \\ 0 & 1 & u_{23} & u_{24} \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}} = \underset{A}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 5 \\ -1 & 1 & -5 & 3 \\ 3 & 1 & 7 & -2 \end{bmatrix}}$$

Step 2: Write the equations:

$l_{11} = 1$

$l_{11} \cdot u_{12} = 1 \rightsquigarrow u_{12} = 1$

$l_{11} \cdot u_{13} = 1 \rightsquigarrow u_{13} = 1$

$l_{11} \cdot u_{14} = 1 \rightsquigarrow u_{14} = 1$

$l_{21} = 2$

$l_{21} \cdot u_{12} + l_{22} = 3$

$\cancel{(2 \cdot 1) + l_{22} = 3}$

$l_{22} = 1$

dot dot dot

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 3 & -2 & 2 & -1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{LUx}_{m} = y \qquad Lm = y \qquad Ux = m$$

$$\begin{array}{ccccc} L & m & & y \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 3 & -2 & 2 & -1 \end{bmatrix} & \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} & = & \begin{bmatrix} 10 \\ 31 \\ -2 \\ 18 \end{bmatrix} \end{array}$$

- $\boxed{m_1} = 10$
- $2 \cdot m_1 + m_2 = 31 \qquad \boxed{m_2} = 11$
- $-1\, m_1 + 2m_2 + (-2)(m_3) = -2 \qquad \boxed{m_3} = 7$
- $\boxed{m_4} = 4$

finally $Ux = m$

$$\begin{array}{ccccc} & U & x & & m \\ U = & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} & = & \begin{bmatrix} 10 \\ 11 \\ 7 \\ 4 \end{bmatrix} \end{array}$$

$x_4 = 4$

$x_3 = 3$

$x_2 = 2$

$x_1 = 1$

⊛ Critical points to note:

- Steps are "mechanical"!