Q-1: [8 + 7 marks]

Given:

- $f(x) = \sqrt[3]{x^2 + 1}$
- $g(x) = x^4 + 1$

(a) Derivatives of $f(x)$ and $g(x)$

Derivative of $f(x) = \sqrt[3]{x^2 + 1}$

Using logarithmic differentiation:

$$\begin{aligned}
\ln f(x) &= \frac{1}{3}\ln(x^2 + 1) \\
\frac{f'(x)}{f(x)} &= \frac{2x}{3(x^2 + 1)} \\
f'(x) &= \frac{2x}{3(x^2 + 1)^{2/3}}
\end{aligned}$$

Derivative of $g(x) = x^4 + 1$

Using limit definition:

$$\begin{aligned}
g'(x) &= \lim_{h \to 0} \frac{(x + h)^4 + 1 - (x^4 + 1)}{h} \\
&= \lim_{h \to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} \\
&= 4x^3
\end{aligned}$$

(b) Derivative of $(f \circ g)(x)$

Let $y = f(g(x)) = (x^4 + 1)^{1/3}$. Using implicit differentiation:

$$\begin{aligned}
y^3 &= x^4 + 1 \\
3y^2 \frac{dy}{dx} &= 4x^3 \\
\frac{dy}{dx} &= \frac{4x^3}{3(x^4 + 1)^{2/3}}
\end{aligned}$$

$$f \circ g)'(x) = \frac{4x^3}{3(x^4 + 1)^{2/3}}$$

Finding the Roots of a Polynomial Function

Problem Statement

Determine all real roots of the function:

$$f(x) = x^4 + 3x^3 - 18x^2$$

Solution

Step 1: Factorization

We begin by factoring out the greatest common factor:

$$f(x) = x^2(x^2 + 3x - 18)$$

Step 2: Solving the Quadratic Equation

We solve the quadratic equation:

$$x^2 + 3x - 18 = 0$$

Using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 1$, $b = 3$, and $c = -18$.
Calculating the discriminant:

$$\Delta = 3^2 - 4(1)(-18) = 9 + 72 = 81$$

Thus, the solutions are:

$$x = \frac{-3 \pm \sqrt{81}}{2} = \frac{-3 \pm 9}{2}$$

Which gives two roots:

$$x_1 = \frac{-3 + 9}{2} = 3$$

$$x_2 = \frac{-3 - 9}{2} = -6$$

Step 3: Complete Solution

Considering all factors:

- From x^2: $x = 0$ (double root, multiplicity 2)
- From the quadratic: $x = 3$ and $x = -6$ (each with multiplicity 1)

Final Answer

The complete set of real roots is:

$$x = -6, \quad x = 0 \text{ (multiplicity 2)}, \quad x = 3$$

Q2: [7 + 4 + 4 marks]

(1) Using the Definition and Power Rule to Differentiate

Given the function:

$$f(x) = x^3 - 14$$

Using the Power Rule:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^3) - \frac{d}{dx}(14) \\ &= 3x^2 - 0 \\ &= 3x^2 \end{aligned}$$

(2) Differentiate the following function:

$$Y = (x^2 + x + 1)^{-1/5}$$

Using Logarithmic Differentiation:

$$\begin{aligned} \ln Y &= -\frac{1}{5}\ln(x^2 + x + 1) \\ \frac{1}{Y} \cdot \frac{dY}{dx} &= -\frac{1}{5} \cdot \frac{2x+1}{x^2+x+1} \\ \frac{dY}{dx} &= -\frac{2x+1}{5(x^2+x+1)} \cdot (x^2+x+1)^{-1/5} \\ &= -\frac{2x+1}{5(x^2+x+1)^{6/5}} \end{aligned}$$

(3) Given the function:

$$G(x) = (x^4 + 1)^{-3}$$

Let:

$$y = (x^4 + 1)^{-3}$$

Raise both sides to the power $-1/3$:

$$y^{-1/3} = x^4 + 1$$

Differentiate both sides implicitly:

$$-\frac{1}{3}y^{-4/3} \cdot \frac{dy}{dx} = 4x^3$$
$$\frac{dy}{dx} = -12x^3 \cdot y^{4/3}$$

Recall $y = (x^4+1)^{-3}$, so:

$$y^{4/3} = (x^4+1)^{-4}$$

Final answer:

$$\frac{dy}{dx} = -\frac{12x^3}{(x^4+1)^4}$$

Q3: [6 + 5 + 4 marks]

1. Monotonicity and Extrema

Given:

$$f(x) = x^4 - 4x^3 + 10$$

First Derivative

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

Critical Points and Behavior

Interval	Test Point	Sign of $f'(x)$	Behavior
$(-\infty, 0)$	$x = -1$	$-$	Decreasing
$(0, 3)$	$x = 1$	$-$	Decreasing
$(3, \infty)$	$x = 4$	$+$	Increasing

- At $x = 0$: No extremum (no sign change)
- At $x = 3$: Local minimum

$$f(3) = 3^4 - 4(3)^3 + 10 = -17$$

Local minimum at $(3, -17)$.

2. Concavity and Inflection Points

Second Derivative

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

Concavity Analysis

Interval	Test Point	Sign of $f''(x)$	Concavity
$(-\infty, 0)$	$x = -1$	$+$	Up
$(0, 2)$	$x = 1$	$-$	Down
$(2, \infty)$	$x = 3$	$+$	Up

Inflection Points

$$f(0) = 10$$
$$f(2) = -6$$

Inflection points at $(0, 10)$ and $(2, -6)$.

3. Motion Problem

Given position function:

$$S(t) = 5t + 3t^2$$

Acceleration at $t = 2$ seconds

$$v(t) = \frac{dS}{dt} = 5 + 6t$$
$$a(t) = \frac{dv}{dt} = 6 \text{ m/s}^2$$
$$a(2) = 6 \text{ m/s}^2$$

Time to reach 29 m/s

$$5 + 6t = 29$$
$$6t = 24$$
$$t = 4 \text{ seconds}$$

Q4: [5 + 5 + 5 marks]

1. Tangent Line to Exponential Function

Given:

$$f(x) = 4xe^{x^2-1} \quad \text{at} \quad (1, 4)$$

Solution:

Using logarithmic differentiation:

$$\begin{aligned}
\ln y &= \ln(4x) + x^2 - 1 \\
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{x} + 2x \\
\frac{dy}{dx} &= y\left(\frac{1}{x} + 2x\right) \\
f'(1) &= 4(1+2) = 12
\end{aligned}$$

Tangent line equation:

$$y - 4 = 12(x-1)$$

$$\boxed{y = 12x - 8}$$

2. Tangent Line to Hyperbola

Given:

$$xy + 18 = 0 \quad \text{at} \quad P(-2, 9)$$

Solution:

$$\begin{aligned}
y &= -\frac{18}{x} \\
\frac{dy}{dx} &= \lim_{h \to 0} \frac{-\frac{18}{x+h} + \frac{18}{x}}{h} \\
&= \frac{18}{x^2} \\
\left.\frac{dy}{dx}\right|_{x=-2} &= \frac{9}{2}
\end{aligned}$$

Tangent line equation:

$$\begin{aligned}
y - 9 &= \frac{9}{2}(x+2) \\
y &= \frac{9}{2}x + 18
\end{aligned}$$

3. Logarithmic Differentiation

Given:

$$f(x) = \frac{x^3(5 - 3x^2)}{\sqrt[5]{x^3 + 6}}$$

Solution:

$$
\begin{aligned}
\ln f(x) &= 3\ln x + \ln(5-3x^2) - \frac{1}{5}\ln(x^3+6) \\
\frac{f'(x)}{f(x)} &= \frac{3}{x} - \frac{6x}{5-3x^2} - \frac{3x^2}{5(x^3+6)} \\
f'(x) &= \frac{x^3(5-3x^2)}{(x^3+6)^{1/5}}\left(\frac{3}{x} - \frac{6x}{5-3x^2} - \frac{3x^2}{5(x^3+6)}\right)
\end{aligned}
$$

$$
\boxed{f'(x) = \frac{x^3(5-3x^2)}{(x^3+6)^{1/5}}\left(\frac{3}{x} - \frac{6x}{5-3x^2} - \frac{3x^2}{5(x^3+6)}\right)}
$$