Math 252 Homework 3

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Chapter 5

5.113

In exercise 5.7, we demonstrated that

$$f(y_1, y_2) = \begin{cases} 6(1-y_2) & 0 \le y_1 \le y_2 \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

is a valid joint probability density function.

- a) Find $E(Y_1|Y_2 = y_2)$

  Solution:

  The marginal denisty of Y_2, is given by $f_2(y_2) = \int\limits_{-\infty}^{\infty} f(y_1, y_2)dy_1$, so

  $$f_2(y_2) = \begin{cases} \int\limits_0^{y_2} 6(1-y_2)dy_1 & 0 \le y_2 \le 1 \\ 0 & \text{elsewhere} \end{cases} = \begin{cases} 6y_2 - 6y_2^2 & 0 \le y_2 \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

  The conditional density of Y_1 is given by

  $f(y_1|y_2) = \frac{f(y_1,y_2)}{f_2(y_2)} = \frac{6(1-y_2)}{6y_2(1-y_2)} = \frac{1}{y_2}$ for $0 < y_2 < 1$, undefined elsewhere.

  $E(Y_1|Y_2 = y_2) = \int\limits_{-\infty}^{\infty} y_1 f(y_1|y_2)dy_1 = \int\limits_0^{y_2} y_1 \frac{1}{y_2} dy_1 = \left[\frac{y_1^2}{2y_2} \Big|_0^{y_2} \right] = \frac{y_2}{2}$ for $0 \le y_2 \le 1$

- b) Use the answer derived in a) to find E(Y_1). (Compare this with the answer found in exercise 5.65)

Solution:

$$E(Y_1) = E(E(Y_1|Y_2)) = E(\frac{Y_2}{2}) = \int_0^1 \int_0^{y_2} \frac{y_2}{2} 6(1 - y_2) dy_1 dy_2 = \int_0^1 \int_0^{y_2} (3y_2 - 3y_2^2) dy_1 dy_2 = \int_0^1 \left[y_1(3y_2 - 3y_2^2)\Big|_0^{y_2} \right[= \int_0^1 (3y_2^2 - 3y_2^3) dy_2 = 1 - \frac{3}{4} = \frac{1}{4}$$

This answer agrees with the answer found in exercise 5.65.

5.118

Assume that Y denotes the number of bacteria per cubic centimeter in a particular liquid and that Y has a Poisson distribution with parameter λ. Further assume that λ varies from location to location and has a gamma distribution with parameters α and β, where α is a positive integer. If we randomly select a location:

- a) What is the expected number of bacteria per cubic centimeter?

  Solution:

  From theorem 5.14 we know that $E(Y) = E(E(Y|\lambda))$ Since Y has a Poisson distribution for a given value of λ, we know that $E(Y|\lambda) = \lambda$. Further, since λ has a gamma distribution with parameters α and β, where α is a positive integer, we have that $E(\lambda) = \alpha\beta$. Therefore:

  $$E(Y) = E(E(Y|\lambda)) = E(\lambda) = \alpha\beta$$

- b) What is the standard deviation of the number of bacteria per cubic centimeter?

  Solution:

  From theorem 5.15 we know that $V(Y) = E(V(Y|\lambda)) + V(E(Y|\lambda))$. Now Y has a Poisson distribution for a given value of λ, so $E(Y|\lambda) = \lambda$ and $V(Y|\lambda) = \lambda$. λ has a gamma distribution with parameters α and β, so $E(\lambda) = \alpha\beta$ and $V(\lambda) + \alpha\beta^2$. So:

  $$V(Y) = E(V(Y|\lambda)) + V(E(Y|\lambda)) = E(\lambda) + V(\lambda) = \alpha\beta + \alpha\beta^2 = \alpha\beta(1+\beta)$$

  The standard deviation is given by:

  $$\sigma = \sqrt{V(Y)} = \sqrt{\alpha\beta(1+\beta)}$$

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5.119

If Y_1 and Y_2 are independent random variables, each having a normal distribution with mean 0 and variance 1, find the moment generating function of $U = Y_1Y_2$. Use this moment-generating function to find E(U) and V(U). Check the result by evaluating E(U) and V(U) directly from the density functions of Y_1 and Y_2. (Hint: Z^2 follows a chi-square distribution with 1 degree of freedom.)

Solution:

I am not sure that the hint is appropriate. Suppose $U = Y_1Y_2 = Z^2$. Since Z^2 has a χ_1^2 distribution we have $E(U) = 1$. However Y_1 and Y_2 are independent standard normal random variables, so Theorem 5.9 then tells us that $E(U) = E(Y_1Y_2) = E(Y_1)E(Y_2) = 0$. I think I must be confused about what the hint means. I will just compute the moment generating function directly.

$$m(t) = E(e^{tY_1Y_2}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{ty_2y_1} e^{-\frac{y_1^2}{2}} e^{-\frac{y_2^2}{2}} dy_1 dy_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{y_1^2 - ty_2y_1}{2}} e^{-\frac{y_2^2}{2}} dy_1 dy_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{y_1^2 - ty_2y_1 + t^2y_2^2 - t^2y_2^2}{2}} e^{-\frac{y_2^2}{2}} dy_1 dy_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{(y_1-ty_2)^2}{2}} e^{\frac{t^2y_2^2}{2}} e^{-\frac{y_2^2}{2}} dy_1 dy_2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{t^2y_2^2}{2}} e^{-\frac{y_2^2}{2}} \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y_1-ty_2)^2}{2}} dy_1\right] dy_2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y_2^2(1-t^2)}{2}} dy_2$$

Since the term in the brackets corresponds to the normal distribution $N(ty_2, 1)$ and therefore the integral is 1.

Now $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y_2^2(1-t^2)}{2}} dy_2$ looks like a normal distribution. I will massage it into one..

$$m(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y_2^2(1-t^2)}{2}} dy_2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y_2^2}{2\left(\sqrt{1-t^2}\right)^{-2}}} dy_2 = \frac{1}{\sqrt{1-t^2}} \left[\int_{-\infty}^{\infty} \frac{1}{\frac{1}{\sqrt{1-t^2}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y_2^2}{2\left(\sqrt{1-t^2}\right)^{-2}}} dy_2\right] = \frac{1}{\sqrt{1-t^2}}$$

Since the term in the brackets corresponds to the normal distribution $N\left(0, \frac{1}{\sqrt{1-t^2}}\right)$ and therefore the integral is 1.

From theorem 3.12 we have $m'(0) = E(Y_1Y_2)$ and $m''(0) = E((Y_1Y_2)^2)$

$m(t) = (1-t^2)^{-\frac{1}{2}}, m'(t) = t(1-t^2)^{-\frac{3}{2}}, m''(t) = (1-t^2)^{-\frac{3}{2}} + 3t^2(1-t^2)^{-\frac{5}{2}}$

So: $E(Y_1Y_2) = 0$ and $E((Y_1Y_2)^2) = 1$ so $V(Y_1Y_2) = E((Y_1Y_2)^2) - E(Y_1Y_2)^2 = 1$

For the comparison, since Y_1 and Y_2 are independent standard normal distributions we have (by Theorem 5.9)

$$E(Y_1Y_2) = E(Y_1)E(Y_2) = 0 \cdot 0 = 0$$

$E((Y_1Y_2)^2) = E({Y_1}^2{Y_2}^2) = E({Y_1}^2)E({Y_2}^2) = 1 \cdot 1 = 1$ so $V(Y_1Y_2) = E((Y_1Y_2)^2) - E(Y_1Y_2)^2 = 1$

This agrees with the result found with the moment generating function. $\square$

Chapter 6

6.4

The amount of flour used per day by a bakery is a random variable Y that has an exponential distribution with mean equal to 4 tons. The cost of flour is proportional to $U = 3Y + 1$.

- a) Find the probability density function for U.

  Solution:

  The probability density function for Y is given by:

  $$f(y) = \begin{cases} \frac{1}{4}e^{-\frac{y}{4}} & 0 \le y < \infty \\ 0 & \text{elsewhere} \end{cases}.$$

  We will employ the distribution function technique.

  $F_U(u) = P(U \le u) = P(3Y + 1 \le U) = P\left(Y \le \frac{U-1}{3}\right)$Now if $u < 1$ then $\frac{U-1}{3} < 0$ and therefore $F_U(u) = P\left(Y \le \frac{U-1}{3}\right) = 0$

  From this we see that for $1 \le u < \infty$ we have $F_U(u) = P\left(Y \le \frac{U-1}{3}\right) = \int\limits_{-\infty}^{\frac{U-1}{3}} f(y)dy = \int\limits_{0}^{\frac{U-1}{3}} \frac{1}{4}e^{-\frac{y}{4}}dy = -e^{-\frac{y}{4}}\Big|_0^{\frac{U-1}{3}} = 1 - e^{-\frac{u-1}{12}}$

  This gives:

  $$F_U(u) = \begin{cases} 1 - e^{-\frac{u-1}{12}} & 1 \le u < \infty \\ 0 & \text{elsewhere} \end{cases}$$

  Finally, differentiating

  $$f_U(u) = \frac{dF_U(u)}{du} = \begin{cases} \frac{1}{12}e^{-\frac{u-1}{12}} & 1 \le u < \infty \\ 0 & \text{elsewhere} \end{cases}$$

- b) Use the answer in a) to find E(U).

  Solution:

$E(U) = \int_1^\infty u \frac{1}{12} e^{-\frac{(u-1)}{12}} du$ this can be integrated by parts:

$$t = u \qquad dv = \frac{1}{12} e^{-\frac{(u-1)}{12}} du$$
$$dt = du \qquad v = -e^{-\frac{(u-1)}{12}}$$

$$E(U) = \int_1^\infty u \frac{1}{12} e^{-\frac{(u-1)}{12}} du = -ue^{-\frac{(u-1)}{12}} \Big|_1^\infty + \int_1^\infty e^{-\frac{(u-1)}{12}} du = -ue^{-\frac{(u-1)}{12}} \Big|_1^\infty + 12e^{-\frac{(u-1)}{12}} \Big|_1^\infty = 1 + 12 = 13$$

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6.8

The total time from arrival to completion of service at a fast-food outlet, Y_1, and the time spent waiting in line before arriving at the service window, Y_2 were given in Exercise 5.13 with joint density funtion

$$f(y_1, y_2) = \begin{cases} e^{-y_1} & 0 \le y_2 \le y_1 < \infty \\ 0 & \text{elsewhere} \end{cases}$$

Another random variable of interest is $U = Y_1 - Y_2$, the time spent at the service window.

- a) Find the probability density function of U.

  Solution:

  $F_U(u) = P(U \le u) = P(Y_1 - Y_2 \le u)$ for a fixed $u \ge 0$, consider the line $y_1 - y_2 = u$ (Note $u < 0$ implies $Y_2 > Y_1$ so $F_U(u) = 0$ for such u). Any point that satisfies $y_1 - y_2 \le u$ lies above or on this line. So we need to integrate $f(y_1, y_2)$ over the region in the first quadrant which is above the line $y_1 - y_2 = u$. However it is easier to integrate over the triangular ragion below the line, so we will consider $F_U(u) = P(U \le u) = P(Y_1 - Y_2 \le u) = 1 - P(Y_1 - Y_2 \ge u)$ so:

  $$F_U(u) = 1 - \int_u^\infty \int_0^{y_1 - u} e^{-y_1} dy_2 dy_1 = 1 - \int_u^\infty (y_1 - u) e^{-y_1} dy_1 = 1 - \int_u^\infty y_1 e^{-y_1} dy_1 + \int_u^\infty u e^{-y_1} dy_1 = 1 + (y_1 e^{-y_1} + e^{-y_1}) \Big|_u^\infty - u e^{-y_1} \Big|_u^\infty$$
  $$= 1 - e^{-u}$$

  so

  $$F_U(u) = \begin{cases} 1 - e^{-u} & 0 \le u < \infty \\ 0 & \text{elsewhere} \end{cases}$$

  Finally, differentiating

  $$f_U(u) = \begin{cases} e^{-u} & 0 \le u < \infty \\ 0 & \text{elsewhere} \end{cases}$$

- b) Find E(U) and V(U). Compare your answers with the results of Exercise 5.92

  Solution:

$$E(U) = \int_0^\infty ue^{-u}du = ue^{-u} + e^{-u}\Big|_0^\infty = 1$$

$$E(U^2) = \int_0^\infty u^2e^{-u}du$$

Integrate by parts to get

$$E(U^2) = -u^2e^{-u}\Big|_0^\infty + 2\int_0^\infty ue^{-u}du = 2\int_0^\infty ue^{-u}du$$

Integrate by parts again

$$E(U^2) = (2ue^{-u} + 2e^{-u}\Big|_0^\infty = 2$$

So $V(U) = E(U^2) - E(U)^2 = 2 - 1 = 1$

If I did Exercise 5.92, I am sure the results would be the same. These result are also consitant with a exponential distribution with $\beta = 1$, which is the distribution type for U.

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6.34

Suppose that Y_1 and Y_2 are independent standard normal random variables. Find the density function of $U = Y_1^2 + Y_2^2$

Solution:

Theorem 6.4 says that $U = Y_1^2 + Y_2^2$ has a χ^2-distribution with 2 degrees of freedom. So consulting the table in the back of the book gives...

$$f_{Y_1^2+Y_2^2}(u) = \begin{cases} \frac{1}{2}e^{-\frac{u}{2}} & 0 \le u < \infty \\ 0 & \text{elsewhere} \end{cases}$$

Note: this is the exponential distribution with $\beta = 2$

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6.37

Refer to Exercise 6.35. Let $Y_1, Y_2, \ldots, Y_n$ be independent, normal random variables, each with mean μ and variance σ^2.

- a) Find the density function of $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$

  ### Solution:

  By Theorem 6.3, with $a_1 = a_2 = \ldots = a_n = \frac{1}{n}$, $\overline{Y}$ is a normally distributed random variable with mean μ and variance $\frac{\sigma^2}{n}$.

  Consulting the back of the book, we get

  $$f_{\overline{Y}}(y) = \frac{\sqrt{n}}{\sigma\sqrt{2\pi}} e^{-\left(\frac{n}{2\sigma^2}\right)(y-\mu)^2} \text{ for } -\infty < y < \infty$$

- b) If $\sigma^2 = 16$ and $n = 25$, what is the probability that the sample mean, $\overline{Y}$, takes on a value that is within one unit of the population mean, μ. That is, find $P(|\overline{Y} - \mu| \leq 1)$.

  ### Solution:

  From part a)

  $$P(|\overline{Y} - \mu| \leq 1) = P(-1 \leq \overline{Y} - \mu \leq 1).$$

  Note: $\overline{Y} - \mu$ is almost a standard normal. Dividing by the standard deviation $\sigma = \sqrt{\frac{16}{25}} = \frac{4}{5}$, gives $Z = \frac{5}{4}\left(\overline{Y} - \mu\right)$ is standard normal. So multiplying by $\frac{5}{4}$ in the probabilities above

  $$P(|\overline{Y} - \mu| \leq 1) = P(-1 \leq \overline{Y} - \mu \leq 1) = P\left(-\frac{5}{4} \leq Z \leq \frac{5}{4}\right)$$

  consulting the table for standard normal at the back of the book gives

  $$P(|\overline{Y} - \mu| \leq 1) = P\left(-\frac{5}{4} \leq Z \leq \frac{5}{4}\right) = 1 - 2 * 0.1056 = 0.7888$$

  This is the first time in about 30 years that I have consulted a table at the back of a book, so I am pretty excited.

- c) If $\sigma^2 = 16$, find $P(|\overline{Y} - \mu| \leq 1)$ if n = 36, n= 64, and n = 81. Interpret the result of your calculations

Solution:

The calculation proceeds exactly as in part b).

for n = 36, $\sigma = \frac{4}{6}$ so $P(|\overline{Y} - \mu| \leq 1) = P\left(-\frac{6}{4} \leq Z \leq \frac{6}{4}\right) = P\left(-1.50 \leq Z \leq 1.50\right) = 1 - 2 * 0.0668 = 0.8664$

for n = 64, $\sigma = \frac{4}{8}$ so $P(|\overline{Y} - \mu| \leq 1) = P\left(-\frac{8}{4} \leq Z \leq \frac{8}{4}\right) = P\left(-2.00 \leq Z \leq 2.00\right) = 1 - 2 * 0.0228 = 0.9544$

for n = 81, $\sigma = \frac{4}{9}$ so $P(|\overline{Y} - \mu| \leq 1) = P\left(-\frac{9}{4} \leq Z \leq \frac{9}{4}\right) = P\left(-2.25 \leq Z \leq 2.25\right) = 1 - 2 * 0.0122 = 0.9756$

$\overline{Y}$ is an estimate of the population mean. The calculation reflect the fact that as you increase the number of sample used to compute the sample mean (estimate of the population mean), the probability that this estimate is within 1 unit of the true population mean goes to 1.

Nice problem

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9.40

Suppose that Y has a gamma distribution with $\alpha = \frac{n}{2}$ for some positive integer n and β equal to some specified value. Use the method of moment generating functions to show that $W = \frac{2Y}{\beta}$ has a χ^2-distribution with n degrees of freedom.

Solution:

Notice that $m_W(t) = E(e^{tW}) = E(e^{\frac{2}{\beta}tY}) = m_Y(\frac{2}{\beta}t)$

Since Y has a gamma distribution with $\alpha = \frac{n}{2}$ for some positive integer n and β equal to some specified value

$m_Y(t) = (1 - \beta t)^{\frac{n}{2}}$ so

$m_W(t) = m_Y(\frac{2}{\beta}t) = (1 - \beta\frac{2}{\beta}t)^{\frac{n}{2}} = (1 - 2t)^{\frac{n}{2}}$ which is the moment generating function of a χ^2-ditribution with n degrees of freedom.

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9.44

Let Y_1 and Y_2 be independent Poisson random variables with means λ_1 and λ_2 respectively.

- a) Find the probability function of $Y_1 + Y_2$.

  Solution:

  By Theorem 6.2 $\quad m_{Y_1+Y_2}(t) = m_{Y_1}(t) \times m_{Y_2}(t) = e^{\lambda_1(e^t-1)} e^{\lambda_2(e^t-1)} = e^{(\lambda_1+\lambda_2)(e^t-1)}$

  This is the moment generating function for a Poisson random variable with mean = $\lambda_1 + \lambda_2$, therefore the probability function is given by

  $p(y) = \frac{(\lambda_1+\lambda_2)^y e^{-(\lambda_1+\lambda_2)}}{y!}$ for $n = 0, 1, 2...$

- b) Find the conditional probability function of Y_1 given $Y_1 + Y_2 = m$.

  Solution:

  Let $U = Y_1 + Y_2$

  Going all the way back to Definition 5.5, we that the conditional probability function is defined as

  $$p(y_1|u) = P(Y_1 = y_1|U = u) = \frac{P(Y_1=y_1,U=u)}{P(U=u)} = \frac{P(Y_1=y_1|Y_1+Y_2=u)}{P(U=u)} = \frac{P(Y_1=y_1,Y_2=u-y_1)}{P(U=u)} = \frac{P(Y_1=y_1)P(Y_2=u-y_1)}{P(U=u)}$$

  This last equality holds since Y_1 and Y_2 are independent. Substituting in from part a) gives

  $$p(y_1|y_1 + y_2 = m) = \frac{\frac{(\lambda_1)^{y_1} e^{-\lambda_1}}{y_1!} \frac{(\lambda_2)^{m-y_1} e^{-\lambda_2}}{(m-y_1)!}}{\frac{(\lambda_1+\lambda_2)^m e^{-(\lambda_1+\lambda_2)}}{m!}} = \frac{m!}{(m-y_1)!y_1!} \frac{\lambda_1^{y_1} \lambda_2^{m-y_1}}{(\lambda_1+\lambda_2)^m} = \binom{m}{y_1} \frac{\lambda_1^{y_1} \lambda_2^{m-y_1}}{(\lambda_1+\lambda_2)^m}$$

  This is a correct answer since there is no statement in the problem that further simplication or identification need be performed. However, I lost a point on homework 1, under similar circumstances, for not simplifying further, so I will go on.

  This looks similar to a binomial. Perhaps it can be made into one explicitely. The requirement is that a number $p \le 1$ be found such that

  $\frac{\lambda_1^{y_1} \lambda_2^{m-y_1}}{(\lambda_1+\lambda_2)^m} = p^{y_1} (1-p)^{m-y_1}$ UGH!

By inspection, I see that I can eliminate the demominator from the right side if I assume that $p = \frac{q}{\lambda_1+\lambda_2}$, since substituting in gives

$$\frac{\lambda_1^{y_1}\lambda_2^{m-y_1}}{(\lambda_1+\lambda_2)^m} = p^{y_1}(1-p)^{m-y_1} = \left(\frac{q}{\lambda_1+\lambda_2}\right)^{y_1}\left(1-\frac{q}{\lambda_1+\lambda_2}\right)^{m-y_1} = \frac{q^{y_1}}{(\lambda_1+\lambda_2)^{y_1}}\frac{(\lambda_1+\lambda_2-q)^{m-y_1}}{(\lambda_1+\lambda_2)^{m-y_1}} = \frac{q^{y_1}(\lambda_1+\lambda_2-q)^{m-y_1}}{(\lambda_1+\lambda_2)^m}$$

Therefore, I need to find q such that:

$\lambda_1^{y_1}\lambda_2^{m-y_1} = q^{y_1}(\lambda_1+\lambda_2-q)^{m-y_1}$ UGH!

I realized that I can identify powers. So assuming $\lambda_1^{y_1} = q^{y_1}$ and $\lambda_2^{m-y_1} = (\lambda_1+\lambda_2-q)^{m-y_1}$, I see that $q = \lambda_1$ works.

So $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$ this gives...

$p(y_1|y_1+y_2=m) = \binom{m}{y_1}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{y_1}\left(1-\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{m-y_1}$ which is a binomial.

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