Math 253 Homework 1

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Chapter 3

3.115

If Y is a binomial distribution with n trials and probability of success p, show that the moment-generating function for Y is $m(t) = (pe^t + q)^n \text{ where } q = 1 - p$

Solution:

Y is given by $p(k) = \binom{n}{k} p^k q^{n-k}$

so $m(t) = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k q^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k q^{n-k}$

the last term is the binomial expansion of $(pe^t + q)^n$

3.116

Differentiate the moment-generating function in Exercise 3.115 to find $E(Y)$ and $E(Y^2)$. Then find $V(Y)$.

Solution:

$\frac{dm(t)}{dt} = \frac{d}{dt}((pe^t + q)^n) = npe^t(pe^t + q)^{n-1}$

$\frac{d^2m(t)}{dt^2} = \frac{d}{dt}(npe^t(pe^t + q)^{n-1}) = npe^t(pe^t + q)^{n-1} + npe^t * (n-1) * pe^t * (pe^t + q)^{n-2}$

so

$$E(Y) = \frac{dm(t)}{dt}\Big|_{t=0} = np(p+q)^{n-1} = np \text{ since p+q} = 1$$

$$E(Y^2) = \frac{d^2m(t)}{dt^2}\Big|_{t=0} = np(1 + (n-1)p) = np(1 - p + np) = np(q + np) \text{ since } 1 - p = q$$

$$V(Y) = E(Y^2) - E(Y)^2 = np(q + np) - n^2p^2 = npq + n^2p^2 - n^2p^2 = npq$$

To summarize

$E(Y) = np$
$E(Y^2) = np(q + np)$
$V(Y) = npq$

3.117

If Y has a geometric distribution with probability of success p, show that the moment generating function for Y is $m(t) = \frac{pe^t}{1-qe^t}$ where $q = 1 - p$

Solution

Y is given by $p(y) = q^{y-1}p$

so

$$m(t) = \sum_{k=1}^{\infty} e^{tk}q^{k-1}p = \sum_{k=0}^{\infty} e^{t(k+1)}q^k p = \sum_{k=0}^{\infty} e^t e^k q^k p = pe^t \sum_{k=0}^{\infty} (qe^t)^k$$

The final sum, $\sum_{k=0}^{\infty} (qe^t)^k$ is a geometric series which converges to $\frac{1}{1-qe^t}$ if $|qe^t| = qe^t < 1$

To summarize

If $t < ln(\frac{1}{q})$ (this is another way of saying $qe^t < 1$)

then $m(t) = pe^t \sum_{k=0}^{\infty} (qe^t)^k = \frac{pe^t}{1-qe^t}$

3.121

Find the distributions of the random variables that have each of the following moment generating functions.

- a) $m(t) = [\frac{1}{3}e^t + \frac{2}{3}]^5$

  Solution

  By looking at problem 3.115, this generating function corresponds to a binomial distribution with 5 trials and $p = \frac{1}{3}$

  so $p(k) = \binom{5}{k}(\frac{1}{3})^k(\frac{2}{3})^{5-k}$

- b) $m(t) = \frac{e^t}{2-e^t}$

  Solution

  Pulling out a 2 from the bottom, this becomes $m(t) = \frac{\frac{1}{2}e^t}{1-\frac{1}{2}e^t}$

  Comparing this to problem 3.117, this corresponds to a geometric distribution with $p = \frac{1}{2}$

  so $p(k) = q^{k-1}p$

- c) $m(t) = e^{2(e^t-1)}$

  Solution

  From example 3.23, the moment-generating function of a Poisson distibuted random variable with mean λ is

  $m(t) = e^{\lambda(e^t-1}$

  so $m(t) = e^{2(e^t-1)}$ corresponds to a Poisson distribution with $\lambda = 2$

  $p(k) = \frac{2^k e^{-2}}{k!}$

3.123

Let $m(t) = \frac{1}{6}e^t + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t}$. Find the following

- a) $E(Y)$

  Solution

  $E(Y) = \frac{d}{dt}(\frac{1}{6}e^t + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t})\Big|_{t=0} = \frac{1}{6}e^t + \frac{4}{6}e^{2t} + \frac{9}{6}e^{3t}\Big|_{t=0} = \frac{1}{6} + \frac{4}{6} + \frac{9}{6} = \frac{14}{6} = \frac{7}{3}$

- b) $V(Y)$

  Solution

  First need to find $E(Y^2)$

  $E(Y^2) = \frac{d^2}{dt^2}(\frac{1}{6}e^t + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t})\Big|_{t=0} = \frac{1}{6}e^t + \frac{8}{6}e^{2t} + \frac{27}{6}e^{3t}\Big|_{t=0} = \frac{1}{6} + \frac{8}{6} + \frac{27}{6} = \frac{36}{6} = 6$

  $V(Y) = E(Y^2) - E(Y)^2 = 6 - \frac{49}{9} = \frac{5}{9}$

- c) The distribution of Y

  Solution

  Since there is a 1-1 correspondence between probability distributions and moment-generating functions, it is enough to write down any distribution with this moment-generating function

  Consider the random variable defined on the set {1,2,3} with probability distribution $p(1) = \frac{1}{6}, p(2) = \frac{2}{6}, p(3) = \frac{3}{6}$

  A formula for this distribution is $p(k) = \frac{k}{6}$ where $k = 1, 2, 3$

  $m(t) = \sum_{k=1}^{3} e^{tk} k \frac{1}{6} = \frac{1}{6}e^t + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t}$

  To summarize

  $p(k) = \frac{k}{6}$ for $k = 1, 2, 3$

Chapter 4

4.22

The proportion of time per day that all checkout counters in a supermarket are busy is a random variable Y with density function

$$f(y) = \begin{cases} cy^2(1-y)^4 & 0 <= y <= 1 \\ 0 & \text{elsewhere} \end{cases}$$

- a) Find the value of c that makes f(y) a probability density function

  Solution

  Need to find c such $\int_{-\infty}^{\infty} f(y)dy = \int_0^1 cy^2(1-y)^4 dy = 1$

  Note: the binomial theorem gives $(1-y)^4 = 1 - 4y + 6y^2 - 4y^3 + y^4$ so

  $\int_0^1 cy^2(1-y)^4 dy = \int_0^1 cy^2(1 - 4y + 6y^2 - 4y^3 + y^4)dy = c\int_0^1 (y^2 - 4y^3 + 6y^4 - 4y^5 + y^6)dy =$
  $c(\frac{1}{3}y^3 - y^4 + \frac{6}{5}y^5 - \frac{4}{6}y^6 + \frac{1}{7}y^7)\Big|_0^1 = c(\frac{1}{3} - 1 + \frac{6}{5} - \frac{4}{6} + \frac{1}{7}) = \frac{c}{105}$ so $\frac{c}{105} = 1 \to c = 105$

  To summarize

  $c = 105$

- b) Find $E(Y)$

  Solution

  $E(Y) = \int_0^1 y * 105y^2(1-y)^4 dy = \int_0^1 105y^3(1-y)^4 dy = \int_0^1 105y^3(1 - 4y + 6y^2 - 4y^3 + y^4)dy =$
  $\int_0^1 105(y^3 - 4y^4 + 6y^5 - 4y^6 + y^7)dy = 105(\frac{1}{4}y^4 - \frac{4}{5}y^5 + y^6 - \frac{4}{7}y^7 + \frac{1}{8}y^8)\Big|_0^1 = 105(\frac{1}{4} - \frac{4}{5} + 1 - \frac{4}{7} + \frac{1}{8}) = \frac{105}{280} = \frac{3}{8}$

  To summarize

  $E(Y) = \frac{3}{8}$

4.89

Suppose that Y has a gamma distribution with parameters $\alpha \text{ and } \beta$.

- a) If a is any positive or negative value such that $\alpha + a > 0$. Show that $E(Y^a) = \frac{\beta^a \Gamma(\alpha+a)}{\Gamma(\alpha)}$

  Solution

  $E(Y^a) = \int_0^\infty y^a \frac{y^{\alpha-1}e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)} = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty y^a y^{\alpha-1} e^{-y/\beta} = \frac{\beta^{a+\alpha}\Gamma(a+\alpha)}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty \frac{1}{\beta^{a+\alpha}\Gamma(a+\alpha)} y^{a+\alpha-1} e^{-y/\beta} = \frac{\beta^a \beta^\alpha \Gamma(a+\alpha)}{\beta^\alpha \Gamma(\alpha)} = \frac{\beta^a \Gamma(a+\alpha)}{\Gamma(\alpha)}$

  Since
  $(*)\ \int_0^\infty \frac{1}{\beta^{a+\alpha}\Gamma(a+\alpha)} y^{a+\alpha-1} e^{-y/\beta} = 1$ because it is the gamma distribution with parameters $a + \alpha \text{ and } \beta$

- b) Why did your answer in a) require that $\alpha + a > 0$.

  Solution

  $\alpha + a > 0$ is needed for $(*)$ to be true. The gamma distribution is only defined for strictly positive parameters.

- c) Show that with $a = 1$, the result in a) gives $E(Y) = \alpha\beta$

  Solution

  If $a = 1$, then from part a) $E(Y) = \frac{\beta\Gamma(\alpha+1)}{\Gamma(\alpha)}$

  Using the property $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ this gives

  $E(Y) = \frac{\beta\Gamma(\alpha+1)}{\Gamma(\alpha)} = \frac{\beta\alpha\Gamma(\alpha)}{\Gamma(\alpha)} = \beta\alpha$

- d) Use the result in a) to give an expression for $E(\sqrt{Y})$. What do you need to assume about α?

  Solution

  Let $a = \frac{1}{2}$ in part a, then $E(\sqrt{Y}) = \frac{\sqrt{\beta}\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha)}$

  The definition of gamma distribution assumes that $\alpha > 0$

The assumption for part a) to apply is that $\alpha + \frac{1}{2} > 0$, this is certainly true for any $\alpha > 0$.

This expression works for any $\alpha > 0$.

- e) Use the result in a) to give an expression for $E\left(Y^{-1}\right), E\left(Y^{-\frac{1}{2}}\right), E\left(Y^{-2}\right)$. What do you need to assume about α in each case?

Solution

$$E\left(Y^{-1}\right) = \frac{\frac{1}{\beta}\Gamma(\alpha-1)}{\Gamma(\alpha)} \quad \text{where } \alpha - 1 > 0 \Rightarrow \alpha > 0$$

$$E\left(Y^{-\frac{1}{2}}\right) = \frac{\frac{1}{\sqrt{\beta}}\Gamma\left(\alpha-\frac{1}{2}\right)}{\Gamma(\alpha)} \quad \text{where } \alpha - \frac{1}{2} > 0 \Rightarrow \alpha > \frac{1}{2}$$

$$E\left(Y^{-2}\right) = \frac{\frac{1}{\beta^2}\Gamma(\alpha-2)}{\Gamma(\alpha)} \quad \text{where } \alpha - 2 > 0 \Rightarrow \alpha > 2$$

4.108

Identify the distributions with the following moment-generating functions

- a) $m(t) = (1-4t)^{-2}$

  Solution

  From example 4.13, we know that the moment-generating function of a gamma distribution with parameters α and β is given by $m(t) = (1-\beta t)^{-\alpha}$.

  So $m(t) = (1-4t)^{-2}$ is the moment-generating function of a gamma distribution function with parameters $\alpha = 2$ and $\beta = 4$

  $$f(y) = \frac{ye^{-y/4}}{16} \quad \text{where } 0 < y < \infty$$

- b) $m(t) = (1-3.2t)^{-1}$

  Solution

  Just as in problem a), $m(t) = (1-3.2t)^{-1}$ is the moment-generating function of a gamma distribution function with parameters $\alpha = 1$ and $\beta = 3.2$

  Note: When $\alpha = 1$, the gamma distribution reduces to exponential distribution with parameter $\beta = 3.2$

  $$f(y) = \frac{e^{-y/3.2}}{3.2} \quad \text{where } 0 < y < \infty$$

- c) $m(t) = e^{-5t+6t^2}$

  Solution

  the moment-generating function of a normal distribution with mean μ and standard deviation σ is given by $m(t) = e^{\mu t + \frac{\sigma^2}{2}t^2}$.

  By comparison we see that $m(t) = e^{-5t+6t^2}$ is the moment-generating function of a normal distribution with $\mu = -5$ and $\sigma = \sqrt{12}$

  $$f(y) = \frac{1}{\sqrt{24\pi}} e^{\left(-\frac{1}{24}(y+5)^2\right)} \quad \text{where } -\infty < y < \infty$$

4.118

Find $P(|Y-\mu| \leq 2\sigma)$ for the exponential random variable. Compare with the corresponding probabilistic statements given by Tchebysheff's theorem and the empirical rule.

Solution

The distribution for an exponential random variable is $f(y) = \frac{1}{\beta}e^{-\frac{y}{\beta}} \quad \text{where } 0 < y < \infty$

In this case

$(*)\ \mu = \beta \text{ and } \sigma = \beta$

$P(|Y-\mu| \leq 2\sigma) = P(\mu - 2\sigma \leq Y \leq \mu + 2\sigma) = P(-\beta \leq Y \leq 3\beta)$ by $(*)$

so, since the exponential distribution is zero for $y \leq 0$

$$P(|Y-\mu| \leq 2\sigma) = \int_{-\beta}^{3\beta} \frac{1}{\beta}e^{-\frac{y}{\beta}} = \int_{0}^{3\beta} \frac{1}{\beta}e^{-\frac{y}{\beta}} = -e^{-\frac{y}{\beta}}\Big|_0^{3\beta} = -e^{-\frac{3\beta}{\beta}} + e^0 = 1 - e^{-3} = 0.9502$$

Comparisons

Tchebysheff's theorem gives $P(|Y-\mu| \leq 2\sigma) \geq 1 - \frac{1}{4} = 0.75$ which is not a very good bound

The emperical says that the $P(|Y-\mu| \leq 2\sigma) \approx .95$ which agrees with the calculated value.