Math 253 Homework 2

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Chapter 5

5.1

Contracts for two construction jobs are randomly assigned to one or more of three firms, A, B, and C. Let Y_1 denote the number of contracts asigned to firm A, and Y_2 the number of contracts assigned to firm B. Recall the each firm can recieve 0, 1, or 2 contracts.

- a) Find the joint probability function of Y_1 and Y_2

  $\textbf{Solution:}$

  I am going to think of this as the contract choose a firm. Then contract 1 can choose A,B,C and contract 2 can choose A,B,C. With this choice the sample space is $(A,A),(A,B),(A,C),(B,A),(B,B),(B,C),(C,A),(C,B),(C,C)$

  Now I will just write down the joint probability function explicitely.

  $$\begin{array}{lll} p(0,0)=\frac{1}{9} & p(0,1)=\frac{2}{9} & p(0,2)=\frac{1}{9} \\ p(1,0)=\frac{2}{9} & p(1,1)=\frac{2}{9} & p(1,2)=0 \\ p(2,0)=\frac{1}{9} & p(2,1)=0 & p(2,2)=0 \end{array}$$

- b) Find F(1,0)

  $\textbf{Solution:}$

  $F(1,0)=P(Y_1\leq 1,Y_2=0)=p(0,0)+p(1,0)=\frac{1}{3}$

5.5

Refer to example 5.4. The joint denisty of Y_1, the proportion of the capacity of the tank that is stocked at the beginning of the week, and Y_2, the proportion of the capacity sold during the week, is given by

$$f(y_1, y_2) = \begin{cases} 3y_1 & 0 \le y_2 \le y_1 \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

- a) Find $F\left(\frac{1}{2}, \frac{1}{3}\right) = P\left(Y_1 \le \frac{1}{2}, Y_2 \le \frac{1}{3}\right)$

  Solution:

  I will break the region in the $y_1 y_2$-plane into two regions:

  Region 1 $= \left(y_1 \le \frac{1}{3}, y_2 \le 3y_1\right)$ and Region 2 $= \left(\frac{1}{3} \le y_1 \le \frac{1}{2}, y_2 \le \frac{1}{3}\right)$

  $$F\left(\frac{1}{2}, \frac{1}{3}\right) = \int_0^{\frac{1}{3}} \int_0^{y_1} 3y_1 dy_2 dy_1 + \int_{\frac{1}{3}}^{\frac{1}{2}} \int_0^{\frac{1}{3}} 3y_1 dy_2 dy_1 = \int_0^{\frac{1}{3}} 3y_1 \left[y_2\Big|_0^{y_1}\right] dy_1 + \int_{\frac{1}{3}}^{\frac{1}{2}} 3y_1 \left[y_2\Big|_0^{\frac{1}{3}}\right] dy_1 = \int_0^{\frac{1}{3}} 3y_1^2 dy_1 + \int_{\frac{1}{3}}^{\frac{1}{2}} y_1 dy_1 = y_1^3\Big|_0^{\frac{1}{3}} + \frac{1}{2} y_1^2\Big|_{\frac{1}{3}}^{\frac{1}{2}}$$

  $$= 0.1064815$$

- b) Find $P\left(Y_2 \le \frac{Y_1}{2}\right)$, the probability that the amount sold is less than half the amount purchased.

  Solution:

  $$P\left(Y_2 \le \frac{Y_2}{2}\right) = \int_0^1 \int_0^{\frac{y_1}{2}} 3y_1 dy_2 dy_1 = \int_0^1 3y_1 \left[y_2\Big|_0^{\frac{y_1}{2}}\right] dy_1 = \int_0^1 \frac{3}{2} y_1^2 dy_1 = \frac{1}{2} y_1^3\Big|_0^1 = \frac{1}{2}$$

5.17

In exercise 5.1 we determined that the joint distribution of Y_1, the number of contracts awarded to firm A, and Y_2, the number of contracts awarded to firm B, is given by the entries in the following table.

$$\begin{array}{lll} p(0,0)=\frac{1}{9} & p(0,1)=\frac{2}{9} & p(0,2)=\frac{1}{9} \\ p(1,0)=\frac{2}{9} & p(1,1)=\frac{2}{9} & p(1,2)=0 \\ p(2,0)=\frac{1}{9} & p(2,1)=0 & p(2,2)=0 \end{array}$$

- a) Find the marginal probability distribution of Y_1

  Solution:

  Add the columns.

  $p_1(0)=\frac{4}{9} \quad p_1(1)=\frac{4}{9} \quad p_1(2)=\frac{1}{9}$

- b) According to result in Chapter 4, Y_1 has a binomial distribution with $n=2$ and $p=\frac{1}{3}$. Is there any conflict between this result and the answer you provided in a).

  Solution:

  The binomial distribution is given by $\sum_{k=0}^{2}\binom{2}{k}\left(\frac{1}{3}\right)^k\left(1-\frac{1}{3}\right)^{2-k}$.

  From this

  $p(0)=\binom{2}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^2=\frac{4}{9}$

  $p(1)=\binom{2}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^1=\frac{4}{9}$

  $p(2)=\binom{2}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^0=\frac{1}{9}$

  There is no conflict.

5.21

In example 5.4 and exercise 5.5 we considered the joint density of Y_1, the proportion of the capacity of the tank that is stocked at the beginning of the week, and Y_2, the proportion of the capacity sold during the week, given by

$$f(y_1, y_2) = \begin{cases} 3y_1 & 0 \leq y_2 \leq y_1 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

- a) Find the marginal density function for Y_2

  Solution:

  $$f_2(y_2) = \begin{cases} \int\limits_{y_2}^{1} 3y_1 dy_1 & 0 \leq y_2 \leq y_1 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

  $\int\limits_{y_2}^{1} 3y_1 dy_1 = \frac{3}{2} y_1^2 \Big|_{y_2}^{1} = \frac{3}{2} - \frac{3}{2} y_2^2$ Notice that this function is 0 at $y_2 = 1$. This will give the strict inequality in the result below.

  $$f_2(y_2) = \begin{cases} \frac{3}{2} - \frac{3}{2} y_2^2 & 0 \leq y_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

- b) For what values of y_2 is the conditional density $f(y_1|y_2)$ defined?

  Solution:

  By Definition 5.7 $f(y_1|y_2) = \frac{f(y_1,y_2)}{f_2(y_2)}$ This will be defined when $f_2(y_2) \neq 0$

  From part a) of this problem, this occurs when $0 \leq y_2 < 1$

  Summary:

  $f(y_1|y_2)$ is defined when $0 \leq y_2 < 1$

- c) What is the probability that more than half a tank is sold given that three-fourths of a tank is stocked?

  Solution:

We need to find $P\left(Y_1 = \frac{3}{4} | Y_2 \geq \frac{1}{2}\right)$

$f(y_2|y_1) = \frac{f(y_1,y_2)}{f_1(y_1)} = \frac{3y_1}{f_1(y_1)}$

So I need to find the marginal density function $f_1(y_1)$

$$f_1(y_1) = \begin{cases} \int_0^{y_1} 3y_1 dy_2 & 0 \leq y_2 \leq y_1 \leq 1 \\ 0 & \text{elsewhere} \end{cases} = \begin{cases} 3y_1^2 & 0 \leq y_2 \leq y_1 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

$f(y_2|y_1) = \frac{3y_1}{3y_1^2} = \frac{1}{y_1}$ Note: $y_2 \leq y_1$ and $\frac{1}{2} \leq y_2$ implies $\frac{1}{2} \leq y_2 \leq \frac{3}{4}$

$$P\left(Y_1 = \frac{3}{4} | Y_2 \geq \frac{1}{2}\right) = \int_{\frac{1}{2}}^{\frac{3}{4}} f(y_2|y_1 = \frac{3}{4}) dy_2 = \int_{\frac{1}{2}}^{\frac{3}{4}} \frac{1}{\frac{3}{4}} dy_2 = 1 - \frac{2}{3} = \frac{1}{3}$$

5.39

In exercise 5.1 we determined that the joint distribution of Y_1, the number of contracts awarded to firm A, and Y_2, the number of contracts awarded to firm B, is given by the entries in the following table.

$$\begin{array}{lll} p(0,0)=\frac{1}{9} & p(0,1)=\frac{2}{9} & p(0,2)=\frac{1}{9} \\ p(1,0)=\frac{2}{9} & p(1,1)=\frac{2}{9} & p(1,2)=0 \\ p(2,0)=\frac{1}{9} & p(2,1)=0 & p(2,2)=0 \end{array}$$

The marginal probability function of Y_1 was derived in Exercise 5.17 to be binomial with $n=2 \text{ and } p=\frac{1}{3}$.
Are $Y_1 \text{ and } Y_2$ independent? Why?

Solution:

By the symmetry in the table $p_1(y_1)=p_2(y_2)$

so

$$p_1(0)=p_2(0)=\frac{4}{9} \quad p_1(1)=p_2(1)=\frac{4}{9} \quad p_1(2)=p_2(2)=\frac{1}{9}$$

Y_1 and Y_2 are independent if and only if $p(y_1,y_2)=p_1(y_1)p_2(y_2)$

This is not the case. Just plugging in $y_1=y_2=0$ gives:

$$(*) \quad p(0,0)=\frac{1}{9} \neq p_1(0)p_2(0)=\frac{4}{9}\cdot\frac{4}{9}=\frac{16}{81}$$

Summary:

$Y_1 \text{ and } Y_2$ are not independent.

Why? $(*)$ above.

The "intuitive" answer here is: awarding a contract to firm A, reduces the number of contracts that can be awarded to firm B and vice versa. There is a dependency between the two.

5.65

In exercise 5.7, we determined that

$$f(y_1, y_2) = \begin{cases} 6(1-y_2) & 0 \le y_1 \le y_2 \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

is a valid probability distribution function.

- a) Find $E(Y_1)$ and $E(Y_2)$.

  Solution:

  $$E(Y_1) = \int_0^1 \int_0^{y_2} 6y_1(1-y_2)dy_1dy_2 = \int_0^1 \left[(3y_1^2 - 3y_1^2y_2)\Big|_0^{y_2}\right] dy_2 = \int_0^1 3y_2^2 - 3y_2^3dy_2 = (y_2^3 - \tfrac{3}{4}y_2^4)\Big|_0^1 = \tfrac{1}{4}$$

  $$E(Y_2) = \int_0^1 \int_0^{y_2} 6y_2(1-y_2)dy_1dy_2 = \int_0^1 \left[(6y_1y_2 - 6y_2^2y_1)\Big|_0^{y_2}\right] dy_2 = \int_0^1 6y_2^2 - 6y_2^3dy_2 = (2y_2^3 - \tfrac{3}{2}y_2^4)\Big|_0^1 = \tfrac{1}{2}$$

  Summary:

  $E(Y_1) = \frac{1}{4}$

  $E(Y_2) = \frac{1}{2}$

- b) Find $V(Y_1)$ and $V(Y_2)$.

  Solution:

  $$E({Y_1}^2) = \int_0^1 \int_0^{y_2} (6y_1^2 - 6y_1^2y_2)dy_1dy_2 = \int_0^1 \left[(2y_1^3 - 2y_1^3y_2)\Big|_0^{y_2}\right] dy_2 = \int_0^1 2y_2^3 - 2y_2^4dy_2 = (\tfrac{1}{2}y_2^4 - \tfrac{2}{5}y_2^5)\Big|_0^1 = \tfrac{1}{10}$$

  $$E({Y_2}^2) = \int_0^1 \int_0^{y_2} (6y_2^2 - 6y_2^3)dy_1dy_2 = \int_0^1 \left[(6y_2^2y_1 - 6y_2^3y_1)\Big|_0^{y_2}\right] dy_2 = \int_0^1 6y_2^3 - 6y_2^4dy_2 = (\tfrac{6}{4}y_2^4 - \tfrac{6}{5}y_2^5)\Big|_0^1 = \tfrac{3}{10}$$

  $$V(Y_1) = E({Y_1}^2) - E(Y_1)^2 = \tfrac{1}{10} - (\tfrac{1}{4})^2 = \tfrac{8}{80} - \tfrac{5}{80} = \tfrac{3}{80}$$

$V(Y_2) = E(Y_2^2) - E(Y_2)^2 = \frac{3}{10} - (\frac{1}{2})^2 = \frac{6}{20} - \frac{5}{20} = \frac{1}{20}$

Summary:

$V(Y_1) = \frac{3}{80}$

$V(Y_2) = \frac{1}{20}$

- c) Find $E(Y_1 - 3Y_2)$.

Solution:

By the linearity of the expectation, $E(Y_1 - 3Y_2) = E(Y_1) - 3E(Y_2) = \frac{1}{4} - \frac{3}{2} = -\frac{5}{4}$

5.75

In exercise 5.1 we determined that the joint distribution of Y_1, the number of contracts awarded to firm A, and Y_2, the number of contracts awarded to firm B, is given by the entries in the following table.

$$\begin{array}{lll} p(0,0)=\frac{1}{9} & p(0,1)=\frac{2}{9} & p(0,2)=\frac{1}{9} \\ p(1,0)=\frac{2}{9} & p(1,1)=\frac{2}{9} & p(1,2)=0 \\ p(2,0)=\frac{1}{9} & p(2,1)=0 & p(2,2)=0 \end{array}$$

Find $Cov(Y_1, Y_2)$. Does it surprise you that $Cov(Y_1, Y_2)$ is negative? Why?

Solution:

$$Cov(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2)$$

$E(Y_1 Y_2) = \sum_{y_1} \sum_{y_2} y_1 y_2 p(y_1, y_2) = p(1,1) = \frac{2}{9}$ All the other terms in the sum are zero.

The marginal probability functions are given by

$$p_1(0) = p_2(0) = \frac{4}{9} \quad p_1(1) = p_2(1) = \frac{4}{9} \quad p_1(2) = p_2(2) = \frac{1}{9}$$

so

$$E(Y_1) = \sum_{y_1} y_1 p_1(y_1) = \frac{4}{9} + \frac{2}{9} = \frac{2}{3}$$

$$E(Y_2) = \sum_{y_2} y_2 p_2(y_2) = \frac{4}{9} + \frac{2}{9} = \frac{2}{3}$$

$$Cov(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2) = \frac{2}{9} - \frac{2}{3} \cdot \frac{2}{3} = -\frac{2}{9}$$

No, I am not surprised that the covariance is negative. There is an inverse relationship between these two random variables; as Y_1 gets larger, Y_2 get smaller. Such a inverse dependence is represented by a negative covariance.

5.78

In exercise 5.7, we determined that

$$f(y_1, y_2) = \begin{cases} 6(1 - y_2) & 0 \le y_1 \le y_2 \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

is a valid probability distribution function. Find $Cov(Y_1, Y_2)$. Are Y_1 and Y_2 independent?

Solution:

$Cov(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2)$

$E(Y_1 Y_2) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} y_1 y_2 f(y_1, y_2) dy_1 dy_2 = \int\limits_{0}^{1}\int\limits_{0}^{y_2} y_1 y_2 \cdot 6(1 - y_2) dy_1 dy_2 = \int\limits_{0}^{1}\int\limits_{0}^{y_2} (6y_1 y_2 - 6y_1 y_2^2) dy_1 dy_2 = \int\limits_{0}^{1} (3y_2^3 - 3y_2^4) dy_2 = \frac{3}{4} - \frac{3}{5} = \frac{3}{20}$

From problem 5.65

$E(Y_1) = \frac{1}{4}$

$E(Y_2) = \frac{1}{2}$

$Cov(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2) = \frac{3}{20} - \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{40}$

Y_1 and Y_2 are not independent sinve the covariance is not zero.

5.90

In exercise 5.7, we determined that

$$f(y_1, y_2) = \begin{cases} 6(1-y_2) & 0 \le y_1 \le y_2 \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

is a valid probability distribution function. In exercise 5.65, we derived the fact that $E(Y_1 - 3Y_2) = -\frac{5}{4}$. In exercise 5.78 we proved that $Cov(Y_1, Y_2) = \frac{1}{40}$. Find $V(Y_1 - 3Y_2)$.

Solution:

From problem 5.65

$V(Y_1) = \frac{3}{80}$

$V(Y_2) = \frac{1}{20}$

By theorem 5.12:

$V(Y_1 - 3Y_2) = V(Y_1) + 9V(Y_2) - 6Cov(Y_1, Y_2) = \frac{3}{80} + 9 \cdot \frac{1}{20} - 6 \cdot \frac{1}{40} = \frac{27}{80}$.